ADDITIONAL MATHEMATICS

CURRICULUM FOR SECONDARY EDUCATION (SHS 1 - 3)

NaCCA

NATIONAL COUNCIL FOR
CURRICULUM & ASSESSMENT
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September, 2023

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FOREWORD

Through the National Council for Curriculum and Assessment (NaCCA), Ghana's Ministry of Education has introduced a series of curriculum reforms to improve the quality and relevance of learning experiences in pre-tertiary schools in the country. These reforms will improve learning through the introduction of innovative pedagogies that encourage critical thinking and problem-solving. For a long time, our learners memorise facts and figures, which does not develop their analytical and practical skills. The Ministry recognises that learners need to be equipped with the right tools, knowledge, skills and competencies to deal with the fast-changing environment and the challenges facing their communities, the nation and the world.

These curriculum reforms were derived from the Education Strategic Plan (ESP 2018-2030), the National Pre-tertiary Education Curriculum Framework (NPTECF) and the National Pre-Tertiary Learning Assessment Framework (NPLAF), which were all approved by Cabinet in 2018. The new standards-based curriculum implemented in 2019 in basic schools, aims to equip learners to apply their knowledge innovatively to solve everyday problems. It also prioritises assessing learners' knowledge, skills, attitudes, and values, emphasising their achievements. The content of the basic school standards-based curriculum was therefore designed to promote a curriculum tailored to the diverse educational needs of the country's youth. It addresses the current curriculum's deficiencies in learning and assessment, especially in literacy and numeracy. These reforms have been carried out in phases. The curriculum for the basic school level – KG, Primary and Junior High School (JHS) – was developed and implemented from 2019 to 2021.

The curriculum for Senior High School (SHS), Senior High Technical School (SHTS) and Science, Technical, Engineering and Mathematics (STEM), which constitutes the next phase, is designed to ensure the continuation of learning experiences from JHS. It introduces flexible pathways for progression to facilitate the choice of subjects necessary for further study, the world of work and adult life. The new SHS, SHTS and STEM curriculum emphasises the acquisition of 21st Century skills and competencies, character development and instilling of national values. Social and Emotional Learning (SEL), Information Communications Technology, Gender Equality and Social Inclusion, have all been integrated into the curriculum. Assessment – formative and summative has been incorporated into the curriculum and aligned with the learning outcomes throughout the three-year programme.

The Ministry of Education's reform aims to ensure that graduates of our secondary schools can successfully compete in international high school competitions and, at the same time, be equipped with the necessary employable skills and work ethos to succeed in life. The Ministry of Education, therefore, sees the Senior High School (SHS) curriculum as occupying a critical place in the education system – providing improved educational opportunities and outcomes for further studies, the world of work and adult life – and is consequently prioritising its implementation.

ACKNOWLEDGEMENTS

This standards-based SHS curriculum was created using the National Pre-Tertiary Learning Assessment Framework (NPLAF), the Secondary Education Assessment Guide (SEAG), and the Teacher and Learner Resource Packs which include Professional Learning Community (PLC) Materials and Subject Manuals for teachers and learners. All the above-mentioned documents were developed by the National Council for Curriculum and Assessment (NaCCA). The Ministry of Education (MoE) provided oversight and strategic direction for the development of the curriculum with NaCCA receiving support from multiple agencies of the MoE and other relevant stakeholders. NaCCA would like to extend its sincere gratitude, on behalf of the MoE, to all its partners who participated in the professional conversations and discussions during the development of this SHS curriculum.

In particular, NaCCA would also like to extend its appreciation to the leadership of the Ghana Education Service (GES), the National School Inspectorate Authority (NaSIA), the National Teaching Council (NTC), the Commission for Technical and Vocational Education and Training (Commission for TVET), West African Examinations Council (WAEC) and other agencies of the MoE that supported the entire process. In addition, NaCCA acknowledges and values the contributions made by personnel from various universities, colleges of education Industry players, Vice Chancellors Ghana, Vice Chancellors Technical Universities as well as educators and learners working within the Ghana education landscape.

Special appreciation is extended to consultants who contributed to development of the curriculum. The development process involved multiple engagements between national stakeholders and various groups with interests in the curriculum. These groups include the teacher unions, the Association of Ghana Industries, and heads of secondary schools.

CONTENTS

THE SHS CURRICULUM OVERVIEW

The vision for this curriculum is to ensure the nation has a secondary education system that enables all Ghanaian children to acquire the 21st Century skills, competencies, knowledge, values and attitudes required to be responsible citizens, ready for the world of work, further studies and adult life. The nation's core values drive the SHS curriculum, and it is intended to achieve Sustainable Development Goal 4: 'Inclusive, equitable quality education and life-long learning for all'. Above all, it is a curriculum enabling its graduates to contribute to the ongoing growth and development of the nation's economy and well-being.

The curriculum is inclusive, flexible, and robust. It was written under the auspices of the National Council for Curriculum and Assessment by a team of expert curriculum writers across Ghana. It reflects the needs of critical stakeholders, including industry, tertiary education, the West African Examination Council, SHS learners, teachers, and school leaders. It has been written based on the National Pre-Tertiary Learning and Assessment Framework and the Secondary Education Policy.

The key features of the curriculum include:

- flexible learning pathways at all levels, including for gifted and talented learners and those with deficiencies in numeracy and literacy, to ensure it can meet the needs of learners from diverse backgrounds and with different interests and abilities.
- the five core learning areas for secondary education: science and technology, language arts, humanities, technical and vocational and business; with emphasis placed on STEM and agriculture as integral to each subject.
- a structured, standards-based approach that supports the acquisition of knowledge, skills and competencies, and transition and seamless progress throughout secondary education, from JHS to SHS and through the three years of SHS.
- a focus on interactive approaches to teaching and assessment to ensure learning goes beyond recall enabling learners to acquire the ability to understand, apply, analyse and create.
- guidance on pedagogy, coupled with exemplars, demonstrating how to integrate cross-cutting themes such as 21st Century skills, core competencies, the use of ICT, literacy and mathematics, Social Emotional Learning, Gender Equality and Social Inclusion as tools for learning and skills for life. Shared Ghanaian values are also embedded in the curriculum.

The curriculum writing process was rigorous and involved developing and using a Curriculum Writing Guide which provided systematic instructions for writers. The process was quality assured at three levels: through (a) evaluation by national experts, (b) trialling curriculum materials in schools and (c) through an external evaluation by a team of national and international experts. Evidence and insights from these activities helped hone the draft's final version. The outcome is a curriculum coherently aligned with national priorities, policies and the needs of stakeholders. A curriculum tailored to the Ghanaian context ensures that all learners benefit from their schooling and develop their full potential.

The following section highlights the details of the front matter of the draft curriculum. The vision, philosophy and goal of the curriculum are presented. This is followed by the details of the 21st Century skills and competencies, teaching and learning approaches, instructional design and assessment strategies. The template for the curriculum frame, which outlines the scope and sequence, the design that links the learning outcomes to particular 21st Century skills and competencies, as well as Gender Equality and Social Inclusion, Social and Emotional Learning and Ghanaian values are presented together with the structure of the lesson frame showing the links between the content standards, learning indicators with their corresponding pedagogical exemplars and assessment strategies.

INTRODUCTION

Effective implementation of this Senior High School (SHS) curriculum is the key to creating a well-educated and well-balanced workforce that is ready to contribute to Ghana's progress by harnessing the potential of the growing youth population, considering the demographic transition the country is currently experiencing (Educational Strategic Plan [ESP] 2018-2030). SHS curriculum aims to expand equitable, inclusive access to relevant education for all young people, including those in disadvantaged and underserved communities, those with special educational needs and those who are gifted and talented. Senior High School allows young people to develop further skills and competencies and progress in learning achievement, building from the foundation laid in Junior High School. This curriculum intends to meet the learning needs of all high school learners by acquiring 21st Century skills and competencies to prepare them for further studies, the world of work and adult life. Changing global economic, social and technological context requires life-long learning, unlearning, and continuous processes of reflection, anticipation and action.

Philosophy of Senior High School Curriculum

The philosophy underpinning the SHS curriculum is that every learner can develop their potential to the fullest if the right environment is created and skilled teachers effectively support them to benefit from the subjects offered at SHS. Every learner needs to be equipped with skills and competencies of interest to further their education, live a responsible adult life or proceed to the world of work.

Vision of Senior High School Curriculum

The vision of the curriculum is to prepare SHS graduates equipped with relevant skills and competencies to progress and succeed in further studies, the world of work and adult life. It aims to equip all learners with the 21st Century skills and competencies required to be responsible citizens and lifelong learners. When young people are prepared to become effective, engaging, and responsible citizens, they will contribute to the ongoing growth and development of the nation's economy and well-being.

Goal of Senior High School Curriculum

The goal of the curriculum is to achieve relevant and quality SHS through the integration of 21st Century skills and competencies as set out in the Secondary Education Policy. The key features to integrate into the curriculum are:

- Foundational Knowledge: literacy, numeracy, scientific literacy, information, communication and digital literacies, financial literacy and entrepreneurship, cultural identity, civic literacy and global citizenship
- Competencies: critical thinking and problem-solving, innovation and creativity, collaboration, and communication
- Character Qualities: discipline, integrity, self-directed learning, self-confidence, adaptability and resourcefulness, leadership, and responsible citizenship.

The JHS curriculum has been designed to ensure that learners are adequately equipped to transition seamlessly into SHS, where they will be equipped with the relevant knowledge, skills and competencies. The SHS curriculum emphasises character building, acquisition of 21st Century skills and competencies and nurturing core values within an environment of quality education to ensure the transition to further study, the world of work and adult life. This requires the delivery of robust secondary education that meets the varied learning needs of the youth in Ghana. The SHS curriculum, therefore, seeks to develop learners to become technology-inclined, scientifically literate, good problem-solvers who can think critically and creatively and are equipped to communicate with fluency, and possess the confidence and competence to participate fully in Ghanaian society as responsible local and global citizens – (referred to as 'Glocal citizens').

The SHS curriculum is driven by the nation's core values of truth, integrity, diversity, equity, discipline, self-directed learning, self-confidence, adaptability and resourcefulness, leadership, and responsible citizenship, and with the intent of achieving the Sustainable Development Goal 4: 'Inclusive, equitable quality education and life-long learning for all'. The following sections elaborate on the critical competencies required of every SHS learner:

Gender Equality and Social Inclusion (GESI)

- Appreciate their uniqueness about others.
- Pay attention to the uniqueness and unique needs of others.
- Value the perspective, experience, and opinion of others.
- Respect individuals of different beliefs, political views/ leanings, cultures, and religions.
- Embrace diversity and practise inclusion.
- Value and work in favour of a democratic and inclusive society.
- Be conscious of the existence of minority and disadvantaged groups in society and work to support them.
- Gain clarity about misconceptions/myths about gender, disability, ethnicity, age, religion, and all other excluded groups in society
- Interrogate and dispel their stereotypes and biases about gender and other disadvantaged and excluded groups in society.
- Appreciate the influence of socialisation in shaping social norms, roles, responsibilities, and mindsets.
- Identify injustice and advocate for change.
- Feel empowered to speak up for themselves and be a voice for other disadvantaged groups.

21st Century Skills and Competencies

In today's fast-changing world, high school graduates must be prepared for the 21st Century world of work. The study of Mathematics, Science, and Language Arts alone is no longer enough. High school graduates need a variety of skills and competencies to adapt to the global economy. Critical thinking, creativity, collaboration, communication, information literacy, media literacy, technology literacy, flexibility, leadership, initiative, productivity, and social skills are needed. These skills help learners to keep up with today's fast-paced job market. Employers want workers with more than academic knowledge. The 21st Century skills and competencies help graduates navigate the complex and changing workplace. Also, these help them become active citizens who improve their communities. Acquisition of 21st Century skills in high school requires a change in pedagogy from the approach that has been prevalent in Ghana in recent years. Teachers should discourage and abandon rote memorisation and passive learning. Instead, they should encourage active learning, collaboration, and problem-solving, project-based, inquiry-based, and other learner-centred pedagogy should be used. As well as aligning with global best practices, these approaches also seek to reconnect formal education in Ghana with values-based indigenous education and discovery-based learning which existed in Ghana in pre-colonial times. This is aligned with the 'glocal' nature of this curriculum, connecting with Ghana's past to create confident citizens who can engage effectively in a global world. Digitalisation, automation, technological advances and the changing nature of work globally mean that young people need a new set of skills, knowledge and competencies to succeed in this dynamic and globalised labour market.

Critical Thinking and Problem-Solving Competency

- Ability to question norms, practices, and opinions, to reflect on one's values, perceptions, and actions.
- Ability to use reasoning skills to come to a logical conclusion.
- Being able to consider different perspectives and points of view
- Respecting evidence and reasoning
- Not being stuck in one position
- Ability to take a position in a discourse
- The overarching ability to apply different problem-solving frameworks to complex problems and develop viable, inclusive, and equitable solution options that integrate the above-mentioned competencies, promote sustainable development,

Creativity

- Ability to identify and solve complex problems through creative thinking.
- Ability to generate new ideas and innovative solutions to old problems.
- Ability to demonstrate originality and flexibility in approaching tasks and challenges.
- Collaborating with others to develop and refine creative ideas
- Ability to incorporate feedback and criticism into the creative process
- Utilising technology and other resources to enhance creativity
- Demonstrating a willingness to take risks and experiment with new approaches
- Adapting to changing circumstances and further information to maintain creativity

- Integrating multiple perspectives and disciplines to foster creativity
- Ability to communicate creative ideas effectively to a variety of audiences

Collaboration

- Abilities to learn from others; to understand and respect the needs, perspectives, and actions of others (empathy)
- Ability to understand, relate to and be sensitive to others (empathic leadership)
- Ability to deal with conflicts in a group
- Ability to facilitate collaborative and participatory problem-solving
- Ability to work with others to achieve a common goal.
- Ability to engage in effective communication, active listening, and the ability to compromise.
- Ability to work in groups on projects and assignments.

Communication

- Know the specific literacy and language of the subjects studied
- Use language for academic purposes
- Communicate effectively and meaningfully in a Ghanaian Language and English Language
- Communicate confidently, ethically, and effectively in different social contexts.
- Communicate confidently and effectively to different participants in different contexts
- Ability to communicate effectively verbally, non-verbally and through writing.
- Demonstrate requisite personal and social skills that are consistent with changes in society
- Ability to express ideas clearly and persuasively, listen actively, and respond appropriately
- Ability to develop digital communication skills such as email etiquette and online collaboration.
- Ability to engage in public speaking, debate, and written communication.

Learning for Life

- Understand subject content and apply it in different contexts
- Apply mathematical and scientific concepts in daily life
- Demonstrate mastery of skills in literacy, numeracy, and digital literacy.
- Develop an inquiry-based approach to continual learning.
- Be able to understand higher-order concepts and corresponding underlying principles.
- Participate in the creative use of the expressive arts and engage in aesthetic appreciation.
- Use and apply a variety of digital technologies
- Be digitally literate with a strong understanding of ICT and be confident in its application.
- Be equipped with the necessary qualifications to gain access to further and higher education and the world of work and adult life
- Ability to apply knowledge practically in the workplace so that they are able to utilise theory by translating it into practice.
- Develop their abilities, gifts and talents to be able to play a meaningful role in the development of the country
- Be able to think critically and creatively, anticipate consequences, recognise opportunities and be risk-takers
- Ability to pursue self-directed learning with the desire to chart a path to become effective lifelong learners.
- Independent thinkers and doers who show initiative and take action.
- Ability to innovate and think creatively, building on their knowledge base so that they take risks to achieve new goals
- Ability to think critically and solve problems so that they become positive change agents at work, in further study and in their personal lives.
- Be motivated to adapt to the changing needs of society through self-evaluation and ongoing training
- Be able to establish and maintain innovative enterprises both individually and in collaboration with others.
- Be able to ethically prioritise economic values to ensure stability and autonomy
- Show flexibility and preparedness to deal with job mobility
- Be committed towards the improvement of their quality of life and that of others
- Feel empowered in decision-making processes at various levels e.g., personal, group, class, school, etc.

- Be able to seek and respond to assistance, guidance and/or support when needed.
- Ability to make and adhere to commitments.
- Adopt a healthy and active lifestyle and appreciate how to use leisure time well.
- Be enthusiastic, with the knowledge, understanding and skill that enable them to progress to tertiary level, the world of work and adult life.
- Ability to transition from school to the world of work or further study by applying knowledge, skills and attitudes in new situations.
- Be independent, have academic and communication skills such as clarity of expression (written and spoken), and the ability to support their arguments.
- Be innovative and understand the 21st Century skills and competencies and apply them to everyday life.

Global and Local (Glocal) Citizenship

- Appreciate and respect the Ghanaian identity, culture, and heritage
- Be conscious of current global issues and relate well with people from different cultures
- Act in favour of the common good, social cohesion and social justice
- Have the requisite personal and social skills to handle changes in society
- Appreciate the impact of globalisation on the society.
- Ability to be an honest global citizen displaying leadership skills and moral fortitude with an understanding of the wider world and how to enhance Ghana's standing.

Systems Thinking Competency

- Ability to recognise and understand relationships
- Ability to analyse complex systems
- Ability to think of how systems are embedded within different domains and different scales
- Ability to deal with uncertainty

Normative Competency

- Ability to understand and reflect on the norms and values that underlie one's actions
- Ability to negotiate values, principles, goals, and targets, in a context of conflicts of interests and trade-offs, uncertain knowledge and contradictions

Anticipatory Competency

- Ability to understand and evaluate multiple futures – possible, probable, and desirable
- Ability to create one's vision for the future.
- Ability to apply the precautionary principle
- Ability to assess the consequences of actions
- Ability to deal with risks and changes

Strategic Competency

- Ability to collectively develop and implement innovative actions that further a cause at the local level and beyond.
- Ability to understand the bigger picture and the implications of smaller actions on them

Self-Awareness Competency

- The ability to reflect on one's role in the local community and (global) society
- Ability to continually evaluate and further motivate one's actions
- Ability to deal with one's feelings and desires

Social Emotional Learning (SEL): Five Core Competencies with Examples

1. *Self-Awareness*

Understanding one's emotions, thoughts, and values and how they influence one's behaviour in various situations. This includes the ability to recognise one's strengths and weaknesses with a sense of confidence and purpose. For instance:

- *Integrating personal and social identities;*
- *Identifying personal, cultural, and linguistic assets;*
- *Identifying one's emotions;*
- *Demonstrating honesty and integrity;*
- *Connecting feelings, values, and thoughts;*

- *Examining prejudices and biases;*
- *Experiencing self-efficacy;*
- *Having a growth mindset;*
- *Developing interests and a sense of purpose;*

2. *Self-Management*

The capacity to control one's emotions, thoughts, and actions in a variety of situations and to realise one's ambitions. This includes delaying obtaining one's desires, dealing with stress, and feeling motivated and accountable for achieving personal and group goals. For instance:

- *Managing one's emotions;*
- *Identifying and utilising stress-management strategies;*
- *Demonstrating self-discipline and self-motivation;*
- *Setting personal and group goals;*
- *Using planning and organisation skills;*
- *Having the courage to take the initiative;*
- *Demonstrating personal and collective agency;*

3. *Social Awareness*

The capacity to comprehend and care for others regardless of their backgrounds, cultures, and circumstances. This includes caring for others, understanding larger historical and social norms for behaviour in different contexts, and recognising family, school, and community resources and supports. For instance:

- *Recognising others' strengths*
- *Demonstrating empathy and compassion*
- *Caring about others' feelings*
- *Understanding and expressing gratitude*
- *Recognising situational demands and opportunities*
- *Understanding how organisations and systems influence behaviour*

4. *Relationship Skills*

The capacity to establish and maintain healthy, beneficial relationships and adapt to various social situations and groups. This includes speaking clearly, listening attentively, collaborating, solving problems and resolving conflicts as a group, adapting to diverse social and cultural demands and opportunities, taking the initiative, and asking for or offering assistance when necessary. For instance:

- *Communicating effectively;*
- *Building positive relationships;*
- *Demonstrating cultural competence;*
- *Working as a team to solve problems;*
- *Constructively resolving conflicts;*
- *Withstanding negative social pressure;*
- *Taking the initiative in groups;*
- *Seeking or assisting when needed;*
- *Advocating for the rights of others.*

5. *Responsible Decision-Making*

The capacity to make thoughtful and constructive decisions regarding acting and interacting with others in various situations. This includes weighing the pros and cons of various personal, social, and group well-being actions. For example:

- *Demonstrating curiosity and an open mind;*
- *Solving personal and social problems;*
- *Learning to make reasonable decisions after analysing information, data, and facts;*
- *Anticipating and evaluating the effects of one's actions;*
- *Recognising that critical thinking skills are applicable both inside and outside of the classroom;*
- *Reflecting on one's role in promoting personal, family, and community well-being;*
- *Evaluating personal, interpersonal, community, and institutional impacts*

Learning and Teaching Approaches

Learning and teaching should develop learners as self-directed and lifelong learners. Learners must be helped to build up deep learning skills and competencies to develop the ability to acquire, integrate and apply knowledge and skills to solve authentic and real-life problems. Learners need to be exposed to a variety of learning experiences to enable them to collaborate with others, construct meaning, plan, manage, and make choices and decisions about their learning. This will allow them to internalise newly acquired knowledge and skills and help them to take ownership of their education. The 21st Century skills and competencies describe the relevant global and contextualised skills that the SHS curriculum is designed to help learners acquire in addition to the 4Rs (Reading, wRiting, aRithmetic and cReativity). These skills and competencies, as tools for learning and teaching and skills for life, will allow learners to become critical thinkers, problem-solvers, creators, innovators, good communicators, collaborators, digitally literate, and culturally and globally sensitive citizens who are life-long learners with a keen interest in their personal development and contributing to national development.

Given the diverse needs of learners, teachers need to have a thorough grasp of the different pedagogies as they design and enact meaningful learning experiences to meet the needs of different learners in the classroom. The teaching-learning techniques and strategies should include practical activities, discussion, investigation, role play, problem-based, context-based, and project-based learning. Active learning strategies have become increasingly popular in education as they provide learners with meaningful opportunities to engage with the material. These strategies emphasise the use of creative and inclusive pedagogies and learner-centred approaches anchored on authentic and enquiry-based learning, collaborative and cooperative learning, differentiated teaching and learning, holistic learning, and cross-disciplinary learning. They include experiential learning, problem-based learning, project-based learning, and talk-for-learning approaches. Some of the pedagogical exemplars to guide learning and teaching of the SHS curriculum include:

- **Experiential Learning:** Experiential learning is a hands-on approach to learning that involves learners in real-world experiences. This approach focuses on the process of learning rather than the result. Learners are encouraged to reflect on their experiences and use them to develop new skills and knowledge. Experiential learning can take many forms, including internships, service learning, and field trips. One of the main benefits of experiential learning is that it allows learners to apply what they have learned in the classroom to real-world situations. This can help them develop a deeper understanding of the material and make connections between different concepts. Additionally, experiential learning can help learners develop important skills such as critical thinking, problem-solving and communication.
- **Problem-Based Learning:** Problem-based learning is an approach that involves learners in solving real-world problems. Learners are presented with a problem or scenario and are asked to work together to find a solution. This approach encourages learners to take an active role in their learning and helps them develop important skills such as critical thinking and problem-solving. One of the main benefits of problem-based learning is that it encourages learners to take ownership of their learning. By working together to solve problems, learners can develop important skills such as collaboration and communication. Additionally, problem-based learning can help learners develop a deeper understanding of the material as they apply it to real-world situations.
- **Project-Based Learning:** Project-based learning is a hands-on approach to learning that involves learners in creating a project or product. This approach allows learners to take an active role in their learning and encourages them to develop important skills such as critical thinking, problem-solving, collaboration, and communication. One of the main benefits of project-based learning is that it allows learners to apply what they have learned in the classroom to real-world situations. Additionally, project-based learning can help learners develop important skills from each other and develop a deeper understanding of the material.
- **Talk for Learning Approaches:** Talk for learning approaches (TfL) are a range of techniques and strategies that are used to encourage learners to talk by involving them in discussions and debates about the material they are learning. This approach encourages learners to take an active role in their learning and helps them develop important skills such as critical thinking, collaboration and communication and also makes them develop confidence. One of the main benefits of TfL is that it encourages learners to think deeply about the material they are learning. By engaging in discussions and debates, learners can develop a deeper understanding of the material and make connections between different concepts.
- **Initiating Talk for Learning:** Initiating talk for learning requires the use of strategies that would encourage learners to talk in class. It helps learners to talk and participate meaningfully and actively in the teaching and learning process. Apart from developing skills such as communication and critical thinking, it also helps learners to develop confidence. Some strategies for initiating talk among learners are Activity Ball; Think-Pair-Share; Always, Sometimes, Never True; Matching and Ordering of Cards.
- **Building on What Others Say:** Building on what others say is an approach that involves learners in listening to and responding to their classmates'

ideas. This approach encourages learners to take an active role in their learning and helps them develop important skills such as critical thinking and communication. One of the main benefits of building on what others say is that it encourages learners to think deeply about the material they are learning. By listening to their classmates' ideas, learners can develop a deeper understanding of the material and make connections between different concepts. Additionally, building on what others say can help learners develop important skills such as collaboration and reflection. Some of the strategies to encourage learners to build on what others say are brainstorming, concept cartoons, pyramid discussion, and 5 Whys, amongst others.

- **Managing Talk for Learning:** Managing talk for learning requires the use of various strategies to effectively coordinate what learners say in class. Effective communication is a crucial aspect of learning in the classroom. Teachers must manage talk to ensure that learners are engaged, learning, and on-task in meaningful and purposeful ways. Some strategies for managing learners' contributions are debates, think-pair-share, sage in the circle etc.
- **Structuring Talk for Learning:** One effective way to shape learners' contributions is to structure classroom discussions. Structured discussions provide a framework for learners to engage in meaningful dialogue and develop critical thinking skills. Teachers can structure discussions by providing clear guidelines, such as speaking one at a time, listening actively, and building on each other's ideas. One popular structured discussion technique is the "think-pair-share" method. In this method, learners think about a question or prompt individually, and then pair up with a partner to discuss their ideas. Finally, the pairs share their ideas with the whole class. This method encourages all learners to participate and ensures that everyone has a chance to share their thoughts. Another effective way to structure talk for learning is to use open-ended questions. Open-ended questions encourage learners to think deeply and critically about a topic. They also promote discussion and collaboration among learners. Teachers can use open-ended questions to guide classroom discussions and encourage learners to share their ideas and perspectives. Other strategies that can be used are Concept/Mind Mapping, "Know," "Want to Know," "Learned" (KWL); Participatory Feedback; and the 5 Whys.
- **Diamond Nine:** The Diamond Nine activity is a useful tool for managing talk for learning in the classroom. This activity involves ranking items or ideas in order of importance or relevance. Learners work in groups to arrange cards or sticky notes with different ideas or concepts into a diamond shape, with the most important idea at the top and the least important at the bottom. The Diamond Nine activity encourages learners to think critically about a topic and prioritise their ideas. It also promotes collaboration and discussion among group members. Teachers can use this activity to introduce a new topic, review material, or assess student understanding.
- **Group Work/Collaborative Learning:** Group work or collaborative learning are effective strategies for managing talk for learning in the classroom. These strategies encourage learners to work together to solve problems, share ideas, and learn from each other. Group work and collaborative learning also promote communication and collaborative skills that are essential for success in the workplace and in life. To implement group work effectively, teachers must provide clear guidelines and expectations for group members. They should also monitor group work to ensure that all learners are participating and on-task. Teachers can also use group work as an opportunity to assess individual student understanding and participation.
- **Inquiry-Based Learning:** Learners explore and discover new information by asking questions and investigating.
- **Problem-Based Learning:** Learners are given real-world problems to solve and must use critical thinking and problem-solving skills.
- **Project-Based Learning:** Learners work on long-term projects that relate to real-world scenarios.
- **Flipped Classroom:** Learners watch lectures or instructional videos at home and complete assignments and activities in class.
- **Mastery-Based Learning:** Learners learn at their own pace and only move on to new material once they have mastered the current material.
- **Gamification:** Learning is turned into a game-like experience with points, rewards, and competition.

These strategies provide learners with opportunities to engage with the material in meaningful ways and develop important skills such as critical thinking, problem-solving, collaboration, and communication. By incorporating these strategies into their teaching, teachers can help learners develop a deeper understanding of the material and prepare them for success in the real world. Effective communication is essential for learning in the classroom. Teachers must manage talk to ensure that learners are engaged in learning and on-task. Strategies such as structuring talk for learning, using Diamond Nine activities, and implementing group work/ collaborative learning can help teachers manage talk effectively and promote student learning and engagement. By implementing these strategies, teachers can create a positive and productive learning environment where all learners can succeed.

Universal Design for Learning (UDL) in the SHS Curriculum

The design of the curriculum uses UDL to ensure the creation of flexible learning environments that can accommodate a wide range of learner abilities, needs, and preferences. The curriculum is designed to provide multiple means of engagement, representation, and action and expression, so teachers can create a more inclusive and effective learning experience for all learners. UDL is beneficial for all learners, but it is particularly beneficial for learners needing special support and learners who may struggle with traditional teaching approaches. The integration of UDL in the pedagogy is aimed at making learning accessible to everyone and helping all learners reach their full potential. For instance, teachers need to:

- incorporate multiple means of representation into their pedagogy, such as using different types of media and materials to present information.
- provide learners with multiple means of action and expression, such as giving them options for how they can demonstrate their learning.
- consider incorporating multiple means of engagement into their choice of pedagogy, such as incorporating games or interactive activities to make learning more fun and engaging.

By doing these, teachers can help ensure that the curriculum is accessible and effective for all learners, regardless of their individual needs and abilities.

Curriculum and Assessment Design: Revised Bloom's Taxonomy and Webb's Depth of Knowledge

The design of this curriculum uses the revised Bloom's Taxonomy and Webb's Depth of Knowledge (DoK) as frameworks to design what to teach and assess.

The Revised Bloom's Taxonomy provides a framework for designing effective learning experiences. Understanding the different levels of learning, informed the creation of activities and assessments that challenge learners at the appropriate level and help them progress to higher levels of thinking. Additionally, the framework emphasises the importance of higher-order thinking skills, such as analysis, evaluation, and creation, which are essential for success in today's complex and rapidly changing world. This framework is a valuable tool for educators who want to design effective learning experiences that challenge students at the appropriate level and help them develop higher-order thinking skills. By understanding the six levels of learning and incorporating them into their teaching, educators can help prepare students for success in the 21st century. The six hierarchical levels of the revised Bloom's Taxonomy are:

1. **Remember** – At the foundation is learners' ability to remember. That is retrieving knowledge from long-term memory. This level requires learners to recall concepts—identify, recall, and retrieve information. Remembering is comprised of identifying, listing, and describing. Retrieving relevant knowledge from long-term memory includes, recognising, and recalling is critical for this level.
2. **Understand** – At understanding, learners are required to construct meaning that can be shown through clarification, paraphrasing, representing, comparing, contrasting and the ability to predict. This level requires interpretation, demonstration, and classification. Learners explain and interpret concepts at this level.
3. **Apply** – This level requires learners' ability to carry out procedures at the right time in a given situation. This level requires the application of knowledge to novel situations as well as executing, implementing, and solving problems. To apply, learners must solve multi-step problems.
4. **Analyse** – The ability to break things down into their parts and determine relationships between those parts and being able to tell the difference between what is relevant and irrelevant. At this level, information is deconstructed, and its relationships are understood. Comparing and contrasting information and organising it is key. Breaking material into its constituent parts and detecting how the parts relate to one another and an overall structure or purpose is required. The analysis also includes differentiating, organising and attributing.
5. **Evaluate** – The ability to make judgments based on criteria. To check whether there are fallacies and inconsistencies. This level involves information evaluation, critique, examination, and formulation of hypotheses.
6. **Create** – The ability to design a project or an experiment. To create, entails learners bringing something new. This level requires generating information—planning, designing, and constructing.

Webb's Depth of Knowledge (DoK) is a framework that helps educators and learners understand the level of cognitive engagement required for different types of learning tasks. The framework includes four levels. By understanding the four DoK levels, educators can design learning activities that challenge students to engage in deeper thinking and problem-solving. DoK is an essential tool for designing effective instruction and assessments. By understanding the different levels of DoK, teachers can design instruction and assessments that align with what they intend to achieve. DoK is a useful tool for differentiating instruction and providing appropriate challenges for all learners. Teachers can use DOK to identify students who need additional support or those who are ready for more advanced tasks. The four levels of Webb's' DoK assessment framework are:

- **Level 1: Recall and Reproduction –** Assessment at this level is on recall of facts, concepts, information, and procedures—this involves basic knowledge acquisition. Learners are asked specific questions to launch activities, exercises, and assessments. The assessment is focused on recollection and reproduction.
- **Level 2: Skills of Conceptual Understanding –** Assessment at this level goes beyond simple recall to include making connections between pieces of information. The learner's application of skills and concepts is assessed. The assessment task is focused more on the use of information to solve multi-step problems. A learner is required to make decisions about how to apply facts and details provided to them.
- **Level 3: Strategic Reasoning** – At this level, the learner's strategic thinking and reasoning which is abstract and complex is assessed. The assessment task requires learners to analyse and evaluate composite real-world problems with predictable outcomes. A learner must apply logic, employ problem-solving strategies, and use skills from multiple subject areas to generate solutions. Multitasking is expected of learners at this level.
- **Level 4: Extended Critical Thinking and Reasoning –** At this level of assessment, the learner's extended thinking to solve complex and authentic problems with unpredictable outcomes is the goal. The learner must be able to strategically analyse, investigate, and reflect while working to solve a problem, or changing their approach to accommodate new information. The assessment requires sophisticated and creative thinking. As part of this assessment, the learner must know how to evaluate their progress and determine whether they are on track to a feasible solution for themselves.

The main distinction between these two conceptual frameworks is what is measured. The revised Bloom's Taxonomy assesses the cognitive level that learners must demonstrate as evidence that a learning experience occurred. The DoK, on the other hand, is focused on the context—the scenario, setting, or situation—in which learners should express their learning. In this curriculum, the revised Bloom's taxonomy guided the design, and the DoK is used to guide the assessment of learning. The taxonomy provides the instructional framework, and the DoK analyses the assignment specifics. It is important to note that Bloom's Taxonomy requires learners to master the lower levels before progressing to the next. So, suppose the goal is to apply a mathematical formula. In that case, they must first be able to identify that formula and its primary purpose (remember and understand). The cognitive rigour is therefore presented in incremental steps to demonstrate the learning progression. When measuring assessments in DoK, learners move fluidly through all levels. In the same example, while solving a problem with a formula, learners recall the formula (DoK 1) to solve the problem (DoK 2 and DoK 3). Depending on the difficulty of the problem to be solved, the learner may progress to DoK 4.

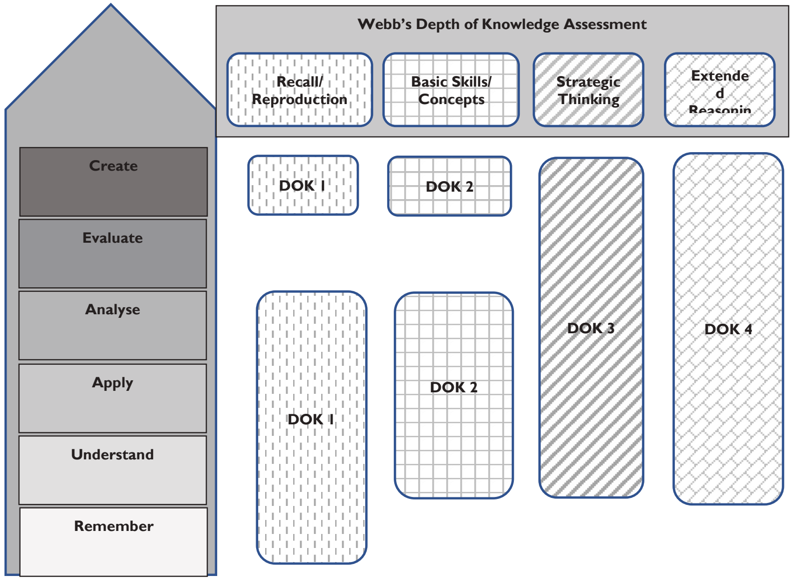

Figure 1: Revised Bloom Taxonomy combined with Webb's Depth of Knowledge for Teaching and Assessment

The structure of teaching and the assessment should align with the six levels of Bloom's knowledge hierarchy and DoK shown in Figure 1. Each level of DoK should be used to assess specific domains of Bloom's Taxonomy as illustrated in the table below:

Depth of Knowledge (DoK) Assessment	**Bloom's Taxonomy applied to DoK**
• Level 1: Recall and Reproduction	• Remembering, Understanding, Application, Analysis and Creation
• Level 2: Basic Skills and Concepts	• Understanding, Application, Analysis and Creation
• Level 3: Strategic Thinking	• Understanding, Application, Analysis, Evaluation and Creation
• Level 4: Extended Reasoning	• Understanding, Application, Analysis, Evaluation and Creation

In line with the National Pre-Tertiary Learning and Assessment Framework, the Secondary Education Assessment Guide (SEAG) requires that classroom assessments should cover **Assessment as learning (AaL), Assessment of learning (AoL) and Assessment for learning (AfL).** Therefore, teachers should align the Revised Bloom's Taxonomy with the DoK framework of assessment. Formative assessments should include classroom discussions, project-based assignments, and self-reflection exercises, while summative assessments should include standardised tests and rubric-based evaluations of learners' work. It is important to seek feedback from learners themselves, as they may have unique insights into how well they are developing these skills in the classroom.

To assess 21st Century skills and competencies in the classroom, teachers will have to use a combination of both formative and summative assessments to evaluate learners' acquisition of these skills and competencies. For instance:

- Identify the specific 21st Century skills and competencies to be assessed. For instance, you might want to assess *critical thinking, problem-solving, or creativity*.
- Align the skills and competencies with the DoK levels. For example, lower DoK levels might be more appropriate for assessing basic knowledge and comprehension, whereas higher DoK levels might be more appropriate for assessing more complex skills such as *analysis, synthesis, and evaluation*.
- Develop assessment items that align with the DoK levels and the skills and competencies you want to assess. These items should be designed to elicit evidence of learning across the different levels of the DoK framework.
- Administer the assessment and collect data. Analyse the data to gain insights into student learning and identify areas where learners may need additional support or instruction.

The DoK framework is a powerful tool for assessing the acquisition of 21st Century skills and competencies in the classroom, helping teachers to better understand how learners are learning and identify areas for improvement.

Educational success is no longer about producing content knowledge, but rather about extrapolating from what we know and applying the knowledge creatively in new situations.

The overall assessment of learning at SHS should be aligned with the National Pre-Tertiary Learning and Assessment Framework and the Secondary Education Assessment Guide. Formative and summative assessment strategies must be used.

Definition of Key Terms and Concepts in the Curriculum

- **Learning Outcomes:** It is a statement that defines the knowledge, skills, and abilities that learners should possess and be able to demonstrate after completing a learning experience. They are specific, measurable, attainable, and aligned with the content standards of the curriculum. It helps the teachers to determine what to teach, how to teach, and how to assess learning. Also, it communicates expectations to learners and helps them to better master the subject.
- **Learning Indicators:** They are measures that allow teachers to observe progress in the development of capacities and skills. They provide a simple and reliable means to evaluate the quality and efficacy of teaching practices, content delivery, and attainment of learning outcomes.
- **Content Standards:** It is a statement that defines the knowledge, skills, and understanding that learners are expected to learn in a particular subject area or grade level. They provide a clear target for learners and teachers and help focus resources on learner achievement.
- **Pedagogical Exemplars:** They are teaching examples used to convey values and standards to learners. Pedagogical Exemplars are usually demonstrated through teacher behaviour.
- **Assessment:** It is the systematic collection and analysis of data about learners' learning to improve the learning process or make a judgement on learner achievement levels. Assessment is aimed at developing a deep understanding of what learners know, understand, and can do with their knowledge because of their educational experiences. Assessment involves the use of empirical data on learners' learning to improve learning. Assessment is an essential aspect of the teaching and learning process in education, which enables teachers to assess the effectiveness of their teaching by linking learner performance to specific learning outcomes.
- **Teaching and Learning Resources:** Teaching and learning resources are essential tools for teachers to provide high-quality education to their learners. These resources can take various forms, including textbooks, audiovisual materials, online resources, and educational software. It is also important to avoid stereotypes and use inclusive language in teaching and learning resources. This means avoiding language that reinforces negative stereotypes and using language that is respectful and inclusive of all individuals regardless of their background. Using a consistent tone, style, and design is very important.

PHILOSOPHY, VISION AND GOAL OF ADDITIONAL MATHEMATICS

Philosophy

Learners can develop their potential in Additional Mathematics through creative, innovative and interactive ways to become lifelong learners, apply mathematical skills and competencies to solve everyday problems, further their education to read mathematics-related courses and/or proceed to the world of work.

Vision

Learners enthusiastic about mathematics, capable of reasoning (quantitatively and abstractly), modelling, representing and using mathematical skills, tools and technology to solve real life problems, further their studies and/or proceed to the world of work.

Goal

The main goal of the Additional Mathematics subject is to guide and train learners to be critical thinkers, proficient in problem-solving, effective in reasoning with concrete and abstract ideas, effectively communicating mathematically and applying mathematical representations to model and interpret practical situations while competently making mathematical connections and explorations, and improving on their 21st century generational skills, competencies and values that are responsive to Gender Equality and Social Inclusion issues to make the world a better place.

Contextual Issues

Over the years, official reports have consistently identified a problem regarding how Mathematics is taught and learnt in Ghana. Consequently, only a few senior high school learners possess the mathematical proficiency level needed to access a myriad of opportunities that the 21st century offers. This has been attributed largely to course overload, lack of Gender Equality and Social Inclusion (GESI) issues in the pedagogy, the relevance of lessons to everyday life and inefficient assessment procedures that have a high tendency to inactivate or de-emphasise the 21st century skills, competencies and values required for the job market, global and local environment. The learning experiences and teaching pedagogies appear to ignore the varied uses of Mathematics in different local contexts to accentuate the beauty of mathematics in solving real life problems and do not consider learners' literacy abilities and differences in native languages.

In addition, respect for culture and diversity and allowing learners to make connections between local and global contexts and then share their understanding with others appear limited in most of the existing mathematics classrooms. The issue of content overload, which is so pronounced in the existing mathematics curriculum in which topics are repeated in Core and Additional Mathematics, cannot be overemphasised. It is these contextual issues that the Additional Mathematics curriculum has been structured to address.

The teacher is uniquely positioned to make a difference in the mathematical conceptual development of learners in this curriculum through innovative pedagogies and learner-centred assessment strategies. The teacher is granted an opportunity to change the narrative in learners using this Additional Mathematics curriculum to inspire and develop highly competent individuals to meet the challenges of the 21st century.

Rationale

Numeracy and literacy are essential skills for work and lifelong learning. Mathematics is an essential tool that enhances progress in the world. Additional Mathematics is a fascinating and useful course to pursue for further studies. Its study is a requirement and a foundation for learners who wish to pursue higher studies in Computing, Engineering, Health Sciences, Research, etc. The strands are designed to equip learners with 21st century skills. Throughout this Additional Mathematics curriculum, there is a strong emphasis on recognising the uses of Mathematics in different local and global contexts as well as exploring learners' appreciation of differentiated approaches.

The first strand, Modelling with Algebra, will equip learners with the ability to represent real life phenomena using mathematical equations. Modelling gives precision to a real life problem, enables a systematic understanding of the problem, and helps discover new features. It will enhance problem-solving and critical thinking skills for decision-making. Mathematical knowledge models situations in natural sciences, physical sciences, social sciences etc. The second strand, Geometric Reasoning and Measurement, creates a foundation for future pursuits in Arts, STEM, Architecture, construction etc., this strand will aid learners' conceptual development, intuition, and estimation and allow them to develop coherent knowledge and apply reasoning skills. The third strand, Calculus, is one of the important branches of Mathematics because of its usefulness for modelling real life data. Calculus is useful for further studies in Engineering, Medicine, Biological research, Statistics, Economics etc. Finally, the strand on Handling Data trains the mind to think about how data is collected, organised, and analysed for decision-making and predictions. The skills practised in Handling Data can be transferred and used in different subject areas such as Data Science and Actuarial Science.

This Additional Mathematics curriculum has taken into account learners' fundamental 21st century generational needs.

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SCOPE AND SEQUENCE

Additional Mathematics Summary

S/N	STRAND	SUB-STRAND	YEAR 1			YEAR 2			YEAR 3		
			CS	LO	LI	CS	LO	LI	CS	LO	LI
1	Modelling with Algebra	Number and Algebraic Patterns	2	4	13	2	7	8	-	-	-
		Applications of Algebra	2	8	21	2	6	19	2	2	8
2	Geometric Reasoning and Measurement	Spatial Reasoning	2	4	11	2	2	8	1	4	11
		Measurement of Triangles	1	2	6	1	2	4	2	1	4
3	Calculus	Principles of Calculus	1	1	6	2	2	9	1	1	4
		Applications of Calculus	1	1	2	1	1	2	1	1	4
4	Handling data	Organising, Representing and Interpreting Data	1	2	8	2	2	4	1	3	9
		Making Predictions with Data	1	2	7	2	2	6	1	2	6
Total			**11**	**24**	**73**	**14**	**22**	**60**	**9**	**14**	**46**

Overall Totals (SHS 1 – 3)

Content Standards	33
Learning Outcomes	61
Learning Indicators	180

YEAR ONE

Subject ADDITIONAL MATHEMATICS
Strand 1. MODELLING WITH ALGEBRA
Sub-Strand 1. NUMBER AND ALGEBRAIC PATTERNS

Learning Outcomes	**21st Century Skills and Competencies**	**GESI[1], SEL[2] and Shared National Values**
1.1.1.LO.1		
Solve problems involving properties of binary operations.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: 1. Respect individuals of different backgrounds in their groups as they solve problems involving properties of binary operations using appropriate technological tools. 2. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to solve problems involving properties of binary operations.

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1 Gender Equality and Social Inclusion
2 Socio-Emotional Learning

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to real life.	3. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. 4. Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies– *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems.

		3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation These may be done through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs and cultures through interactive and collaborative/group work as leaders in their group. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		without unwanted segregation or discrimination among learners as they discuss concepts based on binary operations. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they solve problems involving properties of binary operations.
1.1.1.LO.2		
Model and solve real life problems on sets.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions).	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they inclusively work with each other; cross-sharing knowledge and understanding among groups and individuals lead them to: Respect individuals of different backgrounds in their groups as they model and solve real life problems on sets using appropriate technological tools. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to model and solve real life problems on sets.

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts.	Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. Managing emotions and conflicts as they engage in collaborative group work

		Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, creating a vision board, creating respectful icebreakerss for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values: Leadership and Respect for others’ Views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners.

		Tolerance: Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they model and solve real life problems on sets.
1.1.1.LO.3		
Expand binomials with positive integral indices and simplify coefficients of the terms.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions). **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: Respect individuals of different backgrounds in their groups as they expand binomials with positive integral indices using appropriate technological tools. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to expand binomials with positive indices. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse.

	the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to real life	Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. Managing emotions and conflicts as they engage in collaborative group work. Showing empathy and cooperation: These may be done by the teacher through modelling emotional self-

		regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values: Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-

		ability groups, emphasising differentiated instruction and assessment as they expand binomials with positive integral indices and simplify coefficients of the terms.
1.1.1.LO.4		
Perform basic operations on surds as well as solve simple indicial and logarithmic equations.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions). **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal,	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: Respect individuals of different backgrounds in their groups as they perform basic operations on surds, indicial and logarithmic equations using appropriate technological tools. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to perform basic operations on surds, indicial and logarithmic equations. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. Value and promote justice in the mathematics classroom and beyond.

	take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovations** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to real life	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. Managing emotions and conflicts as they engage in collaborative group work Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a

		vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values: Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising on differentiated instruction and assessment as they perform basic

		operations on surds as well as solve simple indicial and logarithmic equations.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
1.1.1.CS.1	1.1.1.LI.1	1.1.1.AS.1
Demonstrate knowledge and understanding of binary operations, sets and binomial theorem and solve related problems in real life situations.	**Explain binary operations and apply that knowledge in solving related problems.** **Collaborative Learning**: Learners will work in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to identify and define binary operations over given sets and solve related problems. **Talk for Learning Approaches:** Learners will brainstorm by way of **think-pair-share/square and debate** on sets defined by binary operations and establish the rules for such binary operations. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create binary operations, define them over sets and solve related problems. **Activity 1: Define and interpret Binary Operations as a rule** Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to recall the four basic operations and use them to define a binary operation. **Small group discussions/Think-pair-share:** Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) examine binary operations and recognise that binary operations are rules that combine elements from a non-empty set R to produce another element. **Example**: Suppose the operation * is defined on the set of real numbers $\mathbb{R}$ by $a * b = a + b - 2ab$.	**Level 1 Recall** The operation $*$ is defined on the set of real numbers $\mathbb{R}$ by $p * q = 2p + q - 2pq$. a) Find i. $3 * -2$ ii. $3 * 5$ b) If $p * 4 = -2$ find the value of p. **Level 4 Extended critical thinking and reasoning** You have six shirts, two trousers and two pairs of shoes. Explain how you will use the binary operation to determine how many ways a shirt, a trouser and a pair of shoes can be worn

- Find (α)$4 * 2$
- (β) $\frac{1}{3} * -2$
- If $a * 4 = -2$, find the value of a.

Inter-group competition: Learners from one group devise their own binary operations for another group to examine and interpret the rule. Group switch roles.

Work with a partner: Construct tables for a given binary operation.

Examples:

- A binary operation ∇ is defined on the set $S = \{2, 3, 4, 5\}$ by $p\nabla q = p + q - pq$ where p and $q \in S$.
- Construct the table for the operation ∇ on the set S.
- Use your table to evaluate $(2\nabla 3)\nabla 3$
- Which of these can be solved using the operation ∇ on the set S. Give reasons.

(α) $3 \nabla -2$ (β) $4\nabla 3$ (γ) $3\nabla 5$ (δ) $5\nabla 1$

- Given that $m \ominus n = \frac{m+n}{m}$, evaluate $(2 \ominus 3) + (3 \ominus 2)$

- a and b are integers such that $a \diamond b = \frac{ab}{10}$. Find the values of a and b if $a \diamond b = (6 \diamond a) - \frac{1}{2}$

Activity 2: Discuss the history and relevance of learning binary operations.

	1.1.1.LI.2	1.1.1.AS.2
	Describe and interpret the characteristics of commutative, associative, distributive and closure properties of binary operations. **Collaborative Learning**: Learners will work in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to solve problems related to identity and inverse elements and other properties of binary operations. **Talk for Learning Approaches:** Learners will brainstorm through **think-pair-share/square and debate** to discuss and establish the identity, inverse elements, and other properties of a binary operation. **Experiential Learning**: Learners will collaboratively engage in hands-on activities (learning by doing) to create binary operations, define them over sets and determine the inverse and identity elements and other properties of binary operations. **Activity 1**: Identity and inverse elements of a binary operation. Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to investigate the relationship between inverse and identity elements of a binary operation. **Small group discussions/Think-pair-share:** Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) investigate the elements of binary operations to establish that the identity element e of a set S under an operation Δ on S exists if there is an element $e \epsilon S$ such that $a\Delta e = e\Delta a = a$ **Example 1:** A binary operation ∇ is defined on the set R or real numbers by $p\nabla q = p + q - pq$ where p and $q \in R$.Find the identity element under the operation **∇.** **Solution:** Let e be the identity element, where $e \in R$, then for the right identity	**Level 2 Skills of conceptual understanding** A binary operation ∇ is defined on the set $S = \{2,3,4,5\}$ by $p\nabla q = p + q - pq$ where p and $q \in S$. **a)** Construct the table for the operation ∇ on the set S. **b)** Determine whether or not the operation ∇ is **i.** closed under S, **ii.** commutative, **iii.** associative and **iv.** distributive over $*$ if $a * b = ab - 3a, a, b \in S$

$p\nabla e = p + e - pe = p. \Longrightarrow e = 0.$

For the left identity
e∇p = e+$p - ep = p. \Longrightarrow e = 0.$
Since the left identity=right identity, the identity element e of $\mathbb{R}$ under the operation ∇ exists, and it is 0.

Example 2: Suppose the operation $*$ is defined on the set of real numbers R by $a * b = a + b + 2ab$. Find the identity element under the operation $*$.

Small group discussions/Think- pair-share: Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) investigate the elements of binary operation to establish that the inverse of an element a of a set S under an operation Δ on S is an element $a^{-1} \in S$ such that $a\Delta a^{-1} = a^{-1}\Delta a = e,$ where e is the identity element of S under the operation ∇.

