MINISTRY OF EDUCATION

# ADDITIONAL MATHEMATICS

## For Senior High Schools

TEACHER MANUAL

YEAR 1 - BOOK 2

NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

# MINISTRY OF EDUCATION

REPUBLIC OF GHANA

# Additional Mathematics

## For Senior High Schools

## Teacher Manual

### Year One - Book Two

NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

**ADDITIONAL MATHEMATICS TEACHER MANUAL**

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# INTRODUCTION

The National Council for Curriculum and Assessment (NaCCA) has developed a new Senior High School (SHS), Senior High Technical School (SHTS) and Science, Technology, Engineering and Mathematics (STEM) Curriculum. It aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, competencies, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, further their education and enter the world of work.

This is the first time that Ghana has developed an SHS Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

This Book Two of the Teacher Manual for Additional Mathematics covers all aspects of the content, pedagogy, teaching and learning resources and assessment required to effectively teach Year One of the new curriculum. It contains information for the second 12 weeks of Year One. Teachers are therefore to use this Teacher Manual to develop their weekly Learning Plans as required by Ghana Education Service.

Some of the key features of the new curriculum are set out below.

### Learner-Centred Curriculum

The SHS, SHTS, and STEM curriculum places the learner at the center of teaching and learning by building on their existing life experiences, knowledge and understanding. Learners are actively involved in the knowledge-creation process, with the teacher acting as a facilitator. This involves using interactive and practical teaching and learning methods, as well as the learner's environment to make learning exciting and relatable. As an example, the new curriculum focuses on Ghanaian culture, Ghanaian history, and Ghanaian geography so that learners first understand their home and surroundings before extending their knowledge globally.

### Promoting Ghanaian Values

Shared Ghanaian values have been integrated into the curriculum to ensure that all young people understand what it means to be a responsible Ghanaian citizen. These values include truth, integrity, diversity, equity, self-directed learning, self-confidence, adaptability and resourcefulness, leadership and responsible citizenship.

### Integrating 21st Century Skills and Competencies

The SHS, SHTS, and STEM curriculum integrates 21st Century skills and competencies. These are:

- Foundational Knowledge: Literacy, Numeracy, Scientific Literacy, Information Communication and Digital Literacy, Financial Literacy and Entrepreneurship, Cultural Identity, Civic Literacy and Global Citizenship
- Competencies: Critical Thinking and Problem Solving, Innovation and Creativity, Collaboration and Communication
- Character Qualities: Discipline and Integrity, Self-Directed Learning, Self-Confidence, Adaptability and Resourcefulness, Leadership and Responsible Citizenship

### Balanced Approach to Assessment - not just Final External Examinations

The SHS, SHTS, and STEM curriculum promotes a balanced approach to assessment. It encourages varied and differentiated assessments such as project work, practical demonstration, performance assessment, skills-based assessment, class exercises, portfolios as well as end-of-term examinations and final external assessment examinations. Two levels of assessment are used. These are:

- Internal Assessment (30%) – Comprises formative (portfolios, performance and project work) and summative (end-of-term examinations) which will be recorded in a school-based transcript.
- External Assessment (70%) – Comprehensive summative assessment will be conducted by the West African Examinations Council (WAEC) through the WASSCE. The questions posed by WAEC will test critical thinking, communication and problem solving as well as knowledge, understanding and factual recall.

The split of external and internal assessment will remain at 70/30 as is currently the case. However, there will be far greater transparency and quality assurance of the 30% of marks which are school-based. This will be achieved through the introduction of a school-based transcript, setting out all marks which learners achieve from SHS 1 to SHS 3. This transcript will be presented to universities alongside the WASSCE certificate for tertiary admissions.

## An Inclusive and Responsive Curriculum

The SHS, SHTS, and STEM curriculum ensures no learner is left behind, and this is achieved through the following:

- Addressing the needs of all learners, including those requiring additional support or with special needs. The SHS, SHTS, and STEM curriculum includes learners with disabilities by adapting teaching and learning materials into accessible formats through technology and other measures to meet the needs of learners with disabilities.
- Incorporating strategies and measures, such as differentiation and adaptative pedagogies ensuring equitable access to resources and opportunities for all learners.
- Challenging traditional gender, cultural, or social stereotypes and encouraging all learners to achieve their true potential.
- Making provision for the needs of gifted and talented learners in schools.

## Social and Emotional Learning

Social and emotional learning skills have also been integrated into the curriculum to help learners to develop and acquire skills, attitudes, and knowledge essential for understanding and managing their emotions, building healthy relationships and making responsible decisions.

## Philosophy and vision for each subject

Each subject now has its own philosophy and vision, which sets out why the subject is being taught and how it will contribute to national development. The Philosophy and Vision for Additional Mathematics is:

**Philosophy**: Learners can develop their potential in Additional Mathematics through creative, innovative and interactive ways to become lifelong learners, apply mathematical skills and competencies to solve everyday problems, further their education to read mathematics-related courses and/or proceed to the world of work.

**Vision**: Learners enthusiastic about mathematics, capable of reasoning (quantitatively and abstractly), modelling, representing and using mathematical skills, tools and technology to solve real life problems, further their studies and/or proceed to the world of work.

# SUMMARY SCOPE AND SEQUENCE

| S/N | STRAND | SUB-STRAND | YEAR 1 | | | YEAR 2 | | | YEAR 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | CS | LO | LI | CS | LO | LI | CS | LO | LI |
| 1. | Modelling with Algebra | Number and Algebraic Patterns | 2 | 4 | 13 | 2 | 7 | 8 | - | - | - |
| | | Applications of Algebra | 2 | 8 | 21 | 2 | 6 | 19 | 2 | 2 | 8 |
| 2. | Geometric Reasoning and Measurement | Spatial Reasoning | 2 | 4 | 11 | 2 | 2 | 8 | 1 | 4 | 11 |
| | | Measurement of Triangles | 1 | 2 | 6 | 1 | 2 | 4 | 2 | 1 | 4 |
| 3. | Calculus | Principles of Calculus | 1 | 1 | 6 | 2 | 2 | 9 | 1 | 1 | 4 |
| | | Applications of Calculus | 1 | 1 | 2 | 1 | 1 | 2 | 1 | | |
| 4. | Handling data | Organising, Representing and Interpreting Data | 1 | 2 | 8 | 2 | 2 | 4 | 1 | 3 | 9 |
| | | Making Predictions with Data | 1 | 2 | 7 | 2 | 2 | 6 | 1 | 2 | 6 |
| **Total** | | | **11** | **24** | **73** | **14** | **22** | **60** | **9** | **14** | **46** |

**Overall Totals (SHS 1 – 3)**

| | |
|---|---|
| Content Standards | **33** |
| Learning Outcomes | **61** |
| Learning Indicators | **180** |

# SECTION 5: STRAIGHT LINES

Strand: **Geometric Reasoning and Measurement**

**Sub-Strand:** Spatial Reasoning

**Content Standard:** Demonstrate knowledge and understanding in spatial sense in relation to lines and angles between intersecting lines

**Learning Outcomes:**

1. *State the properties of lines including parallel, perpendicular and midpoints.*
2. *Derive the equation of a line in various forms, find the shortest distance between a point and a line and the perpendicular distance from an external point to a line.*
3. *Solve problems on acute angles between two intersecting lines.*

## INTRODUCTION AND SECTION SUMMARY

Have you ever wondered how engineers design bridges, artists create perfect symmetry or even how your phone uses GPS? The answer lies in lines! In this section, we will explore the concept of straight lines. First, we will explore the fundamental properties of lines, understanding what makes a line unique and how different lines can interact. We will then learn how to divide a line into specific parts, a crucial skill for tasks like scaling distances on a map. Next, we will move on to a powerful tool: the equation of a line. This equation acts like a secret code, allowing us to represent and manipulate lines mathematically. We will use these equations to understand the relationships between parallel and perpendicular lines, two fundamental concepts used in everything from building construction to creating geometric patterns. We will then tackle the challenge of measuring the distance between a point and a line. Finally, we will explore the concept of acute angles between two intersecting lines. This seemingly simple task has applications in fields like navigation and designing efficient pathways. By mastering these concepts, you will gain a deeper appreciation for the role lines play in our world.

*The weeks covered by the section are:*

**Week 13:**

1. Properties of lines
2. Division of a line
3. Equation of straight lines
4. Equation of parallel and perpendicular lines
5. Distance between a point and line
6. Interior angles of a triangle
7. Acute angles between two intersecting lines

## SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities in recalling and establishing the concepts of straight lines, properties of straight lines and all the focal areas understudy. Learners should be given the platform to work in groups to develop their real-life questions and find answers. Therefore, Experiential learning activities and Mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities, should be assisted to fully take part in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations and use visual aids to deepen understanding. Then, extend activities for the above average/highly proficient learners to equations, interior angles and acute angles between two points to further deepen comprehension and computer applications to solve problems.

## ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests and homework assignments should be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving properties of straight lines, internal and external division of straight lines, equations of straight, parallel and perpendicular lines, distance between a point and a straight line, interior angles of a triangle as well as the acute angles between two intersecting lines, will also be used to assess learner's application of these mathematical concepts. Teaching and learning materials like straight rules, meter rules etc. will also be incorporated to engage learners in hands-on learning experiences. Use animations or short video clips to visually demonstrate how changing the slope in the equation of a line creates parallel or perpendicular lines. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

# Week 13

**Learning Indicators:**

1. *Describe the properties of lines including parallel, perpendicular and midpoints.*
2. *Work out the midpoint of a line segment given two points and find the generalization of the midpoint of a line segment.*
3. *Apply the knowledge of ratio to divide a line segment in a given ratio either internally or externally.*
4. *Recall the formula for finding the gradient of a line and apply it to find the equation of a straight line in various forms*
5. *Use standard algebraic manipulations to find the equation of parallel and perpendicular lines including the equation of perpendicular bisector of a line*
6. *Deduce the shortest distance between a point and a line and use the knowledge of intercepts and right-angled triangles to find the perpendicular distance from an external point to a line.*
7. *Determine the acute angles between two intersecting lines with the aid of technological tools e.g. GeoGebra*

## Theme or Focal Area: Properties of lines

Straight lines, types and their characteristics

1. A straight line is formed when two points are joined with the shortest distance, it can be extended forever in both directions.
2. A straight line has no curves within
3. A straight line is one-dimensional and has no width
4. Types of straight lines are; horizontal lines, vertical lines, slanted lines, parallel lines and perpendicular lines.

Parallel and perpendicular lines, their characteristics

1. Parallel lines are two or more straight lines that are always the same distance (equidistant) apart. They never intersect. Examples are Railway tracks, edges of a ruler, zebra crossing (parallel white lines)
2. Perpendicular lines are two lines that intersect, but all the angles at that intersection are the same, that is 90°. For example, "T" junctions on roads, corners of a football pitch etc.

**Note:**

1. Distances are always positive.
2. Distance can only be zero if the points coincide.
3. The distance from $P$ to $Q$ is the same as the distance from $Q$ to $P$

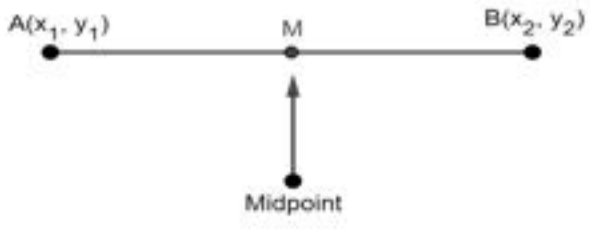
Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$,


the distance between $A$ and $B$, $|AB|$, is given by;

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

## Theme or Focal Area: Finding the Midpoint of a line segment

The midpoint $M$ of the line joining points A$(x_1, y_1)$ and B $(x_2, y_2)$




is given by;

$M = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$

**Examples**

1. Find the length $|AB|$ of the line segment joining the points $A(1, 2)$ and $B(4, 6)$

**Solution**

The length or distance of the line joining points $A$ and $B$ is given by;

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

That is $A(x_1 = 1, x_2 = 2)$ and $B(y_1 = 4, y_2 = 6)$. Thus, by substitution,

$|AB| = \sqrt{(4 - 1)^2 + (6 - 2)^2}$

=5

2. Find the midpoint $M$ between the points $P(4, 3)$ and $Q(12, 5)$

**Solution**

The midpoint $M$ of the line joining points $P$ and $Q$, is given by;

$$M = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$$
$$= \left(\frac{1}{2}(4 + 12), \frac{1}{2}(3 + 5)\right)$$
$$= (8, 4)$$

3. M is the midpoint of $AB$. If the coordinates of $A$ is $(-5, 4)$ and $M$ is $(-2, 1)$, find the co-ordinates of $B$.

**Solution**

Given that $M(-2,1)$ is the midpoint of $\overline{AB}$ and $A$ is located at $(-5, 4)$ from the midpoint formula, $x$ and $y$ coordinates of $B$ can be expressed as:

$x_B = 2(x_M) - x_A$

$y_B = 2(y_M) - y_A$

$x_B = 2(-2) - (-5) = 1$

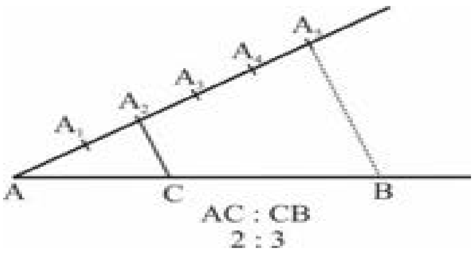
$y_B = 2(1) - 4 = -2$

Therefore, the coordinates of $B$ are $(1, -2)$

## Theme or Focal Area: Division of a line

Dividing a line is all about cutting it into specific parts. We can divide a line segment (a line with two endpoints) into equal parts or unequal parts based on a certain ratio. Imagine a line segment like a ruler. We can divide this line into parts, but there are two ways to think about it:

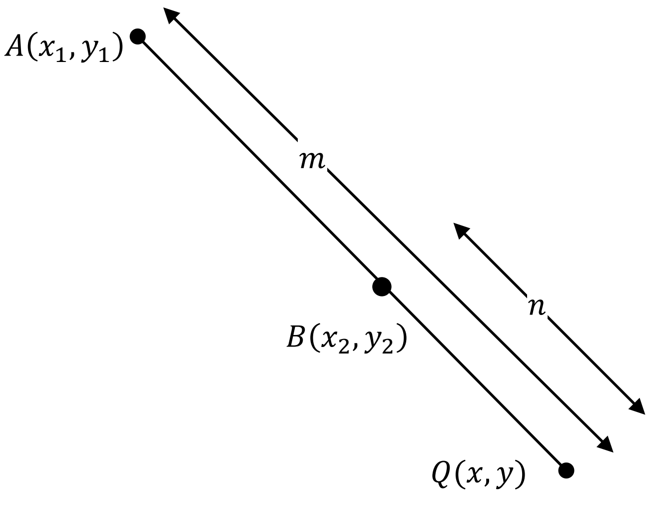
1. **Internal Division:** We divide the line segment itself into two parts by placing a point somewhere between the two endpoints. This point creates a ratio between the two resulting segments. For example, dividing the line in half creates a 1:1 ratio which is the same as finding the midpoint we discussed earlier. Suppose you are asked to divide a line segment $AB$ in the ratio 2:3, it means that you find a point, say $C$ such that $AC:CB = 2:3$ and that $AC:CB = 2:3$ is the same as $\frac{AC}{BC} = \frac{2}{3}$




The coordinates $C$, which divides the line segment $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$ is given by:

$$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

2. **External Division:** Here, the dividing point falls outside the original line segment. It essentially extends the line virtually in one direction and places the dividing point on that extension. This point also creates a ratio but it describes the lengths relative to the original segment not parts within it.




The coordinates $Q$, which divides the line segment $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ is given by:

$$Q = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Understanding both internal and external division of lines is important in various applications like scaling distances on maps or solving geometric problems. We'll explore these concepts further and learn how to calculate the location of dividing points based on desired ratios.

**Examples**

1. Find the coordinates of the point that divides the lines segment $(-4, 3)$ and (6, -12) in the ratio 3: 2,

   i. Internally

   ii. Externally

**Solution**

**i.** for internal division,

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) = \left(\frac{3(6)+2(-4)}{3+2}, \frac{3(-12)+2(3)}{3+2}\right)$$

$$= (2, -6)$$

**ii.** for external division,

$$\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} = \left(\frac{3(6)-2(-4)}{3+2}, \frac{3(-12)-2(3)}{3-2}\right)$$

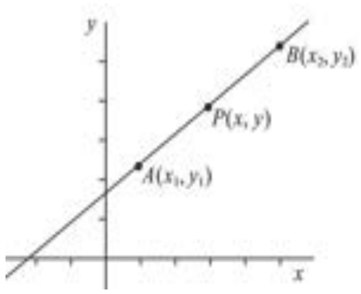
$$= (26, -42)$$

## Theme or Focal Area: Equation of straight lines

In section 3, week 8, we discussed linear functions and established that the graphs of linear functions are straight lines. Recall that the formula for finding the gradient of a line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} or \frac{y_1 - y_2}{x_1 - x_2}$$

If $A(x_1, y_1)$ and $B(x_2, y_2)$, then given any arbitrary point $P(x, y)$ on the line $AB$




Given that $P$ lies on the same line $\overline{AB}$, then the gradient of the line $\overline{AB}$ is equal to the gradient of the line $\overline{AP}$, that is $M_{AB} = M_{AP}$

Then algebraically, we will have $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$ from $M_{AB} = M_{AP}$

Making $y - y_1$ the subject,

$$y - y_1 = (x - x_1) \cdot \frac{y_2 - y_1}{x_2 - x_1}$$

From

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We will have

$$y - y_1 = m\,(x - x_1)$$

Also,

$$y - y_1 = m(x - x_1)$$

$\Rightarrow y - y_1 = mx - mx_1$

$\Rightarrow y = mx - mx_1 + y_1$

Since $m, x_1$ and $y_1$ are constant, $-mx_1 + y_1$ results in a constant which can be represented by $c$ and thus,

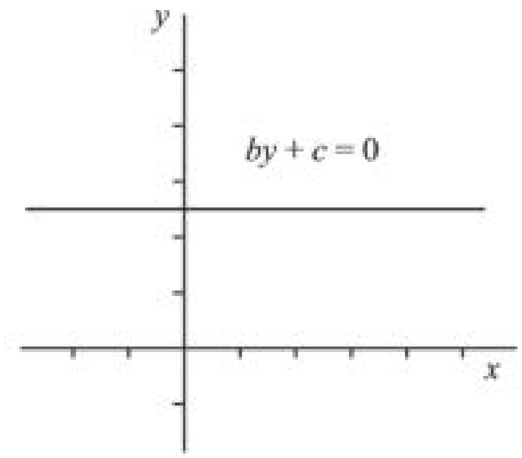
$y = mx + c$

Any of the equations above can be manipulated into the general form $ax + by + c = 0$.

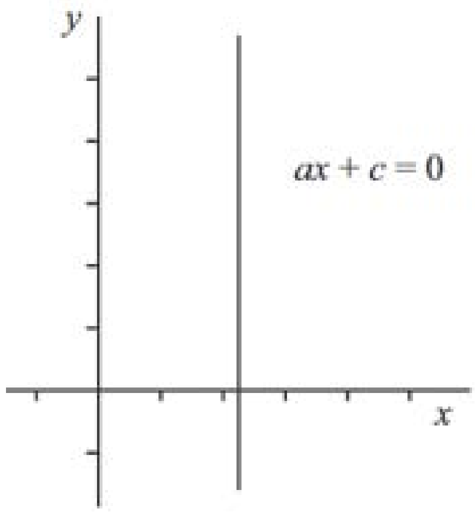
Example, given $y = 3x - 1$, subtracting $3x$ from both sides results in $y - 3x + 1 = 0$

Investigate what happens to the general equation of a line when $a$ or $b$ is zero.

- When $a = 0$, $by + c = 0 \therefore y = \frac{-c}{b}$, these lines are horizontal and parallel to the $x - axis$




- When $b = 0$, $ax + c = 0 \therefore x = \frac{-c}{a}$, these lines are vertical and parallel to the $y - axis$




**NOTE:** the equation of a vertical line cannot be written in the form $y = mx + c$.
The equation $ax + by + c = 0$ is the most general equation for a straight line and can be used where other forms of equation are not suitable.

**Examples**

Find the equation of the line that passes through the points $(-2, 4)$ and $(4, 8)$, leaving your final answer in the form $ax + by + c = 0$.

**Solution**

Using $m = \frac{y_2 - y_1}{x_2 - x_1} or \frac{y_1 - y_2}{x_1 - x_2}$ to find the gradient of the line as $m = \frac{8-4}{4-(-2)} = \frac{2}{3}$

From $y - y_1 = m(x - x_1)$ or $y - y_2 = m(x - x_2)$ using coordinates $(-2,4)$

$y - 4 = \frac{2}{3}(x + 2)$

making $y$ the subject

$y = \frac{2}{3}x + \frac{16}{3}$ multiplying through by 3

$2x - 3y + 16 = 0$

## Theme or Focal Area: Equation of parallel and perpendicular lines

Two lines are perpendicular when they meet at a right angle, therefore, when the slope of one line is $m$, then the slope of the perpendicular line will be $\frac{-1}{m}$ or if two lines are perpendicular then the product of their gradients, $m \times -\frac{1}{m}$ is $-1$

Two lines are said to be parallel when they have the same gradient.

**Example 1**

Find the gradient of $\overline{AB}$ and $\overline{PQ}$ if $A(2, 3)$, $B(5, 6)$ ,$P(-1, 4)$ and $Q(5, 10)$ and state your observations

**Solution**

Gradient of line $AB = \frac{6-3}{5-2} = 1$

Gradient of line $PQ = \frac{10-4}{5-(-1)} = 1$

Observation: Since the gradient of lines $\overline{AB}$ and $\overline{PQ}$ are the same, the two lines are parallel.

**Example 2**

Find the equation of the line that is parallel to the line $y = -2x + 6$ and passes through the point $A(1, 10)$.

**Solution**

The equation of the line given is already in slope-intercept form $(y = mx + c)$, where $m$ is the slope and c is the y-intercept.

Therefore, the slope $(m)$ of the given line is $-2$. Parallel lines have the same slope. So, the line parallel to the given line will also have a slope of $-2$.

The point-slope form is:

$y - y_1 = m(x - x_1)$

$y - (10) = -2(x - 1)$

Simplifying the equation:

$y - 10 = -2x + 2$

$y = -2x + 12$

Therefore, the equation of the line parallel to $y = -2x + 6$ and passing through point $A(1, 10)$ is $y = -2x + 12$.

**Example 3**

Given that line $\overline{AB}$ is such that $A(-1,-1)$ and $B(0,4)$ and $\overline{PQ}$ is such that $P(6,1)$ and $Q(-4,3)$, show that $\overline{AB}$ is perpendicular to $\overline{PQ}$.

**Solution**

Gradient of line $AB = \frac{4-(-1)}{0-(-1)} = 5$

Gradient of line $PQ = \frac{1-3}{6-(-4)} = \frac{-1}{5}$

Observation: Since the product of the gradients of lines $AB$ and $PQ$ i.e., $5 \times \frac{-1}{5} = -1$, that is $m_{AB} \times m_{PQ} = -1$, $AB$ is perpendicular to $PQ$

## Theme or Focal Area: Distance between a point and line

Investigate the shortest distance between a point and a line, as well as the shortest distance between two lines. Establish that the shortest distance between a line and a point is the perpendicular distance, $D$.

**Theorem:** The shortest distance (or the perpendicular distance), $D$, between the point $P\left(x_1, y_1\right)$ and the line $L: ax + by + c = 0$ is given by;

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

**Example**

Find the shortest distance between the perpendicular line drawn from the point $A(1,9)$ to the straight line $-5x + 12y + 13 = 0$.

**Solution**

The shortest distance between a line and a point is the perpendicular distance, $D$, expressed as:

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

From the question $x_1 = 1$, $y_1 = 9$, $a = -5$, $b = 12$ and $c = 13$.

substituting these values into the formula gives

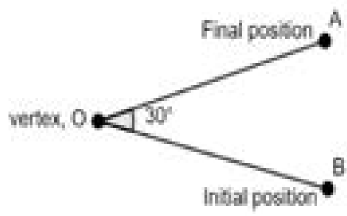
$$D = \frac{|-5(1) + 12(9) + 13|}{\sqrt{5^2 + 12^2}} = \frac{116}{13} = \frac{29}{3} = 9.667\ unit$$

## Theme or Focal Area: Interior angles of a triangle

In geometry, a triangle is a closed, three-sided polygon. Each of the three sides meet at a point called a vertex and the angles formed between these sides within the triangle are called interior angles. Every triangle has exactly three interior angles. The sum of the measures of the three interior angles of any triangle is always equal to $180^o$. This holds regardless of the shape or size of the triangle. Based on the measure of their interior angles, triangles can be classified into three categories:

- **Acute Triangle:** All three interior angles are less than 90 degrees (acute).
- **Right Triangle**: One interior angle is exactly 90 degrees (right angle).
- **Obtuse Triangle:** One interior angle is greater than 90 degrees (obtuse).

Common (mostly used) acute angles (30°,45°, 60° and 75°)




**Example 1**

How many acute angles are found in an acute triangle?

**Solution**

An acute triangle **by definition** has all three angles measuring less than 90° (acute). So, an acute triangle has **3 acute angles**.

**Example 2**

How many acute angles are in an obtuse triangle?

**Solution**

There are two acute angles in an obtuse triangle.

## Theme or Focal Area: Acute angles between two intersecting lines

When dealing with two straight lines, given by equations $y = m_1 x + c_1$ and $y = m_2 x + c_2$, to find their intersection point, we equate the two equations: $m_1 x + c_1 = m_2 x + c_2$

Solve for $x$ to determine the x-coordinate of the intersection point.

Once $x$ is found, substitute it into either of the original equations to obtain the corresponding $y$-coordinate.

**Example**

Given two lines $y = 2x + 3$ and $y = -3x + 5$ find their point of intersection.

**Solution**

**a.** Equate the two equations: $2x + 3 = -3x + 5$

**b.** Solve for $x : 5x = 2 \Rightarrow x = \frac{2}{5}$

**c.** Substitute $x = \frac{2}{5}$ into either of the equations give $\frac{19}{5}$

**d.** Therefore, the point of intersection is $\left(\frac{2}{5}, \frac{19}{5}\right)$

Given that $m_1$ and $m_2$ are gradients of two non-parallel lines such that $m_1 m_2 \neq -1$, then, the acute angle in between these two lines is given by;

$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}$$

**Examples**

**1.** Determine the acute angle between two straight lines having slopes of 5 and $\frac{1}{4}$. Leave your answer to two decimal places.

**Solution :**

$$tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$tan\theta = \frac{5 - \frac{1}{4}}{1 + 5\left(\frac{1}{4}\right)}$$

$tan\theta = \frac{19}{9}$

$\theta = \tan^{-1}\left(\frac{19}{9}\right)$

$\theta = 64.65°$

2. Find the acute angle formed between the lines $y = 7x - 3$ and $y = 2x + 4$

**Solution**

Gradient of line $y = 7x - 3$ is $m_1 = 7$

Gradient of line $y = 2x + 4$ is $m_2 = 2$

The acute angle between two lines with gradient $m_1$ and $m_2$ is given by

$tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \rightarrow tan\theta = \frac{7-2}{1+7(2)} = \theta = \tan^{-1}\left(\frac{5}{15}\right) = 18.435^o$

**Learning Tasks**

Guide learners through tasks aimed at reviewing, understanding and defining properties of straight line, division of lines and equations of straight lines.

Guide learners to:

1. compare acute angles to other types of angles.
2. Verify sizes of acute angles using manipulatives (square cut out)
3. explore common acute angles (30°,45°, 60° and 75°) using conventional methods and ICT tools (where applicable) to ascertain the size of the angle.
    i. angles between two lines
    ii. acute angles in triangles

## Pedagogical Exemplars

The objective of the lessons for week thirteen (13) is for all learners to be able to recall the properties of straight lines, explore the concepts of division of lines and discuss other concepts such as equations of straight, parallel, as well as perpendicular lines. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Collaborative Learning**

    Learners will be working in convenient groups (ability, mixed ability, mixed gender, pairs etc) to:

    a. explore the properties of lines and various ways lines are divided.

    b. discuss the equation of straight lines, parallel lines and perpendicular lines using participatory activities such as **think-pair-share/square and debate**.

    c. create problems involving the distance between a point and line and find the interior angles of a triangle and the acute angle between two intersecting lines using participatory activities such as **think-pair-share/square and debate**

2. **Experiential Learning**

    Learners engage in hands-on activity (learning by doing) to

    a. create/draw straight lines, parallel lines, perpendicular lines, distance between a point and line, interior angles of a triangle and acute angles between two intersecting lines

b. discuss equations of straight lines, parallel lines and perpendicular lines

3. **Enquiry based learning**

   Learners use research resources (textbooks, electronic devices and any additional relevant resources) to deepen their understanding lines, equations of lines and perpendicular and parallel lines, the concept of distance, interior angels and acute angles between two intersecting lines

## Key Assessment

**Assessment Level 1: Recall**

1. Find the midpoint of $A(2, 6)$ and $B(4, 8)$ [**Expected answer:** $\boldsymbol{M}(3, 7)$]
2. What is the midpoint between $P(3, 3)$ and $Q(2, 8)$ $\left[\textbf{Expected answer}: \boldsymbol{M}\left(\frac{5}{2}, \frac{11}{2}\right)\right]$
3. What is the midpoint, $M(x, y)$ the following pair of points?
   i. $P(4, 6)$ and $Q(12, 6)$ [**Expected answer:** $\boldsymbol{M}(8, 6)$
   ii. $R(3,3)$ and $S(2,8)$ $\left[\textbf{Expected answer}: M\left(\frac{5}{2}, \frac{11}{2}\right)\right]$
4. Find the equation of a line passing through the point $(6, -2)$ and has gradient $\frac{-1}{4}$.
   $\left[\textbf{Expected answer}: \boldsymbol{y} = -\frac{1}{4}\boldsymbol{x} - \frac{1}{2}\right]$
5. Find the perpendicular distance from the line $3x + 4y - 5 = 0$ to a point $(5, 1)$
   $\left[\textbf{Expected answer}: \boldsymbol{d} = \frac{3}{5}\right]$
6. Describe any two (2) acute angles [**Expected answer:** ***Any angle between*** $0^o, 90^o$**]**
7. How many acute angles does a right triangle have? [Expected answer: two]
8. Find the coordinates of the point that divides the lines segment $(-1, 2)$ and $(3, 4)$ in the ratio 3: 2,
   i. Internally [**Expected Ans.** $(1.4, 3.2)$]
   ii. Externally. [**Expected Ans.** $(11, 8)$]
9. Find the coordinates of the point that divides the line segment $A(3,1)$ and $B(3, -5)$ in the ratio $2:1$ internally and externally. [**Expected Ans.** *Internally* $(3, -3)$, *Externally* $3, -11$]
10. A line segment AB has a length of 12 cm. Point P divides AB internally in a ratio of 2:3. Find the lengths of segments AP and PB. [**Expected answer:** $AP = 4.8$cm and $PB = 7.2\ cm$]

**Assessment Level 2: Skills of conceptual understanding**

1. $P$, $Q$ and $R$ are the points $(3, 5)$, $(7, 2)$ and $(3, -1)$ respectively. What type of triangle is $\Delta PQR$? [**Expected answer:** Scalene triangle]
2. An engineer is designing a suspension bridge that spans two river banks. The coordinates of the feet of the bridge on the river banks are $A(3,5)$ and $B(9,5)$. Predict the coordinates of the midpoint of the river where the main support pillar will be anchored. [**Expected answer:** $M(6, 5)$
3. $M(-1, 3)$ and $N(6, -11)$ are two points. $P$ is a point on $\overline{MN}$ such that $|MP| : |NP| = 3 : 4$. Find the coordinates of $P$. [**Expected answer:** $\boldsymbol{P}\ (\mathbf{2, -3})$]
4. Find the equation of the line which passes through the point $(1, 3)$ and is perpendicular to the line whose equation is $y = 2x + 1$. $\left[\textbf{Expected answer}: y = -\frac{1}{2}x + \frac{7}{2} \quad or\ x + 2y - 7 = 0\right]$

5. Find the equation of the perpendicular bisector of $AB$, where $A$ and $B$ are the points $(-4, 8)$ and $(0, -2)$ **Expected answer:** $\left[2 - 5y + 19 = 0 \; or \; y = \frac{2}{5}x + \frac{19}{5}\right]$
6. Find the measure of the acute angle between the lines $y = 2x + 3$ and $y = -0.5x - 3$

   **Expected answer:** $90^o$
7. Find the equation of the line passing through the following points

   a. $A(-2, 7)$ and $B(2, -3)$ [**Expected answer**: $5x + 2y - 9 = 0$

   b. $P(-5, 2)$ and $Q(3, 4)$ [**Expected answer:** $x - 4y + 13 = 0$ ]

**Assessment Level 3: Strategic reasoning**

1. Find the coordinates of the point that divides the segment $(2a, 3b)$ to $(12a, -17k)$ in the ratio $3:2$;

   i. Internally $\left[\textbf{Expected answer}: \left(8a, \frac{-51k + 6b}{5}\right)\right]$

   ii. Externally[***Expected answer***: $(32a, -51k - 6b)$ ]
2. Find the gradient of lines $AB$ and $PQ$, If $A(-1, -1)$, $B(0, 4)$ $P(-4, 3)$, $Q(6, 1)$ and state your observation.

