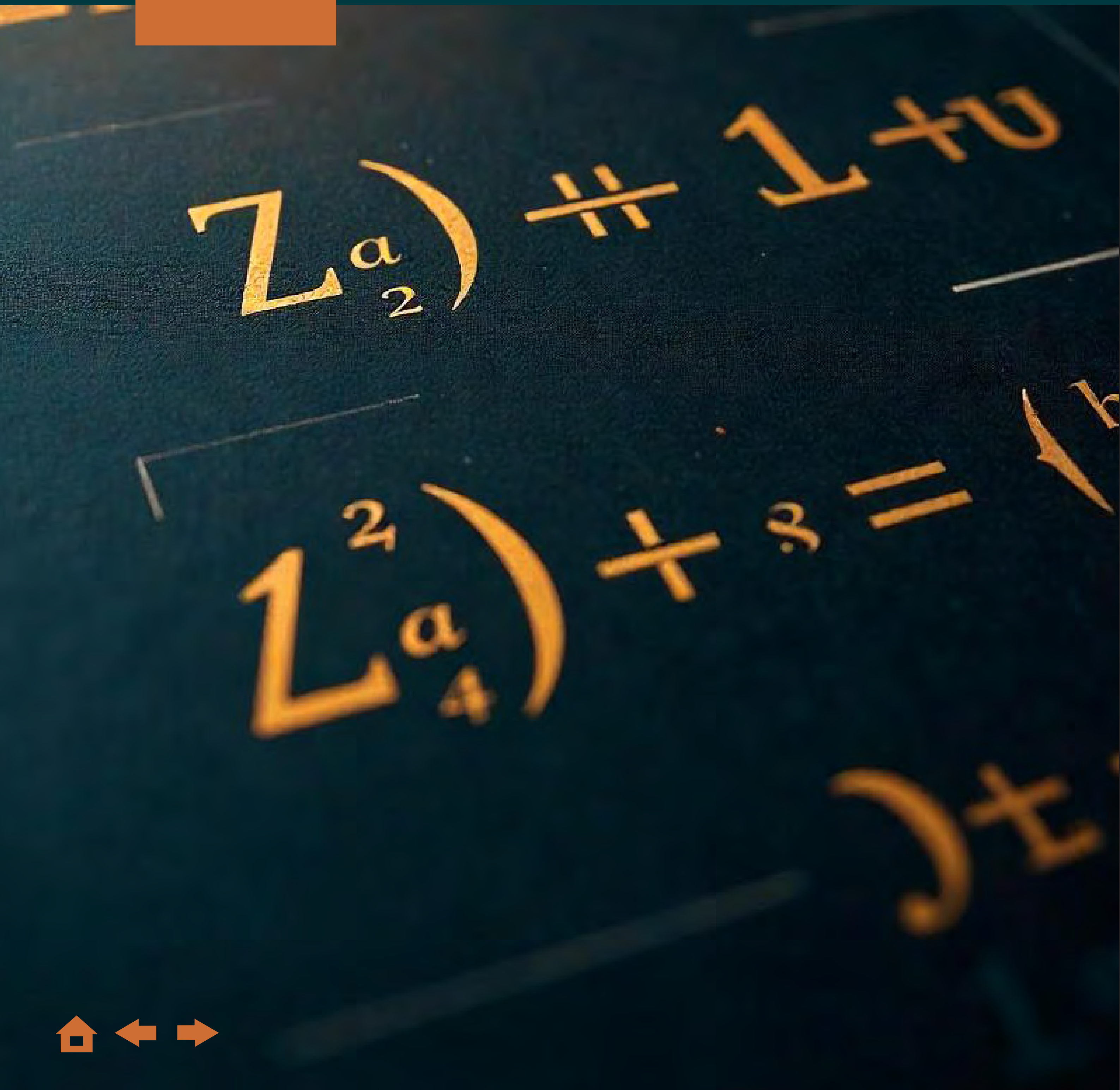
SECTION 1

BINARY OPERATIONS, SETS AND BINOMIAL



MODELLING WITH ALGEBRA

Number and Algebraic Patterns

INTRODUCTION

This section discusses binary operations, sets and binomial expressions.

The part on binary operations, discusses the concept and properties such as closure, associativity, commutativity, distributivity, identity elements and inverse. Binary operations are used extensively in algebra, trigonometry, and calculus, and they form the backbone of numerous calculations.

Sets are essential for organising and classifying information and are used in areas like probability and statistics. The section on sets will focus on operations on three-set problems and set algebra. The Binomial theorem provides a powerful formula to expand expressions of the form $(a + b)^n$, where n is a non-negative integer. It allows us to calculate the coefficients and exponents of the expanded expression. Binomial expansions are crucial for solving various problems in algebra and beyond.

The concepts in this section are essential for science, algebra, trigonometry, and calculus, forming the backbone of numerous calculations in finance to model compound interest, etc.

To understand this section, you will have to rely on your previous knowledge on algebraic expressions, change of subject and substitution at the JHS level. Apart from the four basic operations in arithmetic, other symbols such as Δ, $*$, $\diamondsuit$, etc., can also be used as operations as you find in this statement, $a \Delta b = a + b + (ab)$. You can note how the symbol Δ is interpreted by the use of the two basic operations in arithmetic, $(\times)$ and $(+)$.

At the end of this section, you should be able to:

1. Recognise binary operations, and apply the knowledge in solving related problems
2. Describe and interpret the characteristics of commutative, associative, distributive and closure properties of binary operations.
3. Determine the identity element and use it to find the inverse of a given

element.

4. Establish the properties of operations on set, including commutative, associative, and distributive, sets algebra and apply them to solve problems.
5. Expand Binomial expressions for positive integer indices using Pascal's triangle
6. Use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion.

Key Ideas

- A binary operation is the combination of two ***elements*** at a time under a given rule to produce another element.
- ***Elements*** in binary operations could be the two given *variables*, *terms* or *numbers* which are connected with the operational symbols. For example in $a \Delta b = a + b$, a and b are taken as the elements. Similarly, if we have $m * n = mn + 2$, then the elements are m and n.
- The four basic operations in arithmetic are addition (+), subtraction (–), multiplications (×) and division (÷).
- A set is a *well-defined* collection of objects. Well-*defined* means they have the same features or properties. Examples are a set of footballers, a set of cutleries, and a set of counting numbers.
- A set is usually represented by capital letters and written in a curly or closed { }.
- *Elements* are the individual members in the given set. For example, in a given set $A = \{0,1,2,3,4,5,6,7,8,9,10\}$, the elements in the set A are 0,1,2,3,4,5,6,7,8,9,10.
- A binomial is an expression, which involves ***two terms*** and can be written in the form $(x + y)^n$. For example $(x + y)^2$ is a square of a binomial. For small values of n, it is relatively easy to write the expansion by using multiplication. For example, the $(x + y)^2$

THE CONCEPT OF BINARY OPERATIONS

A binary operation is a rule that combines elements from a non-empty set $\mathbb{R}$ to produce another element. Binary operations form the foundation of digital logic circuits.

Binary operations are used in computer programming, electrical engineering, and number theory. In computer programming, they are used for digital logic and computer arithmetic, while in electrical engineering, they are used in coding theory. In number theory, addition and multiplication are binary operations that define properties of numbers like divisibility and prime factorization. Binary operations are also used to interpret true or false statements in logic.

Binary operations can be defined using symbols such as $\otimes$, $\boxplus$, $\circledast$, * and $\diamond$.

A binary operation is the combination of two elements at a time under a rule to produce another element. For example, $4 + 7 = 11$ is a binary operation, 4 is the first element and 7 is the second element, whereas the $(+)$ is the operation sign and 11 is the third element.

Similarly, $p \, \Delta \, q = p + q + pq$ is a binary operation where the first element is p, q is the second element and the result $p + q + pq$ is the third element, and the operation is Δ.

Furthermore, "An operation is such that the result is the difference of the product of two numbers and the sum of the two numbers". This statement can be written as a mathematical rule thus:

$a * b = ab - (a + b)$, where a and b are the two numbers and * is the symbol that represents the operation

Activity 1.1

1. To simplify $(15 \times 5) + 6$
2. First solve the terms in the bracket
3. Add the result to 6
4. Write down the final answer

Did you get 81? Congratulations!

Note that when you multiplied the two terms in the bracket, you combined the terms based on an operation (multiplication) to get a resulting element (75) as an answer thus you performed a binary operation. Likewise the 75 element was added to 6 to get 81(final resulting element).

Let us further work out the following examples on the concept of binary operations. In pairs or small groups, work through the examples, discussing them as you go.

Worked Examples 1.1

1. If p Δ q= $p + q + pq$, evaluate

i. 3 Δ 4

ii. 4 Δ 3

Solution

In this example, looking at the given definition of the operation Δ, and what we are to evaluate,

p Δ q = 3 Δ 4, where $p = 3$ and $q = 4$. We therefore substitute $p = 3$ and $q = 4$ into the operation p Δ q = $p + q + pq$ for (i),

i. 3 Δ 4 , from $p = 3$, $q = 4$

$3 \,\Delta\, 4 = 3 + 4 + 3(4)$

$= 3 + 4 + 12$

$= 19$

ii. Again, $p \,\Delta\, q = 4 \,\Delta\, 3$

Hence, substituting in the operation $p\Delta q = p + q + pq$,

We have, $4 \,\Delta\, 3 = 4 + 3 + 4(3)$

$= 4 + 3 + 12$

$= \mathbf{19}$

2. The operation ə on the set R of real numbers is defined by $\boldsymbol{x}$ ə $y = \dfrac{4x + 5y + 2xy}{3y + 2x}$ **for all** $\boldsymbol{x}, y \in R$. Evaluate,

i. –2 ə **5**

ii. 4 ə **7**

Solution

i. Given, $x ə y = \frac{4x + 5y + 2xy}{3y + 2x}$

Then, $-2 ə 5 = \frac{4(-2) + 5(5) + 2(-2)(5)}{3(5) + 2(-2)}$

$= \frac{-8 + 25 - 20}{15 - 4}$

$= \frac{-3}{11}$

ii. $4 ə 7 = \frac{4(4) + 5(7) + 2(5)(7)}{3(7) + 2(4)}$

$= \frac{16 + 35 + 70}{21 + 8}$

$= \frac{121}{29}$

$= 4\frac{5}{9}$

3. A binary operation * is defined under the set of real numbers R, by $a * b = a^2 - ab + 4$. Evaluate each of the following equations.

i. $-1 * 2$

ii. $3 * 4$

iii. $3 * -2$

Solution

i. Given $a * b = a^2 - ab + 4$

$-1 * 2 = (-1)^2 — (-1)(2) + 4$

$= 1 + 2 + 4$

$= \mathbf{7}$

ii. $3 * 4 = 3^2 - 3(4) + 4$

$= 9 - 12 + 4$

$= \mathbf{1}$

iii. $3 * -2 = 3^2 - 3(-2) + 4$

$= 9 + 6 + 4$

$= \mathbf{19}$

4. Given that $m*n = \frac{m-n}{n}$, if $3*q = \frac{1}{2}$ find the value of q.

Solution

Given $m*n = \frac{m-n}{n}$, then $3*q = \frac{m-n}{n} = \frac{1}{2}$

Hence, $\frac{1}{2} = \frac{3-q}{q}$

$$q = 2(3 - q)$$
$$q = 6 - 2q$$
$$q + 2q = 6$$
$$3q = 6$$
$$q = 2$$

PROPERTIES OF BINARY OPERATIONS

The properties of binary operations include closure, commutativity, associativity and distributive properties. We will investigate these properties using the four basic operations ($\times$, $\div$, $-$, *and* $+$) on the set of real numbers ($\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{W}$, $\mathbb{N}$) and predefined binary operation on a given set using algebraic manipulations and binary tables.

Verifying Properties of Binary Operations

a) Closure

A binary operation, Δ is closed under a set, A if for all x and y which are elements of A, $x\Delta y$ is also in A. For example, the sum of two natural numbers is a natural number hence it can be said that addition is closed under the set of Natural numbers, $\mathbb{N}$.

The set of natural numbers is also closed under multiplication and addition but not closed under subtraction and division. ie when we add or multiply two natural numbers, the result will always be a natural number, however, when you subtract or divide two natural numbers, your result will not always be a natural number.

Activity 1.2

1.

a) Choose any two even numbers

b) Add them up

c) Is the result an even number?

2. Repeat the steps above with different even numbers

3. Is the sum of two even numbers always an even number?

Yes! The sum of two even numbers is always an even number. This demonstrates the closure property of addition with even numbers.

Activity 1.3

1. Perform the operations on the set of natural numbers

a) $2 + 4 =$

b) $2 \times 4 =$

c) $2 - 4$

d) $2 \div 4$

2. Write down the operations whose results were also natural numbers.

3. Which operations are closed on the set of natural numbers?

Note: If m and n are integers, m + n will be an integer, $m \times n$ will also be an integer, likewise $m - n$. However, $\frac{m}{n}$ will not be integer at all times. Therefore, addition, multiplication and subtraction are said to be closed under the set of integers but division is not.

Can you input real numbers into this example to show that this is true?