Example: A binary operation ∇ is defined on the set R or real numbers by $p\nabla q = p + q - pq$ where p and $q \in R$.Find the inverse element under the operation **∇.**

Solution: Let $p^{-1} \in R$ be the inverse element, then
$p\nabla p^{-1} = p + p^{-1} - pp^{-1} = e. But\ e = 0.$ So $p + p^{-1} - pp^{-1} = 0 \Longrightarrow p^{-1} = \frac{-p}{1-p}$
Similarly, $p^{-1}\nabla p = p^{-1} + p - p^{-1}p = 0 \Longrightarrow p^{-1} = \frac{-p}{1-p}.$
Since the left inverse = right inverse, then the inverse element p of S under the operation ∇ exists, and it is $\frac{-p}{1-p}$

Example: Suppose the operation $*$ is defined on the set of real numbers R by $a * b = a + b + 2ab$. Find the inverse element under the operation $*$.

Activity 2: Commutative, Associative Distributive and Closure properties of binary operations.

Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to explore the commutative, associative, distributive and closure properties of binary operations.

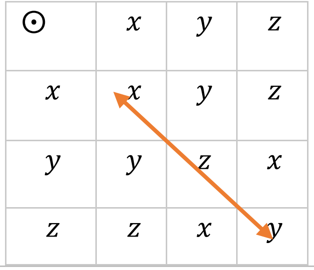
Small group discussions/Think-pair-share: Learners in small convenient groups (e.g., mixed gender, mixed-ability, etc.) investigate the commutative property of a binary operation and recognise from Cayley's tables that a binary operation is commutative if the table is symmetric along its principal diagonal.

Example: A binary operation $\odot$ is defined on the set $P = \{x, y, z\}$ by the table below. Determine whether the operation $\odot$ is commutative.

$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Solution: By inspection we identify that $x \odot y = y \odot x = y$; $x \odot z = z \odot x = z$ and $y \odot z = z \odot y = x$ $\therefore \odot$ is commutative

Alternatively, since the table is symmetric along the principal diagonal, we conclude that the operation $\odot$ is commutative.



$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Example: A binary operation ∇ is defined on the set $S = \{2,3,4,5\}$ by $p\nabla q = p + q - pq$ where p and $q \in S$.

- Construct the table for the operation ∇ on the set S.
- Use your table to determine whether or not the operation ∇ is commutative.

Inter-group competition: Learners from one group create binary operations for the other group to investigate the associative property of the operation. Groups switch roles. Thus, learners recognise that a binary operation $\oslash$ on a set R is associative if $(a \oslash b) \oslash c = a \oslash (b \oslash c)$ for all a, b and $c \in R$.

Inter-group competition: Learners from one group create two binary operations for the other group to investigate the distributive property of the operations. Group switch roles. Thus, learners recognise that two binary operations $\emptyset$ and on a set R are such that associative if.
$(a \oslash b)\Delta c = (a\Delta c) \oslash (b\Delta c)$ for all a, b and $c \in R$

Inter-group competition:
Learners from one group create binary operations over a given set, and the other group investigates the set is closed under the given operation. Group switch roles. Thus, learners recognise that a set S is closed under a binary operation $\divideontimes$ if for any $a, b \in S\, a \divideontimes b \in S$

Example: The table below is defined by the operation $\odot$ on the set $B = \{r, s, t, u\}$

$\odot$	r	s	t	u
r	s	u	r	t
s	u	t	s	r
t	r	s	t	u
u	t	r	u	s

	- Find, giving reasons, whether or not α) B is closed under β) ⊙ is commutative γ) there is an identity element - Find, where possible, the inverse of the elements of set B.	
	1.1.1.LI.3	1.1.1.AS.3
	Determine the identity element and use it to find the inverse of a given element. Think-pair-share, Talk for Learning, Project-Based Learning. **Learning Experience:** Learners in pairs investigate and determine the identity element and use it to find the inverse of a given element **Activity 1:** Learners in pairs investigate the identity element of addition and multiplication and establish the definition of identity element. - Learners identify the identity element of addition and multiplication by solving the following questions: $7 - x = 7, 5 \times y = 5, 12 + z = 12$ - Learners in pairs discuss and conclude that the identity element of addition is zero and the identity element of multiplication is one. - Learners conclude that an identity element or neutral element leaves a combination unchanged or unaffected. Through discussion, learners establish that, for a binary operation $*$, if there exist just one element e such that $a * e = e * a = a$ where $a, e \in \mathbb{R}$ a, then e is called an identity element. **Activity 2:** Learners in groups investigate identity elements in a given table. Example 1: The combination table for the set $Q = \{a, b, c, d\}$ under the operation $*$ is given below	**Level 1 Recall** If $a * b = a + b + 2$, where $a, b \in \mathbb{R}$, find the identity element **Level 3 Strategic reasoning** Given that $m * n = 3m + 2n - mn$**,** investigate whether the operation $*$ has a unique identity

$*$	a	b	c	d
a	c	a	b	d
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

- State the identity element.

Through brainstorming and discussion, learners find out that:

$$a * b = b * a = a$$
$$b * b = b$$
$$c * b = b * c = c$$
$$d * b = b * d = d$$
$$\therefore b \text{ is the identity element}$$

Learners in groups create and solve similar examples.

Learners establish that the clue for identifying the identity element in a table is to look out for where the elements match the outer elements.

Activity 3: Learners in groups investigate how to find the identity element when given a binary definition.

Example 1: If $a * b = a + b + 5$, where $a, b \in R$, find the identify element.

Solution: Definition of identity element:

$$a * e = e * a = a$$
$$\therefore a * e = a + e + 5 = a$$
$$e = -5$$

And

$$e * a = e + a + 5 = a$$
$$e = -5$$

Therefore, the identity element of $a * e = a + b + 5$ is -5

- Using example 2, i.e. $a * b = a - b$, learners in groups discover that the binary operation does not have a unique identity element.
- Learners establish that it is important to check that both the left and right sides give a unique solution because for a binary operation to have an identity $a * e = e * a = a$

Activity 4: Through Talk for Learning, discuss the inverse of a binary operation and solve examples.

- Learners establish that: An element $a \in A$ is called invertible if there exists an element $b \in A$ such that $a * b = e = b * a$, which means
$$a * a^{-1} = e = a^{-1} * a$$
- Learners in groups sole examples to determine the condition for finding the inverse of a binary operation

Examples: Find the inverse of

1. $a * b = a - b$
2. $a * b = a^2 + b^2$,
3. $a * b = \frac{ab}{4}$, if possible.

- Learners establish that it is not possible to find the inverse of examples 1 and 2 since the identity element is not unique.
- In Example 3, Learners discover that the binary operation has a unique identity element; therefore, the inverse can be found.

Solution: The Identity element for $a * b = \frac{ab}{4}$ is 4. If $e = 4$, then let b be the inverse such that $a * b = e = b * a$

$$\therefore a * b = e$$
$$a * a^{-1} = e$$
$$\frac{aa^{-1}}{4} = 4$$

	$(aa)^{-1} = 16$ $\therefore a^{-1} = \frac{16}{a}$ And for $b * a = e$ $a^{-1} * a = e$ $\frac{a^{-1}a}{4} = 4$ $a^{-1}a = 16$ $\therefore a^{-1} = \frac{16}{a}$ Learners in groups research how to find the inverse element in a given table and present findings in class.	
	1.1.1.LI.4	1.1.1.AS.4
	Establish the properties of operations on set, including commutative, associative, and distributive, sets algebra and apply them to solve problems. **Collaborative Learning**: Learners will be working in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to establish set identities, set operations, and apply them to solve problems in different contexts. **Talk for Learning Approaches:** Learners will brainstorm through **think-pair-share/square and debate** to discuss set identities and properties of set operations. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create problems and apply set identities, algebra, and operations to solve.	**Level 1 Recall** Consider the sets: *A={red, green, blue},* *B={red, yellow, orange},* *C={red, orange, yellow, green, blue, purple}* *D={yellow, white} and the universal set* $U = A \cup B \cup C \cup D$ a) find $A' \cap C$ b) find $A \cup C$ and $B' \cap A$. **Level 3 Strategic Reasoning** A survey asks: "Which online social media have you used in the last month: WhatsApp, Facebook, Both". The results show 42% of those surveyed have used WhatsApp, 70% have used Facebook, and

Activity 1: Set identities and set operations.
Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to investigate set identities, set algebra, and set operations and solve related problems.

Small group discussions/Think-pair-share

- Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) investigate and establish set identities and verify these properties.
- Learners use set algebra to verify that for any three Sets A, B and C:

Example 1: Verify that for any three Sets A, B and C:

- Commutative Properties:

$A \cup B = B \cup A$
$A \cap B = B \cap A$

- Distributive Properties:

$A \ \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- Associative properties:

$A \cap (B \cap C) = (A \cap B) \cap C$
$A \cup (B \cup C) = (A \cup B) \cup C$

Activity 2: Use of Venn diagram to verify three set identities/operations.

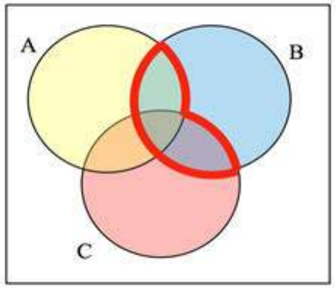
Small group discussions/Think-pair-share: Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) investigate and establish set identities and use Venn diagrams to verify that for any three Sets A, B and C:

- $A \cup B = B \cup A$
- $A \cup (B \cap C) \ = \ (A \ \cup B) \cap (A \ \cup C)$
- $A \cap (B \cup C) \ = \ (A \cap B) \cup (A \cap C)$

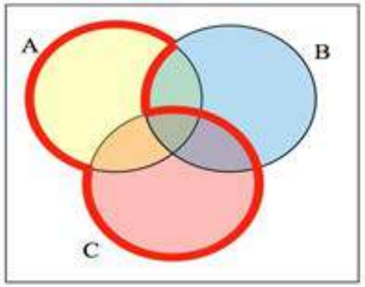
20% have used both. How many people have used neither WhatsApp nor Facebook?

Level 4 Extended critical thinking and reasoning
Write an expression for the regions outlined in red

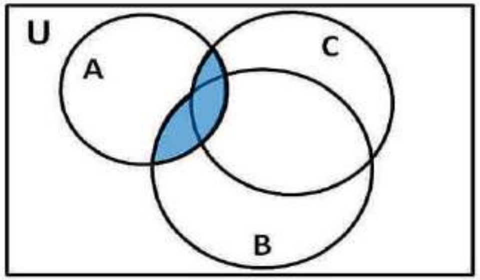
a)



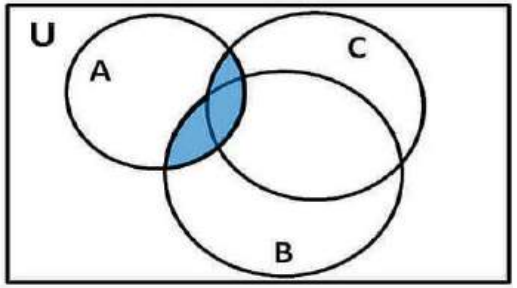
b)



Example of illustration of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in a Venn diagram



$A \cap (B \cup C)$



$(A \cap B) \cup (A \cap C)$

Inter-group competition: Learners from one group create real life problems involving 2 sets whilst the other group applies set algebra, operations, and identities to solve the problems. Group switch roles.

Example: A pharmaceutical company is considering manufacturing new toothpaste. They are considering two charcoal flavours, strawberry and mint. In a sample of 74 people, it was found that 45 liked strawberries, 37 liked mints and 21 liked both types

- Create a Venn diagram to model the information.
- How many liked only strawberries?
- How many liked only mints?
- How many liked exactly one of the two (that is, they liked one but not the other)?

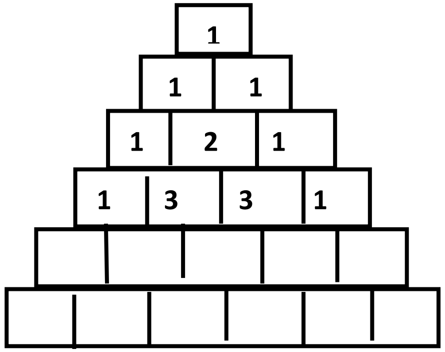
	1.1.1.LI.5	1.1.1.AS.5
	Expand binomial expressions for positive integer indices using Pascal's triangle.	**Level 2 Skills of conceptual understanding**

	Collaborative Learning: Learners will be working in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to establish personal and conventional strategies for determining coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Talk for Learning Approaches:** Learners will brainstorm through think-pair-share/square and debate to discuss strategies for determining coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create set problems and apply personal and conventional strategies to determine the coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Activity 1:** Co-efficient and terms of binomial expansions (Pascal's triangle) Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning and experiential learning approaches to investigate strategies for determining the coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Small group discussions/Think-pair-share:** In small convenient groups (for instance, mixed gender, mixed-ability, etc.) establish the coefficients of binomial expansions involving positive integer powers using personal methods. **Example 1:** Obtain the terms in the binomial expansion of: • $(a + b)^2$ • $(a + b)^4$ • $(a - b)^5$ **Small group discussions/Think-pair-share:** Learners in small convenient groups (i.e. mixed gender, mixed-ability, etc. where appropriate) establish the coefficients of binomial expansions involving positive integer powers using Pascal's triangle.	Obtain the expansion of $\left(2x - \frac{1}{2}\right)^4$ in descending powers of x using Pascal's triangle **Level 3 Strategic reasoning** Using the terms in the expansion of $(x + y)^5$, find $\sum(x + y)^5$, if $x = 1$ and $y = 0.05$ **Level 4 Extended critical thinking and reasoning** Write down the binomial expansion of $(a + b)^4$ and use your expansion to evaluate $(2.01)^4$

Example1: Use Pascal's triangle to expand the following:

- $(a + b)^2$
- $(a + b)^4$
- $(a - b)^5$
- $(2a - 3b)^5$

Differentiation: Support emerging proficiency learners with manipulatives (cut-out paper cards) containing terms of binomial expansions for them to use to build a pyramid and use the addition pyramid to establish the coefficients of given binomial expansions.



Example: Write down the binomial expansion of $(a + b)^4$. Use your expansion to evaluate$(2.01)^4$.

Example: Without using tables and calculators, find the value of $\left(2 + \sqrt{3}\right)^6$ + $\left(2 - \sqrt{3}\right)^6$

	1.1.1.LI.6	1.1.1.AS.6
	Use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion. **Collaborative Learning**: Learners will be working in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to explore the combination strategies for determining various terms of a binomial expansion with positive integer indices in different context. **Talk for Learning Approaches:** Learners will brainstorm using participatory activities such as think-pair-share/square and debate to discuss the combination strategies for determining the various terms in a binomial expansion with positive integer indices in different contexts. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create set problems and apply personal and conventional strategies to determine the coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Activity 1:** Coefficients and terms in binomial expansions Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to explore combination strategies for determining the coefficients of the terms in a binomial expansion with positive integer indices in different contexts. **Small group discussions/Think-pair-share:** Learners in small convenient groups (e.g., mixed gender, mixed-ability, etc.) establish that if n is a positive integer, the expansion $(a + b)^n$ is given by $$(a + b)^n = [nC_0a^nb^0 + nC_1a^{n-1}b^1 + nC_2a^{n-2}b^2 + \cdots + nC_na^{n-n}b^n]$$ which can also be written using the summation	**Level 1 Recall** Provide appropriate responses (True or False) to the following tasks a) For every positive integer n, $(3n)! = (3)!\,(n)!$ b) There are ten terms in the expression $(1 - x)^{10}$ c) The middle term of the expansion of $\left(1 + \frac{1}{x}\right)^8$ is 70. **Level 2 Skills of conceptual understanding** Find the coefficient of the term x^9y^5in the binomial expansion of the expression $(2x + 3y)^{14}$ **Level 4 Extended critical thinking and reasoning** Show that $11^9 + 9^{11}$ is divisible by 10

<table>
<tr>
<td></td>
<td colspan="3">$$(a + b)^n = \sum_{r=1}^{n}\left(\binom{n}{r} a^{n-r} b^r\right)$$
Example 1: Find the coefficient of x^8 in the expansion $(1 + x)^{10}$
Example 2: Find the constant term in the expansion of $\left(x - \frac{2}{x^2}\right)^{10}$</td>
<td></td>
</tr>
<tr>
<td>Teaching and Learning Resources</td>
<td>• Textbooks
• Rectangular paper cut-out
• Addition pyramid</td>
<td>• Curriculum
• Cardboards
• Reading resources</td>
<td>• Colour pens
• Notebook
• Graph sheets</td>
<td>• Mathematical sets
• Technological tools.</td>
</tr>
</table>

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
1.1.1.CS2	1.1.1.LI.1	1.1.1.AS.1
Demonstrate knowledge and understanding of numbers in relation to Surds, Indices and Logarithms.	**Investigate the properties of surds and perform basic arithmetic operations on surds, including rationalisation.** **Collaborative Learning**: Learners will work in convenient groups (e.g., ability, mixed-ability, mixed gender, or pairs etc.) to explore the basic properties and operations on surds in different contexts. **Talk for Learning Approaches:** Learners will brainstorm using participatory activities such as **think-pair-share/square and debate** on various properties of surds. **Experiential Learning**: Learners will work with others in groups and pairs to create surds problems involving basic operations, properties and types of surds. **Activity 1: Definition and Types of Surds** Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to investigate the types of surds**.** Learners in small and convenient groups (i.e., mixed gender, mixed-ability, etc.) identify and work to establish types of surds. **Example:** Learners recognise that: • Surds is used to refer to a number that does not have a root. E.g., $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$ etc. • Surds represent numbers in the form of square roots since these numbers cannot be whole or rational numbers. **Types of surds** Learners investigate and discover the following as types of surds: • **Pure Surds:** A surd having only a single irrational number is called a pure surd. E.g., $\sqrt{7}$	**Level 1 Recall** Given that $\sqrt{a}+\sqrt{b}=7$ and $\sqrt{a}-\sqrt{b}=1,$ find the value of a and b **Level 3 Strategic reasoning** Solve for the value of x in $2\sqrt{3x+5}=4\sqrt{x+1}$ **Level 4 Extended critical thinking and reasoning** What is the square root of $(10+\sqrt{25})(12-\sqrt{49})$?

- **Mixed Surds:** A surd having a mix of a rational number and an irrational number is called a mixed surd. E.g., $5\sqrt{3}$
- **Compound Surds:** A surd composed of two surds or a surd, and a rational number is called a compound surd. E.g., $\sqrt{3}+\sqrt{10}$, 3+ $\sqrt{7}$,
- **Binomial Surd:** When two surds give rise to one single surd, the resultant surd is known as a binomial surd.

Activity 2: Properties of surds

Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to investigate the rules and properties of surds, including conjugates and rationalisation.

- Learners in small convenient groups (i.e., mixed gender, mixed-ability, etc.) work together to establish the properties and rules of surds.

Example: Learners, in groups, establish the rules of surds, including conjugating surds. That is

Rule 1: $\sqrt{(a \times b)} = \sqrt{a} \times \sqrt{b} \qquad a, b \geq 0$

Rule 2: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \qquad b > 0$

Rule 3: $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b} \quad b > 0$

Rule 4: $\sqrt[n]{a} = a^{\frac{1}{n}}$

Rule 5: $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

Rule 6: $\frac{a}{a+\sqrt{b}} = \frac{a}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}}$

Rule 7: $\frac{a}{a-\sqrt{b}} = \frac{a}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}}$

NB:

- $\sqrt{a} + \sqrt{b} \neq \sqrt{(a+b)}$
- $\sqrt{a} - \sqrt{b} \neq \sqrt{(a-b)}$

Activity 3: Simplification of Surds

Inter-group competition: Learners from one group create surd problems whilst the other group applies properties of surds to solve the problem.
Group switch roles.

Example: Simplify $\sqrt{108}$.

Solution

$\sqrt{108} = \sqrt{36 \times 3}$
$= \sqrt{36} \times \sqrt{3}$
$= 6\sqrt{3}$

Example: Simplify the following surd expressions where possible. For those that cannot be solved, state the reasons why**.**

a) $\sqrt{5} + \sqrt{7}$
b) $3\sqrt{2} + 5\sqrt{2}$
c) $\sqrt{7} - \sqrt{5}$
d) $3\sqrt{2} - 5\sqrt{2}$
e) $\sqrt{5} \times \sqrt{7}$
f) $3\sqrt{2} \times 5\sqrt{2}$
g) $\sqrt{15} \div \sqrt{5}$

NB: Learners in their groups create and solve examples and discover that surds

	• cannot be added/subtracted, but similar surds can be added/subtracted. • can be multiplied. • can be divided. • can be written in exponential form.	
	1.1.1.LI.2	1.1.1.AS.2
	Rationalise surds with binomial denominators. **Collaborative Learning**: Learners will be working in convenient groups (e.g., ability, mixed-ability, mixed gender, or pairs etc.) to solve problems involving rationalisation. **Talk for Learning Approaches:** Learners will brainstorm using participatory activities such as **think-pair-share/square and debate** to discuss when and how to rationalise surds. **Experiential Learning**: Learners will work with others in groups and pairs to create surds problems involving basic operations, properties and types of surds. **Activity 1:** Use Talk for Learning Approaches (building on what others say, managing Talk for Learning, structuring Talk for Learning), collaborative learning approaches and experiential learning approaches to investigate conjugates and rationalisation of surds. • In groups, discuss the rationalisation of surds to discover that rationalising surds involves changing the surd denominator to a rational number. That is: $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$ $= \frac{a\sqrt{b}}{b}$ **Example:** $\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ $= \frac{\sqrt{5}}{5}$	**Level 2 Skills of conceptual understanding** Simplify $\frac{8}{1+2\sqrt{3}} \times \frac{8}{1-2\sqrt{3}}$ **Level 3 Strategic reasoning** Express $\frac{1-5\sqrt{5}}{3+\sqrt{5}}$ in the form $m + n\sqrt{5}$; $m, n \in \mathbb{Z}$

- Learners in groups discuss the rationalisation of surds with binomial denominators. Learners discover that to rationalise this type of surds, multiply the fraction by the conjugate of the denominator. That is:

$$\frac{a}{1+\sqrt{b}}=\frac{a}{1+\sqrt{b}}\times\frac{1-\sqrt{b}}{1-\sqrt{b}}$$

Example: Rationalise $\frac{1}{3-\sqrt{5}}$

Solution $=\frac{1}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}$

$\frac{3+\sqrt{5}}{4}$

Example: Simplify $\frac{5}{3+\sqrt{2}}-\frac{2}{3-\sqrt{2}}$

Expected Solution: $\frac{9}{7}-\sqrt{2}$

Activity 2: Learners in groups create and solve problems in surds.

Example:

Express $\sqrt{15\left(\sqrt{27}-\frac{\sqrt{2}}{\sqrt{3}}\right)}$ in the form $p\sqrt{q}$ where p and $q\in R$

Expected Solution: $7\sqrt{5}$

$7\sqrt{5}$

	1.1.1.LI.3	1.1.1.AS.3
	Recollect the initial laws of indices and establish other laws for negative powers and roots. **Collaborative Learning, Talk for Learning, Building on what others say.**	**Level 1 Recall**

Learning Experience: Learners in groups explore the laws of indices.

Activity 1: Learners in pairs share ideas about indices and explain indices to their colleagues.

Expected Responses

- **Index (indices)** is the power or exponent raised to a number or a variable.
- The plural of index is indices.

Activity 1: Learners, in their groups, solve examples and use the examples to explore the derivation of the laws of indices.

Activity 2: Learners in their groups create and solve similar examples to establish the laws of indices.

Activity 3: Learners in their groups establish that:

- $a^m \times a^n = a^{m+n}$, and by extension, once the base is equal $a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\cdots}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$, and by extension
- $(a^m \times b^m \times \ldots)^n = a^{mn} \times b^{mn} \times \ldots$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^n}$
- $\left(\frac{a \times b \times \ldots}{c \times d \times \ldots}\right)^m = \frac{a^m \times b^m \times \ldots}{c^m \times d^n \times \ldots}$
- $(ab)^m = a^m b^n$
- $\left(\frac{a^m}{b^m}\right)^n = \frac{a^{mn}}{b^{mn}}$
- $\left(\frac{a^m \times b^m \times \ldots}{c^m \times d^m \times \ldots}\right)^n = \frac{a^{mn} \times b^{mn} \times \ldots}{c^{mn} \times d^{mn} \times \ldots}$

Other rules are:

- Negative exponent rule:
  $a^{-m} = \frac{1}{a^m}$ by extension $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- Zero power rule: $a^0 = 1$, where $a \neq 0$

Simplify and write $\frac{(a^2b^4)}{(a^5b^3)}$ with positive indices

Level 2 Skills of conceptual understanding

Show that

i. $32^{-\frac{2}{5}} = \frac{1}{4}$

ii. $\left(2x^{-\frac{2}{5}}\right)^5 = \frac{32}{x^2}$

	Activity 4: Learners in their groups create and solve a variety of problems involving the laws of indices. **Example:** Simplify $8^{\frac{2x}{3}}$ **Expected solution =** $8^{\frac{2x}{3}} = 4^{x}$ **Example:** Simplify $2^2 \times 4^{-4} \div 16^{-3}$ **Expected Solution=** 64	
	1.1.1.LI.4	1.1.1.AS.4
	Recognise the relationship between surds and indices and apply laws of indices to simplify expressions. **Research, Collaborative Learning, and Talk for Learning.** **Learning Experience**: Learners in groups conduct research on the relationship between surds and indices. **Activity 1:** Group leads present the findings of the group on the relationship between indices and surds to the whole class. **Activity 2:** Learners apply the laws of surds and the laws of indices to solve problems. **Example:** If $a = 3 - \sqrt{3}$, show that $a^2 + \frac{36}{a^2} = 24$. **Solution** $a = 3 - \sqrt{3}$ (Squaring both sides) $a^2 = 9 - 6\sqrt{3} + 3$	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	$a^2 = 12 - 6\sqrt{3}$ (Squaring both sides) $a^4 - 24a^2 + 144 = 36 \times 3$ $a^4 - 24a^2 + 36 = 0$ (divide through by a^2) $a^2 - 24 + \frac{36}{a^2} = 0$ So, $a^2 + \frac{36}{a^2} = 24$ **Activity 2:** Learners create similar problems involving the application of the laws of surds and the laws of indices to solve real life problems.	
	1.1.1.LI.5	1.1.1.AS.5
	Pose and solve simple equations involving indices. **Collaborative Learning, Talk for Learning, Building on what others say.** **Learning Experience**: Learners in groups solve simple equations involving indices. **Activity 1:** Solving simple indicial equations. Learners in their groups apply their knowledge in solving linear equations and indices to solve problems. **Example:** Given that $y - 2x$ and $3^{x+y} = 27$, Find x **Solution** $3^{x+y} = 27$ $3^{x+2x} = 3^3$ $x + 2x = 3$ $\therefore x = 1$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Solve the simultaneous equations $9^{2a+b} = 2187$ and $3^a \times 9^b = 27$ **Level 4 Extended critical thinking and reasoning**

Activity 2: Learners in groups create similar problems on the application of linear equations and indices to solve problems.

Activity 3: Learners in groups apply their knowledge in solving simultaneous equations and indices to solve problems.

Example: Solve the simultaneous equations

$$9^{2a+b} = \frac{1}{729}$$
$$3^a(9^b) = 27$$

Expected Solution: $a = -3$ and $b = 3$

1.1.1.LI.6

1.1.1.AS.6

Establish the relationship between indices and logarithms and use the properties of logarithms to solve related problems in one base.

Collaborative learning, Talk for Learning, Building on what others say.

Learning Experience: Learners in groups explore the relationship between indices and logarithms.

Activity 1: Relationship between indices and logarithms.
Learners in groups investigate the relationship between indices and logarithms and establish that:
If $N = a^x$, then $\log_a N = x$

$$\log_a a = 1$$
$$\log p^n = n \log P$$
$$\log(NM) = \log N + \log M$$
$$\log\left(\frac{N}{M}\right) = \log N - \log M$$

Where M, N and P are real numbers.

NB: (i) $\frac{\log N}{\log M} \neq \log N - \log M$

Level 4 Extended critical thinking and reasoning

1. For the rule $y = 20 \times 3^t$,
   a) Complete the table of values.

t	0	1	2	3
y				

   b) Plot the graph of y against t.
   c) Find the value y, correct to 2 decimal places, when:
      i. $t = 0.5$
      ii. $t = 2.5$
      iii. $t = 2$

$\log(M+N) \neq \log N + \log M$

Activity 2: Learners in mixed-ability groups solve related problems in one base.
Suppose that GHS 5,000 is invested at 6% interest compounded annually. In t years, an investment will grow to the amount expressed by the function $S(t) = 5000 \cdot 1.06^t$, where t is time (in years).
How long will it take to accumulate GHS 15,000 in the account?

Expected solutions: You should solve an equation $S(t) = 15000$, which is $5000 \cdot 1.06^t = 15000$ for unknown t. Divide both sides of this equation by the initial amount of 5000. You get the equation

$$1.06^t = 3$$

Take logarithm base 10 from both sides. You get the equation,

$$(1.06^t) = 3$$

Apply the power rule to the logarithm. You get the equation,

$$1.06 = 3$$

Therefore, $t = \frac{3}{1.06} = \frac{0.4771}{0.0253} = 18.8542$ (Approximately 19 years)

Further example:
The value V of a Range Rover Velar that is t years old can be modelled by the function:
$V(t) = 25000 \cdot 0.85^t$
What would the car be worth in 3 years?
In how many years will the car be worth GHS 235,000?
Activity 3: Learners in mixed-ability groups solve related problems in different bases.
What should be one important step when solving logarithm equations with different bases?

2. Suppose that GH₵ 5,000 is invested at 6% iinterest compounded annually. In t years an investment will grow to the amount expressed by the function $S(t) = 5000 \cdot 1.06^t$ where t is time (in years).
   a) Explain how you will use the idea of logarithms to calculate the number of years it will take to double the initial investment
   b) How long will it take to accumulate GH₵ 15,000 in the account?

	Note Change of bases: **General rule:** Suppose x, a and b are positive numbers with $a, b \neq 1$, then, $\log_a x = \frac{\log_n x}{\log_n a}$			
Teaching and Learning Resources	- Textbooks - Curriculum	- Cardboards - Reading resource	- Colour pens - Notebook	- Technological tools

Subject ADDITIONAL MATHEMATICS
Strand 1. MODELLING WITH ALGEBRA
Sub-Strand 2. APPLICATIONS OF ALGEBRA

Learning Outcomes	21st Century Skills and Competencies	GESI, SEL and Shared National Values
1.1.2.LO.1		
Examine, analyse, determine and predict other terms in a pattern/sequence.	**Communication:** • Provide learners with the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant and listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. • All learners will be given the opportunity to confidently and effectively participate in different contexts to identify specific literacy skills and adopt appropriate mathematics vocabulary. **Collaboration:** • Create an atmosphere, environment, and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. • The teacher will create an enabling environment and opportunity for learners to understand and respect the needs, perspectives, and actions of others (empathy) fostered through group/individual project works and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: 1. Respect individuals of different backgrounds in their groups as they examine, analyse, determine and predict other terms in a sequence using appropriate technological tools. 2. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to evaluate sequences. 3. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse.

	predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling; and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations.	4. Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. 3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation

	Strategic Competency: Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for life application. **Technology Literacy Skills:** Enable students to think mathematically and solve challenging problems to develop their highest potential in an information and technologically-driven world.	The teacher may do these through modelling emotional self-regulation and decision-making, promoting positive self-talk with self-made portraits, creating a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and

		assessment as they examine, analyse, determine and predict other terms in a pattern/sequence.
1.1.2.LO.2		
Distinguish among various types of relations, find the domain and range of, and evaluate functions.	**Communication:** • Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. • All learners will be given the opportunity to confidently and effectively participate in different contexts to identify specific literacy skills and adopt appropriate mathematics vocabulary. **Collaboration:** • Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. • The teacher will create an enabling environment and opportunity for learners to understand and respect the needs, perspectives, and actions of others (empathy) fostered through group/individual work and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast,	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they distinguish among various types of relations to find the domain and range and evaluate functions using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to work on functions. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to “why, how, when, who, what, and where” questions). **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling; and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Technology literacy Skills:** Enables students to think mathematically and solve challenging problems in an information technology-driven world to develop their highest potential. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for life application.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision- making, the

		promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they distinguish among various types of

		relations, find the domain and range and evaluate functions.
1.1.2.LO.3		
Show that a function is injective (into) and/or surjective (onto). Find the inverse, describe the relationship between two variables and establish composite functions.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they show that a function is injective or surjective using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to show that a function is injective or surjective, • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-*

	classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	*Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for

		healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they show that a function is injective or surjective, find the inverse, describe the relationship between two variables and establish composite functions.

1.1.2.LO.4		
Graph linear and quadratic functions and determine the intercepts.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they graph linear quadratic functions and determine the intercepts using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to graph linear and quadratic functions and determine the intercepts. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-*

	Creativity and Innovation: Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	*Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for

		healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they graph linear and quadratic functions and determine the intercepts.

1.1.2.LO.5		
Find graphical and algebraic solutions to a system of three linear equations in three variables and apply them to solve real life problems.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions). **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they find graphical and algebraic solutions to a system of three linear equations in two variables and apply them to solve real life problems using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to solve a system of three linear equations. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning

	Creativity and Innovation: Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems including drawing graphs up to a system of three linear equations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	Competencies–*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision

		board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they find graphical and algebraic solutions to a system of three linear equations in two variables and apply them to solve real life problems.

1.1.2.LO.6		
Perform algebraic manipulations on polynomial functions and graph polynomial functions.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they perform algebraic manipulations of polynomial functions and graph polynomial functions using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to perform algebraic manipulations on polynomial functions and graph polynomial functions. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	Creativity and Innovation: Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. 3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-

		made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they perform algebraic manipulations on polynomial functions and graph polynomial functions.

1.1.2.LO.7		
Find the domain, range, and zero of a rational function and state when it is undefined.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: 1. Respect individuals of different backgrounds in their groups as they find the domain, range, zero of a rational function, and state when it is undefined using appropriate technological tools. 2. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to find the domain, range ero of rational function and state when it is undefined. 3. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. 4. Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship*

	Creativity and Innovation: Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	*Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. 3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity

		presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they find domain, range, and zero of a rational function and state when it is undefined.
1.1.2.LO.8		
Identify and describe the order of a matrix, the identity matrix and the	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant	**GESI:** Learners having experienced a teaching approach that ensures gender

zero matrix; find the determinant and perform basic arithmetic operations on 2 by 2 matrices (addition and subtraction).	of the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set	equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: 1. Respect individuals of different backgrounds in their groups as they identify and describe the order of a Matrix, the identity Matrix and the zero matrix, find the determinant and perform basic arithmetic operations on 2x2 matrices using appropriate technological tools. 2. Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn to solve problems involving matrices. 3. Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. 4. Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are

	aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Allow learners to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level of life application.	integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. 3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities.

		National Core Values: **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they identify and describe the order of a matrix, the identity matrix and zero matrix, find the determinant and perform basic arithmetic operations on 2 by 2 matrices.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI	Assessment
1.1.2.CS.1	1.1.2.LI.1	1.1.2.AS.1
Demonstrate knowledge and understanding of applying algebraic processes and reasoning involving sequence, functions, and linear programming.	**Recognise sequences in mathematics as an enumerated collection of objects in which repetitions are allowed and classify sequences into linear or exponential.** **Collaborative Learning:** Learners will work in convenient groups (ability, mixed-ability, mixed gender etc.) to solve problems and present answers related to sequence and series. **Talk for Learning Approaches:** Learners will brainstorm through think-pair-square and debate and discuss the types and features of sequences. **Project-based Learning:** Learners will collaboratively embark on a project to develop a sequence that is relevant and applicable to real life and make presentations. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create sequences using objects, symbols and numerals and discuss pattern rules. **Problem-based Learning:** Learners will collaboratively undertake a research inquiry into what happens when terms of sequence are summed up. **Activity 1: Sequence in real life situations** Discuss sequences in the natural environment and create sequences using objects, symbols, and numerals. **Talk for Learning*:*** Initiate TfL approaches for learners to brainstorm, debate, and discuss reasons for learning about sequence: **Example:** Some important facts about sequence and series: • The knowledge of sequence and series helps to shape our logical skills.	**Level 1: Recall** Level 2: Skills of conceptual understanding Level 3: Strategic reasoning Level 4: Extended critical thinking and reasoning

- Sometimes, we want to find out all the possible things out there because we want to know the most probable terms to satisfy our curious minds. Sequence and series help to settle this curiosity in us.
- Knowledge equips us with skills and strategies to find the sum of things going on even up to infinity, which is useful in scientific thesis.

Note: *A Sequence is any set of objects, often numbers, that follow a particular pattern infinitely. A series refers to the description of the operation that would add all of the items in a sequence.*

Collaborative and Problem-based*:* Learners work in small mixed-ability groups to investigate, identify and discuss real life patterns in their environment and talk about the pattern rule. For instance, have learners act out and talk about:

- Stacking cups, chairs, bowls etc., to compare the number of objects to the height of the object.

- Seating around tables: Consider a restaurant with a square table that fits 4 people. What happens when two square tables are put together, now 6 people are seated. Put 3 square tables together, and now 8 people are seated, etc. Learners may visualise this better if they are allowed to model this sequence.

Note: A rectangular table can also be used to model the sequence, but it starts with 6 seats.

- Filling something: Consider the rate at which the object is being filled, using time as the variable. An example could be a sink being filled or a pool being filled. Also, consider draining a liquid, sand, sugar, etc., out of a container with a small hole beneath.

- Fencing and perimeter: Discuss how adding a fence panel to each side of a rectangular fence would change the perimeter. For example, the first figure concept could have one panel on each side (or change it so it is not a square). The next figure concept could have two panels on each side and so on. Each time, find the new perimeter.

- Building pyramid-like patterns, where objects are constantly increasing or decreasing, such as in seats in a stadium or an auditorium. For example, a situation might be that seats in each row decrease by 4 from the previous row.

- Negative number patterns: Think about temperature and places below sea level on Earth. Create situations like, during rainfall, the surface of the water started at 153 feet below sea level and rose at a rate of *such and such* per hour or diving in the ocean.

Activity 2: Modelling instances of sequences

Collaborative, Problem-based and Experiential Learning: Learners work in small mixed-ability groups and model sequences (numerical and non-numerical - using colours, cut-out shapes, objects, fabrics, sound, actions, etc.) and talk about the pattern rule. For instance,

- Learners work in smaller groups/pairs to produce/create a sequence and record their findings in tables,
- Learners in one group create a sequence and then challenge another/other groups to:

  - continue or extend the pattern to include the next term;
  - describe the pattern rule and record them in tables;
  - write a rule that will allow you to find any large shape form.

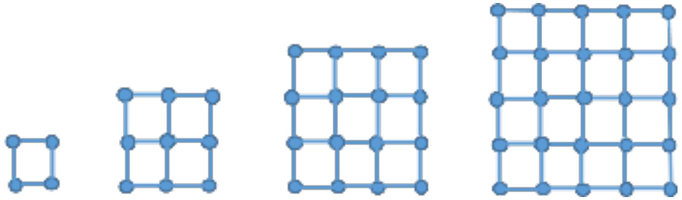
Example 1: Study the sequence, extend it to complete the table below and answer the questions that follow.

Pattern no.	1	2	3	4	5	6	…
No. of bottle caps	3	6	10				

- Describe the pattern rule.
- Write a rule which will allow you to find the number of small pebbles in any large pebbles.

Example 2: Study the pattern, draw a picture of the next likely shapes and complete the table. Write a pattern rule which will allow you to find:

- the number of squares
- the number of joints in any large shape.



Pattern no.	1	2	3	4	5	6	… …
No. of squares	1	4	9				
No. of joints	4	9	16				

	1.1.2.LI.2	1.1.2.AS.2
	Find the nth term of linear and exponential sequences. **Collaborative Learning:** Learners will work in convenient groups (ability, mixed-ability, mixed gender etc.) to solve problems and present answers. **Talk for Learning Approaches:** Learners will be engaged through think-pair-share in brainstorming, build on what others say to justify and provide feedback on sequences created. **Project-based Learning:** Engages learners to come up with a sequence that is relevant and applicable to real life situations. **Experiential Learning**: Engages learners in hands-on activities (learning by doing). **Problem-based Learning:** Learners inquire into what happens when terms of sequences are summed up to infinity. **Activity 1: Use rules to find a specified term of the sequence** **Collaborative Learning:** Have groups work together to recall and explain sequences and their features. Learners investigate linear and exponential: Learners work in pairs or groups to create at least two different numeric and/or non-numeric sequences (creators must know the patterns that define the sequences). Swap between groups and determine the pattern that defines each of the other sequences and give reasons for the pattern form. Pairs or groups work with a sequence of numbers to establish the general rule for arithmetic and geometric sequence, using the first terms and relationship between consecutive terms (simple recursions) and the notations of sequences, and determine if a sequence is arithmetic or geometric and identify key terms of the sequence such as the first, the common difference and/or ratios, and the ***n***th-term.	**Level 2 Skills of conceptual understanding** **1.** Find the $10th$ term of the arithmetic sequence: $2, 5, 8, 11, \ldots$ **2.** Given the arithmetic sequence: $3, 7, 11, 15, \ldots$ Find an expression for the nth term of this sequence. **3.** For the geometric sequence: $4, 12, 36, 108, \ldots$ Find the formula for the nth term of the sequence. **Level 4 Extended critical thinking and reasoning** The set of whole numbers is partitioned into subsets, with the first number in the first subset, the next two numbers in the second subset, and the next three numbers in the

Note:

- In an arithmetic sequence, the difference between one term and the next term is a constant (e.g., $1, 4, 7, 10, 13, 16, 19, \dots$) It has the general rule of $Un = a + d(n-1)$ where ***a*** is the first term, ***d*** is the common difference, and ***n*** is the number of terms.
- For geometric sequence, each term is found by multiplying the previous term by a constant (e.g. 2, 4, 8, 16, 32, 64, ...) it has a general rule, $Un = ar^{(n-1)}$ where $\boldsymbol{a}$ is the first term, $\boldsymbol{r}$**,** the common ratio and *n* is the number of terms in the sequence.

Activity 2: Using a given rule or formula (input-output tables)
Learners determine the first few elements of the sequence given the algebraic relationship.

Example: Complete the table for the sequence given by $u_n = \frac{1}{3}(2)^{n-1}\ ;\ n > 0$

n	1	2	3	4	5	6
$u_n = \frac{1}{3}(2)^{n-1}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$	$\frac{32}{3}$

Alternatively, learners will be required to determine the rule for values given in a table.

Activity 3: Learners use other conventional strategies to find the *n*th term of a sequence.

Activity 4: Learners work in groups to describe linear and exponential sequences and use their understanding to solve real life problems.

Example: The second term of an A.P. is 15, and the fifth term is 21. Find the common difference and the sum of the first 5 terms.

Solution:
$15 = a + d$ ………. (1) and
$21 = a + 4d$ ……… (2)
Solving simultaneously, we have $d = 2$:

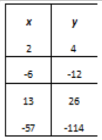
third subset and so on. Find in terms of *n* a formula for the first member of the nth subset.
NB*: Start with a number of whole number(s) in the first partition, second partition, etc.*

	The sum of the first five terms is 85 More examples: • A man starts savings on September 1st. He saves 10 pesewas the first day, 20 pesewas the second day, 40 pesewas the third, and so on, doubling the amount every day. How much would he save if he managed to keep saving under this system until the end of the month (September 30th)? Leave your answer in cedis correct to 2 decimal places. • The set of whole numbers is partitioned into subsets, with the first number in the first subset, the next two numbers in the second subset, and the next three numbers in the third subset and so on. Find, in terms of n, a formula for the last member of the n^{th} subset. **Note:** *Start with the number of whole number(s) in the first partition, second partition, etc.*	
	1.1.2.LI.3	1.1.2.AS.3
	Identify Relations and Functions and describe their differences. **Collaborative Learning:** Learners will be working in convenient groups (ability, mixed-ability, mixed gender etc.) to solve problems and present answers related to functions and relations. **Talk for Learning Approaches:** Learners will brainstorm through think-pair-square and debate and discuss the types and features of functions and relations. **Experiential Learning**: Learners will collaboratively engage in hands-on activity (learning by doing) to create functions. **Problem-based Learning:** Learners will collaboratively undertake research inquiry functions in real life. **Activity 1: Recognising relations and functions in mathematics.** **Talk for Learning:** Learners discuss the historical antecedent of functions, tracing it from Rene Descartes in 1637 through Gottfried Wilhelm Leibniz, Leonhard Euler, Nicolas Bourbáki, John Tate	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

etc. Learners discuss the importance of functions as the building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies, keeping aeroplanes in the air, etc.

Learners work in groups to answer the question, "What is your understanding of a function?" Each group should agree upon and present one meaning on a poster and discuss the concept of function in terms of a function's definition, type, and general properties.

Example 1: Learners discuss within groups which of the relations listed below are examples of functions and justify their answers to the class.



x	y
2	4
-6	-12
13	26
-57	-114

(i)

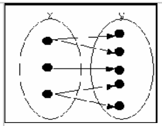
$y = -x^2$

(ii)

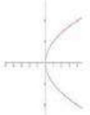
(5, 6)

(3, 2)

(iii)



(iv)



(v)

Example 2: Groups investigate the following problems and make presentations to the class justifying which can be classified as functions.

Let A = (a, b, c}, B = {4, 5, 6}, and f = {(a, 6), (b, 4), (c, 6)}. Is f a function from A to B? Why?

Let A = {1, 2, 3}, B = {c, d, e}, and g = {(1, d), (2, c), (1, e)}. Is g a function from A to B? Why?

Let M be the set of all museums, N the set of all countries, and $L = \{(m, n) \in M \times N$ the museum m is in the country n}. Is L a function from M to N?

Learners reflect on their informal work in trying to describe what a function is and formalize the definition of function, taking notice of the use of the symbols in the definition.

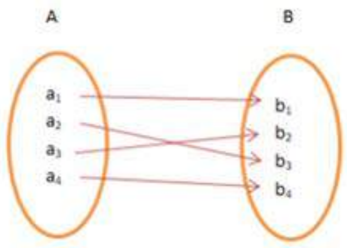
Note: Three examples of how a function might be defined are:

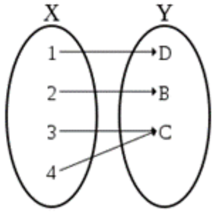
Given two sets A and B, the set $A \times B$ consists of all ordered pairs (a, b) where $a \in A, b \in B$. A subset of $A \times B$ is called a relation. Thus:

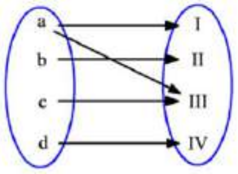
- A function from A to B is a pairing of elements in A with elements in B in such a way that each element in A is paired with exactly one element in B.
- A function f from A to B is a rule or relation between A and B that assigns each element $a \in A$ to a unique element $b \in B$.
- A function f from A to B is a subset of the Cartesian product $A \times B = \{(a, b)\ |a \quad A, b \quad B\}$ such that b is unique for each $a \in A$}.

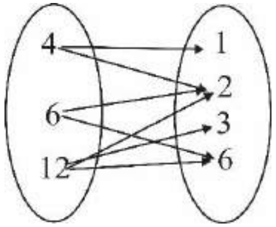
Activity 2: Use of mapping diagrams to establish various types of relations and establish connections between relations and functions.

Learners working in convenient groups observe different graphical representations and explanations for relations and draw conclusions.



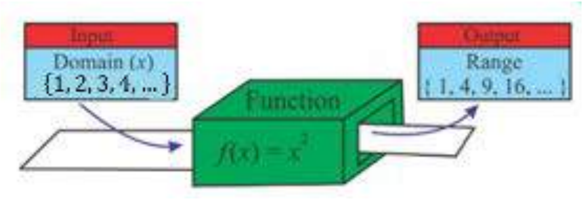




	 **Conclusion:** All functions are relations, BUT not all relations are functions. A function is a type of relation that assigns ONE output to each input. **Problem-based/Research Work:** Have learners investigate the various types of functions (absolute functions, identity function, quadratic, cubic functions etc.) study the nature of their graphs and make presentations on them.	
	1.1.2.LI.4	1.1.2.AS.4
	Use function notation to evaluate functions for inputs in their domain and outputs in the co-domain. **Collaborative Learning:** Learners will be working in convenient groups (ability, mixed-ability, mixed gender etc.) to solve problems and present answers related to functions and relations. **Talk for Learning Approaches:** Learners will brainstorm through Think-Pair-Square and debate and discuss the types and features of functions and relations. **Experiential Learning:** Learners will collaboratively engage in hands-on activity (learning by doing) to create functions. **Problem-based Learning:** Learners will collaboratively inquire into the domain and range of functions in real life. **Activity 1: Determine the domain and range of functions.**	**Level 1 Recall** **Level 2** Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Collaborative Learning, Talk for Learning, Problem-based Learning: Learners work in convenient groups (mixed gender, ability mixed-ability, or pairs) and discuss the domain and range of a function between *A* and *B* as a nonempty relation $f \subseteq A \times B$ such that if $(a, b) \in f$ and $(a, b') \in f$, then b = b′. The domain of 'f 'is the set of all first elements of members of f, and the range of f is the set of all second elements of members of f. Symbolically:

Domain
$f = \{a : \exists b \in B ; (a, b) \in f\}$ Range
$f = \{b : \exists a \in A ; (a, b) \in f\}$
For instance, for the function $f(x) = x^2$,
the values $x = 1,2,3,4, \ldots$ are the inputs and the $f(x) = 1,4,9,16, \ldots$ are the output values:



The set "Out-put values" is referred to as the co-domain of f. If it happens that the domain of f is equal to all of A, then we say f is a function from A into B, and we write $f : A \rightarrow B$.

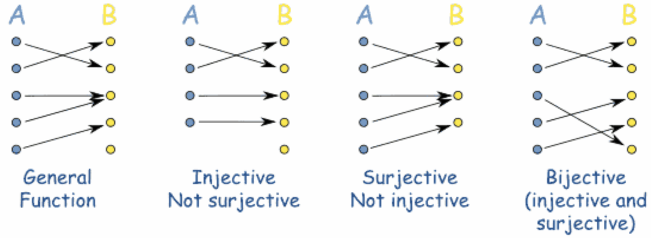
Problem-based Learning: Learners in their various groups investigate the domain of functions such as:

$f = \sqrt{2x - 8}$ and then share findings with the class.

- Learners evaluate functions for output values by substituting giving input values. e.g., Given $f(x) = 2 - 4x$ find $f(-3)$

Note: Learners recognise that evaluating a polynomial function is just a means of substituting whatever is in the bracket of the f of argument for the variable in the function equation.

	Example: Evaluate $f(2)$ for $f(x) = 2x^2 + 2x - 15$ **Solution:** $f(3) = 2(3)^2 + 2(3) - 5$ = $18 + 6 - 15 = 9$	
	1.1.2.LI.5	1.1.2.AS.5
	Establish, describe and determine bijective functions and composite functions. **Collaborative Learning:** Learners will be working in convenient groups (ability, mixed-ability, mixed gender etc.) to solve problems and present answers related to functions. **Talk for Learning Approaches:** Learners will brainstorm through think-pair-square and debate and discuss the bijective functions. **Experiential Learning:** Learners will collaboratively engage in hands-on activity (learning by doing) to create bijective functions. **Problem-based Learning:** Learners will collaboratively investigate one-to-one, onto and invertible functions inquiry functions. **Activity 1: Injective, surjective and bijective functions:** **Collaborative Learning, Talk for Learning, Experiential Learning and Problem-Based Learning**: Learners work in convenient groups (mixed gender, ability mixed-ability or pairs) and investigate the fact that, given two sets A and B the following definitions hold. • **Definition 1:** A function $f: A \rightarrow B$ is called surjective (or is said to map *A* onto *B*) if *B* = range f. • **Definition 2:** A function $f: A \rightarrow B$ is called injective (or one-to-one) if, for all a and a′ in A, $f(a) = f(a')$ implies that $a = a'$. • **Definition 3:** A function $f: A \rightarrow B$ is called bijective if it is both surjective and injective.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**



Collaborative learning, Experiential Learning and Problem-Based Learning: Learners show whether or not a given function is one-to-one or onto or both using mappings, diagrams and algebraic proofs.

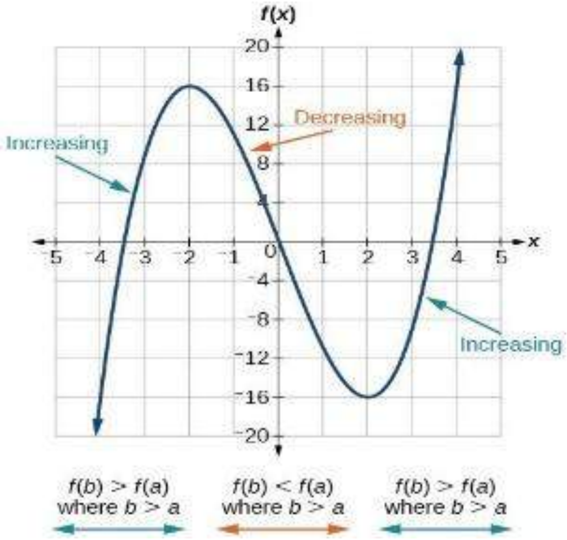
Note: If a function is both surjective and injective, then it is said to be particularly "well behaved".