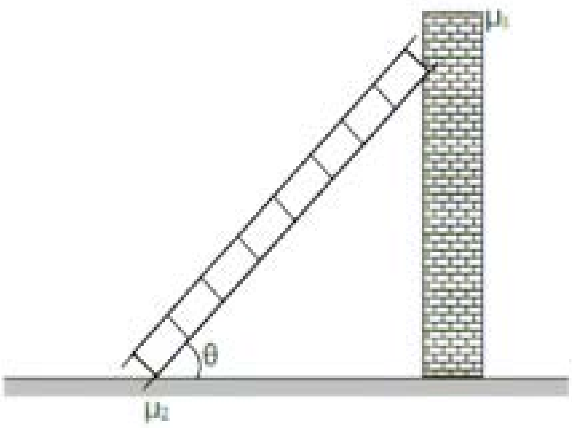
   **Expected answer**: Gradient of $\overline{AB} = 5$, Gradient of $\overline{PQ} = -\frac{1}{5}$, $\overline{AB}$ and $\overline{PQ}$ are perpendicular to each other]
3. $M$ is the intersection of the lines $4x + 5y = 9$ and $x + 2y = 3$. Find

   a. The co-ordinates of $M$ [**Expected answer:** $M(1, 1)$]

   b. The equation of the line through M and is parallel to the $3x = 4y - 7$ $\left[\textbf{Expected answer}: y = \frac{3}{4}x + \frac{1}{4}\right]$
4. The equation of a line passing through the points $A(4, 2)$ and $B(-8, -2)$ is $3y = ax + b$, where $a$ and $b$ are constants. Find the value of $a$ and $b$. [**Expected answer:** $a = \frac{1}{3}, b = \frac{2}{3}$

**Assessment Level 4: Extended thinking**

1. Investigate the behaviour of the acute angles formed when a ladder leaning against a wall is lowered towards the ground or moved upwards towards the wall.




2. Given that the equations of two lines $L$ and $K$ are such that $L$: $tx - y - 3 = 0$, where $t \in \mathbb{R}$ and $K$: $x - 2y - 1 = 0$, the angle between $L$ and $K$ is 45°, find the two possible values of $t$. [**Expected answer:** $t = 0$ *or* $t = 2$]

**3.** Show whether the following three points form the vertices of a right-angle triangle.

a. $A(-4, 3)$, $B(-7, -1)$ and $C(3, -2)$ **[Expected answer:** Points A, B, and C does not form the vertices of a right-angle triangle]

b. $X(-2, 9)$, $Y(-8, 1)$ and $Z(10, 0)$ **[Expected answer:** Points X, Y, and Z does not form the vertices of a right-angle triangle]

**4.** Find the length of the perpendicular line drawn from the point $A(-1, -7)$ to the straight line passing through the points $B(6, -4)$ and $C(9, -5)$. [**Expected Ans.** 2.26]

## Section 5 Review

This section reviews all the lessons taught in section five (5) which consists of week thirteen (13). Section 5 delved into concepts and properties of straight lines which served as the foundation for all the geometric reasoning and measurement strands. This is a summary of what the learner should have learnt. The week began with a recall of the properties of straight lines. It went further to discuss various types of division of lines vis-a-vis internal and external division of lines. It explored the equations of straight, parallel and perpendicular lines, distance between a point and line, interior angles of a triangle and concluded with acute angles.

### Teaching/Learning Resources

Maths posters, Whiteboard Pan balance, Videos, Mini whiteboards or laminated white paper, Dry erase markers and erasers, straight rules and meter rules, Calculator, Traditional Patterns, technological tools such as computer, mobile phone, YouTube videos etc.

### References


**1.** Aufmann, R. N., Barker, V. C., & Nation, R. D. (2011). College algebra and trigonometry. Cengage Learning.

**2.** Baffour, A. (2018). Elective Mathematics for schools and colleges. Baffour Ba Series. SBN: P0002417952

**3.** Lial, M. L., Hornsby, E. J., & McGinnis, T. (2012). Algebra for college students. (7th Ed. Pearson Education, Inc)

**4.** Spiegel, M. R. & Moyar, R. E. (1998). Schaum's outline of theory and problems of college algebra. (2nd Ed. McGraw-Hill).

**5.** Stewart, J., Redlin, L., Watson, S., & Panman, P. (2009). Precalculus Mathematics for Calculus Brooks/Cole Cengage Learning, 1062.

# SECTION 6: VECTORS

Strand: **Geometric Reasoning and Measurement**

**Sub-Strand:** Spatial Reasoning

**Content Standard:** Demonstrate knowledge and understanding of spatial sense relating to vectors in two dimensions and perform algebraic operations on vectors and their geometrical interpretations

**Learning Outcome:** *Perform algebraic manipulations of vectors and resolve vectors using the triangle, parallelogram and polygon laws of addition*

## INTRODUCTION AND SECTION SUMMARY

In this section, we will discuss Vectors. Vectors form a prerequisite knowledge for motion, force, and geometry. The discussion will focus on concepts of vectors as well as the algebraic and geometric operations of vectors which includes resolving vectors and polygon laws of addition. Throughout the discussion, we will discuss Vectors not just mathematical abstractions, but real-world tools with diverse applications. The concepts of vectors are applied in many fields such as physics, engineering and computer science. The geometric interpretation of vectors helps in visualising the behaviour and relationship of vectors. Vectors are mathematical quantities that represent both magnitude and direction, where arrows denote more than just direction on a map and magnitude isn't just a number, but a force to reckon.

*The weeks covered by the section are:*

**Week 14:**

1. Vector forms and Types of vectors
2. Algebraic and Geometric Operations on vectors

## SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on vectors. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, collaborative learning, experiential learning and initiate talk for learning, project-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

## ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests, and homework assignments can be used to evaluate learners' understanding of concepts and their ability to solve problems. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

# Week 14

**Learning Indicators:**

1. *Recognise and explain various forms of vectors and apply the knowledge to find unit vectors.*
2. *Perform algebraic and graphical operations (addition, subtraction, scalar multiplication) and their geometrical interpretation.*
3. *Determine the resultant of vectors using triangle and parallelogram laws of addition.*

## Theme or Focal Area: Forms and types of vectors

In the physical world, certain quantities can be defined mathematically with a single number, which represents their magnitude or size. Mass, volume, distance, and temperature are some examples of such quantities. However, there are many other quantities that require both **magnitude and direction** to be fully described. These quantities are represented mathematically by vectors. For instance, when you exert a force on an object, the direction in which you apply the force is critical. If you push a car forward, backwards, or sideways, you will get different results. Hence, force is an example of a vector quantity. We can represent a vector in many ways. The vector that represents the movement from point $P$ to point $Q$ can be represented graphically as shown in Figure 1, with $\overrightarrow{PQ}$ or $u$.

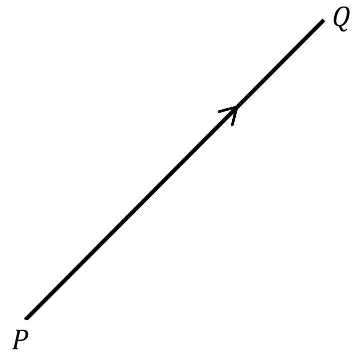



**Figure 1**

The magnitude of a vector is the length or distance of the vector. The Pythagorean theorem $x^2 + y^2 = z^2$ is used to calculate the magnitude of a vector.

**Types of vectors**

- **Position Vector:** A position vector is defined as a vector that indicates the location of any given point concerning any arbitrary reference point like the origin.
- **Free Vectors:** Vectors whose initial point are not located at the origin
- **Unit Vector:** A vector whose magnitude is 1 and is commonly used to indicate the direction of a vector. A unit vector is obtained by dividing the vector by its magnitude, i.e.,$\hat{x} = \frac{\vec{x}}{\|\vec{x}\|}$

- **Co-initial Vectors:** Two or more vectors with the same initial point

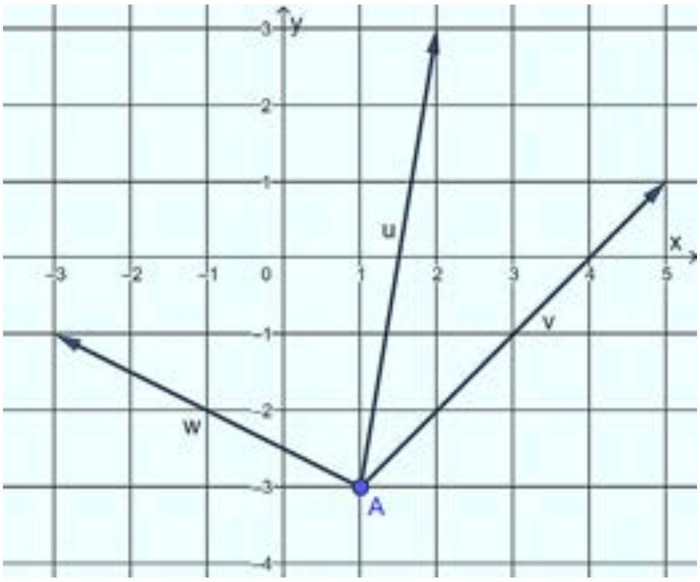



**Figure 2:** *Co-initial Vectors*

From Figure 2: Co-initial Vectors, vectors $w$, $u$ and $v$ share the same initial point, i.e. $A$ and hence are regarded as co-initial

- **Collinear Vectors:** Two or more vectors parallel to the same line irrespective of their magnitude and direction.

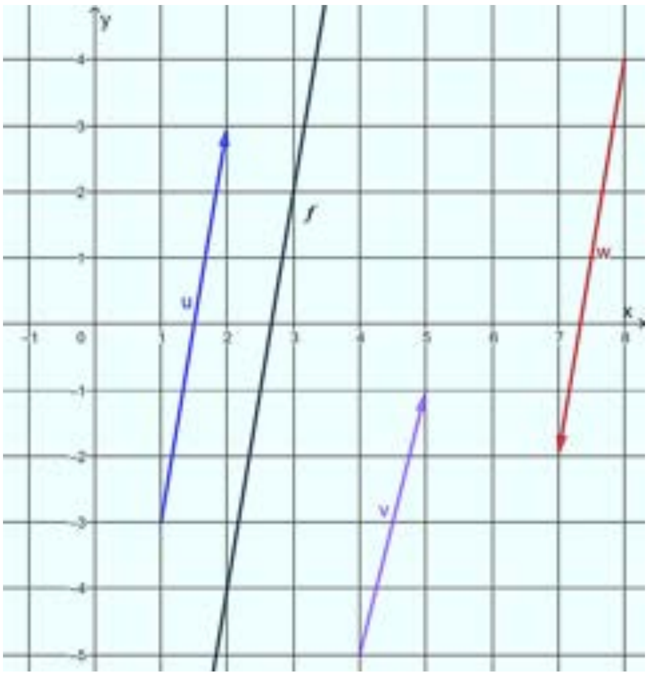



**Figure 3:** *Collinear Vectors*

The vectors $u$, $v$ and $w$ are said to be collinear vectors since they are parallel to the line, $f$. It can be observed that while $u$ and $v$ have the same direction, $w$ has an opposite direction.

- **Equal Vectors:** Two vectors with the same magnitude and direction

- **Negative Vector:** The negative of a vector, say $u$, is the vector say $v$, that has the same magnitude but a different direction as illustrated in Figure 4: Negative Vectors. $v = -u$, or $u = -v$.

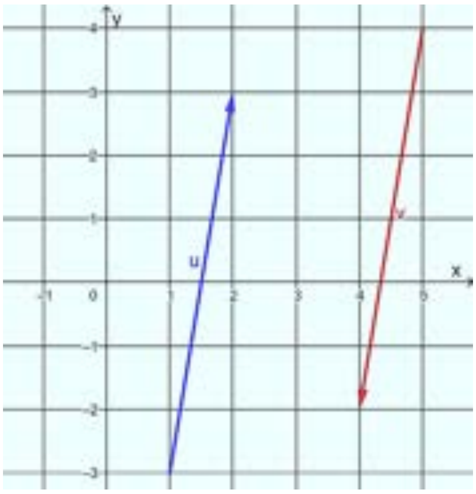



**Figure 4:** *Negative Vectors*

- **Parallel Vectors:** Two vectors are said to be parallel if one is a scalar multiple of the other. Parallel vectors may not have the same magnitude but they must have the same direction

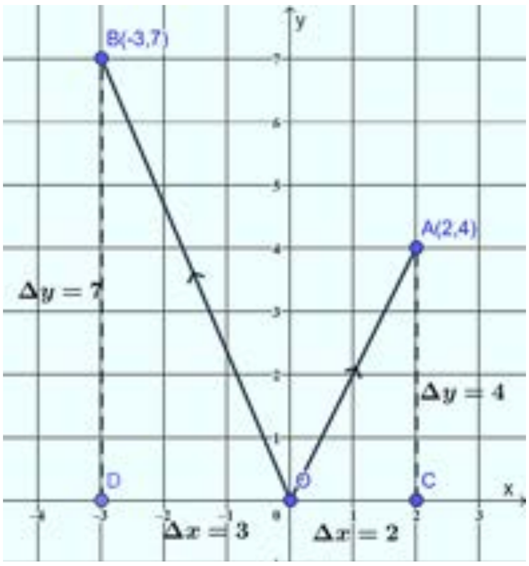
**Forms of vectors**

Vectors can be written as column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, $i, -j$, magnitude and direction form $(r, \theta)$.

**Example 1**

Given the points $A(2,4)$ and $B(-3,7)$, express vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as column vectors.

**Solution**




A particle positioned at $O$ has to be translated 2 units to the right (denoting a positive displacement) and 4 units upwards (positive displacement) to get to $A$. The vector that depicts this movement is thus $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Similarly, a translation of 3 units to the left and 7 units upward will move a point from $O$ to $B$ and hence $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

**Example 2**

The motion represented by $\overrightarrow{PQ} = 2i + 4j$ is the translation of a particle on the $i - j$ plane from $P$ to $Q$ by 2 units in the positive $i$ direction and 4 units in the positive $j$ direction. It can be written in column form as $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

**Example 3**

Aku's house is 40 metres away from her school and the bearing is 150°. Write the location of Aku's house in vectors form.

**Solution**

$(40m, 150°)$

**Example 4**

Identify parallel vectors from the following given vectors

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

**Solution**

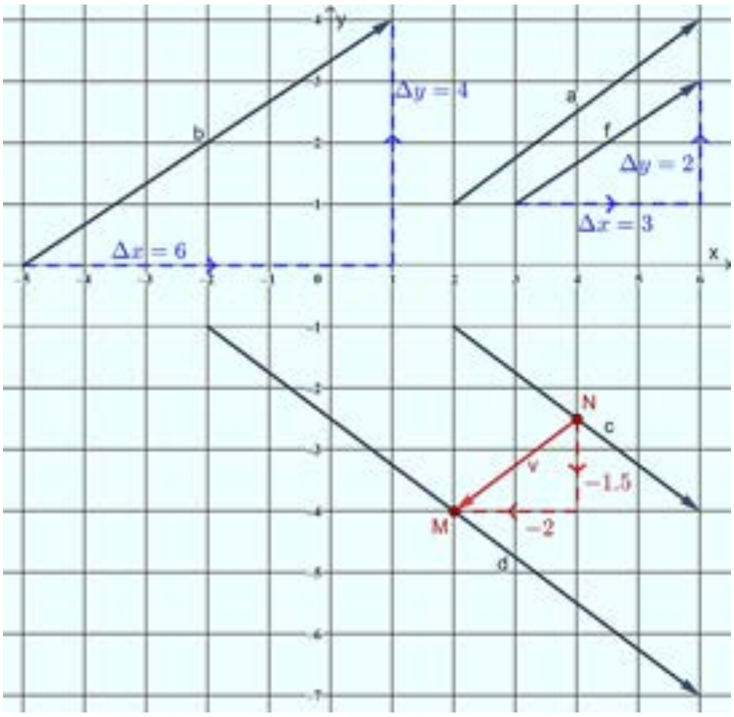



**Figure 5**

Vector $\boldsymbol{a}$ is represented by a horizontal movement of 4 units to the right and a vertical upward movement of 3 units of point (2, 1) to the terminal point, (6, 4). Similarly, $\boldsymbol{b}$ moves (−5, 0) to (1, 4) with a horizontal move of 6 units to the right and an upward movement of 4 units.

The right triangles subtended by the initial and terminal points of $\boldsymbol{b}$ and $\boldsymbol{f}$ are similar. A double of the lengths of the sides of the right triangle formed from $\boldsymbol{f}$ will result in the corresponding lengths of the sides of the right triangle obtained from $\boldsymbol{b}$ and this suggests that $\boldsymbol{b}$ and $\boldsymbol{f}$ must be parallel.

Similar analysis can be done for $\boldsymbol{c}$ and $\boldsymbol{d}$.

From Figure 5, it can be observed that any point of the ray representing $\boldsymbol{c}$ can be translated by the vector $\boldsymbol{v} = \begin{pmatrix} -2 \\ -1.5 \end{pmatrix}$ to obtain its corresponding point on a parallel line that coincides with the ray that represents $\boldsymbol{d}$ as has been illustrated with the midpoints, $N(4, -2.5)$ and $M(2, -4)$. The initial point of $\boldsymbol{c}$ i.e., $(2, -1)$ translated by $\boldsymbol{v}$, is relocated to $(0, -2.5)$, which lies on $\boldsymbol{d}$. The translation vector $\begin{pmatrix} -6.5 \\ 0 \end{pmatrix}$ also moves $\boldsymbol{f}$ to lie perfectly on $\boldsymbol{b}$

Algebraically and by definition,

$\boldsymbol{d} = 4\boldsymbol{c}$ and $\boldsymbol{b} = 2\boldsymbol{f}$, depicting that $\boldsymbol{d}$ is a scalar multiple of $\boldsymbol{c}$ as $\boldsymbol{b}$ is of $\boldsymbol{f}$. Therefore, vectors $\boldsymbol{d}$ and $\boldsymbol{c}$ are parallel and vector $\boldsymbol{b}$ is parallel to vector $\boldsymbol{f}$

**Example 5**

Given that the vector $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find the magnitude of $\overrightarrow{AB}$, $|\overrightarrow{AB}|$

**Solution**

$$|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$$
$$= \sqrt{3^2 + (-4)^2}$$
$$= 5$$

**Example 6**

Find the unit vector, for $a = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

**Solution**

$$\hat{a} = \frac{a}{|a|}$$

But $|a| = \sqrt{(-7)^2 + (3)^2}$

$$= \sqrt{58}$$

Therefore $\hat{a} = \dfrac{\begin{pmatrix} -7 \\ 3 \end{pmatrix}}{\sqrt{58}}$

$$= \begin{pmatrix} \frac{-7}{\sqrt{58}} \\ \frac{3}{\sqrt{58}} \end{pmatrix}$$

## Theme or Focal Area: Algebraic and geometric operations on vectors

**Algebraic operations on vectors**

Algebraic operations that can be performed on vectors are addition, subtraction and scalar multiplication.

1. For addition and subtraction, given that $a = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $b = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$,

   $a + b = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ and $a - b = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$

2. If $k$ is a scalar / constant, then $ka = k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

**Example 1**

Given that $a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, find

- **i.** $a - b$
- **ii.** $a + b$
- **iii.** $3b$

**Solution**

**i.** $a - b = \binom{2}{3} - \binom{6}{2}$

$= \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

**ii.** $a + b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

**iii.** $3b = 3\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 18 \\ 6 \end{pmatrix}$

**Triangular law of vector addition**

Imagine that you travel from a point, $A$ to point $B$, then to point $C$. The vectors, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ can be used to represent the movement. It would be much simpler and straightforward to move from point $A$ to point $C$. This movement can be represented by $\overrightarrow{AC}$. We can infer that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ as shown in Figure 6: Triangular law of vector addition

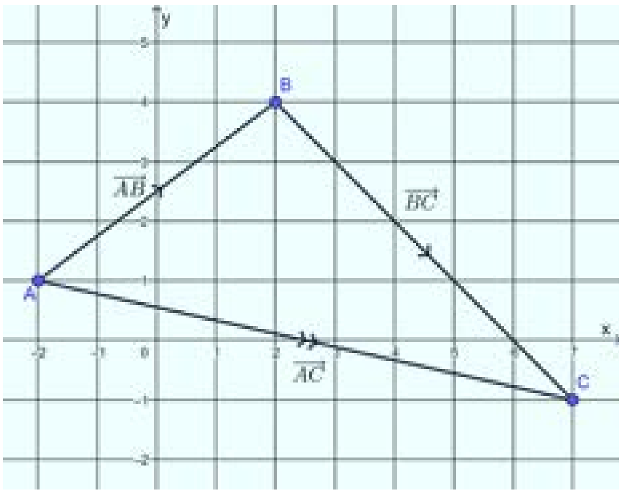



**Figure 6:** *Triangular law of vector addition*

$\overrightarrow{AC}$ is referred to as the resultant vector of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

In Figure 6: Triangular law of vector addition, $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and thus,

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ -2 \end{pmatrix}$

Triangular law of vector addition can be applied to establish the relationship between a free, say $\overrightarrow{AB}$ and the position vectors, $\overrightarrow{OA}$ also written $\boldsymbol{a}$ and $\overrightarrow{OB}$, written as $\boldsymbol{b}$.

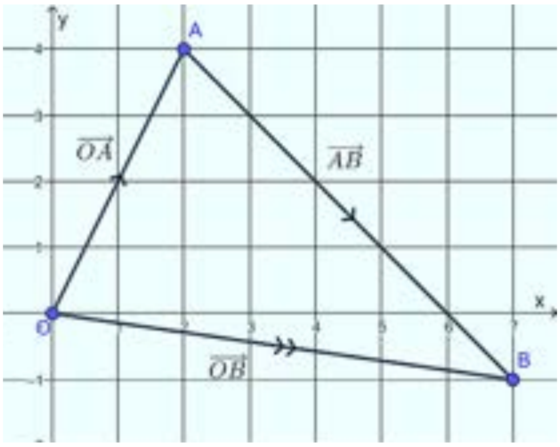



**Figure 7:** *Relationship between free and position vectors*

Considering Figure 7: Relationship between free and position vectors, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

**Parallelogram law of vector addition**

Given two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ sharing the same initial point, $O$, and representing the adjacent sides of a parallelogram, $OACB$, as shown in Figure 8: Parallelogram law of vector addition, the resultant vector, $\overrightarrow{OC}$ can be represented by the diagonal of the parallelogram passing through $O$

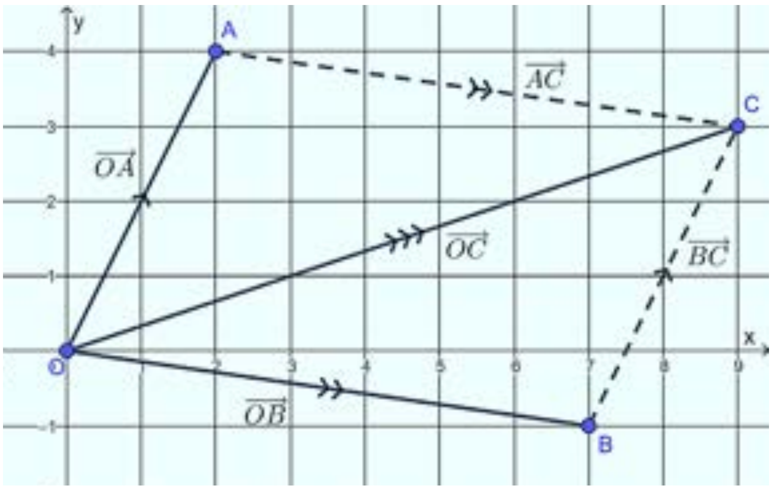



**Figure 8:** *Parallelogram law of vector addition*

It can be inferred that $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are parallel just as $\overrightarrow{OB}$ and $\overrightarrow{AC}$ are also parallel. The triangular law of vector addition can be applied to make the conclusion:

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ and $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$.

Also, since $\overrightarrow{AC} = \overrightarrow{OB}$ and $\overrightarrow{BC} = \overrightarrow{OA}$, $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

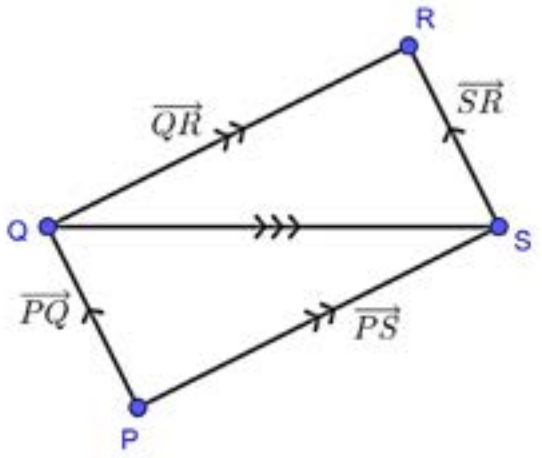
**Example**

The vertices of a parallelogram are at $P(6, -1)$, $Q(5, 1)$, $R(9, 3)$ and $S(x, y)$.

**a)** Find the coordinates of $S$

**b)** Find $|QS|$

**Solution**




**a)** $\overrightarrow{PS} = \overrightarrow{QR}$

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x-6 \\ y+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$x - 6 = 4$

$x = 4 + 6 = 10$

$y + 1 = 2$

$y = 2 - 1 = 1$

$\therefore S = (10, 1)$

**b)** $|QS| = \sqrt{(5 - 10)^2 + (1 - 1)^2}$

$= \sqrt{25} = 5 \; units$

**Learning Tasks**

1. Learners identify vector and scalar quantities from a list of quantities
2. Learners perform addition and subtraction of vectors
3. Learners relate the concept of proportionality to the division of line segments
4. Learners find the magnitude of vectors in a plane
5. Learners prove properties of vectors such as the commutative and associative laws of addition

## Pedagogical Exemplars

**Experiential Learning**

1. Use physical objects like arrows or toy cars to represent vectors
2. Have learners physically move objects to demonstrate vector addition and subtraction Use visual aids such as diagrams and rulers to demonstrate concepts
3. Utilise diagrams, graphs and animations to illustrate vector operations

4. Incorporate interactive simulations or online tools for vector manipulation.
5. Encourage peer collaboration through group problem-solving activities
6. Provide worksheets with varying levels of difficulty to cater for different learning paces

## Key Assessment

**Assessment Level 1: Recall and Reproduction**

1. Given that $\boldsymbol{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find:
   i. $5\boldsymbol{a} + 2\boldsymbol{b}$
   ii. $3\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$
   iii. $\frac{1}{4}\boldsymbol{a} - 6\boldsymbol{ab}$
2. Given that $\boldsymbol{p} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\boldsymbol{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, find
   i. $\boldsymbol{p} + 2\boldsymbol{r}$
   ii. $3\boldsymbol{q} – \boldsymbol{p}$
   iii. $2\boldsymbol{p} – \boldsymbol{q} + 3\boldsymbol{r}$

**Assessment Level 2: Skills and conceptual understanding**

1. Given that $\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ *and* $\boldsymbol{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ find $\boldsymbol{r}$ such that
   $\frac{1}{3}\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
2. Given that $\boldsymbol{m} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\boldsymbol{p} = \frac{1}{4}(m + n)$ and $\boldsymbol{q} = \frac{1}{2}(m - n)$ Show that $|\boldsymbol{p} + \boldsymbol{q}| = |\boldsymbol{p} - \boldsymbol{q}|$

**Assessment Level 3: Strategic Reasoning**

1. Two sides of a triangle are represented by the vectors $\boldsymbol{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. If $\boldsymbol{r} = p\boldsymbol{m} + q\boldsymbol{n}$ where $\boldsymbol{r} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, find the values of the scalars $p$ and $q$.

# Section 6 Review

This section reviews all the lessons taught for week 14. This section discussed vector forms, types of vectors and performing operations on vectors. It is encouraged that portions of the sections which require a review of concepts be administered to learners as presentation tasks. Leaners be provided with adequate learning materials such as graphing utilities, graph books or grid boards to enable them to investigate operations of vectors in a plane.

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1. Adams, R. A. & Essex, C. (2010). *Calculus: A complete course* (7th ed.). Pearson
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5. Stewart, J. (2008). *Calculus: Early transcendentals* (6th ed.). Thomson Brooks/Cole
6. Stewart, J., Redlin, L. & Watson, S. (2009). *Precalculus mathematics for calculus* (5th ed.). Brooks/Cole Cengage learning

# SECTION 7: TRIGONOMETRIC FUNCTIONS AND THEIR APPLICATIONS

Strand: **Geometric Reasoning and Measurement**

**Sub-Strand:** Measurement of Triangles

| |
|---|
| **Content Standard:** Demonstrate understanding of measurement of triangles and radians |

**Learning Outcomes:**

1. *Describe diagrammatically and algebraically ways of representing problems involving angles of elevation and depression and solve related word problems.*
2. *Identify values of the special angles in degrees and radians and solve problems relating to coordinates of the unit circle.*

## INTRODUCTION AND SECTION SUMMARY

In this section, we will investigate some concepts of Trigonometry. Trigonometry is a branch of mathematics that explores the relationships between the angles and sides of triangles. Our discussion will focus on the definition of trigonometry ratios, trigonometric functions and their' application, special angles, quadrantal angles and radian measure. On the definition of trigonometry, we look at the definition of sine, cosine, tangents and their reciprocals. We will use the definition to solve for the unknown sides and angles in a given right triangle. The application of trigonometry will focus on the angles of elevation and depression and solve related problems. Additionally, we will discuss special angles and how to convert between degrees and radians. The concept of trigonometry is used extensively in other fields and in real life. Examples of such applications can be found in physics, arts, design, architecture and construction. Understanding and mastering trigonometry will prepare learners for more advanced topics in years 2 and 3 as well as advanced studies at the tertiary level.

*The week covered by the section is:*

**Week 15:**

1. Definition of trigonometric ratios
2. Trigonometric functions
3. Applications of trigonometric functions (Angles of elevation and depression)
4. Special angles
5. Quadrantal angles
6. Radian measure

## SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on trigonometry. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, collaborative learning, experiential learning and initiate talk for learning, project-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

## ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests and homework can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems in trigonometry will also be used to assess learner's application of these mathematical skills. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

# Week 15

**Learning Indicators:**

1. *Recall basic trigonometric ratios and use the knowledge to solve problems relating to triangles*
2. *Use special triangles and the unit circle to determine the geometrical and functional values of trigonometric ratios including special angles.*
3. *Determine radians measure and apply the knowledge to solve practical problems of arc length.*
4. *Identify the coordinates of the quadrantal angles in a unit circle and use it to find the trigonometric values of quadrantal angles.*

## Theme or Focal Area: Definition of trigonometric ratios

### Definition/Introduction

Trigonometric ratios, namely sine, cosine, tangent, cosecant, secant and cotangent are defined based on the relationships between the angles and sides of a right triangle. The six possible ratios of the sides of a right triangle depend only on the acute angle $\theta$ not on the size of the triangle. In a right-angled triangle, there is a relationship between any of its angles and the sides of that triangle. The values of these ratios at an angle $\theta$ are denoted by $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$ , $\cot(\theta)$, $\sec(\theta)$ and $\csc(\theta)$. From Figure 1

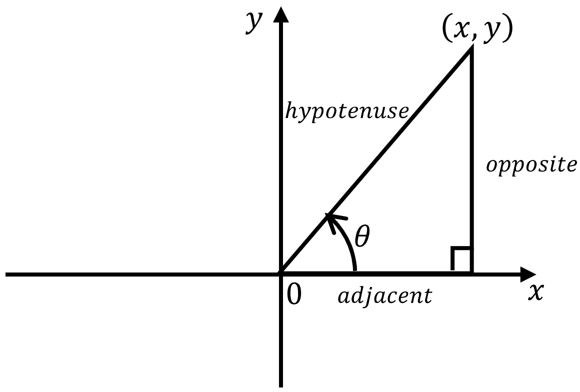



*Figure 1*

$$\sin(\theta) = \frac{opposite}{hypotenuse},\ \cos(\theta) = \frac{adjacent}{hypotenuse},\ \tan(\theta) = \frac{opposite}{adjacent}$$

$$\csc(\theta) = \frac{hypotenuse}{opposite},\ \sec(\theta) = \frac{hypotenuse}{adjacent},\ \cot(\theta) = \frac{adjacent}{opposite}$$

**Example 1**

Identify the six trigonometric ratios in Figure 2

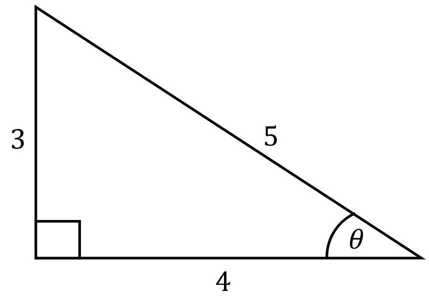


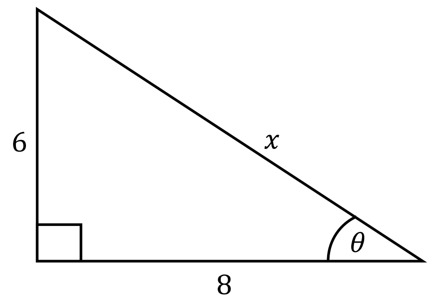

*Figure 2*

**Solution**

$\sin(\theta) = \frac{3}{5}$, $\cos(\theta) = \frac{4}{5}$, $\tan(\theta) = \frac{3}{4}$, $\csc(\theta) = \frac{5}{3}$, $\sec(\theta) = \frac{5}{4}$ , and $\cot(\theta) = \frac{4}{3}$

**Example 2**

Find the value of $x$ and $\theta$.




**Solution**

To find $x$

$x^2 = \sqrt{6^2 + 8^2}$

$x = 10$

To find $\theta$

$\tan(\theta) = \frac{6}{8}$

$\tan(\theta) = \frac{3}{4}$

$\theta = tan^{-1}\left(\frac{3}{4}\right)$

$\theta = 36.9°$

### Theme or Focal Area: Trigonometric functions

Trigonometric functions extend the basic trigonometric ratios to real numbers to help learners to solve problems including real-life problems. Trigonometric function relates an angle in the right-angled triangle to the ratio of lengths of any two sides. Each trigonometric ratio assumes a sign (positive or negative) in each of the four quadrants ( Figure 3).Table 1 shows the sign each trigonometric ratio assumes in the quadrant.