Work Example 1.2

An operation Λ is defined on the set $S = \{0, 2, 3, 4\}$ by $p \Lambda q = p + q - 2pq$. Determine whether or not the set S is closed under the operation Λ.

Solution

Pick two numbers at a time for all members in the set S, and evaluate them with the operation Λ.

Given $p\Lambda q = p + q - 2pq$

$0\Lambda 0 = 0 + 2 - 2(0)(0)$
$= 0$

$0\Lambda 3 = 0 + 3 - 2(0)(3)$
$= 3$

$0\Lambda 4 = 0 + 4 - 2(0)(4)$
$= 4$

$2\Lambda 0 = 2 + 0 - 2(2)(0)$
$= 2$

$3\Lambda 3 = 3 + 3 - 2(3)(3)$
$= -12$

$4\Lambda 4 = 4 + 4 - 2(4)(4)$
$= -24$

$2\Lambda 3 = 2 + 3 - 2(2)(3)$
$= -7$

$2\Lambda 4 = 2 + 4 - 2(2)(4)$
$= -10$

$3\Lambda 4 = 3 + 4 - 2(3)(4)$
$= -17$

From the solution, the numbers 0, –7, –10, –12, –17, and –24 are not in the set S; therefore the set S is not closed undre the operation **Λ.**

Worked Example 1.3

The operation * is defined on the set $M = \{ 1,3,5,7\}$ by $x * y = \frac{x + y}{2}$

i. draw a table for * on the set M

ii. determine whether or not the operation * is closed under M

Solution

i. Combining two elements at a time on the given operation $x * y = \frac{x + y}{2}$,

$1 * 1 = \frac{1 + 1}{2}$

$= 1$

$1 * 3 = \frac{1+3}{2}$

$= 2$

$1 * 5 = \frac{1+5}{2}$

$= 3$

$1 * 7 = \frac{1+7}{2}$

$= 4$

Combine the other pairs to obtain the table below

*	1	3	5	7
1	1	2	3	4
3	2	3	4	5
5	3	4	5	6
7	4	5	6	7

ii. The operation * is not closed because the numbers {2,4, 6} are not members in the set *M*.

Worked Example 1.4

The binary *operation* * is defined on the Real numbers *R*, by $x * y = y - 4xy$. Show whether * is closed under *R* in each of the following;

i. 3 * 4

ii. 4 * 5

iii. 5 * –6

iv. –5 * 7

Solution

i. $3 * 4 = 4 - 4(3)(4)$

$= -44$

ii. $4 * 5 = 4 - 4(4)(5)$

$= -75$

iii. $5 * -6 = -6 - 4(5)(-6)$

$= 114$

iv. $-5 * 7 = 7 - 4(-5)(7)$

$= 147$

The operation * is closed under the set of R because the results for (i) to (iv) are also real numbers.

b) Commutativity

Given that $\diamond$ is a binary operation defined on a set, S which contains a and b, if $a \diamond b = b \diamond a$, for all a and b in S, then $\diamond$ is said to be commutative.

In a **binary table**, the binary operation is only commutative if the results of the operation are symmetrical about the leading diagonal.

*	-1	0	1
-1	1	0	-1
0	0	0	0
1	-1	0	-1

Figure 1: Binary table

Addition and multiplication are commutative. i.e., $4 + 3 = 3 + 4 = 7$ and $4 \times 5 = 5 \times 4 = 20$. Subtraction and division are not commutative. i.e., $4 - 3 \neq 3 - 4$, likewise $\frac{10}{2} \neq \frac{2}{10}$.

To determine whether the operation $*$ defined by $a * b = a + b + 2ab$ is commutative, we need to check if $a * b = b * a$ for all real numbers a and b.

Let us choose any two different numbers, say 3 and 4, to verify whether the operation $*$ is commutative. Compute $3*4$ and $4*3$ separately:

$3*4 = 4 + 2(3)(4)$

$= 4 + 24$

$= 26$

$4*3 = 4 + 2(4)(3)$

$= 4 + 24$

$= 26$

For the operation to be commutative, $3*4$ must equal $4*3$. Since this is true for all real numbers a and b, the operation $*$ is commutative.

Worked Examples 1.5

1. An operation * is defined on the set of real numbers by $a * b = a + 2ab$, evaluate whether * is commutative or not, using the following expressions:
   - **i.** $5 * 7$
   - **ii.** $7 * 5$

Solution

Given, $a * b = a + 2ab$,

i. $5 * 7 = 5 + 2(5)(7)$

$= 5 + 70$

$= 75$

ii. $7 * 5 = 7 + 2(7)(5)$

$= 7 + 70$

$= 77$

Since $5 * 7 \neq 7 * 5$, the operation * is not commutative.

2. The operation ® on the set of real numbers is defined by m® n = m + n – 10mn, evaluate
   - **i.** 2 ® **3**
   - **ii.** 3 ® **2**
   - **iii.** What can you say about the result in i) and ii)?

Solution

i. $2 ® 3 = 2 + 3 - 10(2)(3)$

$= 5 - 60$

$= -55$

ii. 3® 2 = 3 + 2 – 10(3)(2)

= 5–60

= –55

iii. since 2 ® 3= 3 ® 2, the operation ® is **commutative**.

Find out and discuss if this is always going to be commutative no matter what numbers are chosen for m and n.

3. Given that a binary operation, * is defined on a set, $S = \{-2, 3, 5\}$ as $a * b = ab - (a + b)$, show whether or not * is commutative.

Solution

Given that $a * b = ab - (a + b)$, and $S = \{-2, 3, 5\}$

Substituting any two members of S in the equation,

- Using –2 and 3:

$$-2 * 3 = -2(3) - (-2 + 3)$$

$$= -7$$

$$3*(-2) = 3(-2) - (3 + (-2))$$

$$= -7$$

- Using –2 and 5:

$$-2 * 5 = -2(5) - (-2 + 5)$$

$$= -13$$

$$5*(-2) = 5(-2) - (5 + (-2))$$

$$= -13$$

- Using 3 and 5:

$$3 * 5 = 3(5) - (3 + 5)$$

$$= 7$$

$$5 * 3 = 5(3) - (5 + 3)$$

$$= 7$$

Since $a * b = b * a$ for all a and b in S, * is commutative

4. Given that $x * y = x + y + xy$, show that the operation $*$ is commutative.

Solution

For the operation $*$ to be commutative, $x * y$ must be equal to $y * x$

LHS must be equal to RHS

$x * y = x + y + xy$............ (1) LHS

RHS

$y * x = y + x + yx$............ (2) RHS

Comparing equation (1) to equation (2),

LHS = RHS,

Hence, the operation $*$ is commutative.

5. Given that $a * b = a - 2b$, determine whether or not the operation $*$ is commutative.

Solution

For the operation $*$ to be commutative, $a * b = b * a$

LHS

$a * b = a - 2b$............ (1)

RHS

$b * a = b - 2a$............ (2)

Comparing equation (1) to equation (2),

LHS $\neq$ RHS,

Hence, the $*$ is not commutative (unless $a = b$).

6. A binary operation $\odot$ is defined on the set $P = \{x, y, z\}$ by the table below. Determine whether the operation $\odot$ is commutative.

$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Solution

By inspection, we identify that $x \odot y = y \odot x = y$; $x \odot z = z \odot x = z$ and $y \odot z = z \odot y = x$
$\therefore \odot$ is commutative.

Alternatively, since the table is symmetric along the principal diagonal, we conclude that the operation $\odot$ is commutative.

$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Worked Example 1.5

The operation $*$ is defined over the set $M = \{0, 2, 6, 8\}$ as $a * b = a + b + 8$.

a) Construct a table of values

b) Show whether the operation $*$ is commutative or not.

Solution

$0 * 2 = 0 + 2 + 8$
$= 10$

$2 * 0 = 2 + 0 + 8$
$= 10$

$2 * 6 = 2 + 6 + 8$
$= 16$

$6 * 2 = 6 + 2 + 8$
$= 16$

Continue with the rest of the pairs to obtain the table below.

$*$	0	2	6	8
0	8	10	14	16
2	10	12	16	18
6	14	16	20	22
8	16	18	22	24

b) the operation * is commutative because for any two numbers picked in the table, the result for $a * b$ and $b * a$ are equal.

c) Associativity

Given that ⋄ is a binary operation defined on a set, S which contains a, b and c, if $a \diamond (b \diamond c) = (a \diamond b) \diamond c$, for all, a, b and c in S then ⋄ is said to be associative.

Addition and multiplication are associative.

For example, $2 + (3 + 4) = (2 + 3) + 4$

Likewise $2 \times (3 \times 4) = (2 \times 3) \times 4$

Now, let us go through the following activities or steps to prove or otherwise of the associative property using numbers.

To determine whether the operation $*$ defined by $a * b = a + b$ is associative, we need to check if $(a * b) * c = a * (b * c)$ for all real numbers a, b, and c. In this case, let us check with the specific values 2, 3, and 5 using the following steps:

1. Define the operation:

The operation is $\mathrm{a}*\mathrm{b} = \mathrm{a} + \mathrm{b}$

2. Compute $(\mathrm{a}*\mathrm{b})*\mathrm{c}$:

First, find a∗b, substituting the real numbers (e.g. 2 and 3):

$\mathrm{a} * \mathrm{b} = 2 * 3$

$2 * 3 = 2 + 3$

$= 5$

Next, use this result to compute $(\mathrm{a} * \mathrm{b}) * \mathrm{c}$:

$(\mathrm{a} * \mathrm{b}) * \mathrm{c} = 5 * 5$

$= 5 + 5$

$= 10$

3. Compute $\mathrm{a} * (\mathrm{b} * \mathrm{c})$

First, find b∗c: $3*5 = 3 + 5 = 8$

Next, use this result to compute $2 * (3 * 5)$:

$2*(3*5) = 2 * 8$

$= 2 + 8$

$= 10$

4. Compare (a∗b)∗c and a∗(b∗c)

$(2 * 3) * 5 = 10$

$2 * (3*5) = 10$

Since $(2 * 3) * 5 = 2 * (3 * 5)=10$, the operation $*$ is associative for the values 2,3, and 5.

Worked Example 1.6

A binary operation, Δ is defined on the set, R of real numbers as $x\omega y = x + y - 2xy$, where $x, y \in R$. Show whether or not ω is commutative.

Solution

If ω is commutative, it implies $x\omega y = y\omega x$

LHS: $x\omega y = x + y - 2xy$ and

RHS: $y\omega x = y + x - 2yx$

Rearranging, we have

RHS: $x\omega y = x + y - 2xy$

Since $x\omega y = y\omega x$, we conclude that ω is commutative.

Let us narrow it down using the set $\{-1, 0, 1, 2\}$. All possible combinations are shown in the table below.

ω	-1	0	1	2
-1	-4	-1	2	5
0	-1	0	1	2
1	2	1	0	-1
2	5	2	-1	-4

Since the table is symmetrical about the principal diagonal, it confirms the operation, ω is commutative.

Note that we could have used any set containing real numbers.

Worked Examples 1.7

To determine whether the operation $*$ defined by $a * b = a + 2ab$ is associative, using variables (letters), we need to check if $(a * b) * c = a * (b * c)$ for all real numbers a, b, and c.

Here are the steps:

1. Define the operation:

Given $a * b = a + 2ab$

2. Compute $(a * b) * c$:

- First, find $a * b$: $= a + 2ab$
- Let $a * b = m$

  Now, $m = a + 2ab$
- Next, use this result to compute m$*$c:

  $m * c = m * c$

  $m * c = m + 2mc$ …………….. equation (1)

  substitute m = a + $2ab$, into equation (1)

  $= a + 2ab + 2(a + 2ab)c$

  $(a * b)*c = a + 2ab + 2ac + 4abc$

3. Compute $a *(b * c)$:

- First, find $b * c$:

  $b * c = b + 2bc$
- Next, use this result to compute $a * (b * c)$
- Let $n = b + 2bc$

  $= a * n$
- Substitute n into the operation:

  $a * n = a + 2an$ ………………..equation (2)

  substitute n = b + $2bc$ into equation (2)

  $a * (b * c) = a + 2a(b + 2bc)$

  $a * (b * c) = a + 2ab + 4abc$

  o Compare the two expressions: $a + 2ab + 2ac + 4abc$ and $a + 2ab + 4abc$

- Since $(a*b)*c \neq a*(b*c)$, due to the extra $2ac$ term in $(a*b)*c$, the operation $*$ is not associative.

Now solve the following example either in groups, individually, or with the assistance of your teacher.

Worked Example 1.8

A binary operation * defined on the set R of real numbers by $m * n = 2m + 3n - mn$ where $m, n \in R$. Determine whether or not * is associative.

Solution

For $(m*n)*p$

First, compute $m*n$:

$m*n = 2m + 3n - mn$

Now, compute $(m*n)*p$:

Let $k = m*n = 2m + 3n - mn$. Then we need to compute $k*p$:

$$
\begin{aligned}
k*p &= (2m + 3n - mn)*p \\
&= 2(2m + 3n - mn) + 3p - (2m + 3n - mn)p \\
&= 4m + 6n - 2mn + 3p - 2mp - 3np + mnp
\end{aligned}
$$

For m∗(n∗p):

First, compute $n*p$:

$n*p = 2n + 3p - np$

Now, compute $m*(n*p)$:

Let $r = n*p = 2n + 3p - np$.