Activity 2: Investigate increasing and decreasing functions.

Collaborative Learning, Talk for Learning, Experiential Learning and Problem-Based Learning: Learners work in groups/pairs to investigate increasing and decreasing functions by verifying graphically and algebraically:

- A function f is strictly increasing if $f(x) > f(y)$ when $x > y$.
- A function f is strictly decreasing if $f(x) < f(y)$ when $x < y$.
- A function f is increasing if $f(x) \geq f(y)$ when $x>y$.
- A function f is decreasing if $f(x) \leq f(y)$ when $x<y$.

Graphically,

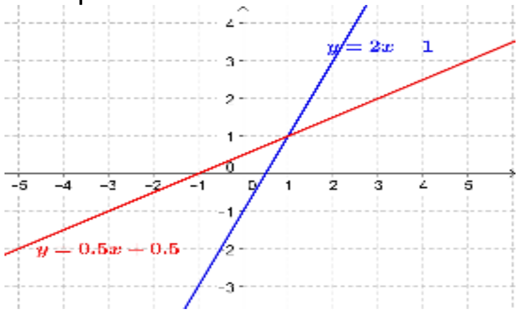


Some conclusions:

- A function is one-to-one if it is either strictly increasing or strictly decreasing.
- A one-to-one function never assigns the same value to two different domain elements.
- For onto functions, range and co-domain are equal.

If a function f is not bijective, the inverse function of f cannot be.

	1.1.2.LI.6	1.1.2.AS.6
	Find the inverse of simple functions. **Collaborative Learning, Experiential Learning and Problem-based Learning:** Learners Recognise bijective functions and determine the inverse of such functions (learners must find the inverse function by using graphic illustrations and through algebraic means). **Activity 1: Inverse of a function** **Note:** A bijection function is also known as an invertible function because they have an inverse function property. The inverse of bijection f denoted as $f^{-1}$is a function which assigns to b, a unique element a, such that $f(a) = b$. Hence $f^{-1}(b) = a$ **Example 1:** Find the inverse of $f(x) = 2x - 1$ By definition, the inverse of a function is that function that takes values of the dependent variable (in this example, y) as arguments and yield values for the independent variable (in this case, x). Based on this definition, we would seek a function with x as the subject and y as the independent variable If we let $f(x) = y$, i.e., $y = 2x - 1$, we can just make x the subject thus: $x = \frac{y+1}{2}$ We need a function in terms of x hence we write it as $f^{-1}(x) = \frac{x+1}{2}$ Alternatively, we could switch the x values and the y values of the given function, then solve for the new "y." Next, replace the new "y" with $f^{-1}(x)$. **Solution:** Let $f(x) = y = 2x - 1$ $x = 2y - 1$ $\Rightarrow 2y = x + 1$ $y = \frac{x+1}{2}$ $\therefore\ f^{-1}(x) = \frac{x+1}{2}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

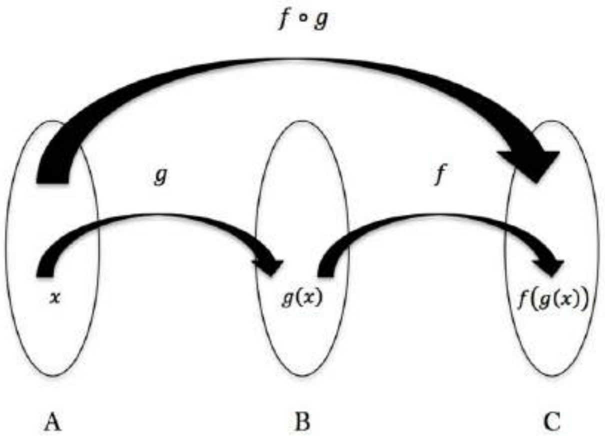
	Activity 1: Investigate the graphs of a function and its inverse **Collaborative Learning, Experiential Learning and Problem-based Learning:** Learners determine the domain and range of inverse functions presented in graphical and algebraic forms. The inverse of a function is visually represented as the original function reflected over the line $y = x$ and passes the vertical line test and the horizontal line test.  The graph shows a function $f(x) = 2x - 1$ and it inverse $f^{-1}(x) = \frac{x+1}{2}$	
	1.1.2.LI.7	1.1.2.AS.7
	Determine the composite of two given functions. **Activity 1:** Investigate common properties of function. **Collaborative Learning:** Learners work in groups/pairs to investigate addition and multiplication properties of function guided to discover that: • Addition: If f_1 and f_2 are two functions from A to B, then $f_1 + f_2$ is defined as:$[f_1 + f_2](x) = f_1(x) + f_2(x)$. • Multiplication: if f_1 and f_2 are any two functions from A to B, then $f_1 \cdot f_2$ is defined as $[f_1 \cdot f_2]\,(x) = f1(x)\,f2(x)$. • Commutativity of functions addition and/or multiplication.	**Level 3 Strategic reasoning** Consider the functions $p(x) = x^3$ and $q(x) = \sqrt{x}$. Find the composite function $(q \circ p)(x)$

- Function Equality: Two functions are equal only when they have the same domain, the same co-domain and same mapping elements from domain to co-domain.

Activity 2: Composite of two functions.

- **Collaborative Learning**: Learners work in groups/pairs and construct two or more functions $[h(x)\ and\ f(x)]$, including scenarios that are relevant and applicable to real life. Then, have them swap with a different pair. The new pair should determine the composite functions. Then, they should swap back and make sure that they were accurate according to the creators)

- **Collaborative Learning, Experiential Learning:** Learners recognise that if g is a function from A to B and f a function from B to C, then the composition of f and g, which is denoted as $f \circ g$ is the function from A to B



Activity 2: I*nvestigate p*roperties of composite functions.

Collaborative Learning: Learners work in groups/pairs to investigate some common properties of composite functions. Properties include:

- $f \circ g \neq g \circ f, except\ g(x) = f(x)$
- $f^{-1} \circ f = f^{-1}(f(a)) = f^{-1}(b) = a.$
- $f \circ f^{-1} = f(f^{-1}(b)) = f(a) = b.$
- If f and g both are one-to-one functions, then fog is also one-to-one.
- If f and g both are onto function, then fog is also onto.
- If f and $f \circ g$ both are one-to-one functions, then g is also one-to-one.
- If f and $f \circ g$ are onto, then it is not necessary that g is also onto.
- $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

1.1.2.LI.8

Recognise and use appropriate algebraic notation properties of linear and non-linear functions and solve them simultaneously.

Activity 1: Investigate situations that lead to linear and non-linear equations.

Collaborative Learning: Present several functions/equations (linear and non-linear equations) in algebraic and graphical forms and have learners investigate and classify them as linear or non-linear and justify their groupings.

Examples: Differences between linear and non-linear equations.

Linear Equations	**Non-Linear Equations**
forms a straight line or represents the equation for the straight line	It does not form a straight line but forms a curve.
It has only one degree.	It has the degree as 2 or more than 2
Lines formed in the X-Y plane can be extended in any direction but in a straight form.	If we increase the value of the degree, the curvature of the graph increases.
The general representation is; y = mx +c	The general representation is; $ax^2 + by^2 = c$

1.1.2.AS.8

Level 1 Recall

Level 2 Skills of conceptual understanding

Level 3 Strategic reasoning

Level 4 Extended critical thinking and reasoning

Where x and y are the variables, m is the slope of the line, and c is a constant value.	Where x and y are the variables and a, b and c are the constant values
Examples: a) $10x = 1$ b) $9y + x + 2 = 0$ c) $4y = 3x$ d) $99x + 12 = 23y$	**Examples:** a) $x^2+y^2= 1$ b) $x^2+12xy+y^2= 0$ c) $x^2+x+2=25$

Activity 2: Solving linear and non-linear equations.

Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning:

Collaborative Learning: Learners work in convenient groups to solve systems of linear and non-linear equations.

Example 1: Solve the linear equation $3x + 9 = 2x + 18$.

Solution: Given, $3x + 9 = 2x + 18 \Rightarrow 3x - 2x = 18 - 9 \quad \Rightarrow x = 9$

Example 2: Solve the nonlinear equation $x + 2y = 1$ and $x = y$.

Solution: Given, $x + 2y = 1; \; x = y$

By putting the value of x in the first equation we get,
$\Rightarrow y + 2y = 1 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3} \;\therefore\; x = y = \frac{1}{3}$

Example 3: Solve the nonlinear equation $x^2 + x = 2$ and $10x = 1$

1.1.2.LI.9

Recognise linear equations in two variables and draw their graphs manually using appropriate technology. Find the area enclosed by the two graphs.

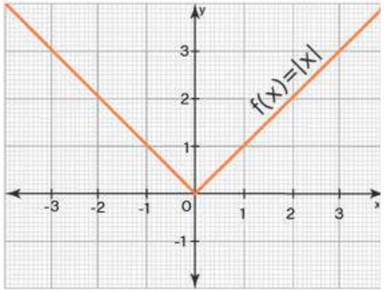
Activity 2: Identifying linear equations and determining areas enclosed by graphs

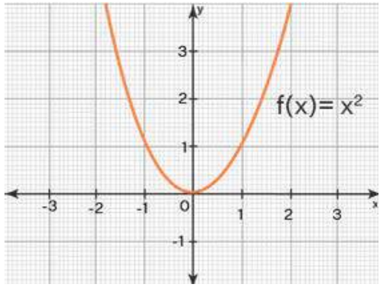
Collaborative Learning, Experiential Learning, and Talk for Learning Approaches:

- **Collaborative Learning:** Learners work in pairs to identify linear and non-linear graphs. With the aid of technological tools by graph paper, learners draw graphs of linear functions and provide reasons why such graphs are linear or curvilinear.
- NB: A linear function is of the form $f(x) = ax + b$. An example of a linear function is $(x) = ax + b$.
- Any function not in $(x) = ax + b$ is a nonlinear function, e.g. $f(x) = x^2$ is nonlinear and is called a quadratic function.

Example:

Which of the following are linear, and which are non-linear?





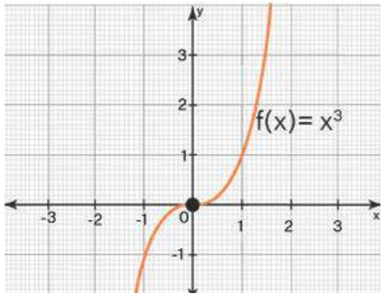
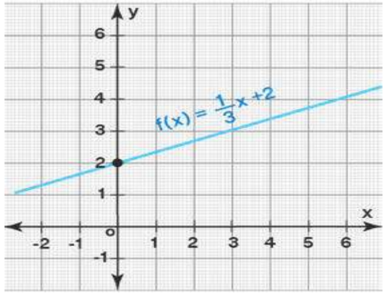
1.1.2.AS.9

Level 1 Recall

Level 2 Skills of conceptual understanding

Level 3 Strategic reasoning

Level 4 Extended critical thinking and reasoning

	 	
	1.1.2.LI.10	1.1.2.AS.10
	Solve up to three systems of linear equations simultaneously by algebraic manipulations. **Activity 2:** Solving systems of linear equations**.** **Collaborative Learning, Experiential Learning, Whole class discussion and Talk for Learning Approaches:** • **Collaborative Learning**: Learners work in groups/pairs with conventional and personal strategies to solve up to three systems of linear equations involving three unknowns. • Learners use the elimination and substitution of other appropriate algebraic methods to solve systems of linear equations. **Example 1:** solve the equations: $2x + y - z = 5$; $4y + 3z = 5$; $y + 2x = 7$ **Example 2:** solve the equations $2x + y = 5$ and $x - y = 3$	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**
	1.1.2.LI.11	1.1.2.AS.11
	Recognise and model statements into linear mathematical equations and solve them simultaneously. **Activity 1:** Model and solve word problems involving a system of linear equations.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning**

	Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning: **Collaborative Learning**: Learners work in groups/pairs to solve real life problems involving simultaneous linear equations. **Example:** The cost of entry to a community zoo was $GH₵\ 100.00$ for 12 children and 3 adults during weekends. The entry fee was $GH₵\ 80.00$ for 8 children and 3adults during public holidays. How much was the entry fee for each child and adult? **Solution** Let x represent the entry cost for each child. Let y represent the entry cost for each adult. That is, $12x + 3y = 100$ $8x + 3y = 80.$ Therefore, the cost of entry for each child is $GH₵\ 5.00$, and for each adult is $GH₵\ 13.33$	**Level 4 Extended critical thinking and reasoning**
	1.1.2.LI.12	1.1.2.AS.12
	Describe polynomial functions and perform the basic arithmetic operations on them. **Activity 1:** Evaluate a polynomial in function notation. **Collaborative Learning**: Learners work in groups/pairs to recall key facts about linear and non-linear functions. • Learners examine different forms of functions, classify them and explain their reasons for the classification. • Learners recognise that polynomial functions comprise various combinations of constants, variables, and exponents and are of the form: • $P(x) = ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots..+k$ where $a, b, c\ and\ d$ are constant and $n \geq 2$	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Example: $g(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$

Note: Examples of types of polynomial functions are linear, quadratic, cubic, quartic and quintic functions. Learners explore the terminology of polynomial functions, including working with vocabularies such as co-efficient, leading co-efficient, term leading term and degree exponents, and constants and then analyse them by putting them all together;

Activity 2: Perform basic arithmetic operations on polynomial functions.

Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based learning:

Collaborative Learning: Learners work in groups/pairs to explore arithmetic operations (addition, subtraction, multiplication and division) on polynomial functions.

Example 1: If $f(x)$ = $4x^4 - x^3 - 9x^2 + 2x - 5$ and $h(x) = x^3 + 3x^2 - 2x + 2$;
- Addition: $f(x)$+ $h(x) = 4x^4 - 6x^2 - 3$
- Subtraction:$f(x) - h(x) = 4x^4 - 2x^3 - 12x^2 + 4x - 7$

Example 2: If $f(x)$ = $x^2 + x - 6 \;\; g(x) = x + 3$
- Multiplication: $f(x)\times g(x) = x^3 + 4x^2 - 3x - 18$
- Division: $\frac{f(x)}{g(x)} = \frac{x^2+x-6}{x+3} = x - 2$

Generally: $\frac{f(x)}{g(x)} = q(x) + Remainder$

Activity 3: Examine the graph of polynomial functions.

	Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning: **Collaborative Learning**: Learners work in convenient groups to identify, examine and construct graphs of polynomial functions up to degree 2 and describe their nature. **Collaborative Learning, Experiential Learning, and Problem-based Learning:** Learners construct functions by hand and with an appropriate technology and investigate the minimum and maximum values and the intercepts. Learners work in groups/pairs to come out with two or more polynomial functions with degrees up to 5. Then, have them swap with a different group/pair. The new group/pair should evaluate the functions at a given value. Then, they should swap back and ensure they were accurate according to the creators. (Learners debate with each other to agree or disagree with the outcome of the task)	
	1.1.2.LI.13	1.1.2.AS.13
	Use the method of completing the square to transform any quadratic equation that has the same solutions and explain how the quadratic formula is derived from this form. **Activity 1*:*** The quadratic formula and its usage. **Collaborative Learning, Experiential Learning, Whole class discussion, and Problem-based Learning:** **Collaborative Learning:** • Learners work in convenient groups/pairs to recall key facts about the quadratic formula $ax^2 + bx + c = 0$. • Learners work in groups/pairs to establish that the general quadratic solution for the quadratic equation $ax^2 + bx + c = 0$ is given as: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **Example:** Find the solution to the equation $2x^2 - x - 4 = 0$ correct to 2 decimal place.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Solution: Comparing $ax^2 + bx + c = 0$ to $2x^2 - x - 4 = 0$

$$a = 2\ b = -1\ and\ c = -4$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2.-4)}}{2(2)}$$

$$x = 1.69\ \ or\ x = -1.19$$

Collaborative Learning, Experiential Learning, Whole class discussion, and Problem-based Learning:

Collaborative Learning:

- Learners work in convenient groups/pairs to recall key facts about the quadratic formula $ax^2 + bx + c = 0$.
- Learners use the method of completing the squares to write the general quadratic functions $ax^2 + bx + c = 0$ in the form
- $A(x \pm B)^2 \pm C = 0$ or $A(x \pm B)^2 = C,$ where A, B and C are constants, and C is the maximum or minimum value of the function, and it occurs at $x \pm B = 0$ or $x \pm B$

Example: Express $3x^2 - 5x - 2 = 0$ in the form $A(x \pm B)^2 = C$ where A, B and C are constants and state the minimum values of the function

Solution: $3x^2 - 5x - 2 = 0 \Rightarrow 3\left(x - \frac{5}{6}\right)^2 - \frac{49}{12} = 0$

The minimum value of the function is $-\frac{49}{12}$

Collaborative Learning, Experiential Learning, Whole class discussion, and Problem-based Learning:

Problem-based learning:

Learners use the discriminant of a quadratic function to establish and determine the nature of the root, solution or zeros of the function. Learners recognise that the expression. $b^2 - 4ac$ of the general quadratic equation $ax^2 + bx + c = 0$ determines the nature of the roots of the function, and the following properties hold:

- If $b^2 - 4ac \geq 0$, the roots are real.

- If $b^2 - 4ac > 0$, the roots are real and different.
- If $b^2 - 4ac < 0$, the roots are complex and imaginary.
- If $b^2 - 4ac$ = 0, the roots are real and equal, or the equation is a perfect square.

Example: Determine the nature of the roots of the function $a^2 - 6a + 4 = 0$

Solution: $b^2 - 4ac = (-6)^2 - 4(1)(4) = 36 - 16 = 20;$
since $b^2 - 4ac > 0$, the roots are real and different.

Activity 1: Use the roots (sum and products) to find other quadratic equations.

Collaborative Learning: Learners work in convenient groups/pairs to explore the relationship between the constants $a, b\ and\ c$ of the general quadratic function $ax^2 + bx + c = 0$ and $\alpha\ and\ \beta$, and use these relations to write other quadratic equations with given roots.

Example 1: Form an equation whose roots are m and 2.

Solution:
Sum of roots = $(m + 2)$
Products of roots = $(2m)$
Equation:

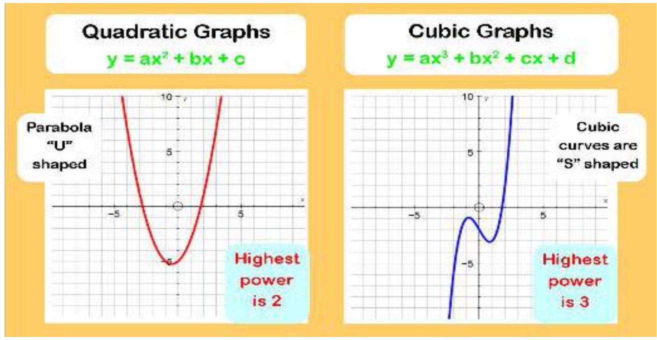
$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$
$$x^2 - (m + 2)x + (2m) = 0$$

Example 2: If $\alpha\ and\ \beta$ are the roots of the equation $3x^2 - x - 3 = 0$, find the value of $\alpha^2 - \beta^2$, if $\alpha > \beta$,

Solution:
The sum of roots $(\alpha + \beta)$=$\frac{1}{3}$
Product of roots $(\alpha.\beta) = -1$

	$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = \frac{1}{9}\sqrt{37}$	
	1.1.2.LI.14	1.1.2.AS.14
	Use the remainder and factor theorems to find the factors and remainders of a polynomial of degree not greater than 4. **Activity 1:** Use the factor and remainder theorems to solve problems of polynomial functions in context. **Collaborative Learning**: • Learners work in groups/pairs to recall key facts about polynomial functions. • Learners explore and recognise that generally, for any polynomial function $P(x)$ division by another polynomial results in: $\frac{Dividend}{Divisor} = quotient + \frac{Remainder}{Divisor}$ and so; $Dividend = Quotient \times Divisor + Remainder$ Learners recognise that for any polynomial function $P(x)$, if $(x - a)$ is a divisor, then, $P(x) = q(x).(x - a) + r;$ where $q(x)$ is the quotient and r is the remainder. Learners establish the Remainder theorem and use it to solve related problems. ***Remainder Theorem*:** Given any polynomial functions $P(x)$, if $(x - a)$ is a divisor, then, $P(a) = q(a).(a - a) + r;$ and $P(a) = r$ **Activity 1:** Apply the remainder theorem to solve related problems. **Example:** find the remainder when the polynomials $f(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$ is divided by $(x - 2)$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Learners investigate and establish the Factor Theorem and use it to solve related problems. **Factor Theorem:** If $(x - a)$ is a factor of a polynomial $P(x)$, then $P(a) = 0$. **Activity 1:** Apply the Factor theorem to solve related problems**.** **Example:** When $P(x) = kx^3 - 3x^2 + 8x + 36$ is divided by $(x + 2)$; it leaves no remainder. • Find the value of k express $P(x)$= $(x + 2)(ax^2 + bx + c)$, where $a, b\ and\ c$ are constants, and find the zeros.	
	1.1.2.LI.15	1.1.2.AS.15
	Draw the graph of a polynomial function with degrees up to 3 by hand and by using technology (e.g., GeoGebra, Demos, PhET Simulations, and Geometer's Sketch Pad) where appropriate. **Activity 1:** Use graphs of polynomial functions to solve problems. **Collaborative Learning**: • Learners work in groups, pairs, and individually to draw graphs of polynomials up to degree 3 by hand and by leveraging technology. • Learners review features of graphs of linear functions and quadratic functions. • Learners draw quadratic and cubic polynomial functions by hand and with the aid of appropriate technology and discuss what happens to the curve as $x - values$ decreases or increases, the point where the curve touches the $y - axis$, axis of symmetry and turning points etc. • Learners (working in groups/pairs and independently) use curves of polynomial functions to find the truth set, minimum and maximum turning points, increasing and decreasing intervals and equations of symmetry.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	 **Example:** Draw the graph of the relation $f(x) \ = x^2 - 4x + 5$ for the interval $-2 \ \leq x \leq 6$. On the same axis, draw the graph of $y = x + 3$. Use your graphs to find: • The equation of the axis of symmetry of the graph $f(x) \ = x^2 - 4x + 5$. • The truth set of $x^2 - 5x + 2 = 0$	
	1.1.2.LI.16	1.1.2.AS.16
	Recognise a rational function and determine the domain and range. **Activity 1:** Domain and zeros of rational function. **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning:** • Learners work in groups/pairs to recall key facts about polynomial functions and recognise that rational function. • Learners examine different forms of functions and classify rational functions as functions of the form, $R(x) \ = \frac{f(x)}{g(x)}; g(x) \neq 0$ (i.e. a ratio of two polynomials for which the divisor is not zero). • Learners examine and explore graphically and by algebraic means the set of values for which a given rational function will exist (domain of the function).	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Example 1: The domain of the function $y = \frac{1}{x-2}$ is the set of all real numbers except $x = 2$ i.e., the domain of y is $\{x: x \in R, x \neq 2\}$ **Example 2:** State the largest possible domain of the function defined by $f: x \to \frac{2x-1}{2x^2-9x-5}$ Learners examine and explore graphically and by algebraic means the values of x of a function for which the function is zero. **Example 1**: The zeros of the function $f: x \to \frac{x+3}{x-2}$ is $x = -3$ - **Example 2:** State the zeros of the function defined by $f: x \to \frac{2x-1}{2x^2-9x-5}$	
	1.1.2.LI.17	1.1.2.AS.17
	Carry out the basic arithmetic operations on rational functions. **Activity 1:** Basic arithmetic operations on rational numbers. **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning:** - Learners work in groups/pairs to perform basic operations on rational functions. - Learners explore factorisation and expansion of denominators and numerators to simplify rational functions. **Example:** Simplify $\frac{2x+6y}{x^2+4xy+3y^2}$ Learners express rational functions into the same denominator to add and subtract. **Example:** Simplify $\frac{2x-1}{x-2} - \frac{x-3}{x+3}$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	1.1.2.LI.18	1.1.2.AS.18
	Apply partial fraction decomposition up to factors with exponents and irreducible quadratic factors. **Activity 1:** Partial Fractions. **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning:** - Learners work in groups/pairs to split rational functions into partial fractions. - Learners apply knowledge of equivalence to determine values of constants of functions. **Example:** Find the values of the constants A, B and C such that $x^2 - 4x + 5 = A(x-1)(x+2) + B(x+2)(x-4) + C(x-4)(x-1)$ Learners resolve rational functions into partial fractions in which: - The denominator is made up of linear factors: $\frac{ax-b}{(x-p)(x+q)(x-r)\ldots} = \frac{A}{x-p} + \frac{B}{x+q} + \frac{C}{x-r}\ldots$ - The denominator is made up of repeated factors: $\frac{ax-b}{(x-p)^3} = \frac{A}{(x-p)} + \frac{B}{(x-p)^2} + \frac{C}{(x-p)^3}$ - The denominator contains irreducible quadratic factors: $\frac{ax-b}{ax^2+bx+c} = \frac{Ax+B}{ax^2+bx+c}$ - The function is an improper rational function. **Example:** resolve $\frac{2x^2+1}{(x-1)(x+2)}$ in partial fractions.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Solution: $\frac{2x^2+1}{(x-1)(x+2)}$ = 2 + $\frac{1}{(x-1)} = \frac{3}{(x+2)}$	
Teaching and Learning Resources	• Cut out shapes of different colours and orientation • Maths Technology learning apps, tools and devices • Chalkboard illustrations • Worksheets • Cut-out geometrical shapes of different colours and orientation	

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
1.1.2.CS.2	1.1.2.LI.1	1.1.2.AS.1
Demonstrate knowledge and understanding of algebraic processes and reasoning in relation to matrix operations.	**Recognise a matrix (including types of matrices) and state its order.** **Collaborative Learning:** Learners will work in convenient groups (ability, mixed-ability, mixed gender, etc.) to solve problems and present answers related to matrices and linear transformation. **Talk for Learning Approaches:** Learners will brainstorm through think-pair-share and debate and discuss the types and features of matrices. **Project-based Learning:** Learners will collaboratively embark on a project to come up with a matrix that is relevant and applicable to real life and make presentations. **Experiential Learning:** Learners will collaboratively engage in hands-on activity (learning by doing) to resolve and manipulate data into a matrix and calculate determinants**.** **Problem-based Learning:** Learners will collaboratively inquire into the importance of matrices in real life. **Activity 1:** Discuss matrices in the natural environment and create matrices using objects, symbols or numerals. **Collaborative Learning, Experiential Learning, and Talk for Learning Approaches:** Learners work in groups/pairs to Identify and describe features of matrices. Learners examine and recognise a matrix as an arrangement of objects or numbers into rows and columns. **Example:** In an $m \; x \; n$ matrix**,** the **m** rows are horizontal, and the **n** columns are vertical. Each element of a matrix is denoted by a variable with subscripts. For example, a_{23} represents the element at the second row and third column of the matrix.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

$$\begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & a_{11} & a_{12} & \cdots & a_{1n} \\ 2 & a_{21} & a_{22} & \cdots & a_{2n} \\ 3 & a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}$$

Example 1: The sitting arrangement in a classroom
Example 2: Provision items in a shop.
Example 3: A pack of bottled water.
Example 4: Suppose that we wish to express the information of possession of pens and pencils by Afiba and his two friends, Enyonam and Yaw, which is as follows:
Afiba has 20 pens and 7 pencils,
Enyonam has 15 pens and, 5 pencils, and
Yaw has 12 pens and 3 pencils.
Now, this could be arranged in tabular form as follows:

	Pens	Pencils
Afiba	2	7
Enyonam	1	5
Yaw	3	2

and this can be expressed as,

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{pmatrix} \text{ or } A = \begin{bmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{bmatrix}$$

	Problem-Based Learning: Learners research the relevance of learning about matrices in mathematics and make presentations. **Activity 2**: Order of a matrix, equality of a matrix, identity matrix and zero matrix. **Collaborative Learning, Experiential Learning, and Talk for Learning Approaches:** Learners recognise the order of a matrix as $m\ x\ n$, where m represents the number of rows and n the number of columns. **Example 1:** $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$ For matrix B, the order is $3\ \times\ 2$ Learners discuss and solve problems related to the order of a matrix, the unit matrix, and matrix equality. **Example:** Given that, $\begin{pmatrix} 3y+1 & 7 \\ 2y+x & 3 \end{pmatrix} = \begin{pmatrix} 10 & 3y-x \\ 8 & 3 \end{pmatrix}$ find the values of x and y.	
	1.1.2.LI.2	1.1.2.AS.2
	Add and subtract matrices (2x2 matrix) and multiply a matrix by a scalar and a matrix by a matrix (2x2 matrices). **Activity 1:** Discuss matrix addition and subtraction. **Collaborative Learning, Experiential Learning, and Talk for Learning Approaches:** • Learners work in groups/pairs to explore commutative and associative properties of addition. • Learners recognise the conditions necessary for matrix addition and subtraction.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Learners verify that: • $A + B = B + A$ • $A + (B + C) = (A + B) + C$ **Activity 2:** Discuss matrix multiplication. **Collaborative Learning, Experiential Learning, and Talk for Learning Approaches:** • Learners work in groups/pairs to explore and solve problems related to scalar and matrix multiplication. • Learners recognise conditions necessary for matrix multiplication. • Learners carry out scalar multiplication of an *m x n* matrix. Investigate the properties of the operations on matrices.	
Teaching and Learning Resources	• Worksheets • Scientific Calculator • Technological tools, apps, etc. • SHS Additional Mathematics Curriculum	

Subject **ADDITIONAL MATHEMATICS**
Strand **2. GEOMETRIC REASONING AND MEASUREMENT**
Sub-Strand **1. SPATIAL SENSE**

Learning Outcomes	**21st Century Skills and Competencies**	**GESI, SEL and Shared National Values**
1.2.1.LO.1		
Describe the properties of lines, including parallel, perpendicular and midpoints.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. Learners respond to ("why, how, when, who, what, and where") questions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they describe the properties of lines, including parallel, perpendicular and midpoints, using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to describe the properties of lines, including parallel, perpendicular and midpoints. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse.

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	• Value and promote justice in the mathematics classroom and beyond. **SEL**: Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation

		These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they describe

		the properties of lines, including parallel, perpendicular and midpoints.
1.2.1.LO.2		
Derive the equation of a line in various forms, find the shortest distance between a point and a line and the perpendicular distance from an external point to a line.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they derive the equation of a line in various forms, find the shortest distance between a point and a line and the perpendicular distance from an external point to a line using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to derive the equation of a line in various forms, find the shortest distance between a point and a line and the perpendicular distance from an external point to a line. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse.

	Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation.

		These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they derive

		the equation of a line in various forms, find the shortest distance between a point and a line and the perpendicular distance from an external point to a line.
1.2.1.LO.3		
Solve problems on acute angles between two intersecting lines.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they solve problems on acute angles between two intersecting lines using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to solve problems on acute angles between two intersecting lines. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work. • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-

		made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they solve problems on acute angles between two intersecting lines.

1.2.1.LO.4		
Perform algebraic manipulations of Vectors and resolve vectors using the triangle, parallelogram and polygon laws of addition.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their collaborative groups as they perform algebraic manipulations of vectors and related concepts using appropriate technological tools in an atmosphere that promotes critical thinking. • Interrogate their stereotypes and biases about the roles and abilities of individuals working in groups as they learn about vectors and other related concepts. • As they examine and begin to dispel misconceptions/ myths about GESI-related issues in their engagement in mathematical discourse based on vectors. • Learners value and promote justice as they take up leadership responsibilities in their groups when learning vectors in the classroom and beyond.

	Creativity and Innovation: Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss concepts based on vectors using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting when using innovative ideas in solving problems based on vectors to promote critical thinking. • Managing emotions and conflicts as they engage in collaborative group work to explore concepts based on vectors • Showing empathy and cooperation in their groups by consciously bearing the diverse nature of group membership.

		These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they perform

		algebraic manipulations of vectors and resolve vectors using the triangle, parallelogram and polygon laws of addition.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
1.2.1.CS.1	1.2.1.LI.1	1.2.1.AS.1
Demonstrate knowledge and understanding of spatial sense in relation to lines and angles between intersecting lines.	**Describe the properties of lines, including parallel, perpendicular and midpoints.** **Learning Experience:** History of and its significance in real life and for further studies and discussion on parallel and perpendicular lines in pairs and present answers **Activity 1:** Initiate Talk for Learning to discuss the history of geometry and its significance. Initiate Talk for Learning on how geometry came into being, its significance and application in real life, as well as further studies. **Activity 2:** Use Think-pair-share approaches to discuss straight lines, their types and characteristics. Learners brainstorm and find out that: • A straight line is formed when two points are joined with the shortest distance, and it can be extended forever in both directions. • A straight line has no curves within. • A straight line is one-dimensional and has no width. • Types of straight lines are horizontal lines, vertical lines, slanted lines, parallel lines and perpendicular lines. **Activity 3:** Use Think-pair-share to discuss parallel and perpendicular lines, their characteristics, and examples in real life. Learners brainstorm and find out that: • Parallel lines are two straight lines that are always the same distance (equidistant) apart. They never intersect. Examples are Railway tracks, edges of a ruler, and zebra crossing (parallel white lines). • Perpendicular lines are two lines that intersect, but all the angles at that intersection are the same, that is, $90°$. For example, "T" junctions on roads, corners of a football pitch, etc. **Activity 4:** Use collaborative learning to find the distance between two points and share. Learners to work in groups to discover that:	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

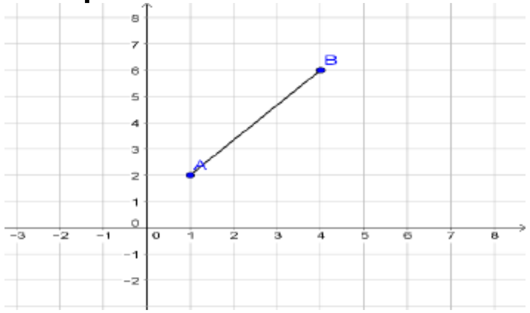
- Distances are always positive.
- Distances can only be zero if the points coincide.
- The distance from P to Q is the same as the distance from Q to P
- The distance between vertical lines is the distance between their x coordinates.

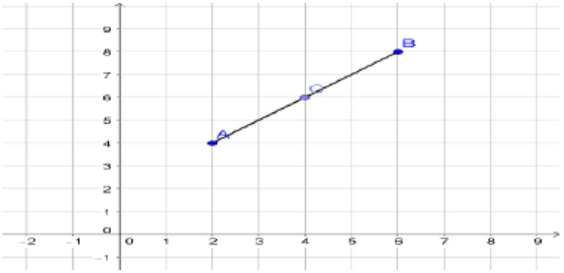
Example 1: Find the distance between the line with coordinates $A(2,4)$ and $B(6,4)$

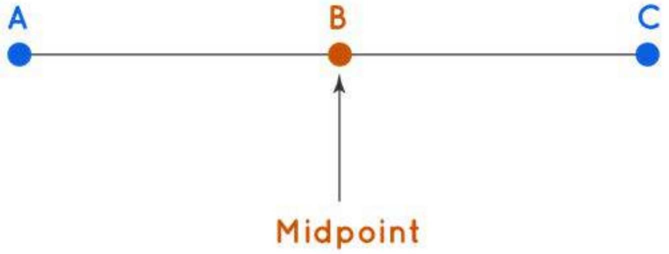
Solution: The distance between A and B is $6 - 2 = 4$

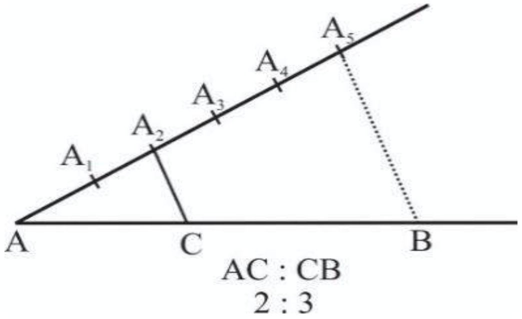
Activity 5: Use collaborative Learning to discover that Pythagoras' theorem calculates the distance between two points when the interval between the points is neither vertical nor horizontal.

Example 1: Find the distance between the line segment joining the points A (1, 2) and B(4, 6)



	Solution: $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\therefore AB = \sqrt{(4 - 1)^2 + (6 - 2)^2}$ $= 5$  **Solution:** $\left(\frac{3}{2}, 6\right)$ **Activity 3:** Learners in collaborative groups generalise that the midpoint of a line segment is the average of the x coordinates and the average of the y coordinates. **Activity 4:** Use collaborative learning to solve direct and indirect questions and explain to the whole group and class.	
	1.2.1.LI.2	1.2.1.AS.2
	Predict the midpoint of a line segment given two points and find the generalisation of the midpoint of a line segment. **Learning Experience:** Learners in pairs will perform class activity and use their information to describe how the midpoint of a line can be obtained from a given line segment and present answers. **Activity 1:** Initiate Talk for Learning to discuss how the midpoint of a line can be obtained from a given line segment and present answers.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	Example: From line segment AC shown below, AB is halfway from AC; hence, point B is the midpoint.  	
	1.2.1.LI.3	1.2.1.AS.3
	Apply the knowledge of ratio to divide a line segment in a given ratio either internally or externally. **Learning Experience:** Learners use the Jigsaw method to investigate the division of line segments in a given ratio. Leader to discuss findings with group members. **Activity 1:** • Use collaborative learning to link the concepts of division by a given ratio algebraically to line segments. • Group leaders to initiate Talk for Learning by discussing division by a ratio generally. **Example 1:** Dividing GHC 50 in the ratio $2:3$. **Solution:** The group leader to build on what others will say to establish that, suppose you are asked to divide a line segment AB in the ratio $2:3$. It means that you to find a point C, such that $AC:CB = 2:3$ $$AC:CB = 2:3$$ Group members to discover that $AC:CB = 2:3$ is the same as $\frac{AC}{BC} = \frac{2}{3}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning



Activity 2: Use collaborative Learning to discuss dividing a line segment in a given ratio internally geometrically. Group leaders to lead discussions on how to geometrically locate point C.

Example 1: Divide a given line segment AB in the ratio $2:3$.
Solution: Learners to experiment together in groups by drawing a line segment of any length

- Draw any ray AX inclined at an acute angle to AB, and mark $2+3=5$ equal intervals on AX with a compass.
- Name the divisions $A_1, A_2, A_3, A_4,$ and A_5.
- Join A_5 to B.
- Through A_2 draw a line parallel to A_5B.
- Learners in groups establish that $\frac{AA_2}{A_2A_7} = \frac{AC}{AB}$.

Activity 3: The group leader initiates talk on dividing a line segment in a given ratio internally algebraically.

Leaders of the jigsaw groups to initiate talks on the division of a line segment in a given ratio algebraically.

Learners in their groups establish the formula $(x, y) = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

	Example 1: Find the coordinates of the point that divides the line segment $A(3,1)$ and $B(3,-5)$ in the ratio $2:1$ internally. $(x,y)=\left(\frac{2(3)+1(3)}{2+1},\frac{2(-5)+1(1)}{2+1}\right)$ $=(3,-3)$ Learners in groups solve direct and indirect questions on internal division and present and solve examples individually. **Activity 4:** Group leaders initiate talk on the division of a line segment in a given ratio externally. • Leaders of the jigsaw group to initiate discussion in their various groups. • Learners in their groups establish the formula $(x,y)=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)$. **Example 1:** Find the coordinates of the point that divides the line segment $A(3,1)$ and $B(3,-5)$ in the ratio $2:1$ externally. $(x,y)=\left(\frac{2(3)-1(3)}{2-1},\frac{2(-5)-1(1)}{2-1}\right)$ $=(3,-11)$ **Activity 5:** Learners in collaborative groups solve direct and indirect questions.	
	1.2.1.LI.4	1.2.1.AS.4
	Recall the formula for finding the gradient of a line and apply it to find the equation of a straight line in various forms. **Learning Experience:** Learners in group members recollect how to find the gradient of a line and apply it to find the equation of a line in various forms. **Activity 1:** Think, pair and share the concept of the gradient of a line.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning**

Learners work in pairs to recollect and state the formula for finding the gradient of a line.
The formula is $m = \frac{y_2 - y_1}{x_2 - x_1}\ or\ \frac{y_1 - y_2}{x_1 - x_2}$.

Learners in pairs solve examples of direct and indirect questions about the gradient of a line and share their solutions with the whole class.

Example 1: Find the gradient of the line passing through the points $A(6, -4)$ and $B(4,2)$.

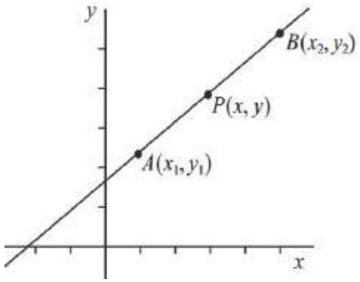
Activity 2: Use collaborative groups to find the Equation of a line passing through two points.

Learners in mixed-ability recollect the equation of a line as $y = mx + c$.

Learners in mixed-ability groups, manipulate the formula for finding a gradient to deduce the equation of a line given two points, say A and B.

Learners discover that:

- If $A(x_1, y_1)$ and $B(x_2, y_2)$, choose any arbitrary point within the line, say $P(x, y)$.



- If P lies on the same line, then the gradient of the line AB is equal to the gradient of the line AP, that is $M_{AB} = M_{AP}$.
- If the relationship is expressed algebraically, we will have $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$.

Level 4 Extended Critical thinking and reasoning

- If $y - y_1$ is made the subject, the formula becomes $y - y_1 = (x - x_1) \cdot \frac{y_2 - y_1}{x_2 - x_1}$
- This can be rewritten as

$$y - y_1 = m(x - x_1)$$

Example 1: Find the equation of the line that passes through the points $(-2,4)$ and $(4,8)$.

Steps:

- In mixed-ability groups, learners use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$ to find the gradient of the line as $m = \frac{8-4}{4-(-2)} = \frac{2}{3}$.
- Group leaders in the various groups lead their members to substitute m and either (x_1, y_1) of (x_2, y_2) into the formula $y - y_1 = m(x - x_1)$ or $y - y_2 = m(x - x_2)$.

  Choching $(-2,4)$,
  The substitution results to

$$y - 4 = \frac{2}{3}(x + 2)$$

- Use the correct algebraic manipulation and make y the subject.

$$y = \frac{2}{3}x + \frac{16}{3}$$

Activity 3: Use collaborative learning to find the Equation of a line given a gradient and one point.

Learners in mixed-ability group use their previous knowledge of finding the equation of a line to find the equation of a line given a gradient and one point.

Learners in groups deduce that when given a gradient and one point, m and (x_1, y_1) will be substituted in the formula
$y - y_1 = m(x - x_1)$ or $y - y_2 = m(x - x_2)$

y is made the subject of the equation or equate the expression to 0.

Example 1: Find the equation of a line passing through the point $(6, -2)$ and has a gradient $\frac{-1}{4}$.

Solution: $y - y_1 = m(x - x_1)$

$$y + 2 = \frac{-1}{4}(x - 6)$$

$4y + x + 2 = 0$ or $y = \frac{-x}{4} - \frac{1}{4}$

Activity 4: In mixed-ability groups learners discuss equation of a line (Intercept Form).

Learners in the mixed-ability group recollect how to find x and y intercepts.

Example 1: Find the x and y intercepts of the equation

$$4y + x + 2 = 0$$

Solution: The y-intercept is found by substituting $x = 0$

$$\therefore y = \frac{-1}{2}$$

x-intercept is found by substituting $y = 0$

$$\therefore x = -2$$

Learners in their groups apply the knowledge of intercepts and the equation of a line, that is $y = mx + c$, to deduce the equation of a given gradient and an intercept.

Learners in groups investigate the conditions that make it impossible to use the intercept form to find the equation of a line. Learners discover that the conditions are:

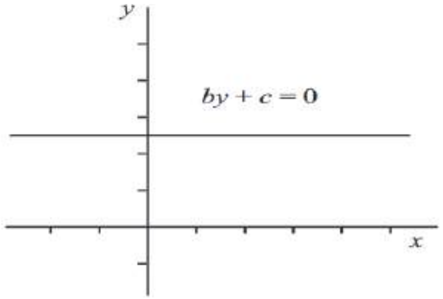
- If the line is parallel to an axis.
- If the line passes through the origin.

Activity 4: General Equation of a line (Standard Form)

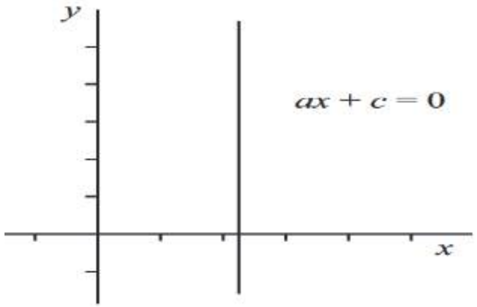
Learners in mixed-ability equate various examples of equations of a line to 0 and establish that the general equation of a line is given by $ax + by + c = 0$.

Learners in groups investigate what happens to the general equation of a line when a or b is zero. Learners discover that:

- When $a = 0$, $by + c = 0 \therefore y = \frac{-c}{b}$, these lines are horizontal and parallel to the $x - axis$



- **When** $b = 0, ax + c = 0 \therefore x = \frac{-c}{a}$**, these lines are vertical and parallel to the** $y - axis$



	Learners in groups confirm that the equation of a vertical line cannot be written in the form $y = mx + c$. The equation $ax + by + c = 0$ is the most general equation for a straight line and can be used where other forms of equation are not suitable.	
	1.2.1.LI.5	1.2.1.AS.5
	Use standard algebraic manipulations to find the equation of parallel and perpendicular lines, including the equation of the perpendicular bisector of a line. **Learning Experience:** Investigate how the formula for finding the gradient of a line can be applied to find the equation of a parallel and a perpendicular line and how to find the equation of a perpendicular bisector. **Activity 1:** Use think pair and share to find the gradient of a parallel line. Learners in pairs recollect the properties of parallel lines and share their findings. **Expected Response to build on:** Parallel lines have the same gradient. Building on the response, learners in pairs find the gradient of given coordinates and state their observations. **Example 1:** Find the gradient of lines AB and PQ if $A(2,3)B(5,6), P(-1,4)\ Q(5,10)$ **Solution:** Gradient of line $AB = \frac{6-3}{5-2} = 1$ Gradient of line $PQ = \frac{10-4}{5-(-1)} = 1$ **Observation:** Since the gradient of lines AB and PQ are the same, they are parallel. Learners, in pairs, solve examples and non-examples to reinforce the concept of parallel lines.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Activity 2: Think-pair-share the concept of the equation of a parallel line.

Learners in pairs apply their knowledge of parallel lines and the gradient of a line to find the equation of a line to find the equation of a parallel line.

Example: Find the equation of the line that is parallel to the line $y = -2x + 6$ and pass through the point $A(1, 10)$.

Solution: The gradient of line $y = -2x + 6$ is -2, if it passes through the point $A(1,10)$ then $y - 10 = -2(x - 1)$
Therefore, the equation of the line is $y = -2x + 12$

Activity 3:

Learners in their mixed-ability groups find the gradient of a perpendicular line.

Learners in mixed-ability groups recollect the properties of perpendicular lines and share their findings.

Expected Response to build on: Two lines are Perpendicular when they meet at a right angle; when the slope of one line is m, the slope of the perpendicular line is $\frac{-1}{m}$. OR If two lines are perpendicular, then the product of their gradients is –1.

Building on the responses, learners in their mixed-ability groups find the gradient of lines and share their observations.

Example 1: Find the gradient of lines AB and PQ, If $A(-1, -1), B\ (0, 4) P(-4, 3), Q(6, 1)$ and state your observation.

Solution:

The gradient of line $AB = \frac{4-(-1)}{0-(-1)} = 5$.

The gradient of line $PQ = \frac{1-3}{6-(-4)} = \frac{-1}{5}$.

Observation: Since the gradient of lines $AB = 5$ and $PQ = \frac{-1}{5}$ then AB is perpendicular to PQ.

Activity 4:

- Use Think-pair-share to deduce the equation of a perpendicular line.
- Learners in pairs apply their knowledge of perpendicular lines, the gradient of a line and finding the equation of a line to find the equation of a perpendicular line.

Example: Find the equation of the line that passes through the point $(1, 3)$ and is perpendicular to the line whose equation is $y = 2x + 1$.

Solution: The gradient of the line $y = 2x + 1$ is 2. Therefore, the gradient of the perpendicular will be $\frac{-1}{2}$. If the line passes through point (1,3), then $y - 3 = \frac{-1}{2}(x - 1)$

Therefore, the equation of the line will be $y = \frac{-x}{2} + \frac{7}{2}$.

Activity 5:

- **In mixed-ability groups,** learners discuss the equation of the perpendicular bisector of a line.
- Learners in their mixed-ability group discuss and extend their knowledge to find the equation of a perpendicular bisector of a line.
- Learners discuss keywords such as perpendicular and bisector. Learners brainstorm on how to find the equation of a perpendicular bisector.

Example 1: Find the equation of the perpendicular bisector of AB, where A and B are the points $(-4, 8)$ and $(0, -2)$

Solution: Gradient of line $AB = \frac{-2-8}{0-(-4)} = \frac{-5}{2}$, then the gradient of the perpendicular bisector is $\frac{2}{5}$

	The midpoint of line $AB = \left(\frac{-4+0}{2}, \frac{8+(-2)}{2}\right)$, which is $(-2{,}3)$ Therefore, the Equation of the perpendicular bisector of AB which passes through the midpoint $(-\ 2, 3)$, is $y - 3 = \frac{2}{5}(x + 2)$ $5y - 2x - 19 = 0$	
	1.2.1.LI.6	1.2.1.AS.6
	Deduce the shortest distance between a point and a line and use the knowledge of intercepts and right-angled triangles to find the perpendicular distance from an external point to a line. **Learning Experience:** Investigate the shortest distance between a point and a line as well as the shortest distance between two lines. Real-life activities to measure these distances in the classroom and around the school **Activity 1:** Use collaborative groups to investigate the shortest distance between a point and a line. • Learners in convenient groups investigate the shortest distance between a point and a line as well as the shortest distance between two lines. • Learners conduct real life activities to measure these distances in the classroom and around the school. • Learners establish that the shortest distance between a line and a point is the perpendicular distance. **Activity 2:** Using a formula for finding the perpendicular distance. **Theorem:** The shortest distance (or the perpendicular distance), D, between the point $P(x_1, y_1)$ and the line L: $ax + by + c = 0$ is given by: $D = \frac{\lvert ax_1+by_1+c\rvert}{\sqrt{a^2+b^2}}$	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

- Learners in convenient groups identify that the numerator of the formula is the equation of a line and the denominator is the Pythagoras theorem for finding the length of the hypotenuse.
- Learners in groups use the formula to solve examples.

Example 1: Find the length of the perpendicular drawn from the point $A(1,9)$ to the straight line $-5x + 12y + 13 = 0$.

Solution:

- Recollect the formula as $D = \frac{|ax_{1+}by_1+c|}{\sqrt{a^2+b^2}}$
- Identify the components of the question. From the question $x_1 = 1$, $y_1 = 9$, $a = -5$, $b = 12$ and $c = 13$.
- Learners substitute these values into the formula and find the solution

, i.e. $D = \frac{|-5(1)+12(9)+13|}{\sqrt{5^2+12^2}} = \frac{116}{13}$

Activity 3:

- Learners in groups apply knowledge in the equation of a line and perpendicular distance.
- Learners in groups extend the knowledge gained to solve application questions.

Example 1: Find the length of the perpendicular drawn from the point $A(-1,-7)$ to the straight line passing through the points $B(6,-4)$ and $C(9,-5)$.

- Learners in their convenient groups discuss and discover that they have to first find the equation of the line using the points $B(6,-4)$ and $C(9,-5)$.

The equation of the line is $x + 3y + 6 = 0$

- Learners in groups use the equation of the line and the point $A(-1,-7)$ to find the perpendicular distance.

	Expected answer $\frac{8\sqrt{10}}{5}$ **Activity 4:** • Learners in groups find the distance between Two Parallel Lines. • Learners in groups investigate and extend their knowledge to find the distance between two parallel lines. **Note:** The distance between two parallel lines can be found as the perpendicular distance between any point on one line and the other line. • Learners in groups discuss and build on what others say to realise that given two parallel lines: • Find one point on one of the lines • Find the distance from this point to the other line	
	1.2.1.LI.7	1.2.1.AS.7
	Determine the acute angles between two intersecting lines with the aid of technological tools, e.g. GeoGebra. **Learning experience:** In pairs, learners manipulate technological tools to ascertain the acute angles of two intersecting lines. **Activity 1: Think pair and explore technological tools.** • Learners in pairs manipulate and explore available ICT tools individually to ascertain the acute angles of two intersecting lines. • Learners share their experiences and findings from using the ICT tools.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	1.2.1.LI.8	1.2.1.AS.8
	Apply knowledge of gradient and the tangent function to find the Acute angle between two intersecting lines. **Learning Experience:** Learners in a mixed-ability group are led by a leader to find the acute angle between two intersecting lines. **Activity 1:** • Use collaborative learning to find the acute angle between two intersecting lines. • The leader of the mixed-ability groups leads by discussing with group members to recollect the slope of lines. • Learners in groups establish that an angle θ between two nonparallel lines in the coordinate plane with slopes m_1 and m_2, such that $m_1 m_2 \neq -1$, is given by $tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$. **Activity 2:** • **In jigsaw groups,** use the formula to find the acute angle between two lines, given their gradient. • Learners in groups solve examples and share the findings with the whole class. **Example 1:** Determine the acute angle between two straight lines having slopes of 5 and $\frac{1}{4}$ to two decimal places. **Solution:** $$tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$ $$tan\theta = \frac{5 - \left(-\frac{1}{4}\right)}{1 + (5)\left(\frac{1}{4}\right)}$$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	$tan\theta = \frac{19}{9}$ $\theta = tan^{-1}\left(\frac{19}{9}\right)$ $\theta = 62.65°$ **Activity 3:** In collaborative groups, apply and Extension of knowledge**.** Learners in their groups find the acute angle between two lines given the equation on the lines.	
Teaching and Learning Resources	• SHS Curriculum • Mathematical set • calculators.	