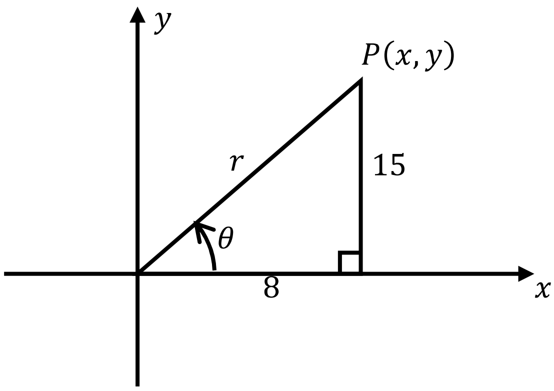
*Table 1*

| Quadrant | Trigonometric Function | | | | | |
|---|---|---|---|---|---|---|
| | sine | cosine | tangent | cosecant | secant | cotangent |
| I | + | + | + | + | + | + |
| II | + | − | − | + | − | − |
| III | − | − | + | − | − | + |
| IV | − | + | − | − | + | − |

**Example 1**

The terminal side of an angle, $\theta$, passes through the point $(8, 15)$. Find the values of the six trigonometric functions of the angle.

**Solution**



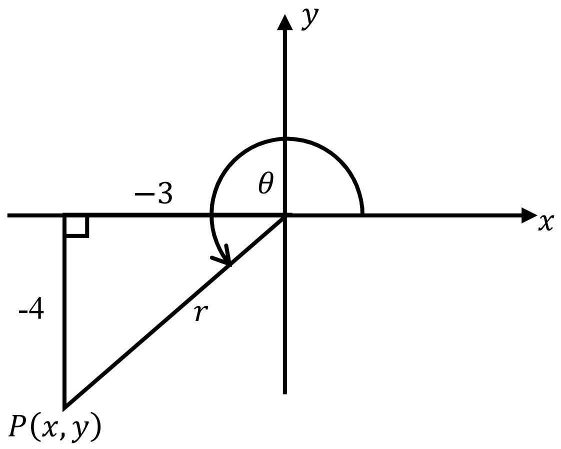

$r = \sqrt{8^2 + 15^2} = 17$

1. $\sin(\theta) = \frac{15}{17}$
2. $\csc(\theta) = \frac{17}{15}$
3. $\cos(\theta) = \frac{8}{17}$
4. $\sec(\theta) = \frac{17}{8}$
5. $tan(\theta) = \frac{15}{8}$
6. $\cot(\theta) = \frac{8}{15}$

**Example 2**

The terminal side of an angle passes through the point $(-3, -4)$. Find the values of the six trigonometric functions of angle $\theta$.

**Solution**



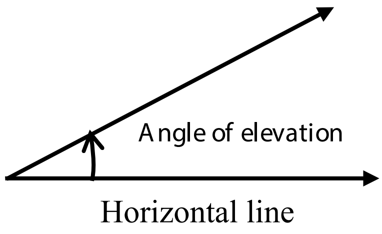

$r = \sqrt{(-3)^2 + (-4)^2} = 5$

1. $\sin(\theta) = \frac{-4}{5}$
2. $\csc(\theta) = -\frac{5}{4}$
3. $\cos(\theta) = -\frac{3}{5}$
4. $\sec(\theta) = -\frac{5}{3}$
5. $\tan(\theta) = \frac{4}{3}$
6. $\cot(\theta) = \frac{3}{4}$

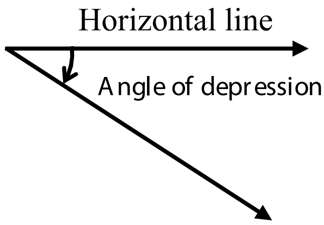
Note: $\tan(\theta)$ and $\cot(\theta)$ are positive in the Quadrant III

**Theme or Focal Area: Applications of trigonometric functions (Angles of elevation and depression)**

Angles of elevation and depression play an important role in solving real-life problems. Understanding of this concept will enable learners to calculate various distances and heights of objects. The angle of elevation is the acute angle formed between a horizontal line and a line of sight above the horizontal line.




Angle of depression is the acute angle formed between a horizontal line and a line of sight below the horizontal line.

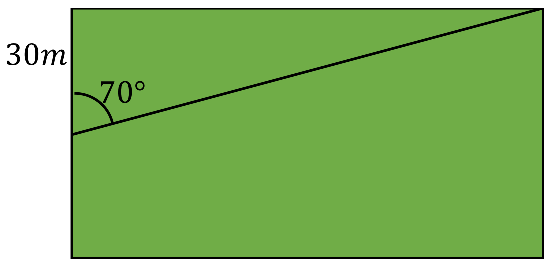



Angles of elevation and depression are useful concepts in the fields of engineering, surveying, measurements etc.

**Example 1**

On a football field, Adjoa Bayo walked from her goal post along the goal line to the corner flag, a distance of 30m. She further walked along the sideline to the next corner flag. The angle between the shortest distance and the goal line was 70°. How long is the side?

**Solution**




$\tan(70°) = \frac{\textit{length of side line}}{30m}$

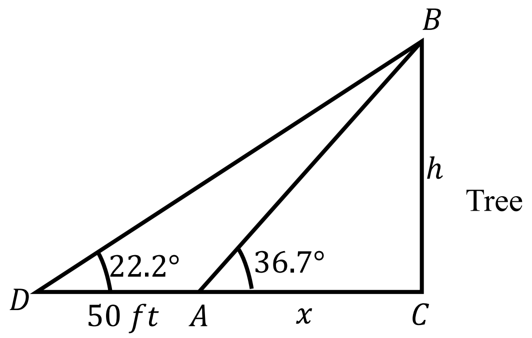
$\textit{length of side line} = 30\tan(70°)$

$\textit{length of side line} = 82.42m$

**Example 2**

Nimatu needs to know the height of a tree. From a given point on the ground, she finds the angle of elevation to the top of the tree is 36.7°. She moves back 50*ft*. From the second point, the angle of elevation of the top of the tree is 22.2°. Find the height of the tree.

**Solution**




From $\Delta ABC$, $tan 36.7° = \frac{h}{x}$

$\therefore\ h = xtan 36.7° \ldots\ldots\ldots\ldots (1)$

From $\Delta BCD$, $\tan\left(22.2°\right) = \frac{h}{50+x}$

$$\therefore\ h = (50+x)\tan(22.2°)$$

Since the heights are equal,

$$xtan 36.7° = (50+x)\ tan 22.2°$$

$$xtan 36.7° = 50 tan 22.2° + xtan 22.2°$$

Solve for $x$

$$x(tan 36.7° - tan 22.2°) = 50 tan 22.2°$$

$$x = \frac{50 tan 22.2°}{tan 36.7° - tan 22.2°} \ldots\ldots\ldots\ldots (2)$$

Substituting (2) into (1),

$$h = \left(\frac{50 tan 22.2°}{(tan 36.7° - tan 22.2°)}\right) tan 36.7°$$

Using a calculator,

$$tan 36.7° = 0.7453,\ \ tan 22.2° = 0.4080$$

$$\Rightarrow h = 45\ ft$$

## Theme or Focal Area: Special Angles

Special angles are angles in trigonometry that have unique characteristics. The special angles that will be discussed are $30°, 45°$ and $60°$. Understanding these angles will enable learners to simplify trigonometric concepts and solve trigonometric equations.

Deriving a $45°-45°-90°$ triangle from a unit square in Figure 4 .

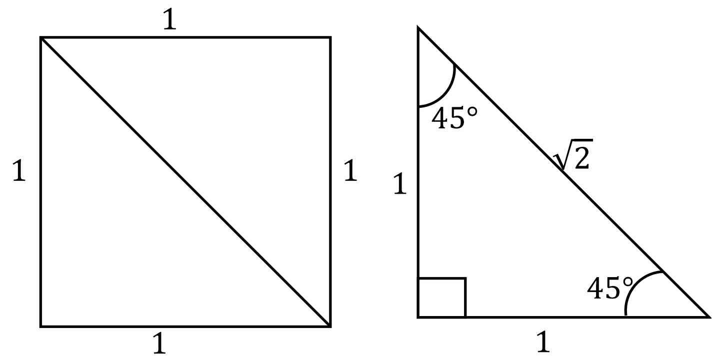



**Figure 4**

From Figure 4

| | |
|---|---|
| $\sin(45°) = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ | $\csc(45°) = \sqrt{2}$ |
| $\cos(45°) = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ | $\sec(45°) = \sqrt{2}$ |
| $\tan(45°) = 1$ | $\cot(45°) = 1$ |

Bisecting an equilateral triangle as shown in Figure 5

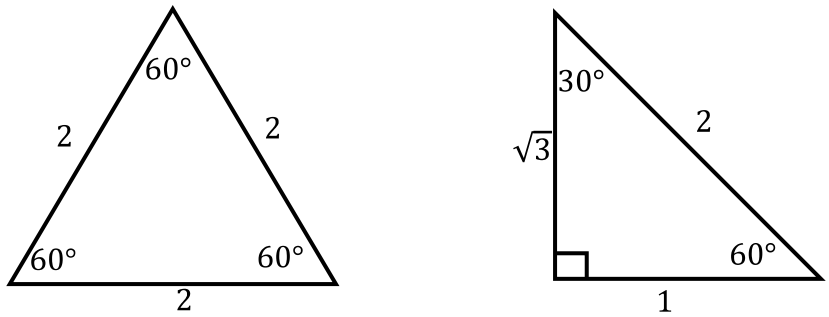



**Figure 5**

From Figure 5

| $\theta$ | **30°** | **60°** |
|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ |
| $\csc(\theta)$ | 2 | $\frac{2\sqrt{3}}{3}$ |
| $\sec(\theta)$ | $\frac{2\sqrt{3}}{3}$ | 2 |
| $\cot(\theta)$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ |

## Theme or Focal Area: Quadrantal Angles

Quadrantal angles have special properties. They are angles that terminate on the $x$ and $y$ axis. Therefore, the quadrantal angles between 0° and 360° are 0°, 90°, 180°, 270° and 360°. The understanding of quadrantal angles will be used in sketching trigonometric graphs and make calculation easy. The sine and cosine values for quadrantal angles are 0, 1 and $-1$. These values form a pattern when sketching trigonometric graphs. The values for the quadrantal angles for the basic trigonometric ratios are found in Table 2

**Table 2**

| $\theta$ | $sin\theta$ | $cos\theta$ | $tan\theta$ |
|---|---|---|---|
| 0° | 0 | 1 | 0 |
| 90° | 1 | 0 | Undefined |
| 180° | 0 | $-1$ | 0 |
| 270° | $-1$ | 0 | Undefined |
| 360° | 0 | 1 | 0 |

## Theme or Focal Area: Radian Measure

Radian measure is a fundamental concept used to express angles. Radian is a real number, and it is used in many calculations in mathematics. To conceptualise radian measure, consider Figure 6 if $\theta = \frac{s}{r}$, when $s = r$, the radian measure is 1. Where $s$ is the arc length and $r$ is the radius of the circle.

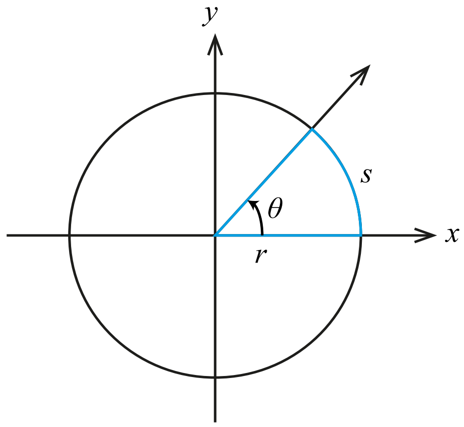



**Figure 6**

From the formula for finding the radian measure, $s = \theta r$ will be used to find arc length. Alternatively, the length of arc when $\theta$ is in degrees can be found using $\frac{\theta}{360°} \times 2\pi r$ or $\frac{\theta}{360°} \times \pi d$. Hence, to convert from degrees to radians, multiply the given degrees by $\frac{\pi}{180°}$ and to convert from radians to degrees, multiply the given radian value by $\frac{180°}{\pi}$.

**Example 1**

Convert 45° to radians.

**Solution**

To convert degrees to radians,

$45 \times \frac{\pi}{180°} = \frac{\pi}{4}$

**Example 2**

Convert 3.5 radians to degrees.

**Solution**

$3.5 \times \frac{180°}{\pi} = 200.63°$

**Example 3**

A bicycle wheel has a radius of 30$cm$. What is the length of the arc travelled by a point on the wheel when it rotates through an angle of 45°.

**Solution**

$radius = 30cm$

$\theta = 45°$

Using $\frac{\theta}{360°} \times 2\pi r$

$\Rightarrow \frac{45°}{360°} \times 2\pi\left(30cm\right)$

$= 7.5\pi cm$

**Learning Tasks**

1. Learners match angles with their corresponding trigonometric ratios (sine, cosine, tangent).
2. Learners calculate missing side lengths or angle measures in basic right-angled triangles using trigonometric functions.
3. Learners solve basic problems involving angles of elevation/depression or distance calculations.
4. Learners solve simple trigonometric equations involving sine, cosine and tangent functions.

## Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to s*olve problems involving trigonometric functions*. The following pedagogical approaches are suggested for facilitators to take learners through.

**Experiential / Exploratory Learning**

1. Utilise visual aids such as diagrams, illustrations and real-world examples to introduce trigonometric concepts
2. Break down concepts into smaller, manageable steps, providing clear explanations and ample opportunities for practice
3. Provide guided practice sessions where students work through problems with the teacher's support, gradually increasing independence
4. Conduct interactive demonstrations using technology or manipulatives to illustrate trigonometric principles
5. Offer problem-solving challenges or puzzles that require application of trigonometric concepts in novel contexts
6. Encourage exploratory learning experiences where students investigate trigonometric functions independently or in small groups.

## Key Assessment

**Assessment Level 1: Recall and reproduction**

1. The terminal side of an angle, $\theta$ passes through the point $(6, 8)$. Find the values of the six trigonometric functions of the angle. [**Expected answer**: radius or hypotenuse is 10, $\sin(\theta) = \frac{4}{5}$, $\csc(\theta) = \frac{5}{4}$, $\cos(\theta) = \frac{3}{5}$, $\sec(\theta) = \frac{5}{3}$, $\tan(\theta) = \frac{4}{3}$, $\cot(\theta) = \frac{3}{4}$]
2. Convert $\frac{5\pi}{4}$ to degrees. [**Expected Answer**: 225° ]
3. A sector of a circle has a central angle of 120° and a radius of $8m$. What is the length of the arc formed by this sector? Take $\pi$ as $\frac{22}{7}$ **[Expected answer:** 16.8***m*** **]**

**Assessment Level 2: Skills and conceptual understanding**

1. Find the three basic trigonometric function values of the angle $\theta$ if the terminal side of $\theta$ is defined by $2x + y = 0, x < 0$. [**Expected Answer**: $\sin(\theta) = \frac{2}{\sqrt{5}}, \cos(\theta) = -\frac{1}{\sqrt{5}}, \tan(\theta) = -2$].
2. A ladder is leaning against an outside wall of a building. If the angle of elevation at the base of the ladder to the wall is 55° and the ladder is 5 metres long, how far up the wall does the ladder reach? [**Expected Answer**: $4.1m$]

**Assessment Level 3: Strategic Reasoning**

1. If a regular pentagon (a five-sided regular polygon) is inscribed in a circle of radius 5.35 centimetres, find the length of one side of the pentagon. [**Expected answer**: 6.28 cm]
2. Without using calculators, find cos(210°), leaving your answer in surd form.

   [**Expected answer** $= -\frac{\sqrt{3}}{2}$]
3. Two points, $A$ and $B$ are on the same level ground as the foot of a pole, $B$. The angles of elevation of the top of the pole, $D$ from $A$ and $B$ are 35° and 62° respectively. If the distance between $A$ and $C$ is 20 $m$, find the height of the pole if $A$ and $B$ are on the same side of the pole. [**Expected Answer**: 22.3$m$]

# Section 7 Review

This section reviews the lesson taught in week 15. This week discussed concepts of trigonometry which will also serve as the foundation for trigonometry concepts for year 2 and year 3. We explored concepts on real-life applications for trigonometry. The following learning resources are recommended to facilitate teaching and learning:

## Teaching/Learning Resources

Dry erase markers and erasers, straight edge, calculator, technological tools such as computers, mobile phones, YouTube videos etc.

## References


1. Aufmann, R. N., Barker, V. C., & Nation, R. D. (2011). College algebra and trigonometry. Cengage Learning.
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# SECTION 8: LIMITS AND DIFFERENTIATION

Strand: **Calculus**

**Sub-Strand:** Principles of Calculus

**Content Standard:** Demonstrate understanding of limit of a function, investigate the behaviour of a function near a value in its domain and establish the derivative of a function

**Learning Outcomes:**

1. *Describe graphically and algebraically behaviour of the function about an input value and determine its' derivative*
2. *Describe graphically and algebraically behaviour of the function about an input value and determine its' derivative*
3. *Expand binomials with positive integral indices and simplify coefficients of the terms*

## INTRODUCTION AND SECTION SUMMARY

Understanding limits and differentiation is essential for learners in senior high school as they delve deeper into calculus. Limits serve as the foundation upon which calculus is built, enabling analysis of function behaviour as they approach specific values. Differentiation provides insights into rates of change and slopes of curves, crucial for understanding function behaviour. The development of calculus, pioneered by mathematicians like Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, revolutionised mathematics and science. Their ground-breaking work laid the groundwork for modern calculus, with limits being a fundamental concept. In this module, we will explore the concept of limits, learn to compute them algebraically and graphically and understand their significance in calculus. We will then transition into differentiation, where we'll learn techniques to find derivatives and apply them to solve real-world problems.

*The weeks covered by the section are:*

**Week 16:**

1. Idea of limits of a function
2. Left-hand and right-hand limits
3. Properties of limits of a function
4. Behaviour of functions
5. Continuity of functions

**Week 17:**

1. Concept of differentiation
2. Differentiating polynomial functions from the first principle
3. Differentiating polynomial functions using the power rule

**Week 18:**

1. Gradient of curves
2. Equation of tangents and normal to curves
3. Rate of change of a function

## SUMMARY OF PEDAGOGICAL EXEMPLARS

To effectively teach limits, differentiation and their applications in high school, it is important to have a versatile approach that can be adapted to meet the needs of diverse learners and work within resource constraints. Interactive visuals like graphs and animations, along with real-world examples, can be useful in illustrating abstract concepts and benefit students who learn visually or kinaesthetically. To break down complex ideas into manageable steps, scaffolded learning can be used, which progressively increases difficulty and provides ample practice opportunities. Cooperative learning strategies like peer teaching and group problem-solving can help foster collaboration and reinforce understanding. Differentiated instruction can also be used to offer multiple entry points to cater for various learning styles and abilities. For advanced students, extension activities can be provided, while additional support can be provided for struggling learners. Technology integration, through online resources and educational apps, can be used to supplement teaching, especially in resource-limited settings. Real-world applications can also be used to connect mathematical concepts to practical scenarios, motivating student engagement and highlighting relevance. By using these pedagogical approaches, educators can create an inclusive learning environment that promotes conceptual understanding and proficiency regardless of students' diverse needs and classroom constraints.

## ASSESSMENT SUMMARY

Assessing high school students' understanding of limits, differentiation and their practical applications requires flexible methods that cater to diverse learning abilities and resource constraints. Traditional assessments like quizzes and tests evaluate procedural knowledge, assessing students' ability to solve limit problems and differentiate functions. Performance tasks such as real-world problem-solving exercises, gauge students' capacity to apply concepts in practical contexts, promoting deeper understanding. To accommodate varying learning needs, it is important to incorporate peer assessment and collaborative activities which enable students to evaluate one another's work and engage in discussions, fostering a supportive learning community. Additionally, formative assessments like exit tickets or concept maps can provide ongoing feedback, guiding instructional adjustments and offering support to struggling learners. In resource-limited settings, prioritising low-tech options such as whiteboard exercises and manipulatives, along with leveraging open educational resources and digital platforms can enrich learning experiences despite constraints. Employing a balanced approach that integrates various assessment methods ensures inclusive learning environments and comprehensive mastery of limit concepts.

# Week 16

**Learning Indicators:**

1. *Describe and interpret the meaning of limit of a function through graphics and algebraic approaches*
2. *Classify left-hand and right-hand limits algebraically and if possible, with the aid of technology or any creative means*
3. *Distinguish between continuous and discontinuous function near an input value on its domain and investigate them with the use of technology or any other means appropriate*

## Theme or Focal Area: Idea of limits of a function

The concept of limit is the basis for a solid understanding of calculus. For example:

Frequently when studying (mostly when trying to sketch) a function, say $f(x)$, we are interested in the function's behaviour near a particular point $x_0$, but not at $x_0$. For instance, if we seek to evaluate a function at $x$ equals an irrational number, say $\pi$ (when the value of $x$ can only be approximated), we might evaluate the function at an $x$-value that is *very close* to our required $x$ i.e., $\pi$ say 3.14159 instead and thus conclude that the value of $f(\pi)$ is *very close* or *approximately equal to* $f(3.14159)$.

Another situation occurs when trying to evaluate a function at $x_0$ leads to division by zero, which is undefined. Here's a specific example where we explore numerically how the graph of a function looks near a particular point at which we cannot directly evaluate the function

**Example**

What is the height of the graph of the function $g(x) = \frac{x^2 - 2}{x + 2}$ at $x = -2$?

**Solution**

$g(-2)$ is undefined as the denominator, $x + 2$ will be zero for $x = -2$ so we can not evaluate the function at $x = -2$ and presume that the result is the height of the graph. We can rather evaluate the function at values of $x$ which are very close to $x = -2$ and predict the height of the graph thus.

**Table 1**

| Values of $x$ around $x = 1$ | $g(x)$ |
|---|---|
| -2.1 | -4.1000 |
| -2.01 | -4.0100 |
| -2.001 | -4.0010 |
| -2 | |
| -1.999 | -3.9990 |
| -1.99 | -3.9900 |
| -1.9 | -3.9000 |

From Table 1, it can be observed that the values of $g(x)$ get closer to $-4$. We can thus assert that the height of the graph of $g(x)$ at $x = 2$ is $-4$. The graph of $g(x)$ as illustrated in Figure 1 confirms it. There is a hole at $(-2, -4)$ since $g(-2)$ is undefined.

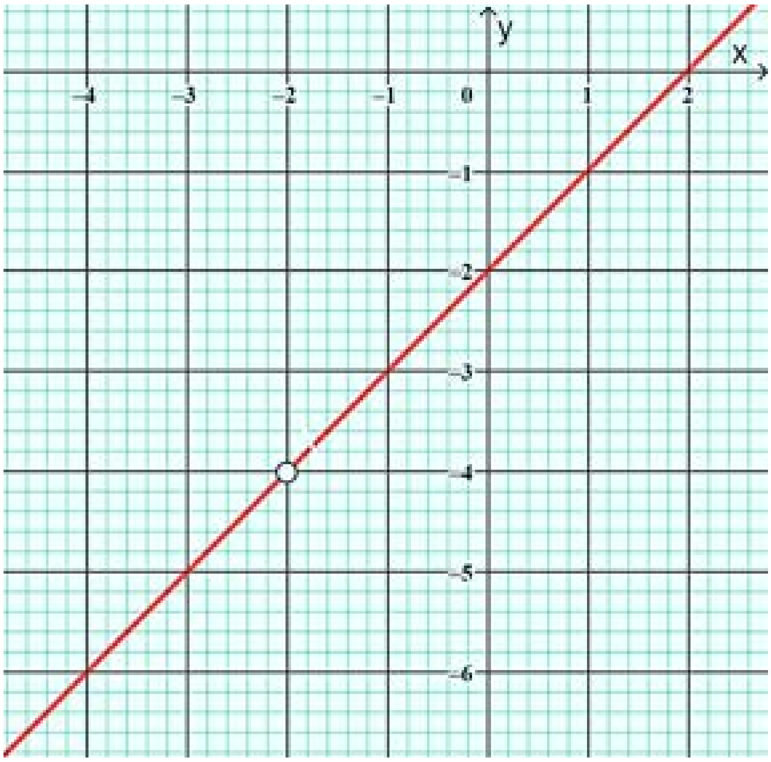


**Figure 1**

Also, consider a circle $C$ of radius $r$, its area, $A$, is $\pi r^2$ and circumference, $C$, is $2\pi r$. One can approximate the area, $A$ and circumference of $C$, by a region with an area and perimeter respectively that we do know. One approach is to inscribe an equilateral triangle (a regular 3-gon) in $C$. We add sides, one at a time, to the inscribed figure to create inscribed polygons. The area and circumference of the inscribed figure get closer and closer to the area and circumference of the circle in which it is inscribed respectively as shown in Figure 2: The area problem.

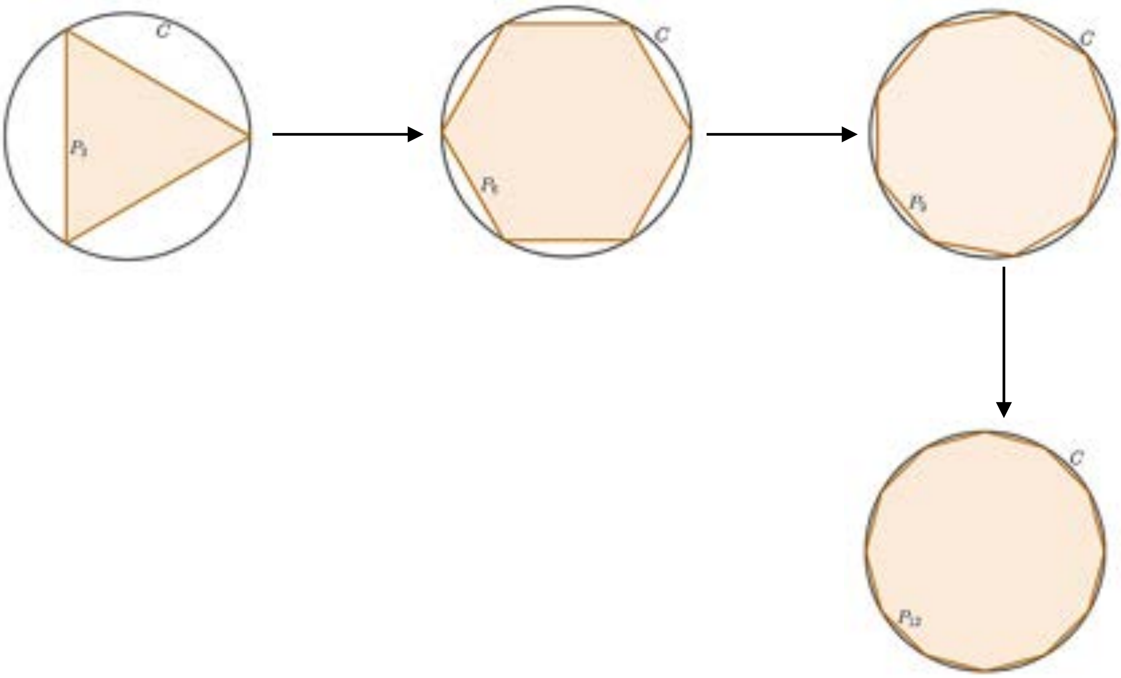


**Figure 2:** *The area problem*

The more the number of sides the inscribed polygon, $n$ becomes, the shorter the sides become. This means as the number of sides, $n$ gets bigger, the more the inscribed polygon will resemble the circle and the area of the inscribed polygon can be used to approximate the area of the circle just as the perimeter can be used as an approximation for the circumference of the circle. We could also say that as $n$ tends to infinity, the limit of the area, $A_n$, inside the polygon, $P_n$ equals the area, $A$ inside the circle and the limit of the perimeter $C_n$ of $P_n$ equals the circumference $C$ of the circle. So, we write this as $A = A_n$ and $C = C_n$

The statement can also be written as $\lim_{n \to \infty} A_n = A$ read as "*limit of $A_n$ as n approaches infinity equals A*" and $\lim_{n \to \infty} C_n = C$ read as "*limit of $C_n$ as n approaches infinity equals C*"

**Example**

Given that $f(x) = 2x$, find $\lim_{x \to 2}(f(x))$

**Solution**

A table of values can be constructed with values of $x$ that are very close to 2 as shown

| $x$ | $f(x)$ |
|---|---|
| 2.1 | 4.2000 |
| 2.01 | 4.0200 |
| 2.001 | 4.0020 |
| ↓ | |
| 2 | |
| ↑ | |
| 1.999 | 3.9980 |
| 1.99 | 3.9800 |
| 1.9 | 3.8000 |

It can be observed that the values of $f(x)$ approach 4 as the values of $x$ approach 2.

Hence $\lim_{x \to 2}(f(x)) = 4$

## Theme or Focal Area: Left-hand and right-hand limits

Consider the graph of $f(x) = \frac{x+2}{x-1}$ as shown in Figure 3. The height of $f(x)$ decreases to negative infinity as the graph is traced from the left side of 1 on the $x-axis$ towards $x = 1$ (values of $x$ that approach $x = 1$ from the left) i.e., for values such as 0.9, 0.99, 0.999 and 0.9999 which are less than but very close to 1 as shown in Table 2. On the other hand, the height of the graph increases infinitely (to positive infinity) as the values of $x$ approach $x = 1$ from the right i.e., for values such as 1.1, 1.01, 1.001 and 1.0001. We say the left-hand limit of $f(x)$ as $x$ approaches 1 is negative infinity and this can be written as $\lim_{x \to 1^-}(f(x)) = -\infty$ and the right-hand limit of $f(x)$ as $x$ approaches $x = 1$ which is written as $\lim_{x \to 1^+}(f(x)) = \infty$

**Table 2**

| $x$ | $f(x)$ |
|---|---|
| 0.9 | -29.0000 |
| 0.99 | -299.0000 |
| 0.999 | -2999.0000 |
| 0.9999 | -29999.0000 |
| ↓ | |
| 1 | |
| ↑ | |
| 1.0001 | 30001.0000 |
| 1.001 | 3001.0000 |
| 1.01 | 301.0000 |
| 1.1 | 31.00000 |

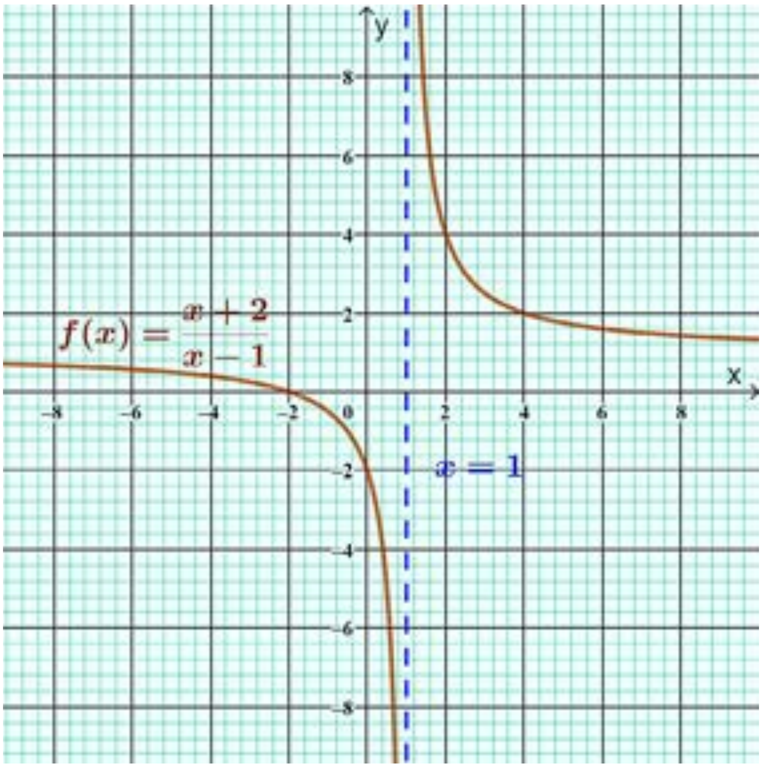


**Figure 3**

The left-hand and the right-hand limits are concepts used to describe the behaviour of a function as it approaches a particular point.

Generally, the left-hand limit of a function f($x$) as $x$ approaches a point $c$ from the left side (i.e. as $x$ approaches $c$ from the values less than $c$ is denoted as: $\lim_{x \to c^-} f(x)$

The right-hand limit of a function $f(x)$ as $x$ approaches a point $c$ from the right side (i.e. as $x$ approaches $c$ from the values greater than $c$ is denoted as: $\lim_{x \to c^+} f(x)$

**Existence of a limit**

It is important to note that for a limit to exist at a specific point, the left-hand limit and the right-hand limit must be equal.

The limit of a function describes the behaviour of the function as the independent variable approaches a particular value.

The limit of a function, $f(x)$ as $x$ approaches a specific point, $c$ is the same as the functional value at $c$ and can be obtained when the independent variable $x$ is substituted with the value $c$ in the function $f(x)$

**Example 1**

The graph of $k(x)$ is obtained from the combination of the graphs of $f(x)$, $g(x)$, $h(x)$ and $j(x)$ as shown in $r$.

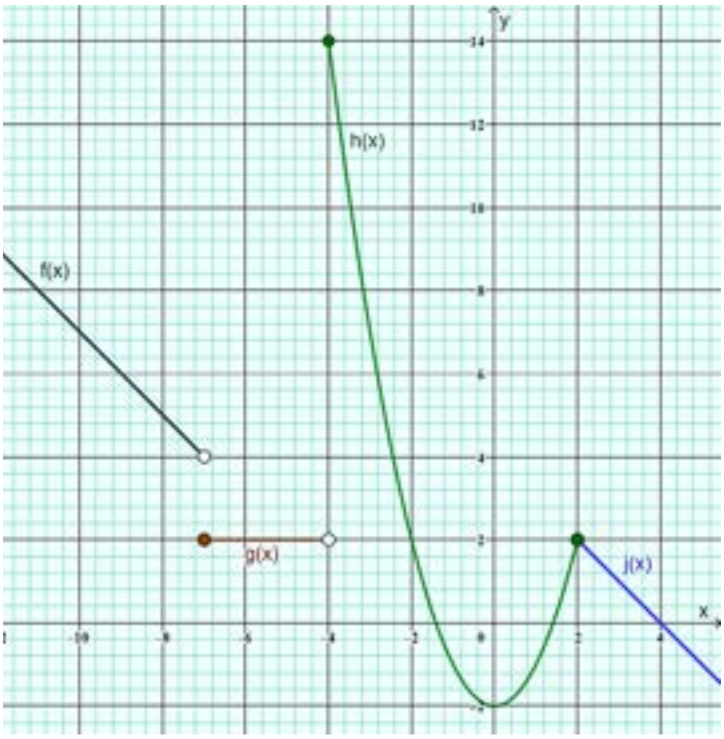


**Figure 4**

The value $k(x)$ approaches when $x$ approaches the number $-7$ from the left is 4 and from the right is 2. Since the two limits are different, we conclude that the limit does not exist for $k(x)$ as $x$ approaches $-7$. Also, $\lim_{x\to -4^-}(k(x)) = 2$ and $\lim_{x\to -4^+}(k(x)) = 14$ and since $\lim_{x\to -4^-}(k(x)) \neq \lim_{x\to -4^+}(k(x))$, $\lim_{x\to -4^-}(k(x))$ does not exist. Contrarily, $\lim_{x\to 2^-}(k(x)) = \lim_{x\to 2^+}(k(x)) = 2$ and thus, $\lim_{x\to 2^-}(k(x))$ exists and it has a value of 2

**Example 2**

What is the meaning of $\lim_{x\to c}(f(x)) = L$?