Then we need to compute $m*r$:

$$
\begin{aligned}
m*r &= m*(2n + 3p - np) \\
&= 2m + 3(2n + 3p - np) - m(2n + 3p - np) \\
&= 2m + 6n + 9p - 3np - 2mn - 3mp + mnp
\end{aligned}
$$

From the computations above, we have:
$(m*n)*p = 4m + 6n - 2mn + 3p - 2mp - 3np + mnp$

m∗(n∗p) = $2m + 6n + 9p - 3np - 2mn - 3mp$ + mnp

Since the expressions (m∗n)∗p and m∗(n∗p)m are not equal, the operation ∗ defined by m∗n = $2m + 3n$−mn is not associative.

Worked Example 1.9

Determine whether or not the operation * is associative given that $x * y = x + y + xy$.

Solution

For the operation *to be associative, $x * (y * z) = (x * y) * z$

LHS:

$x * (y * z)$

$(y * z) = y + z + yz$

$x * (y + z + yz) = x + y + z + yz + x(y + z + yz)$

$x * (y * z) = x + y + z + xy + yz + xz + xyz$ ……….(1)

RHS:

$(x * y) * z$

$(x * y) = x + y + xy$

$(x + y + xy) * z = (x + y + xy) + z + (x + y + xy)z$

$x * (y * z) = x + y + z + xy + xz + yz + xyz$ ……….(2)

LHS = RHS,

Hence, the operation * is associative.

d) Distributive Property

Given that * and ⊗ are binary operations defined on the set, $S = \{a, b, c\}$ then * is distributive over ⊗ if $a * (b \otimes c) = (a * b) \otimes (a * c)$. Example. $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$. The distributive property is applied when we are opening brackets in mathematics. For example, $2(x + y) = 2x + 2y$.

Worked Example 1.10

If two binary operations * and Δ are defined as $x * y = 2xy$ and $x \Delta y = 2x + 3y$ on the set R of real numbers. Show whether or not

a) Δ is distributive over *

b) * is distributive of Δ

Solution

a) If Δ is distributive of *, then for all $x, y, z \in R$

We should have $x \Delta (y * z) = (x \Delta y)*(x \Delta z)$

LHS: $x \Delta (y * z) = x \Delta (2yz)$

$= 2x + 3(2yz)$

$= 2x + 6yz$

RHS: $(x \Delta y)*(x \Delta z) = (2x + 3y)*(2x + 3z)$

$= 2(2x + 3y)(2x + 3z)$

$= (4x + 6y)(2x + 3z)$

$= 8x^2 + 12xz + 12xy + 18yz$

Since LHS≠RHS, the operation Δ is not distributive over *.

b) If * is distributive of Δ, then for all $x, y, z \in R$

We should have $x*(y \Delta z) = (x * y) \Delta (x * z)$

LHS: $x*(y \Delta z) = x*(2y + 3z)$

$= 2x(2y + 3z)$

$= 4xy + 6xz$

RHS: $(x * y) \Delta (x * z) = (2xy) \Delta (2xz)$

$= 2(2xy) + 3(2xz)$

$= 4xy + 6xz$

Since LHS = RHS, the operation * is distributive over Δ.

Worked Example 1.11

Given that *Δ* and + are binary operations defined on the set, $S = \{ x, y, z\}$ as $x \Delta y = x + 2xy$ determine whether Δ is distributive over + for $x \Delta (y + z) = (x \Delta y) + (x \Delta z)$

Solution

LHS:

$x \, \Delta \, (y + z) = x + 2(x)(y + z)$

$= x + 2xy + 2xz$ ……….(1)

RHS:

$(x\Delta y) + (x\Delta z) = x + 2xy + x + 2xz = 2x + 2xy + 2yz$……………….. (2)

Since $x \, \Delta \, (y + z) \neq (x \, \Delta \, y) + (x \, \Delta \, z)$, Δ is not distributive over +

Worked Example 1.12

Two binary operations * and Δ are defined as a $* \, b = 2ab$ and $a \, \Delta \, b = 2a + 3b +$ 2 for all $a, b \in R$.

Evaluate:

i. $2 * (3 \, \Delta \, 4)$

ii. $(2 * 3) \, \Delta \, (2 * 4)$

iii. What can you say about (i) and (ii)?

Solution

i. $3 \, \Delta \, 4 = 2(3) + 3(4) + 2$

$= 20$

$2 * 20 = 2(2)(20)$

$= 80$

ii. $2 * 3 = 2(2)(3)$

$= 12$

$2 * 4 = 2(2)(4)$

$= 16$

$12 \, \Delta \, 16 = 2(12) + 3(16) + 2$

$= 74$

iii. Comparing (i) and (ii):

$2 * (3 \, \Delta \, 4) = 80$ and

$(2 * 3) \, \Delta \, (2 * 4) = 74$

These two expressions are not equal. This means that the operation * is not distributive over Δ

Since $2 * (3 \,\Delta\, 4) \neq (2 * 3) \,\Delta\, (2 * 4)$

IDENTITY AND THE INVERSE OF AN ELEMENT

Identity / Neutral Element

The identity element, e of a set, S under an operation, Δ on S exists if there is an element $e \in S$ such that $a \,\Delta\, e = e \,\Delta\, a = a$. If a binary operation is not commutative, it cannot have an identity element. Under the operation of addition, (+), the identity element is 0 ie $6 + 0 = 6$, $14 + 0 = 14$. Also, under the operation of multiplication, the identity element is 1, i.e., $3 \times 1 = 3$, $100 \times 1 = 100$.

Worked Example 1.13

A binary operation ∇ is defined on the set R or real numbers by $p \nabla q = p + q - pq$ where p and $q \in \mathbb{R}$. Find the identity element under the operation ∇.

Solution

Check for commutativity:

$p \nabla q = p + q + pq$

$q \nabla p = q + p + qp$

Since $p \nabla q = q \nabla p$, ∇ is commutative and hence could have an identity element.

Let e be the identity element, where $e \in \mathbb{R}$, then by definition, $p \nabla e = p$

But $p \nabla e = p + e + ep$

$\implies p + e + ep = p$

$p + e(1 + p) = p$

$e(1 + p) = p - p$

$e(1 + p) = 0$

$e = 0$

Worked Example 1.14

A binary operation ∇ is defined on the set R of real numbers by $m \nabla n = m + 2mn$ where m and $n \in \mathbb{R}$. Find the identity element under the operation ∇.

Solution

Check for commutativity:

$m\nabla n = m + 2mn$

$n\nabla m = n + 2nm$

Since $m\nabla n \neq n\nabla m$, ∇ is not commutative and hence does not have an identity element.

Worked Example 1.15

A binary operation $\Diamond$ is defined on the set R or real numbers by $a \Diamond b = a - 2b + ab$ where a and $b \in \mathbb{R}$. Find the identity element under the operation $\Diamond$

Solution

Check for commutativity:

$a \Diamond b = a - 2b + ab$

$b \Diamond a = b - 2a + ba$

Since $a \Diamond b \neq b \Diamond a$, $\Diamond$is not commutative and hence would not have an identity element.

Worked Example 1.16

The binary operation ∋ is defined on the set $\phi = \{u, v, w, x, y\}$ by the table below.

∋	***u***	***v***	***w***	***x***	***y***
u	v	w	y	u	x
v	w	y	x	v	u
w	y	x	u	w	v
x	u	v	w	x	y
y	x	u	v	y	w

Find the identity element.

Solution

If $e \epsilon \phi$ is the identity element, then for all $R \in \phi$, $R \ni e = e \ni R = R$.

By inspection, we have

$u \ni x = x \ni u = u$

$v \ni x = x \ni v = v$

$w \ni x = x \ni w = w$

$x \ni x = x \ni x = x$

$y \ni x = x \ni y = y$

Therefore, the identity element is x.

The Inverse of an Element

The inverse of an element a of a set S under an operation Δ on S is an element $a^{-1} \in S$ such that $a \,\Delta\, a^{-1} = a^{-1} \,\Delta\, a = e$, where e is the identity element of S under the operation, Δ. A binary operation with no identity has no inverse for the general elements. Under the operation of addition, $2 + (-2) = 0$, therefore the additive inverse of 2 is –2.

Under the operation of multiplication, $2 \times \frac{1}{2} = 1$, therefore the multiplicative inverse of 2 is $\frac{1}{2}$. Remember that if a binary operation is not commutative it cannot have an identity element.

Worked Example 1.17

A binary operation ∇ is defined on the set R or real numbers by $p\nabla q = p + q - pq$ where p and $q \in R$. Find the inverse element under the operation ∇.

Solution

Let $p^{-1} \in R$ be the inverse element, then

$p\nabla p^{-1} = p + p^{-1} + pp^{-1} = e$

Check for commutativity:

$p\nabla q = p + q + pq$

$q\nabla p = q + p + qp$

Since $p\nabla q = q\nabla p$, ∇ is commutative and hence could have an identity element

Let e be the identity element, where $e \in \mathbb{R}$, then by definition, $p\nabla e = p$

But $p\nabla e = p + e + ep$

$\Longrightarrow p + e + ep = p$

$p + e(1 + p) = p$

$e(1 + p) = p - p$

$e(1 + p) = 0$

$e = 0$

$\Rightarrow p + p^{-1} + pp^{-1} = 0$

$\Longrightarrow p^{-1} = \dfrac{-p}{1 + p}$

The inverse element, p^{-1} of p under the operation ∇ exist and it is $\dfrac{-p}{1+p}$

Worked Example 1.18

A binary operation ∇ is defined on the set R or real numbers by $a\nabla b = a + b + 2ab$ where a and $b \in R$.

i. Find the inverse element under the operation ∇.

ii. Hence determine the inverse of 4.

Solution

Let $a^{-1} \in R$ be the inverse element, then

$a \nabla a^{-1} = a + a^{-1} + 2aa^{-1} = e$

Check for commutativity:

$a \nabla b = a + b + 2ab$

$b \nabla a = b + a + 2ba$

Since $a \nabla b = b \nabla a$, ∇ is commutative and hence could have an identity element

Let e be the identity element, where $e \in \mathbb{R}$, then by definition, $a \nabla e = a$

But $a \nabla e = a + e + 2ea$

$\Longrightarrow a + e + 2ea = a$

$a + e(1 + 2a) = a$

$e(1 + 2a) = a - a$

$e(1 + 2a) = 0$

$e = 0$

$a + a^{-1} + 2aa^{-1} = e$

$\Rightarrow a + a^{-1} + 2aa^{-1} = 0$

$a^{-1} + 2aa^{-1} = -a$

$a^{-1}(1 + 2a) = -a$

$\Longrightarrow a^{-1} = \dfrac{-a}{1 + 2a}$

The inverse element, p^{-1} of p under the operation ∇ exist and it is $\dfrac{-a}{1 + 2a}$

The inverse of 4 is $\dfrac{-4}{1 + 2(4)} = \dfrac{-4}{1 + 8}$

$= \dfrac{-4}{9}$

Worked Example 1.19

1. The operation * is defined on the set {1, 3, 5} as shown in the table below

*	1	3	5
1	3	1	5
3	1	3	5
5	5	5	5

a) Evaluate (3 * 1)*(5 * 3)

b) State the identity element for *

c) Which of the elements has no inverse

Solution

a) From the table

$3 * 1 = 1$

$5 * 3 = 5$

$1 * 5 = 5$

b) For identity element, e, $a * e = a$

From the table, the identity element for * is 3

c) 5 has no inverse under *

Worked Example 1.16

2. The operation * is defined over the set of real numbers by $a * b = a + b + 3ab$

a) Show that *

i. is commutative

ii. is associative

b) Find the identity element for the operation *

c) Find the inverse and the value for which a has no inverse solution.

d) Determine whether or not $a * (b + c) = (a * b) + (a * c)$

Solution

2a.

i. for the operation * to be commutative,

$a * b = b * a$

LHS $a * b = a + b + 3ab$ ....(1)

RHS $b * a = b + a + 3ba$..........(2)

$a * b = b * a$, therefore the operation * is commutative.

ii. For the operation * to be associative,

$a * (b * c) = (a * b) * c$

$a * [b + c + 3bc]$

$a + b + 3bc + 3[(a)(b + c + 3bc]$

$a + b + 3bc + 3[ab + ac + 3abc]$

$a + b + 3bc + 3ab + 3ac + 9abc$

$a * b = a + b + 3ab$

$(a * b) * c = a + b + 3ab + c + 3[(a + b + 3ab)c]$

$a + b + 3ab + 3[ac + bc + 3abc]$

$a + b + 3ab + 3ac + 3bc + 9abc$

Since LHS = RHS , the operation * is associative.

b) Let e be the identity element

$a * e = a$

$a + e + 3ae = a$

$e + 3ae = a - a$

$e(1 + 3a) = 0$

$e = \frac{0}{1 + 3a}$

$e = 0$

c) Let a^{-1} be the inverse

$a * a^{-1} = e$

$a + a^{-1} + 3aa^{-1} = 0$

$a^{-1} + 3aa^{-1} = -a$

$a^{-1}(1 + 3a) = -a$

$a^{-1} = \frac{-a}{1 + 3a}$

For a to have an inverse, $1 + 3a \neq 0$

$1 + 3a = 0$

$3a = -1$

$a = \frac{-1}{3}$, hence a $\neq \frac{-1}{3}$ in order for a to have an inverse

d) $a * (b + c) = (a * b) + (a * c)$

LHS:

$a * (b + c) = a + b + c + 3[(a)(b + c)]$

$a + b + c + 3[ab + ac)]$

$a + b + c + 3ab + 3ac$

$(a * b) + (a * c) = a + b + 3ab + a + c + 3ac$

$2a + c + 3ab + 3ac$

Therefore $a * (b + c) \neq (a * b) + (a * c)$

Worked Example 1.20

The operation * is defined over the set of real numbers by $m * n = m + n + 10$

Find

i. the identity element

ii. the inverse of –3 and 7 under *

Solution

i. Check for commutativity:

$m\nabla n = m + n + 10$

$n\nabla m = n + m + 10$

Since $m\nabla n = n\nabla m$, ∇ is commutative and hence could have an identity element.