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
1.2.1.CS.2	1.2.1.LI.1	1.2.1.AS.1
Demonstrate knowledge and understanding of spatial sense relating to vectors in two dimensions and perform algebraic operations on vectors and their geometrical interpretations.	**Recognise and explain various forms of vectors and apply the knowledge to find unit vectors.** **Learning Experience:** Learners in a mixed-ability group conceptualise the idea of vectors and, perform algebraic manipulations on vectors, and interpret it geometrically. **Activity 1:** Using Think-Pair-Share, learners recollect the definition of vector, vector notation, types and examples of vectors. **Activity 2:** - Use collaborative learning to express vectors in a Cartesian plane, express vectors in magnitude and direction form, and calculate the magnitude of a vector. - Learners in groups express vectors. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ from $A(2,4)$ and $B(-3,7)$ as column $(x\ y\ )$, or $x_i + y_i$. **Example:** $A(2,4)$ will become $\overrightarrow{OA} = (2\ 4\ )$ and similarly $\overrightarrow{OB} = (-3\ 7\ )$. - Learners in groups brainstorm to express a vector in magnitude and direction form. **Example:** Aku's house is 40 metres away from her school, and the bearing is $150°$. Write the location of Aku's house in vector form. **Answer:** $(40m, 150°)$ - Learners in groups estimate distances of their sitting positions and the directions in relation to their teacher's table and write in vector forms. **Example:** Esi's position is $(5m, 060°)$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Edem's position is $(8m, 120°)$ etc.

Activity 3: In collaborative groups**,** Learners identify parallel vectors from given vectors and make generalisations of parallel vectors.

Example 1: Identify parallel vectors from the following given vectors $a = (4\ 3\ )$, $b = (6\ 4\ )$, $c = (2\ \ -3\ )$, $d = (8\ \ -12\ )$,$e = (3\ 2\ )$

Expected Responses: $d = 4c$ and $b = 2e$. Therefore, vectors d and c are parallel, and vectors b is parallel to vector d

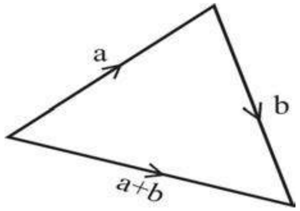
Generalisation: Two vectors are parallel when one vector can be expressed as a scalar multiple of the other.

Learning Experience: Through collaborative learning and activities, learners determine the resultant vectors using the polygon laws of addition.

Activity 1:

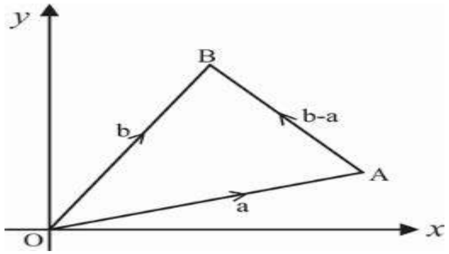
- **In collaborative groups,** learners discuss triangle laws of addition.
- The group leader initiates Talk for Learning on the triangle law of addition, and Learners in groups represent the triangle law of addition with a diagram.

Addition of vectors



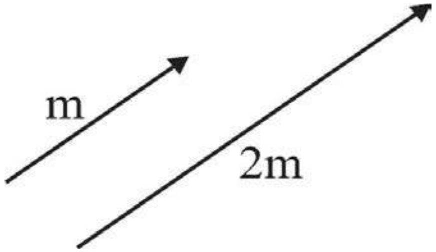
- Learners in groups apply the triangle law of addition to subtraction of vectors and represent it with a diagram.

Subtraction



- Learners represent multiplication of a vector by a scalar diagrammatically.

Scalar multiplication



- Learners in collaborative groups perform algebraic manipulations on vectors.

Example 1:

Given the $m = (3 \ -4\ )$ and $= (5\ 0\ )$, $p = \frac{1}{4}(m+n)$ and $q = \frac{1}{2}(m-n)$ Show that $|p+q| = |p-q|$.

Solution

$$p = \frac{1}{4}[(3\ \ -4\,) + (5\ 0\,)]$$

$$p = (2\ \ -1\,)$$

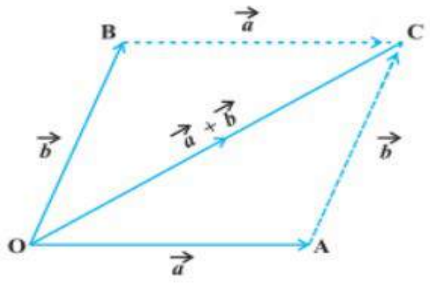
$$q = \frac{1}{2}[(3\ \ -4\,) - (5\ 0\,)]$$

$$= (-1\ \ -2\,)$$

$(p+q) = (1\ \ -3\,)$ and $(p-q) = (3\ 1\,)$
$|p+q| = \sqrt{10}$ and $|p-q| = \sqrt{10} \therefore |p+q| = |p-q|$

Activity 2:

- In collaborative groups, learners discuss parallelogram laws of addition and represent it in a diagram.
- The group leader initiates discussion on parallelogram and polygon laws of addition and represents it in a diagram.



- Through brainstorming, learners in groups establish that the triangle and parallelogram laws of addition give equal resultants.

	1.2.1.LI.2	1.2.1.AS.2
	Perform algebraic and graphical operations (addition, subtraction, scalar multiplication) and their geometrical interpretation. **Collaborative Learning, Talk for Learning Approaches, Project-based Learning, Experiential Learning:** Learners in a mixed-ability group discuss the concept of vectors, perform algebraic manipulations on vectors and make geometrical interpretations. **Activity 1:** Learners will work in convenient groups (ability, mixed-ability, mixed gender, etc.) to perform algebraic and graphical operations (addition, subtraction, scalar multiplication) and their geometrical interpretation. **Activity 2:** Learners will collaboratively embark on a project on the applicability of vectors in real life and make presentations. Examples: • Calculate the angle between two vectors; • Determine a vector normal to a plane; • Calculate the moment of a force about a point; and • Calculate the moment of a force about a line.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning
	1.2.1.LI.3	1.2.1.AS.3
	Determine the resultant of vectors using triangle and parallelogram laws of addition. **Activity 1:** In collaborative groups, learners review and calculate the magnitude of vectors. **Example 1:** Determine the magnitude of given vectors if $\overrightarrow{AB} = (3 \ \ -4)$, find $\left\|\overrightarrow{AB}\right\|$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning**

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Solution: $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + (-4)^2}$
$= \sqrt{25}$
$= 5$

Activity 2:
1. In collaborative groups, students discuss unit vectors and solve related problems.
2. Learners in groups brainstorm and establish that a unit vector is a vector with a magnitude of 1 unit denoted by, $\hat{a}$, and a vector is divided by its magnitude to find the unit vector, i.e. $\hat{a} = \frac{a}{|a|}$

Example 1: Find the unit vector for $b = (-2\ 1\ )$

Solution:
$\hat{b} = \frac{(-2\ 1\ )}{\sqrt{5}}$ or $\frac{1}{\sqrt{5}}(-2i + j)$

Activity 3: Using Collaborative Learning, learners determine a vector a in the direction of another vector b.

For $a = b$, then $\frac{a}{|a|} = \frac{b}{|b|}, a = |a| \times \frac{b}{|b|}$

Example 1: Given that $m = (-3\ 4\ )$ and $n = (3\ \ -5\ )$, find a vector r, such that $|r| = 25$ and r in the direction of $2m + n$.

Solution:
$2m + n = 2(-3\ 4\ ) + (3\ \ -5\ )$
$(3\ 3\ )$
$\therefore |2m + n| = \sqrt{18}$
$3\sqrt{2}$
</td>
<td>Level 4 Extended critical thinking and reasoning</td>
</tr>
</table>

	So $r = 25 \times \frac{(3\ 3\ )}{3\sqrt{2}} = \frac{25\sqrt{2}}{6}(3\ 3\ )$ or $\frac{25\sqrt{2}}{6}(3i + 3j)$	
Teaching and Learning Resources	• Graph board or paper • Compasses • Mathematical tools • Dynamic tools like GeoGebra • Video clips	

Subject ADDITIONAL MATHEMATICS
Strand 2. GEOMETRIC REASONING AND MEASUREMENT
Sub-Strand 2. MEASUREMENT OF TRIANGLES

Learning Outcomes	21st Century Skills and Competencies	GESI, SEL and Shared National Values
1.2.2.LO.1		
Describe diagrammatically and algebraically ways of representing problems involving angles of elevation and depression and solve related word problems. Identify values of the special angles, explain the radian measure orally and mathematically and find coordinates of the unit circle and their corresponding angles in the quadrants.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions).	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they describe diagrammatically and algebraically ways of representing problems involving angles of elevation and depression and solve related word problems using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to describe diagrammatically and algebraically ways of representing problems involving angles and related concepts.

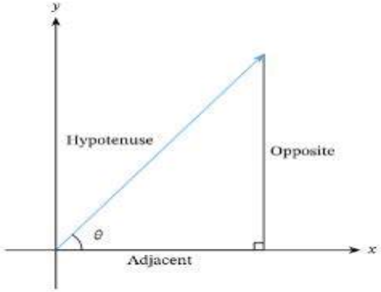
	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills**: Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to real life.	• Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems.

		• Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability

		groups, emphasising differentiated instruction and assessment as they describe diagrammatically and algebraically ways of representing problems involving angles of elevation and depression and solving related word problems.
1.2.2.LO.2		
Identify values of the special angles in degrees and radians and solve problems relating to the coordinates of the unit circle.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions).	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they identify values of the special angles in degrees and radians using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to special angles and related concepts. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling; and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to real life.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies–*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the

		creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they identify values of the special angles in degrees and in radians and solve problems relating to coordinates of the unit circle.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
1.2.2.CS.1	1.2.2.LI.1	1.2.2.AS.1
Demonstrate understanding of the measurement of triangles and radians.	**Recall basic trigonometric ratios and use the knowledge to solve problems relating to triangles.** **Collaborative learning, initiate Talk for Learning.** **Learning Experience:** Learners in groups recollect how to apply Pythagoras theorem to solve practical problems in trigonometry. **Activity 1:** Using Think-Pair-Share, learners review trigonometric ratios, their reciprocals, and sketch right triangles and state the relationship among the sides and angles.  - Learners establish in their groups that $sin\theta = \frac{opposite}{hypotenuse}, cos\theta = \frac{adjacent}{hypotenuse}$ and $tan\theta = \frac{opposite}{adjacent}$ - Learners in pairs establish the reciprocals as $csc\theta = \frac{hypotenuse}{opposite}, sec\theta = \frac{hypotenuse}{adjacent}$ and $cot\theta = \frac{adjacent}{opposite}$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Activity 2: Using collaborative learning, learners solve right triangles and state their observations about the sign $(+,-)$ of the values.

Activity 3: Using collaborative learning, learners recall the signs each trigonometric ratio assumes in each of the four quadrants.

Quadrant	**Trigonometric Function**					
	$sine$	$cosine$	$tangent$	$cotangent$	$secant$	$cosecant$
I	+	+	+	+	+	+
II	+	-	-	-	-	+
III	-	-	+	+	-	-
IV	-	+	-	-	+	-

Activity 4: With the use of collaborative learning, learners solve word problems involving angles of elevation and depression and share their solutions with the class.

1.2.2.LI.2

1.2.2.AS.2

Use special triangles and the unit circle to determine the geometrical and functional values of trigonometric ratios, including special angles.

Collaborative learning, experiential learning, and initiate Talk for Learning, Project-Based Learning.

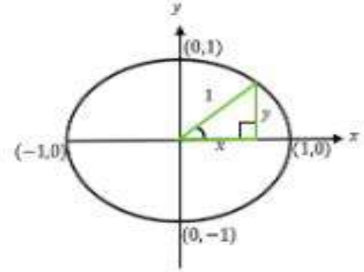
Learning Experience: Use collaborative learning and experiential learning to conceptualise the functional values of trigonometric ratios, including special angles.

Activity 1:

- With the use of experiential learning, learners investigate the trigonometric functional values of angles.

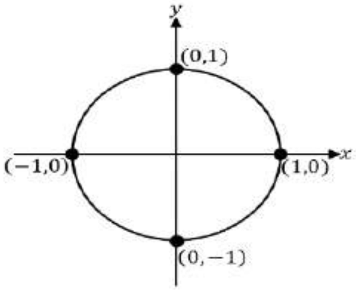
Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

- In collaborative groups, construct circles, assume that the radius is one (1) and label the coordinates of the terminal sides angle as (x, y)



From the unit circle, learners discover that the trigonometric ratios of the unit circle are $sin\theta = y$, $cos\theta = x$, $tan\theta = \frac{y}{x}$, $csc\theta = \frac{1}{y}$, $sec\theta = \frac{1}{x}$, and $cot\theta = \frac{x}{y}$

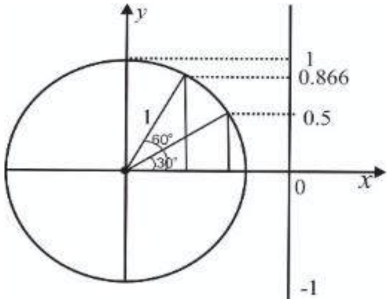
- Learners, in collaborative groups, state the functional values of trigonometric ratios using the unit circle.



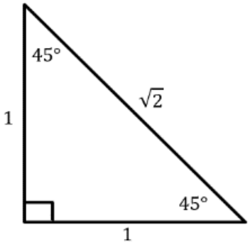
Activity 2:

- Through collaborative learning, experiential learning, and brainstorming, learners identify the values of other trigonometric ratios.

- Learners in groups explore values trigonometric ratios using a circular model (Unit circle).

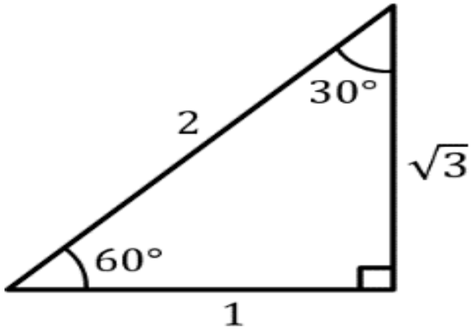


Activity 3: Learners work in groups to establish the trigonometric function values for $45°$ using a $45 - 45 - 90$ triangle.

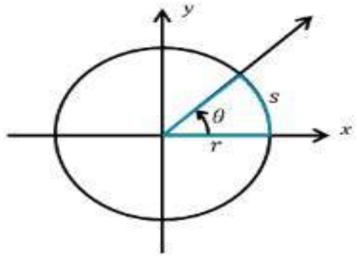


$sin45° = \frac{\sqrt{2}}{2}$	$csc45° = 2$
$cos45° = \frac{\sqrt{2}}{2}$	$sec45° = 2$
$tan45° = 1$	$cot45° = 1$

Activity 4: Learners work in groups to establish the trigonometric values for 30° and 60° using a $30 - 60 - 90$ triangle.



θ	$30°$	$60°$
$sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$tan\theta$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$csc\theta$	2	$\frac{2\sqrt{3}}{3}$
$sec\theta$	$\frac{2\sqrt{3}}{3}$	2
$cot\theta$	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$

	Activity 4: **Research and group work:** Learners to find alternative ways to deduce the functional values of the trigonometric ratios and share them with the class.	
	1.2.2.LI.3	1.2.2.AS.3
	Determine radian measure and apply the knowledge to solve practical problems of arc length. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Use collaborative learning and experiential learning and initiate Talk for Learning to discuss radian measure, unit circle and arc length. **Activity 1:** Using Experiential and collaborative learning, learners discuss and share ideas on unit circle, radian measure and arc length. • Learners in collaborative groups conceptualise 1 radian measure.  $\theta = \frac{s}{r}$, when $s = r$, the radian measure is 1 • Learners use $s = \theta r$ to find arc length and solve practical problems.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

	• Learners in mixed-ability group investigate and brainstorm on how to find the length of the arc when θ is in degrees, that is $=\frac{\theta}{360^\circ}\times 2\pi r$ or $\frac{\theta}{360^\circ}\times \pi d$ **Activity 2:** • Using collaborative learning, learners convert degrees to radians and vice versa. • Learners recollect the relationship between 360° and 2π, as well as 180° and π. That is $360^\circ = 2\pi$ and $180^\circ = \pi$ **Through brainstorming and building on what others say,** learners convert degrees to radians and vice versa. That is, to convert from degrees to radians, multiply the given degrees by $\frac{\pi}{180^\circ}$, and to convert from radians to degrees, multiply the given radian value by $\frac{180^\circ}{\pi}$.	
	1.2.2.LI.4	1.2.2.AS.4
	Identify the coordinates of the quadrantal angles in a unit circle and use them to find the trigonometric values of quadrantal angles. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners in groups use collaborative learning and experiential learning and initiate Talk for Learning to discuss ways of identifying the coordinates of the quadrantal angles in a unit circle and using the idea to find the trigonometric values of quadrantal angles. **Activity 1:** Learners discuss and share ideas for ways of identifying the coordinates of the quadrantal angles in a unit circle and, using the idea, obtain the trigonometric values of these quadrantal angles.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

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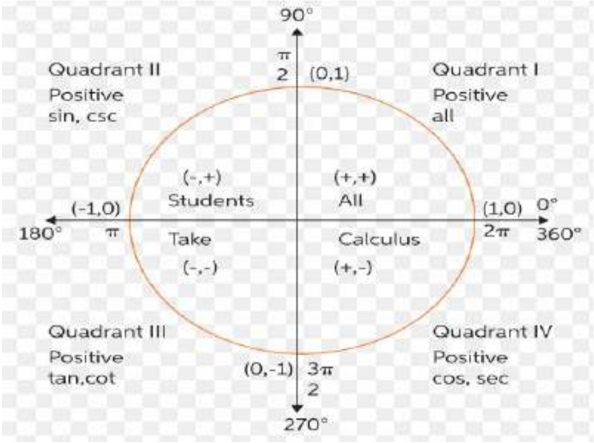
NB: A quadrantal angle is an angle whose terminal side coincides with the x-axis or y-axis. These angles are in multiple of $\pi/2$ or 90° and include 0°, ±90 (π/2), ±180°(π), ±270(3π/2), ±360°(2π), …



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Teaching and Learning Resources

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<td colspan="2">

- Inclinometer Protractor
- rule
- tape measure
- SHS mathematics curriculum
- Video clips

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</table>

Subject **ADDITIONAL MATHEMATICS**
Strand **3. CALCULUS**
Sub-Strand **1. PRINCIPLES OF CALCULUS**

Learning Outcomes	**21st Century Skills and Competencies**	**GESI, SEL and Shared National Values**
1.3.1.LO.1		
Describe graphically and algebraically the behaviour of the function about an input value and determine its derivative.	**Communication and Collaboration:** • As they work in groups, learners learn to share ideas on limits. • As they work in groups, learners learn to share ideas, improve on teamwork and accommodate shades of understanding on left and right side limits. • As they work in groups, learners learn to share ideas and improve on teamwork to understand continuous and discontinuous functions using the concepts of limits. • As they work in pairs, learners learn to share ideas and be sensitive to each other's views. • As they work in groups, learners learn to share ideas and improve their problem-solving skills and critical thinking. **Problem-Solving Skills and Critical Thinking:** • The activities that require learners to critique others' work and perform limits in a natural environment. • The activities that require learners to critique others' work and perform limits in a natural environment will eventually develop their problem-solving skills on the left- and right-hand limits. • The activities require learners to critique others' work and investigate the continuity and discontinuity of a function. • The activities that require learners to critique others" work on the rate of change along a curve.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they describe graphically and algebraically the behaviour of the function about an input value and determine its derivative using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to describe graphically and algebraically the behaviour of the function about an input value and determine its derivative. • Examine and dispel misconceptions/ myths about

	• The activities that require learners to brainstorm and critique others' work. • The activities that require learners to critique others work on investigating the behaviour of objects moving along a curve. **Digital literacy:** • Learners acquire skills in interpreting graphs (on graph sheets and from GeoGebra, etc.) with respect to the meaning of limits. • Learner's use of technology to investigate and interpret graphs (on graph sheets and from GeoGebra, etc.) distinguish between continuous and discontinuous functions. • Learners use technology to investigate the rate of change (on graph sheets and from GeoGebra), etc. • Digital literacy: Learners are to interpret graphs (on graph sheets and from GeoGebra, etc.	gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning, **Competencies** – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work, • Showing empathy and cooperation.

		These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they describe graphically and algebraically the behaviour

		of the function about an input value and determine its derivative.

<table>
<tr><th>Content Standards</th><th>Learning Indicators and Pedagogical Exemplars with 21st Century and GESI</th><th>Assessment</th></tr>
<tr><td>1.3.1.CS.1</td><td>1.3.1.LI.1</td><td>1.3.1.AS.1</td></tr>
<tr><td>Demonstrate understanding of the limit of a function, investigate the behaviour of a function near a value in its domain and establish the derivative of a function.</td><td>

Describe and interpret the meaning of the limit of a function through graphical and algebraic approaches.

Talk for Learning, Collaborative learning.

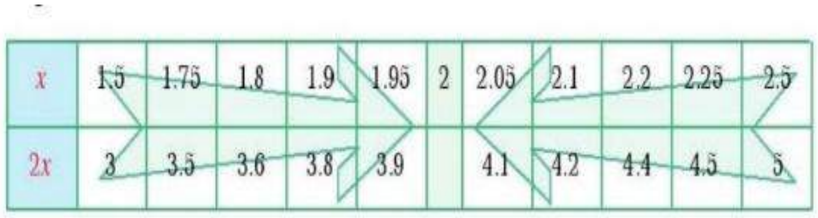
Learning Experience: Learners working in mixed-ability groups to consolidate and foster their understanding of concepts of limits in mathematics and in daily life using both algebraic and graphical means.

Activity 1: Learners should work in mixed-ability groups to demonstrate the concepts of limits in mathematics and in daily life using both algebraic and graphical means.

Example 1: Investigate and interpret the limit of $g(x)$ as x approaches 2 (*refer to Fig. 1*)

Solution: $The\ function\ g(x)$ approaches 2
Note: You can approach 2 from the left or right.

Example 2: Given a function, $f(x) = 2x$



<table>
<tr><td>x</td><td>1.5</td><td>1.75</td><td>1.8</td><td>1.9</td><td>1.95</td><td>2</td><td>2.05</td><td>2.1</td><td>2.2</td><td>2.25</td><td>2.5</td></tr>
<tr><td>2x</td><td>3</td><td>3.5</td><td>3.6</td><td>3.8</td><td>3.9</td><td></td><td>4.1</td><td>4.2</td><td>4.4</td><td>4.5</td><td>5</td></tr>
</table>

Display the table above to learners to investigate the value of the function as x gets closer to 2.

</td><td>

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

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</table>

Solution: $f(x)$ gets closer to 4 as x approaches 2.

Activity 2: Learners are to work in groups with different tasks to investigate the behaviour of a function for different intervals for the input values and share their observations.

Example 1: Investigate the behaviour of $g(x)$ on the interval input values on the intervals (-6,-4), (-3.5, 0) and (0,4).

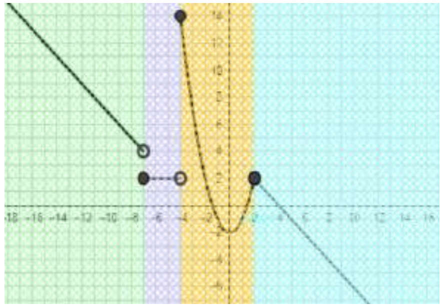
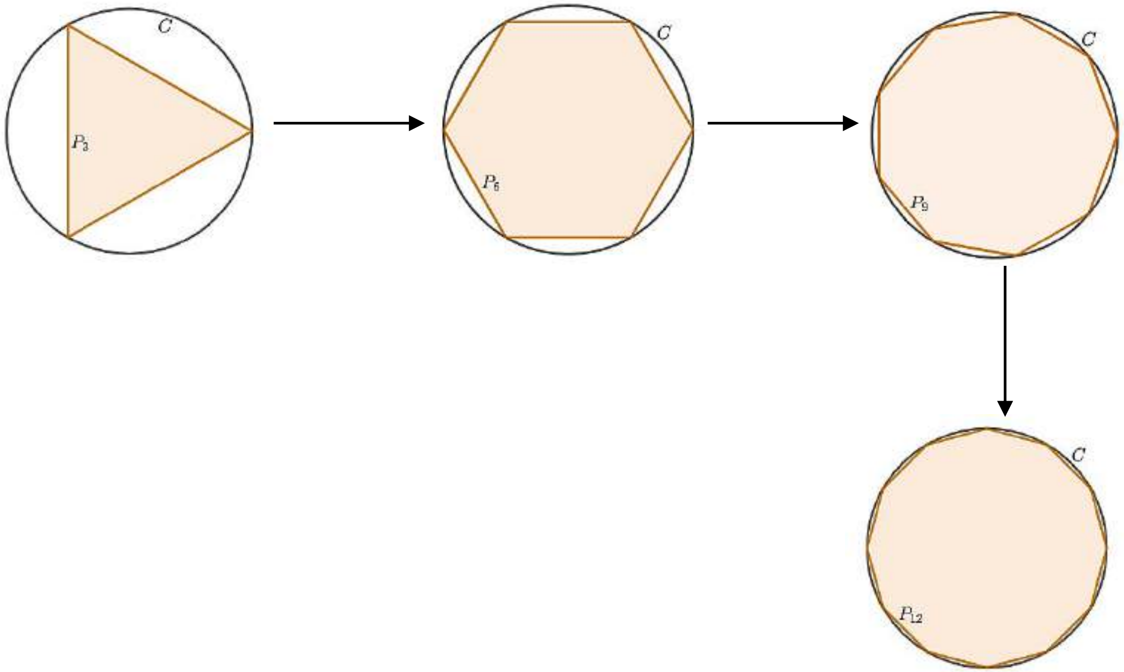


Fig. 1: Graph of a piecewise function $g(x)$ in Geogebra.

Activity 3: Think pair share within and across groups to investigate and discuss the value $g(x)$ approaches as x approaches the number -7 on the horizontal axis. (*Refer to Fig. 1)*

Note: The concept of limit is the basis for a solid understanding of calculus. For example, consider a circle C of radius r - its area A is πr^2 and circumference C is $2\pi r$. One can approximate the area and circumference of C by a region with an area and perimeter that we do know. One approach is to inscribe an equilateral triangle (a regular 3-gon) in C. We add sides, one at a time, to the inscribed figure to create inscribed polygons. The area and circumference of the inscribed figure get closer and closer to the area and circumference, respectively.



No matter how small the sides of a polygon become, the polygon will have many small equal lengths, even though the circle is round. This means as the number of sides n tends to infinity, the limit of the area, A_n, inside the polygon P_n equals the area A inside the circle and the limit of the perimeter C_n of P_n equals the circumference C of the circle. So we write this as

$$A = A_n \ \ and \ \ C = C_n$$

	1.3.1.LI.2	1.3.1.AS.2
	Classify left-hand and right-hand limits algebraically and with the aid of technology where available or through any creative means. **Group discussion:** **Learning Experience:** Learners in their mixed-ability groups are to initiate a discussion on the meaning of approaching a number on a real number line from the left or from the right of the number. **Activity 1: (left- and right-hand limits)** - Learners work in mixed groups and present across groups on left- and right-hand limits of a function and represent each of them algebraically. In addition, learners are to use graphs to distinguish the value a function approaches when an input approaches a number from the left or right. **Example 1:** What is the value of $g(x)$ approaches as *x* approaches the number -7 from the left or from the right on the horizontal axis? (*Refer to Fig. 1*). Learners are to share any observations made across groups. NB: **The value $g(x)$ approaches when *x* approaches the number -7 from the left is 4 and from the right is 2. This means the value $g(x)$ approaches depend on the direction (left or right) x approaches -7.** **Example 2 (notation of limits of a function):** Write down the mathematical notation; What value will $g(x)$ approach as x approaches -7 from the left? **Solution**: $g(x)$? Note*:* The negative sign on the superscript -7 means x approaching -7 from the left. And the value $g(x)$ approaches is 4, so we write, $g(x) = 4$.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

- **-What is the limit as x approaches -7 from the right?**

Solution:

$g(x) = 2$

Note: The positive sign on the superscript -7 means x approaching −7 from the right. And the value is 2.

Example 3 (Generalisation):

Type equation here.What is the meaning of $f(x) = L$?

Solution:

If f is a function, then the $f(x) = L$ means the value of $f(x)$ approaches L as x gets very close to a from the left and right.

Example 4 (Existence of a limit): Determine whether $g(x)$ exists?

Solution:

From Fig. 1, Note that $g(x) = g(x) = 2$

Hence the $g(x)$ exists.

Example 5 (limit that does not exist): Verify whether $g(x)$ exists.

Solution:

Since

$g(x) \neq g(x)$

So $g(x)$ does not exist. (Refer to Fig. 1)

Example 6 (limit of a function and function value):

Find the function value f(x) approaches as x approaches 8 and the function value at x=8, thus f (8). What is the difference? (Use fig. 2)

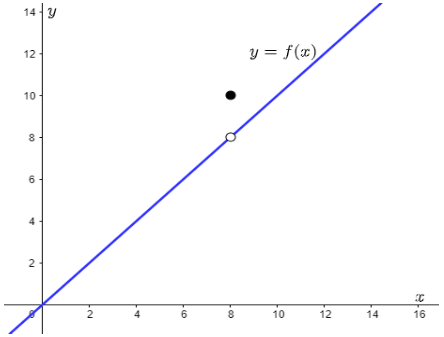


Fig. 2

Solution

The function value when $x = 8, f(8) = 10$; however, as x approaches 8, the function approaches 8.

Mini project

Learners are to work in groups to construct a table and graph f. Use technology or any innovative ways to investigate the behaviour of $f(x) = \frac{1}{x+2}$ **as** x **approaches** 2**.**

NB: x can approach 2 from the left or right with small step sizes; hence, you may use 3 input values for each side (left and right) to undertake the investigation. Thus, approaching 2 from the left, use: 1.97, 1.988, 1.9998.

And from the right use: 2.03, 2.002, 2.0001. You increase the input values.

NB: Unbounded behaviour of a limit refers to a function growing without bound (in other words, to infinity) at the limit point.

Limit Properties:

Suppose that $f(x)$ and $g(x)$ both exist, then we have the following results.

- If k is a constant, then $k \cdot f(x) = k \cdot f(x)$

- If r is a positive constant, then $[f(x)]^r = [\lim_{x\to a} f(x)]^r$
- $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) \cdot g(x)] = \left[\lim_{x\to a} f(x)\right] \cdot [\lim_{x\to a} g(x)]$
- $if \lim_{x\to a} g(x) \neq 0, then \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Limit of a polynomial function: Let $p(x)$ be a polynomial function, a of any number. Then

$$\lim_{x\to a} p(x) = p(a)$$

Limit of a rational function: Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)\ and\ q(x)$ are polynomials. Let a be a number such that $q(a) \neq 0$. Then

$$\lim_{x\to a} r(x) = r(a)$$

Example 6

Given$f(x) = -9$, $g(x) = 2$ and $h(x) = 4$

Use the limit properties to compute each of the following limits. If it is not possible to compute any of the limits, clearly explain why not. Find

a. $[f(x) - g(x) + h(x)]$

Solution:

$$\lim_{x\to 8}[f(x) - g(x) + h(x)] = \lim_{x\to 8} f(x) - \lim_{x\to 8} g(x) + \lim_{x\to 8} h(x)$$
$$= -9 - 2 + 4$$
$$= -7$$

b. $[3h(x) - 6]$

$$\lim_{x\to 8}[3h(x) - 6] = 3\lim_{x\to 8} h(x) - \lim_{x\to 8} 6$$

$$= 3(4) - 6$$
$$= 6$$

Example 7

Given $f(x) = 1,\ \ g(x) = 10$ and $h(x) = -7$. Use the limit properties to compute each of the following limits. If it is not possible to compute any of the limits, clearly explain why not.

a. $\lim\limits_{x\to -4}[f(x)g(x)h(x)]$

Solution:

$$\lim_{x\to -4}[f(x)g(x)h(x)] = \left[\lim_{x\to -4} f(x)\right]\left[\lim_{x\to -4} g(x)\right]\left[\lim_{x\to -4} h(x)\right]$$
$$= (1)(10)(-7)$$
$$= -70$$

b. $\lim\limits_{x\to -4}\left[\frac{1}{h(x)} + \frac{3-f(x)}{g(x)+h(x)}\right]$

Solution:

$$Lim_{x\to -4}\left[\frac{1}{h(x)} + \frac{3-f(x)}{g(x)+h(x)}\right]$$
$$= \lim_{x\to -4}\frac{1}{h(x)} + \lim_{x\to -4}\frac{3-f(x)}{g(x)+h(x)}$$
$$= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)} + \frac{\lim\limits_{x\to -4}[3-f(x)]}{\lim\limits_{x\to -4}[g(x)+h(x)]}$$

$$= \frac{\lim\limits_{x\to -4} 1}{\lim\limits_{x\to -4} h(x)} + \frac{\lim\limits_{x\to -4} 3 - \lim\limits_{x\to -4} f(x)}{\lim\limits_{x\to -4} g(x) + \lim\limits_{x\to -4} h(x)}$$
$$= \frac{1}{-7} + \frac{3-1}{10-7}$$
$$= \frac{11}{21}$$

Example 8

Given that $f(x) =\ 3x + 4$ and $g(x) = 7 - 4x$, verify the following limits properties for $k = 6, a = 1$ and $= 2$.

Solution:

- If k is a constant, then

$$k \cdot f(x) = k \cdot f(x)$$

$$k \cdot f(x) = 6 \cdot (3x + 4)$$

$$= (18x + 24)$$

$$= 42$$

$$k \cdot f(x) = 6 \ (3x + 4) = 6(7) = 42$$

Hence $k \cdot f(x) = k \cdot f(x)$

- If r is a positive constant, then $[f(x)]^r = [\lim_{x \to a} f(x)]^r$
- $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- $\lim_{x \to a}[f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right] \cdot [\lim_{x \to a} g(x)]$

NB:

- Some forms of limits are called **indeterminate** if the limiting behaviour of individual parts of the given expression cannot determine the overall limit. Learners must be aware of such limits and how to overcome them without necessarily going through L'Hôpital's Rule.
- **Determinate Forms**: An undefined expression involving some operation between two quantities is called a determinate form if it evaluates to a single number value or infinity.
- **Indeterminate Forms**: An undefined expression involving some operation between two quantities is called an indeterminate form if it does not evaluate to a single number value or infinity.

- *the indeterminate forms are* $\frac{0}{0}$**,** $\frac{\infty}{\infty}$**,** 0^0**,** $\infty^0, \infty - \infty, 1^\infty$ **and** $\infty - \infty$

Example 9: Find the following

a. $\frac{\sqrt{x+4}-2}{x}$

Solution:

Note that applying the properties may yield an indeterminate form; thus,

$\frac{\sqrt{x+4}-2}{x} = \frac{0}{0}$ Hence, you may have to rationalise the numerator

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{(x+4)-4}{x(\sqrt{x+4}+2)}$$
$$= \frac{x}{x(\sqrt{x+4}+2)}$$
$$= \frac{1}{\sqrt{x+4}+2}$$ the as x approaches 1

$\frac{1}{\sqrt{x+4}+2}$ approaches $\frac{1}{4}$.

b. $\frac{x^3+8}{x^2-4}$

Solution

Note: $\frac{x^3+8}{x^2-4} = \frac{0}{0}$ so we have indeterminate form and their square roots, so from the remainder theorem, there are common factors; hence

$$\frac{x^3+8}{x^2-4} = \frac{(x+2)(x^2-2x+4)}{(x-2)(x+2)}$$
$$= \frac{(x^2-2x+4)}{(x-2)}$$

Thus

$$\frac{x^3+8}{x^2-4} = \frac{x^2-2x+4}{x-2}$$

$$= -\frac{12}{4} = -3$$

Activity 2: (limit at infinity)

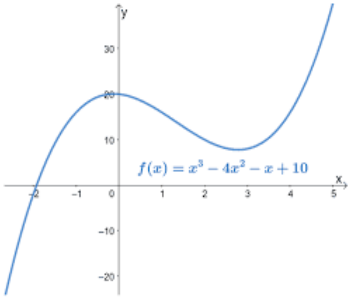
-Learners are to work in mixed groups to discuss and present across groups to find limits at infinity.

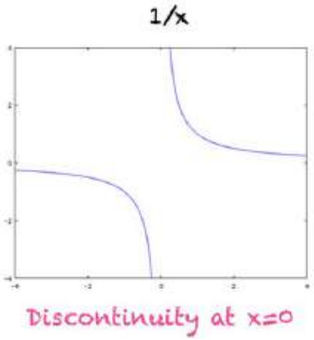
Example 1 (limit at infinity)

$$\frac{x+1}{x-1}$$

	Solution: $\lim\limits_{x\to\infty} \frac{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)}$ $\frac{1}{1} = 1$ E.g. $\lim\limits_{x\to\infty}(2x^4 - x^2 - 8x)$ Solution: $Lim_{x\to\infty}(2x^4 - x^2 - 8x) = \infty$	
	1.3.1.LI.3	1.3.1.AS.3
	Distinguish between continuous and discontinuous functions near an input value on its domain and investigate them with the use of technology or any other means appropriate. Collaborative learning, Discussions, Experiential Learning, and initiate Talk for Learning. **Learning Experience:** Learners working in mixed-ability groups investigate and classify various functions into continuous and discontinuous functions and relate these to things around them. **Activity 1 (Continuous and Discontinuous functions)** Learners are to work in groups to identify the point(s)/interval(s) of continuity or discontinuity of a function on a graph and algebraically. **Definition (Continuous at a Point)** A function f is continuous at a point a if $f(x) = f(a)$ Theorem: If f(x) is continuous at a, then the following three conditions hold; • f(a) is defined, • $f(x)$ must be defined (thus, the left and right limits must be the same and • $f(x) = f(a)$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Example:





Example 1: Find the interval(s)/point(s) $g(x)$ (Refer to Fig. 1) is(are) continuous or discontinuous
Solution: The function g is discontinuous at x= -7 or x=-4 because there are jumps at these points.

Example 2: Determine whether the following functions are continuous at x = 2. Why?
a. f(x)= 3x + 1
b. $f(x) = \frac{x-4}{x^2-4}$

- $h(x) = |x|,\ x \in R$

Solution: h is continuous for all, $x \in R$

Activity 2:
Learners are to work in groups and discuss how to use GeoGebra/technological tools or any appropriate means to investigate and identify the points of discontinuity:

- $f(x) = \frac{1}{x^2-3x-4}\ for\ \ 5 \leq x \leq 6$

NB:

- Learners should be made to understand that from the concepts of limit, the value $\mathrm{f}(c)$ plays no role in the definition of the limit of $f(x)$ as x tends to c. We do not even assume that f is defined at c.

	• A definition of “limit” that requires f to be defined at c would not apply to many important applications. Even when $f(c)$ is defined, the value $f(c)$ does not have to be related to $f(x)$ in any way. In a sense, $f(x)$ is what we anticipate that $f(x)$ will equal at $x = c$, not necessarily what $f(x)$ actually equals when $x = c$. • When we drive along a high-speed road, we frequently cannot see very far ahead. We learn to drive as though the portion of the road immediately beyond our vision is the obvious extension of the portion of the road that we can see. What we are doing is computing the limit of what we see. Usually, the actual state of the road coincides with what we have anticipated it will be (that is, the road is continuous). If the road has been damaged, or if a bridge is out, then the actual state of the road does not coincide with what we have anticipated (the road is discontinuous). In this section, we develop mathematical analogues of these ideas.	
	1.3.1.LI.4	1.3.1.AS.4
	Use the limits of a function to find its derivative. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners working in mixed-ability groups investigate how the limit of a function relates to its derivative. **Activity 1:** Learners working in mixed-ability groups investigate both algebraically and graphically how the limits of a function relate to its derivative. **Example**: Imagine that a training film is taken of a runner. The film shows elapsed time and distance markers that allow us to measure the distance $s(t)$ the athlete has run in any given time t. However, the speed of the runner will vary from one instant of time to another. How can the runner’s speed $v(t)$ at a given instant of time t be calculated? $\frac{s(u) - s(t)}{u - t}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

For *u* close to *t*, this will give a good approximation to $v(t)$ because the runner's speed does not change much in the small-time interval $[t, u]$. If we press this point further, then we intuitively arrive at the concept of instantaneous velocity at *t*:

$$v(t) = \lim_{u \to t} \frac{s(u) - s(t)}{u - t}$$

The number u cannot actually be equal to t because that results in the meaningless fraction $0/0$. The key is first to understand exactly what is meant by the limits in these equations and then to learn methods for computing them.)

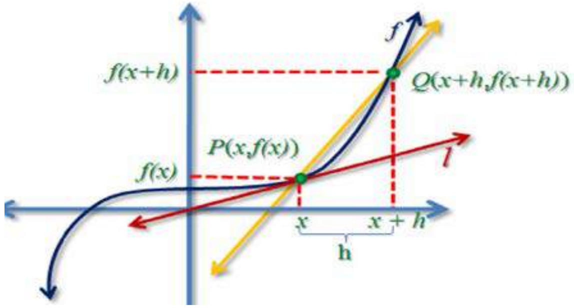


Fig 3

Activity 2: Learners are tasked in groups to brainstorm on Fig 3 to establish the link between limits and derivative of a function given by

$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h}$$

NB:

- There are different notations used for the derivative of a function. For example, the derivative for a function $y = \mathrm{f(x)}$ could be written as $f'(x)$, $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$.
- The slope of the tangent line is also referred to as the instantaneous rate of change of F at x.

Using limits to find a derivative of $y = f(x)$:

- Find $f(x+h)$.
- Write the difference quotient, $\frac{f(x+h)-f(x)}{h}$.
- Simplify the difference quotient.
- Find the limit as $h \to 0$.

Example 1: Learners are to discuss how to find the derivative of the functions$f$, g and p using the first principle:

- $t(x) = 2x^2$
- $g(x) = 6x + 4$
- $p(x) = 3x^2 - 6x + 7$

Solution: (i)

$t(x) = 2x^2$.

Let h be a very small change which approaches 0

At point $x + h$**,** $t(x+h) = 2(x+h)^2 = 2(x^2 + h^2 + 2hx) = 2x^2 + 2h^2 + 4hx$

The link between the limit of a function and the derivative of a function $y = t(x)$ is given by

$\frac{dy}{dx} = \frac{t(x+h)-t(x)}{h} = \lim\limits_{h\to 0} \frac{(2x^2+2h^2+4hx)-(2x^2)}{h} = \lim\limits_{h\to 0} \frac{(2h^2+4hx)}{h} = \lim\limits_{h\to 0} (2h + 4x) = 4x$

Example 2: (Continuous function that is not differentiable) - Learners are to work in groups and use limits to find the derivative of $f(x) = |x|$,

Solution:

$f(x) = |x|$ is the distance from the origin

$f(x) = x \;\; for\; x \geq 0$ and

$f(x) = -x \;\; for\; x < 0$

then

$f(x+h) = |x+h|$ means

$f(x+h) = x + h \;\; for\; x + h \geq 0$ and

$f(x+h) = -(x+h) \; for\; x + h < 0$ so for x>0

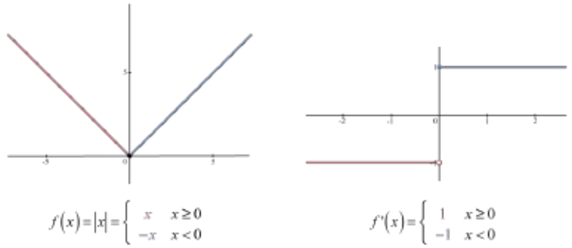
$$\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{x+h-x}{h} = 1$$

$$\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-}\frac{-(x+h)-(-x)}{h}$$

$\lim\limits_{h\to 0^-}\frac{-(x+h)-(-x)}{h} = -1$

Since the slope of the left side equals -1 and the slope of the right side equals +1, they disagree; hence, the function is not differentiable at x = 0.

Illustration:



$f(x)=|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$ $f'(x)=\begin{cases} 1 & x\geq 0 \\ -1 & x<0 \end{cases}$

Fig. 4

NB:

- Learners are confronted with the conceptual understanding of why to connect the limits of a function to its derivative. Example 2 reinforces the need to connect limits to the derivatives.
- All differentiable functions are continuous. However, not all continuous functions are differentiable; hence, differentiability is stronger than continuity.

Example 3: Identify the values of x if the graph in Fig 5 is not differentiable.

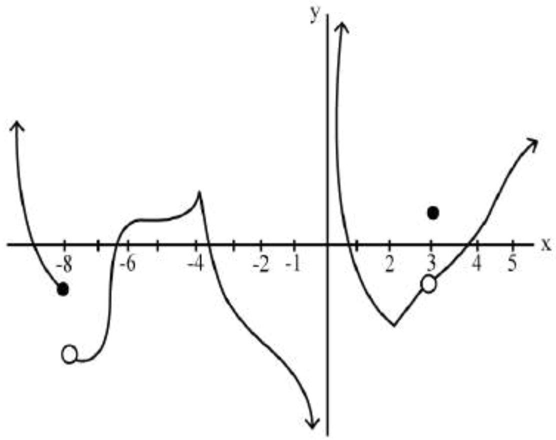


Fig. 5

Solution
Values for x=-8, 0 and 3 are not differentiable and are discontinued, and values x=-4 and 2 are also not differentiable; however, they are continuous.
Activity 4 (Generalization)
Learners are to work in mixed-ability groups to establish the general formula for the function $f(x) = x^n$ where n is a natural number.

Recall that $f(x+h) = (x+h)^n$.
From the binomial expansion, we have
$f(x+h) = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 \ldots\ldots\ldots\ldots.. + h^n$
$f(x+h) - f(x) = nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 \ldots\ldots\ldots\ldots.. + h^n$

$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h} = nx^{n-1}$$

Example 1

Find the derivative of the following;

- $t(x) = 2x^2$
- $g(x) = 6x + 4$
- $p(x) = 3x^2 - 6x + 7$
- $f(x) = -x^3 + 4x^2 + 9x$
- $h(x) = 3x^{-2} - 4x^{-3} + 7$
- $h(t) = 5t^{-2} + t^6$
- $f(x) = \sqrt{x}$

Solution

- $\frac{dt(x)}{dx} = 4x$
- $\frac{dg(x)}{dx} = 6$
- $\frac{dp(x)}{dx} = 6x - 6$

Derivative of sin and cosine

Theorem: (Trigonometry)

$\frac{d}{dx} sin\, sin\, (x) \ = cos\, (x)$ and $\frac{d}{dx} cos\, cos\, (x) \ = -sin\, (x)$

Proof

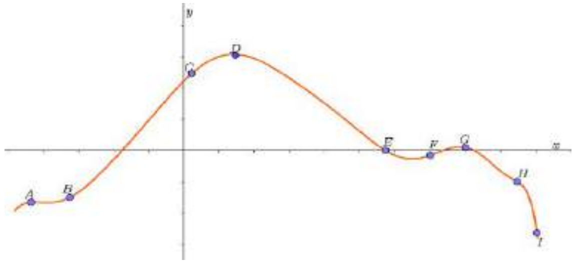
$$\frac{d}{dx} sin\, sin\, (x) = \frac{sin\, sin\, (x+h)\ - sin\, (x)}{h}$$

$$= \frac{sinsin\, (x) coscos\, (h) + sinsin\, (h)\, cos\, (x) - sin\, (x)}{h}$$

$$= \frac{sinsin\, (x)[coscos\, (h) - 1] + sinsin\, (h)\, cos\, (x)}{h}$$

$$\frac{sin\, sin\, (x)[cos\, cos\, (h) - 1]}{h} + \frac{sin\, sin\, (h)\, cos\, (x)}{h}$$

	$\frac{cos\,(h)-1}{h}\, sin\,(x) + \frac{sin\, sin\,(h)\, cos\,(x)}{h}$ Note that $\frac{cos\,(h)-1}{h} = 0$ and $\frac{sin(h)}{h}$=1 Hence $\frac{d}{dx} sin\, sin\,(x) = cos\,(x)$ Similarly, learners should work in pairs to establish $\frac{d}{dx} cos\, cos\,(x) = -sin\,(x)$	
	1.3.1.LI.5	1.3.1.AS.5
	Use technology or any innovative ways to investigate the rate of change of a function, h (u), with respect to u. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners working in mixed-ability groups investigate how the limit of a function relates to its derivative. **Activity 1:** Learners working in mixed-ability groups and deliberate on the link between the rate of change of a function and derivative. For example, in pairs, discuss the rate of change at a point for a given function $h(u)$ and compare the rate of change at different points using a technological tool, software or any innovative way. **Example 1:** On a graph sheet/technological tools/software or graphical calculator, draw the graph $h(u) = u^2 - 6u + 5$ use u values between 0 and 8 inclusive and determine the rate of change of the function at u= 0.5,3,4 and 2.5. **Example 2:** Assuming you are walking along the curve below, then ask learners to discuss the rate of change at a point on the curve. What point will it be the same?	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

		
	1.3.1.LI.6	1.3.1.AS.6
	Generalise the behaviour of a moving object along a path or curve. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners working in mixed-ability groups investigate the behaviour of a moving object along a path or curve and provide their generalisations for whole class discussions. **Activity 1** Think-pair share: Ask Learners to draw a graph and investigate the behaviour of an object moving along it with respect to derivatives **Activity 2:** In pairs, discuss and find the domain's value(s)/interval(s) of the function f increases, decreases or no change as x increases. **Example 1:** Is the function $f(x) = 3x + 4, x \in R$ increases, decreases or no change? **Solution:** $\frac{df(x)}{dx} = 3 > 0$, so the function f increases. **Example 2:** Is the function $f(x) = -5x + 4,\ x \in R$ increases, decreases or no change as x increases?	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Solution:
$\frac{df(x)}{dx} = -5 < 0$ so, the function decreases as x increases.

Example 3: Is the function $f(x) = 7$ increases, decreases or no change as x increases?

Solution:
$\frac{df(x)}{dx} = 0$. This means it is momentarily at rest at that x value.

NB: The following three (3) are true for a given function, f, as x increases:

- f(x) increases for all values of x for which $\frac{df(x)}{dx} > 0$.
- f(x) decreases for all values of x for which $\frac{df(x)}{dx} < 0$.
- f(x) momentarily at rest for all values of x for which $\frac{df(x)}{dx} = 0$.

Example 4: Find the value(s) of x for which $f(x) = 2x^3 - 9x^2 - 24x + 56$ increases, decreases or momentarily at rest

Solution:
Step 1. Differentiate the given function:
$f(x) = 2x^3 - 9x^2 - 24x + 56$
That is,

$$\frac{df(x)}{dx} = 6x^2 - 18x - 24$$

Step 2. Find the values of x for which f is momentarily at rest:

$$\frac{df(x)}{dx} = 6x^2 - 18x - 24 = 0$$

$$6x^2 - 18x - 24 = 0$$

$$(x-4)(x+1) = 0$$

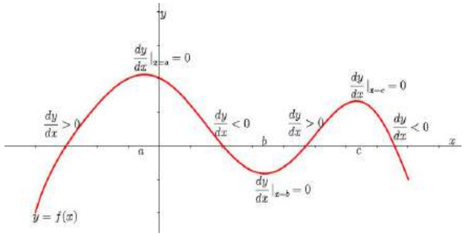
Hence, at x=4 or -1, f is momentarily at rest.