**Solution**

If $f(x)$ is a function, then $\lim_{x\to c}(f(x)) = L$ means the value of $f(x)$ approaches $L$ as $x$ gets very close to $c$ from the left and right.

**Example 3**

From Figure 5 . what is $\lim_{x\to 8}(f(x))$

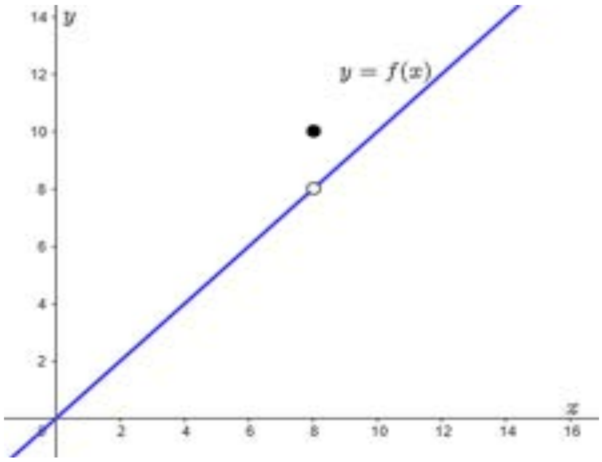


**Figure 5**

**Solution**

The function value when $x = 8$, i.e., $f(8) = 10$ ,however, as $x$ approaches 8 the function approaches 8 and hence $\lim_{x\to 8}(f(x)) = 8$

## Theme or Focal Area: Properties of limits of a function

Suppose that $\lim_{x\to a}(f(x))$ and $\lim_{x\to a}(g(x))$ both exist, then we have the following results:

1. If $k$ is a constant, then $k \times \lim_{x\to a}(f(x)) = \lim_{x\to a}(k \times f(x))$
2. If $r$ is a positive constant, then $[f(x)]^r = [\lim_{x\to a} f(x)]^r$
3. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
5. $\lim_{x\to a}[f(x) \times g(x)] = [\lim_{x\to a} f(x)] \times [\lim_{x\to a} g(x)]$
6. $if\ \lim_{x\to a} g(x) \neq 0, then\ \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

The following statements about limits should be noted:

1. If $p(x)$ be a polynomial function and $a$ is any real number, then

   $\lim_{x\to a} p(x) = p(a)$

2. Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ *and* $q(x)$ are polynomials. Let $c$ be a real number such that $q(c) \neq 0$. Then

   $\lim_{x\to a} r(x) = r(a)$

**Example 1**

Given $f(x) = -9$, $g(x) = 2$ and $h(x) = 4$ use the limit properties to compute each of the following limits. If it is not possible to compute any of the limits, clearly explain why .

**a.** $[f(x) - g(x) + h(x)]$

**b.** $[3h(x) - 6]$

**Solution**

**a.** $[f(x)\text{-}g(x)\text{+}h(x)] \quad = \underset{x\to 8}{(lim)}\, f(x) \; - lim\; g\,(x) + \underset{x\to 8}{lim}\; h(x)$

$= -9 -2 + 4$

$= -7$

**b.** $[3h(x) - 6] \quad = 3\underset{x\to 8}{lim}\; h(x) \; - \underset{x\to 8}{lim}\; 6$

$= 3(4) - 6$

$= 6$

**Example 2**

Given $f(x) = 1$, $g(x) = 10$ and $h(x) = -7$, use the limit properties to compute each of the following limits. If it is not possible to compute any of the limits, clearly explain why.

**a.** $\lim\limits_{x\to -4}[f(x) \times g(x) \times h(x)]$

**b.** $\lim\limits_{x\to -4}\left[\frac{1}{h(x)} + \frac{3 - f(x)}{g(x) + h(x)}\right]$

**Solution**

**a.** $\underset{x\to\text{-}4}{lim}[f(x) \times g(x) \times h(x)] \quad = [\underset{x\to\text{-}4}{(lim)}\, f(x)]\; [\underset{x\to\text{-}4}{(lim)}\, g(x)]\; [\underset{x\to\text{-}4}{(lim)}\, h(x)]$

$= (1)(10)(-7)$

$= -70$

**b.**

$$\lim_{x\to -4}\left[\frac{1}{h(x)} + \frac{3 - \mathrm{f(x)}}{g(x) + h(x)}\right] = \lim_{x\to -4}\left[\frac{1}{h(x)}\right] + \lim_{x\to -4}\left[\frac{3 - f(x)}{g(x) + h(x)}\right]$$

$$= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)} + \frac{\lim\limits_{x\to -4}\,[3 - f(x)]}{\lim\limits_{x\to -4}\,[g(x) + h(x)]}$$

$$= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)} + \frac{\lim\limits_{x\to -4}\,[3 - f(x)]}{\lim\limits_{x\to -4} g(x) + \lim\limits_{x\to -4} h(x)}$$

$$= \frac{1}{-7} + \frac{3 - 1}{10 - 7}$$

$$= \frac{11}{21}$$

**Example 3**

Given that $f(x) = 3x + 4$ and $k$ is a constant, verify that for $k = 6$, $a = 1$,

$k \times \lim\limits_{x\to a}(f(x)) = \lim\limits_{x\to a}(k \times f(x))$

**Solution**

$\lim\limits_{x\to a}(f(x)) = \lim\limits_{x\to 1}((3x + 4))$

$= 7$

$$k \times \lim_{x\to a}(f(x)) = 6(7) = 42$$

$$k \cdot f(x) = 6 \cdot (3x+4)$$

$$= 18x + 24$$

$$\lim_{x\to a}(k \times f(x)) = \lim_{x\to 1}(18x+24)$$

$$= 42$$

Hence $k \times \lim_{x\to a}(f(x)) = \lim_{x\to a}(k \times f(x))$

**Determinate Form**

An undefined expression involving some operation between two quantities is said to be in a determinate form if it evaluates to a single number value or infinity.

**Indeterminate Forms**

An undefined expression involving some operation between two quantities is in indeterminate form if it does not evaluate to a single number value or infinity. The indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0^0$, $\infty^0$ , $\infty - \infty$ , $1^\infty$and $\infty - \infty$

**Example 4**

Find the following

a. $\lim_{x\to 0}\left(\frac{\sqrt{x+4}-2}{x}\right)$

b. $\lim_{x\to -2}\left(\frac{x^3+8}{x^2-4}\right)$

**Solution**

Note that applying the properties i.e., evaluating the function at $x = 0$, may yield an indeterminate form thus: $\frac{\sqrt{x+4}-2}{x} = \frac{0}{0}$

but rationalising the numerator makes it determinate

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{x+4-4}{x(\sqrt{x+4}+2)}$$

$$= \frac{x}{(\sqrt{x+4}+2)}$$

$$= \frac{1}{\sqrt{x+4}+2}$$

$$\lim_{x\to 0}\left(\frac{\sqrt{x+4}-2}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sqrt{x+4}+2}\right)$$

$$= \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$$

Similarly, evaluating $\frac{x^3+8}{x^2-4}$ at $x = -2$ yields an indeterminate form $\left(\frac{0}{0}\right)$

Since both the numerator and denominator are factorable polynomial expressions, the expression can be simplified thus

$$\frac{x^3+8}{x^2-4} = \frac{(x+2)(x^2-2x+4)}{(x-2)(x+2)} = \frac{(x^2-2x+4)}{(x-2)}$$

$$\lim_{x\to -2}\left(\frac{x^3+8}{x^2-4}\right) = \lim_{x\to -2}\frac{(x^2-2x+4)}{(x-2)}$$

$$= -\frac{12}{4} = -3$$

## Theme or Focal Area: Behaviour of functions

### Limit at infinity

Consider the graph of $f(x) = \frac{x+1}{x-1}$, $x \neq 1$ as shown in Figure 6 and some points that lie on the graph in Table 3. As the values of $x$ get negatively bigger, i.e., approaches $-\infty$, the value of $f(x)$ increases towards 1. Of course, since the inverse of $f(x)$ is $f^{-1}(x) = \frac{x+1}{x-1}$, $x \neq 1$, the height of $f(x)$ cannot be 1 hence there is a horizontal asymptote at $y = 1$ and thus, no matter how negatively big the value of $x$ will be, $f(x)$ can only get closer 1. We say then that the limit of $f(x)$ as $x$ approaches negative infinity (our idea of a very big number), written as $\lim\limits_{x \to -\infty}(f(x))$ is 1.

Likewise, as the value of $x$ get positively larger, the value of $f(x)$ and in effect, the height of its graph decreases to 1 and thus, $\lim\limits_{x \to \infty}(f(x)) = 1$

**Table 3**

| $x$ | $f(x)$ |
|---|---|
| $-\infty$ | 1 |
| $\uparrow$ | $\uparrow$ |
| -1000 | 0.998001998 |
| -100 | 0.980198020 |
| -10 | 0.818181818 |
| -1 | 0.000000000 |
| 10 | 1.222222222 |
| 100 | 1.020202020 |
| 1000 | 1.002002002 |
| 1000 | 1.002002002 |
| $\downarrow$ | $\downarrow$ |
| $\infty$ | 1 |

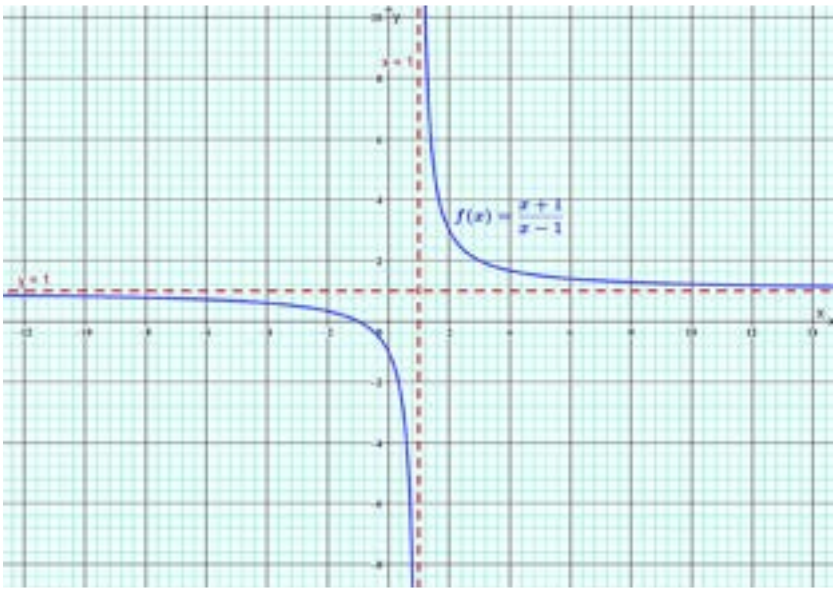

**Figure 6**

The idea of limits at infinity can be depended on to predict the nature or behaviour of functions for extreme values of the independent variables which may be difficult or impractical to evaluate and thus make sketching of such functions possible.

It must be noted that infinity is not a real number. It is only an idea of a certain very huge number. This understanding of infinity helps us to evaluate limits of functions (without having to graph the functions or evaluate functions at very huge values for the independent variable).

**Example 1**

$\lim_{x\to\infty}(2x^4 - x^2 - 8x)$

**Solution**

Let $g(x) = 2x^4 - x^2 - 8x$

$g(x)$ is a fourth-degree polynomial function and since the leading term ($2x^4$) has a positive coefficient, the bigger the value of $x$, the bigger the value of $y$ will be too. This is confirmed by the shape of the graph of the function: an expected ∪-shape as shown in Figure 7. The height increases as the graph is traced from the turning point towards the right. From these analyses, it can be concluded that $\lim_{x\to\infty}(2x^4 - x^2 - 8x) = \infty$

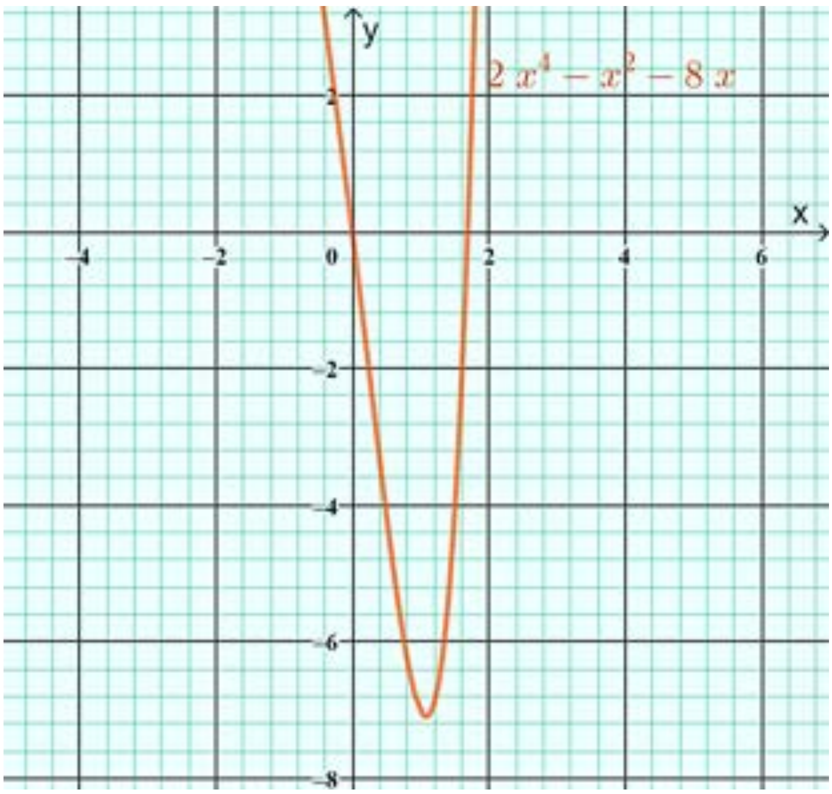


**Figure 7**

Algebraically

"Twice the fourth power of a very big number" i.e., ∞ into $2x^4$ is still a big number, say ∞

"The square of a very big number" i.e., ∞ into $x^2$ is still a big number (even though, it will be smaller than that for $2x^4$), say ∞

"Eight times a very big number is also big

Consequently, performing the combined operation $2x^4 - x^2 - 8x$ on that very big number still results in a big number, $\infty$ and hence, it can be concluded that

$\lim\limits_{x\to\infty}(2x^4 - x^2 - 8x) = \infty$

**Example 2**

Evaluate $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)$

**Solution:**

$\frac{x+1}{x-1}$ is a rational expression and so the algebraic analysis done in example 1 above does not necessarily apply

$\frac{x+1}{x-1}$ remains the same if it is multiplied by 1 in the form, $\frac{1}{x} \div \frac{1}{x}$

The result of that division is

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

And thus,

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)$$

And

$$\lim_{x\to\infty}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) = \left(\frac{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)}\right)$$

As the value of $x$ becomes bigger, the value of $\frac{1}{x}$ becomes smaller i.e., as $x \to \infty$ , $\frac{1}{x} \to 0$ and thus, $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$ and $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right) = 1$

Hence, $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right) = \frac{1}{1} = 1$

## Theme or Focal Area: Continuity of functions

A function $f$ is continuous at a point "$a$" if $\lim_{x\to a} f(x) = f(a)$

**Theorem:**

If $f(x)$ is continuous at "$x = a$", then the following three conditions hold;

a. $f(a)$ is defined,

b. $\lim_{x\to a} f(x)$ must be defined (thus the left and right limits must be the same) and

c. $f(x) = f(a)$

**Example 1**

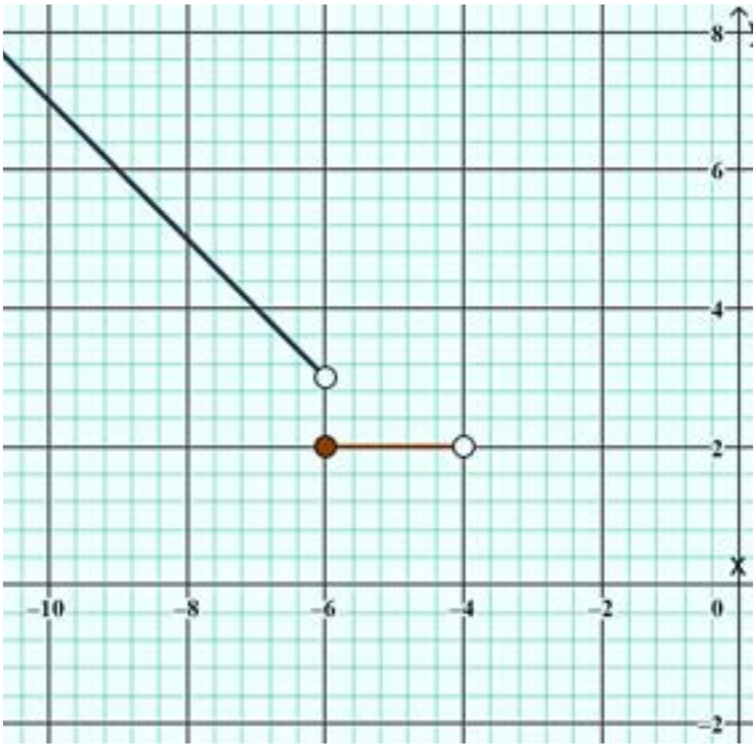


**Figure 8**

Provided that a function, $h(x)$ is defined as

$$h(x) = \begin{cases} -x-3, \ when -\infty < x \leq -6 \\ 2, \ when -6 \leq x < -4 \end{cases}$$

determine, with the aid of the graph in Figure 8, the interval(s) / point(s) for which $h(x)$ is (are) continuous or discontinuous

**Solution**

The function $g(x)$ is continuous on the intervals, $-\infty < x < -6$ and $-6 \leq x < -4$ which can also be written as and $(-\infty, -6)$ and $[-6, -4)$ respectively. $h(x)$ is however discontinuous at $x = -6$ because there is a jump there. In the interval $-4 \leq x < \infty$, the graph discontinues since the graph does not exceed $x = -4$

**Example 2**

Determine whether the following functions are continuous at $x = 2$. Why?

**a)** $f(x) = 3x + 1$

**b)** $g(x) = \frac{x-4}{x^2-4}$

**Solution**

**a)** $f(x)$ is a polynomial function and hence is continuous for all real numbers including 2 and hence, $f(x)$ is continuous at $x = 2$

**b)** $g(x)$ is a rational function defined for all real numbers except values of $x$ for which $x^2 - 4$ equals zero. When $x = -2$ or $x = 2$, $x^2 - 4 = 0$ and hence $g(2)$ is undefined and $g(x)$ is not continuous at $x = 2$

***Note to teachers:***

*Learners should be made to understand that from the concepts of limit, the value $f(c)$ plays no role in the definition of the limit of $f(x)$ as $x$ tends to $c$. We do not even assume that $f$ is defined at $c$. A definition of "limit" that requires $f$ to be defined at $c$ would not apply to many important applications. Even when $f(c)$ is defined, the value $f(c)$ does not have to be related to $f(x)$ in any way. In a sense, $f(x)$ is what we anticipate that $f(x)$ will equal at $x = c$, not necessarily what $f(x)$ actually equals when $x = c$.*

*When we drive along a high-speed road, we frequently cannot see very far ahead. We learn to drive as though the portion of the road immediately beyond our vision is the obvious extension of the portion of the road that we can see. What we are doing is computing the limit of what we see. Usually, the actual state of the road coincides with what we have anticipated it will be (that is, the road is continuous). If the road has been damaged or if a bridge is out, then the actual state of the road does not coincide with what we have anticipated (the road is discontinuous). In this section, we develop mathematical analogues of these ideas.*

**Learning Tasks**

1. Individual learners state the value of the limit of a constant function
2. Learners individually mention the conditions for the continuity of a function
3. Learners calculate the limits of given functions using the properties of limits
4. Learners are tasked in groups to determine the continuity of given polynomial and rational functions at given points and in some intervals
5. Learners in pairs use the limit of a function at a point, the nature of continuity of the function in an interval and its behaviour to sketch rational functions
6. Learners use the conditions of continuity, the properties of limits, the graph of a function and any relevant concepts to justify the prediction of the continuity or discontinuity of a function in a given interval to the larger group
7. Learners working in mixed-ability groups investigate and classify various functions into continuous and discontinuous functions and relate them to real-life situations
8. Learners investigate and identify the points of discontinuity

## Pedagogical Exemplars

1. **Collaborative Learning**: In mixed-ability groups, learners work to consolidate and foster their understanding of concepts of limits in mathematics and daily life using both algebraic and graphical means.
   a. Based on their past knowledge and learning requirements, learners should be divided into groups. For example, individuals who need additional scaffolding ought to be placed in the same group while others who are up for a challenge are also grouped separately
   b. Learners work in groups with different tasks to investigate the behaviour of a function for different intervals for the input values and share their observations. This will promote critical thinking and communication skills
   c. Pair students with different levels to encourage peer teaching, allowing stronger students to reinforce their knowledge while supporting their peers.
   d. Learners in their mixed ability groups are to initiate a discussion on the meaning of approaching a number on a real number line from the left or the right of the number
2. **Experiential Learning**: learners find functional values and the limits of functions around given points to investigate the continuity of functions at the given points
3. **Enquiry-based learning:** learners research the applications of limits in their home settings and other subject areas and industries
4. **Exploratory learning:** Learners use dynamic tools and calculators to explore the limits and behaviour of the graphs of functions at a point.

## Key Assessment

**Assessment Level 1: Recall and reproduction**

1. What are the conditions that should be met for a function to be continuous?

**Assessment Level 2: Skills and conceptual understanding**

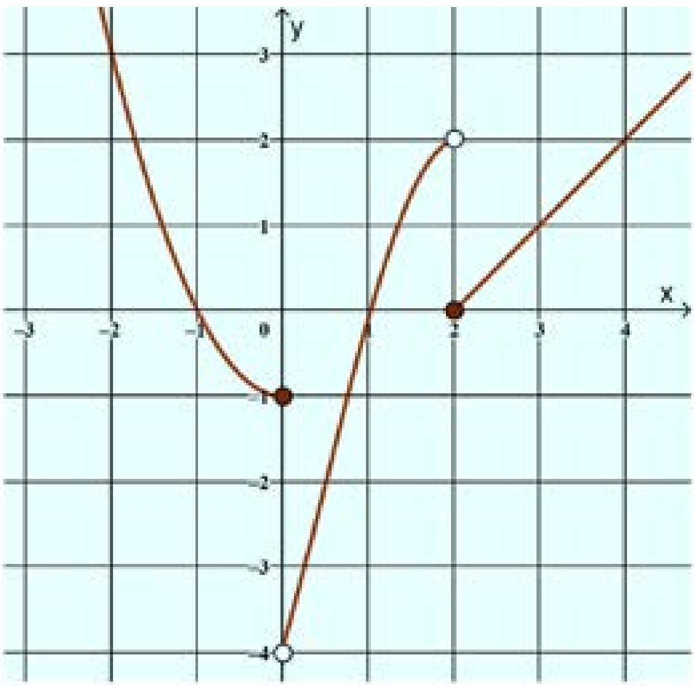


**Figure 9**

**1.** For the function $g(x)$ whose graph is given in Figure 9, state the value of each quantity, if it exists. If it does not exist, explain why

a) $\lim_{x \to 0^-} g(x)$

b) $\lim_{x \to 0^+} g(x)$

c) $\lim_{x \to 0} g(x)$

d) $\lim_{x \to 2^-} g(x)$

e) $\lim_{x \to 2^+} g(x)$

f) $\lim_{x \to 2} g(x)$

g) $g(2)$

h) $\lim_{x \to 4} g(x)$

**2.** Evaluate $\lim_{x \to \infty} \left( \frac{2x^2 - x + 3}{3x^2 + 5} \right)$

# Week 17

**Learning Indicator:** *Use limits of a function to find its derivative*

## Theme or Focal Area: The concept of differentiation

Consider the problem of finding the gradient of a non-linear function whose graph is shown below in figure 10. Gradient or slope as we know it is the ratio of the rise (the change in the height of the graph) to the run (the corresponding change in the horizontal). To find the gradient, we would need the coordinates of two points on the curve say $P(x, f(x))$ and $Q((x + h), f(x + h))$ to obtain

$$\frac{f(x+h)-f(x)}{h}$$

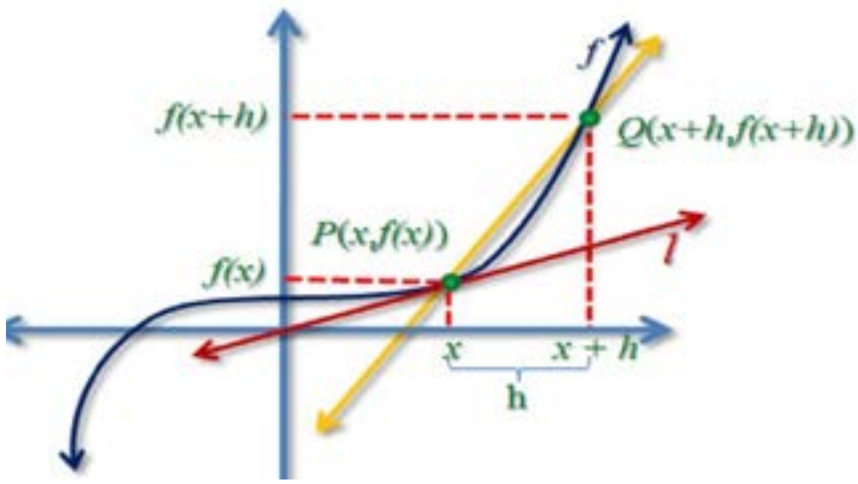


**Figure 10:** *The gradient problem*

It must be noted, however, that the gradient which will be obtained as a result, will be the gradient of the straight line that passes through points $P$ and $Q$. Unfortunately, that line does not coincide with the graph of $f(x)$. It is a secant line, hence its gradient will be a very bad approximation for the gradient of the curve. We can make better our approximation by decreasing the value of $h$, which serves as the difference in the two $x$-coordinates of the points selected for the approximation i.e., $P$ and $Q$, gradually until we get $P$ and $Q$ to be so close to each other that their coordinates are approximately equal and thus, the gradient that will be obtained thence, sufficiently approximates the gradient of $f(x)$ at $P(x, f(x))$. This can be done by getting the difference closer and closer to zero i.e., finding

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

It can be inferred that the value for the gradient that will be obtained at different points on the curve will also be different since the tangent line to the curve at different points on the curve will have different slopes.

The new function that models the gradient of the curve at points on the curve is called the *derivative* of the function. This new function, as it is derived from $f(x)$, can be written as $f'(x)$ or $\frac{dy}{dx}$ as it is a ratio of the change in $y$ values, representing the change in the height ($\Delta y$), to the change in the $x$ values, representing the horizontal change ($\Delta x$) or $\frac{d}{dx}(f(x))$ as differentiation can be thought of as an operator. It can also be called the instantaneous rate of change of $f$ at $x$

Imagine also, that a training film is taken of a runner. The film shows elapsed time and distance markers that allow us to measure the distance $s(t)$ the athlete has run in any given time t. However, the speed of the runner will vary from one instant of time to another. How can the runner's speed $v(t)$ at a given instant of time t be calculated?

$$\frac{s(u)-s(t)}{u-t}$$

For $u$ close to $t$, this will give a good approximation to $v(t)$ because the runner's speed does not change much in the small-time interval $[t, u]$. If we press this point further, then we intuitively arrive at the concept of instantaneous velocity at $t$:

$$v(t)=\lim_{u\to t}\left(\frac{s(u)-s(t)}{u-t}\right)$$

The number $u$ cannot be equal to $t$ because that results in the meaningless fraction $0/0$. The key is first to understand exactly what is meant by the limits in these equations and then to learn methods for computing them.)

**NB:** *It is emphatic to state that a function can only be differentiable at a point if it is continuous at that point and the limit of the function as x approaches that point exists*

### Theme or Focal Area: Differentiating polynomial functions from first principle

As discussed earlier in this section, the derivative of a function can be obtained by limiting the change in the independent variable to zero. Thus,

$$\frac{dy}{dx}=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

This process, known as *differentiation from first principle*, is broken down into the following steps:

1. Find $f(x+h)$
2. Write the difference quotient $\frac{f(x+h)-f(x)}{h}$
3. Simplify the difference quotient
4. Find the limit as $h \to 0$.

**Example 1**

Find the derivative of the functions$f$, $g$ and $p$ using the first principle.

**a)** $t(x)=2x^2$

**b)** $g(x)=6x+4$

**c)** $p(x)=3x^2-6x+7$

**Solution**

**a)** $t(x)=2x^2$

Let $h$ be a very small change which approaches 0

At point $x+h$,

$$\begin{aligned} t(x+h) &= 2(x+h)^2 \\ &= 2(x^2+h^2+2hx) \\ &= 2x^2+2h^2+4hx \end{aligned}$$

The link between limit of a function and derivative of a function $y=t(x)$ is given by

$$\frac{dt}{dx} = \lim_{h\to 0}\left(\frac{t(x+h)-t(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{2x^2+2h^2+4hx-2x^2}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{2h^2+4hx}{h}\right)$$
$$= \lim_{h\to 0}(2h+4x)$$
$$= 4x \text{ since as } h \to 0,\ 2h \to 0$$

**b)** $g(x) = 6x + 4$

Let $h$ be a very small change which approaches 0

At point $x + h$,

$$g(x+h) = 6(x+h)+4$$
$$= 6x + 6h + 4$$
$$\frac{d}{dx}(g(x)) = \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{6x+6h+4-(6x+4)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{6h}{h}\right)$$
$$= \lim_{h\to 0}(6)$$
$$= 6$$

**c)** $p(x) = 3x^2 - 6x + 7$

Let $h$ be a very small change which approaches 0

At point $x + h$,

$$p(x+h) = 3(x+h)^2 - 6(x+h) + 7$$
$$= 3x^2 + 6hx - 6x + 3h^2 - 6h + 7$$
$$p'(x) = \lim_{h\to 0}\left(\frac{p(x+h)-p(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{3x^2+6hx-6x+3h^2-6h+7-3x^2-6x+7}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{6hx+3h^2-6h}{h}\right)$$
$$= \lim_{h\to 0}(6x+3h-6)$$
$$= 6x - 6$$

**Example 2**

Find the derivatives of at $x = 0$

**a)** $f(x) = |x|$ and

**b)** $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

**Solution**

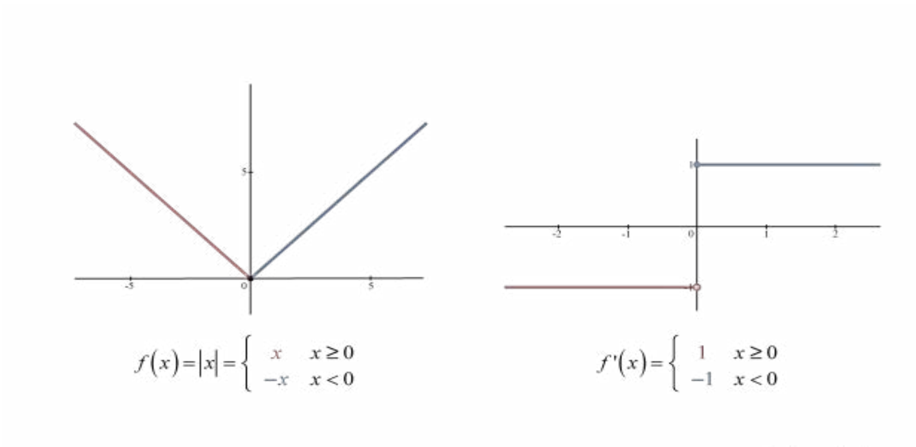


**Figure 11:***Graphs of signum and piecewise defined function*

**a)** $f(x) = |x|$ is called a signum function and it is defined as

$$f(x) = |x| = \begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x < 0 \end{cases}$$

And it is interpreted thus: "the value of $f(x)$ is $x$ (remains same) if the value of $x$ is greater than or equal to zero (positive) and the value of $f(x)$ is $-x$ (the negative of the original $x$ value or expression) if the value of $x$ is less than zero (negative)". It is essential to recall that on the real number line, negative values lie to the left of zero, while positive values lie to the right. The function gives the distance from a point to the origin and hence, its value would always be positive

$f(x + h) = |x + h|$ means

$f(x + h) = x + h \text{ for } x + h \geq 0$ and

$f(x + h) = -(x + h) \text{ for } x + h < 0$

for $x > 0$ where the values are on the right side of zero and we approach zero from the right, i.e., $h \to 0^+$

$$\lim_{h\to 0^+}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0^+}\left(\frac{x+h-x}{h}\right)$$
$$= \lim_{h\to 0^+}(1) = 1$$

and for $x < 0$ and as we approach zero from the left, i.e., $h \to 0^-$

$$\lim_{h\to 0^-}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0^-}\left(\frac{(-x+h)-(-x)}{h}\right)$$
$$= \lim_{h\to 0^-}\left(\frac{-x-h+x}{h}\right)$$
$$= \lim_{h\to 0^-}(-1)$$

The slope of the left-side equals -1 and the slope of the right-side equals +1, so they disagree hence the function is not differentiable at $x = 0$.