Let e be the identity element, where $e \in \mathbb{R}$,

Then by definition, $m\nabla e = m$

But $m\nabla e = m + e + 10$

$\Longrightarrow m + e + 10 = m$

$e = m - m - 10$

$e = -10$

ii. Let a^{-1} be the inverse:

$m * m^{-1} = e$

$m + m^{-1} + 10 = -10$

$m^{-1} = -10 - 10 - m$

$m^{-1} = -20 - m$

The inverse of –3 under *:

$m^{-1} = -20 - (-3)$

$= -17$

The inverse of 7 under *:

$m^{-1} = -20 - (7)$

$= -27$

SETS, PROPERTIES OF SETS AND OPERATIONS ON SETS

In your class, consider the following scenarios:

i. A girl using a cutlery set for the first time in boarding school

ii. The football team of your school

iii. Natural numbers less than 10

From the above scenarios, the following can be deduced;

- Members of the cutlery sets are fork, spoon and knife which can conveniently be used for eating.
- There are eleven members in a football team playing different roles of defending, midfield and attacking to win a game
- The members of natural numbers less than 10 are 1,2,3,4,5,6,7,8 and 9.
- All these groups belong to common features called a set.

Therefore a set is a well–defined collection of objects. Sets are usually denoted by capital letters such as A, B, *C*, D…etc

REPRESENTING SETS

Sets are commonly represented by any of the following methods:

1. Statement form (word description).
2. Tabular or Roster form (Listing).
3. Rule or set builder notation form.

Suppose you want to describe the following situations in statement form:

1. Set of trees in the forest that are taller than 20 feet.
2. Set of birds in your region that move north to south in harmattan.
3. Positive integers less than 10.

From the above, you can deduce the following as statement form of the sets:

1. $\{trees\ in\ the\ forest\ that\ are\ taller\ than\ 20\ feet\}$
2. $\{b{:}b\ is\ a\ bird\ in\ your\ region\ that\ moves\ north\ to\ south\ in\ harmattan\}$
3. $\{postive\ integers\ less\ than\ 10\}$

1. The Statement Form (Word Description)

The statement form is a method used to describe a set using natural language. We use words to convey the characteristics or elements of the set. For example, the set of even numbers less than 8 is written as: {even numbers less than 8}

2. The Tabular or Roster Form

The roster notation involves listing all the elements of a set. We enclose the elements within curly brackets { } and separate them with commas.

For example, let N denote the set of the first five natural numbers. Therefore, N = {1, 2, 3, 4, 5}. Similarly, if *M* denotes the set of the months beginning with the letter "J", then *M* = {January, June, July}.

3. The Rule or Set Builder Form

In this method, the elements of the set are described by using the variable "*x*" or any other variable, followed by a colon. The symbol ":" or" "/" as used in this case is read "such that".

Thereafter, the property possessed by the elements is written in brackets (called "set of all"). For example, *P* is the set of counting numbers greater than 12. In set builder notation or form, P is expressed as P = {x: x is a counting number greater than 12}. This is read as "P is the set of elements of x such that x is a counting number greater than 12. The set builder notation also involves the use of inequality symbols to describe the range of data.

Likewise, if A is the set of even numbers between 6 and 14, then A is written in set builder notation as A = {x / x is even, $6 < x < 14$} or A = {x: x in P, $6 < x < 14$, P is an even number}

Also, if K = {4, 5, 6, 7} then set builder form of K is: K = {x: x is a natural number and $3 < x < 8$}

Also, if K = {4, 5, 6, 7} then set builder form of K is: K = {x: x is a natural number and $3 < x < 8$}.

In pairs or small groups, read and discuss the following, explaining the concepts to one another.

TYPES OF SET

1. The Null Set or Empty Set

Consider the following statements:

- A set of students in your school who are 100 years old.
- A set of dogs that can fly.

The two statements above would contain nothing. Thus a set which does not contain any elements is called a *null set*. It is denoted by ∅ (read as *phi*). The null set can also be denoted by { }.

It can be concluded that the number of elements in an empty set is 0. An empty set is therefore an example of a finite set (see below). For example, the set of whole numbers less than 0. Since there is no whole number less than 0, the set is said to be a null set.

Likewise M = {x: x is a composite number less than 4}. M is an empty set because there is no composite number (remember that a composite number is a number which has more than two factors) less than 4.

Note: $\varnothing \neq \{0\}$ because {0} is a set which has one element 0.

2. The Unit Set or Singleton Set

A Unit set is a set which contains only one element. For example, the set of Headmasters in your school could be a unit set. Why?

Take $R = \{x{:}x$ is neither prime nor composite$\}$, then $R = \{1\}$

Other examples are:

- $S = \{x{:}x$ is an even prime$\} = \{2\}$, $n(S) = 1$
- $T = \{x{:}x$ is a natural number and $x^2 = 4\} = \{2\}$, $n(T) = 1$

3. Finite Set

This is a set which contains a definite number of elements. A finite set can contain all its elements. For example, A = {set of days of the week} B = {2, 3, 5, 7….19}

$C = \{x{:}x$ is a natural number and $x < 7\}$

4. Infinite Set

It is a set whose elements cannot be listed completely. In other words it is a set containing never–ending elements. For example, $A = \{x : x$ N, $x > 1\}$ $B = \{$Set of all prime numbers$\}$

Cardinality of Set

The total number of elements in a set P, is called the cardinal number of P, denoted by $n(P)$. For example,

1. $A = \{x{:}x$ is a natural number, $x < 5\}$ $A = \{1, 2, 3, 4\}$ $n(A) = 4$

2. $B = \{$set of elements in the word {"MATHEMATICS"}$\}$ $n(B) = 11$

OPERATIONS ON SETS

Operations on sets which will be discussed in this section are Union, Intersection and Complements of sets. Read the information below so you can investigate them as a class and see how they are represented on Venn diagrams.

1. Intersection of Sets

Activity 1.4

Consider three students, Kojo (K), Amina (A) and Joseph (J) who selected their favourite fruits as follows:

K ={ orange, mango, pawpaw}

A={orange, strawberries, pawpaw}

J={mango, pawpaw, banana, pineapple, orange}

Is there any fruit(s) that were selected by Kojo, Amina and Joseph?

Yes, all of them selected pawpaw.

We therefore say that the intersection of K, A and J is pawpaw. This is written mathematically as $K \cap A \cap J = \{\text{pawpaw}\}$.

Now, describe intersection of sets in your own words.

The *intersection* of sets A, B and C is all the elements that are common for both A, B and C. Symbolically, we denote the intersection of sections A, B and C as $A \cap B \cap C$.

For example the intersection for the sets, A ={2, 4, 6, 8}, B ={4, 8, 12, 16, 20}, and C ={4, 8, 12, 16, 20, 24, 28}is

$$A \cap B \cap C = \{4, 8\}$$

Worked Example 1.21

1. Three learners were asked to select their four best subjects (not in any particular order) by their class teacher. The responses were as follows:

   A = {Science, Computing, Mathematics and English}

   B = {Mathematics, Computing, French and English}

   C = {Ghanaian Language, Computing, Agriculture and Mathematics}

If the teacher wanted to find out the subject that learners like most in his school, then we can use

$A \cap B \cap C =$ {Mathematics, Computing}

2. If U, the universal set, = {positive integers less than 20}

$M =$ {factors of 18}

$N =$ {multiples of 3}

$Q =$ {prime numbers}

M, N and Q are subsets of the universal set U. Find $M \cap N \cap Q$

Solution

$U = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19\}$

$M = \{1,2,3,6,9,18\}$

$N = \{3,6,9,12,15,18\}$

$Q = \{2,3,5,7,11,13,17,19\}$

$M \cap N \cap Q = \{3\}$

2. Union of Sets

If a set is the union set of two or more sets, then it consists of all elements that are in the individual sets and those that are common to the sets. Note that the common elements to the set are written once.

For example, if set $P = \{1,2,3,6,12\}$, $Q=\{2,4, 6,8,10\}$ and $R = \{5, 10,15,20\}$ then

$P \cup Q \cup R = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20\}$

Again, if $P = \{a, b, c, d, e\}$, $Q =\{a, e, i, o, u\}$ and $R = \{g,i,r,l\}$

Then $P \cup Q \cup R=\{a,b,c,d,e,g,i,l,o,r,u\}$

Worked Example 1.22

A, B and C are integers and $A = \{-6, -5, 0, 3, 4\}$,

$B = \{b{:}b \textit{ is an odd number}, 3 \leq b < 12\}$ and

$C = \{\textit{positive prime numbers less than } 8.\}$

Find $A \cup B \cup C$

Solution

$A = \{-6, -5, 0, 3, 4\}$

$B = \{3, 5, 7, 9, 11\}$

$C = \{2, 3, 5, 7\}$

$A \cup B \cup C = \{-6, -5, 0, 2, 3, 4, 5, 7, 9, 11\}$

3. Complements of Sets

The complement of a set is the set of all elements in a universal set that are **not** in the given set. Given a universal set U and a set A, the complement of A, denoted as A' or $\overline{A}$, includes all elements in U that are not in A.

Here are a few examples to illustrate the concept of the complement of a set:

1. Universal Set $U=\{1,2,3,4,5\}$, Set $A=\{1,2,3\}$
   Complement of A: $A'=\{4,5\}$
2. Universal Set: $U=\{a,b,c,d,e,f\}$, Set $B=\{a,c,e\}$
   Complement of B: $B'=\{b,d,f\}$
3. Universal Set: $U = \{apple,orange,banana,grape,mango\}$,
   Set $C = \{apple,mango\}$
   Complement of C: $C'=\{orange,banana, grape\}$
4. Universal Set: $U=\{red,blue,green,yellow,purple\}$, Set $D=\{red,yellow\}$
   Complement of D: $D'=\{blue,green,purple\}$

SETS ON VENN DIAGRAMS

Activity

Go through the following activities to identify various regions in a Venn diagram

1. Use three different writing materials of different colours (crayons, markers, pens) to draw the three intersecting circles in a Venn diagram and label them A, B and C.

2. Identify the various regions that have;

a) only one of the colours

b) two different colours meeting

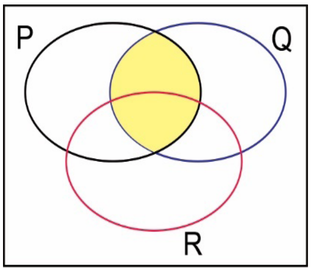
c) all three colours meeting

d) only two colours meeting

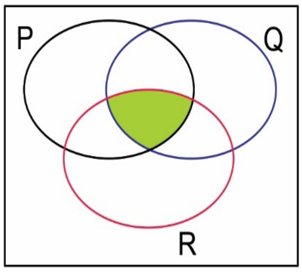
Now, study the diagrams below in groups or individually and verify the shaded regions.

Regions of a Three-Set Venn Diagram

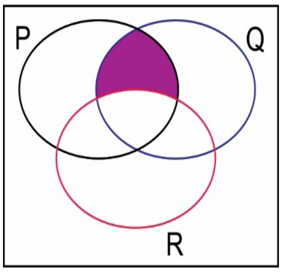
The various operations on sets can be represented in a Venn diagram as depicted below.