Step 3. Use the values of x in step 2 to partition the real number line to obtain a real interval. Note that values that are momentarily at rest should not be part of the intervals constructed:
Partitioned intervals: $(-\infty,-1)$ $(-1,4)$ $(4,+\infty)$

Step 4. Test $\frac{df(x)}{dx}$ on the intervals in step 3 to arrive at the desired results:
Select any value in the interval $(-\infty,-1)$ to test. Set x=-2 and test
$\frac{df(-2)}{dx}=6(-2)^2-18(-2)-24=36>0$, which implies f is increasing on $(-\infty,-1)$.
Select any value in the interval $(-1,4)$ to test. Set x=3 and test
$\frac{df(3)}{dx}=6(3)^2-18(3)-24=-26<0$, which implies f is decreasing on $(-1,4)$.
Select any value in the interval $(4,+\infty)$ to test. Set x=5 and test.

$$\frac{df(5)}{dx}=6(5)^2-18(5)-24=36>0$$

which implies that f is increasing on $(4,+\infty)$.
NB: One may use a graphical approach



Teaching and Learning Resources

- GeoGebra
- PhET
- Technology tools
- Mathematical sets
- Calculators
- Learners textbooks
- Graph sheets

Subject ADDITIONAL MATHEMATICS
Strand 3. CALCULUS
Sub-Strand 2. APPLICATIONS OF CALCULUS

Learning Outcome	21st Century Skills and Competencies	GESI, SEL and Shared National Values
1.3.2.LO.1		
Determine the equation of tangents and normal to a curve at a given point	**Communication and Collaboration:** • As they work in groups, learners learn to share ideas and improve their problem-solving and critical thinking skills: • The activities that require learners to critique others work on investigating the behaviour of objects moving along a curve. **Digital literacy: Learners** are to interpret graphs (on graph sheets and from GeoGebra, etc. **Critical thinking** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they determine the equation of tangents and normal to a curve at a given point. Using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to determine the equations of tangents and normal. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-*

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions.	*Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and

		cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they determine the equation of tangents and normal to a curve at a given point.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
1.3.2.CS.1	1.3.2.LI.1	1.3.2.AS.1
Demonstrate the ability to apply differentiation to find equations of tangents, normal to a curve and rate of change.	**Use knowledge of differentiation to determine the equation of tangents and normal to curves at a given point.** **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners working in mixed-ability groups explore ways of finding the equation of the tangent to a curve. **Activity 1:** **Think-pair share:** Learners in their mixed-ability groups use their previous knowledge of finding the equation of a line to investigate and discuss how to determine the equation of a line tangent to a curve. **Example 1:** Find the equation of the tangent line to the curve $y=6x^2$ at the point (1,6) **Solution:** **Step 1:** Find the slope of the tangent: $\frac{dy}{dx} = 12x$, so at (1,6), the slope of the tangent will be $m = 12(1) = 12$ **Step 2:** Using the knowledge in finding the equation of a line, we have, $y = 12x - 6$ **Activity 2:** Group work/Collaborative learning: Task learners in their mixed-ability groups to use their previous knowledge of finding the equation of a line to find the equation of a line normal to a curve $y=6x^2$ at the point (1,6) **Note:** Learners must understand that the tangent line to a circle is always perpendicular to the radius of a circle at a point. If the tangent line passes through a point, then the circle lies on one side of the circle. The tangent line passes through a point and intersects at only that point.	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	1.3.2.LI.2	1.3.2.AS.2
	Apply differentiation to find the rate of change. **Collaborative learning, Experiential Learning, and initiate Talk for Learning.** **Learning Experience:** Learners working in mixed-ability groups explore ways of finding the rate of change of a given function or phenomenon at certain points. **Activity 1:** **Think-pair share:** Learners in a mixed-ability group use their previous knowledge of finding the slope between two points to find the rate of change at a point. **Example 1:** A dropped ball has height (m) $h(t)=100-4.9t^2$, t seconds after it is released. How fast is the ball going at time t=2? **Solution:** At time t=2 height is $100-4.9*(2^2)= 80.4$. A second later, that is, at t=3, what will be the height? **Solution:** The height, h (3) =100-4·9*(9)=55.9, so in that second, the ball has travelled 80.4-55.9=24.5 meters. This means that the average speed during that time was 24.5 meters per second. **Note:** If Δt is some tiny amount of time, what we want to know is what happens to the average speed (h(2)-h(2+Δt))/Δt as Δt gets smaller and smaller. Doing a bit of algebra: $\frac{(h(2)-h(2+\Delta t))}{\Delta t}=\frac{(80.4-(100-4.9(2+\Delta t)^2)}{\Delta t}$ $=\frac{(80.4-100+19.6+19.6\Delta t+4.9\Delta t^2}{\Delta t}$ $=\frac{(19.6\Delta t+4.9\Delta t^2}{\Delta t}$ $=19.6+4.9\Delta t$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

When Δt is very small, this is very close to 19.6, and indeed, it seems clear that as Δt goes to zero, the average speed goes to 19.6, so the exact speed at t=2 is 19.6 meters per second.

When the COVID-19 hit Accra, Public Health Officials estimated that the number of persons having the COVID-19 at time t (measured in days from the beginning of the COVID-19) is approximated by $P(t)=45t^2- t^3$, provided that $0\leq t\leq 50$.
At what rate is the COVID-19 spreading when t=10?
When is the COVID-19 spreading at the rate of 600 per day?

Solution: The rate at which the COVID-19 is spreading is given by the derivative; $P'(t)=90t-3t^2$ $(0\leq t\leq 50)$
Since P(t) is measured in people and time is measured in days, the rate P'(t) is measured in people per day. When t=10
$P'(10)=90(10)-3(10)^2 =600$
Thus, 10 days after the beginning of COVID-19, it spreads at the rate of 600 people per day.

In this case, we are given the rate of change of P(t), and we must find the time corresponding to that rate. We set the expression for P'(t) equal to 600 and solve for t;
$90t-3t^2 = 600$
$90t-3t^2 -600 = 0$
Dividing by -3 and factoring, we have
$t^2-30t+200=0$
$(t-10)(t-20)=0$
Then t=10 or t=20. At both times COVID-19 is spreading at the rate of 600 people per day.

Example 2: Suppose that $f(x)=x^{2.}$ Calculate the average rate of change of f(x) over the intervals 1 to 2, 1 to 1.1, and 1 to 1.01
Determine the (instantaneous) rate of change of f(x) when x=1

Solution: The intervals are of the form 1 to $1+\Delta x$ for $\Delta x=1,0.1$, and 0.01. The average rate of change is given by the ratio $\Delta y/\Delta x=(f(1+\Delta x)-f(1))/\Delta x=(〖(1+\Delta x)^2 -1)/\Delta x$
For the three given values of Δx, this expression has the following respective values

	Δx=1: Δy/Δx= $(2^2-1^2)/1=(4-1)/1=3$ Δx=0.1: Δy/Δx= (〖1.1〗^2-1^2)/0.1=(1.21-1)/0.1=2.1 Δx=0.01: Δy/Δx= (〖1.01〗^2-1^2)/0.01=(1.0201-1)/0.01=2.01 Thus, the average rate of change for Δx=1,0.1, and 0.01 is 3,2.1,2.01 units per unit change in x, respectively The instantaneous rate of change of f(x) at x=1 is equal to f'(1). We have f^' (x)=2x f^' (1)=2·1=2 That is, the instantaneous rate of change is 2 units per unit change in x. **Example 3:** Let the production function p(x) give the number of units of goods produced when employing x units of labour. Supposed 4000 units of labour are currently employed, p(4000)=200, and p'(4000)=4. Estimate the number of additional units of goods produced when employing: One additional unit of labour An additional 1/4 unit of labour One less unit of labour.	
Teaching and Learning Resources	• GeoGebra and PhET • Technology tools, mathematical sets and calculators • Learners' textbooks and graph sheets • curriculum	

Subject ADDITIONAL MATHEMATICS
Strand 4. HANDLING DATA
Sub-Strand 1. ORGANISING, REPRESENTING AND INTERPRETING DATA

Learning Outcomes	**21st Century Skills and Competencies**	**GESI, SEL and Shared National Values**
1.4.1.LO.1		
Collect quantitative and qualitative data, and organise and present data using graphs.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI**: Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they collect, organise and present data using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of individuals in the different groups as they learn about data collection, organisation and presentation. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse.

	phenomena. (Learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** • Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. • Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. Cut out geometrical shapes of different colours and orientations. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling; and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life	• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies *–Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-

		regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they collect quantitative and qualitative data and organise and present data using graphs.

1.4.1.LO.2		
Calculate the measures of central tendencies, and measures of dispersion and use simple language to interpret the results.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** • Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they calculate measures of central tendency, measures of dispersion and interpret the results using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to calculate the various measures. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are

	groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. • Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. Cut out geometrical shapes of different colours and orientation. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life	integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities.

		National Core Values: **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they calculate the measures of central tendencies, and measures of dispersion and use simple language to interpret the results**.**

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
1.4.1.CS.1	1.4.1.LI.1	1.4.1.AS.1
Investigate techniques for collecting data and determine measures of central tendency and dispersion.	**Identify and present appropriate ways of collecting and representing data.** **Project-based learning, Talk for Learning, Group work, Presentation and Building on what others say.** **Learning Experience:** Learners conduct research on the importance of statistics in real life, the importance and uses of data, and the process of collecting data from a survey or census. Learners to present their findings to the class. **Activity 1:** Learners in groups make a presentation on the importance of statistics in real life. **Activity 2:** Learners present their findings on the importance of data and their uses to make informed decisions and policies. **Activity 3**: Learners present the processes of obtaining data using different data collection processes such as interviews, questionnaires, observation, etc. **Activity 4: Learners** present their findings on sampling techniques such as random, convenient, purposive, etc. and state an ideal situation where each technique can be used.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning
	1.4.1.LI.2	1.4.1.AS.2
	Categorise data and determine which scale of measurement describes the data. **Project-Based Learning, Think-pair-share, Group discussions, Talk for Learning** **Learning Experience:** Learners in groups categorise data and determine the appropriate scales of measurement for describing the data.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Activity 1:

- Learners in groups discuss the types of data, that is, quantitative and qualitative data and share their findings with the class
- Learners in groups explain quantitative data and qualitative data.
- Learners in groups state examples of quantitative and qualitative data.

Example of qualitative data: Interview data gathered from aspiring prefects in school, gender of a student, a person's nationality, colour preference, etc.

Examples of quantitative data: Test scores, age of students, time for completing a race, number of students reading various courses in school, etc.

Activity 3:

- Learners' groups discuss special features of qualitative and quantitative data in order to categorise them under nominal, ordinal, interval and ratio.
- Through brainstorming and discussion, as well as building on what others say, learners in groups discover that qualitative data can be further classified under nominal and ordinal and give examples
- Learners discover that nominal scale is used to classify data without any order or quantitative value.

Examples of nominal scale: Gender (Male, Female), Position on a ballot for an election, ethnic group, etc.

- Learners discover that ordinal scale is used to classify data with an order; that is, it shows a sequence.

Example of ordinal scale: Grades in exams (A1, B2, B3,…), Level of Education (Primary, JHS, SHS), Level of happiness (Very happy, Not happy, A little happy).

	• Through brainstorming and discussion, as well as Building on what others say, learners in groups discover that quantitative data can be further classified under interval and ratio scales and state-specific examples of each. • Learners in groups discover that the interval scale involves order and difference between variables, and their values can be added or subtracted, though it does not have a true zero. **Examples of Interval scales:** Temperature, and test scores. • Learners in their groups discover that the ratio scale involves the order of variables and the differences between them, and they have absolute zero but can't have a negative value. **Examples of Ratio scales**: are height of an object, weight, age, etc.	
	1.4.1.LI.3	1.4.1.AS.3
	Organise data into appropriate frequency distribution tables manually, and with Microsoft Excel. **Group discussions, Talk for Learning and experiential learning.** **Learning Experience:** Learners in mixed-ability groups discuss grouped and ungrouped and how to organise data by hand and/or with the use of Excel spreadsheet. **Activity 1:** • Learners review ungrouped (raw) data and organise it using frequency tables • Learners in their groups discover that ungrouped data is the data you gather from an experiment that is not sorted into categories or classified. **Example 1:** The ages of some randomly selected football players are as follows 24,23,25,23,30,24,37,25,23, 22,25,22,31,29, 22,25,21,25,24,24,22 1. Make a frequency table for the data. 2. Represent this information on a graph or chart.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Solution:

- Frequency Table

Age	**Tally**	**Frequency**
21	/	**1**
22	////	**4**
23	///	**3**
24	///	**3**
25	////	**5**
29	/	**1**
30	/	**1**
31	/	**1**
37	/	**1**
Total		**20**

- Learners in their groups discuss the importance of organising data into a frequency table.
- Learners in their groups solve a practical question and share their findings with the class.

Activity 2:

- Learners in their groups discuss grouped data.
- Learners in their groups discover that grouped data is data that has been bundled together in categories; that is, when given a large data, it is better to group them using intervals.

Example of interval for grouped data: 1-9, 10-19, 20-29 etc.

- Learners discover through interaction that the groups into which the entries are put are referred to as classes.

Example 2: 1-9, 10-19 and 20-29 are the classes.

- Learners in their groups discuss class limits, that is, lower- and upper-class limits.

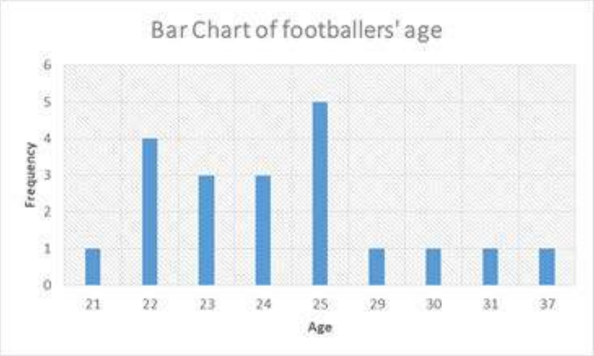
	Example 3: Taking 1-9 as the class, 1 represents the lower-class limit, and 9 represents the upper-class limit. • **Learners discover that the formula used to calculate class boundaries is given as** $\frac{1}{2}(upper\ limits\ of\ one\ class + lower\ limit\ of\ the\ next\ class)$, is. **Example 4:** • Learners in their groups apply their knowledge in finding midpoints to find the class midpoint, which is the average between the lower- and the upper-class limits. • Learners in their groups discuss class width/size. Learners discover that class width/size is the difference between the upper- and lower-class boundaries of a class.	
	1.4.1.LI.4	1.4.1.AS.4
	Present data using appropriate graphs by hand and/or by technology and justify why a particular representation is more suitable than others for a given situation. **Group discussions, Talk for Learning, experiential learning, building on what others say.** **Learning Experience:** Learners in mixed-ability groups discuss how to represent data. **Activity 1:** • Learners review various ways of representing data in statistics and solve practical examples**.** • Learners in groups represent a given data using pie charts, bar charts, line graphs, etc. Each group uses a different graph or chart to represent data and explain it to the class. **Example 1:** The ages of some randomly selected football players are as follows 24,23,25,23,30,24,37,25,23, 22,25,22,31,29,22,25,21,25,24,24,22. Represent this information on a graph or chart.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Solution:

Frequency Table

Age	Tally	Frequency
21	/	1
22	////	4
23	///	3
24	///	3
25	////	5
29	/	1
30	/	1
31	/	1
37	/	1
		$\sum f = 20$

Bar Chart



- Other groups represent data using a pie chart and line chart.
- Learners discuss which graph/chart is more appropriate for the representation of data.
- Learners discover that:
- **Bar charts** are used to show numbers that are independent of each other. For example, ages of footballers, preference for a particular food.

	• **Pie Charts** are used to show how a whole is divided into different parts. For example, how a budget is spent on items in a home within a month. • **Line graphs** are used to show trends over time between numbers that are connected. For example, temperature in each month of the year. **Activity 2:** Learners in groups explore how to input and analyse data using software such as Excel or any other spreadsheet programme or software.	
	1.4.1.LI.5	1.4.1.AS.5
	Present grouped data using appropriate graphs by hand and/or by appropriate technology and justify why a particular representation is more suitable than others for a given situation. **Group discussions, Talk for Learning, experiential learning, building on what others say.** **Learning Experience:** Learners in mixed-ability groups discuss how to represent data using histograms and cumulative curves. **Activity 1**: • Learners in groups recall how to represent data using histogram with equal intervals. • Learners in groups collaborate and solve practical questions by representing data using a histogram with equal intervals. • Learners recollect how to estimate the measures of central tendencies using the histogram. **Activity 2:** • Learners in their groups brainstorm to discover how to represent data using a histogram with unequal intervals. • Learners in groups recognise that data given for such histograms have unequal intervals.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Example 1: 1-5, 5-15, 15-20, 20-35 etc.

- Learners in groups discover that in order to represent data with unequal intervals, it is necessary for the area of each bar in a histogram, rather than the height, to represent the frequency.
- Learners discover that to draw a histogram for unequal class intervals, you need to adjust the heights of the bars so the area is proportional to the frequency.

NOTE: The height of the bar, called the **frequency density**, is found by dividing the frequency by the class width, i.e. $frequency\ density = \frac{Frequency}{class\ width}$

- Learners solve a practical example of data with unequal intervals.

Example 1: The table gives information about the speed of cars in km/h of 81 cars

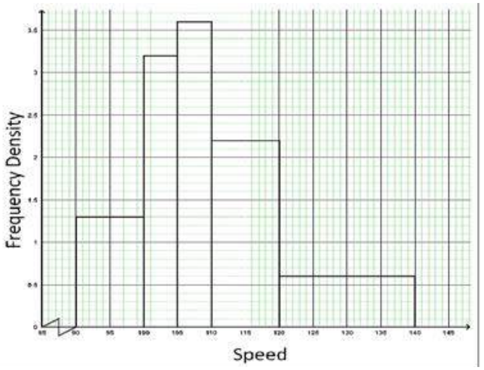
Speed(s) km/h	**Frequency**
$90 < s \leq 100$	13
$100 < s \leq 105$	16
$105 < s \leq 110$	18
$110 < s \leq 120$	22
$120 < s \leq 140$	12

Draw a histogram to represent the information in the table.

Solution:
Frequency table

Speed	Class boundaries	Frequency	Class width	Frequency density
$90 < s \leq 100$	90	13	10	1.3
$100 < s \leq 105$	100	16	5	3.2

$105 < s \leq 110$	105	18	5	3.6
$110 < s \leq 120$	110	22	10	2.2
$120 < s \leq 140$	120	12	20	0.6



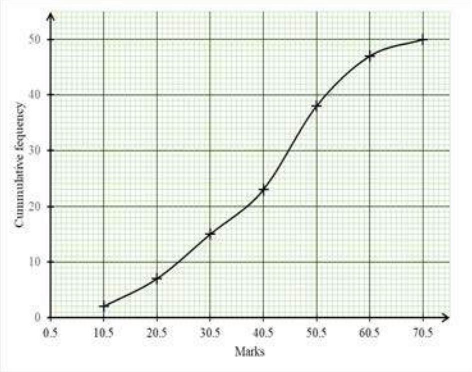
- Learners in groups create practical examples and solve them in their various groups.

Activity 3:

- Learners in their collaborative groups construct cumulative frequency curves (ogive) to represent data.
- Learners in their collaborative groups recollect how to construct cumulative frequencies to represent data.
- Learners solve practical examples.

Example 1: The distribution of marks for 50 students in a test is given in the table. Draw a cumulative frequency curve to the distribution.

Marks	Frequency
1-10	2
11-20	5
21-30	8
31-40	8
41-50	15
51-60	9
61-70	3



	1.4.1.LI.6	1.4.1.AS.6
	Compare various statistical representations and justify why a particular representation is more suitable than others for a given situation. **Group discussions, Talk for Learning and research.** **Learning Experience:** Learners in mixed-ability groups conduct research on the various statistical representations discussed in the classroom and justify why a particular representation is more suitable for a particular data	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	Activity 1: • Group leaders to lead discussions on the comparison of the various statistical representations. • Groups present their findings to the whole class.	
	1.4.1.LI.7	1.4.1.AS.7
	Calculate measures of central tendencies (mode, mean and median) for a given data by formulas or other techniques and establish which is appropriate to report on a given data. **Think-pair-share, Group discussions, Talk for Learning and experiential learning.** **Learning Experience:** Learners in mixed-ability groups discuss measures of central tendencies, how to calculate them given grouped and ungrouped data, and discover which is appropriate to report on a given data. **Activity 1:** • Learners in pairs identify the measures of central tendencies they have encountered previously as mode, median and mean. • Learners in pairs establish that mode median and mean are also known as averages and can be used to describe data. • Learners in pairs recollect the concept of mode, median and mean and how to calculate the three measures of central tendencies given raw data. **Example 1:** Find the mode, median and mean of $3,20,5,7,18,16,5,11,4,5,3$. • Learners in pairs recollect that **mode** is the value that occurs most frequently in a given data. Therefore, the mode in Example 1 is 5. • Learners in pairs recollect the **median** is the middle value when data is arranged in ascending or descending order, and the number of values is odd. Therefore, the data in example 1 needs to be arranged as $3,3,4,5,5,5,7,11,16,18,20$. Therefore, the median value is 5. • Learners further discover that when the number data is even, then the **median**= $\frac{sum\ of\ middle\ two\ values}{2}$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Example 2: Find the median of $3,5,7,9,11,13$.

Solution: Median= $\frac{7+9}{2} = 9$

- Learners in pairs recollect that the **mean,** for **Example 1** is calculated as $\frac{3+3+4+5+5+5+7+11+16+18+20}{11} = 8.818$. Therefore, the **mean** is approximately 8.8
- Learners in pairs solve practical examples of measures of central tendencies.

Example 1: Akua is an Art student, The following are her scores for the first term examinations: $58,67,60,84,93,97,98$. Calculate her mean score.

Solution: Akua's mean score $= \frac{58+67+60+84+93+97+98}{7}$

$= 79.6$

Activity 2:

- Learners in groups extend their knowledge in finding measures of central tendencies of raw data to grouped and ungrouped data.
- Learners in groups first brainstorm how to calculate measures of central tendencies of ungrouped data where some of the values are repeating.
- Learners to solve practical examples using real data.

Activity 3:

- Learners in groups extend their knowledge and conceptualise finding measures of central tendencies of grouped data.
- Learners apply their knowledge and understanding of grouped data to construct a frequency table and calculate the measures of central tendency.

Example 1:

The data below shows the mass of 40 students in a class. The measurement is to the nearest kg.

55	70	57	73	55	59	64	72
60	48	58	54	69	51	63	78
75	64	65	57	71	78	76	62
49	66	62	76	61	63	63	76

52	76	71	61	53	56	67	71

- Construct a frequency table for the data using the 45-49, 50-54, etc.
- Calculate the mean, mode and median.

Solution

- Learners in groups construct a frequency table with the details of grouped data.

Mass (kg)	**Frequency**	**Class boundaries**	**Class mark** (x)	fx
45 –49	2	44.5-49.5	47	94
50 –54	4	49.5-54.5	52	208
55 –59	7	54.5-59.5	57	399
60 –64	10	59.5-64.5	62	620
65 –69	4	64.5-69.5	67	268
70 –74	6	69.5-74.5	72	432
75-79	7	74.5-79.5	77	539
	$\sum f = 40$			$\sum fx = 2560$

- Learners calculate the mean (direct mean) using the formula $Mean\ (\underline{x}) = \frac{\Sigma fx}{\Sigma f}$
- $\underline{x} = \frac{2560}{40}$
- $\underline{x} = 64\ kg$
- Learners in groups identify the mode by inspection and by using a formula.
- By inspection identify 62 as the mode.
- By formula
- $mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) C$

Where

L_1 =Lower class boundary of the modal class,

Δ_1=excess frequency of modal class over the frequency of the next lower class.
Δ_2=excess frequency of modal class over the frequency of the next higher class, and
C=size of modal class

- Applying the formula to solve the mode,
- Mode $= 59.5 + \left(\frac{3}{3+4}\right) \times 4$
- ≈ 61.2143

Therefore, the mode is 62 to the nearest whole number.

- Learners in groups brainstorm to find the median. Learners apply the formula $median = L_1 + \left(\frac{\frac{1}{2}N - \sum F_1}{F_m}\right) C$

Where,
L_1 = the lower class boundary of the median class;
N = total frequency;
$\sum F_1$ = sum of frequencies of all classes lower than the median class;
F_m = frequency of the median class;
C = the size of the median class.

- Applying the formula in their collaborative group

$$Median = 59.5 + \left(\frac{\frac{1}{2}(40) - 13}{10}\right) \times 4$$

$$Median = 62.3$$

- Learners in groups brainstorm to calculate the assumed mean formula, which is

$\underline{x} = A + \frac{\sum fd}{\sum f}$

where,
A =the assumed mean
$d = x - A$, which is the deviation from the assumed mean.

Mass (kg)	f_i	**Class mark** (x_i)	**Deviation** (d) $x_i - A$	$f_i d_i$
45 –49	2	47	$47 - 62 = -15$	-30
50 –54	4	52	$52 - 62 = -10$	-40
55 –59	7	57	$57 - 62 = -5$	-35
60 –64	10	62	$62 - 62 = 0$	0
65 –69	4	67	$67 - 62 = 5$	20
70 –74	6	72	$72 - 62 = 10$	60
75-79	7	77	$77 - 72 = 5$	35
	$\sum f$=40			$\sum fd$=10

- Learners in groups select the central value as the assumed mean. With regard to the question, the assumed mean is 62.
- Learners in groups determine the values in the table and find the totals needed.
- Once the totals are found, learners go ahead and calculate the assumed mean
  1. $\underline{x} = 62 + \frac{10}{40}$
  2. $\underline{x} = 62.4$

	Activity 3: • Research work on how to select the assumed mean. • Learners in groups research how to select the assumed mean and share findings with the whole class.	
	1.4.1.LI.8	1.4.1.AS.8
	Work out simple measures of dispersion (range, quartile and, inter-quartile, etc.) for raw data and interpret them in context. **Group discussions, Talk for Learning and experiential learning.** **Learning Experience:** Learners in mixed-ability groups discuss estimation measures of dispersion. **Activity 1:** • Learners in groups estimate quartiles of a given data. • Learners recollect and construct cumulative frequency curves of a given data. • Learners brainstorm and discover how to estimate quartiles from a cumulative frequency curve. • Learner in their groups discover that; • The **lower quartile** corresponds to the **25th percentile,** i.e. 25% of the total frequency. • **The median** corresponds to the **50th percentile,** i.e. 50% of the total frequency. • The **upper quartile** corresponds to the **75th percentile,** i.e. 75% of the total frequency. • The interquartile range = $upper\ quartile - lower\ quartile$ • Learners solve practical examples and estimate the quartiles by plotting them on the cumulative frequency curve. **Example 1:** The distribution of marks for 50 students in a test is given in the table. Draw a cumulative frequency curve to the distribution and use your graph to find the median, semi-interquartile range and pass mark if 30% of the students passed the test.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Marks	**Frequency**
1-10	**2**
11-20	**5**
21-30	**8**
31-40	**8**
41-50	**15**
51-60	**9**
61-70	**3**

Activity 2:

- Learners in the collaborative groups work out measures of dispersion (Standard deviation).
- Learners in their collaborative groups conceptualise standard deviation and work out some examples.
- Learners recollect in their groups and share their ideas on standard deviation.

Expected Responses

- Standard deviation is a statistic that tells us the spread of data around the mean.
- Standard deviation tells the variation in the data
- The higher the standard deviation, the higher the spread
- If the data is close to the mean, the lower the standard deviation.
- Learners in groups use the formula for standard deviation, which is $\sqrt{\frac{\sum(x-\underline{x})^2}{n}}$, to calculate the standard deviation for raw data.

Example 1: The ages in years of 8 students are: 14, 14, 15, 15, 12, 11, 13, 10. Calculate the standard deviation

Solution:

Mean $(\underline{x}) = \frac{14+14+15+15+12+11+13+10}{8} = 13$

x	$x - \underline{x}$	$(x - \underline{x})^2$
14	1	1
14	1	1
15	2	4
15	2	4
12	-1	1
11	-2	4
13	0	0
10	-3	9
Total		24

- $\therefore \delta = \sqrt{\frac{24}{8}} = 1.7$
- Learners in groups use the formula for standard deviation, which is $\sqrt{\frac{\Sigma f x^2}{\Sigma f} - (\underline{x})^2}$, to calculate the standard deviation for ungrouped data
- Learners in groups use the formula for standard deviation, which is $\sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2}$, to calculate the standard deviation for grouped data.

Activity 3:

Learners in the collaborative groups work out measures of dispersion (Variance).

- Learners in their collaborative groups discuss variance and work out some examples.
- Learners establish that squaring the standard deviation gives the variance.

Example: Draw a frequency distribution table for the ages of some 50 citizens in a community below and calculate the standard deviation and variance of the distribution.

21	35	52	70	55	48	42	09	48	57
36	46	15	35	12	60	29	61	48	22
43	58	25	42	1	45	60	44	38	54
47	69	30	47	18	16	35	32	21	50
11	29	41	50	53	33	30	54		
	47	34							

Marks	**Midpoint,** x	x^2	$freq.(f)$	fx	fx^2
$1-10$	5.5	30.25	2	11	60.5
$11-20$	15.5	240.25	5	77.5	1201.25
$21-30$	25.5	650.25	8	204	5202
$31-40$	35.5	1260.25	8	284	10082
$41-50$	45.5	2070.2	15	682.5	31053.75
$51-60$	55.5	3080.25	9	499.5	27722.25
$61-70$	65.5	4290.25	3	196.5	12870.75
			$\Sigma f = 50$	$\Sigma fx = 1955$	$\Sigma fx^2 = 88192.5$

$$Standard\ Deviation = \sigma = \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2}$$

$$= \sqrt{\frac{88192.5}{50} - \left(\frac{1955}{50}\right)^2}$$

$= \sqrt{235.04}$

$= 15.33$ years

$Variance = \sigma^2 = 235.04$ years

- Learners in groups discuss deviation taken from an assumed mean for grouped and ungrouped data.

	- Learners recollect the idea of assumed mean and apply it in finding the actual mean and standard deviation. - Learners discover the standard deviation given the assumed mean: - **Actual mean**$= A + \frac{\Sigma fd}{\Sigma f}$ - **Standard deviation is** $\delta = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$ **Activity 4:** Learners, in their collaborative work, research how to interpret measures of dispersion with respect to real data and present their findings in the classroom	
Teaching and Learning Resources	- SHS curriculum - Research journals - Internet or e-books - Real-life examples - Software (Excel, SPSS)	

Subject ADDITIONAL MATHEMATICS
Strand 4. HANDLING DATA
Sub-Strand 2. MAKING PREDICTIONS WITH DATA

Learning Outcomes	21^{st} Century Skills and Competencies	GESI, SEL and Shared National Values
1.4.2.LO.1		
Explain combination and permutation, state their difference and solve basic problems related to permutation and combination.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they explain combination and permutation by solving related problems using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to explain combination and permutation. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	Personal and Leadership Skills: • Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. • Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. Cut out geometrical shapes of different colours and orientations. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates,

		encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they explain combination and permutation, state their differences and solve basic problems related to permutation and combination.

1.4.2.LO.2		
Explain the terminologies in probability orally and find the relative frequency in a given experiment.	**Communication:** Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration:** Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others while they embark on project works, classroom activities and presentations. **Critical Thinking:** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** • Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they explain terminologies in probability orally and find the relative frequency in a given experiment using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to use probability and related concepts. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage

	in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. • Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. Cut out geometrical shapes of different colours and orientation. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities.

		National Core Values: **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they explain the terminologies in probability orally and find the relative frequency in a given experiment.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
1.4.2.CS.1	1.4.2.LI.1	1.4.2.AS.1
Demonstrate knowledge of basic principles of permutation and combination and interpret probability in everyday life.	**Use the fundamental counting principle to identify and determine the number of ways an event can occur.** **Collaborative Learning, Initiate Talk for Learning, Talk for Learning Approaches, Problem-based learning, Experiential Learning:** **Learning Experience:** Learners to establish different ways of counting, inquire into what happens when elements are repeated in an arrangement and explore different ways of counting **Activity 1:** Application of fundamental principles of counting ordered and unordered arrangements. **Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the fundamental principles of counting to solve problems.** **Learners think-pair-share to brainstorm and collaboratively work with examples to recognise that if a task can be accomplished in m different ways and following this task, a second task can also be accomplished in n ways, then the first task followed by the second task can be accomplished in m×n ways.** **Example 1:** Akwesi has 5 pairs of trousers and 8 shirts. Assuming that each pair of trousers can be worn with each shirt, how many trousers-shirt outfits does he have? **Solution:** For each pair of trousers, Akwesi has 8 shirts. Therefore, he has 5×8 = 40 different trousers-shirt outfits to choose from. **Example 2:** An eatery serves three different corn dishes (banku, akple and kenkey) and four different types of cold local drinks (asana, lamugin, soobolo, and pitoo). How many different different meal-with-a-drink can you order as a customer?	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Example 3: How many numbers of 3 different digits can be formed by choosing from the digits 1,2,3,4 and 5? **Solution:** Task 1: Choosing the hundred's digit, we can do this in 5 ways. Task 2: Choosing the ten's digit, we can do this in 4 ways since one digit is fixed as a hundred digit. Task 3: Choosing the one's digit, this can be done in 3 ways because two digits are fixed as hundred and ten digits, respectively. Therefore task 1, followed by task 2, followed by task 3 can be accomplished in 5×4×3=60 **Activity 2:** Arrangements with repetitions • Problem-based learning-learners in their groups work together to investigate and report on what happens if elements of sets are repeated in an arrangement. • Here are some stimuli to guide learners. **Example 1:** How many numbers of 3 digits can be formed by choosing from the digits 1,2,3,4 and 5 if the digits can be repeated? **Example 2:** The ID numbers of WASSCE candidates consist of one capital letter (from the English alphabet – i.e. A -Z), followed by 3-digit number containing repeated digits; for example, A-000 is an ID number. How many such ID numbers can be formed?	
	1.4.2.LI.2	1.4.2.AS.2
	Discuss the concepts of permutation and combination and use them to solve real life problems. **Collaborative Learning, Initiate Talk for Learning and Talk for Learning Approaches, Problem-based learning,** **Learning Experience:** Learners to discuss the types and features of different permutations (circular arrangements, identical elements etc.), and what happens when terms of ordered and un-ordered arrangements and special arrangements are used.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Activity 1:
Ordered arrangements: Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to explore how ordered arrangements can be counted.

Learners think-pair-square to brainstorm and discuss situations where they have to arrange a set of objects in which the order of appearance of elements of the set matters. For instance, APC differs from PAC.

Example: Three students, Esi, Abiba and Makafui, are contesting for 1st, 2nd and 3rd positions in a marathon race.

Using the first letters of their names, the possible arrangements are: EAM, AEM, EMA, MEA, MAE, and AME.
Learners, in convenient groups, work together on different examples to establish that permutation is an ordered arrangement of objects in a row. It is a selection of groups of objects in which order is important.

Example: Kweku has 4 different mathematisc books on his shelf. Find the number of orders in which the 4 books can be arranged on the shelf.

Solution:
There will be 4 choices in the first slot.
There will be 3 choices in the first slot.
There will be 2 choices in the first slot.
There will be 1 choice in the first slot.
$\therefore$ There are $4 \times 3 \times 2 \times 1 = 24$ possible arrangements of Kweku's books on the shelf.

Learners recognise that the number of permutations of n objects is denoted $n!$ and use it to solve related problems.

Example: Kweku has 4 different mathematics books on his shelf. Find the number of orders in which the 4 books can be arranged on the shelf.

Solution: There are 4! Ways to arrange the four books hence
$4! = 4 \times 3 \times 2 \times 1 = 24$ ways Kweku can arrange the mathematics books.

Activity 2: Permutation formula
Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the permutation formula to solve related problems.
Learners think-pair-square to brainstorm and discuss facts about permutation, selection of items out of a given set and unordered arrangement of things.
Learners establish that the number of ways in which r items can be selected out from a set n items in which order matters is given by the formula:
$nPr = P(n,r) = n!/(n-r)! = n(n-1)(n-2)(n-3)\ldots[n-(r-1)]$

Example: Evaluate the permutation ${}_5P_3$
Solution:
${}_5P_3 = P(5,3) = 5!/(5-3)! = 60$

Activity 3: Permutation with identical/repeated objects
Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the permutation formula to solve related problems.
Learners establish that the number of ways of arranging n objects where different r's of them are of the same kind is given by: $n!/(r1!.r2!.r3!\ldots..rk!)$

Example: In how many ways can the letters in the word SECRETS be arranged?

Solution: The number of letters is 5, so n=5; S and E are repeated, so the permutation will be affected by 2×2!
Therefore, the required number of ways is $5!/(2! \times 2!) = 30$ ways.

Activity 4: Permutation with identical/repeated Objects
Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the permutation formula to solve related problems.

	Learners work in groups and, through role play (acting out), establish/discover that the number of ways of arranging objects in a circle is (n-1)! Example: In how many different ways can 6 people sit at a circular table? Solution: The number of persons is 6, so n=6; since they are to sit in a circle, we have (6-1)!=5!=5×4×3×2×1=120 ways.	
	1.4.2.LI.3	1.4.2.AS.3
	Distinguish between the concepts of permutation and combination and establish the relationship between them. **Collaborative Learning, Initiate Talk for Learning and Talk for Learning Approaches, Problem-based learning, Experiential Learning** **Learning Experience:** Learners to discuss the types and features of different permutations (circular arrangements, identical elements, etc.). What happens when terms of ordered and un-ordered arrangements and special arrangements. **Activity 1: Ordered and un-ordered arrangements** • Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to distinguish between ordered and un-ordered choices. • Learners think-pair-square to brainstorm and discuss situations where they have to arrange a set of objects in which the order of appearance of elements of the set does not matter. For instance, APC is not different from PAC, because both sets have the same elements. • Learners work in their groups to make presentations by comparing, contrasting and debating with other groups on ordered arrangements of items and un-ordered arrangements of items • Learners discover/establish that the number of subsets of r elements that can be formed from elements is nCr =(n¦r)= n!/r!(n-r)!	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Example 1: In how many ways can a committee of 3 members be formed from a team of 8 people?

Solution

$_{8}C_{3}$ =(8¦3)= 8!/3!(8-3)!=56

Example 2: There are 6 different coloured cubes in a bag. In how many ways can Mawuli draw a set of 3 cubes from the bag?

Activity 2: Combination as a way of counting

- Initiate Talk for Learning, problem-based, Experiential and Collaborative learning Approaches to support learners to use combination formulas in different contexts.
- Learners explore situations where fundamental principles of counting along with the combination formula are used to solve counting problems.
- Learners find the number of ways of selecting r objects from a collection with a given condition.

Example: How many committees consisting of 2 women and 1 man can be formed from a group of 4 women and 2 men?

Solution:

Task 1: Selecting 2 women from 4 women can be done in 4C2 ways.

Task 2: Selecting 1 man from 2 men can be done in 2C1 ways.

Task 1 followed by task 2 can be accomplished in

$$_{4}C_{2} \times {}_{2}C_{1} = 6\times2 = 12 \text{ ways}$$

Activity 3: Relationship between combination and permutation formulae

Initiate Talk for Learning, problem-based, Experiential and Collaborative learning Approaches to support learners in using the relationship between combinations and permutation formulas to Solve problems

Learners discover/establish that nCr =(n¦r)= n!/r!(n-r)!= nPr/r! and use it to solve problems.

	Example 1: Find the value of n if i) $nP_3 / nC_4 = 24/5$ ii) $nC_5 / nP_4 = 1/4$	
	1.4.2.LI.4	1.4.2.AS.4
	Simplify permutation and combination expressions and solve related problems. **Collaborative learning, Initiate Talk for Learning and Talk for Learning Approaches, Experiential Learning.** **Learning Experience:** Learners discuss the relationship between Combinations and Permutations. **Activity: Permutation and combinations theorems and formulae** - Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to use the relationship between permutation and combinations theorems and formulae to solve related problems - Learners work in their groups to revise combinations and permutation theorems and formulae and use them to solve related problems **Example 1:** Solve for n if ${}_{(n-1)}P_3 / {}_{n}P_4 = 1/9$ **Solution** ${}_{(n-1)}P_3 = (n-1)!/(n-4)!$ and ${}_{n}P_4 = n!/(n-4)!$ ${}_{(n-1)}P_3 / {}_{n}P_4 = (n-1)!/(n-4)! / (n!/(n-4)!)$ $= 1/n$ $1/n = 1/9 \quad \therefore n = 9$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	1.4.2.LI.5	1.4.2.AS.5
	Recall the basic terminologies such as experiment, events, outcome, trial, sample space, etc. as used in the context of probability and give examples. **Collaborative Learning, Initiate Talk for Learning and Talk for Learning Approaches, Experiential Learning**	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning

	Learning Experience: Learners to engage in brainstorming to discuss the various terminologies associated with probability and experiments **Activity:** Initiate Talk for Learning, Experiential and Collaborative learning Approaches to support learners to revise their understanding of basic probability terminologies and give example Learners working in groups debate each other on terminologies such as experiment, probability, events, outcome, trial, sample space, etc. and provide everyday examples. **Example:** Undertake the following experiments and discuss and identify the various basic probability terminologies in relation to experiments such as: i) tossing a coin three times ii) rolling a dice iii) drawing a card from a set of playing cards iv) drawing items from objects in a bowl.	Level 4 Extended critical thinking and reasoning
	1.4.2.LI.6	1.4.2.AS.6
	Work collaboratively on an experiment (e.g. tossing of coins or dice) to determine the relative frequencies of events and interpret them. **Collaborative Learning, Initiate Talk for Learning and Talk for Learning Approaches, Problem-based learning, Experiential Learning** **Learning Experience:** Learners discuss the relative frequencies of events, compare results with theoretical definitions of probability and what happens when terms of ordered and un-ordered arrangements and special arrangements. **Activity 1: Relative frequencies of events** **Initiate Talk for Learning, problem-based and Collaborative learning Approaches to support learners in solving problems of relative frequencies of events.**	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Learners work in groups to perform experiments using coins, ludo dice, playing cards etc, up to about 10 trials in each case and record their findings.

Example 1: The table shows the number of times a coin is tossed and the outcome of each trial.

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13
Outcome	T	T	H	T	H	T	H	H	T	T	H	T	H

What is the relative frequency of (observing) heads after

- each trial ii) the experiment

Solution:

•

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13
Outcome	T	T	H	T	H	T	H	H	T	T	H	T	H
RF	0	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{3}{7}$	$\frac{4}{8}$	$\frac{4}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	$\frac{5}{12}$	$\frac{6}{13}$

- $\frac{6}{13}$**= 0.46**

Example 2: The table shows the number of times a coin is tossed and the outcome of each trial.

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13
Outcome	T	T	H	T	H	T	H	H	T	T	H	T	H
Trial	14	15	16	17	18	19	20	21	22	23	24	25	26
Outcome	T	H	T	T	H	H	T	H	H	T	T	T	H

What is the relative frequency of (obtaining)

- Tails after the experiment
- Tails after each trial, and how does it compare to the theoretical probability of obtaining tails?
- Draw a graph to show how the relative frequencies of tails vary in the experiment.

	Activity 2: Theoretical probability of an event • Learners to establish and use the theoretical probability of an event to solve simple problems • Learners discover the theoretical probability of an event as the likeliness of the event happening based on all the possible outcomes given by: • Probability of Event = $\frac{Number\ of\ favourable\ outcomes}{Number\ of\ possible\ outcomes}$ **Example:** A fair coin is tossed twice, and the result is recorded. What is the probability that (a) the first toss shows a head; (b) the second toss shows a head.	
Teaching and Learning Resources	• Ludo dice, coins, playing cards, objects in a sac or bowl.	

YEAR TWO

Subject ADDITIONAL MATHEMATICS

Strand 1. MODELLING WITH ALGEBRA

Sub-Strand 1. APPLICATION OF ALGEBRA

Learning Outcomes	**21st-Century Skills and Competencies**	**GESI[3], SEL[4] and Shared National Values**
2.1.1.LO.1		
Investigate De Morgan's law on sets algebraically and graphically, formulate and solve real life problems up to three sets.	**Communication**: Provide learners the opportunity to engage and participate in mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they investigate De Morgan's Laws on sets using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they investigate de Morgan's Laws and related concepts.

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[3] Gender Equality and Social Inclusion

[4] Socio-Emotional Learning

	information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena. (Learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident and happy to participate in all activities; have all learner taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (inspirators and mentors) to learn from them and their actions. **Creativity and Innovation:** Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling and set aside a dedicated time of mindfulness each school day. Ask open-ended questions and set problem-finding contexts. **Strategic Competency:** Conscious efforts will be made to enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to the national level.	• Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies *–Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems.

		• Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they investigate De Morgan's law on sets algebraically and graphically, formulate and solve real problems in real life up to three sets**.**
2.1.1.LO.2		
Model sequence recursively and explicitly, and establish the relationship between the two forms, as well as solve real life problems involving linear and exponential sequences and series.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they model sequences recursively and explicitly using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about sequences and related concepts.

		• Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems.

		- Managing emotions and conflicts as they engage in collaborative group work - Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they model sequence recursively and explicitly, and establish the relationship between the two forms, as well as solve real life situations involving linear and exponential sequences and series.
2.1.1.LO.3		
Apply indices and logarithms to solve real life problems, including logarithms with different bases, and sketch and interpret logarithmic functions.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they apply indices and logarithms to solve real life problems and related concepts using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn to apply the concepts.

		• Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems.

		• Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they apply indices and logarithms to solve real life problems, including logarithms with different bases, and sketch and interpret logarithmic functions**.**
2.1.1.LO.4		
Formulate and derive appropriate strategies to solve quadratic inequalities.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they formulate and derive appropriate strategies to solve quadratic inequality using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to describe graphically and

		algebraically, the behaviour of the function about an input value and determine its derivative. • Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss. mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use

		innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work. • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment.

		Equity: Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they formulate and derive appropriate strategies to solve quadratic inequality using appropriate technological tools**.**
2.1.1.LO.5		
Graph systems of given inequality and identify the region that provides the feasible solution and apply it to real life situations.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other inclusively; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they used different creative approaches to graph systems of given inequality and identify the region that provides the feasible solution and apply it to real life situations.

		• Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of graph systems and inequality to solve real life situations and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and finding confidence

		• Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values: Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they used different creative approaches to graph systems of given inequality and identify the region that provides the feasible solution and apply it to real life situations.
2.1.1.LO.6		
Determine the set of values for which a rational function is defined and resolve rational functions into partial fractions.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they use innovative strategies to determine the set of values for which a rational function is defined and resolve rational functions into partial fractions. • Interrogate their stereotypes and biases about the roles and abilities of different groups in

		learning and applying their knowledge of rational functions and resolve rational functions into partial fractions in mathematics and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and finding confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts

		• Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners.

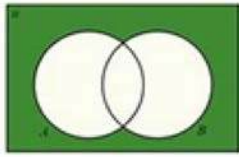
		Tolerance: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they determine the set of values for which a rational function is defined and resolve rational functions into partial fractions.
2.1.1.LO.7		
Multiply matrices, determine the inverse of a 2 x 2 matrix, find the determinant up to a 3 x 3 matrix and represent matrices in linear transformations.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they multiply matrices, determine the inverse of a 2 x 2 matrix, find the determinant up to a 3x3 matrix and represent matrices in linear transformations. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of Multiplying matrices, determining the

		inverse of a 2 x 2 matrix, finding the determinant up to a 3x3 matrix and representing matrix in linear transformation in mathematics and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and finding confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts

		- Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners.

		Tolerance: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they multiply matrices, determine inverse of a 2 x 2 matrix, find the determinant up to a 3x3 matrix and represent matrix in linear transformation.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
2.1.1.CS.1	2.1.1.LI.1	2.1.1.AS.1
Demonstrate the ability to use sets properties and theories to solve real life problems and apply binomial theorem to approximate numbers to a given index.	**Establish De Morgan's laws of set theory and use it to solve related problems.** **Pedagogy and Learning Experience** **Collaborative Learning**: Learners work in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) and use analytic and graphic illustrations to verify De Morgan's laws and other laws of set algebra. **Talk for Learning Approaches:** Learners brainstorm using strategies such as think-pair-share/square, debates and discussions to explain why a proof of a set property or laws of set algebra is true or false. **Experiential Learning**: Learners collaboratively (pair and group work) verify and use graphic illustrations and algebraic manipulations to establish properties of sets and laws of set algebra. **Activity 1:** Use graphical representations (e.g., Venn diagrams) and algebraic methods (reasoning) to establish De Morgan's Laws and other laws of set algebra focusing on the complement, union, and intersection of sets. **Using Talk for Learning Approaches in collaborative groups,** learners investigate and establish the laws of set algebra, including De Morgan's Laws, using Venn diagrams and by algebraic analysis. **Example** • Show by graphical and algebraic analysis that for any non-empty sets A, B and C, (i)$(A \cup B)'$=$A' \cap B'$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning



⇔

Algebraically

Let, then thus or

and x ∈

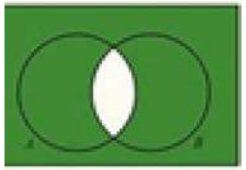
and so

Again, let x . Then and x ∈, thus and

,

Hence =

(ii)$(A \cap B)' = A' \cup B'$



⇔

Algebraically

Let x . Then or x ∈, thus and

,

Hence =

Let, and x ∈, then thus or
and so

- Using a venn diagram, shade the regions represented by:
- $(A \cup B) \cap C'$

	• $(A \cup B)' \cap C'$ • $(A \cap B)' \cup C'$ • Rewrite the following: • (A’ ∪ B’)’ • (A’ ∩ B ‘)’	
	2.1.1.LI.2	2.1.1.AS.2
	Describe the set theory as a foundation for many subfields of mathematics, and create and model set problems in the areas pertaining to industry, commerce, sports, etc. **Pedagogy and Learning Experience** **Collaborative Learning**: Learners work in convenient groups (ability, mixed-ability, mixed gender, or pairs, etc.) and use analytic and graphic illustrations to apply properties of sets and laws of set algebra to solve set problems in real life contexts. **Talk for Learning Approaches:** Learners brainstorm using strategies such as think-pair-share/square, debates and discussions to explain and talk about the relevance of learning set properties and algebra and how they are useful in our daily lives. **Experiential Learning**: Learners collaboratively (pair and group work) work together, create and solve sets problems in real life situations by applying laws of set algebra and other properties of sets **Activity 1:** Utilisation of ideas of Laws of set algebra and properties of sets. **Using Talk for Learning Approaches in collaborative groups, learners research and discuss the relevance of learning about the properties and laws of set algebra and how they are utilised in real life contexts.** **Example** Discuss the relevance of learning about sets and their application in real life contexts.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Set theory is used in almost every discipline, including engineering, business, medical and related health sciences, along with the natural sciences. In business operations, it can be applied at every level where intersecting and non-intersecting sets are identified. For example, the sets for warehouse operations and sales operations are both intersected by the inventory set. To improve the cost of goods sold, the solution might be found by examining where inventory intersects both sales and warehouse operations.

Set theory can assist in planning and operations. Every element of business can be grouped into at least one set, such as accounting, management, operations, production and sales. Within those sets are other sets. In operations, for example, there are sets of warehouse operations, sales operations and administrative operations. In some cases, sets intersect - as sales operations can intersect the operations set and the sales set.

In the kitchen, sets of similar utensils are kept separately.

School bags: sets of notebooks and textbooks are kept separately in the divisions in the school bags.

Music Playlist: Most of us have different playlists of songs on our smartphones and computers. Afrobeat songs are often separated from high-life or any other genre. Hence, playlists also form an example of sets.

Activity 2: Apply set theory to solve real life problems.

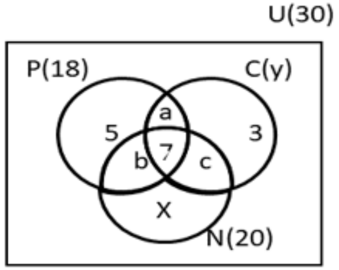
Using collaborative learning groups, learners challenge each other in an Inter-group competition. Thus, learners from one group create real life set problems for the members of the other group to investigate, prove and or answer.