**b)** $f(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$

for $x > 0$

$\lim_{h\to 0^+}(f(x)) = 1$

for $x < 0$

$\lim_{h \to 0^-} (f(x)) = -1$

Since the slope of the left-side and the slope of the right-side are not equal, the function is not differentiable at $x = 0$.

**NB:** *Learners are confronted with the conceptual understanding of why to connect limits of a function to its derivative. Example 2 reinforces the need to connect limits to the derivatives.*

*All differentiable functions are continuous, however, not all continuous functions are differentiable, hence differentiability is stronger than continuity.*

**Example 3**

Identify the values of $x$ *in* the graph in Figure 12 for which the function whose graph is illustrated is not differentiable and give reasons for your responses

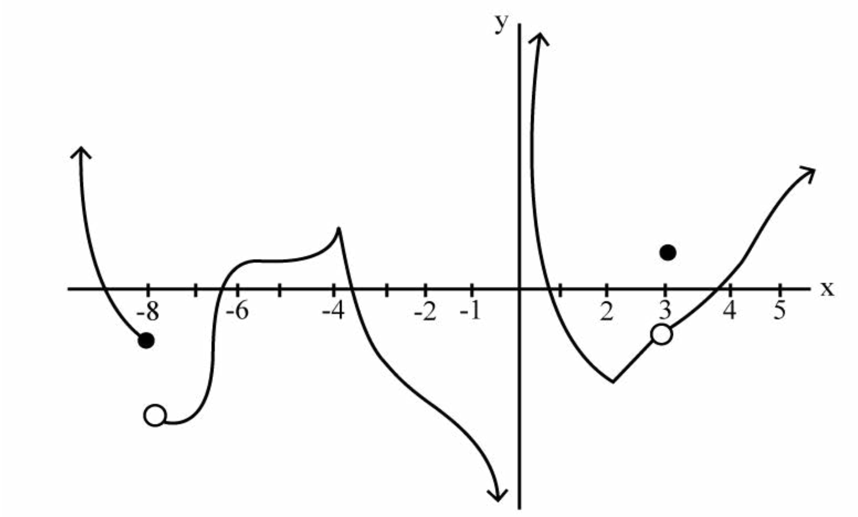


**Figure 12**

**Solution**

When $x = -8$, there is a jump, the graph is not continuous, hence the function is not differentiable

When $x = 0$, the limit of the function does not exist as the left-hand $(-\infty)$ and right-hand limit $(\infty)$ around $x = 0$ are different and thus, the function is not differentiable at $x = 0$

At $x = 3$, the graph is discontinuous as there is a hole and thus, the function is not differentiable

At $x = -4$ and $x = 2$, there are kinks in the graph and so even though the graph is continuous at those points, the function is not differentiable at those points.

## Theme or Focal Area: Differentiating functions using the power rule

Given that $f(x)$ is a polynomial function of degree $n$ say $f(x) = x^n$, differentiating same from first principle requires that $f(x + h)$ be found.

$f(x + h) = (x + h)^n$.

From the binomial expansion, we have

$$f(x + h) = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + h^n$$

$$f(x + h) - f(x) = nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + h^n$$

$$\frac{f(x+h)-f(x)}{h} = \mathrm{n}x^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots + h$$

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots + h\right)$$

$$= nx^{n-1}$$

Generally, given a polynomial function of degree $n$, for example, $f(x) = x^n, f'(x) = nx^{n-1}$. i.e., to obtain the derivative of such functions, "bring the exponent of the independent variable in a term down to multiply the original term and reduce the original exponent by one. Repeat the process for all the terms and the combination of the new terms would represent the derivative of the function"

**Example 1**

Find the derivative of the following;

**a)** $t(x) = 2x^2$

**b)** $g(x) = 6x + 4$

**c)** $p(x) = 3x^2 - 6x + 7$

**Solution**

**a)** $\frac{d}{dx}(t(x)) = 2(2)x^{2-1} = 4x$

**b)** $\frac{dg}{dx} = 1(6)x^{1-1} + 0(4)x^{0-1} = 6$

**c)** $p'(x) = 6x - 6$

**Example 2**

Find the derivative of the following;

**a)** $h(x) = 3x^{-2} - 4x^{-3} + 7$

**b)** $f(x) = \sqrt{x}$

**Solution**

Even though these functions are not polynomial functions, they are in the form $f(x) = x^n$, where $n$ is a real number and thus, the power rule can be applied

**a)** $h(x) = 3x^{-2} - 4x^{-3} + 7$

$h'(x) = -2(3)x^{-2-1} - 3(-4)x^{-3-1} = -6x^{-3} + 12x^{-4}$

**b)** $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

**Learning Tasks**

1. Learners are provided with worksheets with basic polynomial functions and with some guidance, are tasked to differentiate them
2. Learners are offered a set of standard practice problems that require applying the power rule to polynomial functions of varying degrees
3. Learners individually apply the 1st principle of differentiation to find the derivative of polynomial functions

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## Pedagogical Exemplars

1. **Collaborative Learning**: In mixed-ability groups, learners work to consolidate and foster their understanding of concepts of limits in mathematics and daily life using both algebraic and graphical means.
   a. Learners working in mixed-ability groups investigate how the limit of a function relates to its derivative
   b. Learners at varying ability levels should be paired. Assist peers in understanding topics and problem-solving techniques by having more capable students explain them. Promote peer teaching by involving peers in peer feedback sessions and cooperative problem-solving exercises.
   c. Break down the process of differentiating polynomial functions into smaller, more manageable steps for learners who might need more assistance and provide ample support and guidance as learners progress through each step
   d. Pair students with different levels to encourage peer teaching, allowing stronger students to reinforce their knowledge while supporting their peers.
2. **Problem-based Learning**: learners work in groups and individually on tasks to differentiate polynomial functions of varying degrees from first principles and with the power rule
   a. Create a series of tasks that range in difficulty, similar to different routes through a polynomial garden. Based on their degree of readiness, learners select one of three pathways: “Advanced Lane,” “Intermediate Avenue” or “Beginner’s Boulevard.” Every path includes tasks and challenges catered to the relevant degree of difficulty.
   b. Create tiered worksheets on polynomial differentiation. For struggling students, provide step-by-step guided examples with simplified problems. For average learners, offer standard practice problems. For advanced students, include challenging problems that require deeper critical thinking and application skills
3. **Experiential learning:**
   a. Learners apply the idea of limits, skills learnt in evaluating functions and any relevant skills and concepts to find the derivative of a function by 1st principle or by the power rule
   b. Set up learning stations around the classroom with different activities. Station 1 could involve watching a video tutorial on polynomial differentiation, Station 2 could be a hands-on activity with algebra tiles to visualise differentiation and Station 3 could be an interactive computer simulation for practicing differentiation
4. **Enquiry-based learning:** Learner’s research to discover the link between limits and derivative of a function

## Key Assessment

**Assessment Level 2: Skills and conceptual understanding**

1. Find the derivative, $\frac{dy}{dx}$ of $3x^2 + 4x - y = 4$ **[Expected answer: $\frac{dy}{dx} = 6x + 4$]**
2. Find the derivative of the following;

   a) $f(x) = -x^3 + 4x^2 + 9x$ **[Expected answer: $f'(x) = 9 + 8x - 3x^2$]**

   b) $h(t) = 5t^{-2} + t^6$ **[Expected answer: $h'(t) = 6t^5 - 10t^{-3}$]**

**Assessment Level 3: Strategic Reasoning**

Given that $f(x) = -4x^2 + 2x$, $g(x) = 5x^3 + x$. If $h(x) = f(x) + g(x)$, find $h'(x)$ using

1. differentiation by first principles
2. the power rule **[Expected answer: $h'(x) = 15x^2 - 8x + 3$]**

# Week 18

**Learning Indicators:**

1. *Use technology or innovation ways to investigate the rate of change of a function, h(u), with respect to u*
2. *Generalise the behaviour of a moving object along a path or curve*
3. *Use knowledge of differentiation to determine the equation of tangents and normal to curves at a given point*
4. *Apply differentiation to find the rate of change*

## Theme or Focal Area: Gradient of curves

As discussed in the previous week, the derivative of a function which can be graphed on a plane represents the slope or gradient of the tangent line to the curve at points that lie on the curve. Also, as discussed earlier in section 3, the value of the gradient of a linear graph (straight line) gives information on the behaviour of the line (whether it is increasing or decreasing). In this week, we will apply that knowledge to predict the behaviour of the graph of curves for functions, at least for some intervals of the independent variable.

NB: The following three (3) are true for a given function, $f(x)$, as $x$ increases;

- $f(x)$ increases for all values of $x$ for which $f'(x) > 0$,
- $f(x)$ decreases for all values of $x$ for which $f'(x) < 0$
- $f(x)$ momentarily at rest for all values of $x$ for which $f'(x) = 0$

**Example 1**

Does the function $f(x) = 3x + 4$, $x \in R$ increase, decrease or does not change?

**Solution**

By observation, it can be deduced that $f(x)$ is a linear function so we do not expect any turning points. The leading coefficient is greater than 0 (being positive). This bit of information suggests that the graph of $f(x)$ strictly increases over its entire domain (the set of all real numbers). The derivative of the function confirms this assertion as shown below:

$\frac{d}{dx}(f(x)) = 3 > 0$ so, the function $f$ increases

**Example 2**

Is the function $f(x) = -5x + 4$ , $x \in R$ increasing, decreasing or has no change as $x$ increases?

**Solution**

$f'(x) = -5 < 0$ so, the function decreases as $x$ increases

**Example 3**

Is the function $f(x) = 7$ increases, decreases or no change as x increases?

**Solution**

$\frac{df(x)}{dx} = 0$ this means momentarily at rest at that $x$ value. We can also infer that the graph of $f(x)$ is a horizontal line as horizontal lines have their gradients to be 0. As the value of $x$ changes, there is no change in the $y$ values

**Example 4**

Find the value(s) of x for which $f(x) = 2x^3 - 9x^2 - 24x + 56$ increases, decreases or momentarily at rest

**Solution**

Step 1. Differentiate the given function.

$f(x) = 2x^3 - 9x^2 - 24x + 56$

That is

$\frac{d}{dx}(f(x)) = 6x^2 - 18x - 24$

Step 2. Find the values of $x$ for which $f$ is momentarily at rest,

$$\frac{d}{dx}(f(x)) = 0$$

$$\frac{df(x)}{dx} = 6x^2 - 18x - 24 = 0$$

$$6x^2 - 18x - 24 = 0$$

$$(x - 4)(x + 1) = 0$$

Hence at $x = 4$ or $x = -1$, $f(x)$ is momentarily at rest. The tangent to the curve at those points are horizontal and there are turning points at those points

Step 3. Use the values of $x$ in step 2 to partition the real number line to obtain a real interval. Note that values that are momentarily at rest should not be part of the intervals constructed.

Partitioned intervals: $(-\infty, -1)$, $(-1, 4)$ $(4, +\infty)$

Step 4: Test $\frac{d}{dx}(f(x))$ on the intervals in step 3 to arrive at the desired results.

| Interval | Test value | Derivative | Conclusion |
|---|---|---|---|
| $(-\infty, -1)$ | $-2$ | $6(-2)^2 - 18(-2) - 24 = 36 > 0$ | Increasing |
| $(-1, 4)$ | 3 | $6(3)^2 - 18(3) - 24 = -26 < 0$ | Decreasing |
| $(4, \infty)$ | 5 | $6(5)^2 - 18(5) - 24 = 36 > 0$ | Increasing |

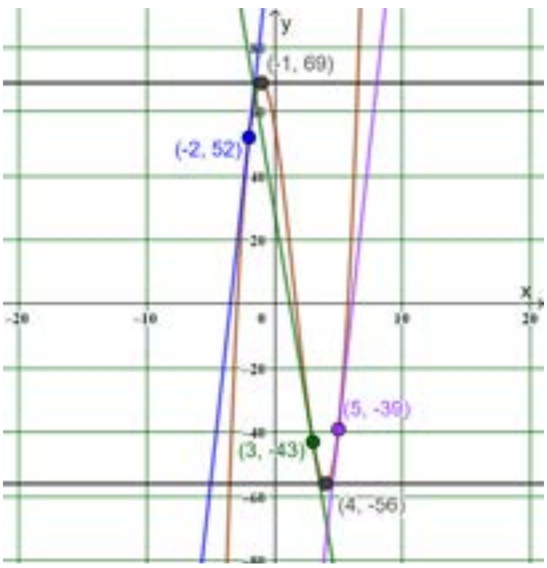


**Figure 13**

Figure 13 shows the graph of $f(x) = 2x^3 - 9x^2 - 24x + 56$ with the points at which a point moving along it will be momentarily at rest: $(-1, 69)$ and $(4, -56)$, the test points and the tangent lines that pass through the test points and the turning points. It can be observed that the tangents at the turning points are both horizontal lines, the tangent lines that pass through $(-2, 52)$ and $(5, -39)$ strictly increase while the tangent that passes through $(3, -43)$ strictly decreases

## Theme or Focal Area: Equation of tangents and normal to curves

We will also apply the knowledge learnt in section 5 to find the equation of tangents and normal top curves through some given points on the curve. It is important to remember the following statements:

1. The equation of any straight line is of the form $y - y_1 = m(x - x_1)$ where $(x_1, y_1)$ is a point that lies on the line and the gradient of the line is $m$
2. If $m_1$ is the gradient of a line, then the gradient of any straight line perpendicular to it, $m_2 = \frac{1}{m_1}$

**Example 1**

Find the equation of the tangent line to the curve $y = 6x^2$ at the point $(1,6)$

**Solution:**

**Step 1:** Find slope of the tangent

$y = 6x^2$

$\frac{dy}{dx} = 12x$ so at $(1,6)$ the slope of the tangent will $m = 12(1) = 12$

**Step 2:** Using $y - y_1 = m(x - x_1)$ and the point $(1, 6)$,

$y - 6 = 12(x - 1)$

we have, $y = 12x - 6$

**Example 2**

Find the equation of the normal line to the curve $y = 6x^2 + 3x - 2$ at the point $(1,7)$

**Solution:**

$y = 6x^2 + 3x - 2$

$\frac{dy}{dx} = 12x + 3$ so at $(1,7)$ the slope of the tangent will $m = 12(1) + 3 = 15$

And the gradient of the normal will be $-\frac{1}{15}$

**Step 2:** Using $y - y_1 = m(x - x_1)$ and the point $(1, 7)$,

$y - 7 = -\frac{1}{15}(x - 1)$

we have, $x + 15y + 106 = 0$

## Theme or Focal Area: Rate of change of a function

**Example 1:**

A dropped ball has height in *metres* modelled by $h(t) = 100 - 4.9t^2$, $t$ seconds after it is released. How fast is the ball going at time $t = 2$ *s*?

**Solution:**

At time $t = 2$, height is $100 - 4.9*(22) = 80.4$

A second later, that is at $t = 3$, what will be the height?

The height, $h(3) = 100 - 4.9*(9) = 55.9$, so in that second the ball has traveled $80.4 - 55.9 = 24.5$ *meters*. This means that the average speed during that time was 24.5 metres per second.

**Note:** If $\Delta t$ is some tiny amount of time, what we want to know is what happens to the average speed $\frac{h(2) - h(2 + \Delta t)}{\Delta t}$ as $\Delta t$ gets smaller and smaller. Doing a bit of algebra:

$$\begin{aligned}\frac{h(2) - h(2 + \Delta t)}{\Delta t} &= \frac{80.4 - (100 - 4.9(2 + \Delta t)^2)}{\Delta t} \\ &= \frac{80.4 - 100 + 19.6 + 19.6\Delta t + 4.9\Delta t^2}{\Delta t} \\ &= \frac{19.6\Delta t + 4.9\Delta t^2}{\Delta t} \\ &= 19.6 + 4.9\Delta t\end{aligned}$$

When $\Delta t$ is very small, this is very close to 19.6, and indeed it seems clear that as $\Delta t$ goes to zero, the average speed goes to 19.6, so the exact speed at $t = 2$ is 19.6 metres per second.

**Example 2**

When the COVID-19 hit Accra, Public Health Officials estimated that the number of persons having the COVID-19 at time t (measured in days from the beginning of the COVID-19) is approximated by

$P(t) = 45t^2 - t^3$, provided that $0 \leq t \leq 50$.

**a)** At what rate is the COVID-19 spreading when $t = 10$?

**b)** When is the COVID-19 spreading at the rate of 675 per day?

**Solution:**

**a)** The rate at which the COVID-19 is spreading is given by the derivative;

$$P'(t) = 90t - 3t^2 \ (0 \leq t \leq 50)$$

Since $P(t)$ is measured in people and time is measured in days, the rate $P'(t)$ is measured in people per day. When $t = 10$

$P'(10) = 90(10) - 3(10)^2 = 600$

Thus, 10 days after the beginning of the COVID-19, it spreads at the rate of 600 people per day.

**b)** In this case, we are given the rate of change of $P(t)$ and we must find the time corresponding to that rate. We set the expression for $P'(t)$ equal to 600 and solve for $t$;

$90t - 3t^2 = 675$

$90t - 3t^2 - 675 = 0$

Dividing by $-3$ and factoring, we have

$t^2 - 30t + 225 = 0$

$(t - 15)^2 = 0$

$t = 15$. After 15 days, COVID-19 is spreading at the rate of 675 people per day.

**Example 3**

**a)** Suppose that $f(x) = x^2$. Calculate the average rate of change of $f(x)$ over the intervals 1 to 2, 1 to 1.1, and 1 to 1.01

**b)** Determine the (instantaneous) rate of change of $f(x)$ when $x = 1$

**Solution**

**a)** The intervals are of the form 1 to $1 + \Delta x$ for $\Delta x = 1, 0.1$ and 0.01. the average rate of change is given by the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(1+\Delta x) - f(1)}{\Delta x} = \frac{(1+\Delta x)^2 - 1}{\Delta x}$$

For the three given values of $\Delta x$, this expression has the following respective values

$\Delta x = 1$ for which

$\frac{\Delta y}{\Delta x} = \frac{2^2 - 1}{1} = \frac{4 - 1}{1} = 3$

$\Delta x = 0.1$ for which

$\frac{\Delta y}{\Delta x} = \frac{1.1^2 - 1}{0.1} = \frac{1.21 - 1}{0.1} = 2.1$

When $\Delta x = 0.01$,

$\frac{\Delta y}{\Delta x} = \frac{1.01^2 - 1}{0.01} = \frac{1.0201 - 1}{0.01} = 2.01$

Thus, the average rate of change for $\Delta x = 1$, 0.1 and 0.01 is 3, 2.1 and 2.01 units per unit change in $x$ respectively

**b)** The instantaneous rate of change of $f(x)$ at $x = 1$ is equal to $f'(1)$. We have

$f'(x) = 2x$

$f'(1) = 2 \cdot 1 = 2$

That is, the instantaneous rate of change is 2 units per unit change in $x$.

**Learning Tasks**

1. Learners solve real-world scenarios or advanced mathematical problems that require applying polynomial differentiation in context
2. Task learners in pairs to use the graphing tool to draw the graph of functions and deduce the rate of rate or certain points
3. Task learners in groups to create real-life problems involving rates of change
4. Task learners in groups to sketch cubic functions and rational functions

## Pedagogical Exemplars

1. **Collaborative Learning**: Learners working in mixed-ability groups explore ways of finding the rate of change of a given function or phenomenon at certain points.
2. **Individualised Learning:** tailor learning tasks to suit individual learners

a. Offer students a menu of learning tasks or assignments related to polynomial functions and derivatives. Allow them to choose tasks based on their interests, learning preferences or proficiency levels. Tasks can include problem-solving activities, investigations, presentations or creative projects

b. Develop learning contracts that outline specific learning objectives, tasks and assessment criteria for each student. Students can negotiate their learning goals and select activities that align with their strengths and areas for growth

3. **Problem-based learning:** Present real-world problems or scenarios that require the application of derivative concepts. Allow students to work collaboratively to solve these problems, encouraging critical thinking and problem-solving skills

4. **Exploratory learning:** Use knowledge of finding the slope between two points to find the rate of change at a point. Explore ways of finding the rate of change of a given function or phenomenon at certain points

## Key Assessment

**Assessment Level 1: Recall and reproduction**

1. State the conditions that are true for a given function as:

   a) it increases **[Expected answer: When the derivative is positive]**

   b) decreases **[Expected answer: When the derivative is negative]**

   c) it is momentarily at rest **[Expected answer: When the derivative is zero]**

2. What is the rate of change of the volume of a ball ($V = \frac{4}{3}\pi r^3$) with respect to the radius when the radius, $r$ is 2? **[Expected answer: 16$\pi$]**

**Assessment level 2: Skills of conceptual understanding**

Find an equation for the tangent and normal line to the curve at the given point

a) $y = (x - 1)^2 + 1$, $(1, 1)$**[Expected answer: Tangent: $y = 1$, Normal: $x = 1$]**

b) $y = 4 - x^2$, $(-1, 3)$ **[Expected answer: Tangent: $2x - y - 5 = 0$, Normal: $x + 2y + 5 = 0$]**

**Assessment level 3: Strategic reasoning**

A water tank is being filled with water at a variable rate. The amount of water in the tank, in cubic metres, at time $t$ minutes is given by the equation $V(t) = 2t^3 - 5t^2 - 10t$. At time $t = 4$ minutes, what is the rate of change of the water in the tank. Interpret your results

**[Expected answer: 46 $m^3$ *per minute*]**

## Section 8 Review

In the current sections on limits and differentiation, we focused on essential themes that are crucial for understanding these fundamental mathematical concepts. We emphasised the significance of building a strong foundation through intuitive explanations and real-life applications.

To cater for different learning styles and abilities, we utilised diverse strategies such as tiered assignments tailored to individual needs, flexible grouping to promote collaboration and scaffolded tasks to support learner progression. Additionally, we used visual aids such as graphs and diagrams to enhance comprehension and foster engagement.

Assessment methods played a crucial role in measuring learner understanding and progress. Formative assessments such as quizzes and peer reviews provided valuable feedback for both learners and teachers, allowing for adjustments in instruction as needed. Furthermore, collaborative activities and problem-solving tasks encouraged active participation and critical thinking skills development.

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# SECTION 9: STATISTICS

Strand: **Handing Data**

**Sub-Strand:** Organising, representing and interpreting Data

**Content Standard:** Investigate techniques for collecting data and determine measures of central tendency and dispersion

**Learning Outcomes:**

1. *Collect quantitative and qualitative data, organise and present data using graphs*
2. *Calculate the measures of central tendencies, and measures of dispersion and use simple language to interpret the results.*

## INTRODUCTION AND SECTION SUMMARY

For centuries, Ghanaians have relied on data collection and analysis to make informed decisions. From tracking cocoa yields to monitoring rainfall patterns, understanding this information is key to our success. This section begins with week 19 by examining how data is collected, organised and classified. Imagine sorting different types of fish at the Makola Market – categorising them by size, colour or even the region they were caught. This is similar to how data can be grouped based on its characteristics. Week 20 focuses on visualising continuous data, i.e., data that can take on any value within a range. Think about the daily high temperatures recorded throughout the year in Accra. We can use graphs and charts to represent these continuous changes, revealing trends and patterns over time. Finally, Week 21 delves into measures of central tendency and dispersion. Just as knowing the average height of people in your village gives you a general idea, central tendency statistics summarise a dataset's " centre." Dispersion tells us how spread out the data is – are most people close to the average height or is there a wide range? Throughout this section, we will explore real-life applications relevant to Ghana. From analysing market trends to understanding crop yields, you will gain valuable skills to make informed choices for yourself and your community.

*The weeks covered by the section are:*

**Week 19:**

1. Data collection and representation methods
2. Data categorisation

**Week 20**

1. Graphical Representation
2. Comparative statistical representations

**Week 21:**

1. Measures of central tendency
2. Measures of dispersion

## SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities in collection, categorisation and representation of data, graphical representation of data, as well as measures of central tendencies and dispersion. Learners should be given the platform to work in groups to develop their real-life questions and find answers. Therefore, experiential learning activities and mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to fully take part in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

## ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests and homeworks can be used to evaluate learners' understanding of concepts and their ability to solve problems. Performance tasks such as solving real-world problems involving collection, categorisation and representation of data, graphical representation of categorical and continuous data, as well as measures of central tendencies and dispersion will also be used to assess learner's application of these mathematical skills. Role playing will also be incorporated to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment record.

# WEEK 19

**Learning Indicators:**

1. *Identify and present appropriate ways of collecting and representing data.*
2. *Categorise data and determine which scale of measurement describes the data.*
3. *Organise data into appropriate frequency distribution tables manually and with Microsoft Excel.*

## Theme or Focal Area: Collecting and representing data

**Data collection**

Data is everywhere! It is the information we gather to understand the world around us. This process starts with data collection, where we use various tools such as:

i. *Interviews:* One-on-one conversations asking targeted questions.

ii. *Questionnaires:* Surveys with written questions for a wider audience.

iii. *Observation:* Recording and analysing behaviours or events systematically.

Once we have data, we need to make it clear and understandable. Here's where data representation comes in where we use tools like:

i. *Tables:* Organise data in rows and columns for easy comparison.

ii. *Graphs:* Visualise trends and patterns using charts like bar graphs or line graphs.

By collecting and representing data effectively, we can turn information into knowledge, helping us make informed decisions and solve real-world problems.

**Importance of statistics in real-life**

Statistics might seem like just calculations and graphs but they are the power behind understanding the world around you! Some important statistics in your everyday life include:

i. *Making Informed Decisions:* From choosing the best mobile network to comparing exam results, statistics help us analyse data and make informed choices.

ii. *Success in Many Careers:* Whether you dream of working in finance, marketing, medicine or even sports analysis, statistics are a valuable skill that are highly employable. Data Scientists, data analysts, data engineers and biostatisticians are among the high sought-after experts in our world.

iii. *Understanding News and Information:* News reports are full of statistics – percentages, averages and graphs. Statistics help us interpret this information critically.

iv. *Solving Real-World Problems*: From improving crop yields to managing traffic flow, statistics are used to analyse situations and develop solutions. For instance, statistics can help farmers understand weather patterns and optimise planting seasons for better harvests.

v. Statistics are used in government reports, public health initiatives and social development programmes.

**Importance of data and their uses to make informed decisions and policies**

From crop yields, medical records to exam scores, data holds immense power! Data helps us make informed decisions in all aspects of life. Data is important in the following ways:

- **i.** *Better Policies:* Imagine policymakers using data on school enrolment rates to target areas needing more classrooms. Data helps identify problems and design effective solutions for education, healthcare and infrastructure.
- **ii.** *Smarter Businesses:* Local businesses can analyse customer preferences (like popular fabric patterns at a market) to stock what sells best. Data empowers businesses to make informed choices and thrive in the Ghanaian market.
- **iii.** *Efficient Resource Management:* Data on water usage in communities can help identify areas with water scarcity.
- **iv.** *Improved Service Delivery:* Data on traffic patterns can guide traffic light placement, reducing congestion and improving commute times.
- **v.** *Empowering Individuals:* Farmers can use data on weather patterns to plan planting seasons for higher yields.

**Sampling techniques**

During data collection for study, one cannot study or interview everyone. Sampling helps us choose a representative group to learn about a larger population. Some common sampling techniques include:

- **i.** *Simple Random Sampling*: Imagine writing your classmate's names on pieces of paper and picking them out of a hat to give the selected ones a gift of black chocolate. This ensures everyone has an equal chance of being chosen.
- **ii.** *Systematic Sampling*: List everyone in order and choose every $k^{th}$ person, (where $k = \frac{N}{n}$ ($k$ = *interval*), *N is the population size and n is the sample size required*) until you have your sample size required. Works well for large, ordered and finite lists.
- **iii.** *Stratified Sampling:* Divide the population into subgroups (strata) based on a key characteristic (e.g., region, age group). Randomly select participants from each subgroup to ensure your sample reflects the population proportions. Useful when studying diverse populations.
- **iv.** *Cluster Sampling:* Instead of individuals, we randomly select groups (clusters) from the population. Useful in geographically spread-out populations or when reaching individuals within groups is easier (e.g., studying health practices in randomly chosen villages).
- **v.** *Convenience Sampling:* You choose the easiest people to access (e.g., classmates, neighbours). Not ideal for generalisable research (results may not represent the whole population).
- **vi.** *Purposive Sampling (Judgmental Sampling):* Participants are selected based on their specific characteristics relevant to the research question. Ideal for in-depth information from a specific group with expertise or experience (e.g., interviewing experienced cocoa farmers about sustainable practices).
- **vii.** *Snowball Sampling:* Start with a few participants who meet your criteria and ask them to refer others in their network who share similar characteristics. Useful for studying hard-to-reach populations.
- **viii.** *Quota Sampling:* Set quotas (target numbers) for different subgroups within the population and then choose participants until those quotas are filled.

The best sampling technique depends on the research question and the population you are studying

## Theme or Focal Area: Data Categorisation

In any research, we collect data. But data comes in different forms. The two main types of data are quantitative and qualitative data.

**i.** *Quantitative Data:* This is all about numbers and measurements! We can count it, measure it or express it as a numerical value. Examples include:

- Number of mangoes harvested this season.
- Heights of students in your class.
- Scores on a math test.
- Prices of different fabrics at a market stall.
- Temperatures recorded throughout the day in Accra.
- Age of students
- Time for completing a race
- Number of students reading various courses in school

**ii.** *Qualitative Data:* This data describes experiences, qualities or opinions. It's more about words and stories. Examples are;

- Reasons why students choose a particular secondary school.
- Students' favoured food
- Descriptions of traditional Ghanaian festivals.
- Opinions on the effectiveness of a new government policy.
- Experiences of taxi drivers in navigating rush hour traffic.
- Observations about the different types of music played at a wedding.
- Gender of a student
- A person's nationality

Understanding the difference between these data types is crucial. Quantitative data allows you to calculate statistics and identify trends. Qualitative data provides details and helps understand the "why" behind things.

**Categorisation of qualitative and quantitative data**

Quantitative data may be continuous or discrete. Examples of continuous quantitative data include age, weight, height or temperature of students as these quantities are never constant and must be measured to obtain. Discrete quantitative data include the number of light bulbs, the number of desks, the number of students etc in a classroom. Discrete data are simply counted to obtain, thus are also known as count data.

Qualitative and quantitative data are further divided into nominal, ordinal, interval and ratio scales of measurement. Qualitative data are categorised into nominal and ordinal scales of measurement. Quantitative data are grouped into interval and ratio measurement scales. Measurement scales tell us what kind of information the numbers we assign represent.

**i.** *Nominal:* These are simply categories or labels with no inherent order or ranking. Examples are Favourite Ghanaian Dish: Jollof Rice, Waakye, Fufu with Light Soup, gender (Male, Female), Position on a ballot for an election, ethnic group etc.

*Use case:* Grouping preferences or classifying objects

**ii.** *Ordinal*: These categories have a clear order but the differences between them are not necessarily equal. Example: Movie Rating: 1 star (worst), 2 stars, 3 stars, 4 stars, 5 stars (best), grades in exams (A1, B2, B3,…), Level of Education (Primary, JHS, SHS), Level of happiness (Very happy, Not happy, A little happy).

*Use case***:** Ranking items based on opinion or quality (e.g., customer satisfaction surveys).

**iii.** *Interval:* The intervals between units on the scale are fixed and equal. However, there's no true zero point. Example: Temperature in Celsius: The difference between 20°C and 30°C is the same as the difference between 10°C and 20°C. But 0°C doesn't represent the absolute absence of heat.

*Use case:* Measuring temperature, calendar years, IQ scores (though intelligence itself can't be directly measured on a true zero-based scale).

**iv.** *Ratio:* These scales have a fixed zero point that represents a complete absence of the quantity being measured. Ratios can be formed between values. Example: Height in centimetres: 0 cm represents no height, and you can say someone is twice as tall as another person (e.g., 180 cm vs. 90 cm).

*Use case:* Measuring weight, length, age, etc.

Understanding these scales is crucial for interpreting data correctly. Imagine comparing exam scores on a "star rating" system (ordinal) vs actual marks (ratio).

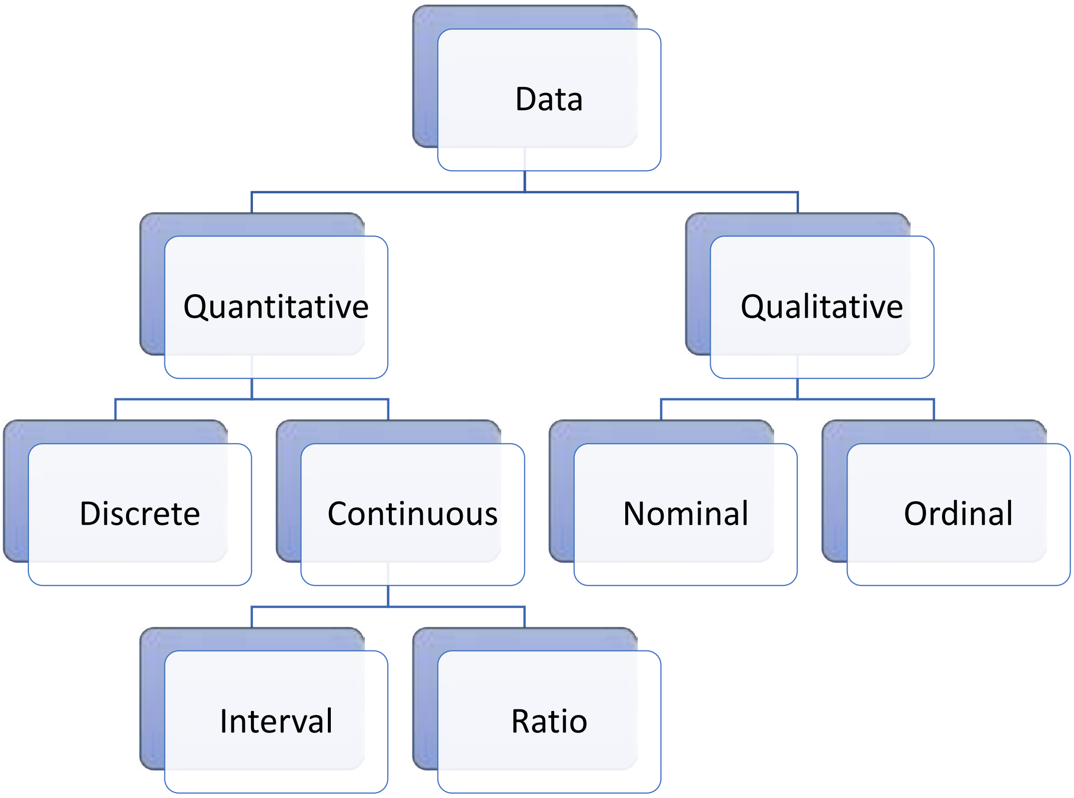



**Figure 1:** *Summary of data hierarchy*

## Theme or Focal Area: Tabular representation of data

**Definition/Introduction**

**a.** Representing ungrouped (raw) data on frequency tables

Ungrouped data is the data you gather from an experiment that is not sorted into categories or classified.