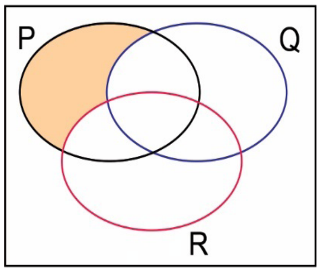
$P \cap Q$



$P \cap Q \cap R$



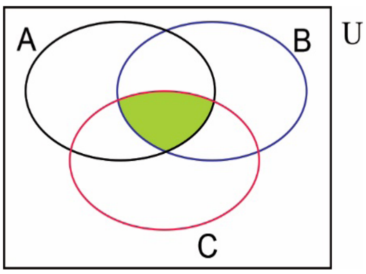
$P \cap Q \cap R'$ (P and Q only)



$P \cap (Q \cup R)'$ (P only)

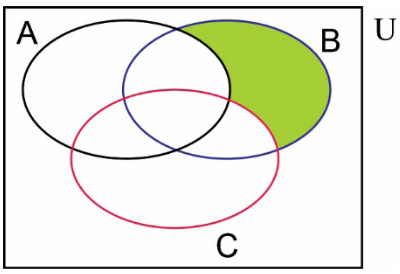
1. Study the shaded portions of the three set Venn diagram

(a)



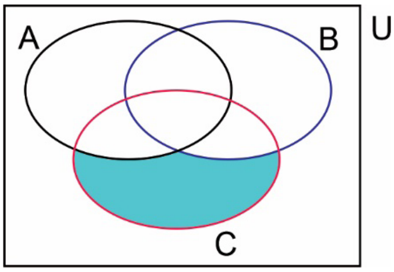
$A \cap B \cap C$

(b)



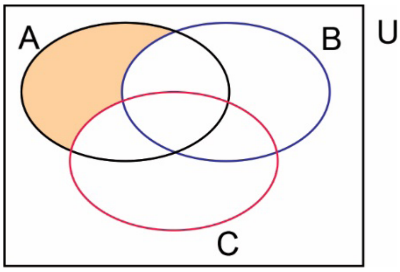
$A' \cap B \cap C'$

(c)



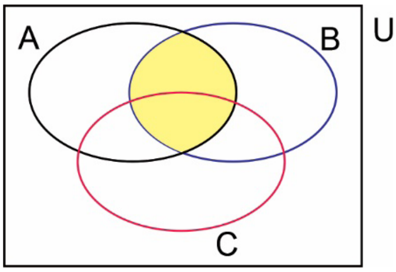
$A' \cap B' \cap C$

(d)



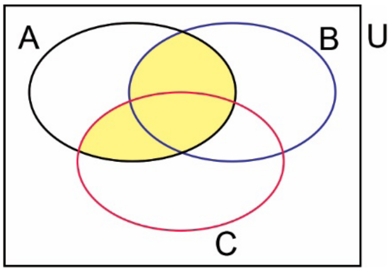
$A \cap B' \cap C'$

(e)



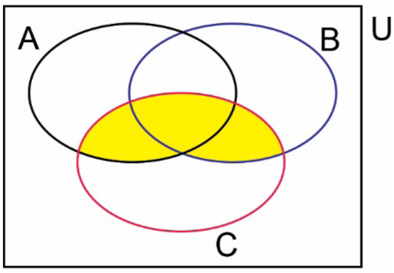
$A \cap B$

(f)



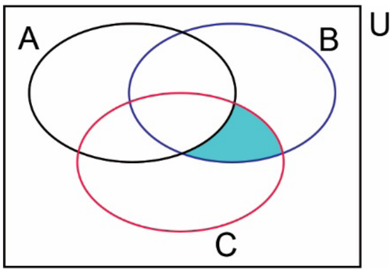
$A \cap (B \cup C)$

(g)



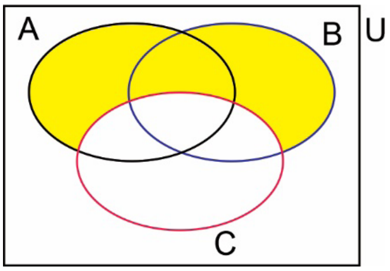
$(A \cup B) \cap C$

(h)



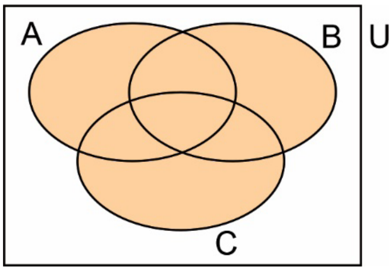
$A' \cap B \cap C$

(i)



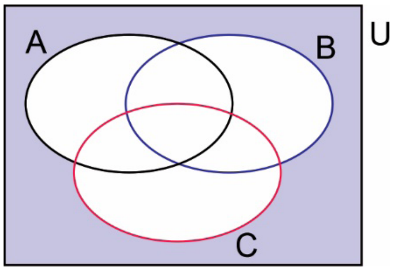
$(A \cup B) \cap C'$

(j)



$A \cup B \cup C$

(k)



$(A \cup B \cup C)'$

Let us go through the following examples to enable us illustrate three-sets problems in a Venn diagram.

Worked Example 1.23

The universal set, $U = \{1,2,3,4,5,6,7,8,9,10,11\}$, A, B and C are subsets of U such that A={factors of 6}, B={multiples of 3}, C= {odd numbers greater than 1}

Find

a) **i.** $(A \cap B)$

ii. $(B \cap C)$

iii. $(A \cap C)$

iv. $A \cap B \cap C$

b) Illustrate the above information on a Venn diagram.

Solution

$A = \{1,2,3,6\}$, $B = \{3, 6, 9\}$, $C = \{3, 5, 7, 9, 11\}$

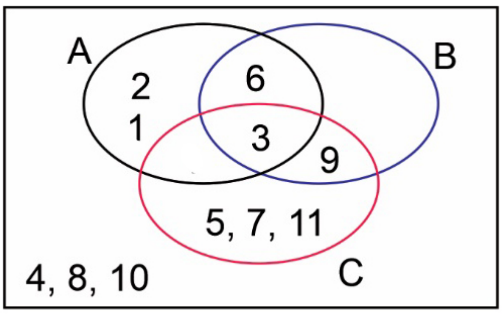
a) **i.** $A \cap B = \{3, 6\}$

ii. $B \cap C = \{3,9\}$

iii. $A \cap C = \{3\}$

iv. $A \cap B \cap C = \{3\}$

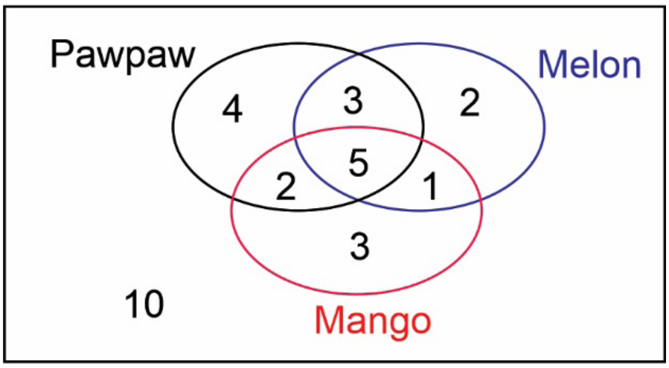
b) $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$



Worked Example 1.24

A group of students were asked to say which type of fruits they like. Some students like more than one fruit and some did not like any of the three types of fruits.

Explain what each of the numbers in the diagram represents;



4 students like pawpaw only.

3 students like pawpaw and melon but do not like mango.

2 students like melon only.

2 students like pawpaw and mango but do not like melon.

5 students like all three fruits.

1 student likes melon and mango but does not like pawpaw.

3 students like mango only.

10 students do not like any of the three fruits.

Find the number of students who like;

i. Melon

ii. Only one of the three fruits

Solution

i. $3 + 5 + 2 + 1 = 11$

ii. $4 + 3 + 2 = 9$

Worked Example 1.25

In a class of 55 students, some students study at least one of the following subjects: Manufacturing, Robotics and Aviation. 20 students study none of them. The following gives details of the subjects:

Manufacturing only = 4

Robotics only = 5

Aviation only = 7

All three subjects = 3

Manufacture and Aviation = 7

Robotics and Aviation = 8

a) Illustrate the above data in a Venn diagram

b) Find the number of students who study

   i. Manufacturing or Aviation or both, but not Robotics

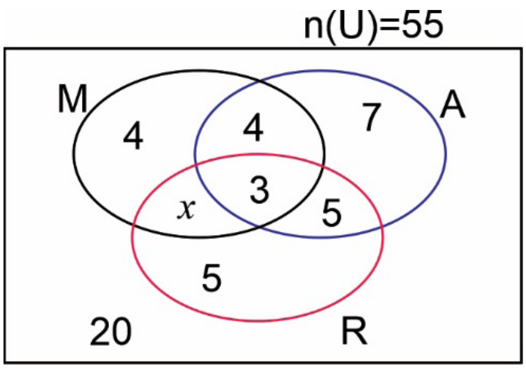
   ii. Robotics

Solution

Let Manufacturing be denoted by M, Robotics by R and Aviation by A and the class of students by U

n(M only) = 4, n(R only) = 5, n (A only) = 7, n($M \cap R \cap$ A) = 3, n($M \cap$ A) = 7, n($R \cap$ A) = 8,   n(U) = 55

a) In a Venn diagram the information can be represented as shown, letting x represent those who study M and R only.



Given n (U) = 55

$4 + 4 + 3 + x + 5 + 5 + 7 + 20 = 55$

$48 + x = 55$

$x = 55 - 48$

$x = 7$

b)

i. Manufacturing or Aviation or both but not Robotics refer to all elements in the circle of both M and A but outside R

$\therefore$ required number $= 4 + 4 + 7$

$= 15$

ii. Number of students who study Robotics are in all the elements in the circle of R

$\therefore$ required number $= 5 + 5 + 3 + 7$

$= 20$

Worked Example 1.26

The Ghana Tourism Authority conducted a survey to find out the number of students who had visited various tourism centres in Ghana. The result of the survey revealed the following information: 48 had visited Kakum National Park, 50 had visited Cape Coast Castle, and 59 had visited Mole National Park. 33 had visited Kakum National Park and Cape Coast Castle 29 had visited Cape Coast Castle and Mole National Park, 36 Kakum and Mole National parks. 24 visited all the places. They also observed that 2 students had not visited any of the three places.

a) Illustrate the above information on a Venn diagram.

b) How many students

i. were surveyed?

ii. had visited Mole National Park, but not Cape Coast nor Kakum?

iii. had visited Cape Coast or Mole National Park?

iv. had visited exactly one of the tourism sites?

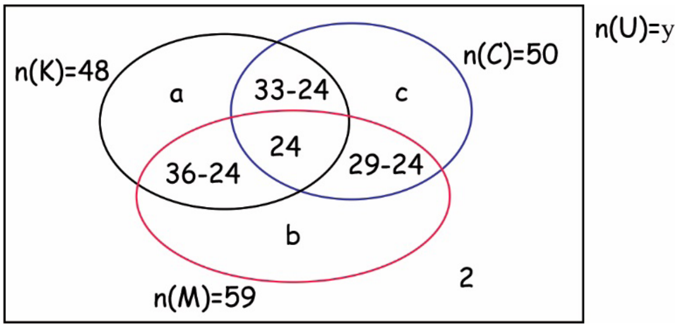
v. had visited exactly two of the tourism sites?

vi. had visited at least two tourism sites?

Solution

Let K = {Kakum National Park}, C = {Cape Coast Castle}, M = {Mole National Park}

$n(U) = y$, $n(K) = 48$, $n(C) = 50$, $n(M) = 59$, $n(K \cap C) = 33$, $n(C \cap M) = 29$, $n(K \cap M) = 36$, $n(K \cap C \cap M) = 24$, $n(C \cup P \cup M)' = 2$



$n(K) = 48$

$a + (33–24) + 24 + (36–24) = 48$

$a + 9 + 24 + 12 = 48$

$a = 48 – (9 + 24 + 12)$

$a = 3$

$n(C) = 50$

$c + (33–24) + 24 + (29–24) = 50$

$c + 9 + 24 + 5 = 50$

$c = 50 – (9 + 24 + 5)$

$c = 12$

$n(M) = 59$

$b + (36–24) + 24 + (29–24) = 59$

$b + 12 + 24 + 5 = 59$

$b = 59 – (12 + 24 + 5)$

$b = 18$

i. $n(U)=3 + 9 + 12 + 5 + 24 + 12 + 18 + 2 = 85$

85 students were surveyed

ii. 18 students

iii. $9 + 12 + 5 + 24 + 12 + 18 = 80$ students

iv. $18 + 12 + 3 = 33$ students

v. $36 – 24 + 33 – 24 + 29 – 24 = 26$ students

vi. $36 – 24 + 33 – 24 + 29 – 24 + 24 = 50$ students

Properties of Operation on Sets

1. Commutative Property

i. $A \cup B = B \cup A$ Hence, Union is commutative.

ii. $A \cap B = B \cap A$ Thus, intersection is commutative.

Verification

If A = {1, 2, 3, 4, 5, 6, 8}, and B = {2, 3, 5, 7, 8, 9, 10}

i. $A \cup B$={1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and $B \cup A$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Therefore $A \cup B = B \cup A$

ii. $A \cap B$= {2, 3, 5, 8} and $B \cap A$ = {2, 3,5, 8}

Therefore $A \cap B = B \cap A$

2. Associative property

i. $(A \cap B) \cap C = A \cap (B \cap C)$ Hence, intersection is associative.

ii. $A \cup (B \cup C) = (A \cup B) \cup C$ Thus, Union is associative.

3. Distributive property

i. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, Intersection distributes over union.

ii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Union distributes over intersection.

Worked Example 1.27

Now, in pairs, do the same to verify these properties, the solution is below.