Example

- The sets L, M and N are subsets of a universal set consisting of the first 10 lower-case letters of the alphabet, if L = {a, b, c}, M = {b, c, a, e} and N = {a, d, e, f},

Determine the members of the following sets: i) $M \cup N$ ii) $L \cup N$ iii) L' iv) $L \cap M \cap N'$ v) $(L \cup M \cup N)'$ vi) $M \cap N$

- A sample of 100 Students' Representative Council voting delegates revealed the following concerning three candidates: Akayuure, Manukre and Odonti, who were running for the SRC Chairman, Secretary and Treasurer, respectively. 14 delegates preferred both Akayuure and Manukre, 49 preferred Akayuure or Manukre but not Odonti. 21 preferred Manukre but not Odonti or Akayuure. 61 preferred Manukre or Odonti but not Akayuure. 32 preferred Odonti but

	not Akayuure or Manukre. 7 preferred Akayuure and Odonti but not Manukre. With the aid of a Venn diagram, determine the number of voters that were in favour of all three candidates. Assume that every member of the SRC voted for at least one candidate. Determine the candidate that went unopposed if a rule of 50% majority was used in such a decision. • Find the values of a, b, c, x and y. 	
	2.1.1.LI.3	2.1.1.AS.3
	Use the expansion for (1-x)^n or (1+x)^n to approximate exponential numbers. **Pedagogy and Learning Experience** **Collaborative Learning**: Learners work in convenient groups (ability, mixed-ability, mixed gender, or pairs etc.) to determine the terms in a given binomial expansion. **Talk for Learning Approaches:** Learners brainstorm using think-pair-share/square and debates to talk about and justify the nature and or signs of the terms for a given binomial expansion. **Experiential Learning**: Learners collaboratively (pair and group work) work together, create and solve binomial expansions and apply them in real life contexts. **Activity**: Expand binomial expressions with a given exponent. **Using Talk for Learning Approaches** (Building on what others say, managing Talk for Learning, Structuring Talk for Learning) and Collaborative learning approaches, learners adopt the combination	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	method and other methods to determine the terms, coefficient and exponent of a given term in an expansion. **Examples** • Expand $(1-2x)^8$ and use the expansion to approximate $(0.98)^8$ correct to 3 decimal places **Solution:** $(1-2x)^8 = 1 - 8(2x) + \frac{(8)(7)(-2x)^2}{2}$ $+ \frac{(8)(7)(6)(-2x)^3}{5} + \cdots + (-2x)^8$ $(0.98)^8 = (1-0.02)^8$ • Expand $(1+x)^{12}$ in ascending powers of x up to the fifth term.	
Teaching and Learning Resources	• ICT tools and resources • Worksheets • Scientific calculator • Technological tools, apps, etc. • SHS Additional Mathematics Curriculum	

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
2.1.1.CS.2	2.1.1.LI.1	2.1.1.AS.1
Demonstrate the ability to apply algebraic processes and reasoning to model and solve real life situations involving sequences, and linear programming and use appropriate techniques to solve quadratic inequalities, as well as resolve rational functions.	**Generate the terms of a recurrence sequence and find an explicit formula for the sum of the sequence.** **Collaborative Learning**: Learners to work in convenient groups (ability, mixed-ability, mixed gender, etc.) to solve problems and present answers **Talk for Learning Approaches:** Learners brainstorm and discuss the types and features of sequences. **Project-based Learning:** Engage learners to come up with a sequence that is relevant and applicable to real life. **Experiential Learning**: Engage learners with hands-on activity (learning by doing). **Problem-based learning**: Learners inquire into what happens when terms of sequences are summed up to infinity. **Activity 1:** Facts about Geometric and Arithmetic sequences. Initiate Talk for Learning Approaches (building on what others say, managing Talk for Learning, Structuring Talk for Learning); Experiential and Collaborative learning approaches to support revised facts about arithmetic and geometric sequences. Starting with whole group discussion and transiting through small convenient groups (e.g., mixed-ability/mixed gender) or stations/centres, then pair work (e.g., using think-pair-share) and finally individual (e.g., using give-one take-one) learners recall, assess and debate each other on what they know about sequences including how to find the nth term, means of sequences and solve problems related to sequence and series in different contexts. **Example** Learners discuss, debate, assess and challenge colleagues on: i) examples of geometric and arithmetic sequences (both mechanical and real life examples)	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

ii) how arithmetic and geometric sequences are similar;
iii) how arithmetic and geometric sequences are different;
iv) examples of sequences that are neither geometric nor arithmetic and explain how to solve such sequences.

The first four terms of a geometric sequence are given: $-\frac{1}{32}, \frac{1}{8}, -\frac{1}{2}, 2$ ... what is the 5th term in the sequence?

Activity 2: Sums in a Sequence

Use Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning Approaches:
Learners work in convenient groups to study, interpret and use summation notation/symbols to generate terms in a sequence.

Example: Indicate the first three terms in the series
$\sum_{n=1}^{7}(4n-5)$**.**

Solution: To find the first three terms, replace n with 1, 2 and 3: $\therefore$ The first three terms are -1, 3 and 7

- $\sum_{n=1}^{10}(4n-5)$.
- $\sum_{k=-2}^{9}(2)^{k}$.
- $\sum_{a=2}^{11}\frac{1}{2}(4)^{a-2}$.

Activity 3: Method of undetermined coefficient

Use Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning Approaches:
Learners use the Method of Undetermined Coefficient to find the sum of the first n terms of other series by equating identically the given series to a series of the form $A + Bn + Cn^2 + \ldots\ldots\ldots$, and then determining the values of the constants A, B, C

Example

Find the sum of the first n natural numbers.

Solution:

$$n = 1, 2, 3, 4, \dots \dots \dots \dots \infty$$

The sum of the first n natural number can be written as:

$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = A + Bn + Cn^2$ ...(1)

Adding $(n + 1)$ natural numbers gives:

$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n + (n + 1) = A + Bn + 1 + C(n + 1)^2$ …(2)

Solving equation (2)-(1) and comparing coefficients we have $\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

- Evaluate $\sum_{r=1}^{n} \frac{2}{(2r+1)(2r-1)}$;

Hint: Split into partial fractions and solve.

Activity 4: Sum of AP and GP

Use Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning Approaches:

Learners work collaboratively to investigate, establish, and use the formula $Sn = \frac{n}{2}(a_1 + a_{n)})$ and $Sn = \frac{n}{2}[(2a_1 + (n - 1)d]$ to find the sum of an AP and solve related problems; where n is the number of terms in the series, a_1 is the first term and a_n is the last term, and d is the common difference of an AP.

Example

Find the sum of all even integers from 250 to 350.

Solution: $a_1 = 250$; $a_n = 350, n = 50$

$S_{50} = \frac{50}{2}(250 + 350)$ = 15,000

A supermarket displayed milk tins piled up in the form of a pyramid. The bottom layer has 25 tins, and each successive layer has 2 fewer tins. How many milk tins are displayed?

Learners work collaboratively to investigate, establish, and use the formula $S_\infty = \frac{a_1}{1-r}$; $S_n = \frac{a_1(r^n-1)}{r-1}; r >$ 1 or $S_n = \frac{a_1(1-r^n)}{1-r}; r < 1$ to find the sum of a GP and solve related problems, where n is the number of terms in the series, a_1 is the first term and r is constant ratio.

Example

Evaluate: $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} +$ ...

Indicate the sum of the following series:

i) $\sum_{k=-2}^{9}(2)^k$.

ii) $\sum_{m=2}^{11}\left(\frac{1}{2}(4)^{m-2}\right)$.

Activity 5: Convergence or divergence of a series:

Use Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning Approaches: Learners work in convenient groups, investigate whether a series converges, and justify their claims.

Example:

Consider the series $\sum_{r=1}^{\infty}\frac{1}{2^r} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$

Adding the terms in the series gives:

$S_1 = \frac{1}{2}$; $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$; $S_4 = \frac{15}{16}$;

$S_5 = \frac{31}{32}$; $S_6 = \frac{63}{64}$; $S_7 = \frac{127}{128}$; ... and so on.

It is obvious the sum of the terms in the series (S_n), always less than 1 but gets closer to 1 as we take more and more terms. It is reasonable to claim that $\sum_{r=1}^{\infty}\frac{1}{2^r} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$= 1

	Note that the above series is a G. P. with a = $\frac{1}{2}$ and common ratio r =$\frac{1}{2} < 1$. Verify that $S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ **Note:** S_1, S_2, S_3, S_4…are the partial sums of the series. If there is a number L such that $S_n = \sum_{r=1}^{\infty} S_n = L$. The number L is called the sum of the infinite series. If there is no such number, then the series is said to diverge. Find the sum, if possible: 1−3+9−27+…	
	2.1.1.LI.2	2.1.1.AS.2
	Use recursive and explicit formulae of sequences to model situations and translate between the two forms. **Collaborative Learning:** Talk for Learning Approaches, Project-based Learning **Experiential Learning and Problem-based Learning** **Activity:** Recursive sequences **Use Talk for Learning Approaches, Problem-based Approaches, Experiential and Collaborative Learning Approaches to solve problems related to recursive sequences.** Learners recognise recursive sequences, work in their groups to investigate recurrence sequences and use initial conditions to generate terms of the sequence as well as iterative rules. **Example** Write the recurring decimals as a series. Identify the common ratio and the first and an explicit formula for the nth term. • 0.3333... • 0.54545454… • 3.212121…	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Solution • 0.3333... = 0.3 + 0.3(0.1) + 0.3(0.1)^2 + 0.3(0.1)^3 + .. The first term is 0.3, and the constant ratio between each successive term is 0.1. The formula for the nth term will be $U_n = 0.3(\frac{1}{10})^{n-1}$ Find the recurrence relation and the initial conditions for the sequence: 1, 5, 17, 53, 161, 485…. **Solution**: The recurrence formula is $a_n = 3a_{n-1} + 2$, and the initial condition is $a_0 = 1$ A recursive rule for an AP is $a_1 = -3;\ a_n = a_{n-1} + 7$. What is the iterative rule for this sequence? • $a_n = 7n - 10$ • $a_n = -3n + 7$ Find the next two terms in $(a_n), n \geq 0$ for the sequence 3, 5, 11, 21, 43, 85…… the recurrence relation and initial conditions for the sequence	
	2.1.1.LI.3	2.1.1.AS.3
	Determine the arithmetic and geometric means of linear and exponential sequences and apply linear and exponential sequences to solve real life problems. **Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning approaches** **Activity:** Arithmetic and geometric means of sequences. **Use Collaborative Learning, Experiential Learning, Problem-based Learning and Talk for Learning approaches to support learners in solving problems related to geometric and arithmetic means of sequences:** Learners in small mixed-ability/gender groups discuss arithmetic and geometric means outlining their differences and similarities and extending the conversation to include;	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	• how to insert means between given sequences; • when to use either means in contexts. **Example** • The three numbers, $n-2, n\ ,\ \ n+3$, are consecutive terms of a G.P. find n, and the term after n+3. • Insert three arithmetic means between 25 and 9. • Insert three geometric means between 243 and 48.) The geometric mean between 2 and 8 is 4, and two arithmetic means between 2 and 8 are 4 and 6.	
Teaching and Learning Resources	• Worksheets • Scientific Calculator • Technological tools, apps, etc. • SHS Additional Mathematics Curriculum	

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
2.1.1.CS.3	2.1.1.LI.1	2.1.1.AS.1
Demonstrate understanding of the laws and properties of indices and apply the ideas to solve problems.	**Review indices and apply the idea to create and solve real life problems involving indices.** **Collaborative Learning**: Learners work in convenient groups (ability, mixed-ability, mixed gender, etc.) to model real life problems involving indices and logarithms and present answers. **Talk for Learning Approaches:** Learners engage in brainstorming, think-pair-share, building on what others say, debates and discussions to explain how to solve problems involving indices and logarithms and the relationship between the two concepts. **Project-based Learning:** Engage learners to come up with graphs and solutions of logarithmic functions modelled from real life contexts. **Experiential Learning**: Learners practice hands-on activities (learning by doing) by manipulating symbols, using any spreadsheet programme such as Microsoft Excel and ICT tools such as GeoGebra to draw graphs of logarithmic functions and explain their solution process. **Problem-based earning**: Learners to investigate and explain real life models of logarithmic functions and talk about what happens when specific conditions are varied/imposed. **Activity:** Application of the Laws of indices **Use Talk for Learning Approaches, Problem-based, Experiential and Collaborative Learning Approaches to solve problems related to the application of the laws of indices.** Learners working with a partner or in small groups recall and investigate properties of indices and make deductions from the laws.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

	Example Simplify $\left(\frac{9}{25}\right)^{-\frac{1}{2}}$ Simplify $\frac{(1+x)^{\frac{1}{2}}-2x(1+x)^{-\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$ Learners working with a partner or in small groups apply the laws of indices to solve problems. **Example** • Solve $3^{2n}=\frac{1}{81}$ • $16^{\frac{1}{5}} \times 2^{x} = 8^{\frac{3}{4}}$ • Solve the simultaneous equations: $3^{x-1} = 27^{y-1}$ and $\left(\frac{1}{3}\right)^{2x} = \left(\frac{1}{9}\right)^{y+2}$	
	2.1.1.LI.2	2.1.1.AS.2
	Use the properties of logarithms and indices to solve equations involving logarithms, including changing the base. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning** **Activity 1:** Rules of logarithms Using Talk for Learning, Problem-based, Experiential and Collaborative learning approaches, learners state and prove the rules of logarithms and make deductions. Learners working in pairs and/or small groups discuss and investigate the relationship between indices and logarithms, state the laws of logarithms and make deductions from the laws. **NB**: A logarithm is the power to which a number must be raised in order to get some other number.	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Example

- The logarithm of a number x to the given base b is the power to which b will be raised such that it equals x. i.e., if $x = b^x$, then $x = r$
- If $e^x = 8$, then $x = 8$
- Express in terms of y if $\frac{1}{2}(y - 5) = x$
- Rewrite $V = \frac{\pi h}{3}(l + h)(l - h)$ in the simplest log form.

Activity 2: Apply the laws of logarithms to solve problems.

Using Talk for Learning, problem-based, Experiential and Collaborative Learning Approaches, learners use a combination of rules of logarithms to solve problems.

- Learners working with a partner or in small groups apply a combination of rules of logarithms to solve problems.

Example

- x and y are real numbers. If $3x = y$ and $4x = y + 4$, find the values of x and y
- Find, simultaneously, the solution to the system of equations:
  1. $2x - 3y = 7$
  2. $x - 2y = 4$
- Solve the equation $6.9^x + 3^x - 2 = 0$ and justify your solution.
- Learners solve problems involving change of the base of logarithms.

Example

- If $n = \frac{1}{3}p$, $3n = q$, and $p - q = 2$, find the possible values of n
- Find, correct to three significant figures the value of x, if $3^{x+1} = 4^{2x-1}$

If x and y are any two positive real numbers. Show that $(ab) \cdot (ab) = ab + \quad ab$

2.1.1.LI.3

Draw graphs of logarithmic functions using an appropriate technology and by hand, and interpret them.

Collaborative Learning, Experiential Learning, Problem-based Learning, Project-Based Learning and Talk for Learning

Activity: Graphs of Logarithmic Functions

Using Talk for Learning Approaches, Problem-based Approaches, Project-based, experiential, Experiential Learning (use of ICT tools) and collaborative learning approaches, learners construct graphs of logarithms and describe the behaviour of the graph when a restriction/ condition is imposed.

Learners recognise that any relation of a non-linear form $y = ax^n$, where x and y are variables and a being constant can be reduced to a linear form $y = mx + c$, by first taking the logarithm of both sides of the relation, then applying the addition and the power rule, rearrange and compare with $y = mx + c$

Example

Draw the graphs by hand, then photograph or scan your graph; use GeoGebra to create the graph; then insert a screenshot.

Imagine that you are asked to graph $log6(x + 3) - 2$. How does this graph compare to the graph of $log6(x)$?

The table presents the value of two related variables a and b, as obtained in an experiment:

a	1	2	3	4	5
b	3	10	51	190	760

If it is true that a and b are connected by a scientific law given by $b = kc^a$, where c and k are constants,

- Plot a suitable graph and verify that the law is valid.
- Use your graph to estimate

2.1.1.AS.3

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
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	• the values of c and k correct to one decimal place • the value of b when $a = 3.5$ (Hint: take the log to the base 10 of both sides and proceed).	
	2.1.1.LI.4	2.1.1.AS.4
	Describe the processes of solving quadratic equations by graphical method, factorisation, and inspection for quadratic functions, including functions of the form ($x^2 = 49$) where possible. **Collaborative Learning**: Learners to work in convenient groups (ability, mixed-ability, mixed gender, etc.) to solve problems and present answers. **Talk for Learning Approaches:** Learners brainstorm through think-pair-share, debates and discussions to explain the features of quadratic inequalities. **Project-based Learning:** Engage learners to come up with graphs of quadratic inequalities indicating when and what to choose as members of a solution set. **Experiential Learning**: Engage learners in hands-on activity (learning by doing) by manipulating symbols, using excel spreadsheets and other ICT tools such as GeoGebra to sketch and determine points on a graph. **Problem-based learning**: learners inquire into what happens when the equality sign of an equation changes to an inequality. **Activity 1**: Recall quadratic solutions to functions. **Use Talk for Learning Approaches, Problem-based Approaches, Project-based Learning, Experiential and Collaborative Learning approaches to review concepts about quadratic functions.**	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Example

Learners revise features of quadratic functions, including their graphs, turning points, nature of roots, inverse, minimum and maximum values, increasing and decreasing values, domains and range using algebraic reasoning and graphical processes.

i.e.,

Learners working in groups/pairs establish that for the quadratic equation,

$ax^{2} + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Learners understand and use the expression $b^2 - 4ac$ and any other methods to determine the nature of the roots of the function.

Learners use completing squares, graphs, and other methods to write functions in the form $A(x \pm B)^2 \pm C = 0$ or $A(x \pm B)^2 = C,$ where A, B and C are constants, and C is the maximum and minimum values of the function, and the maximum and minimum occur at

$x \pm B = 0$ or $x \pm B$

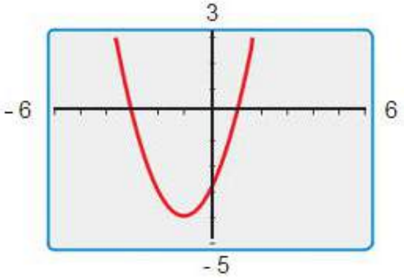
Activity 2: Apply knowledge of solutions to quadratic functions to solve quadratic inequalities.

Use Talk for Learning Approaches, Problem-based Approaches, Project-based Learning, Experiential and collaborative learning approaches to graph quadratic inequalities in two variables and solve quadratic inequalities in one variable.

Learners work in groups/pairs, research how to solve quadratic inequalities, and make presentations to the class in the plenary.

Learners work in pairs to solve quadratic inequalities by way of using:

- graphs and illustrations (number line or coordinate plane); and
- algebraic reasoning.

	Example • Solve $x^2 \ - 3x - 4 < 0$ algebraically. **Solution** So, the solution is $-1 < x < 4$. • The diagram or figure shows the graph of $f(x) = x^2 + 2x - 3$.  Explain how you can use the graph to solve the inequality, $x^2 + 2x - 3 \leq 0$	
	2.1.1.LI.5	2.1.1.AS.5
	Solve quadratic inequalities involving real life problems. **Collaborative Learning**: Learners to work in convenient groups (ability, mixed-ability, mixed gender, etc.) to model real life problems involving quadratic inequalities and present answers. **Talk for Learning Approaches:** learners brainstorm through think-pair-share, building on what others say, debates and discussions to explain how they model real life quadratic inequalities. **Project-based Learning:** Engage learners to come up with graphs and solutions of quadratic inequalities modelled from real life contexts.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Experiential Learning: Engage learners in hands-on activity (learning by doing) by manipulating symbols, using excel spread sheets and ICT tools such as GeoGebra to sketch real life quadratic inequalities and explain their solution process.

Problem-based Learning: Learners investigate and explain real life models of quadratic inequalities and talk about what happens when specific conditions are varied/imposed.

Activity: Revise solutions to quadratic inequalities.

Use Talk for Learning Approaches, Problem-based Approaches, Experiential and Collaborative Learning Approaches to solve and revise solutions to quadratic inequalities in one and two variables.

Example:

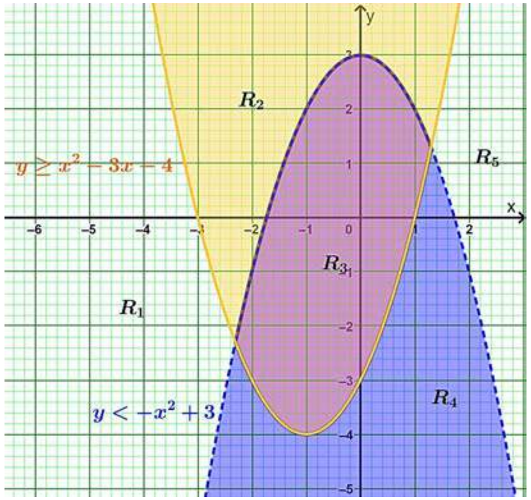
- Graph the system of quadratic inequalities.

$y < -x^2 + 3$ and $y \geq x^2 + 2x - 3$

Solution

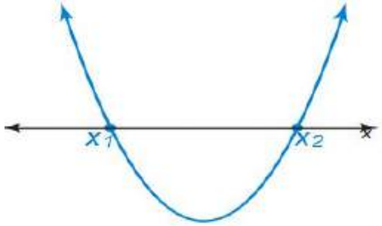
The solution region could also be found algebraically by substituting corresponding values for x and y in possible solution regions into both inequalities for testing
We can use the origin, $(0, 0)$ as a test point
For $y < -x^2 + 3$,
$-x^2 + 3 = -0^2 + 3 = 3$
$0 < 3$ satisfying $y < -x^2 + 3$ and thus, $(0, 0)$ falls in the solution region of $y < -x^2 + 3$

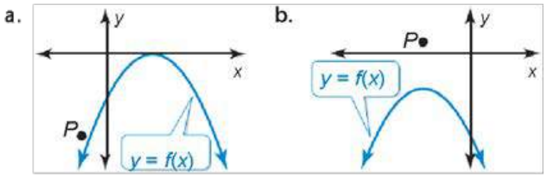
$y \geq x^2 + 2x - 3$
$x^2 + 2x - 3 = 0^2 + 2(0) - 3 = -3$
$0 \geq -3$ satisfying $y \geq x^2 + 2x - 3$ and thus, $(0, 0)$ falls is in the solution region
Since $(0, 0)$ lies in the solution region of both inequalities, the range of values of x and y in which $(0, 0)$ falls in the solution region for the system



R_3 is the only solution region common to both inequalities. This implies that all points in that region, including but not exclusive to $(0, 0), (-1.4, 0.6)$ $(-1, 1), (0.5, 2), (1, 1)$ and $(1, 1.6)$ are in the solution set for the system

- Use the graphs ***a*** and ***b*** to write an inequality in terms of $f(x)$ so point P is a solution.
- Consider the graph of the function $f(x) = ax^2 + bx + c$.



	 **1.**What are the solutions of $ax^2 + bx + c < 0$? **2.**What are the solutions of$ax^2 + bx + c > 0$? **3.**The graph of g represents a reflection in the x-axis of the graph of f. For which values of x is $g(x)$ positive? **Activity 2:** Real-life Problems Involving Quadratic Inequalities. **Use Talk for Learning Approaches, Problem-based Approaches, project-based learning, Experiential and Collaborative Learning Approaches to solve problems contextual real life problems.** **Example** Ama is instructed to make a garden plot which has an area less than 18m^2. The length should be 3m longer than the width. • How would you represent the width of the garden plot? • What would be the mathematical sentence? • What are the possible dimensions of the garden plot?	
	2.1.1.LI.6	2.1.1.AS.6
	Use a graphical approach (by hand and technology) to solve simultaneous linear inequalities. **Collaborative Learning:** Learners to work in convenient groups (ability, mixed-ability, mixed gender, etc.) to model real life problems involving linear inequalities and present answers. **Talk for Learning Approaches –**learners engage in brainstorming, think-pair-share, building on what others say, debates and discussions, etc., to explain how they model real life linear inequalities. **Project-based Learning:** Learners to come up with graphs and solutions of a system of linear inequalities modelled from real life contexts.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Experiential Learning: Engage learners in hands-on activity (learning by doing) by manipulating symbols, using Microsoft Excel spreadsheets and ICT tools such as GeoGebra to sketch real life linear inequalities and explain their solution process.

Problem-based Learning: Learners investigate and explain real life models of linear inequalities and talk about what happens when specific conditions are varied/imposed

Activity: Solutions of systems of linear inequalities.

Collaborative Learning, Problem-based Learning, Experiential Learning and Talk for Learning Approaches:

Learners work together from whole class discussions to small convenient groups, then pair work:

- recognise and classify linear inequality as a mathematical statement containing the inequality signs $(>, <, \geq, or \ \leq)$ instead of an equal sign $(=)$ and create real life context that models linear inequalities.

Example

- A phone company charges 50 pesewas per minute during the daytime and 10 pesewas per minute at night. How many daytime minutes and night-time minutes could you use in one week if you wanted to pay less than GHC 20? Write a mathematical sentence to model the word problem
- Formulate a word sentence that models the inequalities:

(i) $4x + 3 < 15$ (ii) $2x \ + 3y \ \geq 12$

- Solve systems of linear inequalities by algebraic means and graphical solutions using appropriate technology and manually.

Example

Find the integer values of x satisfying the inequality: $6 < 2 – 3x < 14$

	Solution: Write $6 < 2 - 3x < 14$ as two separate inequalities and solve them separately. $6 < 2 - 3x$ $3x < 2 - 6$ $3x < -4$ $x < -\frac{4}{3}$; also $2 - 3x < 14$ $-3x < 14 - 2$ $-3x < 12$ $3x > -12$ $x > -4$ Common values: $-4 < x < -\frac{4}{3}$: The integer values of x are -3 and -2	
	2.1.1.LI.7	2.1.1.AS.7
	Predict and identify the region representing the solution to systems of linear inequality. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-Based Learning and Talk for Learning** **Activity:** Regions representing solutions of systems of equations and inequalities. **Project-based Learning: Collaborative Learning, Problem-based Learning, Experiential Learning and Talk for Learning Approaches:** Learners work together (from whole class discussion to small convenient groups, then in pairs) and explore by analytical methods and then manually (by hand) and using appropriate technology and ICT tools (e.g., GeoGebra), the solutions for systems of linear and quadratic equations and inequalities in two variables identify and state the region that the systems of equations and/or inequalities satisfy.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Example

Solve $x + 2 < 6$ and $x - 3 > -1$ using algebraic manipulation and show the region satisfying both inequalities.

Solution

$x + 2 < 6$

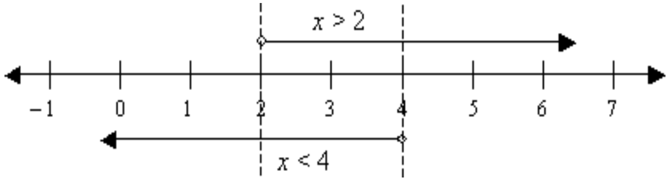
$x < 6 - 2$

$x < 4$

Also: $x - 3 > -1 \Leftrightarrow x > -1 + 3 \Leftrightarrow x > 2$

Common values: $2 < x < 4$

Graphically, we have:



Example

Given $3x + y \geq 1, x - y < 4$ and $3x + 2y \leq 6$

Show that $-\frac{4}{3} \leq x \leq \frac{14}{5}$ and $-\frac{11}{4} \leq y \leq 5$

Solution:

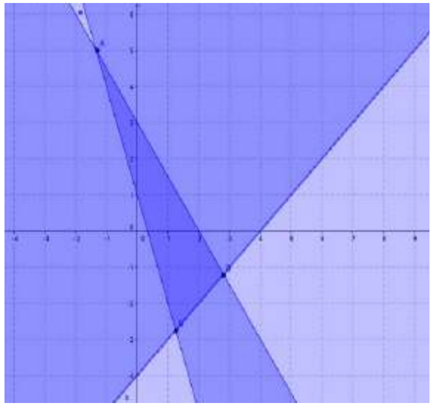
2(i) + (ii): gives $-\frac{4}{3} \leq x$; and

(ii) + 2(iii) gives $x \leq \frac{14}{5}$ $\therefore -\frac{4}{3} \leq x \leq \frac{14}{5}$

Also

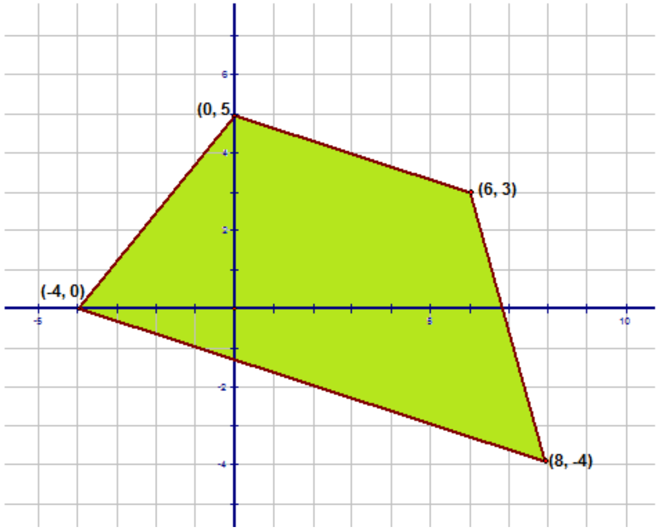
(i) + (ii) gives; $y \leq 5$; and

(i) + 3(iii) gives; $-\frac{11}{4} \leq y$; $\therefore -\frac{11}{4} \leq y \leq 5$

	Graphically,  Hence, the set of points that satisfy the system of inequalities is $\left(-\frac{4}{3},5\right)$, $\left(\frac{14}{5},-\frac{6}{5}\right)$ and $\left(\frac{5}{4},-\frac{11}{4}\right)$	
	2.1.1.LI.8	2.1.1.AS.8
	Determine the maximum/minimum values within the given constraints. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-Based Learning and Talk for Learning** **Activity:** Feasible regions of systems of linear inequalities. **Project-based Learning: Collaborative Learning, Problem-based Learning, Experiential Learning and Talk for Learning Approaches:** Learners work together (from whole class discussions to working in small convenient groups, then pair work) to explore by analytical methods and by hand and use of appropriate ICT tools (e.g., GeoGebra) the solutions for systems of linear and quadratic equations and inequalities to establish maximum and or minimum values within the given constraints.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Example

- Evaluate the expression $l = 3x + 4y$ for the given feasible region to determine the point at which 'l' has a maximum value and the point at which 'l' has a minimum value.



Solution

At (−4,0) $l = 3x + 4y \rightarrow l = 3(-4) + 4(0) \rightarrow l = -12 + 0 \rightarrow l = -12$

Therefore $3x + 4y = -12$

At (0,5): $l = 3x + 4y \rightarrow l = 3(0) + 4(5) \rightarrow l = 0 + 20 \rightarrow l = 20$

Therefore $3x + 4y = 20$

At (6,3): $l = 3x + 4y \rightarrow\ l = 3(6) + 4(3)\ l = 18 + 12 \rightarrow l = 30;$

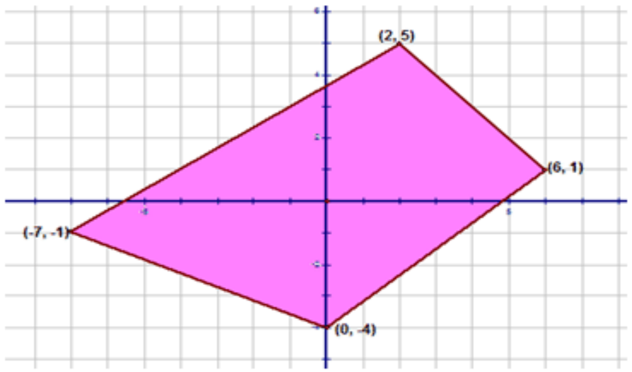
Therefore $3x + 4y = 30$

At (8,−4): $l = 3x + 4y \rightarrow l = 3(8) + 4(-4) \rightarrow l = 24 - 16 \rightarrow l = 8$

Therefore $3x + 4y = 8$

The maximum value of 'l' occurred at the vertex $(6, 3)$.

The minimum value of 'l' occurred at the vertex $(-4, 0)$

	- For the following graphed region and the expression $z = 5x + 7y - 1$, find a point where 'z' has a maximum value and a point where 'z' has a minimum value. 	
	2.1.1.LI.9	2.1.1.AS.9
	Solve real life problems involving linear inequalities. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-Based Learning and Talk for Learning** **Activity:** Real-life applications of linear programming. **Collaborative Learning, Problem-based Learning, Experiential Learning, and Talk for Learning Approaches:** Learners work together (from whole class discourse, to small convenient groups, then pair work as well as individual work) to create and solve real life problems involving the application of linear programming. **Example 1** A company that produces football jerseys makes two sets of jerseys for Accra Hearts of Oak: the traditional rainbow jerseys for home matches and the white jerseys with rainbow for away matches. To produce each jersey, two types of material – nylon and cotton – are used.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

The company has 450 units of nylon in stock and 300 units of cotton. The traditional rainbow jersey requires 6 units of nylon and 3 units of cotton. The white 'away' jersey requires 5 units of nylon and 5 units of cotton. Each white 'away' jersey that is made realises a profit of GHC 12.00 for the company, whereas each rainbow jersey realises a profit of GHC15.00. For the nylon and cotton that the company currently has in stock, how many jerseys should the company make to maximise their profit?

Solution

Let 'x' represent the number of blue flags.

Let 'y' represent the number of green flags

	Units required per rainbow jersey	Units required per white jersey	Units Available
Nylon,	6	5	450
Cotton	3	5	300
Profit per jersey	$GH₵\ 12.00$	$GH₵\ 15.00$	

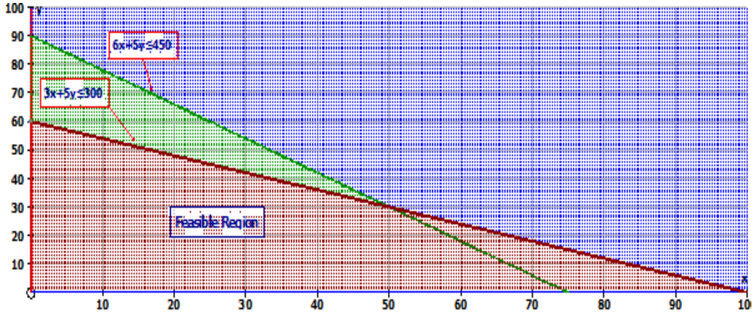
The number of rainbow jerseys and/or white jerseys that are produced must be either zero or greater than zero. Therefore, the constraint $is\ x \geq 0$ and $y \geq 0$, respectively.

The total number of units of nylon and/or cotton required to make both types of jerseys cannot exceed 450. Therefore, the constraint is

$6x + 5y \leq 450$ and $3x + 5y \leq 300$ respectively. The equation to identify the profit.

$P = 12x + 15y$

Graphically, we have:



By determining the exact point of intersection between the constraints at $(50, 30)$, the *x*-intercept of the feasible region $(75, 0)$. The *y*-intercept is $(0, 60)$ and using the profit equation, for each vertex of the feasible region, the maximum profit occurred at the vertex $(50, 30)$. This means with the supplies in stock, the company should make 50 rainbow jerseys and 30 white jerseys.

Example 2

A cocoa processing factory can provide its customers with chocolate, cocoa paste and cocoa butter by processing either of two cocoa bean varieties – Forastero or Criollo. The cocoa beans arrive at the company in railroad trucks. Each railroad truck of Forastero cocoa beans can be processed into 3 tons of chocolate, 3 tons of paste and 1 ton of butter. Each railroad truck of Criollo cocoa beans can process 1 ton of chocolate, 4 tons of paste and 3 tons of butter. The factory received an order for 7 tons of chocolate, 19 tons of cocoa paste and 8 tons of cocoa butter. The cost to purchase and process a carload of Forastero beans is GHC7000.00 while the cost for Criollo beans is GHC6000.00 If the company wants to fill the order at a minimum cost, how many carloads of each variety must be bought?

2.1.1.LI.10	2.1.1.AS.10
Find the factors and zeros of a polynomial function using conventional and personal strategy. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-Based Learning and Talk for Learning**	Level 1 Recall **Level 2 Skills of conceptual understanding**

Activity : Zeros of a Polynomial function.

Use Problem-based Learning, Collaborative Learning and Talk for Learning Approaches:
Working with a partner using think-pair-share, learners recall and discuss some facts about polynomial functions.

Example

- ***Factoring Polynomials:*** Terms are factors of a polynomial if, when they are multiplied, they equal that polynomial: e.g.,

$x^2 + 2x - 15 = (x - 3)(x + 5)$ and so, $(x - 3)$ and $(x + 5)$ are Factors of the polynomial $x^2 + 2x - 15$.

- ***Solving a Polynomial Equation***

By rearranging the terms to have zero on one side: e.g.,
$x^2 + 2x = 15 \Rightarrow x^2 + 2x - 15 = 0$ therefore, $(x + 5)(x - 3) = 0$
and $(x + 5) = 0$ and $(x - 3) = 0$;
so $x = -5$ or $x = 3$

- ***Solutions/Roots a Polynomial:*** Setting the Factors of a Polynomial Expression equal to ***zero*** gives the ***Solutions*** to the Equation when the polynomial expression equals zero. Another name for the Solutions of a Polynomial is the ***Roots*** of a Polynomial!

- The Zeros of a Polynomial Function are the solutions to the equation you get when you set the polynomial equal to zero.

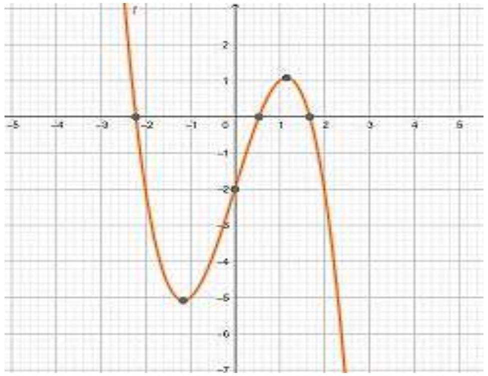
Activity 2: Graphical solutions: Factors, Roots and Zeros of Polynomial functions.

Collaborative Learning, Experiential Learning and Talk for Learning Approaches:
Learners work in pairs: explore by hand and by use of appropriate ICT tools (e.g. use of GeoGebra) to establish and recognise that the *Solutions/Roots* of Polynomial Equations are the x-coordinates for the x-intercepts of the Polynomial Graph.

Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

Example

1. Here is a graph of our polynomial function:



Conclusion

For our Polynomial Function: $y = x^2 + 2x - 15$
The Factors are: $(x + 5)$ & $(x - 3)$
The Roots/Solutions are: $x = -5$ and 3

The Zeros are at: $(-5, 0)$ and $(3, 0)$

2. The graph shows a polynomial function. Study it carefully and answer the questions that follow.
   a. Write down the zeros of the polynomial.
   b. Identify the roots or solution for the polynomial.
   c. State the factors of the polynomial.
   d. Write the polynomial function for the graph.

Activity 3: Application of Rational Zeros Theorem

Collaborative Learning, Experiential Learning and Talk for Learning Approaches:
Work with a partner to investigate the possible rational to zeros of a polynomial function and use the theorem to establish the zeros, factors and roots of a polynomial function.

Note: If the polynomial
$P\ (\ x\ )\ = an\ x^{n} + \ an - 1\ x^{n-1} + \ an - 2\ x^{n-2} + \ ... \ + \ a1 + \ a0 +$ has integer coefficients, then every rational zero of $P(x)$ is of the form $\frac{p}{q}$; where p is a factor of the constant coefficient $a0$; and q is a factor of the leading coefficient an (Rational Zeros Theorem)

Example

- List all possible rational zeros given by the Rational Zeros Theorem of $P(x)\ =\ 6x^4\ +\ 7x^3\ -\ 4$ (but don't check to see which actually are zeros)

Solution:
Factors of $a0$ =4= $\pm 1, \pm 2, and\ \pm 4$
Factors of an =6 = ±1, ±2, ±3, ±6
Possible $\frac{p}{q} = \frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 2}, \frac{\pm 1}{\pm 3}, \frac{\pm 1}{\pm 6}, \frac{\pm 2}{\pm 1}, \frac{\pm 2}{\pm 2}, \frac{\pm 2}{\pm 3}, \frac{\pm 2}{\pm 6}, \frac{\pm 4}{\pm 1}, \frac{\pm 4}{\pm 2}, \frac{\pm 4}{\pm 3}, \frac{\pm 4}{\pm 6}$
By simplifying the fractions and eliminating duplicates, we get the following list of possible values for $\frac{p}{q}$
$\pm 1, \pm 2, \pm 4, \frac{\pm 1}{\pm 2}, \frac{\pm 1}{\pm 3}, \frac{\pm 2}{\pm 3}, \frac{\pm 4}{\pm 3}, \frac{\pm 1}{\pm 6}$

Find all real zeros of the polynomial $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$.

Solution:
Applying the Rational Zeros Theorem, we have $\pm 1, \pm 2, , \frac{\pm 1}{\pm 2}$ as the possible rational zeros.
Using synthetic division,
We now check if 1 is a root by dividing $P(x) = 2x^4 + x^3 – 6x^2 – 7x – 2$ by $x - 1$

$$\begin{array}{r|l} -1 & 2 \quad 1 -6 -7 -2 \\ & \ \ -2 \quad 1 \quad 5 \quad 2 \\ \hline & 2 - 1 -5 -2 \quad 0 \end{array}$$

Since the remainder is zero, 1 is a zero
This also tells us that *P* factors as:
$2x^4 + x^3 - 6x^2 – 7x – 2 = (x+1)(2x^3 – x^2 – 5x – 2)$

Applying the Rational Zeros Theorem again, and again P factor as:
$2x^4 + x^3 - 6x^2 - 7x - 2 = (x + 1)^2(x - 2)(2x + 1)$
Thus, the zeros of $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$ are: (–1,0), (2, 0) and $(-\frac{1}{2}, 0)$

Activity 4: Application of Descartes' rule of signs Theorem.

Collaborative Learning, Experiential Learning and Talk for Learning Approaches:
Work with a partner: investigate using Descartes' rule of signs Theorem to the nature of the roots of a polynomial function.

Note: Let *P* be a polynomial with real coefficients:

- The number of positive real zeros of *P*(*x*) is either equal to the number of variations in sign in *P*(*x*) or is less than that by an even whole number.
- The number of negative real zeros of *P*(*x*) is either equal to the number of variations in sign in *P*(–*x*) or is less than that by an even whole number.

(Missing terms (those with 0 coefficients) are counted as no change in sign and can be ignored.

Example

- Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 6x^3 + 17x^2 - 31x - 12$ can have. Then determine the possible total number of real zeros

Solution:
$P(x) = 6x^3 + 17x^2 - 31x - 12$.

P(*x*) has one positive real zero.
Also; $P(-x) = -6x^3 + 17x^2 + 31x - 12$.

P(–*x*) has two variations in sign.
Combining the findings, P(x) has either one or three real zeros

Positive	Negative	Real
1	0	1
1	2	3

- Find the maximum number of positive and negative real zeros of the polynomial function $f(x) = 2x^4 + x^3 - 6x^2 - 7x + 1$.

Activity 5: Application of the Fundamental Theorem of Algebra.

Collaborative Learning, Experiential Learning and Problem-based Learning Approaches:
Learners work in pairs to **i**nvestigate and apply The Fundamental Theorem of Algebra to establish the zeros, factors and roots of a polynomial function

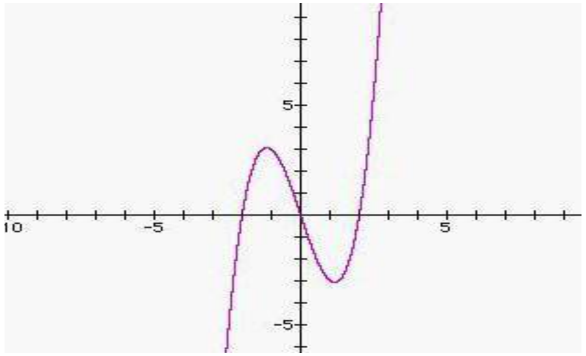
Note 1:

- Every polynomial equation with a degree higher than zero has at least one root in the set of complex numbers.
- A polynomial equation of the form $P(x) = 0$ of degree 'n' with complex coefficients has exactly 'n' Roots in the set of complex numbers.

Example

- Investigate whether if a polynomial has 'n' complex roots, will its graph necessarily have 'n' x-intercepts?

For instance:



- In this example, the degree *n* = **3**; if we factor the polynomial, the roots are *x* = -2, 0, 2. We can also see from the graph that there are three x-intercepts.

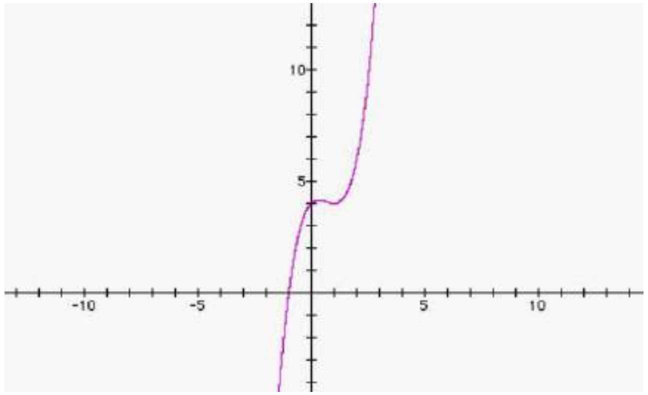
$$y = x^3 - 4x$$

- In this example, however, the degree is still n = **3**, but there is only **one** Real x-intercept or root at x = -1. The other **2** roots must have ***imaginary*** components.

$$y = x^3 - 2x^2 + x + 4$$

Conclusion:

Just because a polynomial has ‘n’ complex roots does not mean they are all Real!



Note 2:

Every polynomial $P\,(\,x\,)\;=an\,x^{n}+\,an-1\,x^{n-1}+\,an-2\,x^{n-2}+$ ... + $a1+\,a0$ (n ≥ 1, $an\;\neq 0$) with complex coefficients has at least one complex zero, hence for every polynomial $P\,(\,x\,)$, there is a complex number c_1 such that $P\,(c_1\,) = 0$. From the Factor Theorem, this tells us that $x – c_1$ is a factor of $P(x)$. Thus, we can write $P\,(\,x\,) = (x − c_1\,)\,Q\,(\,x)$, where $Q(x)$ has degree $n – 1$.

Example:

- Factor the polynomial $P(x) = 4x^5 – 324x$ completely and find all its zeros. State the multiplicity of each zero.

Solution:

$$P\,(\,x\,)\;=\;4x\,(\,x\;-\;3)(\,x\;+\;3)(x2\;+\;9)$$
$$=\;4x\,(\,x\;-\;3)\,(\,x\;+\;3)\,(\,x\;-\;3i\,)\,(\,x\;+\;3i)$$

Therefore, the zeros of P are $0, 3, – 3, 3i$ and $– 3i$. Since each factor occurs only once, all the zeros are of multiplicity 1, and the total number of zeros is five.

- Find all zeros of the polynomial $P(x) = x^4 + x^3 + 3x^2 – 5x$.

Activity 6: Complex Conjugates Theorem

Collaborative Learning, Experiential Learning and Problem-based learning Approaches:
Work with a partner to investigate and apply the Complex Conjugate Theorem to solve problems related to the zeros, factors and roots of a polynomial function.

Note:

- Roots/Zeros that are not Real are Complex with an Imaginary component. Complex roots with Imaginary components always exist in Conjugate Pairs.
- If a $+ bi$ (b ≠ 0) is a zero of a polynomial function, then its conjugate, a $- bi$, is also a zero of the function.

Example
Find all the roots of $f\ (x) = x^3 - 5x^2 - 7x + 51$. If one root is 4 - i.

Solution:
Applying the Complex Conjugate Theorem, and Descartes' rules we have the factor $[x - (4 - i)]$, and $[xc - (4 + i)]$.
The product of factors gives

$$[x - (4 - i)].[x - (4 + i)] = x^2 - 8x + 17$$

Since the product of the two non-real factors is $x^2 - 8x + 17$, then the third factor (that gives us the negative real root) is the quotient of $P(x)$ divided by $x^2 - 8x + 17$, which is $x = -3$
Therefore, the roots of P(x) are -3, -4i and 4i.

Activity 7: Linear and Quadratic Factors Theorem.

Collaborative Learning, Experiential Learning and Problem-based Learning Approaches:
Learners work in pairs to investigate and establish that every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

	Example: Given the polynomial $P(x) = x^4 + 2x^2 - 63$. • Factor *P* into linear and irreducible quadratic factors with real coefficients. • Factor *P* completely into linear factors with complex coefficients. **Solution:** a) $P(x) = (x - \sqrt{7})(x + \sqrt{7})(x^2 + 9)$ $b)$ $P(x) = (x - \sqrt{7})(x + \sqrt{7})(x - 3i)(x + 3i)$	
	2.1.1.LI.10	2.1.1.AS.10
	Recall the processes of finding the domain, range, zero of a rational function and state when it is undefined. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning** **Activity:** Recall basic facts about rational functions. **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based learning**: Learners work in pairs; one of the pairs decomposes the irrational function into partial terms, whiles the other composes the fractions back into a single function. Learners recall facts using graphs and by algebraic methods, the domains, range and zeros of rational function. **Example** • The domain of the function $y = \frac{1}{x-2}$ is the set of all real numbers except $x = 2$ i.e., the Domain of y is $\{x: x \in R, x \neq 2\}$ • State the largest possible domain of the function defined by $f: x \to \frac{2x-1}{2x^2-9x-5}$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	• Learners recall processes for adding, subtracting, multiplying, dividing and simplifying rational function. **Example:** Simplify $\frac{2x+ 6y}{x^2+4xy+3y^2}$	
	2.1.1.LI.11	2.1.1.AS.11
	Resolve rational functions into partial fractions, specifically quadratic denominators and improper rational functions. **Activity1:** Working with Partial Fractions **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning** Learners work in groups/pairs to revise processes for resolving rational functions containing linear factors and or repeated factors as denominators. Learners revise resolving rational functions whose denominator is made up of linear factors and repeated factors using different strategies: **Examples** Split $\frac{5(x + 2)}{(x + 1)(x + 6)}$ into partial fractions. Split $\frac{x - 2}{(x + 1)(x-1)^2}$ into partial fractions. **Activity 2:** Improper Rational functions and functions with irreducible quadratic factors. **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for learning Approaches and Problem-based learning**:	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Work with a partner: One of the pairs decomposes the irrational function into partial terms whiles the other composes the fractions back into a single unit.

Learners recognise that a rational function with the denominator containing irreducible quadratic factors is resolved using the process:

$$\frac{ax-b}{ax^2+bx+c}=\frac{Ax+B}{ax^2+bx+c}$$

Example

Express $\frac{5x}{(x^2+x+1)(x-2)}$ as the sum of its partial fractions. Reverse the process to verify your answer.

Solution:

$\frac{5x}{(x^2+x+1)(x-2)} = \frac{5(1-2x)}{7(x^2+x+1)}+\frac{10}{7(x-2)}$

Express $\frac{x^2+4x\ _{-}2}{(x-1)(x^2+1)}$ in partial fractions.

Learners recognise that an improper rational function is resolved by first reducing it to a proper rational function:

Example: Resolve $\frac{2x^2+1}{(x-1)(x+2)}$ in partial fractions. Reverse the process to verify your answer.

Solution:

$\frac{2x^2+1}{(x-1)(x+2)} = 2+\frac{1}{(x-1)}-\frac{3}{(x+2)}$

Teaching and Learning Resources	• Graph paper • Ruler • A scientific calculator

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
2.1.1.CS.4	2.1.1.LI.1	2.1.1.AS.1
Demonstrate the ability to carry out matrix operations, determine the inverse of a linear transformation and represent real life situations in matrix forms.	**Distinguish between 'singular' and 'non-singular' square matrices (2 x 2** and **3 x 3) and evaluate determinants.** **Collaborative Learning, Experiential Learning, Talk for Learning, problem-based Learning** **Activity 1:** Revise Types of Matrices **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.** Learners work in pairs; one of the pairs identifies a matrix they know whiles the other partially writes an example (learners' reverse roles). **Example:** Revise special types of Matrices with examples (i.e. Square matrix, Diagonal matrix, Identity Matrix, Upper Triangular matrix, Lower Triangular Matrix, Symmetric Matrix, Skew-Symmetric Matrix, Zero Matrix, Row Vector, Column Vector) **Activity 2:** Revise Matrices Algebra **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning**. Learners work in pairs: one pair identifies and creates a problem involving operation on a matrix and tasks the other pair to solve, justify, prove or investigate the context (learners' reverse roles). **Example** • Learners revise Equality of matrices, Scalar multiple of a matrix, Addition of two matrices, Multiplication of two matrices and Properties of Matrices (commutative, Associative and Distributive law). **Activity 3**: Determinant, transpose of a matrix, singular and non-singular matrix, Minor Cofactors of a square matrix**.**	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.

- Work with a partner: one of the pair identifies a create a 2x2 and 3x3 matrix whiles the other pair writes the transpose of the matrix and verify the transpose properties. Learners change roles.

- Learners establish that the transpose of matrix A, written A^t is the matrix obtained by writing the rows of A in order as columns.

Example

If $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ then $A^t = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$

- Work with a partner: one of the pair identifies and creates a 2x2 matrix while the other pair calculate the determinant of the matrix. Learners change roles.

- Establish that; If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of matrix A, denoted by det(A) or $|A|$ is given by $|A| = ad - bc$.

Note: If the determinant of a matrix is **zero,** then that matrix is called a singular matrix.

Example

Find the determinant of $A = \begin{pmatrix} 3 & -5 \\ 2 & -4 \end{pmatrix}$

Solution:

$|A| = 3 \times (-4) - (-5) \times 2 = -12 + 10 = -2$

Example

Find the determinant of $B = \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}$

Solution:

$|B| = 4 \times (6) - 8 \times (3) = 24 - 24 = 0$

Matrix B is singular because its determinant is zero.

- If A is a square matrix, verify the following properties of the Determinant:
  i) The determinant of a square matrix A and its transpose are equal. $|A| = \lfloor A^t \rfloor$.
  ii) If A has a row (column) of zeros, then $|A| = 0$.
  iii) If A has two identical rows (or columns), then $|A| = 0$.
  iv) If A is a triangular matrix, then $|A|$ is a product of the diagonal elements.
  v) If A is a square matrix of order n and k is a scalar, then $|kA| = k^n \lfloor A \rfloor$.