**Example:**

The ages of some randomly selected football players are as follows:

24, 23, 25, 23, 30, 24, 37, 25, 23, 22, 25, 22, 31, 29, 22, 25, 21, 25, 24, 24, 22.

**a.** Make a frequency table for the data

**Solution**

| Age | Tally | Frequency |
|---|---|---|
| 21 | / | 1 |
| 22 | //// | 4 |
| 23 | /// | 3 |
| 24 | /// | 3 |
| 25 | ~~////~~ | 5 |
| 29 | / | 1 |
| 30 | / | 1 |
| 31 | / | 1 |
| 37 | / | 1 |
| Total | | 20 |

**b.** Grouped data is data that has been bundled together in categories, that is when given a large data, it is better to group them using intervals.

**Example**

The data below shows the mass of 40 students in a class. The measurement is to the nearest kg.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 55 | 70 | 57 | 73 | 55 | 59 | 64 | 72 |
| 60 | 48 | 58 | 54 | 69 | 51 | 63 | 78 |
| 75 | 64 | 65 | 57 | 71 | 78 | 76 | 62 |
| 49 | 66 | 62 | 76 | 61 | 63 | 63 | 76 |
| 52 | 76 | 71 | 61 | 53 | 56 | 67 | 71 |

**1.** Construct a frequency table for the data using the intervals 45-49, 50-54 etc.

**Solution**

| Mass (kg) | Frequency | Class boundaries | Class midpoint ($x$) | $fx$ |
|---|---|---|---|---|
| 45 – 49 | 2 | $44.5 \leq x < 49.5$ | 47 | 94 |
| 50 – 54 | 4 | $49.5 \leq x < 54.5$ | 52 | 208 |
| 55 – 59 | 7 | $54.5 \leq x < 59.5$ | 57 | 399 |
| 60 – 64 | 10 | $59.5 \leq x < 64.5$ | 62 | 620 |
| 65 – 69 | 4 | $64.5 \leq x < 69.5$ | 67 | 268 |
| 70 – 74 | 6 | $69.5 \leq x < 74.5$ | 72 | 432 |
| 75 - 79 | 7 | $74.5 \leq x < 79.5$ | 77 | 539 |

**Learning task**

Guide learners to collect data such as age, height, favourite food, ethnicity, region of origin etc., from their schoolmates.

Learners to:

**i.** Discuss the importance of statistics and data to the nation and individuals

**ii.** Group data collected into quantitative, qualitative and levels of measurement

**iii.** Construct ungrouped and grouped frequency tables for data collected

## Pedagogical Exemplars

1. **Group Discussion**

   Learners will work in collaborative groups (based on ability, mixed ability, mixed gender or pairs) to:

   - collaborate on fundamental data collection tasks, ensuring active participation from all members.
   - categorise simple data sets and present their methods, discussing the basic rationale behind their choices.

2. **Learning Experience**

   Learners engage in hands-on activity (learning by doing) to

   - identify appropriate methods for collecting and representing data, solving basic hands-on tasks like surveys or simple data collection using elementary tools like basic graphs.
   - categorise data based on simple criteria and complete exercises or projects with basic organisation.

## Key Assessment

**Assessment Level 1: Recall**

1. Research the importance of statistics in real-life, the importance and uses of data and the process of collecting data from a survey or census.
2. discuss the meaning of class width/size.
3. What is the difference between the upper- and lower-class boundaries of a class?

**Assessment Level 2: Conceptual understanding**

1. Assess the nature of the following data as either continuous or discrete:
   a. Counting the number of children in different families in the community.
   **b.** Calculating the monthly income of workers in a factory.
   **c.** Recording the marks obtained by a student in an examination.
   **d.** Measuring the distance travelled by a day student to school each school day.
   **e.** Tracking the amount of pocket money given to students.

2. The heights (in inches) of 30 students in a classroom were measured and recorded as follows:
   64 66 68 70 64 62 66 66 64 68 70 68 66 62

   64 70 68 64 66 68 64 66 62 70 64 66 68 70

   Construct a frequency table for the data, showing the height range and its corresponding frequency count.

3. The weight in kg of some randomly selected football players are as follows
   55, 52, 51, 58, 57, 54, 51, 52, 50, 53, 54, 55, 50, 59, 56, 53, 52, 55, 54, 55, 52.

   With the aid of a technological device (software), draw a pie chart to represent the given information

**Assessment Level 3: Strategic thinking**

1. Consider the following data collected from a survey about the ages of students in a school club: 12, 5, 14, 13, 16, 12, 13, 14, 17, 15.
   a. Determine which scale of measurement (nominal, ordinal, interval, ratio) is most appropriate for categorising the ages of students in the club.
   b. Justify your choice of scale by explaining the characteristics of the data and how they align with the chosen scale.

**Assessment Level 4: Extended reasoning**

1. You are conducting a survey among your classmates to understand their study habits. You asked 40 students about the number of hours they spend studying each week. These are their responses:
   4, 8, 10, 12, 5, 9, 11, 7, 6, 8,

   10, 12, 5, 9, 11, 7, 6, 8, 10, 12,

   5, 9, 11, 7, 6, 8, 10, 12, 5, 9,

   11, 7, 6, 8, 10, 12, 5, 9, 11, 7

   Organise this data into an appropriate frequency distribution table using intervals of 2 hours. Include on the table the range for the study hour and its corresponding frequency count.

2. A survey was conducted among employees of a company to gather data on their monthly salaries in thousands of dollars. The results are as follows:
   28, 35, 41, 47, 51, 54, 56, 57, 58, 60,

   61, 62, 62, 64, 64, 66, 67, 68, 69, 70,

   70, 72, 73, 74, 75, 76, 77, 78, 80, 81,

   83, 88, 90, 94

   a. Create a grouped frequency table for this salary data, using classes such as $19.5 \leq x < 29.5$, $29.5 \leq x < 39.5$ and so on.

# WEEK 20

**Learning Indicators:**

1. *Present data using appropriate graphs by hand and/or by technology and justify why a particular representation is more suitable than others for a given situation.*
2. *Present grouped data using appropriate graphs by hand and/or by appropriate technology and justify why a particular representation is more suitable than others for a given situation.*
3. *Compare various statistical representations and justify why a particular representation is more suitable than others for a given situation.*

## Theme or Focal Area: Graphical Representation of categorical data

### Definition/Introduction

Graphical representation of data is a crucial aspect of statistics. The popular saying that "picture no lie" holds here. Numerical summaries and numbers are great but sometimes they don't tell the whole story. That is where graphical representation comes in! Although graphs can mislead, using graphs and diagrams such as bar charts, pie charts, line graphs, histograms, Ogive curves etc., one can see patterns, identify trends and make comparisons that might be hidden in raw data. Choosing the right graph depends on the type of data one has and what is to be learnt from it. Graphs find their applications in business reports and presentations, market research and consumer behaviour analysis, public policy and government reports, health care, medical data etc.

Note that:

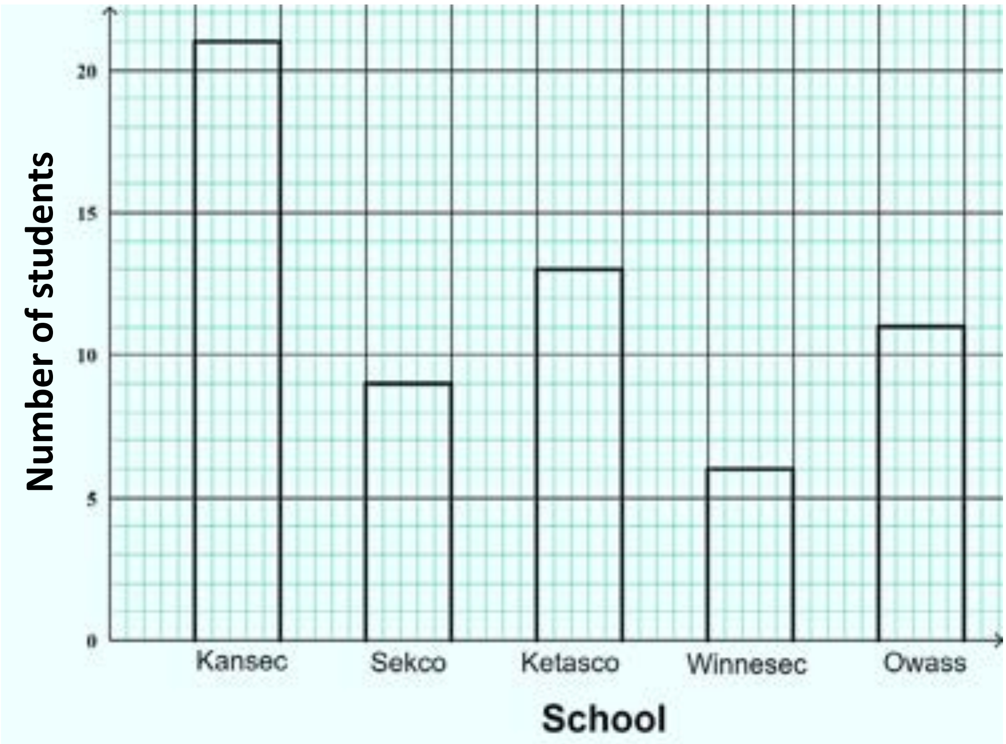
a. **Bar charts** are used to show numbers that are independent of each other. For example the ages of footballers, preference for a particular food, university, programme etc. Bar charts are mostly used for comparisons. Each bar represents a category, and its height shows the value. They work best for comparing things that don't change over time (e.g., student test scores in different subjects).

### Example

1. The data below shows a number of final-year science students in the Tutco Demonstration School and their first-choice Senior High Schools. Represent the data on a bar chart.

| School | Kansec | Sekco | Ketasco | Winnesec | Owass |
|---|---|---|---|---|---|
| No. of Students | 21 | 9 | 13 | 6 | 11 |

**Solution**




**2.** The ages of some randomly selected football players are as follows: 24, 23, 25, 23, 30, 24, 37, 25, 23, 22, 25, 22, 31, 29, 22, 25, 21, 25, 24, 24, 22

a. Represent the data on a frequency distribution table

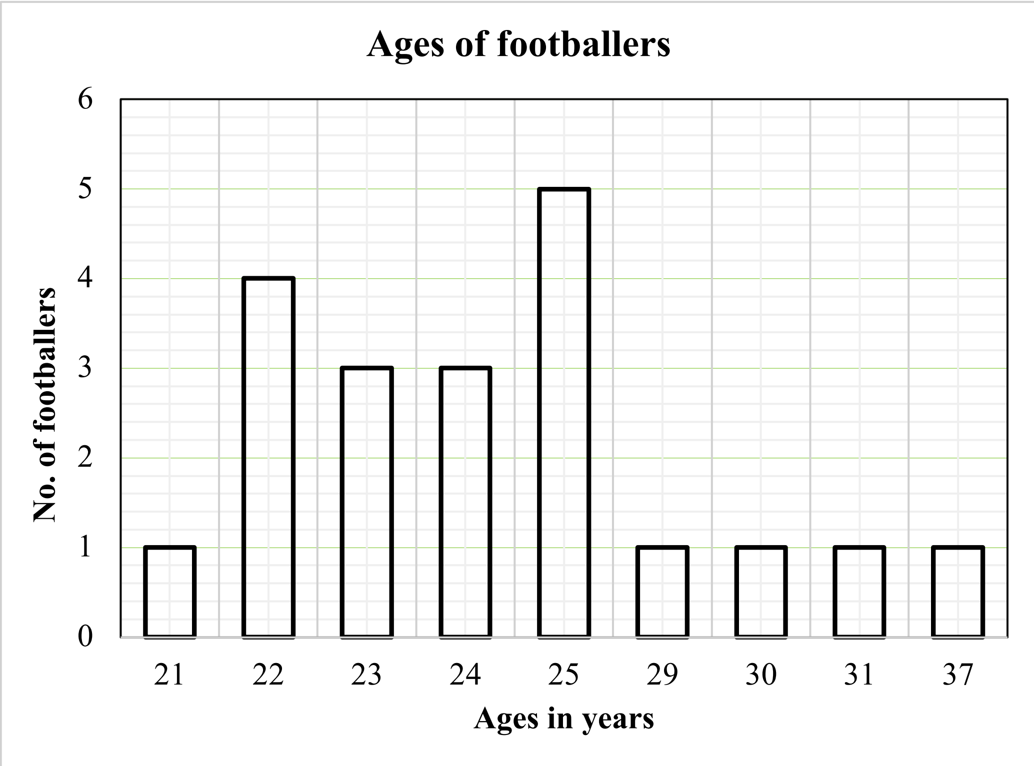
b. Represent this information on a bar chart.

**Solution**

**a.** Frequency table

| Age | Tally | Frequency |
|---|---|---|
| 21 | / | 1 |
| 22 | //// | 4 |
| 23 | /// | 3 |
| 24 | /// | 3 |
| 25 | $\cancel{////}$ | 5 |
| 29 | / | 1 |
| 30 | / | 1 |
| 31 | / | 1 |
| 37 | / | 1 |
| | | $\sum f = 20$ |

**b.** Bar chart

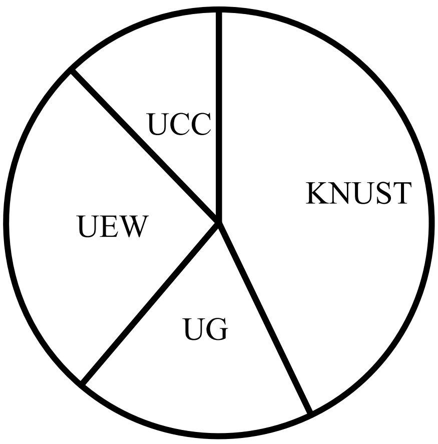



**b.** **Pie Charts** are used to show how a whole is divided into different parts. For example: how a budget is spent on items in a home within a month. They are mostly used to represent distinct or categorical data.

**Example**

Represent the data set on students' university preference in example 1 above on a pie chart

**Solution**



## Theme or Focal Area: Graphical representation of continuous data

**Definition/Introduction**

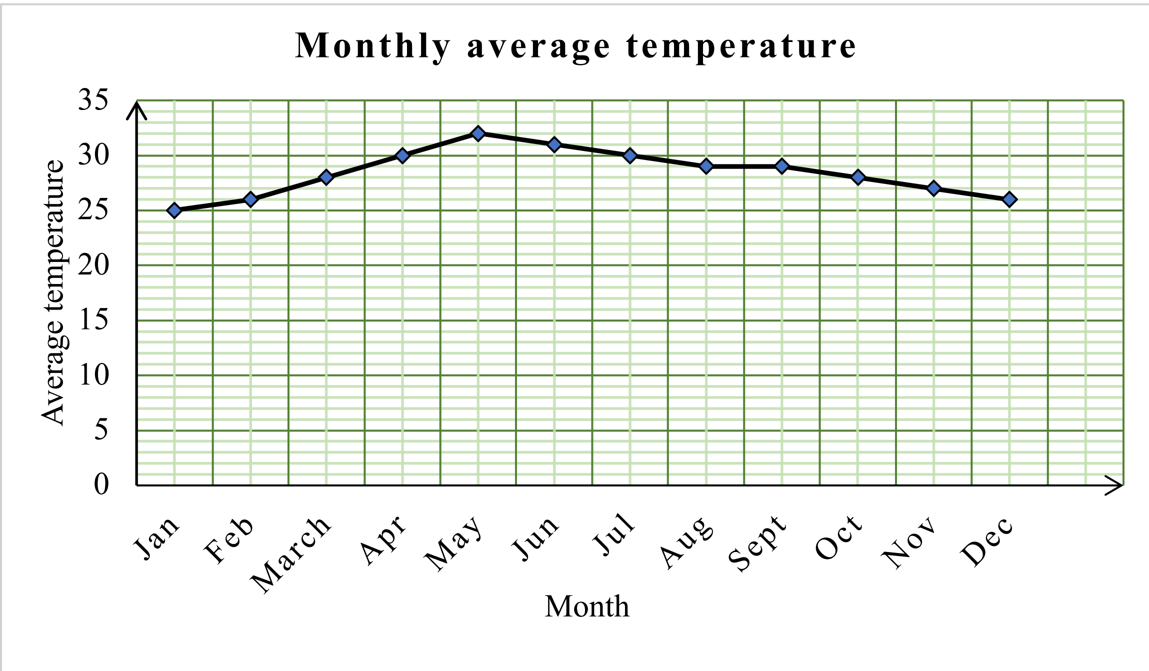
Graphical representation of continuous data is vital for visualising measurements like height or temperature within a range. Techniques include line graphs, histograms and cumulative frequency curves. Understanding this helps interpret trends in science, economics and decision-making field.

**a.** **Line graphs** are used to show trends over time between numbers that are connected. For example: the temperature in each month of the year.

**Example**

The average monthly temperature of a region in Ghana is given in the table below. Fit a line graph to represent the data.

| **Month** | Jan | Feb | March | Apr | May | Jun | Jul | Aug | Sept | Oct | Nov | De |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Average Temp. (°C)** | 25 | 26 | 28 | 30 | 32 | 31 | 30 | 29 | 29 | 28 | 27 | 26 |



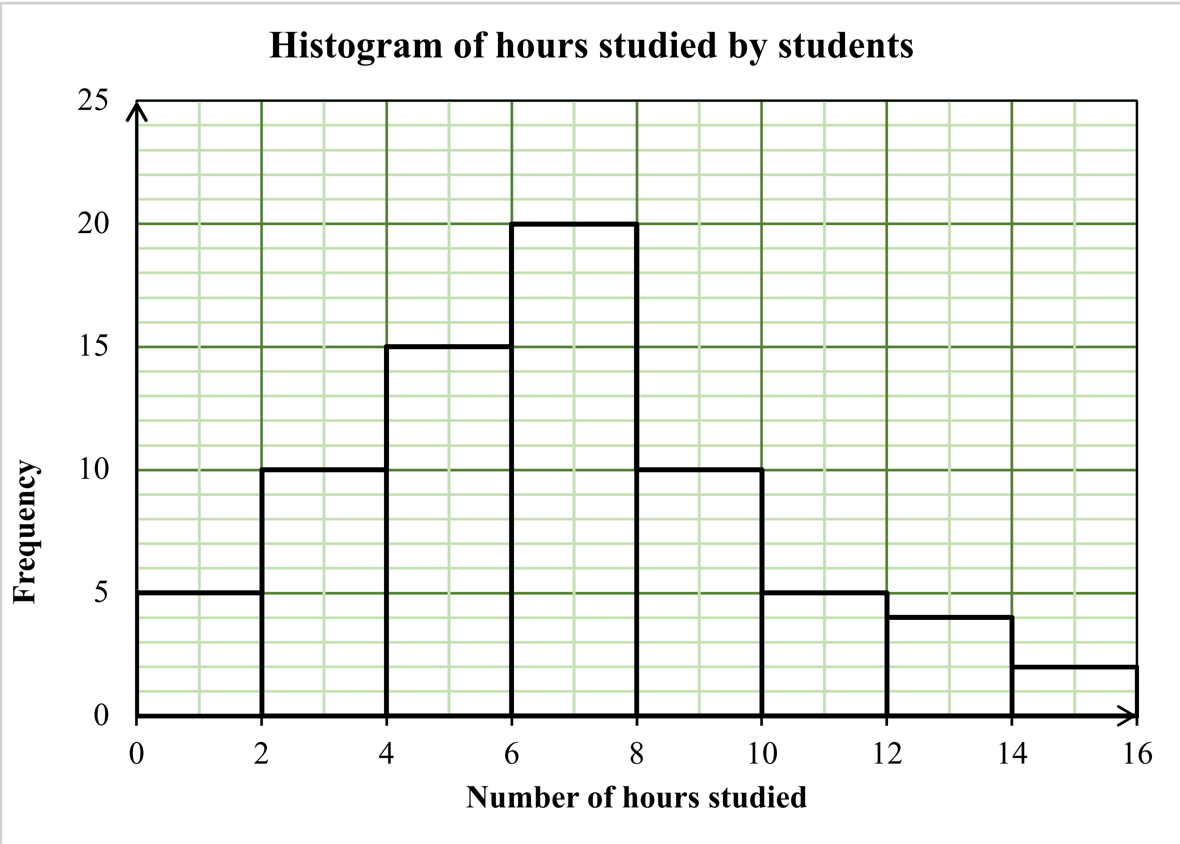

**b.** represent data using histogram with equal intervals

**Example**

The dataset below represents the number of hours studied by students, draw a histogram representing the information

| Hours Studied ($x$) | Number of Students |
|---|---|
| $0 \le x < 2$ | **5** |
| $2 \le x < 4$ | **10** |
| $4 \le x < 6$ | **15** |
| $6 \le x < 8$ | **20** |
| $8 \le x < 10$ | **10** |
| $10 \le x < 12$ | **5** |
| $12 \le x < 14$ | **4** |
| $14 \le x < 16$ | **2** |

**Solution**




c. to represent data with unequal intervals, it is necessary for the area of each bar in a histogram, rather than the height, to represent the frequency. To draw a histogram for unequal class intervals, you need to adjust the heights of the bars so the area is proportional to the frequency. In general, the area of a bar is proportional to the frequency of that particular class interval.

**NOTE:** The height of the bar, called the frequency density, is found by dividing the frequency by the class width, i.e. frequency density $= \frac{\textit{frequency}}{\textit{class width}}$

**Example**

The table gives information about the speed of cars in $km/h$ of 81 cars

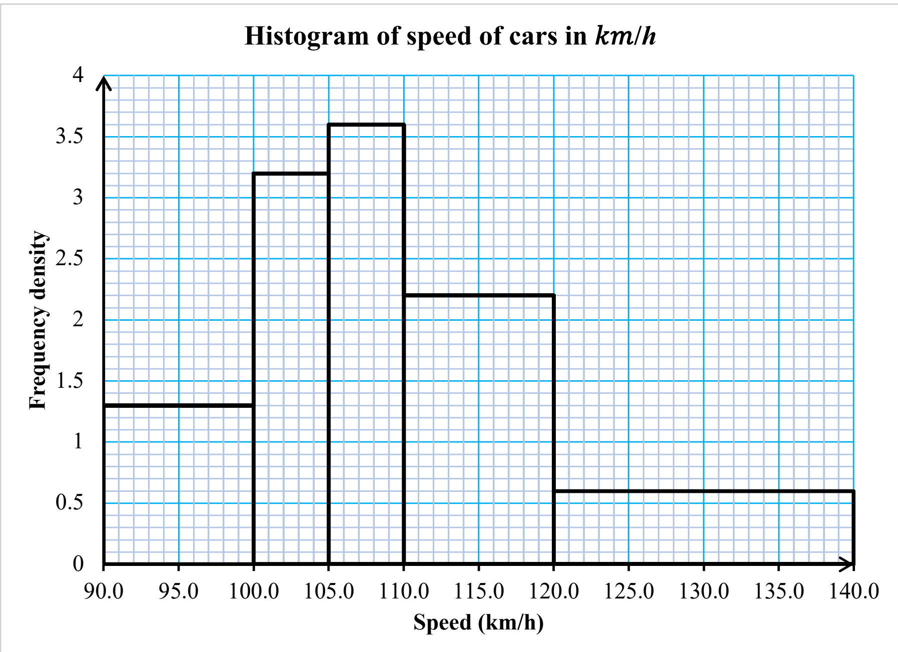
| Speed(s) $km/h$ | Frequency |
|---|---|
| $90 < s \le 100$ | 13 |
| $100 < s \le 105$ | 16 |
| $105 < s \le 110$ | 18 |
| $110 < s \le 120$ | 22 |
| $120 < s \le 140$ | 12 |

Draw a histogram to represent the information in the table.

**Solution:**

**Frequency table**

| Speed | Class boundaries | Frequency | Class width | Frequency density |
|---|---|---|---|---|
| $90 < s \le 100$ | 90 and 100 | 13 | 10 | $13/10 = 1.3$ |
| $100 < s \le 105$ | 100 and 105 | 16 | 5 | $16/5 = 3.2$ |
| $105 < s \le 110$ | 105 and 110 | 18 | 5 | $18/5 = 3.6$ |
| $110 < s \le 120$ | 110 and 120 | 22 | 10 | $22/10 = 2.2$ |
| $120 < s \le 140$ | 120 and 140 | 12 | 20 | $12/20 = 0.6$ |




**d.** Draw a cumulative frequency curve for a given data

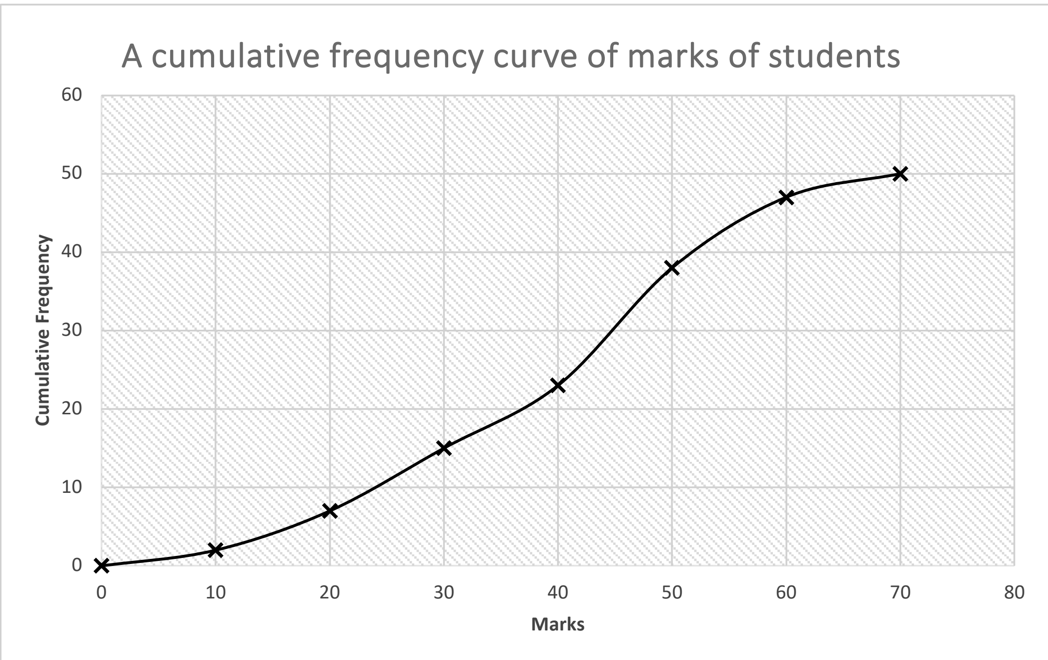
**Example**

The distribution of marks for 50 students in a test is given in the table. Draw a cumulative frequency curve to represent the distribution.

| Marks | Frequency |
|---|---|
| 0-9 | 2 |
| 10-19 | 5 |
| 20-29 | 8 |
| 30-39 | 8 |
| 40-49 | 15 |
| 50-59 | 9 |
| 60-70 | 3 |

**Solution**

| Marks | Frequency | Cumulative Frequency Boundaries | Cumulative frequency |
|---|---|---|---|
| $0 \le x < 10$ | 2 | $0 \le x < 10$ | 2 |
| $10 \le x < 20$ | 5 | $0 \le x < 20$ | 7 |
| $20 \le x < 30$ | 8 | $0 \le x < 30$ | 15 |
| $30 \le x < 40$ | 8 | $0 \le x < 40$ | 23 |
| $40 \le x < 50$ | 15 | $0 \le x < 50$ | 38 |
| $50 \le x < 60$ | 9 | $0 \le x < 60$ | 47 |
| $60 \le x < 70$ | 3 | $0 \le x < 70.$ | 50 |



## Theme or Focal Area: Justifying the choice of a particular representation

**Definition/Introduction**

Comparison of the various graphical representations based on data type, variable type and levels of measurement.

Below is a breakdown of choosing the right graphical representation based on data type, variable type and level of measurement:

a. **Data Type:**
   i. Categorical Data (Nominal or Ordinal): Use bar charts for comparisons between categories (e.g., customer satisfaction). Pie charts can work for proportions within a single dataset (e.g., budget breakdown).
   ii. Numerical Data (Interval or Ratio): Line graphs showcase trends over time (e.g., stock prices). Histograms visualise the distribution of continuous data (e.g., test scores).

b. **Variable Type:**
   i. Independent Variable (Categorical): Bar charts or grouped bar charts are ideal for comparing effects across categories (e.g., plant growth with different fertilizers).
   ii. Dependent Variable (Numerical): Line graphs, scatter plots (for relationships between two numerical variables), or histograms (for single numerical variables) are suitable choices.

c. **Level of Measurement:**
   i. Nominal: Use bar charts (e.g., eye colour distribution). The order of bars does not matter.
   ii. Ordinal: Bar charts or ordered bar charts work well (e.g., customer satisfaction ratings). The order has meaning.
   iii. Interval/Ratio: Line graphs, scatter plots and histograms are appropriate. The Order and equal intervals/ratios matter. Cumulative frequency curves (for ordered data) show the percentage of data points below a specific value.

**Learning Tasks for practice**

Guide learners to explore how to input and analyse data using software such as Excel or any other software

Research the various statistical graphical representations discussed in the classroom and justify why a particular representation is more suitable for a particular data

## Pedagogical Exemplars

The objective of this week's lessons is to help learners explore and understand various graphical representations of data and for learners to be able to justify the choice of a particular graphical representation. The following pedagogical approaches are suggested for effective class facilitation:

1. **Group Discussion:** Learners will be working in convenient groups (ability, mixed ability, mixed gender or pairs etc) to:
   a. discuss different types of graphs and their suitability for specific data presentation.
   b. discuss and present grouped data effectively through graph selection

2. **Talk for Learning:** Learners will be working in convenient groups (ability, mixed ability, mixed gender or pairs etc.) to:
   a. conduct talk sessions where learners share their experiences in selecting and justifying graph types
   b. use real-life examples to demonstrate the impact of choosing the right graph for effective data presentation
   c. facilitate discussions where learners justify why a specific representation is more suitable based on factors like data distribution outliers and audience comprehension
3. **Experiential Learning**: Learners engage in hands-on activity (learning by doing) to
   a. choose and create appropriate graphs to present their data, emphasising the justification for their choices
   b. create experiential tasks where learners analyse the same dataset using different statistical representations
   c. provide opportunities for learners to apply their knowledge in practical situations, reinforcing the importance of choosing the right statistical representation

## Key Assessment

**Assessment Level 1: Recall and reproduction**

1. Use the following data sets to answer the following questions:

   Eye colour of 100 Students: Black (30), Brown (15), Blue (30), Green (15), Hazel (10)

   Test Scores (out of 100): 85, 92, 78, 88, 90, 75, 82, 80, 72, 87

   **a.** Which graph is best suited to represent the number of students with different eye colours?

   **[Expected ans:** Bar Graph or Pie Chart**]**

   **b.** What type of graph would you use to show the distribution of test scores?

   **[Expected ans:** Histogram or Line Graph**]**

   **c.** What does the x-axis typically represent in a bar graph?

   **[Expected ans:** Categories (e.g., eye colour types)**]**

   **d.** What does the y-axis typically represent in a histogram? **[Expected ans:** Frequency (number of data points)**]**

   **e.** A pie chart can only represent one data set at a time. True or False: **[Expected ans:** True**]**

**Assessment Level 2: Skills and conceptual understanding**

1. Given the following list of data types or variables types: Height, IQ Score, School rating, Shirt size, country of origin, educational level, student satisfaction in a teacher, age. Temperature, weight, exam score, hair colour
   a. Classify each variable under nominal, ordinal, interval and ratio level of measurement
   b. Complete the table below with the correct variable type under each graph type

| Graphical representation | Variable type (Select one or more from given list) |
|---|---|
| **1.** Histogram | |
| **2.** Pie Chart | |
| **3.** Bar Chart | |
| **4.** Line graph | |

**Assessment Level 3: Strategic Reasoning**

1. An educationist wants to represent the university preferences of senior high school students in a survey. The survey results show a preference for four universities (KNUST, UEW, UG, UCC) along with a “no preference” category.
   a. What type of graph would be most suitable for this data? Explain. **[Expected ans:** A bar graph is suitable. It effectively compares university preferences (categorical data) side-by-side.**]**
   b. Why might using two different graph types (e.g., a bar graph and a pie chart) be a good approach for this data set? **[Expected ans:** A pie chart could be a good secondary visualisation to show the overall proportion of customers who have a university preference vs those with no preference**]**

2. A die was thrown 40 times and the following results were recorded

   6 5 4 1 2 6 1 3 2 6 4 1 3 5 6 3 2 4 6 4

   3 1 1 6 1 2 4 2 5 6 5 6 6 2 1 3 4 2 5 1

   Use the data to draw a histogram and estimate the mode from the histogram

**Assessment Level 4: Extended thinking**

1. The table shows the marks scored by 285 students in a test.

| Marks | Frequency |
|---|---|
| 6-70 | 9 |
| 71-75 | 42 |
| 76-80 | 69 |
| 81-85 | 66 |
| 86-90 | 81 |
| 91-95 | 15 |
| 96-100 | 3 |

   a. Draw a cumulative frequency curve for the distribution.
   b. Use the curve in (a) to estimate;
      i. the pass mark if 60% of the students passed,
      ii. the probability of selecting a student who scored between 72% to 90%.