If A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {2, 4, 8} are subsets of the universal set U = {1, 2, 3. . . 10}, list the elements of the sets

a) **i)** $(A \cap B) \cap C$

ii. $A \cap (B \cap C)$

iii. What can you say about your result in **a**. i) and ii)?

b) **i)** $A \cup (B \cup C)$

ii. $(A \cup B) \cup C$

iii. What can you say about your result in **b** i) and ii)

Solution

a) **i)** $(A \cap B) \cap C = \{2,4\}$

ii. $(A \cap B) \cap C = \{2,4\}$,

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, showing associative property

b) **i)** $A \cup (B \cup C) = \{1,2,3,4,6,8\}$

ii. $(A \cup B) \cup C = \{1,2,3,4,6,8\}$,

iii. $A \cup (B \cup C) = (A \cup B) \cup C$, showing associative property

Worked Example 1.28

Let $U = \{1, 2, 3, \ldots, 12\}$ and $U \in \mathbb{Z}$

$A = \{x \in U : x \textit{ is a prime number}\}$

$B = \{x \in U : x \textit{ is an even number}\}$

$C = \{x \in U : x \textit{ is divisible by } 3\}$.

Find the following sets:

a) $A \cup B$

b) $A \cap C$

c) $(A \cap C) \cup (B \cap C)$

d) $A \cap B$

Solution

$A = \{2, 3, 5, 7, 11\}$

$B = \{2, 4, 6, 8, 10, 12\}$

$C = \{3, 6, 9, 12\}$

a) $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12\}$

b) $A \cap C = \{3\}$

c) $(A \cap C) \cup (B \cap C) = \{3\} \cup \{6, 12\} = \{3, 6, 12\}$

d) $A \cap B = \{2, 3, 5, 7, 11\} \cap \{2, 4, 6, 8, 10, 12\} = \{2\}$

THE CONCEPT OF BINOMIAL THEOREM

A binomial is a simplified polynomial with two terms i.e., of the form $(p + q)$. $(2x^2 + 3)$, $(y - 2x)$ and $(-5 + 0.2x)$ are all examples of binomials since they have exactly two terms only but $-\frac{1}{4}y^3$, $(2x^2 + 3y - 4)$ and $(y + 3 - 0.1x)$ are not binomials since in their simplest forms, do not have exactly two terms.

In certain situations, in mathematics, it is necessary to write $(p + q)^n$ as the sum of its terms. Because $p + q$ is a binomial, this process is called expanding the binomial. For small values of n, it is relatively easy to write the expansion by using multiplication. For example, $(p + q)^2$ can be written as $(p + q)(p + q)$ and expanded to obtain $p^2 + 2pq + q^2$ and $(p + q)^3$, written as $(p + q)^2(p + q)$ and so on. We could continue to build on previous expansions and eventually have quite a comprehensive list of binomial expansions. Instead, we could look for a theorem that will enable us to expand directly. This theorem is the binomial theorem.

BINOMIAL EXPANSION AND PASCAL'S TRIANGLE

One great French mathematician and philosopher, Blaise Pascal who lived from 1623 to 1662 helped to develop a triangular array of numbers and it was named after him, hence Pascal's triangle**.** The Pascal triangle depicts the coefficients of binomial expansions based on the given power.

Let us now go through the following activities to help us generate the coefficient of the terms by Pascal's triangle.

The Coefficient of Terms

Suppose you buy a number of snacks at the school canteen, biscuits(b) and popcorn(p).

How will you write out the combination of these snacks mathematically?

Did you get $(b + p)$? Good!

Given a flat square tray, with dimension $(b + p)$, how many snacks can be arranged on the tray. Did you get $(b + p)^2$? Congratulations!

How many snacks can you pack into a cube-shaped box of dimension $(b + p)$?

Is it $(a + b)^3$? Your guess is as good as mine!

In Algebra, it must be recalled that the letters of the alphabet are used to represent items, objects, things or numbers to make problem-solving easier. Also, the number in front of a letter is called its **coefficient**.

Terms that are like terms can be grouped together or added. For example, adding five mangoes to three mangoes will result in eight mangoes. However, one mango and four oranges will remain the same because they are not alike.

Worked Example 1.29

Expand and simplify the two brackets $(2x + 1)(3x - 4)$.

If your answer is $6x^2 - 5x - 4$, then you are correct.

If not this is one way of doing it:

$$
\begin{aligned}
(2x + 1)(3x - 4) &= (2x + 1)(3x - 4) \\
&= 2x(3x - 4) + 1(3x - 4) \\
&= 6x^2 - 8x + 3x - 4 \\
&= 6x^2 - 5x - 4
\end{aligned}
$$

The coefficient of x^2 is 6, that of x is –5 and –4 is the constant.

Similarly, in the expression, $x^3 + 7xy^2 - 9y^3$, the coefficient of x^3 is 1, that of xy^2 is 7, and it is –9 for y^3.

Worked Example 1.30

Given the expression $x^2 + 10xy + 24y^2$, find the coefficient of i) x^2 ii) y^2.

Solution

(i) Coefficient of x^2 is 1

(ii) Coefficient of y^2 is 24

BINOMIAL EXPANSION AND PASCAL'S TRIANGLE

Now, let us consider the expansion of the following consecutive powers of a binomial,

The expression a + b is actually a shortened version for $1a + 1b$. the next power of a + b is (a + b) (a + b), which is also written as $(a + b)^2$.

Using Pascal's triangle, this can be expanded as follows:

$$(a+b)^2 = (a+b)(a+b)$$
$$= a(a+b) + b(a+b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

Similarly, $(a + b)^3$ can be expanded as follows:

$$(a+b)^3 = (a+b)(a+b)^2$$
$$= (a+b)(a^2 + 2ab + b^2) \text{ (by substitution)}$$
$$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = (a+b)(a+b)^3$$
$$= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a(a^3 + 3a^2b + 3ab^2 + b^3) + b(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Example 1.31

1. Use the pattern to obtain the coefficient for the following expression:

i.. $(a + b)^5$

You may have noticed a pattern in the coefficients of the expansions. Writing the coefficients of the expansions of $(a + b)^2$, $(a + b)^3$ $(a + b)^4$, we get

n	coefficients
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

You may have observed that

1. Every subsequent row is obtained by starting with 1 and ending with 1.
2. Each interior member is the sum of the two elements directly above it. For example $1 + 4 = 5$, $4 + 6 = 10$, $6 + 4 = 10$, $4 + 1 = 5$

For completeness, it may be observed that

$(a + b)^0 = 1$ and $(a + b)^1 = 1a + 1b$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

When an expression is written as a series of terms, it is said to be **expanded,** and the series is its **expansion**.

This pattern created by the coefficients is referred to as Pascal's triangle.

Pascal's triangle therefore is the triangular arrangement of numbers that represent the coefficients in the expansion of any binomial expression such as $(p + q)^n$. The numbers are so arranged that they reflect as a triangle.

The numbers in Pascal's triangle are placed in such a way that each number is the sum of two numbers just above it.

Take note of the following characteristics with respect to the expansion of $(a + b)^n$, where n is a positive integer.

i. Reading from either end of each row, the coefficients are the same.
ii. There are (n + 1) terms.

iii. Each term is of degree (sum of the powers of the two terms) n.

iv. The coefficients are obtained from the row in Pascal triangle.

v. The exponent on a decreases by 1 for each successive term.

vi. The exponent on b increases by 1 for each successive term.

Go through the steps below to expand a binomial expression

Step 1: Write down the binomial coefficients:

Step 2: Write the expression with the binomial coefficients:

Step 3: Simplify the expression:

Step 4: Simplify further:

Worked Example 1.32

Expand $(a + b)^6$ in descending powers of a.

Solution

Write down the binomial coefficients

1, 6, 15, 20, 15, 6, 1

Write the expression with the binomial coefficients:

$1(a)^6(b)^0 + 6(a)^{6-1}(b)^{0+1} + 15(a)^{6-2}(b)^{0+2} + 20(a)^{6-3}(b)^{0+3} + 15(a)^{6-4}(b)^{0+4} + 6(a)^{6-5}(b)^{0+5} + 1(a)^{6-6}(b)^{0+6}$

Simplify the expression

$1(a^6) + 6(a^5)(b^1) + 15(a)^4(b)^2 + 20(a)^3(b)^3 + 15(a)^2(b)^4 + 6(a)(b)^5 + 1(a)^0(b)^6$

Therefore, the expansion of $(a + b)^6$ in descending powers of a is

$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Worked Example 1.33

Expand $(x + 3y)^3$ in descending powers of x.

Solution

Here, a = x, b = $3y$ and there will be four terms involving $(3 + 1 = 4)$

$x^3, (x^2)(3y), (x)(3y)^2, (3y)^3$

Their coefficients obtained from Pascal's triangle are respectively

1, 3, 3, 1

Therefore, the expansion of $(x + 3y)^3$ in descending powers of x is

$x^3 + \quad 3(x^2)(3y) + 3(x)(3y)^2 + (3y)^3$

Which simplifies to $x^3 + 9x^2y + 27xy^2 + 27y^3$

Worked Example 1.34

Expand $(2x + 3y)^5$ in descending powers of x

Solution

$(2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$

$32x^5 + 5(16x^4)(3y) + 10(8x^3)(9y^2) + 10(4x^2)(27y^3) + 5(2x)(81y^4) + (243y^5)$

$32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

Worked Example 1.35

Expand $\left(x + \frac{1}{x}\right)^4$

Solution

Step 1: Write down the binomial coefficients:

1, 4, 6, 4, 1

Write the expression with the binomial coefficients:

$\left(x + \frac{1}{x}\right)^4 = 1(x)^4\left(\frac{1}{x}\right)^0 + 4(x)^{4-1}\left(\frac{1}{x}\right)^{0+1} + 6(x)^{4-2}\left(\frac{1}{x}\right)^{0+2} + 4(x)^{4-3}\left(\frac{1}{x}\right)^{0+3} + 1(x)^{4-4}\left(\frac{1}{x}\right)^{0+4}$

Simplify the expression

$(x)^4\left(\frac{1}{x}\right)^0 + 4(x)^3\left(\frac{1}{x}\right)^1 + 6(x)^2\left(\frac{1}{x}\right)^2 + 4(x)^1\left(\frac{1}{x}\right)^3 + (x)^0\left(\frac{1}{x}\right)^4$

$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

Simplify further:

$= x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4}$

Worked Example 1.36

a) Expand $\left(\frac{1}{x} - \sqrt{x}\right)^5$

b) Expand $\left(\frac{1}{x} + 1\right)^5$

Solution

a) From the expansion of $(a + b)^5$ in the previous example and by comparison i.e.,

$a = \frac{1}{x}$ and $b = \sqrt{x}$,

$\left(\frac{1}{x} - \sqrt{x}\right)^5$

$= \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4(-\sqrt{x}) + 10\left(\frac{1}{x}\right)^3(-\sqrt{x})^2 + 10\left(\frac{1}{x}\right)^2(-\sqrt{x})^3 + 5\left(\frac{1}{x}\right)(-\sqrt{x})^4$

$+ (-\sqrt{x})^5$

$= \frac{1}{x^5} - \frac{5\sqrt{x}}{x^4} + \frac{10}{x^2} - \frac{10\sqrt{x}}{x^2} + 5x - \sqrt{x^5}$

$= x^{-5} - 5x^{-\frac{7}{2}} + 10x^{-2} - 10x^{-\frac{1}{2}} + 5x - x^{\frac{5}{2}}$

b) From the expansion of $(a + b)^5$ in the previous example and by comparison i.e., $a = \frac{1}{x}$ and $b = 1$,

$\left(\frac{1}{x} + 1\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4(1) + 10\left(\frac{1}{x}\right)^3(1)^2 + 10\left(\frac{1}{x}\right)^2(1)^3 + 5\left(\frac{1}{x}\right)(1)^4 + (1)^5$

$= \frac{1}{x^5} + \frac{5}{x^4} + \frac{10}{x^3} + \frac{10}{x^2} + \frac{5}{x} + 1$

$= x^{-5} + 5x^{-4} + 10x^{-3} + 10x^{-2} + 5x^{-1} + 1$

Worked Example 1.37

Follow the steps below to complete the tasks use the binomial expansion: find the value of

a) $(1.01)^3$

b) $(2.1)^4$ without using a calculator.

Solution

a) Step 1: write 1.01 as a binomial and since 1.01 is closer to 1, write it as 1 + 0.01

Step 2: Expand the expression:

$(1 + 0.01)^3 = 1^3 + 3(1^2)(0.01) + 3(1)(0.01^2) + 1(1)^0(0.01^3)$

Step 3: Simplify each term:

$1 + 3(1)(0.01) + 3(1)(0.0001) + 1(1)(0.000001)$

$1 + 0.03 + 0.0003 + 0.000001$

1.030301

Step 4: Write the final answer:

1.0303 (4 decimal places)

b) go through the steps in (a) to solve the question (b) and verify your answer with the calculator.

Congratulations on obtaining 19.4481 as your answer. You are good to move on.