- Work with a partner: One pair identifies and creates a 2x2 matrix while the other pair investigates whether the matrix is singular or non-singular. Learners change roles and summarise their findings.

Example: The table shows a comparison of singular and non-singular matrix

	Non-singular	Singular
A is	invertible	not invertible
Columns	independent	dependent
Rows	independent	dependent
det(A)	≠ 0	= 0
Ax = 0	one solution x = 0	infinitely many solutions or
Ax = b	one solution	no solution

	2.1.1.LI.2	2.1.1.AS.2
	Multiply an m x n matrix by an n x 1 matrix. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning** **Activity:** Exploring the possibilities of multiplying given matrices**.** **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.** Work with a partner: Learners explore matrix multiplication, multiplying $m \times n$ matrix by an $n \times 1$ matrix. Learners recognise that two matrices, *A* and *B,* are said to be confirmable for product AB if the number of columns in *A* equals the number of rows in matrix *B*. **Example:** Let $A = \begin{pmatrix} 1 & 2 \\ -3 & 0 \\ -5 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & -5 \\ 0 & 6 & -2 \end{pmatrix}$ Calculate (i) AB (ii) BA (iii) is AB = BA?	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**
	2.1.1.LI.3	2.1.1.AS.3
	Transform systems of linear equations into a Matrix form and state the matrix representing a linear transformation. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning** **Activity:** Application of Matrices to Simultaneous Equations**.** **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.**	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	Work with a partner: One pair identifies and creates a simultaneous linear equation while the other pair transforms it into a Matrix and vice versa. Learners change roles and summarise their findings. **Example** Write the simultaneous equations into a matrix of the form $AX = B$: $2x + 3y = 13$ and $5x + 2y = 16$ **Solution:** $\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$ **Example** Transform the matrix into a system of linear equations: $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ -3 \\ 7 \end{pmatrix}$ • Write the following word problem as a matrix in the form $AX = B$: • There are 5,500 men, women and children altogether at the swimming pool. There are twice as many women as men and four times as many children as women. How many men, women and children are at the swimming pool? • If one side of the triangle increases by 11 cm and the other side decreases by the same value, we get an equilateral triangle. When the first side is multiplied by four, it is 10 cm longer than three times the third side. Find the lengths of the sides of the triangles. • How many kilograms of iron and how many kilograms of sulphur contain 100 kg of FeS if the relative atomic weight of iron is 56 and the relative atomic weight of Sulphur is 32?	
	2.1.1.LI.4	2.1.1.AS.4
	Find the inverse of a matrix using linear transformation. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning**	Level 1 Recall **Level 2 Skills of conceptual understanding**

Activity: Inverse of a matrix

Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.

Work with a partner: One pair identifies and creates a square matrix while the other pair identifies the minors and co-factors matrix. Then, the pair work together to determine the adjoint matrix. Learners change roles and summarise their findings.

Example

- Establish that the adjoint of a matrix (also called the adjugate of a matrix) is defined as the transpose of the cofactor matrix of that particular matrix. For a matrix A, the adjoint is denoted as $adj\ (A)$.

Thus, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then the matrix formed by the cofactors of the elements is

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

and the adjoint of $A, \mathrm{a}dj(A) = \begin{pmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{pmatrix}$ where

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ \ A_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ \ A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

$$A_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},\ \ A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},\ \ A_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$A_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},\ \ A_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix},\ \ A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

- Find the adjoint matrix of the matrix.

  i. $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$

  ii. $B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$

- Verify that
  1. for any square matrix A, then
     $A(adj\ A) = (adj\ A)A = |A|I$
     where I is the identity matrix of the same order.
  2. If *A* is a non-singular matrix of order n, then
     $|adjA| = |A|^{n-1}$
  3. If *A* and *B* are two square matrices of order n, then $(adj\ (AB) = (adj\ B)(adj\ B)$
  4. For any square matrix A, $(adj\ A)^t = \ adj\ A^t$.
  5. The adjoint of an identity matrix is the identity matrix.
  6. The adjoint of a symmetric matrix is a symmetric matrix.

Work with a partner: learners explore with several pairs of matrices to establish that if $AA^{-1} = I = A^{-1}A$, then A^{-1} is called the multiplicative inverse of A.

Example

1. Verify which pair of matrices are inverses of each other.

   i. $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ and $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

   ii. $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

   iii. $\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

	2. What can you say about (i) and (ii) - Learners establish that the necessary and sufficient condition for a square matrix A to have an inverse is that $A \neq 0$ (That is, A is non-singular). - Learners establish that If $A = (a\ b\ c\ d\ )$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Where $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the adjoint matrix of A $(adj(A))$, and $ad - bc$ is the determinant of A. **Note**: that the inverse of a matrix can be easily found using the calculator. Simply raise the matrix to the power of -1. **Work with a partner:** Learners explore several matrices to establish that; 1. If $AX = B$, where A, X and B are matrices, then $A^{-1}B = X$ 2. If $XA = B$, where A, X and B are matrices, then $BA^{-1} = X$	
	2.1.1.LI.5	2.1.1.AS.5
	Model and solve problems based on real life situations using matrices. **Collaborative Learning, Experiential Learning, Problem-based Learning, Project-based Learning and Talk for Learning** **Activity:** Application of Matrices to Simultaneous Equations **Collaborative Learning, Experiential Learning, Whole class discussion, Talk for Learning Approaches and Problem-based Learning.** **Work with a partner:** Solve problems systems of linear simultaneous equations using matrix properties. - Solve the following simultaneous equations using matrix methods: $2x + 3y = 13$ and $5x + 2y = 16$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Solution: $\begin{bmatrix}2 & 3\\5 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}13\\16\end{bmatrix}$ Notice that $A=\begin{bmatrix}2 & 3\\5 & 2\end{bmatrix}$ is the matrix of the coefficients, $X=\begin{bmatrix}x\\y\end{bmatrix}$ is a column matrix of the pronumerals *x* and *y*, and $B=\begin{bmatrix}13\\16\end{bmatrix}$ is a column matrix of the values on the right-hand side of the equations. Using the relation $A^{-1}B=X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2 & 3\\5 & 2\end{bmatrix}^{-1}\times\begin{bmatrix}13\\16\end{bmatrix}$, so $x=2$ and $y=3$ It is worth checking your answers by substituting the values in the original equations. • Solve $2x+y+4z=17$ $3x-2y=-6$ $2x+y+5z=7$ • Apply matrix methods to solve real life problems: There are 5,500 men, women and children altogether at the swimming pool. There are twice as many women as men and four times as many children as women. How many men, women and children are at the swimming pool?	
Teaching and Learning Resources	• Worksheets • Scientific Calculator • Technological tools, apps, etc.	

Subject ADDITIONAL MATHEMATICS
Strand 2. GEOMETRIC REASONING AND MEASUREMENT
Sub-Strand 1. SPATIAL SENSE

Learning Outcomes	21st Century Skills and Competencies	GESI, SEL and Shared National Values
2.2.1.LO.1		
Deduce the equation of a circle and find its centre and radius.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** • Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. • Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they deduce the equation of a circle and find its centre and radius using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to deduce the equation of a circle and find its centre and radius. • Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning

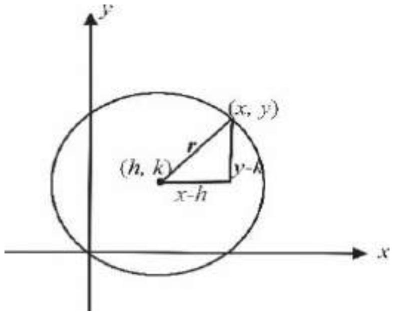
	interpret phenomena (learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals, encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: 1. Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills 2. Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. 3. Managing emotions and conflicts as they engage in collaborative group work 4. Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs,

		religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they deduce the equation of a circle and find its centre and radius**.**
2.2.1.LO.2		
Determine the equation of a locus under a given condition.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they determine the equation of a locus under a given condition. • Interrogate their stereotypes and biases about the roles and abilities of different

		groups as they learn how to determine the equation of a locus under a given condition. • Examine and dispel misconceptions/ myths about GESI as they engage in finding the equation of a locus under a given condition in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work. • Showing empathy and cooperation.

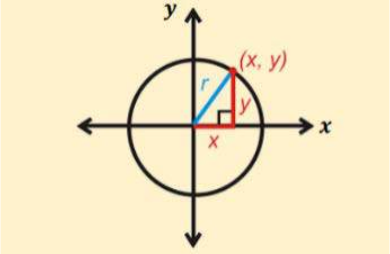
		These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they determine the equation of a locus under a given condition.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
2.2.1.CS.1	2.2.1.LI.1	2.2.1.AS.1
Demonstrate understanding of loci and their applications.	**Explore the equation a circle and its properties using technological tools, e.g. GeoGebra, Geometer's sketchpad, paper cutting, Geoboard and other realia.** **Experiential Learning, Think-pair-Share.** **Learning Experience**: Learners in pairs manipulate a technological tool (GeoGebra) to construct a circle and discover its properties. **Activity 1:** Learners in pairs observe a circle, identify its properties and describe them. • Learners recollect the concept of circles. • Learners identify and explain the properties of a circle (centre, radius, diameter, chord, sector, arc, and segment). **Activity 2:** Learners in pairs construct a circle using GeoGebra and realia. E.g., Learners in pairs construct a circle using GeoGebra/drawing instruments • Learners in pairs to describe the geometric construction of a circle and state the equation of a circle (algebraic form). In pairs, learners share a variety of circles constructed and present the various equations derived. E.g. $r^2 = (x-h)^2 - (y-k)^2$	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**
	2.2.1.LI.2	2.2.1.AS.2
	Apply knowledge of the distance between two points and the Pythagoras theorem to describe a circle in the algebraic form. Experiential Learning, Collaborative Learning, Talk for Learning **Learning Experience**: Learners in a mixed-ability group apply the idea of distance between two points and its equation to derive the equation of a circle.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning

	Activity 1: Learners in a mixed-ability group discuss the equation derived through construction with ICT tools like GeoGebra. Learners in groups brainstorm the equation derived. Learners in mixed-ability groups brainstorm on how the equation of the circle can be derived. **Activity 2:** Learners in a mixed-ability group discuss how to derive the standard form of the equation of a circle. Learners in a mixed-ability group construct a circle and label the centre as $O(h,k)$ and any point $P(x,y)$on the circumference of the circle. • Learners then apply their understanding of distance between two points to state the equation of distance between O and P, which is the radius of the circle.  • Learners in their groups derive the equation O and P as $$r = \sqrt{(x-h)^2 - (y-k)^2}$$ • Learners in their groups simplify this expression as $$r^2 = (x-h)^2 - (y-k)^2$$ • Learners compare the equation derived from the distance between OP and the equation of the circle derived by GeoGebra to establish that the two equations are the same. • Learners in their mixed-ability groups recognise the equation of a circle in standard form as $r^2 = (x-h)^2 - (y-k)^2$ where (h, k) is the centre and (x, y) is a point on the circumference.	Level 4 Extended critical thinking and reasoning

Activity 3: Learners in a mixed-ability group discuss how to deduce the equation of a circle with a centre at the origin.

- Learners in their groups investigate the behaviour of the equation of a circle if the centre is at the origin $O(0,0)$.



E.g., At $(0,0)$, we have

$$r^2 = (x-h)^2 - (y-k)^2$$
$$r^2 = (x-0)^2 - (y-0)^2$$

Therefore, the equation of the circle turns to $r^2 = x^2 + y^2$.

Activity 4: Learners in their mixed-ability group model practical problems for other groups to derive the equation of a circle.

Example 1: A goat is tethered by a rope of length 2 meters away from a peg. If the peg is at the point (3, 2). Find the equation to describe the area the goat can go round the peg.

Example 2: Write the standard form of the equation of a circle with centre $(2,4)$ and radius of 10.

Solution:

$$(x-2)^2 + (y-4)^2 = 10^2$$

Activity 5: Learners in their mixed-ability group derive the general equation of a circle.

Learners in their groups expand the standard form of the equation of a circle to derive the general equation of a circle.

That is, expanding

$$r^2 = (x-h)^2 - (y-k)^2$$

to obtain

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Hence, the general equation of the circle is $x^2 + y^2 + 2gx + 2fx + c = 0$,
Where g replaces $-h$ with f replaces $-k$ and $c = h^2 + k^2 - r^2$ since $h, k\ and\ r$ are numbers.

Example 1: Write the general equation of a circle with a centre $(2,4)$ and a radius of 10.

Solution:

$$x^2 + y^2 - 4x - 8y - 80 = 0$$

Activity 6: Learners create and solve practical problems.

Example:

Learners establish that the general equation of a circle is

$$x^2 + y^2 + 2gx + 2fx + c = 0$$

with centre $(-g, -f)$ and radius $r^2 = g^2 + f^2 - c$

Learners in their mixed-ability groups create and solve practical problems for other groups to derive the equation of a circle.

Example 1: The point $(4,6)$ is on a circle whose centre is $(1,2)$. Write a standard equation of the circle.

Solution:

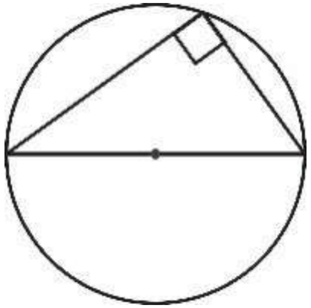
$$r = \sqrt{(4-1)^2 + (6-2)^2}$$
$$r = 5$$

	Using $r = 5$ and centre $(1,2)$, the equation of the circle is $(x-1)^2 + (y-2)^2 = 5^2$ **Example 2:** Find the centre and the radius of the circle with the equation $x^2 + y^2 - 14x + 16y - 12 = 0$ Learners apply their knowledge in completing the square or any other method to solve the question. **Solution:** **Using completing the square method:** $x^2 + y^2 - 14x + 16y - 12 = 0$ $x^2 - 14x + y^2 + 16y - 12 = 0$ $(x-7)^2 - 7^2 + (y+8)^2 - 8^2 - 12 = 0$ $(x-7)^2 + (y+8)^2 = 125$ Therefore, the centre is $(7,-8)$, and the radius is $r^2 = 125 = 5\sqrt{5}$ **Alternative method:** $2g = -14, \therefore g = -7$ $2f = 16, \therefore f = 8$ Since the centre of the circle is $(-g,-f)$, the centre of this circle is $(7,-8)$ The radius of the circle is found by $r^2 = g^2 + f^2 - c$ $r^2 = 7^2 + (-8)^2 + 12$ $r = \sqrt{125}$	
	2.2.1.LI.3	2.2.1.AS.3
	Apply knowledge of properties of lines to derive the equations of a circle, a tangent and normal to a circle. **Experiential Learning, Collaborative Learning, Talk for Learning and building on what others say**	Level 1 Recall **Level 2 Skills of conceptual understanding**

Learning Experience: Learners in collaborative groups deduce the equation of a circle given the endpoints of a diameter.

Activity 1: Learners in mixed-ability groups brainstorm and derive various ways to find the equation of the circle given the endpoints of its diameter, and share their findings with the whole class.

- Learners recollect that a diameter subtends an angle of $90°$ at the circumference of a circle.



If the point on the diameter is A and B, and the perpendicular point is P, then AP is perpendicular to BP.

- Learners recollect that the product of the gradient of AP and BP is -1.
- Learners brainstorm and establish that simplifying the product of the gradient of $AP\ and\ BP = -1$ will result in the formula for finding the equation of a line.

That is, If $A(x_1, y_1), B(x_2, y_2)$ and $P(x, y)$, then product of the gradient of $AP\ and\ BP = -1$ is given as

$$\frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

$$(y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0$$

- Learners in groups extend their knowledge and understanding to find the centre and the radius.

Level 3 Strategic reasoning

Level 4 Extended critical thinking and reasoning

Example 1: Find the equation of a circle through the ends $(5,7)$ and $(1,3)$ of its diameter and find the centre and radius.

Solution:
Equation of the circle:

$$(x-5)(x-1)+(y-7)(y-3)=0$$

$$x^2+y^2-6x-10y+26=0$$

Centre:
$2g=-6$ and $2f=-10$, Therefore the centre of the circle is $(3,5)$

Radius:

$$r^2=3^2+5^2-26$$
$$r=\sqrt{8}$$

- Learners apply the concept of midpoint to find the radius of the circle and the equation of the circle using the idea of distance between two points.

Equation of a Circle given three points.

Activity 2: Learners brainstorm finding the equation of a circle given three points using the general equation of the circle or the equation of a circle in standard form.

- Learners discover that, to find the equation of a circle given three points, they have to substitute the three given coordinates in the general equation of a circle and solve the resulting equations simultaneously.

Example: Find the equation of the circle passing through the points; $(1,3)$, $(-1,5)$and $(-1,1)$.

Solution: General equation

$$x^2+y^2+2gx+2fx+c=0$$

Substituting the coordinates will result into

$$10 + 2g + 6y + c = 0$$
$$26 - 2g + 10y + c = 0$$
$$2 - 2g + 2y + c = 0$$

Solving it simultaneously,

$y = -3, g = 1, c = 6$

Therefore, the general equation of the circle is $x^2 + y^2 + 2x - 6y + 6 = 0$

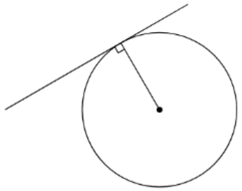
- Learners in other groups discover that points can be substituted in the standard form of a general equation, and since the radius of the circle is equal, equation 1 can be equated to equation 2, and equation 2 can be equated to equation 3 to obtain two equations that can be solved simultaneously.

Activity 3: Learners in their groups create practical problems, and other groups solve and present their solutions to the class.

Activity 4: Research Work: In groups, learners research and find alternative ways to find the equation of the circle and present the findings in class.

Activity 5: Learners in their collaborative groups recollect what a tangent is and the theorem between the tangent and the radius of a circle.

- Learners recollect that a tangent to a circle is a straight line that touches the circumference of a circle at one point only.
- Learners in their groups establish that a tangent is always perpendicular to the radius at the point of contact (point of tangency).



- Learners establish that a point lies on the circle if the point (x, y) satisfies the equation of a circle.

Activity 6: Learners in their collaborative groups discuss and share ideas on how to find the equation of a tangent to a given circle.

- Learners in their collaborative groups share ideas on how to solve a practical question.

Example 1: Find the equation of the tangent through $(3,4)$ and on the circle $x^2 + y^2 = 25$

Solution

- Learners verify that the point lies on the circle.

$$3^2 + 4^2 = 25$$

Therefore, $(3,4)$ lies on the circle.

- From the equation, learners deduce that the circle has its centre at the origin, and the radius is 5.

Coordinates to be used to find the slope of the radius of the circle are $(0,0)$ and $(3,4)$

The slope of the radius $=\frac{4-0}{3-0} = \frac{4}{3}$

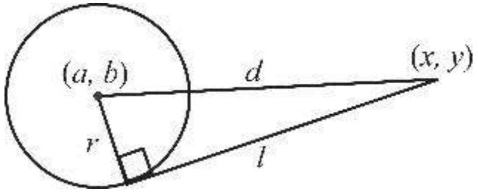
Since the tangent is perpendicular to the radius, the radius of the tangent is $-\frac{3}{4}$

The equation of the tangent is given as $y = mx + c$, $(3,4)$

Therefore, the equation of the tangent is $y = -\frac{3}{4}x + \frac{25}{4}$

Activity 7: Learners in their collaborative groups discuss and brainstorm to find the length of a tangent to a given circle from an external point.

- Group leaders lead their members to sketch a diagram to illustrate the length of a tangent to a given circle from an extended point.



- Learner groups brainstorm and apply their knowledge of Pythagoras theorem and find the distance between two points to find the length of the tangent.

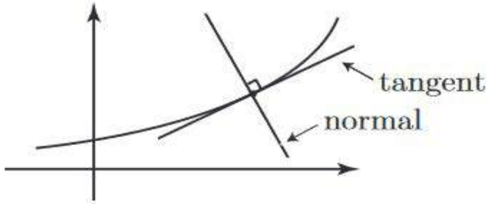
Example 1: Find the length of the tangent from the point $(2,8)$ to the circle $x^2 + y^2 + 4x - 10y + 20 = 0$.

Solution:

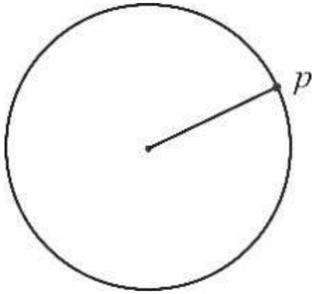
- Learners collaborate in their groups to find the centre and radius of the circle as $(-2,5)$ and 3, respectively.
- Learners use $(-2,5)$ and $(2,8)$ to find the length of the hypotenuse as 5.
- Learners apply Pythagoras theorem to find the length of the tangent as 4.

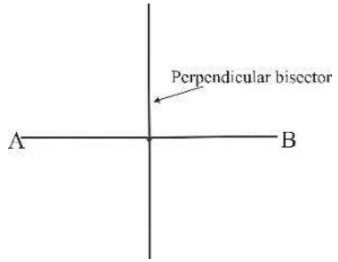
Activity 8: Learners in a collaborative group discuss the equation of a normal to a given circle

- Through interactions in learners' collaborative groups, learners discover that a normal is a line perpendicular to the tangent.



	• Learners establish that since the normal and tangent lines are perpendicular to each other, the products of their gradient is -1 and the equation of the normal is of the form $y = mx + c$. • Learners solve practical examples to consolidate the concept. **Example 1:** Find the equation of the normal to the circle $x^2 + y^2 = 5$. **Solution:** • Through interaction in their collaborative groups, learners establish that the circle has its origin at the centre; therefore, the coordinates at the centre is $(0,0)$. • The gradient of the normal/radius is $\frac{2-0}{6-0} = \frac{1}{3}$. • To find c $y = mx + c$ $2 = \frac{1}{3}(6) + c$ $c = 0$ Therefore, the equation of the circle is $y = \frac{1}{3}x$ • Learners in their groups create questions for other groups to solve and present their solutions to the class.	
	2.2.1.LI.4	2.2.1.AS.4
	Deduce relations of various loci under given conditions. **Experiential Learning, Collaborative Learning, Talk for Learning and building on what others say.** **Learning Experience**: Learners in collaborative groups deduce the equation of a locus under given conditions.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

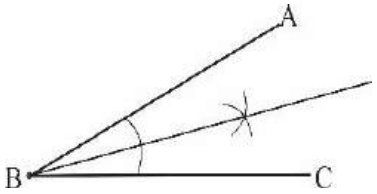
	Activity 1: Learners in groups collaborate and: • discuss the meaning of locus and share their thoughts with the whole class. • deduce that a set of points that satisfy a given condition is a circle. • establish locus as a geometric path that can be written algebraically. **Activity 2:** Learners in groups collaborate to construct loci under given conditions. **Condition 1: Locus equidistant from the centre** Learners in mixed-ability groups: • construct a locus whose distance is equidistant from the centre. • establish that the locus of a fixed point is the circumference of a circle, as shown.  **Condition 2: Locus equidistant from two fixed points** Learners in mixed-ability groups brainstorm and discuss: • a locus equidistant from two fixed points • establish that a locus from two fixed points is a perpendicular bisector, as shown.	



Condition 3: Locus from two fixed lines

Learners in mixed-ability groups brainstorm and discuss:

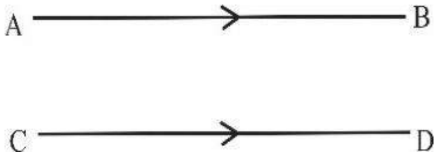
- how to construct a locus equidistant from two fixed lines, say AB and BC
- establish that a locus from two fixed lines is an angle bisector, and for any point lying on the angle bisector, the perpendicular distance is equidistant from the two lines.



Condition 4: Locus equidistant from a line

Learners in mixed-ability groups brainstorm and discuss:

- how to construct a locus equidistant from a line, say AB
- deduce that a locus from a line AB is a parallel line.



Condition 5: The equation of a locus from a fixed point

Learners in groups deduce that since the locus of a point fixed point is the circumference of a circle, the equation of this locus is the equation of a circle.

That is, the equation of a locus from a fixed point is

$$r^2 = (x - h)^2 - (y - k)^2$$

Example 1: Find the equation of the locus of a moving point $P(x, y)$, which is always at a distance of 3 units from a fixed point $Q(1,2)$.

Solution:

$$(x - 1)^2 - (y - 2)^2 = 3$$

$$x^2 + y^2 - 2x + 4y - 12 = 0$$

Learners in mixed-ability groups:

establish that since, a locus P from two fixed points (AB) is a perpendicular bisector, then $|AP| = |BP|$

apply their knowledge in finding the distance between two points to find the equation of this loci

Example 2: Find the equation of a locus of a point that is equidistant from the points $A(-2,0)$ and $B(3,2)$.

Solution

$$\sqrt{(x + 2)^2 + (y - 0)^2} = \sqrt{(x - 3)^2 + (y - 2)^2}$$

$$10x + 4y - 9 = 0$$

Learners in mixed-ability apply their knowledge of dividing a line in a given ratio to find the equation of a locus involving a constant distance r from two fixed points in the ratio $m:n$

Example 3: If $A(2, 0)$ and $B\ (0, -2)$ are two fixed points and point P moves with a ratio so that $AP : BP = 1 : 3$. Find the equation of the locus of point P

Solution:

$$\frac{AP}{BP} = \frac{1}{3}$$

$$(3AP)^2 = BP^2 \Longrightarrow 9AP^2 = BP^2$$

$$9[(x-2)^2+(y)^2]=[(x)^2+(y+2)^2]$$
$$8x^2+8y^2-36x-4y+32=0$$

Other conditions

Learners in groups apply their knowledge in equations of perpendicular lines to find the equation of locus of a Point $P(x, y)$ such that AP is Perpendicular to BP where A and B are constants.

Learners in mixed-ability groups brainstorm to solve an example.

Example 4: A point $P(x, y)$ moves so that AP and BP are perpendicular. Given that $A(1,2)$ and $B(2,4)$ find the equation of the locus P.

Solution:

The product of slopes of perpendicular lines is -1, therefore $\left(\frac{y-2}{x-1}\right)\left(\frac{y-4}{2-x}\right)=-1$

The equation of the locus P is

$x^2+y^2-3x-6y+10=0$

Learners in groups collaborate to find the locus of a point $P(x, y)$ that is always at a constant distance of units from a given Line.

Learners work in mixed-ability groups to solve examples.

Example 5: What is the locus of point $P(x, y)$ that is always 2 units from the line $x=4$?

Solution

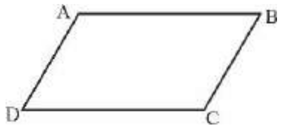
$A(4, y), P(x, y)$ and the constant distance $(d)=2$

$$\underline{PA}=\sqrt{(x-4)^2+(y-y)^2}=2$$
$$x^2-8x+12=0$$
$$x=2 \text{ or } x=6$$

Learners in their groups create questions involving other conditions to be solved.

Example 6: $A(4,5)$ and $B(-2,7)$ are given points. Find the equation of the line such that $2PA=PB$.

	Solution: $P(x, y), A(4,5)$ and $B(-2,7)$ Given condition is $2PA = PB$ $\therefore 4PA^2 = PB^2$ $4[(x-4)^2 + (y-5)^2] = (x+2)^2 + (y-7)^2$ The equation of the locus is $3x^2 + 3y^2 - 36x - 26y + 111 = 0$	
Teaching and Learning Resources	• Worksheets • Scientific Calculator • Technological tools, apps, etc. • SHS Additional Mathematics Curriculum	

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
2.2.1.CS.2	1.2.1.LI.1	1.2.1.AS.1
Demonstrate knowledge and understanding of spatial sense in relation to related problems.	**Apply the knowledge of operations of vectors to solve simple geometric problems, including the position vector of a point that divides a vector internally and externally in a given ratio.** Experiential Learning, Collaborative Learning, Talk for Learning and building on what others say **Learning Experience**: Learners in collaborative groups apply knowledge of vectors to solve geometric problems, including deducing the position vector of a point that divides a vector internally and externally in a given ratio. **Activity 1**: Learners brainstorm using examples to recollect the idea of transposition of vectors and unit vectors in the direction of vectors, e.g. as shown in Example 1. **Example 1:** $ABCD$ is a parallelogram such that $\overrightarrow{AB} = 3i + j$ and $\overrightarrow{AC} = 5i - 3j$. Find: i) $\overrightarrow{BC}$ ii) $\overrightarrow{BD}$ iii) the unit vector in the direction of $\overrightarrow{BC}$ **Solution**  **i**. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ $(5i - 3j) - (-3i - j)$ $\overrightarrow{BC} = 2i - 4j$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

ii. $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

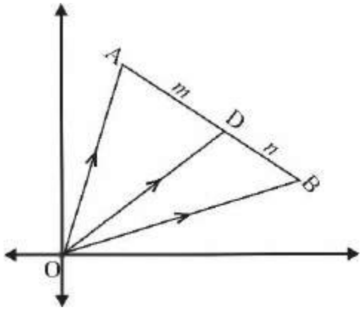
$$= \overrightarrow{BC} - \overrightarrow{AB}$$
$$= (2i - 4j) - (3i + j)$$
$$= -i - 5j$$

iii. $\left|\overrightarrow{BC}\right| = \sqrt{2^2 + (-4)^2}$

$$= 2\sqrt{5}$$
$$\overrightarrow{BC} = \frac{1}{2\sqrt{5}}(2i - 4j)$$
$$= \frac{\sqrt{5}}{5}(i - 2j)$$

Activity 2: Find the position vector $\overrightarrow{OD}$ that divides a given vector (or line) in a given ratio.

- Learners to study the diagram below carefully and use it to arrive at a solution when line AB is divided in the ratio $m:n$.



D is a point on $\overrightarrow{AB}$ such that $\overrightarrow{AD} : \overrightarrow{DB} = m:n$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$$= \overrightarrow{OA} + \left|\overrightarrow{AD}\right| \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

$= \overrightarrow{OA} + \frac{AD}{AB} \cdot \overrightarrow{AB} \ldots \ldots . e$quation (1)

But $\overrightarrow{AD} : \overrightarrow{DB} = m : n$, i.e $AB = m + n$

So $d = a + \frac{m}{m+n}(b - a)$

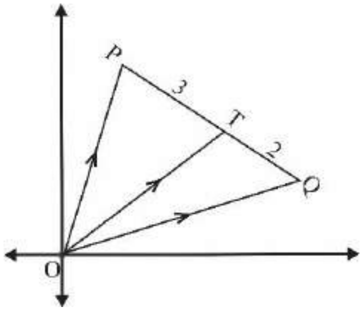
$d = \frac{na+mb}{m+n}$ … …. equation (2)

Note: The vector $\overrightarrow{AD}$ is equal to the product of its magnitude and the unit vector in the same direction as $\overrightarrow{AD}$ i.e

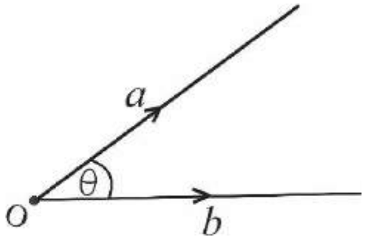
$$AD \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

Equation (2) gives the required position vector whether the line or vector is divided internally or externally. In the internal case, both m and n are positive. However, in the external case, any one of the ratios is taken as negative.

Example: P and Q are points on the position vectors $2i - 3j$ and $2j - i$, respectively, on a vector $\overrightarrow{PQ}$. T is a point on $\overrightarrow{PQ}$ such that $PT : TQ = 2 : 3$. Find the position vector of the point T.



	Solution: Finding position vector $\overrightarrow{OT}$:: $\overrightarrow{OT} = \frac{2\overrightarrow{OQ} + 3\overrightarrow{OP}}{2 + 3}$ $= \frac{2(-i + 2j) + 3(2i - 3j)}{5}$ $= \frac{4i - 5j}{5}$ $= \frac{1}{5}(4i - 5j)$	
	2.2.1.LI.2	2.2.1.AS.2
	Derive the rule for scalar (dot) product and use it to solve problems relating to angles between two vectors. **Collaborative Learning, Talk for Learning and building on what others.** **Learning Experience:** Learners in collaborative groups derive the scalar (dot) product and solve problems relating to angles between two vectors. **Exemplars:** **Activity 1: Scalar (dot) product** Learners in groups deduce the scalar (dot) product for any two vectors a and b as $\|a\|\|b\|cos\theta$, as shown in the figure	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning



From the figure, $a \cdot b$ (read as a dot b) is the product of the magnitude of a and b and the cosine of the angle between the vectors a and b.
Symbolically:

$$a \cdot b = |a||b|cos\theta$$

Therefore

$$cos\theta = \frac{a \cdot b}{|a||b|}$$

Note:

- $a \cdot b$ is a scalar and not a vector, and
- $a \cdot b = b \cdot a$

Activity 2: Properties of the scalar (dot) product.
Learners in groups discuss and establish the properties of the scalar (dot) product.

Example 1: If $a = (p\ q\ )$ and $b = (r\ s\ )$ then $a \cdot b = pr + qs \ldots\ldots\ldots (1)$
Also

$$cos\theta = \frac{pr + qs}{|a||b|}$$

Hence

$$\theta = cos^{-1}\left(\frac{pr + qs}{|a||b|}\right)$$

Putting $p = r$ and $q = s$ in equation (1) gives

$$a \cdot a = p^2 + q^2$$
$$p^2 + q^2 = |a|^2$$

Property 1: The scalar product of a vector with itself equals the square of its magnitude. That is $\mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2$.

Activity: Learners in their groups discuss what happens when two vectors a and b have the same direction.

Property 2: Parallelism property. When two vectors a and b have the same direction, then $\widehat{\theta} = 0°$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 0°$

$$a \cdot b = |a||b|$$

Activity: Learners explore using the formula of the dot product to ascertain what happens if two vectors a and b are perpendicular.

Property 3: *If two vectors a and b are perpendicular, then $\theta = 90°$ and $a \cdot b = |a||b|cos90°$*
So that $a \cdot b = 0$

Example: Given that $r = 3i + 4j$ and $t = mi - 6j$ are perpendicular. Find the value of the constant m. Thus r and t are perpendicular, so $r \cdot t = 0$.

$$(3\;4) \cdot (m\;\; -6) = 0$$
$$3m - 24 = 0$$
$$m = 8$$

Activity 3: Learners in groups collaborate and draw pairs of vectors and measure the angles between them. Learners then discuss how their answers relate to the scalar dot product.

Other properties:

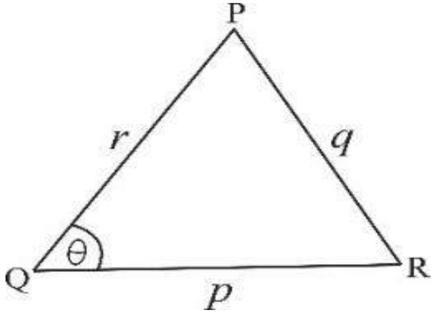
- Commutativity: The Dot product of two vectors is commutative

$$a \cdot b = b.a = |a||b|$$

- Multiplication by constant:

$pa \cdot qb = pq|a||b|$ where *p* and *q* are constants, and *a* and *b* are vectors

	• The Dot product follows the distributive law $a \cdot (b + c) = a.b + b.c$ where a, b and c are vectors **Example:** Find the angle between the vectors $a = 2i + 3j$ and $b = 4i - j$ **Solution:** $a \cdot b = (2i + 3j) \cdot (4i - j)$ $= 8 - 3$ $= 5$ $\lvert a \rvert = \sqrt{4 + 9} = \sqrt{13}$ $\lvert b \rvert = \sqrt{16 + 1} = \sqrt{17}$ So $\theta = cos^{-1} \frac{5}{\sqrt{13} \times \sqrt{17}}$ $\theta = cos^{-1} 0.3363$ $\theta = 70.4°$ The angle between the vectors $2i + 3j$ and $4i - j$ if 70.4°	
	2.2.1.LI.3	2.2.1.AS.3
	Use vectors to establish the sine and cosine rules and solve problems involving areas of polygons. Collaborative learning, Talk for Learning and building on what others say **Learning Experience:** Learners in their groups discuss their understanding of vectors and use the idea to establish the cosine and sine rules. **Exemplars** **Activity 1: The Cosine rule**: Learners in groups refer to triangle PQR with the sides p, q and r, respectively and use the diagram to establish the cosine rule given as $p^2 = q^2 + r^2 - 2qrcos\theta$, where $\theta \leq P$	Level 1 Recall **Level 2 Skills of conceptual understanding:** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning:



From the diagram

$$\overrightarrow{QR} = q - r$$

$$\overrightarrow{QR} \cdot \overrightarrow{QR} = (q - r) \cdot (q - r)$$

$$= q^2 + r^2 - 2qr$$

$$= |q^2| + |r^2| - 2|q||r|cos\hat{P}$$

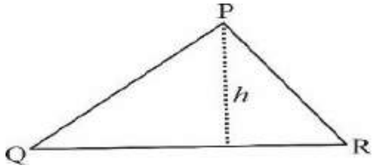
Thus $|p|^2 = |q|^2|r|^2 - 2|q||r|cos\hat{P}$

$$\therefore p^2 = q^2 + r^2 - 2qrcos\hat{P}$$

Also $q^2 = p^2 + r^2 - 2prcos\hat{Q}$
And $r^2 = p^2 + q^2 - 2pqcos\hat{R}$
Where $\hat{P}$, $\hat{Q}$ and $\hat{R}$ are angles at points P, Q and R.

Activity 2: Area of a triangle using the Sine rule

Learners refer to, draw or construct triangle PQR as shown below. In mixed-ability groups, learners discuss how to determine the area of a triangle.



Area of triangle PQR is given by
Area= $\frac{1}{2} \times |PQ| \times h$

$$= \frac{1}{2} r \times h$$

But $h = |PR|sinP$

$$= |q|sinP$$

So

Area $= \frac{1}{2}|q||r|sin\theta$

Example 1: $A(1,-2), B(3,0)$ and $C(1,2)$ are vertices of triangle ABC

- Express $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$ as column vectors
- Use scalar dot product to calculate angle ABC
- Find the area of triangle ABC

Solution

i. $\overrightarrow{AB} = (3\ 0\ ) - (1\ \ -2\ ) = (2\ 2\ )$

$$\overrightarrow{BC} = (1\ 2\ ) - (3\ 0\ ) = (-2\ 2\ )$$
$$\overrightarrow{CA} = (1\ \ -2\ ) - (1\ 2\ ) = (0\ \ -4\ )$$

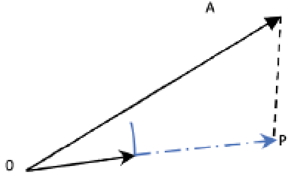
ii. Finding angle $ABC = \frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}, \overrightarrow{BA} = -\overrightarrow{AB}$

$$\left|\overrightarrow{BA}\right| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$
$$\left|\overrightarrow{BC}\right| = 2\sqrt{2}$$
$$cos < ABC = \frac{(-2\ 2\ ) \cdot (-2\ \ -2\ )}{2\sqrt{2} \times 2\sqrt{2}}$$
$$cos < ABC = 0$$
$$< ABC = cos^{-1}0$$
$$\therefore < ABC = 90°$$

iii. Area of Triangle

$ABC = \frac{1}{2}|AC|ABsin45°$

But $|AC| = \sqrt{8^2 + 8^2}$

	$= 8 + 8$ $= \sqrt{16} = 4$ Area of triangle ABC $= \frac{1}{2} \times 4 \times \sqrt{8} sin45°$ $= 2 \times 2\sqrt{2} \times \frac{\sqrt{2}}{2}$ $= 4\ sq\ units$	
	2.2.1.LI.4	2.2.1.AS.4
	Determine the projection of one vector on a given vector. **Collaborative Learning, Talk for Learning** **Learning Experience:** Learners in groups discover how to find the projection of a vector on a given vector. **Activity 1: Scalar product with a unit vector** (projection of OA on the direction û). Learners, in pairs, recall scalar product with a unit vector. Let vector a make an angle θ with the unit vector $\hat{u}$.  $a \cdot \hat{u} = \lvert a\rvert\lvert u\rvert cos\theta$, but $\lvert\hat{u}\rvert = 1$ $= acos\theta$ $= OP$ = projection of OA on the direction û	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	Example 1: Find the projection of a vector $a = 2i + 3j$ in the direction of vector $b = i + 2j$. **Solution:** Find $\lvert b\rvert = \sqrt{1^2 + 2^2} = \sqrt{5}$ $b = \lvert b\rvert\hat{b} = \sqrt{5}\hat{b}$ Now, the projection of a in the direction of $\hat{b}$ is given by $a \cdot \hat{b}$ $a \cdot \hat{b} = (2i + 3j) \cdot \frac{1}{\sqrt{5}}(i + 2j)$ $= \frac{2+6}{\sqrt{5}} = 8\frac{\sqrt{5}}{5}$ So the projection of a in the direction of b is $8\frac{\sqrt{5}}{5}$ units long.	
Teaching and Learning Resources	• SHS curriculum, Mathematical set, ICT apps	

Subject ADDITIONAL MATHEMATICS

Strand 2. GEOMETRIC REASONING AND MEASUREMENT

Sub-Strand 2. MEASUREMENT OF TRIANGLES

Learning Outcomes	21st Century Skills and Competencies	GESI, SEL and Shared National Values
2.2.2.LO.1		
Find trigonometric values using compound, multiple and half angles and prove the sine and cosine rule.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they determine trigonometric values under related scenarios using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to find trigonometric values. • Examine and dispel misconceptions/ myths about GESI as they engage trigonometric values using compound, multiple and half angles and prove the sine and cosine rule in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities.

		National Core Values: **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they determine trigonometric values under related scenarios using appropriate technological tools**.**
2.2.2.LO.2		
Verify whether or not a given trigonometric equation is an identity, and solve trigonometric equations.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they verify

		whether a given trigonometric equation is an identity or not and solve trigonometric equations. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to verify whether a given trigonometric equation is an identity and solve trigonometric equations. • Examine and dispel misconceptions/ myths about GESI as they verify whether a given trigonometric equation is an identity or not and solve trigonometric equations. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills

		• Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners.

		Tolerance: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they verify whether a given trigonometric equation is an identity or not and solve trigonometric equations**.**

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
2.2.2.CS.1	2.2.2.LI.1	2.2.2.AS.1
Demonstrate understanding of trigonometric identities and apply algebraic techniques to verify identities and solve Trigonometric problems on them.	**Prove and apply compound angles to derive the identities for multiple angles and half angles.** **Project-based Learning, Collaborative Learning, Talk for Learning** **Learning Experience**: Learners in mixed-ability groups research how to derive compound angles from basic trigonometric ratios and share their findings in class. **Activity 1: Proofs of compound angles** Group leader shares their findings on compound angles with the class as follows: 1. $(A+B) = sinAcosB + cosAsinB$ - $sin\,sin\,(A-B) = sinAcosB - cosAsinB$ - $cos\,cos\,(A+B) = cosAcosB - sinAsinB$ - $cos\,cos\,(A-B) = cosAcosB + sinAsinB$ - $tan\,tan\,(A+B) = \frac{tanA+tanB}{1-tanAtanB}$ - $tan\,tan\,(A-B) = \frac{tanA-tanB}{1+tanAtanB}$ **Activity 2**: **Application of identities of compound angles** Learners in groups state the importance of the identities in Activity 1 and apply their knowledge of special angles to solve practical examples without using calculators. **Example** Express $cos150°$ in a surd form. **Solution:** $cos150° = cos\,(90 + 60)°$ $cos\,(A+B) = cosAcosB - sinAsinB$ $cos(90+60)° = cos\,90° cos\,60° – sin\,90° sin\,60°$ $= 0\left(\frac{1}{2}\right) - (1)\left(\frac{\sqrt{3}}{2}\right)$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

$$cos150° = -\frac{\sqrt{3}}{2}$$

Learners in their mixed-ability group create questions on compound angles for other groups to solve and present their solutions to the class.

Activity 3: Proofs of Multiple angles
Learners in their mixed-ability research on multiple angles and group leaders share their findings in class.

Condition
If two angles are such that $A = B$ then

- $sin2A = 2sinAcosA$
  Or $sin2B = 2sinBcosB$
- $cos2A = cos^2A - sin^2A$
- $tan2A = \frac{2tanA}{1-tan^2A}$

Example: Given that $B = 30°$, find $sin2A$ if angle A is equal to angle$B$.

Solution:

$$sin2A = 2sin30°cos30°$$
$$= 2\frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

- Learners in their mixed-ability groups create questions on multiple angles for other groups to solve and present their solutions to the class.

Activity 4: Proofs of half angles
Learners in their mixed-ability groups research on half angles, and group leaders share their findings in class.

Condition: If A is an angle, then half of A is represented by $\frac{A}{2}$ and hence

Half angle formula for the sine function

$$sin\frac{A}{2} = \pm\sqrt{\frac{1 - cosA}{2}}$$

Half angle formula for the cosine function

$$cos\frac{A}{2} = \pm\sqrt{\frac{1 + cosA}{2}}$$

Half angle formula for the tangent function

$tan\frac{A}{2} = \pm\sqrt{\frac{1-cosA}{1+cosA}}$ or $\frac{sinA}{1+cosA}$ or $\frac{1-cosA}{sinA}$

Example: Find the value of $sin15°$ using the half-angle formula.

Solution:

$$\frac{A}{2} = 15° \Longrightarrow A = 30°$$

$$sin\frac{30°}{2} = \pm\sqrt{\frac{1 - cos30°}{2}}$$

$$sin15° = \pm 0.2588$$

Learners, in their mixed-ability groups, create questions on half angles for other groups to solve and present their solutions to the class.

	2.2.2.LI.2	2.2.2.AS.2
	Derive the sine and cosine rules and apply them to solve problems. **Project-based Learning, Collaborative Learning, Talk for Learning**	Level 1 Recall **Level 2 Skills of conceptual understanding**

Learning Experience: Learners in mixed-ability groups research how to derive the sine and cosine rules and share their findings in class.

Activity 1: The sine and cosine rules

Learners in their mixed-ability groups research on half angles, and the group leaders share their findings on sine and cosine rules with the class.

Sine rule:

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$$ or

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c},$$ and

- **Cosine rule**

$$a^2 = b^2 + c^2 - 2bccos(A)$$

or $$cos\, cos\, (A) = \frac{b^2+c^2-a^2}{2bc}$$

Conditions

The sine rule is used if:

Two angles and one side.

Two sides and an angle opposite one of these sides.

The cosine rule is used if:

Two sides and an included angle.

The lengths of the three sides.

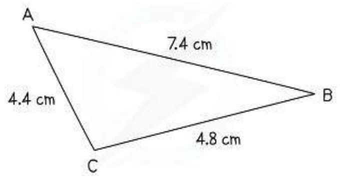
Activity 2: Application of sine and cosine rules

Learners apply sine and cosine rules to solve problems.

Learners in their mixed-ability groups solve questions on sine and cosine rules and present their solutions to the class.

Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

Example 1: In the following



Find the size of angle ABC.

Given that angle ACB is obtuse, use the Sine rule and your answer from (1) to find the size of angle ABC.

Solution:

To find angle B

$$b^2 = a^2 + c^2 - 2accos(B)$$
$$4.4^2 = 4.8^2 + 7.8^2 - 2(4.8)(7.8)cosB$$
$$cosB = \frac{4.8^2 + 7.8^2 - 4.4^2}{2(4.8)(7.8)}$$

Angle $B \approx 34.7°$

To find angle C

$$\frac{sinB}{b} = \frac{sinC}{c}$$
$$\frac{sin34.7}{4.4} = \frac{sinC}{7.4}$$

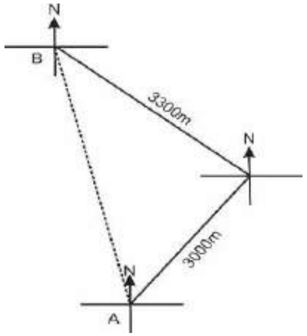
Angle $C \approx 107°$

Example 2: A yacht starts from a point A and sails on a bearing of $038°$ for $3000\ m$. It then alters its course to a bearing of $318°$◦, and after sailing for 3300m it reaches a point B.

1. Find the distance AB correct to the nearest meter.

2. Find the bearing of B from A correct to the nearest degree.

Solution



1. Angle $C = 180° - (38 + 42)] = 100°$
Let the distance of $AB = c, AC = b$ and $BC = a$

$$\therefore c^2 = a^2 + b^2 - 2abcosC$$
$$= 3300^2 + 3000^2 - 2(3300)(3000)\ cos\ cos\ 100$$
$$c = 4830m$$

2. To find the bearing of B from A,
$\frac{3300}{sinA} = \frac{4830}{sin100}$

$$A = 42.8°$$
$$\therefore The\ bearing\ of\ B\ from\ A =$$
$$[360 - (42.3 - 38°)] = 355.7°$$

	2.2.2.LI.3	2.2.2.AS.3
	Use correct algebraic techniques to isolate the trigonometric functions and find the values for the angle that makes the equation true. **Collaborative Learning, Talk for Learning and building on what others say.** **Learning Experience**: Learners work in mixed-ability groups to evaluate trigonometric equations. **Activity 1: Solving linear trigonometric equations** Learners in mixed-ability groups solve linear trigonometric equations. **Example 1:** Solve for the values of x, $2sinx - 1 = 0, 0^\circ \leq x \leq 360^\circ$ **Solution** $2sinx - 1 = 0$ $sinx = \frac{1}{2}$ $x = sinx^{-1}\left(\frac{1}{2}\right)$ $x = 30^\circ \; and \; 150^\circ$ **Activity 2:** Learners in their mixed-ability groups create linear trigonometric equations for other groups to solve and share their solutions. **Activity 3:** Learners recollect how to solve quadratic equations and apply this knowledge to solve quadratic trigonometric equations. **Example 1:** Solve for values of x, $2sin^2x + sinx - 1 = 0, 0^\circ \leq x \leq 360^\circ$. **Solution** $2sin^2x + sinx - 1 = 0$ $(2sinx - 1)(sinx + 1) = 0$ $2sinx - 1 = 0$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	$sinx = \frac{1}{2}$ $x = sin^{-1}\left(\frac{1}{2}\right)$ $x = 30° \; and \; 150°$ OR $sinx + 1 = 0$ $sinx = -1$ $x = sin^{-1}(-1)$ $x = 270°$ $\therefore x = 30°, 150°, 270°$	
Teaching and Learning Resources	• Graph paper • Ruler • Scientific calculator • Protractor • Computer	

Subject ADDITIONAL MATHEMATICS
Strand 3. CALCULUS
Sub-Strand 1. PRINCIPLES OF CALCULUS

Learning Outcomes	21st Century Skills and Competencies	GESI, SEL, and Shared National Values
2.3.1.LO.1		
Determine the appropriate rule and use it to find the derivative of a function.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical thinking** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they determine the appropriate rule and use it to find the derivative of a function. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn about how to determine the appropriate rule and use it to find the derivative of a function. • Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each

	Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations.	learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment.

	Strategic Competency: Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	**Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance**: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability groupings within a differentiated Additional Mathematics classroom instruction and assessment as they determine the appropriate rule and use it to find the derivative of a function**.**
2.3.1.LO.2		
Estimate the area under a curve using the trapezoid rule.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they estimate the area under a curve using the trapezoid rule using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to estimate the area under a curve using the trapezoid rule. • Examine and dispel misconceptions/ myths about gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

		SEL: Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills. • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work. • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for

		individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they estimate the area under a curve using the trapezoid rule using appropriate technological tools**.**

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
2.3.1.CS.1	2.3.1.LI.1	2.3.1.AS.1
Determine the appropriate rule to use in finding the derivative of a function and relations.	**Identify the rules of differentiation.** **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Activity 1: Group Work/Collaborative Learning** Work in mixed-ability and gender-balanced or sensitive groups to arrive at situations where differentiation is necessary and distinguish between the rules of differentiation. **Rules for Differentiation** • **Derivative of a constant function** If α is a real number and if $f(x) = \alpha$, then $f'(x) = 0$ for all x • **Derivative of a sum of two or more functions** If $f'(c)$ and $g'(c)$ exist, then so do $(f + g)'(c)$. Moreover$(f + g)'(c) = f'(c) + g'(c)$ • **Derivative of a difference between two functions** If $f'(c)$ and $g'(c)$ exist, then so do $(f - g)'(c)$. Moreover $(f - g)'(c) = f'(c) - g'(c)$ • **Derivative of a constant times a function** If $f'(c)$ exist, and if α is any constant, then $(\alpha \cdot f)'(c) = \alpha \cdot f'(c)$. • **Derivative of a linear combination of functions** If $f'(c)$ and $g'(c)$ exist, and if α and β are any constants, then $(\alpha \cdot f + \beta \cdot g)'(c)$ exists. Moreover $(\alpha \cdot f + \beta \cdot g)'(c) = \alpha \cdot f'(c) + \beta \cdot g'(c)$	**Level 1 Recall** Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

- **Product rule**
  If $f'(c)\ and\ g'(c)$ exist, then $(f \cdot g)'(c)$ also exist and
  $(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$

- **Quotient rule**
  If $f'(c)\ and\ g'(c)$ exist and if $g(c) \neq 0, (f/g)'(c)$ exists.
  Moreover, $\left(\frac{f}{g}\right)'(c) = \frac{g(c) \cdot f'(c) - f(c) \cdot g'(c)}{g^2(c)}$

- **The power rule**
  If $y = [g(x)]^n$ then $y' = n[g(x)]^{n-1} g(x)'$

- **Chain rule**
  If $y = f[g(x)]$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ where $u = g(x)$

Example:

Identify which rule(s) of differentiation will be more appropriate for the given functions and why.