# Week 21

**Learning Indicators:**

1. *Calculate measures of central tendencies (mode, mean and median) for a given data by formulas or other techniques and establish which is appropriate to report on a given data.*
2. *Work out simple measures of dispersion (range, quartile and, inter-quartile etc) for raw data and interpret them in context.*

## Theme or Focal Area: Measures of central tendency

**Definition/Introduction**

Central tendency describes how data points cluster around a middle value or the centre. There are three main measures:

i. **Mode:** The most frequent value in the data. Imagine a "fashion" - the mode is the most popular choice. In social science, the mode can be used to determine the most common response or opinion in a survey

ii. **Median:** The middle value when the data is ordered from least to greatest. Think of a balanced seesaw - the median is the " centre point" or the fulcrum. the median income is used in economics to represent the typical income of a population, as it is less influenced by extreme high or low incomes

iii. **Mean:** The average of all the data values is calculated by adding all values and dividing by the number of data points. It's like sharing a cake equally - the mean is the amount each person gets. The mean is used in financial analysis to calculate average returns on investments

Mode, median and mean are also known as **averages** and can be used to describe data.

These measures can be used to describe data and provide a quick summary of how the data is distributed.

**Calculating measures of central tendencies**

i. Given raw data
   - Mode: identify the value(s) with the highest frequency or the most occurring value
   - Median: arrange the data in ascending or descending order and locate the middle value.
     - If the number of observations ($n$) is odd, the median is the middle value of the ordered data.
     - If $n$ is even, the median is the average of the two middle values.
     - For odd $n$, the median is the value at position $\frac{n+1}{2}$
     - For even $n$, the median is the average of the values at positions $\frac{n}{2}$ and $\frac{n}{2}+1$.
   - Mean: sum all values and divide by the total number of observations ($n$), $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

   **Examples**

   Find the mode, median and mean of the following dataset; 3, 20, 5, 7, 18, 16, 5, 11, 4, 5, 3

   **Solution**

   Arranging data in ascending order: 3, 3, 4, 5, 5, 5, 7, 11, 16, 18, 20

   Total number of values or observations, $n = 11$, which is odd

   **Mode:** Value with highest frequency: 5

**Median:** Central value: $\left(\frac{n+1}{2}\right)^{th}$ $position = \left(\frac{11+1}{2}\right)^{th}$ $position = 6^{th}$ $so\ median = 5$

**Mean:** $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{3+3+4+5+5+5+7+11+16+18+20}{11} = 8.82$

**ii.** Given an ungrouped tabular data =

- **Mode**: identify value(s) with the highest frequency
- **Median**:
    - If the sum of frequencies $(\sum f)$ is odd, the median i$\left(\frac{\sum f+1}{2}\right)^{th}$ $position$ adding the frequencies from top until the $\left(\frac{\sum f+1}{2}\right)^{th}$ $position$.
    - If $(\sum f)$ is even, the median is the average of the values at positions $\left(\frac{(\sum f)}{2}\right)^{th}$ and $\left(\frac{(\sum f)}{2}+1\right)^{th}$ position.
- Mean: $\bar{x} = \frac{(\sum fx)}{(\sum f)}$

**Example**

The data below is the number of hectares of cocoa plantations owned by farmers in a farming community in the Western Region.

1, 4, 3, 4, 1, 5, 4, 4, 2, 3, 3, 3, 5, 4, 3, 6, 3, 2, 4, 5, 1, 3, 4, 3, 2, 6, 5, 2, 5, 5

Construct a frequency table for the data and find the averages

**Solution**

| No. of Hectares ($x$) | Tally | Frequency ($x$) | $fx$ |
|---|---|---|---|
| 1 | /// | 3 | 3 |
| 2 | //// | 4 | 8 |
| 3 | ~~////~~ /// | 8 | 24 |
| 4 | ~~////~~ // | 7 | 28 |
| 5 | ~~////~~ / | 6 | 30 |
| 6 | // | 2 | 12 |
| | | $\sum f = 30$ | $\sum fx = 105$ |

Finding the averages

**Mode:** Highest frequency = 3 *hectares*

**Median**: $(\sum f)$ is even, thus median is the average of the values at positions $\left(\frac{(\sum f)}{2}\right)^{th}$ and $\left(\frac{(\sum f)}{2}+1\right)^{th}$ position. That is, $15^{th}$ and $16^{th}$ positions. Adding the frequencies from top, $15^{th}$ $position = 3$ and $16^{th}$ position = 4, Thus median 3.5 *hectares*

**Mean:** $\bar{x} = \frac{(\sum fx)}{(\sum f)} = \frac{105}{30} = 3.5$ *hectares*

**iii.** Given a grouped data

- **Mode:** identify the mode by inspection and by using a formula.

  By inspection, the mode is the value with the highest frequency.

  By formula $mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$

  Where:

  $L_1$ = Lower class boundary of modal class,

  $\Delta_1$= excess frequency of modal class over the frequency of the next lower class.

  $\Delta_2$= excess frequency of modal class over the frequency of the next higher class, and

  $C$ = size of modal class

- $\boldsymbol{median} = L_1 + \left(\frac{\frac{1}{2}N - \sum F_1}{F_m}\right)C$

  Where:

  $L_1$ = the lower class boundary of the median class;

  $N$ = total frequency;

  $\sum F_1$ = sum of frequencies of all classes lower than the median class;

  $F_m$ = frequency of the median class;

  $C$ = the size of the median class.

- $\boldsymbol{Mean}\ (\bar{x}) = \frac{\sum fx}{\sum f}$

  Or use the assumed mean formula which is

  $\bar{x} = A + \frac{\sum fd}{\sum f}$

  Where:

  $A$ = the assumed mean

  $d = \bar{x} - A$, which is the deviation from the assumed mean

**Example**

The data below shows the mass of 40 students in a class. The measurement is to the nearest *kg*.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 55 | 70 | 57 | 73 | 55 | 59 | 64 | 72 |
| 60 | 48 | 58 | 54 | 69 | 51 | 63 | 78 |
| 75 | 64 | 65 | 57 | 71 | 78 | 76 | 62 |
| 49 | 66 | 62 | 76 | 61 | 63 | 63 | 76 |
| 52 | 76 | 71 | 61 | 53 | 56 | 67 | 71 |

1. Construct a frequency table for the data using the intervals 45-49, 50-54 etc.
2. Calculate the mean, mode and median.

**Solution**

**1.**

| Mass (kg) | Frequency | Mid-Class mark $(x)$ | $fx$ |
|---|---|---|---|
| $45 \le x < 50$ | 2 | 47.5 | 95 |
| $50 \le x < 55$ | 4 | 52.5 | 220 |
| $55 \le x < 60$ | 7 | 57.5 | 402.5 |
| $60 \le x < 65$ | 10 | 62.5 | 625 |
| $65 \le x < 60$ | 4 | 67.5 | 270 |
| $70 \le x < 75$ | 6 | 72.5 | 435 |
| $75 \le x < 80$ | 7 | 77.5 | 542.5 |
| | $\sum f = 40$ | | $\sum fx = 2590$ |

**2.** $Mean\ (\bar{x}) = \sum\frac{fx}{\sum f}$

$\bar{x} = 2\frac{590}{40}$

$\bar{x} = 64.75\ kg$

identify the mode by inspection and by using a formula.

By inspection identify 62 as the mode.

By formula

$$mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

Where:

$L_1$ = Lower class boundary of modal class,

$\Delta_1$ = excess frequency of modal class over the frequency of the next lower class.

$\Delta_2$ = excess frequency of modal class over the frequency of the next higher class and

$C$ = size of modal class

Applying the formula to solve the mode

$$\text{Mode} = 60 + \left(\frac{3}{3+4}\right) \times 4$$

$$\approx 61.7143$$

Therefore, the mode is 61 to the nearest whole number.

$$median = L_1 + \left(\frac{\frac{1}{2}N - \sum F_1}{F_m}\right)C$$

Where:

$L_1$ = the lower class boundary of the median class;

$N$ = total frequency;

$\sum F_1$ = sum of frequencies of all classes lower than the median class;

$F_m$ = frequency of the median class;

$C$ = the size of the median class.

Applying the formula,

$$Median = 60 + \left(\frac{\frac{1}{2}(40) - 13}{10}\right) \times 4$$

$Median = 62.8$

Using the assumed mean formula which is

$$\bar{x} = A + \frac{\sum fd}{\sum f}$$

Where:

$A$ =the assumed mean

$d = \bar{x} - A$, which is the deviation from the assumed mean.

Using an assumed mean of 62, calculate the mean mass of the students from the student data above.

| **Mass (kg)** | $f_i$ | **Mid-Class mark** $(x_i)$ | **Deviation ($d$)** $x_i - A$ | $f_i d_i$ |
|---|---|---|---|---|
| $45 \le x < 50$ | 2 | 47.5 | $47.5 - 62 = -14.5$ | $-29$ |
| $50 \le x < 55$ | 4 | 52.5 | $52.5 - 62 = -9.5$ | $-38$ |
| $55 \le x < 60$ | 7 | 57.5 | $57.5 - 62 = -4.5$ | $-31.5$ |
| $60 \le x < 65$ | 10 | 62.5 | $62.5 - 62 = 0.5$ | 5 |
| $65 \le x < 60$ | 4 | 67.5 | $67.5 - 62 = 5.5$ | 22 |
| $70 \le x < 75$ | 6 | 72.5 | $72.5 - 62 = 10.5$ | 63 |
| $75 \le x < 80$ | 7 | 77.5 | $77.5 - 72 = 5.5$ | 38.5 |
| | $\sum f = 40$ | | | $\sum fd = 30$ |

the assumed mean

$\bar{x} = 62 + 3\frac{0}{40}$

$\bar{x} = 62.75$

## Theme or Focal Area: Measures of dispersion

### Definition/Introduction

Measures of dispersion quantify the extent to which data values vary from the central tendency, providing insight into the spread or distribution of the data. They complement measures of central tendency by revealing the degree of variability within a dataset. Key measures of dispersion include the range, variance, standard deviation and quartiles. Quartiles and inter-quartile ranges are used to compare the variability of two or more data sets, allowing the analyst to make relative assessments of the spread. Variance and standard deviation are widely used in finance to measure the volatility of asset returns by helping investors assess the risk associated with different investment options. In manufacturing and industrial processes, the range is used for quality control to monitor the variability of product specifications and ensure consistent quality by setting an acceptable range for certain dimensions or parametres.

**a.** *Estimate quartiles of a given data.*

Quartiles divide a dataset into four equal parts, each containing 25% of the data. The **lower quartile** corresponds to the 25$^{th}$ **percentile** i.e. 25% of the total frequency.

The lower quartile or first quartile $(Q_1)$ is the median of the lower half of the dataset.

$Q_1 = \left(\frac{n+1}{4}\right)^{th}$ *position* if $n$ is odd and $Q_1 = \left(\frac{n}{4}\right)^{th}$ *position* if $n$ is even

The second quartile $(Q_2)$ is the median of the entire dataset. **Median** corresponds to the **50$^{th}$ percentile** i.e. 50% of the total frequency.

$Q_2 = \left(\frac{n+1}{2}\right)^{th}$ *position* if $n$ is odd and $Q_2 = \left(\frac{n}{2}\right)^{th}$ *position* if $n$ is even

The upper quartile or third quartile $(Q_3)$ is the median of the upper half of the dataset. The **upper quartile** corresponds to the **75$^{th}$ percentile** i.e. 75% of the total frequency.

$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th}$ *position* if $n$ is odd and $Q_3 = \left(\frac{3n}{4}\right)^{th}$ *position* if $n$ is even

Interquartile range = $\textit{upper quartile} - \textit{lower quartile} = Q_3 - Q_1$

Semi-interquartile range $= \frac{\text{Interquartile range}}{2} = \frac{Q_3 - Q_1}{2}$

### Examples

1. Determine the quartiles and interquartile range for the following set of data: 2, 6, 8, 4, 7, 5, 3

### Solution

Arrange the values in order: 2, 3, 4, 5, 6, 7, 8,

There are 7 values or observations or number of items $n = 7$

**i.** Lower quartile $Q_1 = \left(\frac{7+1}{4}\right)^{th} = \left(\frac{8}{4}\right)^{th} = 2^{nd}$ value. $Q_1 = 3$

**ii.** Second quartile $Q_2 = \left(\frac{7+1}{2}\right)^{th} = \frac{8}{2} = 4^{th}$ *value*. $Q_2 = 5$

**iii.** Third quartile $Q_3 = \left(\frac{3(7+1)}{4}\right)^{th} = 6^{th}$ *value*. $Q_3 = 7$

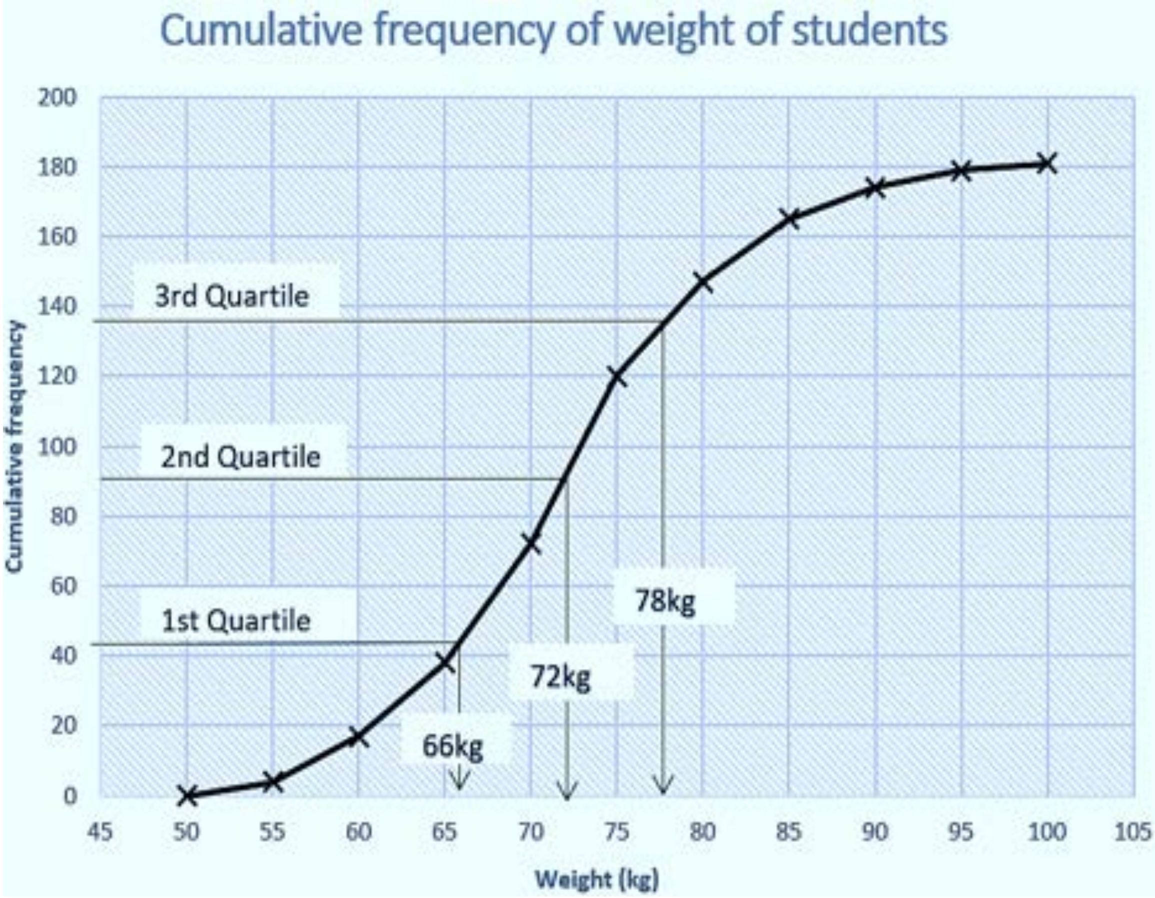
**iv.** Interquartile range $= Q_3 - Q_1 = 7 - 3 = 4$

**2.** Construct a cumulative frequency curve for the data below and estimate the quartiles by plotting it on the cumulative frequency curve.

| Weight (kg) | Frequency |
|---|---|
| $50 \leq w < 55$ | 4 |
| $55 \leq w < 60$ | 13 |
| $60 \leq w < 65$ | 21 |
| $65 \leq w < 70$ | 34 |
| $70 \leq w < 75$ | 48 |
| $75 \leq w < 80$ | 27 |
| $80 \leq w < 85$ | 18 |
| $85 \leq w < 90$ | 9 |
| $90 \leq w < 95$ | 5 |
| $95 \leq w < 100$ | 2 |

**Solution**

| Weight (kg) | Frequency | Lower boundaries | Cumulative frequency |
|---|---|---|---|
| $50 \leq w < 55$ | 4 | 50 | 4 |
| $55 \leq w < 60$ | 13 | 55 | 17 |
| $60 \leq w < 65$ | 21 | 60 | 38 |
| $65 \leq w < 70$ | 34 | 65 | 72 |
| $70 \leq w < 75$ | 48 | 70 | 120 |
| $75 \leq w < 80$ | 27 | 75 | 147 |
| $80 \leq w < 85$ | 18 | 80 | 165 |
| $85 \leq w < 90$ | 9 | 85 | 174 |
| $90 \leq w < 95$ | 5 | 90 | 179 |
| $95 \leq w < 100$ | 2 | 95 | 181 |




From the curve:

Cumulative frequency $N = 181$

Thus, $1st\ Quartile\ (Q_1) = \frac{181 + 1}{4} = 45.5^{th}\ observation,\ Q_1 = 66kg$

$2nd\ Quartile\ (Q_2) = \frac{181 + 1}{2} = 91^{st}\ observation,\ Q_2 = 72kg$

$3rd\ Quartile\ (Q_3) = \frac{3(181 + 1)}{4} = 136.5^{th}\ observation,\ Q_3 = 78kg$

**Note:** The frequency (cumulative frequency, CF) is the location where the quartiles are traced from and are not the actual quartiles. The CF is the number of persons, thus we divide the frequency by 4 and not the weight.

**b.** Standard deviation is a statistic that tells us the spread of data around the mean. Standard deviation tells the variation in the data. The higher the standard deviation, the higher the spread. If the data is close to the mean, the lower the standard deviation.

i. The formula for standard deviation, is given by $\sqrt{\frac{\sum(x - \bar{x})^2}{n}}$, to calculate the standard deviation for raw data.

**Example**

The ages in years of 8 students are: 14, 14, 15, 15, 12, 11 13, 10. Calculate the standard deviation

**Solution:**

**Mean** $(\overline{x}) = \frac{14 + 14 + 15 + 15 + 12 + 11 + 13 + 10}{8} = 13$

| $x$ | $x - \overline{x}$ | $(x - \overline{x})^2$ |
|---|---|---|
| 14 | 1 | 1 |
| 14 | 1 | 1 |
| 15 | 2 | 4 |
| 15 | 2 | 4 |
| 12 | -1 | 1 |
| 11 | -2 | 4 |
| 13 | 0 | 0 |
| 10 | -3 | 9 |
| Total | | 24 |

$\therefore \delta = \sqrt{\frac{24}{8}} = 1.7\ years$

**ii.** the formula to calculate the standard deviation for ungrouped data is $\sqrt{\frac{\sum fx^2}{\sum f} - (\overline{x})^2}$

**Example**

Find the standard deviation of the data set below

| Age ($x$) | Frequency ($f$) |
|---|---|
| 21 | 1 |
| 22 | 4 |
| 23 | 3 |
| 24 | 3 |
| 25 | 5 |
| 29 | 1 |
| 30 | 1 |
| 31 | 1 |
| 37 | 1 |
| | $\sum f = 20$ |

**Solution**

| Age ($x$) | Frequency ($f$) | $fx$ | $x^2$ | $fx^2$ |
|---|---|---|---|---|
| 21 | 1 | 21 | 441 | 441 |
| 22 | 4 | 88 | 484 | 1936 |
| 23 | 3 | 69 | 529 | 1587 |
| 24 | 3 | 72 | 576 | 1728 |
| 25 | 5 | 125 | 625 | 3125 |
| 29 | 1 | 29 | 841 | 841 |
| 30 | 1 | 30 | 900 | 900 |
| 31 | 1 | 31 | 961 | 961 |
| 37 | 1 | 37 | 1369 | 1369 |
| | $\sum f = 20$ | $\sum fx = 502$ | | $\sum fx^2 = 12888$ |

The mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{502}{20} = 25.1$

Standard deviation $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{12888}{20} - 25.1^2} = 3.79$

**iii.** the formula to calculate the standard deviation for grouped data is $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

**Example**

From previous grouped table above on student mass, calculate the standard deviation of student masses

**Solution**

| Mass (kg) | Frequency ($f$) | Mid-Class mark ($x$) | $fx$ | $fx^2$ |
|---|---|---|---|---|
| $45 \leq x < 50$ | 2 | 47.5 | 95 | 4512.5 |
| $50 \leq x < 55$ | 4 | 52.5 | 220 | 11025 |
| $55 \leq x < 60$ | 7 | 57.5 | 402.5 | 23143.75 |
| $60 \leq x < 65$ | 10 | 62.5 | 625 | 39062.5 |
| $65 \leq x < 60$ | 4 | 67.5 | 270 | 18225 |
| $70 \leq x < 75$ | 6 | 72.5 | 435 | 31537.5 |
| $75 \leq x < 80$ | 7 | 77.5 | 542.5 | 42043.75 |
| | $\sum f = 40$ | | $\sum fx = 2590$ | $\sum fx^2 = 169550$ |

*Mean* $(\bar{x}) = \frac{\sum fx}{\sum f}$, $\bar{x} = \frac{2590}{40}$, $\bar{x} = 64.75\ kg$

The standard deviation, $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{169550}{40} - \left(\frac{2590}{40}\right)^2} = 6.79kg$

**iv.** Given the assumed mean A, Standard deviation is $\delta = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$

**Example**

Find the standard deviation of student masses from the students' mass data above

| Mass (kg) | $f_i$ | Mid-Class mark ($x_i$) | Deviation ($d$) $x_i$ — $A$ | $f_i d_i$ | $fd^2$ |
|---|---|---|---|---|---|
| $45 \le x < 50$ | 2 | 47.5 | $47.5 - 62 = -14.5$ | $-29$ | 420.5 |
| $50 \le x < 55$ | 4 | 52.5 | $52.5 - 62 = -9.5$ | $-38$ | 361 |
| $55 \le x < 60$ | 7 | 57.5 | $57.5 - 62 = -4.5$ | $-31.5$ | 81 |
| $60 \le x < 65$ | 10 | 62.5 | $62.5 - 62 = 0.5$ | 5 | 2.5 |
| $65 \le x < 60$ | 4 | 67.5 | $67.5 - 62 = 5.5$ | 22 | 121 |
| $70 \le x < 75$ | 6 | 72.5 | $72.5 - 62 = 10.5$ | 63 | 661.5 |
| $75 \le x < 80$ | 7 | 77.5 | $77.5 - 72 = 5.5$ | 38.5 | 211.75 |
| | $\sum f = 40$ | | | $\sum fd = 30$ | $\sum fd^2 = 1859.25$ |

Standard deviation, $\delta = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} = \sqrt{\frac{1859.25}{40} - \left(\frac{30}{40}\right)^2} = 6.78kg$

***Note:*** *squaring the standard deviation gives the variance*

**Learning Tasks**

Guide learners to solve tasks related to measures of central tendencies and measures of dispersion. Provide support systems for learners who may encounter difficulties.

Learners to

1. review and explain the concept measures of central tendencies
2. find mean, median and mode of raw data, ungrouped data and grouped data
3. find quartiles and standard deviation of raw data, ungrouped data and grouped data
4. estimate quartiles from cumulative frequency curve.

## Pedagogical Exemplars

1. **Group Discussion**

   Learners will be working in convenient groups (ability, mixed ability, mixed gender or pairs etc) to:

   - discuss and compare results, identifying which measure of central tendency is most appropriate to report based on the characteristics of the data (e.g., skewed data, outlier)
   - calculate these measures using appropriate formulas or techniques
   - interpret the measures of dispersion in the context of the data, discussing what they reveal about the variability or consistency of values.

2. **Learning Experience**

   Learners will collaboratively be engaged in hands-on activity (learning by doing) to

   - engage in a design experiential learning activities where they work with real or simulated data to calculate mode, mean and median

- present their calculated measures and justify their choice of measure of central tendency for reporting based on the nature of the data
- apply these measures in real-life contexts, reinforcing their understanding through practical application.

## Key Assessment

**Assessment Level 1: Recall:**

1. What are the three measures of central tendencies? [**Expected ans:** mode, median and mean**]**
2. What does the median represent in a dataset? **[Expected ans:** The middle value]
3. Find the mode, median, mean and standard deviation of the following datasets:

   i. 3, 20, 5, 7, 18, 16, 5, 11, 4, 5, 3

   **[Expected ans:** $Mode = 5$, $median = 6$, $\bar{x} = 8.82$, $\sigma = 6.06$]

   ii. 7, 10, 15, 20, 12, 15, 18, 20, 25

   **[Expected ans:** $Mode = 15$ *and* $20$, $median = 15$, $\bar{x} = 12.91$, $\sigma = 7.74$]

   iii. 4, 6, 8, 12, 15, 10, 12, 14, 16, 18

   **[Expected ans:** $Mode = 12$, $median = 12$, $\bar{x} = 11.5$, $\sigma = 4.27$]

**Level 2 Skills of conceptual understanding:**

1. The following table shows the exam scores of 10 students.

| Score | Frequency |
|---|---|
| 50 | 2 |
| 60 | 3 |
| 70 | 4 |
| 80 | 1 |

   a. Calculate the mean score. **[Expected answer: 64]**

   b. Calculate the median score. **[Expected answer: 65]**

**Assessment Level 3: Strategic thinking**

1. The distribution of marks for 50 students in a test is given in the table. Draw a cumulative frequency curve to the distribution and use your graph to find the median, semi-interquartile range and pass mark if 30% of the students passed the test.

| Marks | Frequency |
|---|---|
| 1-10 | 2 |
| 11-20 | 5 |
| 21-30 | 8 |
| 31-40 | 8 |
| 41-50 | 15 |
| 51-60 | 9 |
| 61-70 | 3 |

2. Draw a cumulative frequency distribution table for the ages of 50 citizens in a community below

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 35 | 57 | 52 | 70 | 55 | 48 | 23 | 42 | 09 | 48 | 36 | 46 |
| 15 | 35 | 12 | 60 | 29 | 61 | 48 | 22 | 43 | 58 | 25 | 42 | 34 |
| 01 | 45 | 19 | 60 | 44 | 38 | 54 | 47 | 69 | 30 | 47 | 18 | |
| 35 | 32 | 21 | 50 | 11 | 29 | 41 | 50 | 53 | 33 | 47 | 30 | |

**i.** From the table

i. Mean
ii. Median
iii. Mode
iv. Standard deviation
v. Variance.

**Assessment Level 4: Extended thinking and reasoning**

1. A company is surveying customer satisfaction with their new product. They receive ratings from 1 (very dissatisfied) to 5 (very satisfied) on a 100-response survey. However, 20 of the responses were accidentally lost.
   a. Is it possible to calculate the mean and median for this data set? Explain. **[Expected ans:** No, the mean and median cannot be calculated **accurately** without the missing data because the mean uses all data values and the position of the missing data is not known, so the median or the middle value if calculated, would be inaccurate]
   b. How could you estimate the range? **[Expected ans:** We can estimate the range by finding the highest and lowest values in the remaining 80 responses. This would not be the exact range though**.**

2. Suppose you have a dataset with outliers. Which measure of central tendency would you use to describe the central value of the dataset and why? **[Expected ans:** The median is the best central tendency to use because it is the most robust against outliers]

3. Below is a record of the marks obtained by 30 students in a midterm test. Marks are out of 50

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 12 | 8 | 19 | 13 | 27 | 22 | 21 | 32 | 30 | 35 | 49 | 37 | 28 | 43 |
| 39 | 36 | 37 | 42 | 45 | 40 | 41 | 44 | 46 | 44 | 46 | 31 | 38 | 20 | 18 |

   a. Prepare a group frequency table for the data using the intervals 5 – 9, 10 – 14 …
   b. From the table, determine the mean mark.
   c. Draw a cumulative frequency curve to represent the data.
      i. Estimate the quartiles from the cumulative frequency curve
      ii. How many students scored between 37 and 38

## Section 9 Review

This section reviews all the lessons taught in section nine (9) which consists of weeks nineteen (19), twenty (20) and twenty-one (21). The section delved into collection and organisation of data, representation of data, measures of central positions and measures of dispersion. This is a summary of what the learner should have learnt. In week nineteen (19), we dealt with data collection and representation methods and data categorisation. Week twenty (20) delved into the graphical representation of categorical and continuous data. The section ends at week twenty-one (21) where we reviewed and delved into measures of central location or tendencies and measures of dispersion.

## Teaching/Learning Resources

Maths posters, White board, dice, coins, graph books, Videos, Mini whiteboards or laminated white paper, Dry erase markers and erasers, straight rules and meter rules, Calculator, technological tools such as computers, mobile phones, YouTube videos etc.

## References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3. Salkind, N. J. (2017). S*tatistics for people who (think they) hate statistics (6th ed.)*. SAGE Publications, Inc.

# SECTION 10: COMBINATIONS, PERMUTATIONS AND PROBABILITY

**Strand: Handling Data**

**Sub-Strand:** Making Predictions with Data

**Content Standard:** *Demonstrate knowledge of basic principles of permutation and combination and interpret probability in everyday life*

**Learning Outcomes:**

1. *Explain combination and permutation, state their difference and solve basic problems related to permutation and combination*
2. *Explain the terminologies in probability orally and find the relative frequency in a given experiment*

## INTRODUCTION AND SECTION SUMMARY

In this section, we focus on counting rules, permutations, combinations and the concept of probability. These topics, with roots tracing back to the 17th-century mathematicians Pierre de Fermat and Blaise Pascal, provide a powerful language for understanding and analysing chance and possibility in everyday life. Beginning with the fundamental principles of counting, imagine arranging delicious jollof rice and kelewele for a market. Counting rules equip us to determine the total number of unique ways we can display these items, considering factors like order and selection. Permutations come into play when order matters. Like forming a football team from a group of talented players. The order you choose your starting eleven significantly impacts the game's outcome. Combinations focus on selections where order becomes irrelevant, like choosing different colours for a traditional kente – picking gold first or blue – doesn't affect the final design. Probability helps us quantify the likelihood of events occurring, from predicting the outcome of a penalty shootout in a Ghanaian football match to determining the chances of rain affecting a farmer's harvest. Throughout this section, we explore not only the theoretical foundations of these concepts but also their practical applications in everyday Ghanaian life.

*The weeks covered by the section are:*

**Week 22:**

1. Fundamental Counting Principle (Multiplication Rule):
2. Permutation and combination

**Week 23:**

1. Difference and relationship between permutations and combinations
2. Solving problems involving permutations and combinations

**Week 24:**

1. Probability in everyday life
2. Probability of given events

## SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities in *counting rules, permutations and combination and* theoretical probabilities of events. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, experiential learning activities and mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

## ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests and homeworks can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving counting rules, permutation and combination, as well as theoretical probabilities of events will also be used to assess learner's application of these mathematical skills. Counting rules will make use of various visual aids, teaching aids and activities such as local Ghanaian fabrics like Ankara, GTP or Kente design patterns, traditional games like Oware, Ayo, bead puzzles, role-playing etc will also be incorporated to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

# Week 22

**Learning Indicators:**

1. *Use the fundamental counting principle to identify and determine the number of ways an event can occur*
2. *Understand the concept of permutation and combination and use it to solve related problems.*

## Theme or Focal Area: Fundamental Counting Principle (Multiplication Rule)

### Definition/Introduction

**Application of fundamental principles of counting Ordered and unordered arrangements**

Have you ever wondered how many different outfits you can create with your wardrobe or how many ways you can arrange your favourite books on a shelf? The answer lies in counting rules. Its fundamental principle states that if event A can occur in $m$ different ways and event B can occur in $n$ different ways, then the total number of ways for both events to happen together is $m \times n$. Think of flipping a coin (heads/tails) and throwing a die (six outcomes). The total number of possible outcomes for both events (coin toss and die thrown) is:

$2\ (coin) \times 6\ (die) = 12$ combinations. This concept is applied in:

1. menu and recipe planning
2. scheduling and time management
3. voting systems
4. password and security
5. genetics

**Ordered Arrangements**

When the order of events matters, we use permutations. Imagine arranging four friends in a line for a photo. Each person has different positions they can occupy. The number of arrangements, considering order, is calculated using factorials ($n!$). For *four*(4) friends, there would be $4! = 24$ unique orderings.

**Unordered Arrangement**

If the order doesn't matter, we use combinations. Choosing a fruit salad from a variety of fruits is an example. Whether you pick an apple first or a banana doesn't change the combination. Combinations are calculated using a formula involving factorials and selections. Selecting 2 fruits from 5 options would be $^5C_2 = 10$ combinations

**Examples**

1. Twum has 5 pairs of trousers and 8 shirts. Assuming that each pair of trousers can be worn with each shirt, how many trousers-shirt outfits does he have?

   **Solution**

   For each pair of trousers, Twum has 8 shirts, therefore, he has $5 \times 8 = 40$ different trousers-shirt outfits to choose from.

2. How many numbers of 3 different digits can be formed with no repetition by choosing from the digits 1,2,3,4 and 5?

**Solution**

**Task 1**: choosing the hundred's digit: we can do this in 5 ways

**Task 2**: choosing the ten's digit: we can do this in 4 ways since one digit is fixed as a hundred digit.

**Task 3**: choosing the one's digit: this can be done in 3 ways because two digits are fixed as hundred and ten digits respectively

Therefore, task 1, followed by task 2, followed by task 3 can be accomplished in $5 \times 4 \times 3 = 60$ ways.

3. A committee needs to select a president, vice-president and secretary from a group of 10 candidates. How many different ways can the committee choose these positions.