Worked Example 1.38

Use Pascal's triangle of binomial expansion to obtain the value of $(3.004)^4$, correct to six decimal places

Solution:

$$\begin{aligned}(3.004)^4 &= (3 + 0.004)^4 \\ &= 3^4 + 4(3^3)(0.004) + 6(3^2)(0.004^2) + 4(3)(0.004^3) + (0.004^4) \\ &= 81 + 4(27)(0.004) + 6(9)(0.000016) + 4(3)(0.000000064) + \\ &\quad (0.000000000256) \\ &= 81 + 0.432 + 0.0000864 + 0.000000768 + 0.000000000256 \\ &= 81.43208717 \\ &\approx 81.432087 (6\ dp)\end{aligned}$$

THE COMBINATION APPROACH

Combination is the selection in which the order of selection is not important. Combination is used in lottery in calculating the probability of winning numbers. It also used in choosing committee members from a large group of members. It is not practical to use Pascal's triangle to expand binomial expressions with larger positive integer values as exponents since the method is recursive. It requires that to obtain the coefficients of the expansion of $(p + q)^n$, we would need to find the coefficients of the expansion of $(p + q)^{n-1}$.

Thus, to find the 100*th* row of the Pascal's triangle, we must first find the preceding 99 rows. The coefficients in the Pascal's triangle however, can be generated using the combination approach.

Activity 1.5

1. Look for n_{C_r} on your calculator.
2. Now use n_{C_r} on your calculator to evaluate i) 3_{C_0} ii) 3_{C_1} iii) 3_{C_2} iv) 3_{C_3}

Did you obtain? 1 3 3 1.

You are right, congratulations!

What do these numbers represent? They represent the coefficients of n = 3 as obtained in the Pascal triangle. This means there are other ways to obtain the coefficients in binomial expansions.

Using n_{C_r} to obtain the coefficients in binomial expansion is the **combination method.**

If n is a positive integer, the expansion $(a + b)^n$ is given by:

$$(a+b)^n = \left[{}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^{n-n} b^n\right]$$

The combination of n items taking r at a time is given by:

$$\binom{n}{r} = n_{C_r} = \frac{n!}{(n-r)!r!}$$

Where n! is read as n factorial and is explained as

$$n! = n(n-1)(n-2)(n-3)\ldots 2 \times 1$$

Worked Examples 1.40

1. $5! = 5(5-1)(5-2)(5-3)(5-4) = 5 \times 4 \times 3 \times 2 \times 1 = 120$
2. $7! = (7-1)(7-2)(7-3)(7-4)(7-5)(7-6)$
   $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Example 1.41

Study the computation of the following factorials to appreciate the derivation of the Binomial Theorem:

- **i.** $6_{C_3} = 20$
- **ii.** $4_{C_2} = 6$
- **iii.** $7_{C_4} = 35$

Confirm your answers with a calculator.

Note: It must be noted that $n_{C_0} = 1$ *and* $0! = 1$

Binomial Expansion of Positive Integers Using the Combination Method

Using the combination method, the binomial $(a + b)^n$, n being a positive integer, is expanded as:

$(a + b)^n = (a + b)(a + b)\ldots\ldots(a + b)$ to n factors.

- **(a)** Choosing an **a** from each bracket we obtain a^n
- **(b)** The term a^{n-1} is obtained by choosing one **b** from one bracket and **'a's** from the other n–1. This can be done in n_{C_1} ways, giving $n_{C_1 a^{n-1} b}$.
- **(c)** The term a^{n-2} is obtained by choosing one **b** from two brackets and **'a's** from the other n–2 . This can be done in n_{C_2} ways, giving $n_{C_2 a^{n-2} b^2}$
- **(d)** The term in a^{n-r} is obtained by choosing one **b** from r brackets and **'a's** from other n-r. This can be done in n_{C_r} ways, giving $n_{C_r a^{n-r} b^r}$.
- **(e)** Choosing a **b** from each bracket we obtain b^n.

Worked Example 1.42

Use the binomial theorem to write out the **first four** terms of the binomial expansion and simplify $(2x+ y)^{10}$

$$(3x + y)^{10} = x^{10} + \binom{10}{1}(2x)^9(y)^1 + \binom{10}{2}(2x)^8(y)^2 + \binom{10}{3}(2x)^7(y)^3$$

Technology Tip (use of calculator evaluation)

$$\binom{10}{1} = \frac{10}{1} = 10$$

$$\binom{10}{2} = \frac{10\ .\ 9}{1\ .\ 2} = 45$$

$$\binom{10}{3} = \frac{10\ .\ 9\ .\ 8}{1\ .\ 2\ .\ 3} = 120$$

$$(2x+ y)^{10} = (2x)^{10} + \binom{10}{1}(2x)^9(y)^1 + \binom{10}{2}(2x)^8(y)^2 + \binom{10}{3}(2x)^7(y)^3$$

$$(2x+ y)^{10} = (2x)^{10} + \binom{10}{1}(2x)^9(y)^1 + \binom{10}{2}(2x)^8(y)^2 + \binom{10}{3}(2x)^7(y)^3$$

$$+ \binom{10}{4}(2x)^6(y)^4 w$$

Worked Example 1.43

By using the combination approach, expand completely the expression $(2 + x)^5$.

Solution

$(2+x)^5 = 5_{C_0}\, 2^5 + 5_{C_1} \times 2^4 \times x + 5_{C_2} \times 2^3 \times x^2 + 5_{C_3} \times 2^2 \times x^3 + 5_{C_4} \times 2 \times x^4 + x^5$

$= 1 \times 2^5 + 5 \times 2^4 \times x + 10 \times 2^3 \times x^2 + 10 \times 2^2 \times x^3 + 5 \times 2 \times x^4 + x^5$

$= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

Worked Example 1.44

Expand $(x–2y)^6$ using the combination method, and hence find the value of $(1.06)^6$, correct to 5 decimal places.

Solution

$(a+b)^n = (a+b)^n = [{}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^{n-n} b^n]$

$(x–2y)^6 = \binom{6}{0} x^6(-2y)^0 + \binom{6}{1}(x)^{6-1}(-2y) + \binom{6}{2}(x)^{6-2} \times (-2y)^2 + \binom{6}{3}(x)^{6-3} \times (-2y)^3 + \binom{6}{4} \times (x)^{6-4} \times (-2y)^4 + \binom{6}{5} \times (x)^{6-5} \times (-2y)^5 + \binom{6}{6}(-2y)^6$

$= (1)x^6(1) + 6(x)^5(-2y) + 15(x)^4 \times 4y^2 + 20(x)^3 \times -8y^3 + 15 \times (x)^2 \times 16y^4 + 6 \times (x)^1 \times -32y^5 + 1 \times 64y^6$

$= x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

$(1.06)^6 = (1+0.06)^6$

Comparing

$(x–2y)^6 = (1+0.06)^6$

$x = 1, –2y = 0.06$

$y = –0.03$

$(1.06)^6 = (1)^6 - 12(1)^5(-0.03) + 60(1)^4(-0.03)^2 - 160(1)^3(-0.03)^3 + 240(1)^2(-0.03)^4 - 192(1)(-0.03)^5 + 64(-0.03)^6$

$= 1 + 0.36 + 0.054 + 0.00432 + 0.0001944 + 0.0000046656 + 0.000000046656$

$(1.06)^6 = 1.41851911$

$= 1.41852(5d.p)$

Worked Example 1.45

The coefficient of x^3 in the expansion of $(k+3x)^4$ is 54. Find the value of k.

Solution

$4_{C_3} k^1 (3x)^3 = 54x^3$

$4k.27x^3 = 54x^3$

$\therefore 4k.27 = 54$

$k = \frac{54}{4 \times 27}$

$k = \frac{1}{2}$

Worked Example 1.46

Find the numerical coefficient of x^{17} in the expansion of $(x + y)^{20}$

Solution

$^{20}C_3\, x^{17}y^3 = 1140\, x^{17}y^3$

Therefore, the numerical coefficient is 1140.

Worked Example 1.47

Without using tables or calculator, evaluate $\frac{(\sqrt{2} + 1)^4 - (\sqrt{2} - 1)^4}{2}$

Solution

$(\sqrt{2} + 1)^4 = (\sqrt{2})^4 + 4.(\sqrt{2})^3 1^1 + 6(\sqrt{2})^2 1^2 + 4(\sqrt{2})^1 1^3 + 1^4$

$= 4 + 8\sqrt{2} + 12 + 4\sqrt{2} + 1$

$= 17 + 12\sqrt{2}$

$(\sqrt{2} - 1)^4 = (\sqrt{2})^4 - 4.(\sqrt{2})^3 1^1 + 6(\sqrt{2})^2 1^2 - 4(\sqrt{2})^1 1^3 + 1^4$

$= 4 - 8\sqrt{2} + 12 - 4\sqrt{2} + 1$

$= 17 - 12\sqrt{2}$

$\frac{(\sqrt{2} + 1)^4 - (\sqrt{2} - 1)^4}{2} = \frac{17 + 12\sqrt{2} - (17 - 12\sqrt{2})}{2}$

$= \frac{24\sqrt{2}}{2}$

$= 12\sqrt{2}$

Worked Example 1.48

a) Write down the binomial expansion of $(2 + x)^4$.

b) Use your expansion to evaluate $(1.97)^4$, correct to two decimal places.

Solution

a) $(2 + x)^4 = 2^4 + 4 \times 2^3 \times x^1 + 6 \times 2^2 \times x^2 + 4 \times 2^1 \times x^3 + x^4$

$= 16 + 32x + 24x^2 + 8x^3 + x^4$

b) Comparing $(2 + x)^4 \equiv (1.97)^4$,

$2 + x = 1.97$

$x = 1.97 - 2$

$= -0.03$

$\therefore (1.97)^4 = 16 + 32(-0.03) + 24(-0.03)^2 + 8(-0.03)^3 + (-0.03)^4$

$= 16 - 0.96 + 0.0216 - 0.000216 + 0.00000081$

$= 15.06138481$

$= 15.06$ (2 d.p)

REVIEW QUESTIONS

Review Question 1

1. State with reason whether the following statements are true or false:

a) The set of odd integers are closed under multiplication.

b) The set of rational numbers are closed under addition.

c) The set of whole numbers are closed under subtraction.

2. A binary operation ∇ is defined under the set of real numbers $R = \{2, 3, 4, 5\}$ by m ∇ n = m + n + mn.

Use this definition to copy and complete the table below.

∇	2	3	4	5
2		11	14	17
3	11	15	19	
4		19		
5	17		29	35

From the table, by giving reason(s) determine whether

a) the operation ∇ is commutative

b) the operation is closed under the set of real numbers

3. Study the table below under * and answer the questions that follow:

*	*p*	*q*	r	s
P	*q*	*p*	s	r
q	*p*	*q*	r	s
r	s	r	*q*	*p*
s	r	s	*p*	*q*

By giving reason(s) determine whether the operation *

i. is closed

ii) is commutative

   iii. has an identity element

   iv. has identity element, and find the inverses of p, q, r *and* s.

4. The operation ∘ is defined under the set of real numbers R by $p \circ q = \frac{p+q}{3}$. Evaluate

   i. $3 \circ 2$

   ii. $2 \circ 3$

   iii. What can be said about the results in i) and ii)?

   iv. if $p \circ 4 = -1$, *find the value of* p.

5. A baker produces two types of bread: wheat and potato. The wheat bread has a production cost of *GH¢* 250.00, and potato bread has a production cost of *GH¢* 300.00. The baker sells the wheat bread to retailers for GH¢ 6.00 each and the potato bread for *GH¢* 7.50 each. Last month, the baker produced 500 loaves of wheat bread and 300 pieces of potato bread.

   a) Explain how you will use binary operation to find the total revenue generated from the sale of bread last month and determine the overall profit or loss.

   b) Calculate the total revenue generated from the sale of bread last month and determine the overall profit or loss.

6. You have six shirts, two trousers and two pairs of shoes. Explain how you will use a binary operation to determine how many ways a shirt, a trouser and a pair of shoes can be worn.

7. A set $P = \{1, 4, 7, 8, 10, 11\}$. The operation ® is defined as follows: if a and b are any elements of P, then a ® b denotes the remainder when the results of adding a to b is divided by 5; For example, $4 * 10 = 4$

   a) Draw a table for the above operation ®

   b) Determine if the binary operation:

      i. is closed

      ii. is commutative

      iii. has an identity element

   c) Solve the equation $11 * x = 4$

8. The operation * is defined on the set of real numbers such that $d * e = d^2 + 2de + 5e$. If d = $2x + 1$ and e = $x - 1$, evaluate d * e in terms of x.

9. Two students were asked if $a, b \in R$, such that $a * b = 2a - 3b$, to find the identity element.

Ama's solution was:

$a * e = a$

$2a - 3e = a$

$-3e = -a$

$e = \frac{a}{3}$

Her study partner, Kwame solved it as

$a * b = b * a$

$a * b = 2a - 3b$

$b * a = 2b - 3a$

$a * b \neq b * a$, therefore, the identity element did not exist.

State with reasons who is correct.