**Solution**

**Task 1**: Out of 10 candidates, a president can be selected in 10 ways

**Task 2**: A vice-president can be selected in 9 ways out of the remaining 9 people after selecting a president

**Task 3**: A secretary can be selected in 8 ways out of the remaining 8 people after selecting a president and a vice.

Therefore, selecting a president, vice-president and secretary from a group of 10 candidates can be done in $10 \times 9 \times 8 = 720$ different ways.

4. Kwesi is setting a combination lock with 3 dials. Each dial has 5 numbers (0-4). How many unique 3-digit code combinations are possible?

**Solution**

Since each dial has 5 options, the total number of unique codes is achieved by multiplying the number of options for each dial:
5 (*choices for dial)* 1 × 5 (*choices for dial* 2) × 5 (*choices for dial* 3) = 125 *possible codes*.

## Theme or Focal Area: Permutation and Combination

### Definition/Introduction

Permutations come into play when order matters. Imagine selecting your starting line-up for a crucial inter-school football match. The order you choose your players – striker first, midfielder next – significantly impacts your team's strategy. Permutations help us calculate the total number of unique ways to arrange players based on their positions. For instance, *AB* is not the same as *BA* and *APC* differs from *PAC*. Combinations, on the other hand, focus on situations where order doesn't matter. In the traditional kente scenario given earlier, the final design of the kente is not affected by picking the colour gold first or the colour blue second and vice versa. Combinations help us determine the total number of unique colour combinations possible for the kente, regardless of the order you choose the colours. The concept is applied in:

1. game theory
2. password generation
3. cryptography
4. committee formation

**Explore how the ordered arrangement can be counted and the arrangement of a set of objects in which the order of appearance of elements of the set matters.**

The general rule for the number of permutations of $n$ different items taken all at once is $n!$

**Examples**

1. Find the number of permutations or unique arrangements of the letters in the following words
   a. AYO
   b. OWARE
   c. CANOE

   **Solution**
   a. AYO $= 3! = 3 \times 2 \times 1 = 6$
   b. OWARE $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   c. CANOES $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

2. Kweku has 4 different mathematics books on his shelf. Find the number of orders in which the 4 books can be arranged on the shelf.

   **Solution**
   There will be 4 choices in the first slot
   There will be 3 choices in the first slot
   There will be 2 choices in the first slot
   There will be 1choices in the first slot
   $\therefore$ There are $4 \times 3 \times 2 \times 1 = 24$ permutations

3. Akosua has 3 different necklaces (A, B, C) to wear. How many unique ways can she order them if order matters?

   **Solution**
   The 1st one can be chosen in 3 different ways, the 2nd in two ways and the third in one way thus, $3! = 3 \times 2 \times 1 = 6$

**Establish that the number of ways in which *r* items can be selected out from a set *n* items in which order of arrangement matters is given by the formula:**

$$^{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!}$$

Now if we were selecting $n$ items from a set $n$ item in which order matters, then:

$$^{n}P_{n} = P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \text{ since } 0! = 1$$

**Example**

1. In how many ways can 9 different objects be arranged taking 5 at a time.

   **Solution**
   The number of ways $^{9}P_{5} = 15120$ ways

**Permutation with identical/repeated Objects: establish that the number of ways of arranging n objects where different r of them are of the same kind is given by:**

$$\frac{n!}{(r_1!.r_2!.r_3!.....r_k!)}$$

**Example**

1. In how many ways can the letters of the word **MATHEMATICS** be arranged?

**Solution**

The word **MATHEMATICS** has 11 letters with 2 **M's,** 2 **A's** and 2 **T's**

Thus, the number of ways the letters in MATHEMATICS can be arranged $\frac{11!}{2!\times 2!\times 2!} = 4989600$

**Establish/discover that the number of ways of arranging n objects in a circle is $(n-1)!$**

**Examples**

1. Ama wants to create a necklace using 12 identical glass beads. How many unique circular arrangements (necklace designs) are possible?

   **Solution**

   In a circular arrangement, rotating the entire necklace doesn't create a new design. Total arrangements $= (n-1)! = (12-1)! = 11!$

**Establishes that the number of subsets of r elements that can be formed from *n* elements when order doesn't matter is**:

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

**Example**

1. Kwame needs to choose 3 passengers from a group of 8 friends to fill the front seats of his trotro. How many unique seating arrangements are there for the 3 passengers?

   **Solution**

   We have 8 choices for the first seat, 7 choices remaining for the second seat and 6 choices remaining for the third seat. However, this overcounts arrangements since order doesn't matter (Kwame can seat Kofi first or Ama first). Therefore, we need to divide by the number of ways to arrange 3 things (3!).

   Total arrangements $= \frac{(8\times 7\times 6)}{3!} = 56$ or $^8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56$

**Worked examples**

1. A school band has 5 musical instruments (drums, flute, xylophone, trumpet and bells). Due to instrument size, the drums must always be positioned at the back. How many unique arrangements are there for the remaining 4 instruments in the front row?

   **Solution**

   Treat the drums as one fixed unit. Now, we have 4 instruments (drums plus flute, xylophone, trumpet, and bells) to arrange in 4! ways.

   Total arrangements $P(4, 4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4\times 3\times 2\times 1 = 24$

2. A restaurant offers jollof rice with various protein options (chicken, fish, shrimp or none). They always include a side of fried plantains. How many unique meal combinations are possible if a customer chooses one protein and keeps the plantains?

   **Solution**

   There are 4 protein options (chicken, fish, shrimp or none). Since plantains are always included, we simply consider the protein selection.

   Total combinations = 4

3. During a traditional ceremony, the chief is always accompanied by his linguist. 5 other attendants can stand in any order behind them. How many unique standing arrangements are there?

**Solution**

Treat the chief and linguist as one fixed unit. We now have 5 attendants to arrange in 5! ways.

Total arrangements = 5!

**Learning Tasks**

Guide learners to solve tasks related to understanding fundamental principles in counting, permutation and combination. Provide support systems for learners who may encounter difficulties.

Learners to

**a.** review and explain the fundamental principles in counting

**b.** perform operations like finding the number of possible ways of arranging items in ordered and unordered manner.

**b.** identify and understand the concept of permutations and combinations.

**c.** solve real-world situations involving counting rules, permutations and combinations.

## Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to understand and appreciate the fundamental counting principles and demonstrate knowledge of the basic principles of permutation and combination. The following pedagogical approaches are suggested for effective class facilitation:

1. **Experiential Learning**
   a. Begin with concrete examples using objects like marbles, coins or coloured blocks to demonstrate basic counting principles
   b. Introduce permutations and combinations using real-life scenarios such as arranging seats at a table or selecting items from a menu
   c. Provide ample opportunities for practice with guided exercises
   d. Introduce more complex scenarios involving multiple sets of objects or conditions
   e. Encourage students to use systematic approaches such as tree diagrams or organised lists etc.

2. **Enquiry-based learning:** Guide learners:
   a. to establish different ways of counting, undertake an inquiry into what happens when elements are repeated in an arrangement and explore different ways of counting
   b. to use research resources (textbooks, electronic devices and any additional relevant resources) to appreciate the counting principle and its applications in real life.
   c. to discuss the types and features of different permutations (circular arrangements, identical elements etc.) and what happens when terms of ordered and un-ordered arrangements and special arrangements.
   3. Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the permutation formula to solve related problems

4. **In a well-regulated class discussion**, summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignments or take-home tasks could be given to learners.

## Key Assessment

**Assessment Level 1: Recall and reproduction**

1. A restaurant offers a menu with 3 appetisers, 5 main courses and 4 desserts. How many different meal combinations can be created by choosing one appetiser, one main course and one dessert? [**Expected answer:** 60 different meal combinations**]**
2. The representatives of 10 regions attend a conference. In how many ways can they be seated at a round table? [**Expected answer: 362,880 ways]**
3. Kwesi has 4 different shirts (red, blue, green, yellow) and wants to choose 2 to wear on a trip. How many unique outfits can he create if the order he wears them matters? [**Expected answer:** Kwesi can create 12 unique outfits if the order of the shirts he wears matters.]

**Assessment Level 2: Skills and conceptual understanding**

1. In how many ways can the letters of the word SWALLOW be arranged if
   a. there are no restrictions
   b. the L's must be together
   c. the L's must not be together

   **[Expected answer: a.** 5040 **b.** 720 **c.** 4320]
2. You and your 6 friends want to take a group photo. How many unique arrangements are there if:
   a. You stand in a straight line?
   b. You stand in a circle?

   **[Expected answer:** a. 5040 b. 720]
3. Esi is baking cupcakes and wants to decorate them with sprinkles. She has 3 different coloured sprinkles (red, yellow, green) and decides to use all 3 on each cupcake, creating layers of colour. How many unique colour combinations are possible for a single cupcake if she desires some specific pattern of colours?

   **Expected answer:** there are 6 unique colour combinations possible for a single cupcake if Esi desires a specific pattern of colours using all three different coloured sprinkles.

**Assessment Level 3: Strategic Reasoning**

1. There are three (3) bus lines between town A and B and three (3) bus lines between town B and C.
   a. In how many ways can a person travel from town A to town C by way of town B?
   b. In how many ways can a person travel by bus in a return trip by bus from town A to town C by way of town B?
   c. In how many ways can a person travel by bus on a return trip from town A to C, if this person doesn't want to use a line more than once?

   **Expected answer**: a. 9 ways b. 81 ways c. 36 ways
2. A school has 6 students, and the teacher wants to assign them to 3 different projects, with 2 students in each project. In how many ways can the teacher assign the students to the project?

   **Expected answer**: there are 90 ways for the teacher to assign the 6 students to 3 different projects with 2 students in each project**.**
3. There are 6 students on a debate team. How many unique ways can a coach choose 2 students to give a presentation if the order they speak in matters (e.g., John then Ama vs. Ama then John)?

   **Expected answer:** there are 30 unique ways the coach can choose 2 students out of 6 to give a presentation, considering the order in which they speak matters.

4. Kwame wants to invite 3 friends out of his group of 8 for a movie. How many different groups of 3 can he invite if the order they go in doesn't matter?

   **Expected answer:** Kwame can invite 56 different groups of 3 friends from his group of 8, where the order they go in doesn't matter.

**Assessment Level 4: Extended thinking**

1. Ama is creating a bead bracelet using 4 different coloured beads (red, blue, green, yellow). If repetition is allowed, and the bracelet needs 6 beads, how many unique designs can she create?

   **Expected answer:** Ama can create 84 unique designs for her bead bracelet

2. John has 3 different shirts (red, blue, green) and 2 pairs of jeans (dark wash, light wash). How many unique outfit combinations are there if he can choose one shirt and one pair of jeans?

   **Expected answer:** John has 6 unique outfit combinations he can create with his 3 shirts (red, blue, green) and 2 pairs of jeans (dark wash, light wash).

3. A bookshelf has 5 different history books. How many unique ways can they be arranged on the shelf if they must be arranged by their year of publication and no two books were published in the same year?

   **Expected answer**: there are 120 unique ways to arrange the 5 history books on the shelf if they must be arranged by their year of publication, and no two books were published in the same year.

4. How many ways could 12 teachers be divided into three groups of 6, 4 and 2 people?

   **Expected answer:** there are 13,860 ways to divide the 12 teachers into three groups of 6, 4, and 2 people.

5. A restaurant offers 5 different appetisers and 7 main courses. How many unique 2-course meal combinations can a customer create if they can choose any appetiser and any main course?

   **Expected answer:** the customer can create 35 unique 2-course meal combinations at the restaurant.

# Week 23

**Learning Indicators:**

1. *Distinguish between the concepts of permutation and combination and establish the relation between them.*
2. *Simplify permutation and combination expressions and solve related problems.*

## Theme or Focal Area: Difference and relationship between permutation and combination

### Definition/Introduction

Permutations are used when order matters. For instance, when arranging trophies on a shelf – where one place each trophy (1st, 2nd, etc.) matters. Combinations, on the other hand, focus on selections or arrangements when order does not matter. Choosing ingredients for a dish (onions first or tomatoes?) doesn't affect the final recipe. These concepts are applied in the following fields:

1. game theory
2. password generation
3. cryptography
4. committee formation

a. Establishes that the number of subsets of $r$ elements that can be formed from $n$ elements is

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

b. Establish that the number of ways in which $r$ items can be selected out from a set $n$ items in which order of arrangement matters is given by the formula:

$$^{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!}$$

c. Establish the relationship between combination and permutation formulae:

Establish that the relationship between combination and permutation is expressed as:

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{^{n}P_{r}}{r!}$$

### Examples

1. There are 6 different coloured cubes in a bag. How many ways can Mawuli draw a set of 3 cubes from the bag?

   **Solution**

   $^{6}C_{3} = 20\ ways$

2. How many committees consisting of 2 women and 1 man can be formed from a group of 4 women and 2 men?

   **Solution**

   Task 1: selecting 2 women from 4 women can be done in $^{4}C4_{2}$ ways

   Task 2: selecting 1 man from 2 men can be done in $^{2}C_{1}$ ways

   Task 1 followed by task 2 can be accomplished in $^{4}C4_{2} \times {}^{2}C_{1} = 6 \times 2 = 12$ ways

**3.** Find the value of $n$ if

a. $\frac{^nP_3}{^nC_4} = \frac{24}{5}$

b. $\frac{^nC_5}{^nP_4} = \frac{1}{4}$

**Solution**

**a.**

$$\frac{^nP_3}{^nC_4} = \frac{\frac{n!}{(n-3)!}}{\frac{n!}{4!(n-4)!}} = \frac{4!(n-4)!}{(n-3)!} = \frac{24}{n-3}$$

But

$$\frac{^nC_5}{^nP_4} = \frac{24}{5}$$

$$\Rightarrow \frac{24}{n-3} = \frac{24}{5}$$

$$120 = 24n - 72$$

$$n = 8$$

**b.**

$$\frac{^nC_5}{^nP_4} = \frac{\frac{n!}{5!(n-5)!}}{\frac{n!}{(n-4)!}} = \frac{(n-4)}{5!} = \frac{n-4}{120}$$

But

$$\frac{^nC_5}{^nP_4} = \frac{1}{4}$$

$$\frac{n-4}{120} = \frac{1}{4}$$

$$4n - 16 = 120$$

$$n = 34$$

## Theme or Focal Area: Solving problems involving permutation and combinations

### Definition/Introduction

We can use the relationship between permutation and combinations theorems and formulae to solve related problems

### Examples

**1.** Solve for $n$ if $\frac{^{(n-1)}P_3}{n\,^nP_4} = \frac{1}{9}$

**Solution**

$$^{(n-1)}P_3 = \frac{(n-1)!}{(n-4)!} \text{ and } ^nP_4 = \frac{n!}{(n-4)!}$$

$$\Rightarrow \frac{^{(n-1)}P_3}{^nP_4} = \frac{\frac{(n-1)!}{(n-4)!}}{\frac{n!}{(n-4)!}}$$

$$= \frac{1}{n}$$

$$\frac{1}{n} = \frac{1}{9}$$

$$\therefore n = 9$$

**2.** If ${}^{3a}C_2 = 15$, find the value of $a$

**Solution**

$${}^{3a}C_2 = \frac{3a!}{2!3a-2!} = 15$$

$$\frac{3a \times 3a-1 \times 3a-2 \times 3a-3 \times \ldots \times 3a-k}{2 \times 3a-2 \times 3a-3 \times \ldots \times 3a-k} = 15$$

$$\frac{3a \times 3a-1}{2} = 15$$

$$3a \times (3a-1) = 30$$

$$9a^2 - 3a - 30 = 0$$

Solving quadratically, $a = 2$ *or* $a = -\frac{5}{3}$

**Learning Tasks for practice**

Learners to:

1. review and explain permutation and combination
2. identify the total number of elements and the number of elements to be selected or arranged given a permutation or combination expression
3. establish the connection between combinations and permutations
4. present a variety of permutation and combination problems of moderate complexity for students to solve independently or in pairs etc.

## Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to find the difference and relationship between permutation and combination, as well as solve problems involving permutation and combinations. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Collaborative Learning:** Learners in mixed ability, gender groups discuss the relationship between combinations and permutations
2. **Experiential Learning**:
   a. Use physical objects like beads or coloured tokens to represent the elements in permutation and combination expressions
   b. Guide students through physically arranging and rearranging the objects to understand the concepts of arrangement and selection
   c. Break down the process of simplifying expressions into smaller, manageable steps
   d. Provide visual aids and cue cards to help students remember each step
   e. Present real-life scenarios where permutation and combination principles are applied, such as seating arrangements or lottery probabilities etc.
3. **In a well-regulated class discussion**, summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignments or take-home tasks could be given to learners.

## Key Assessment

**Assessment level 1: Recall:**

1. A group of 10 friends wants to go on a trip with a car which can take only 5 people. In how many ways can they choose the 5 friends to go on the trip?

   **Expected answer**: there are 252 ways for the group of 10 friends to choose 5 friends to go on the trip.

2. Simplify the expression: ${}^{10}C_3 + {}^{9}C_2$ **[Expected answer: 156]**

**Level 2 Skills of conceptual understanding:**

1. Ama has 5 different coloured beads (red, blue, green, yellow, purple) and wants to create a bracelet using 3 of them. How many unique colour combinations are there if:
   a. the order she places the beads on the bracelet doesn't matter?
   b. the order the beads are placed matter?

   **Expected answer:** a. There are 10 unique colour combinations for Ama's bracelet if the order of the beads doesn't matter. b. There are 60 unique colour combinations for Ama's bracelet if the order of the beads matters.

2. A bookshelf has space for $y$ books. Ama has a collection of $z$ unique number of novels. How many unique combinations of novels can she choose to display on the bookshelf if the order she places them doesn't matter?

   **[Expected answer: $C(z, y) = \frac{z!}{y!(z-y)!}$ ]**

**Level 3 Strategic reasoning:**

1. Out of 9 Biologists and 6 Chemists, a committee of 3 Biologist and 4 Chemists is to be formed. In how many ways can this be done if:
   a. any Biologist and Chemist can be selected
   b. two particular Biologist cannot be on the committee
   c. three particular Chemist must be selected

   **Expected answer:** a. 1260 ways b. 525 ways c. 252 ways

2. Find the value of $m$ if ${}^{m}P_4 = 24$ **[Expected answer: the value of m is 4.]**
3. If ${}^{n}C_2 = 10$ find the value of $n$ **[Expected answer: The value of n is 5.]**

**Level 4 Extended critical thinking and reasoning:**

1. Given that ${}^{n}C_4 = 210$, find the value of $n$. **Expected answer: the value of *n* is 10**
2. Given that ${}^{n}P_r = 90$ and ${}^{n}C_r = 15$, find the value of $r$
   **Expected answer: the value of *r* is 3.**
3. There is a total of $x$ students in a class. In how many unique ways can a teacher choose a class prefect, an assistant class prefect and a welfare head to lead the class if the order the students are chosen matters? **Expected answer:** there are $x \times (x - 1) \times (x - 2)$ unique ways for the teacher to choose a class prefect, an assistant class prefect and a welfare head from a class of $x$ students where the order matters.
4. A sports team has a roster of $w$ number of players. The coach needs to choose 5 players for the starting lineup and then select a captain from those 5 starters. How many ways can the coach decide on a starting lineup with a captain? **Expected answer:** the coach can decide on a starting lineup with a captain in $\binom{w}{5} \times 5$ ways.

# Week 24

**Learning Indicators:**

1. *Recall the basic terminologies such as experiment, events, outcome, trial, sample space etc. as used in the context of probability and give examples*
2. *Work collaboratively on an experiment (e.g. tossing of coins or dice) to determine the relative frequencies of events and interpret them*

## Theme or Focal Area: Probability in everyday life

**Definition/Introduction**

Probability is a branch of mathematics which equips us to quantify the likelihood of events. When predicting the outcome of a penalty shootout, predicting rain chances for a farmer's harvest or analysing the likelihood of drawing a specific card. By expressing chance as a numerical value (strictly between 0 and 1 inclusive), probability empowers you to make informed decisions and navigate the fascinating world of uncertainty. The concept of probability finds its application in numerous fields including weather forecasting, game theory, insurance and risk management, sports analysis and prediction etc. Importance terms in probability are defined below:

a. **Experiment:** An operation or process performed to observe a result or an outcome or to obtain measurements like flipping a coin (experiment) to see if it lands on heads or tails (outcome). The outcome of an experiment is not certain and thus cannot be predicted. Examples include:
   i. Tossing a coin
   ii. Rolling a die
   iii. Drawing a card from a set of playing cards
   iv. Drawing items from a bag or a bowl.

b. **Random experiment:** An experiment whose outcome or results cannot be predetermined. Each outcome has an equal chance of occurring. Imagine drawing a bead from a bag full of different coloured beads (random experiment) – you don't know which colour (outcome) you'll get beforehand.

c. **Trial:** A single performance of an experiment. Each kick in the penalty shootout is a separate trial. Selecting a single fruit from the market stall is one trial

d. **Event (E)**: A specific set of outcomes you're interested in. In a football penalty shootout (experiment), scoring a goal (outcome) is an event you might be interested in.

e. **Outcome:** The specific result of a single trial in an experiment. Each kick in the penalty shootout (experiment) has a specific outcome - goal (success) or miss (failure).

f. **Sample Space (S)**: The collection of all possible outcomes for an experiment. For tossing a coin, the sample space is {heads, tails}. At a market stall selling apples, oranges and pineapples (experiment), the sample space is {apple, orange, pineapple}.

g. **Mutually exclusive events:** Two events $A$ and $B$ are said to be mutually exclusive if event $A$ and event $B$ cannot happen simultaneously

h. **Independent events:** Event $A$ is said to be independent of event $B$ if the occurrence of $A$ does not affect the probability of the $B$

**Examples**

1. An experiment is performed by tossing a coin. What is the sample space?

   **Solution**

   The sample space, $S = \{Head\ (H),\ Tail\ (T)\}$

2. An experiment is performed in a school to record the ages of teachers in the staff common room. The following event may be recorded

   Event $A = \{A\ teacher\ is\ 29\ years\ old\}$

   Event $B = \{A\ teacher\ is\ older\ than\ 46\ years\}$

3. A die is tossed in an experiment. List the sample space and the following events.
   a. Event A: An odd number is obtained
   b. Event B: A number greater than 3 is observed
   c. Event C: The outcome is a prime number

   **Solution**

   The sample space $S$ obtained when a die is tossed is $S = \{1, 2, 3, 4, 5, 6\}$

   a. $Event\ A = \{1,\ 3,\ 5\}$
   b. $Event\ B = \{4,\ 5,\ 6\}$
   c. $Event\ C = \{2,\ 3,\ 5\}$

4. Ama is picking mangoes from her favourite mango tree. There are 3 large mangoes and 2 small mangoes on the tree. She reaches up and blindly picks one mango. Indicate the following from the statement:
   **a.** The experiment
   **b.** The outcome
   **c.** Sample space
   **d.** Event

   **Solution**

   a. Experiment: Picking a single mango from the tree.
   b. Outcome: The size (large or small) of the mango Ama picks.
   c. Sample Space: {Large, Small} (all possible sizes of the mango she can pick).
   d. Event: Picking a large mango. (This is a specific set of outcomes you're interested in).

5. State whether the following pairs of events are mutually exclusive events, independent events or neither of them.
   **a.** Event A: It rains today. Event B: The sun shines today.
   **b.** Event A: You draw a red marble from a bag. Event B: The marble you draw is larger than 1cm.
   **c.** Event A: You roll a die and get an even number. Event B: You roll a die and get a number greater than 3.
   **d.** Event A: A coin toss lands on heads. Event B: You win the lottery.
   **e.** Event A: Today is Monday. Event B: Tomorrow is Tuesday.
   **f.** Event A: You roll a die and get a number greater than 5. Event B: You roll a die and get a prime number

**Solution**

a. Neither

b. Independent events

c. Neither mutually exclusive nor independent

d. Independent events

e. Neither mutually exclusive nor independent

f. Mutually exclusive events

## Theme or Focal Area: Probability of given events

Probability of events could be obtained in three ways, namely:

**Relative frequencies of events**

Relative frequency informs one of what occurs in an experiment. Experiments are performed using coins, ludo dice, playing cards etc. up to about 10 trials in each case and record the findings. The relative frequency of an event is defined as:

$$Relative\ frequency = \frac{Frequency\ of\ the\ event}{Total\ Frequncy}$$

**Examples**

1. The table shows the number of times a coin is tossed and the outcome of each trial.

| **Trial** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Outcome** | T | T | H | T | H | T | H | H | T | T | H | T | H |

What is the relative frequency of (observing) heads after

a. each trial

b. the experiment

**Solution**

1. Relative frequency (RF) of heads after each trial

i.

| **Trial** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Outcome** | T | T | H | T | H | T | H | H | T | T | H | T | H |
| **RF** | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{2}{5}$ | $\frac{2}{6}$ | $\frac{3}{7}$ | $\frac{4}{8}$ | $\frac{4}{9}$ | $\frac{4}{10}$ | $\frac{5}{11}$ | $\frac{5}{12}$ | $\frac{6}{13}$ |

ii. Relative frequency of heads after the experiment$= \frac{Number\ of\ heads\ (H)}{Total\ number\ of\ trials} = \frac{6}{13} = 0.46$

2. The table shows the number of times a fair coin is tossed and the outcome of each trial.

| **Trial** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Outcome** | T | T | H | T | H | T | H | H | T | T | H | T | H |
| **Trial** | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| **Outcome** | T | H | T | T | H | H | T | H | H | T | T | T | H |

What is the relative frequency of (obtaining)

a. tails after 10 trials?

b. tails after 20 trials?

c. tails after the experiment and how does it compare to the theoretical probability of obtaining tails?

**Solutions**

$$Relative\ frequency\ (RF) of\ tails = \frac{Number\ of\ tails}{Total\ number\ of\ trials}$$

**a.** Number of tails after 10 trials = 6, Number of trials = 10

RF $= \frac{6}{10} = 0.6$

**b.** Number of tails after 20 trials = 11, Number of trials = 20

RF $= \frac{11}{20} = 0.55$

**c.** Number of tails after the experiment = 14, Number of trials = 26

RF $= \frac{14}{26} = 0.54$

When a coin is tossed, sample space $S = \{Head\ (H), Tail (T)\}$, $n(S) = 2$, $n(T) = 1$

Theoretical probability of obtaining tails, $P(T) = \frac{n(T)}{n(S)} = \frac{1}{2} = 0.50$

Therefore, it is observed that as the number of trials increases, the relative frequencies get closer and closer to the theoretical probability.

**Theoretical probability of events**

In an experiment, theoretical probability informs one of what should happen. Theoretically, the probability of an event is the likelihood of the event happening based on all the possible outcomes and is given by:

$$Probability\ of\ Event\ E,\ P(E) = \frac{Number\ of\ favourable\ or\ desired\ outcomes\ n(E)}{Number\ of\ possible\ outcomes\ n(S)}$$

$$P(E) = \frac{n(E)}{n(S)}$$

**Examples**

**1.** A die is rolled once. What is the probability of obtaining

a. an even number?

b. a number greater than 4?

**Solutions**

Sample space, $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$

**a.** Event of an even number, $E_e = \{2, 4, 6\}$, $n(E_e) = 3$

$P(even\ number) = \frac{n(E_e)}{n(S)} = \frac{3}{6} = 0.5$

**b.** Numbers greater than 4 = $\{5, 6\}$, $n(greater\ than\ 4) = 2$

$P(greater\ than\ 4) = \frac{2}{6} = 0.33$

**Subjective probability of events**

This is based on human opinion due to individual observations or available information. For instance, one may observe the clouds and conclude that there is an 80% likelihood that it will rain. A person may say there is a 90% chance that Ghana would score against Nigeria in a friendly match. A farmer may assert she would have 100% crop yield. All these instances are subjective to individual opinions which we might not be able to verify.

**Learning Tasks**

Guide learners through tasks aimed at understanding and defining the basic terms in probability such as experiments, events, sample space, etc. Offer support systems to learners facing difficulties in grasping these concepts, ensuring a comprehensive understanding of the probability of events.

Learners to

1. review and explain the basic terms of probability.
2. Identify experiments, trials, random experiments, events and sample space from everyday life.
3. find the relative frequencies and theoretical probabilities of events from everyday life.
4. engage in group activities to explore the relationship between relative frequencies and theoretical probabilities of events.

## Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to recall the basic terminologies in probability, work collaboratively on experiments to determine the relative frequencies of events and interpret them. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Experiential Learning**
   a. Provide hands-on activities using dice, spinners, and cards to demonstrate probability concepts
   b. Break down complex concepts into smaller, more manageable parts using visual aids and real-life examples
   c. Scaffold learning by providing step-by-step guidance through problem-solving exercises
   d. Offer frequent opportunities for practice with immediate feedback to reinforce understanding, etc.
2. **Collaborative Learning:** In groups conveniently based on ability, mixed ability, mixed gender, or pairs, guide learners to recall basic probability terminologies using dice, coins, cards, etc. to enhance their understanding and application of probability in everyday life in diverse mathematical scenarios.
3. **In a well-regulated class discussion**, summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignments or take-home tasks could be given to learners.

## Key Assessment

**Assessment Level 1: Recall:**

1. Give a brief definition of each of the following terms:
   a. Random sampling
   b. Mutually exclusive events
   c. Independent events
   d. Probability

**Expected answer:**

a. Random sampling is selecting a sample from a population with equal chances for each item, ensuring the sample represents the entire population for concluding.

b. Mutually exclusive events are those that cannot happen together; for instance, when tossing a coin, getting heads and tails are mutually exclusive outcomes.

c. Independent events occur where one event's outcome doesn't influence the probability of another event; for example, rolling dice twice, the first roll's outcome doesn't affect the second roll's probability.

d. Probability is the measure of an event's likelihood, ranging from 0 (impossible) to 1 (certain), and is used in various fields to predict outcomes and analyse uncertainty

**2.** A fair die is tossed twice.

a. What is the sample size?

b. What is the probability of obtaining

i. a sum of 7?
ii. a product of 15?
iii. the same outcomes?

**Expected answer:** **a**. sample size is 36 **b.** i. $\frac{1}{6}$ ii. $\frac{1}{18}$ iii. $\frac{1}{6}$

**Assessment Level 2 Skills of conceptual understanding:**

**1.** Kwesi want to decorate his shop by hanging 4 different coloured flags (red, yellow, green, blue). He randomly chooses one flag to put up first. Indicate the following from this scenario:

a. The experiment

b. The outcome

c. Sample space

d. Event

**Expected answers:**

a. The experiment is the action of Kwesi randomly choosing one flag to put up first.

b. The outcome refers to the specific colour of the flag that Kwesi chooses to put up first.

c. The sample space ($S$) is the set of all possible outcomes of the experiment.

d. An event is a subset of the sample space, representing one or more outcomes of interest.

**2.** A fair coin is tossed twice and the result is recorded. What is the probability that:

**a.** the first toss shows a head

**b.** the second toss shows a head.

**Expected answer:**

a. Probability of the first toss showing head: $\frac{1}{2}$ or 0.5

b. Probability of the second toss showing head: $\frac{1}{2}$ or 0.5

**3.** A bag contains 3 red marbles, 2 blue marbles and 1 yellow marble. What is the probability of drawing a red marble randomly?

**Expected answer:** the probability of drawing a red marble randomly from the bag is $\frac{1}{2}$ or 0.5

**Assessment level 3: Strategic reasoning**

1. Two identical dice are tossed once. If each face of each dice is labelled in the manner 1, 2, 3, 4, 5, 6. What is the probability of obtaining a sum equal to 7?

   **Expected answer:** the probability of obtaining a sum equal to 7 when two identical dice are tossed once is $\frac{1}{6}$ or 0.16

2. Suppose the probability of your school football team winning their match in the inter-school football competition is $P(A) = 0.70$ (70% chance of winning) and the probability of the Ghana national football team winning their match is $P(B) = 0.60$ (60% chance of winning). Are these events independent or mutually exclusive? Explain.

   **Expected answer:** $P(A \cap B) = 0.70 \times 0.60 = 0.42$

   This means that there is a 42% chance that both the school football team and the national football team will win their respective matches. These events are independent because the outcome of one event (the school football team winning) does not affect the probability of the other event (the national football team winning).

3. Consider the probability of flipping a fair coin twice and getting heads both times such that;

   Event A: Getting heads on the first coin flip.

   Event B: Getting heads on the second coin flip, are these events independent? Explain.

   **Expected answer:** If these events are independent, the probability of both events happening should be the product of their probabilities

   $P(A \cap B) = 0.50 \times 0.50 = 0.25$

   This means that there is a 25% chance of getting heads on both coin flips. These events are independent because the outcome of the first coin flip does not affect the probability of the outcome of the second coin flip

**Assessment level 4 Extended critical thinking and reasoning:**

1. A fair coin is tossed 24 times and the outcomes are observed. The results of the trials are listed below.

   H, T, H, T, T, H, H, H, H, H, T, T, H, T, T, H, T, T, T, H, T, T, H, H

   a. If you work out the relative frequency after the first 10 trials, what is the relative frequency of observing heads?
   b. What is the relative frequency of observing heads after 15 trials?
   c. What is the relative frequency of observing heads after 20 trials?
   d. Calculate the theoretical probability of observing heads.
   e. How does the relative frequency compare with the theoretical probability?

   **Expected answer:**

   **a.** Relative frequency of heads after 10 trials: 0.7
   b. Relative frequency of heads after 15 trials: 0.53
   c. Relative frequency of heads after 20 trials: 0.5
   d. Theoretical probability of heads: 0.5
   e. Comparison: The relative frequency tends to approach the theoretical probability as the number of trials increases.

# Section 10 Review

This section reviews all the lessons taught for the last three (3) weeks. This is a summary of what the learner should have learnt. These first three weeks provided a strong foundation for learner's journey into the world of uncertainties and the fundamental principles of counting. We explored counting principles (multiplication rules), concept of permutation and combination, the difference and relationship between permutation and combination, simplified problems relating to permutation and combination, probability in everyday life, as well as the probability of simple events. The following learning resources are recommended to facilitate teaching and learning:

## Teaching/Learning Resources

Maths posters, White board Pan balance, Videos, Mini whiteboards or laminated white paper, Dry erase markers and erasers, local games such as Oware and Ayo, traditional kente patterns, dice, coins, uniquely coloured balls, calculator, technological tools such as computers, mobile phones, YouTube videos etc.

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