10. The operation * is defined over the set of real numbers by $p * q = \frac{p + 2q}{3}$

Investigate if *

a) **i.** is commutative

ii. is associative

b) Determine if the operations has an identity and an inverse.

c) determine whether or not $p * (q + r) = (p * q) + (p * r)$

11. An operations ⊙ is defined on the set of real numbers by $\frac{1}{2}m + \frac{1}{3}n + 5mn$, evaluate

a) $(4 \odot 5)$

b) $3 \odot (4 \odot 5)$

12. A mathematical set costs GH¢ 30.50 and a calculator costs GH¢ 120.00. How much will Ms Ekua Bonney pay for 5 mathematical sets and 7 calculators she procured for her bookshop.

13. m, n ∈R, the operation *is defined by $m * n = \frac{2m + n}{4n - m}$, evaluate

i. $-8 * 5$

ii. $4 * (5 * 6)$

14. A binary operation * is defined on the set of real numbers R, by $p * q = \frac{pq}{p + q}$, evaluate

i. $\frac{5}{7} * \frac{2}{3}$

ii. If $(x + 1) * 3 = -5$, find the value of x.

15. Kofi had 20 GH¢ 100.00 notes and 25 GH¢ 200.00 notes in his wallet. How much is in the wallet?

16. A binary operation * is defined on the set of real numbers R, by $a * b = \frac{a}{2b} + \frac{b}{7a}$, find 3*–5

17. The relation for paying for total cost of a number of books(x) and a number of pencils(y) was $x * y = 15x + 5y + 10$. If the total payment made was GH¢ 300 and 5 books were bought, how much was a pencil?

18. An operation * is defined on the T = {1, 2, 3, 4, 6} by $a * b = a + 2b + 1$.

a) Copy and complete the table below

*	1	2	3	4	6
1		6			14
2					
3			10		
4	7				
6		11			

i. Is the operation * closed?

ii. Is the operation * commutative?

19. An operation * is defined on the set of real numbers by $p * q = \frac{2p + q}{p}$, determine whether or not, the operation * is commutative.

20. An operation ɵ is defined on the set of real numbers by $x ɵ y = x + y + \frac{1}{2}xy$, determine whether or not the operation ɵ is associative

Review Question 2

1. P and Q are subsets of U such that; $U = \{12, 13, 14, \ldots 21\}$ $P=\{X{:}12 \leq X < 16\}$ $Q = \{\text{multiples of } 3 \leq 18\}$, find

a) $P \cap Q$

b) $n(P \cap Q^1)$

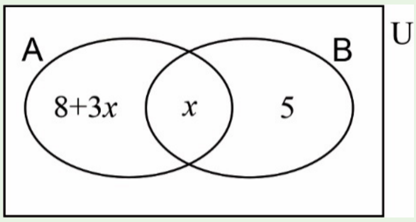
c) $Q \cup (P^1 \cap Q^1)$

2. Given that $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, d, g\}$ and $B = \{b, d, e, h\}$

a) Illustrate U, A and B in a Venn diagram

b) Write down the elements of the following sets: (i) $A \cap B$ (ii) $A \cup B$

3. In the diagram $n(A) = 2n(B)$. Find x



4. Consider the sets:

$A = \{red, green, blue\}$,

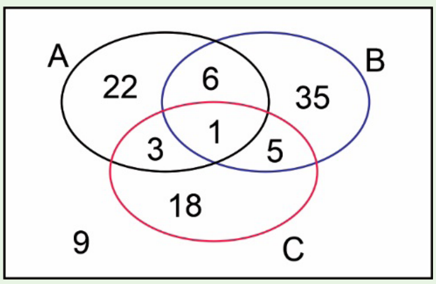
$B = \{red, yellow, orange\}$,

$C = \{red, orange, yellow, green, blue, purple\}$

$D = \{yellow, white\}$ *and the universal* set, $U = A \cup B \cup C \cup D$

a) Find $A' \cap C$

5. In the Venn diagram below. The numbers indicate the number of elements in the region



Determine the following

a) $n(A)$

b) $n(C)$

c) $n(B)$

d) $n(A \cap C)$

e) $n(B \cup C)$

f) $n((B \cap C) \cup A)$

6. If A = {x:x is a factor of 72}, B = {x:x < 20}, C= {multiples of 3}, and A,B,$C \subset U$, where U = {integers}

List the elements of $A \cap B \cap C$

7. Given $\mu = \{1, 2, 3, 4,\ldots,10\}$, $A = \{1, 4, 9\}$, $B = \{1, 3, 5, 7, 9\}$ *and* $C = \{1, 3, 6,10\}$, show that:

a) $(A\cup B)\cup C = A\cup(B\cup C)$ **b)** $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

c) $(A\cap B)\cap C = A\cap(B\cap C)$ **d)** $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

8. Write the following in roster form:

a) The set of whole numbers less than 20 and divisible by 3.

b) The set of integers greater than –2 and less than 4.

c) The set of integers between - 4 and 4, including both –4 and 4.

d) The squares of the first four natural numbers.

e) The set of prime factors of 36.

f) The set of counting numbers between 15 and 35, each of which is divisible by 6.

9. Write each of the following in set builder form:

a) A = {16, 25, 36, 39, 64}

b) B = {2, 4, 6, 8, 10}

c) C = {2, 3, 5, 7, 11}

d) D = {1, 3, 5, 7, 9}

e) E = {a, e, i, o, u}

10. The universal set, U = {1,2,3,4,5,6,10,20,50,100}, P, Q and R are subsets of U, P={1,2,5,10,20} Q={2,5,10,20,50} and R= {5,10,20,50,100}

Find

i. $n(P \cap Q)$ **ii)** $P \cup (Q\cap R)$

iii. $(P \cup R) \cap Q$

11. U = {m,h,e,a,t,c,s,i} A= {m,e,s,t,i,c} B = {s,e,m,t,c,} C= {m,e,a,t}, A,B,C are subsets of U. List the elements of

a) $A \cup (B \cap C)$ **b)** $(A \cap B') \cup C$

c) $(A \cup B) \cap (B \cup C)'$

12. $U = \{x{:}1 \leq x \leq 20\}$, where x is an integer and P, Q and R are subsets of U such that

$P = \{x{:}x$ is multiple of $2\}$,

$Q = \{x{:}x$ a multiple $3\}$

$R = \{x{:}x$ is multiple of $9\}$

i. What is the relationship between Q and R?

ii. Show the relationship between P, Q and R in Venn diagram, listing the elements of each region.

iii. List the elements of

a) PÇQÇR^{1}

b) P^{1}ÇQÇR^{1}

13. Given $U = \{1, 2, 3, 4, \ldots, 15\}$

$A = \{1, 4, 9, 15\}$

$B = \{1, 3, 5, 7, 9, 11, 13\}$

$C = \{1, 3, 6, 10, 15\}$

Show that:

a) $(A \cup B) \cup C = A \cup (B \cup C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

c) $(A \cap B) \cap C = A \cap (B \cap C)$

d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e) $(A \cup B) \cap (B \cup C) = (C \cup B) \cap (B \cup A)$

14. Let $P = \{0, 2, 4, 6, 8, 10\}$, $Q = \{0, 1, 2, 3, 4, 5, 6\}$, and $R = \{4, 5, 6, 7, 8, 9, 10\}$. Find

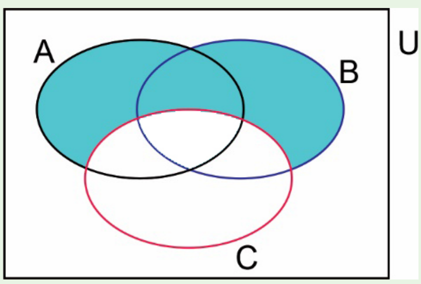
a) $P \cap Q \cap R$.

b) $P \cup Q \cup R$.

c) $(P \cup Q) \cap R$.

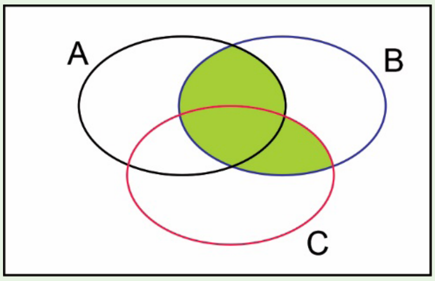
d) $(P \cap Q) \cup R$.

15. Write an expression for the shaded region.

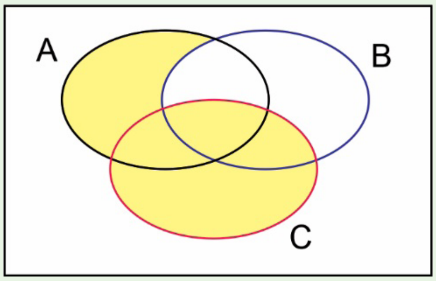


16. Write an expression for the shaded regions in the diagrams below:

a)



b)



17. A survey asked people what alternative transportation modes they use. 30% use the bus; 20% ride a bicycle, 25% walk; 5% use the bus and ride a bicycle 10% ride a bicycle and walk; 12% use the bus and walk; 2% use all three. Use the data to complete a Venn diagram, then determine:

a) What percent of people only ride the bus

b) How many people don't use any alternate transportation

18. A survey asks: "Which online social media have you used in the last month: WhatsApp, Facebook, Both". The results show 42% of those surveyed have used WhatsApp, 70% have used Facebook, and 20% have used both. How many people have used neither WhatsApp nor Facebook?

19. In a survey of 115 pet owners, 26 said they own a dog, and 64 said they own a cat. 5 said they own both a dog and a cat. Use a Venn diagram to determine how many of the pet owners surveyed owned neither a cat nor a dog?

20. A pharmaceutical company is considering manufacturing new toothpaste. They are considering two charcoal flavours, strawberry and mint. In a sample of 74 people, it was found that 45 liked strawberries, 37 liked mint and 21 liked both types

a) Create a Venn diagram to model the information.

b) How many liked only strawberry?

c) How many liked only mint?

d) How many liked exactly one of the two (that is they liked one but not the other)?

Review Questions 3

1. Indicate which of the following statements is true and which is false.
   a) Pascal's triangle is a triangular array of numbers in which the first and the last numbers in each row are 1.
   b) In Pascal's triangle, each number is the two numbers above it added together.
2. Use the combination formula to expand the following:
   a) $(2a + 3b)^3$
   b) $(3x - 5y)^4$
3. Obtain the expansion of $\left(2x + \frac{1}{2}\right)^4$ in descending powers of x.
4. Use Pascal's triangle to expand the following:
   a) $(2a + 3b)^2$
   b) $\left(4x + \frac{1}{2}\right)^4$
5. Obtain the expansion of $\left(2x - \frac{1}{2}\right)^4$ in descending powers of x.
6. Find the coefficient of the term $(x)^{10}$ in the binomial expansion of the expression: $(1 + x)^{25}$
7. Find the coefficient of the term $x^9y^5 in$ the binomial expansion of the expression:

   $(2x + 3y)^{14}$.
8. Find the x^3 term in the expansion of $(x + 3)^{12}$.
9. Find the expression for the $4th$ term of the expression $\left(\frac{2}{3x} - \sqrt{x}\right)^6$.
10. i) Using the binomial theorem, expand $(1 + 2x)^5$, simplifying all terms.
    ii. Use your expansion to calculate the value if 1.05^5, correct to six decimal places.
11. If the first three terms of the expansion $(1 + px)^n$ in ascending powers of x are $1 + 20x + 160x^2$, find the values of n and p.
12. Kwame was granted a loan of GH¢ 5000.00 at a rate of 5% for 3 years from a bank at compound interest (CI). The bank used the relation CI= $P(1 + r\%)^n$, where P is the principal, r is the rate and n is the number of years. Find the total amount he had paid at the end of third year using the binomial theorem.

13. The population of a senior high school in Ghana increases according to the formula $N = P(1 + 0.025)^t$, where t is the number of years, P is the current population, and N is the total population. If the current population for that school is 1400 which is almost full capacity of the school, what steps should government take for the school in 5 years' time.

14. Use the Pascal triangle to expand and simplify all terms in $(2 - 3y)^4$, hence find the value of $(1.97)^4$.

15. Expand

    i. $(2x - 4)^4$

    ii. $(3y + 1)^5$, using Pascal triangle.

16. Use binomial theorem to expand $(1-2x)^6$ and simplify all coefficients.

17. Find the numerical coefficient of x^4 in the expansion $(2-x)^{10}$.

18. Find the x^5 term in the expansion $(2x-y)^7$.

19. The first three terms of the binomial expansion of $(1 + bx)^n$ are $1 + 2x + \frac{5}{3}x^2$.

    a) Using this information, work out the values of b and n.

REFERENCES

1. J B Ofosu 'O' Level Additional Mathematics Study Guide and Worked Examples, Volume1
2. Tuttuh-Adegun, M.R & Adegoke, D. G. New (2011) *Further Mathematics Project* Bounty Press Limited, Ibadan
3. Otoo, D.K. (2005) *Concise Elective Mathematics*, ISBN 9988–0–2259 – *X*
4. Backhouse, J.K & Houldsworth, S.P.T. (1985) *Pure mathematics 1*
5. Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952

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Ghana Education Service (GES)

List of Contributors

Name	Institution
Yaw Efa	Ghana National College
Stella Awinipure	Walewale SHTS
Joseph Fancis Kittah	Bompeh SHTS, Takoradi
Mpeniasah Kwasi Christopher	Three-Town SHTS