- $f(x) = \frac{1}{(x-3)}$
- $f(x) = 4x - 7x^2$
- $f(x) = (2x + 5x^5)(x - 2)$
- $f(x) = \frac{(2x+5x^5)(x-2)}{(x-3)}$
- $f(x) = (4x - 7x^2)(3x - 9)^{10}$
- $f(x) = (3x - 4\sqrt{x})^{10}$
- $f(x) = (3x - 9)^{10}$
- $f(x) = sin\ (2x)$
- $f(x) = sin\ (2x)$
- $f(x) = sin\ (2x - x^2)$
- $f(x) = cos\ (x^2 + 3x)$
- $f(x) = \frac{coscos\ (x) + cos\ (2x)}{(x-3)}$

	• $f(x) = sin^2(x)$ • $f(x) = cos^2(x) + \sqrt{1 - x}$ • $f(x) = cos^2(x) + \sqrt{sin\ (1 - x)}$	
	2.3.1.LI.2	2.3.1.AS.2
	Apply the product and quotient rules to differentiate functions. **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work /Collaborative Learning.** **Activity 1: Product and Quotient Rule** In mixed-ability groups, learners explore the product and quotient rules to differentiate functions, including composite functions. **Example:** Use an appropriate rule of differentiation to find the derivative of the following functions; • $f(x) = \frac{1}{(x-3)}$ • $f(x) = 4x - 7x^2$ • $f(x) = (2x + 5x^5)(x - 2)$ • $f(x) = \frac{(2x+5x^5)(x-2)}{(x-3)}$ • $f(x) = (4x - 7x^2)(3x - 9)^{10}$ • $f(x) = (3x - 4\sqrt{x})^{10}$ • $f(x) = (3x - 9)^{10}$ • $f(x) = sin\ (2x)$ • $f(x) = sin\ (2x)$ • $f(x) = sin\ (2x - x^2)$ • $f(x) = cos\ (x^2 + 3x)$ • $f(x) = \frac{coscos\ (x) + cos\ (2x)}{(x-3)}$	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

- $f(x) = sin^2(x)$
- $f(x) = cos^2(x) + \sqrt{1-x}$
- $f(x) = cos^2(x) + \sqrt{sin\,(1-x})$

Activity 2: Learners to Think-pair-share with another learner how to use specific rules on a specific function. E.g.

- $f(x) = cos^2(x) + \sqrt{1-x}$
- $f(x) = cos^2(x) + \sqrt{sin\,(1-x})$
- $f(x) = \left(3x - 4\sqrt{x}\right)^{10}$
- $f(x) = (3x-9)^{10}$

Example: Find the derivative of the following:

i. $f(x) = 5\sqrt{3x^2 + x}$

Solution:

$$f(x) = 5(3x^2 + x)^{\frac{1}{2}}$$
$$f'(x) = \frac{5(6x+1)}{2} \cdot \frac{1}{\sqrt{3x^2+1}}$$
$$= \frac{5(6x+1)}{2\sqrt{3x^2+1}}$$

ii. $f(x) = x^5 + \frac{1}{x}$

Solution:

$$f(x) = x^5 + x^{-1}$$
$$f'(x) = 5x^4 - x^{-2}$$
$$= 5x^4 - \frac{1}{x^2}$$

	iii. $y = 2\sqrt{x}$ **Solution:** $y = 2(x)^{\frac{1}{2}}$ $\frac{dy}{dx} = 2 \cdot \frac{1}{2}(x)^{-\frac{1}{2}}$ $= \frac{1}{\sqrt{x}}$ iv. $y = (\sqrt{-2x+1})(x^2+4)$ **Solution:** $y = (-2x+1)^{\frac{1}{2}}(x^2+4)$ $\frac{dy}{dx} = (-2x+1)^{\frac{1}{2}}(2x) + (x^2+4)\frac{1}{2}(-2)(-2x+1)^{-\frac{1}{2}}$ $= 2x\sqrt{(-2x+1)} - \frac{(x^2+4)}{\sqrt{(-2x+1)}}$ $= \frac{2x(-2x+1)-(x^2+4)}{\sqrt{(-2x+1)}}$ $= \frac{-5x^2+2x-4)}{\sqrt{(-2x+1)}}$	
	2.3.1.LI.3	2.3.1.AS.3
	Find derivatives of functions and relations that are not functions (Implicit differentiation). **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Activity 1: Derivative Applications** In pairs or groups, task learners to discuss how to find a derivative of a relation that is not a function and implement it on a given function. $2x^3 + 3y^3 = 9xy$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Solution:

$$6x^2 + 3y^2\frac{dy}{dx} = 9(x\frac{dy}{dx} + y)$$

$$6x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y$$

$$6x^2 - 9y = \frac{dy}{dx}(9x - 3y^2)$$

$$\frac{dy}{dx} = \frac{6x^2-9y}{9x-3y^2}$$

$$\frac{dy}{dx} = \frac{2x^2-3y}{3x-y^2}$$

Activity 2: Implicit Differentiation

Learners will be working in convenient groups (ability, mixed-ability and mixed gender) to solve problems on implicit differentiation and share it across groups. E.g. In convenient groups, learners find the derivative of:

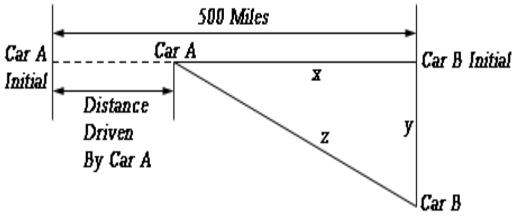
i. $x^2y + 2 = 5y^4 - xy$
ii. $e^{2y}x + 2\, sin\, sin\, (y) \ = 5x + 2 - 3xy$

Activity 3: Everyday Problems Involving Derivatives

Learners in specific groups use chain notes to solve everyday problems involving derivatives of relations that are not functions. Learners discuss and demonstrate their understanding of the problem and the solutions arrived at.

Example 1: Two cars start out 500 miles apart. Car A is to the west of Car B and starts driving to the east (*i.e.,* towards Car B) at 35 mph, and at the same time, Car B starts driving south at 50 mph. After 3 hours of driving, at what rate is the distance between the two cars changing? Is it increasing or decreasing?

Solution:



In this figure, y represents the distance driven by Car B, x represents the distance separating Car A from the initial position of Car B, and z represents the distance separating the two cars. After 3 hours of driving time, we have the following values of x and y:

$$x = 500 - 35(3) = 395$$

$$y = 50(3) = 150$$

By Pythagoras theorem,

$$z^2 = x^2 + y^2 = 395^2 + 150^2$$

$$= 178525$$

$$z = \sqrt{178525} = 422.5222$$

Now to determine z', we apply implicit differentiation as all variables change with time. Given $x' = -35 \ and \ y' = 50$

$$z^2 = x^2 + y^2$$

$$\Longrightarrow 2zz' = 2xx' + 2yy'$$

$$z'(422.5222) = (395)(-35) + (150)(50)$$

$$z' = \frac{-6325}{422.5222} = -14.9696$$

So, after three hours, the distance between them is decreasing at a rate of 14.9696 mph.

Example 2: Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution: The volume $V(t)$ and radius $r(t)$ are varying with time.

$$V'(t) = 5, \;\; r'(t) = ? \;\;\; when\, r(t) = \frac{d}{2} = 10cm$$

Volume of a sphere is given by;

$V(t) = \frac{4}{3}\pi[r(t)]^3$

By implicit differentiation;

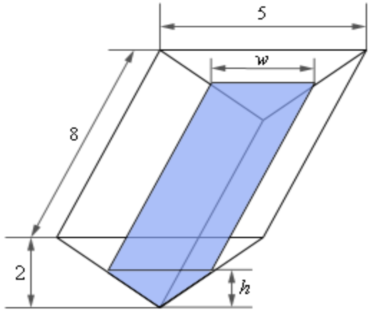
$V' = 4\pi r^2 r'$

$5 = 4\pi(10^2)r'$

$\Longrightarrow \quad r' = \frac{1}{80} cm/min$

Example 3: A trough of water is 8 meters in length, and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is pumped in at a constant rate of 6m^3/sec, at what rate is the height of the water changing when the water has a height of 120 cm? At what rate is the width of the water changing when the water has a height of 120cm?

Solution:



$V' = \frac{6m^3}{sec} \quad h' = ? \quad and \;\; h = 1.2m$

$V = (Area\ of\ End)(depth)$

$\;\;\; = \left(\frac{1}{2} base \times height\right)(depth)$

	$= \frac{1}{2} hw(8)$ $= 4hw$ $\frac{w}{5} = \frac{h}{2} \quad \Longrightarrow \quad w = \frac{5}{2}h$ $V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$ By implicit differentiation; $V' = 20hh'$ $6 = 20(1.2)h'$ $\Longrightarrow \quad h' = 0.25\ m/sec$ So, the height of the water is rising at a rate of 0.25 m/sec $w = \frac{5}{2}h$ $\Longrightarrow w' = \frac{5}{2}h'$ $w' = \frac{5}{2}(0.25) = 0.625\ m/sec$ Therefore, the width is increasing at a rate of 0.625 m/sec.	
	2.3.1.LI.4	2.3.1.AS.4
	Identify and apply techniques of differentiation to solve problems involving transcendental functions. **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Activity 1: Derivative of specific transcendental functions** Each group identifies suitable techniques for finding a derivative of a specific transcendental function and shares their techniques with other groups. Learners to research how exponential and	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

logarithmic functions are applied in technology, business, biology, chemistry, physics and engineering and present findings to the whole class.

NB: Growth and decay models are investigated using such functions.

Activity 2: Derivative of Exponential and Logarithm functions
Learners work in convenient groups (ability, mixed-ability and mixed gender) to find the derivative of exponential and natural logarithm functions.
Hint:

- If $g(x) = e^{f(x)}$ then $\frac{d(g(x)}{dx} = f'(x)e^{f(x)}$

and

- if $f(x)$ then $\frac{dg(x)}{dx} = \frac{f\prime(x)}{f(x)}$

Example 1:
Find the derivative of the following;

- $g(x) = e^{x}$
- $g(x) = e^{3x}$
- $g(x) = xe^{x}$
- $g(x) = e^{x^2-6x}$
- $g(x) = ln\ (x)$
- $g(x) =ln\ ln\ (x+2)$
- $g(x) = ln\ (x+4x^2)$
- $g(x) =ln\ ln\ (x+4x^2)\ + e^{3x}$
- $g(x) = xln\ (x+4x^2)$

Example 2:
Find the derivative of the following:
i. $y = \sqrt[3]{lnx}$

	Solution: $y = (lnx)^{\frac{1}{3}}$ $\frac{dy}{dx} = \frac{1}{3}(lnx)^{-\frac{2}{3}} \cdot \frac{1}{x}$ $= \frac{1}{3x} \cdot \frac{1}{\sqrt[3]{(lnx)^2}}$ $= \frac{1}{3x\sqrt[3]{(lnx)^2}}$ ii. $y = \frac{e^x+e^{-x}}{e^x-e^{-x}}$ **Solution:** $\frac{dy}{dx} = \frac{(e^x-e^{-x})(e^x-e^{-x}) - (e^x + e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}$ $= \frac{(e^x-e^{-x})^2-(e^x+e^{-x})^2}{(e^x-e^{-x})^2}$ $= \frac{(e^x-e^{-x}-e^x-e^{-x})(e^x-e^{-x}+e^x+e^{-x})}{(e^x-e^{-x})^2}$ $= \frac{(-2e^{-x})(2e^x)}{(e^x-e^{-x})^2}$ $= -\frac{4}{(e^x-e^{-x})^2}$	
	2.3.1.LI.5	2.3.1.AS.5
	Generalise the behaviour with respect to the slope of a moving object along a curve. **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Activity 1: Classify the behaviour of a curve** In pairs or groups task, learners to discuss how to determine the slope of a curve at a point and use it to classify its behaviour.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

Activity 2: Derivative of Exponential and Logarithm function

Learners work in convenient groups (ability, mixed-ability and mixed gender) to find and classify slopes of a curve at a point.

Hint:

Given a function $y = f(x)$ if,

- $\frac{dy}{dx}|_{x=c} > 0$ then the slope of $y = f(x)$ is positive at $x = c$
- $\frac{dy}{dx}|_{x=b} < 0$ then the slope of $y = f(x)$ is negative at $x = b$
- $\frac{dy}{dx}|_{x=d} = 0$ then the slope of $y = f(x)$ is zero and $(\mathrm{d}, \mathrm{f}(\mathrm{d}))$ is a turning point or a saddle point

Example 1

Find the slope and classify the behaviour of the curve at the point. State whether the slope is positive or negative or a turning point.

- $f(x) = \frac{1}{(x-3)}$ at $x = 4$
- $f(x) = 4x - 7x^2$ at $x = -3$
- $f(x) = (2x + 5x^5)(x - 2)$ at $x = 1$
- $f(x) = \frac{(2x+5x^5)(x-2)}{(x-3)}$ at $x = -2$
- $f(x) = (4x - 7x^2)(3x - 9)^{10}$ at $x = 0$
- $f(x) = (3x - 4\sqrt{x})^{10}$ at $x = 2$
- $f(x) = (3x - 9)^{10}$ at $x = 1.5$

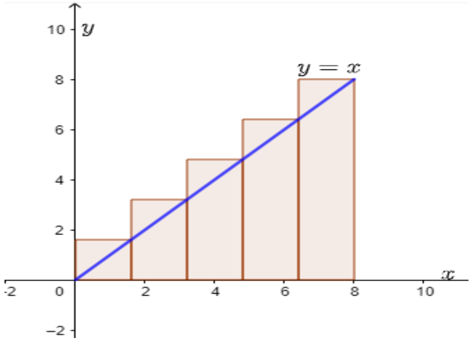
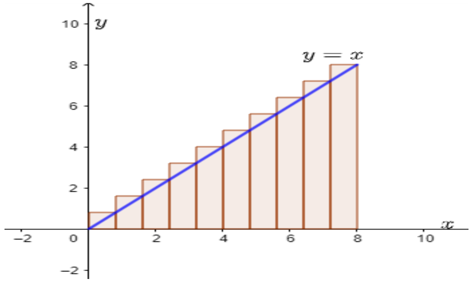
Example 2

Given that the slope of the curve

$f(x) = \mathrm{cx}^2 + dx + 4$ at $x = 2$ is 5 and turns at x=1, find the zeros and the turning point of f

Teaching and Learning Resources	• Reading resource • Colour pens • Notebook	• Graph sheets • Mathematical sets • Technological tools

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
2.3.1.CS.2	2.3.1.LI.1	2.3.1.AS.1
Demonstrate a conceptual understanding of the connection between integration and limits and integration as a reverse process of differentiation.	**Distinguish between partitioning an interval for a given step size and the number of subintervals.** **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Activity 1: Partitioning of an area within an interval** In mixed-ability groups, learners investigate the process of partitioning an area within a given interval into subintervals of equal size. **Example:** Construct N subinterval on the interval $[a, b]$ Step 1: Compute the step $h = \frac{b-a}{N}$ Step 2: Get the subintervals as follows $[a, a+h), [a+h, a+2h), \ldots\ldots\ldots, [a+(n-1)h, a+nh]$ **NB:** • The length of each subinterval is the step size. • The relationship between the step size and the number of subintervals allows for partitioning once we know the step size or the number of subintervals. **Example:** How many subintervals will be from [0,10] for a step size of 2?	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	2.3.1.LI.2	2.3.1.AS.2
	Partition intervals for a given step size or number of sub-intervals and find area under curves through graphics.	Level 1 Recall Level 2 Skills of conceptual

	Activity 1: Area under a curve In small groups, learners construct a specified number of subintervals for an interval of a given function and find the sum areas of the rectangles. Within each group, Learners share their results with those in other groups. **Example:** Investigate the area under the curve, $y = x$ for $x \in [0,8]$ when the number of intervals N is 5, 10 and 20. Try to approximate the area of the rectangles constructed under subintervals. N=5, step size h=1.6  N=10, h=0.8 	understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

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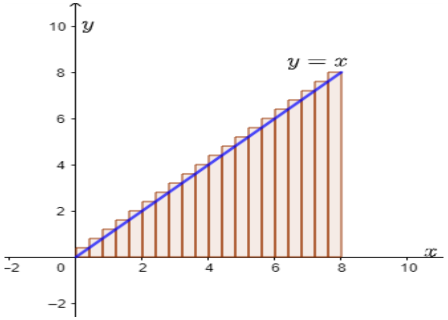
N=20, h=0.16



Note that since the area of the rectangles is
$Area = h * \sum_{j=1}^{n} \quad f(a+jh)$
Learners are to explore and compare the effect of step size of the sum of areas under the rectangle and the actual area:

$$= \frac{1}{2} base \times height$$
$$= \frac{1}{2} \times 8 \times 8 = 32$$

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<tr><td></td><td>2.3.1.LI.3</td><td>2.3.1.AS.3</td></tr>
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Identify and write a definite integral notation and its connection to the limits of the partial sum of areas. Identify integration as a reverse process of differentiation.

Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.

Activity 1: Indefinite and definite integrals
In mixed-ability groups, learners distinguish between indefinite and definite integrals and discuss the definition of anti-derivative.

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Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

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- **Definition of anti-derivative**

An **anti-derivative**, $F(x)$, of a function, $f(x)$, can be defined as a function that can be differentiated to obtain the original function, $f(x)$. Mathematically,

$\int \quad f(x)dx = F(x) + C$, where

the derivative of $F(x)$ is $f(x)$. i.e., $F'(x) = f(x)$ and
C is the integration constant

- **Definite and indefinite Integrals**

Given $f(x)$, a definite integral is a number that represents the area under the curve $f(x)$ from $x = a \, to \, x = b$.

Mathematically,

- A **definite integral** is denoted by

$$\int_{x=a}^{x=b} \quad f(x)dx$$

Where $x = a$ is the lower limit, and $x = b$ is the upper limit.

- An **indefinite integral** is a function denoted by

$$\int \quad f(x)dx$$

Example:

What is the lower and upper limit of the following integrals?

- $\int_0^2 \quad (x^2 + 1)dx$
- $\int_{-3}^7 \quad (x^2 - 8x)dx$

Activity 2: Sum of areas of rectangles

Using think pair share, learners are to compare the sum of areas of rectangles constructed as the number of subintervals gets larger (or step size gets to zero) and the definite integral.

$$h * \sum_{j=1}^{n} \quad f(a+jh)] \cong \int_{a}^{b} \quad f(x)dx$$

Or

$$\int_{a}^{b} \quad f(x)\, dx \cong [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Where, $\Delta x = \frac{b-a}{n}$,

$$n\ (number\ of\ subintervals)$$

Fundamental Theorem of Calculus

Suppose that $f(x)$ is continuous on the interval $a \leq x \leq b$, and let $F(x)$ be an antiderivative of $f(x)$.

Then $\int_{a}^{b} \quad f(x)dx = F(b) - F(a)$

Suppose that f is a positive continuous function. To f we associate an area function F that is defined by

$F(x) = \int_{a}^{x} \quad f(t)\, dt \qquad (a < x < b)$

Understand integration as an inverse process of differentiation

	2.3.1.LI.4	2.3.1.AS.4
	Compare and judge the effect of reduction in step size and increase in the number of subintervals in an interval for a given function on the area under a curve. **Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.**	Level 1 Recall Level 2 Skills of conceptual understanding

	Activity 1: Area of rectangles constructed for different numbers of subintervals In mixed-ability groups, learners find the sum of the area of rectangles constructed for different numbers of subintervals or step sizes on a close interval of a function. Learners record their findings and present them to the whole class for discussion. Using $f(x) = x^2$ on the interval $0 \leq x \leq 3$, find • The definite integral $\int_0^3 \quad f(x)dx$ • On the interval $[0,3]$, get three subintervals with their corresponding rectangles and estimate the integral by summing the areas. Repeat the process 10, 15 and 20 subintervals.	Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**
Teaching and Learning Resources	• Reading resource • colour pens • Notebook • Graph sheets • Mathematical sets • Technological tools	

Subject **ADDITIONAL MATHEMATICS**
Strand **3. CALCULUS**
Sub-Strand **2. APPLICATIONS OF CALCULUS**

Learning Outcomes	**21st Century Skills and Competencies**	**GESI, SEL and Shared National Values**
2.3.2.LO.1		
Investigate the turning point of a function.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they Investigate the turning point of a function. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn how to Investigate the turning point of a function. • Examine and dispel misconceptions/ myths about GESI as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond.

	phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work • Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the

		promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. Tolerance: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they investigate the turning point of a function.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
2.3.2.CS.1	2.3.2.LI.1	2.3.2.AS.1
Find maximum and minimum values and points of a function, sketch the functions, and Solve some real life problems.	**Find the maximum and minimum values and points of a function, sketch the functions and solve some real life problems.** **Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.** **Activity 1**: **Maximum values and points** Learners in their small mixed-ability/gender groups discuss how to find maximum values and points. **Theorem:** Let f be a differentiable function on an open interval and suppose that $f'(c) = 0$ at a point c inside this interval: If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c If f' does not change sign at c for $x < c$ or $x > c$. Even though f(c)=0), f has neither a local minimum nor maximum at c. **Example:** Find the turning point(s) and sketch of the following • $y = x^2 - 3x - 4$ • $y = x^2 - x - 6$ • $y = x^3 - 9x^2 - 21x - 4$ **Solution** $$y = f(x) = x^3 - 9x^2 - 21x - 4$$ Step 1 Find the derivative of y $\frac{dy}{dx} = 3x^2 - 18x - 21$	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Step 2 Equate $\frac{dy}{dx}$ to zero and solve for x.

$\frac{dy}{dx} = 3x^2 - 18x - 21 = 0$

$$3x^2 - 18x - 21 = 0$$

Solving the above, we have $x = 7$ or -1

Step 3 Find the turning points

At $x = -1, \quad f(-1) = 7$ is a critical value; hence $(-1,7)$ is a turning/critical point.

At $x = 7, \quad f(7) = -249$ is a critical value; hence $(7, -249)$ is a turning point.

Sketching

Step 1: Find the zeros of the function; find the values of x for which f(x)=0.

Step 2: Find the turning points.

Step 3: Investigate the turning points, whether it is maximum or minimum.

Step 4: Sketch.

Activity 2: Application of differentiation

Learners in mixed-ability groups discuss and apply differentiation to solve life problems.

- Suppose that a ball is thrown straight up into the air and its height after t seconds is $4 + 48t - 16t^2$ feet. Determine how long it will take for the ball to reach its maximum height and determine the maximum height.

Solution: Let h be the height of the ball. Such that $h = f(t), t$ in seconds

$f(t) = 4 + 48t - 16t^2 = 48 - 32t$

At turning point, $f'(t) = 0$

$$48 - 32t = 0$$

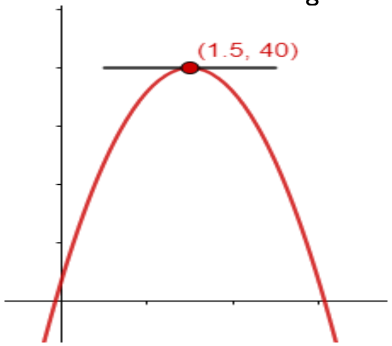
$$t = \frac{3}{2} = 1.5\, seconds$$

Maximum height of the ball

$$f(1.5) = 4 + 48(1.5) - 16(1.5)^2$$

$$= 40\, feet$$

The ball reaches its maximum height of 40 feet in 1.5 seconds



- A person wants to plant a rectangular garden along one side of a house, with a picket fence on the other three sides of the garden. Find the dimensions of the largest garden that can be enclosed using 40 feet of fencing.

Solution: Let $w \text{ and } x$ denote the dimensions of the rectangular garden, the area, $A = wx$

The perimeter of the rectangular garden fencing 3 sides:

$$2x + w = 40$$

$$w = 40 - 2x$$

$$\therefore A = (40 - 2x)x$$

$$= 40x - 2x^2$$

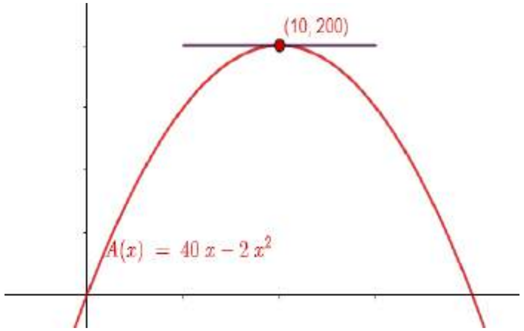
At turning point, $A' = 0$

$$40 - 4x = 0$$

$$x = 10 \text{ feet}$$

The maximum area, $A = 40(10) - 2(10)^2 = 200 \; square \; feet$

$$w = 40 - 2(10) = 20 \text{ feet}$$



- The manager of a department store wants to build a 600-square-foot rectangular enclosure on the store's parking plot in order to display some equipment. Three sides on the enclosure will be built of redwood fencing at a cost of GHC14 per running foot. The fourth side will be built of cement blocks at a cost of GHS28 per running foot. Find the dimensions of the enclosure that will minimize the total of the building materials.

Solution: Let $x\ and\ y$ be the length of the side built out of cement blocks and the adjacent side, respectively.
Cost of redwood= $(x + 2y) \cdot 14 = 14x + 28y$
Cost of cement blocks $= 28x$

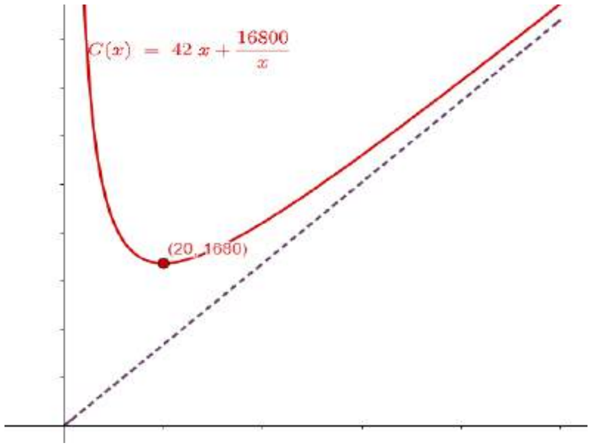
If C denotes the total cost of the materials, then

$$C = (14x + 28y) + 28x$$
$$C = 42x + 28y$$

Area, $A = xy = 600$

$$y = \frac{600}{x}$$
$$C = 42x + 28\left(\frac{600}{x}\right)$$
$$C = 42x + \frac{16{,}800}{x}$$

By sketching the curve:



The minimum total cost of GHC1680 occurs at $x = 20$

By this, $y = \frac{600}{20} = 30$

$\therefore x = 20\ feet, y = 30\ feet$

- Ghana's regulations on posting parcels state that packages must have a length plus girth of no more than 84 inches. Find the dimensions of the cylindrical package of the greatest volume that is mailable by parcel post.

Solution:

Let l and r denote the length of the package and radius of the circular end, respectively.

The volume,

$V = \pi r^2 l$The girth is the circumference of the circular end $= 2\pi r$

$length + girth = 84$

$l + 2\pi r = 84$

$l = 84 - 2\pi r$

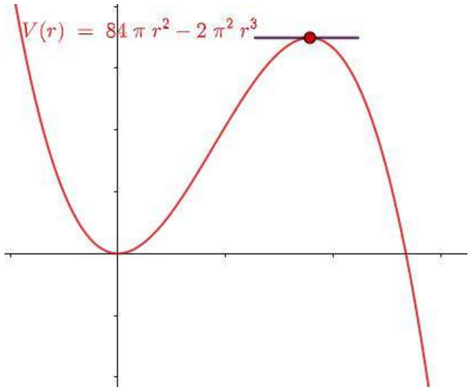
$$\Longrightarrow V = \pi r^2(84 - 2\pi r)$$
$$= 84\pi r^2 - 2\pi^2 r^3$$

At maximum, $V' = 0$

$$\therefore V' = 168\pi r - 26r^2 = 0$$
$$r = \frac{28}{\pi}$$

The maximum volume,

$$V = 84\pi\left(\frac{28}{\pi}\right)^2 - 2\pi^2\left(\frac{28}{\pi}\right)^3$$
$$= \frac{28^3}{\pi}$$
$$l = 84 - 2\pi\left(\frac{28}{\pi}\right) \quad = 28$$
$$girth = 2\pi\left(\frac{28}{\pi}\right) \quad = 56$$
$$\therefore l = 28\ inches, r = \frac{28}{\pi}\ inches$$
$$girth = 56\ inches$$



- Suppose that, on a certain route, an airline carries 8000 passengers per month, each paying GHC50. The airline wants to increase the fare. However, the market research department estimates that for each GHC1 increase in fare, the airline will lose 100 passengers. Determine the price that maximizes the airline's revenue.

Solution:

Let x and n be the price per ticket and number of passengers, respectively.

Revenue, $R = nx$

Number of passengers

=original number of passengers-(number of passengers lost due to fare increase)

$n = 8000 - (x - 50) \cdot 100$

$\quad = 13{,}000 - 100x$

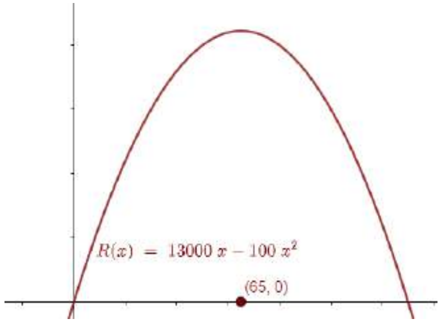
$\therefore R = (13{,}000 - 100x)x = 13{,}000x - 100x^2$

At maximum, $R' = 0$

$13{,}000 - 200x = 0$

$\qquad x = 65$

The maximum revenue occurs when the price per ticket is GHC65.



- Suppose that, from Takoradi to Ho, a VIP bus carries 8000 passengers per month, each paying GHC50. The VIP wants to increase the fare. However, the market research department estimates that for each GHS1 increase in fare, the airline will lose 100 passengers. Determine the price that maximizes the VIP's revenue.

Solution:

Let x and n be the price per ticket and number of passengers, respectively.

Revenue, $R = nx$

Number of passengers

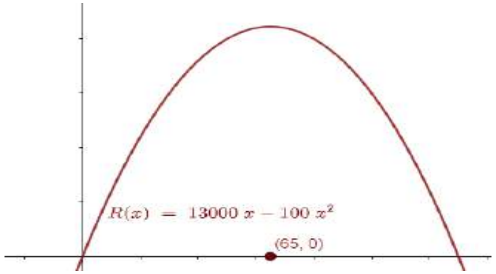
=original number of passengers-(number of passengers lost due to fare increase)

$$n = 8000 - (x - 50) \cdot 100$$
$$= 13{,}000 - 100x$$
$$\therefore R = (13{,}000 - 100x)x = 13{,}000x - 100x^2$$

At maximum, $R' = 0$

$$13{,}000 - 200x = 0$$
$$x = 65$$

The maximum revenue occurs when the price per ticket is GHC65.



- After an injection, the concentration of the drug in a muscle varies according to a function of time, $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Determine the time when the maximum concentration of the drug occurs.

Solution:

The maximum time concentration occurs when $f'(t) = 0$

$$\Rightarrow f'(t) = -0.02e^{-0.02t} + 0.42e^{-0.42t} = 0$$
$$0.42e^{-0.42t} = 0.02e^{-0.02t}$$
$$21e^{-0.42t} = e^{-0.02t}$$
$$21 = \frac{e^{-0.02t}}{e^{-0.42t}}$$

$$21 = e^{(-0.02+0.42)t}$$
$$ln21 = 0.4t \quad (t \approx 7.6)$$
$$t \approx 8\ hours$$

- After an injection, the concentration of the drug in a muscle varies according to a function of time, $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Determine the time when the maximum concentration of the drug occurs.

- Freddy Phones Company limited market its product in Kumasi and Accra and can charge different amounts in each city. Let x be the number of units to be sold in Kumasi and y the number of units to be sold in Accra. Due to the law of demand, Freddy phones must set the price at $97 - (x/10)$ Ghana cedi in Kumasi and $83 - (y/20)$ Ghana cedi in Accra in order to sell all the units. The cost of producing these units is $20{,}000 + 3(x + y)$. Find the values of x and y that maximise the profit.
- Find the path to minimise pigeon flight energy if a pigeon released 1 mile from the shore needs to reach a point on the shore 2 miles from the closest shore point, and the pigeon needs $4/3$ more energy to fly over water.

- A tetramer is a protein with four subunits. In the study of tetramer binding, the equation

$$Y = \frac{x + 3x^2 + 3x^3 + 10x^4}{1 + 4x + x^2 + 4x^3 + 10x^4}$$

expresses a typical relationship between saturation Y and ligand concentration $x(x \geq 0)$. Ordinarily, the variable Y/x is plotted as a function of x. Explain why Y/x an absolute maximum value has. Find this value to three decimal places.

- Suppose a large computer file is sent over the internet. If the probability that it reaches its destination without any errors is x, then the probability that an error is made is $1 - x$. The field of Information Theory studies such situations. An important quantity is entropy (a measure of unpredictability), defined by

$H = -xlnx - (1 - x)ln(1 - x), for\ 0 < x < 1$. Find the value of x that maximises this quantity. Explain why this value makes sense to the probability that maximises entropy.

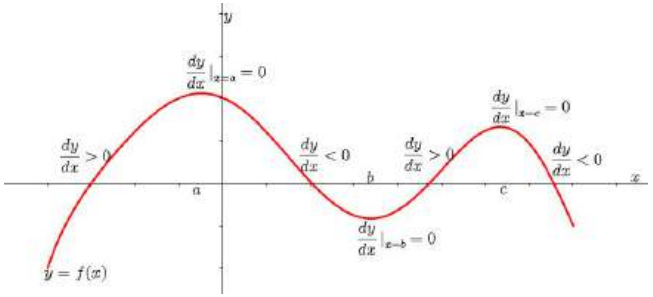
2.3.2.LI.2

Use the second derivative of a function to classify the maximum, minimum and saddle point of that function and perform curve sketching.

Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.

Activity 1: Turning points

Learners are to work in pairs or small groups to discuss how to classify turning points.



From the first derivative test, if f′ moves from positive to negative about a point, then that point is local maximum. Likewise, if f′ moves from negative to positive about a point, then that point is local minimum.

Theorem (Second derivative Test)

Given a continuous function f if $f''(x)|_{x=c} > 0$ then the point $(c, f(c)$ is a minimum and if $f''(x) < 0$
the point $(c, f(c)$ is maximum
if $f''(x) = 0$ then $(c, f(c)$ a saddle point

2.3.2.AS.2

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

	Example: Classify the turning point(s) of the following $y = x^3 - 9x^2 + 24x$ $y = x^4 - 9x^2$	
Teaching and Learning Resources	Cardboards Reading resource Colour pens Notebook Graph sheets Mathematical sets Technological tools	

Subject ADDITIONAL MATHEMATICS

Strand 4. HANDLING DATA

Sub-Strand 1. ORGANISING, REPRESENTING AND INTERPRETING DATA

Learning Outcomes	21^{st} Century Skills and Competencies	GESI, SEL and Shared National Values
2.4.1.LO.1		
Conduct research on a given situation, collect, organise, and represent the data graphically and explain the findings of the research using the graphs.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as they conduct research on a given situation, collect, organise and represent the data graphically and explain the findings of the research based on the graphs using appropriate technological tools. • Interrogate their stereotypes and biases about the roles and abilities of different groups as they learn to conduct research and related scenarios. • Examine and dispel misconceptions/ myths about

	phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	gender as they engage in mathematical discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence as they discuss mathematical ideas using ICT tools to promote the development of digital literacy skills • Exhibiting motivation and SMART goal-setting as they use innovative ideas in solving mathematical problems. • Managing emotions and conflicts as they engage in collaborative group work

		• Showing empathy and cooperation. These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the Additional Mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners.

		Tolerance: Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated Additional Mathematics classroom instruction and assessment as they conduct research on a given situation, collect, organise and represent the data graphically and explain the findings of the research based on the graphs using appropriate technological tools.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century and GESI**	**Assessment**
2.4.1.CS.1	2.4.1.LI.1	2.4.1.AS.1
Apply techniques for collecting data, including census, sampling and observation in real life.	**Undertake research to gather data using appropriate tools.** **Project-based Learning, Talk for Learning, Group work, Presentation and Building on what others say.** **Learning Experience**: Learners undertake research on a chosen phenomenon in school and/or the community. **Activity 1 Choosing a research area or problem.** Learners in groups choose a topic of interest for their research and discuss in detail how to conduct research on it. Examples are average bills paid by households of students, favourite musicians of students, a suitable method for domestic waste disposal, solutions to water or air pollution in the mining area, etc. **Activity 2 Choose the appropriate method of data collection** Learners in groups choose a suitable method of data collection to address a given research problem (e.g. survey, interview or observation) and decide how to sample people and design appropriate tools (e.g. questionnaire, interview guide or observation guide) for collecting such data. **Example**: Using a survey questionnaire or interview guide to collect data on ways to reduce air or water pollution. Using observation to collect data on the amount of electricity current consumed in a week by households of students. **Activity 3 Initial organisation and presentation of raw data** Group leaders make a presentation of their raw data to the whole class.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning
	2.4.1.LI.2	2.4.1.AS.2
	Perform analysis on a data set using appropriate techniques and tools and interpret findings.	Level 1 Recall Level 2 Skills of conceptual

	Project-based Learning, Talk for Learning, Group work, Presentation and Building on what others say. **Learning Experience:** Learners in groups analyse data collected from their research. **Activity 1 Planning for data analysis** Learners in their groups use data gathered from their research (e.g. waste air pollution, electricity or water billing, favourite food, etc.) and plan how to process the data. **Activity 2 Processing and analysing data** Learners in groups process the data gathered, represent it graphically and interpret the data. Examples of processing data include defining variables, categorising data, giving codes to variables, and removing missing data. Data analysis includes making tally tables, frequency counts or percentages, drawing graphs, computing means, mode, standard deviations, etc. **Activity 3 Use of central tendencies to interpret data** Learners in groups calculate measures of central tendencies and interpret their results. Learners in groups decide on an appropriate central measure of tendencies for analysing their data. **Activity 4**: **Use of measure of dispersion to interpret data** Learners in groups calculate measures of dispersion (standard deviation, variance, range, interquartile) for the data they collected and discuss their suitability for analysing their data and interpreting their results.	understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**
Teaching and Learning Resources	• Real-life data samples from both the school community and/or outside the school community. • ICT tools	

Subject ADDITIONAL MATHEMATICS
Strand 4. HANDLING DATA
Sub-Strand 2. MAKING PREDICTIONS WITH DATA

Learning Outcomes	**21st Century Skills and Competencies**	**GESI, SEL and Shared National Values**
2.4.2.LO.1		
Solve problems using the axioms and the laws of probability.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups to solve problems using the axioms and the laws of probability. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying creative strategies to solve problems using the axioms and the laws of probability and related fields. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse.

	phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity

		presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics-learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they solve problems using the axioms and the laws of probability.
2.4.2.LO.2		
Solve real life problems using combination and permutation.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they

		work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups to solve real life problems using combination and permutation. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying innovative strategies in solving real life problems using combination and related fields. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the

		teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics-learning environment.

		Equity: Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they solve real life problems using combination and permutation.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
2.4.2.CS.1	2.4.2.LI.1	2.4.2.AS.1
Apply the addition and multiplication laws and axioms of probability to solve real life problems.	**Use De Morgan's law in set theory to establish the addition and multiplication laws of probability and apply them to solve real life problems.** **Group work, Talk for Learning, and Building on what others say.** **Learning Experience:** Learners in groups discuss De Morgan's law and use it to solve problems. **Activity 1** Learners in groups give real life examples of independent variables in an experiment and debate on the probability of their occurrence. Learners extend the discussion to De Morgan's law in relation to two independent events, A and B from the same experiment. Mathematically, the probability of independent variables occurring as A or B or both is given by $P(A \cup B) = P(A) + P(B) - P(A \cup B)$ **Example 1** Two events R, and Q are such that $(R \cup Q) = \frac{23}{30}$, $P(R) = 0.6$ and $P(R \cap Q) = 0.5$. Find $P(Q)$. **Solution** $P(R \cup Q) = P(R) + P(Q) - P(R \cap Q)$ $\frac{23}{30} = 0.6 + P(Q) - 0.5$ $P(Q) = \frac{2}{3}$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	2.4.2.AS.2	2.4.2.AS.2
	Investigate the axioms of probability: For any event A, there is $0 \leq P(A) \leq 1$; that is, the probability of an event A is a number between 0 and 1 inclusive. **Group work, Talk for Learning, and Building on what others say.**	Level 1 Recall **Level 2 Skills of conceptual understanding**

Learning Experience: Learners in groups investigate axioms of probability.

Activity 1 Properties and axioms of probability

Learners brainstorm in groups to investigate in an experiment the properties and axioms of probability. Thus, for any outcome or event, A of an experiment;

- $P(S) = 1$, where $A = S$, the sample space.
- $P(\emptyset) = 0, A = \emptyset$, A is a null set
- $0 \leq P(A) \leq 1$, where A is a proper of subsets of S
- a) $P(A) = 1 - P(A')$
  b) $P(A') = 1 = P(A)$

Example: The probabilities of Ben and Alice solving a mathematics problem correctly are 0.7 and 0.8, respectively. Find the probability that

- only Alice solves it correctly.
- both Ben and Alice fail to solve it.
- only one of them solves it correctly.
- What is the probability of at least one of them solving the problem correctly?

Solution

Let P(Alice solving correctly)= $P(A)$ and P(Ben solving correctly) $= P(B)$

$$P(A) = 0.8, P(A') = 0.2$$
$$P(B) = 0.7, P(B') = 0.3$$

a) P(only Alice Solving) $= P(A) \times P(B')$

$$= 0.8 \times 0.3$$
$$= 0.24$$

b) P(both failed to solve)$= P(B') \times P(A')$

$$= 0.3 \times 0.2$$
$$= 0.06$$

c) P(only one solved it)$= P(A) \cdot P(B') + P(A') \cdot P(B)$

$$= 0.8 \times 0.3 + 0.2 \times 0.7$$
$$= 0.24 + 0.14$$

Level 3 Strategic reasoning
Level 4 Extended critical thinking
and reasoning

	$= 0.38$ d) P(at least one of them solving it correctly)$= 1 - P$(none solving) $= 1 - P$(None solving) $= 1 - P(A') \cdot P(B')$ $= 1 - 0.06$ $= 0.94$	
	2.4.2.AS.3	2.4.2.AS.3
	Create and solve problems using the axioms and the laws of probability. **Group work, Talk for Learning, and Building on what others say, Project-based Learning.** **Learning Experience:** Learners in groups create problems using the axioms and laws of probability. **Activity 1: Multiplication and addition laws** Learners in groups lead each other groups to discuss and recall the axioms of probability and the multiplication and addition laws, create some probability problems and solve them. **Example:** The probabilities that two friends, Kojo and Yayra will be in school are $\frac{2}{3}$ and $\frac{3}{5}$ respectively. Find the probability that on a given day: • Kojo is absent from school, but Yayra is present. • both are absent from school • at least one of them will be in school. **Solution** P(Kojo)$= \frac{2}{3}$, P(Kojo Absent)$= \frac{1}{3}$ P(Yayra)$= \frac{3}{5}$, P(Yayra absent)$= \frac{2}{5}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

	a) P(Kojo absent, Yayra present)$= \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$ b) P(both absent)$= \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$ c) P(at least one of them will be in school)$= 1 - P$(none in school) $= 1 - \frac{2}{15} = \frac{13}{15}$ **Activity 2: Axioms and laws of probability** Learners in groups discuss alternative ways of solving problems using the axioms and laws of probability and present their findings in class.	
Teaching and Learning Resources	• Real-life data samples from both the school community and/or outside the school community. • ICT tools	

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
2.4.2.CS.2	2.4.2.LI.1	2.4.2.AS.1
Apply the concepts of permutation and combination to solve real life problems.	**Describe the number of ways objects can be arranged using fundamental counting principles.** **Group work, Talk for Learning, and Building on what others say.** **Learning Experience:** Learners in groups create problems using the axioms and laws of probability. **Activity 1: Fundamental counting principles** Learners in pairs recollect the fundamental counting techniques and principles for the outcomes of some experiments and state the number of ways each can be performed. **Example 1:** If a sports club decides to elect a chairman and a secretary from 5 members, assuming the same person cannot hold both positions, find the number of ways this can be done. **Solution:** Thus, number of ways of choosing a chairman $= 5$ And the number of ways to choose a secretary $= 4$ $= 5 \times 4$ $= 20$ **Example 2:** The Department of Vehicle and Licensing Authority designs the registration of each car with 2 letters from E, R, G, T and 3 digits from $1,4,5,6$ on a given day. How many cars can be registered on that day under this arrangement? **Solution:** Firstly, how many 2-letter arrangements can be formed from the 4 letters, E, R, G, T? This is $4P_2 = \frac{4!}{(4-2)!}$ $= \frac{4!}{2!} = 12$ Similarly, arranging 3 digits from the digits $1,4,5,6$ is $4P_3$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

	$4P_3 = \frac{4!}{(4-3)!}$ $= \frac{4!}{1!}$ $= 24$	
	2.4.2.LI.2	2.4.2.AS.2
	Solve real life problems involving permutations. **Group work, Talk for Learning, and Building on what others say.** **Learning Experience:** Learners in groups solve problems involving permutations. **Activity 1: Permutation** Learners in groups recall the concepts of arrangements or permutation as arranging objects where order is very important, i.e. ab is different from ba and abc, acb, bac etc., are all different arrangements. **Example 1:** How many numbers can be formed from 2,3,4 and 5, if each number must be divisible by 5? **Solution** Think through the problem as follows: For 2,3,4,5: • The digit is divisible by $5 = 1P_1 = 1$ • Two digits ending $5 = 3P_1 = 3$ • Three digits ending $5 = 3P_2 = 6$ • Four digits ending $5 = 3P_3 = 6$ ∴Total number of arrangements$= 1 + 3 + 6 + 6$ $= 16$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	2.4.2.LI.3	2.4.2.AS.3
	Model and solve real life problems involving combination. **Group work, initiate Talk for Learning, and building on what others say, Project-based Learning.** **Learning Experience:** Learners in groups create and solve real life problems involving combination. **Activity 1: Combinations** Learners in groups apply the knowledge of combination where order does not matter to solve real life problems. **Example 1:** In a certain examination, a candidate is to answer 6 out of 8 questions. How many choices does the candidate have if the first 2 questions must be answered at least three of the first four questions must be answered. **Solution:** If the first two questions must are selected, then there are $6C_4$ ways of selecting the next 4. $6C_4 = \frac{6!}{4!2!} = 15$ ways At least 3 out of the first 4 questions gives $4C_3 \times 4C_3$ or $4C_4 \times 4C_2$ $i.e. (4 \times 4) + (1 \times 6) = 22$ **Example 2:** A committee of 5 teachers is to be formed from 5 males and 4 females. Find the number of ways of forming the committee if there must be only 2 females one particular man must be included.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**

	Solution The committee must comprise of 3 men and 2 women Thus giving us $5C_3 \times 4C_2 = 60\ ways$ If we select that particular man, there are 8 persons remaining, and we need any 4 to make up the 5. $So\ 1 \times 8C_4 = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2} = 70$ ways **Activity 2 Modelling and solving Combination problems** Learners work in groups to create real life problems involving combination and solve them.	
Teaching and Learning Resources	• Real-life data samples from both the school community and/or outside the school community. • ICT tools	

YEAR THREE

Subject ADDITIONAL MATHEMATICS
Strand 1. MODELLING WITH ALGEBRA
Sub-Strand 2 APPLICATIONS OF ALGEBRA

Learning Outcomes	**21st Century Skills and Competencies**	**GESI[5], SEL[6] and Shared National Values**
3.1.2.LO.1		
Construct compound statements and truth tables using connectives.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners construct compound statements and truth tables using connectives. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of the construction of compound

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[5] Gender Equality and Social Inclusion

[6] Socio-Emotional Learning

	Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	statements and truth tables using connectives. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-

		regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they construct compound

		statements and truth tables using connectives.
3.1.2.LO.2		
Apply linear transformation in: - finding images of points and object points. - finding reflections and rotations of points and plane figures.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: - Respect individuals of different backgrounds in their groups to appreciate various ways of applying linear transformation to find images of points and objects, and reflections and rotation of points and plane figures. - Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying linear transformation to find images of points and objects, and reflection and rotation. - Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. - Value and promote justice in the mathematics classroom and beyond.

	a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	**SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and finding confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership

		through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they apply linear transformation to find images of points and object and reflections and rotation of points and plane figure.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
3.1.2.CS.1	3.1.2.LI.1	3.1.2.AS.1
Appreciate the concepts of logic and apply the concept to draw valid conclusions and deductions from arguments.	**Form statements including negation statements and comment on them.** **Talk for Learning, Think-Pair-Share, building on what others say.** **Learning Experience**: Learners, in pairs, discuss statements and the associations in those statements. **Activity 1: Mathematical statements** Learners in groups bring out statements for the group to comment whether they are indeed statements and also whether those statements are true or false. A mathematical statement is a sentence which is either true or false. It may contain words and symbols. **Example** • Accra is the capital of Ghana. • There are three raining seasons in a year in the whole of Ghana. • Are there factories in every district of Ghana? **Comments:** From the example above, sentences **a** and **b** are statements, but sentence b is not true. Sentence **c,** however, is not a statement. **Notation of Statements:** In mathematics, statements are denoted by letters such as P, Q and R. **Example** P: $7 + 9 = 15$, simply means P is the statement $7 + 9 = 15$. P is obviously a false statement because $7 + 9 \neq 15$. **Activity 2: Negation of a Statement** Learners in squares discuss and construct negation of a statement and share with the group for comments.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Notes: The negation uses the word **no**, **not**. For a statement p, its negation is ~p. The negation of a given statement is the denial of a given statement.

Example 1: Let us take a simple example of the negation of a statement:
P: Accra is the capital of Ghana.
~P: Accra is **NOT** the capital of Ghana.

Example 2:
P: The earth is round in shape.
~P: The earth is **NOT** round in shape.

Activity 3: Learners discuss the difference between the opposite of a statement and the denial of a statement. Learners to come to the conclusion that the negation of a statement is a denial, not the opposite of a given statement.

Example 1
Kofi Yesu is tall.
Kofi Yesu is short. (opposite, therefore not a negation statement).
Kofi Yesu is **not** tall. (negation statement).

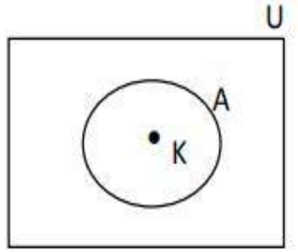
Activity 4: Statements and Venn Diagrams
Learners in convenient groups discuss the representation of statements in Venn diagrams.

Note: Similar to sets, statements can be represented in Venn diagrams.
Example: Identify the following from the given statements:
1. Universal set. 2. Subsets. 3. Complement.

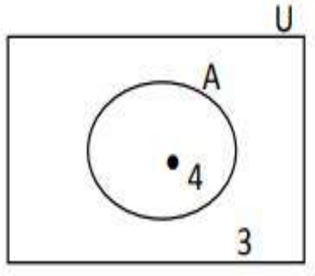
Example 1: Represent each of the following statements on a Venn diagram:

1. P: Kofi is a good boy.
2. Q: 4 is a factor of 100, but 3 is not.

Solution
U = {boys}
A = {Good boys}
k = Kofi
k {Good boys}



ii. U = {Integers}
A = {factors of 100}
4 ∈ factors of 100}
A^{I} = {3}



Activity 4: Compound statements
Learners in squares discuss and construct compound statements and share them with the group for comments.

3.1.2.LI.2

Draw implications from given statements and their converses.

Talk for Learning, Think-Pair-Share, building on what others say.

Learning Experience: Learners, in pairs, discuss statements and draw implications from those statements.

Activity 1: Implicative statements
Learners in groups discuss the concept of implications.

Implication: Consider the following statements;

- If it is a dog, then it is an animal with four legs.
- If two triangles are congruent, then the two triangles are similar.
- If a = b, then $a^2 = b^2$.
- If he has passed the examination, then he is promoted.

Each of these sentences consists of two clauses – the "if clause" and the "then clause". Each of the two clauses is a statement in itself.
Let us consider the "if clause" as statement P and the "then clause" as statement Q. It is observed that in each case, if statement P is true, then it implies that statement Q is also true. This is expressed as P implies Q, symbolically P ⇒ Q Sentences of the type P ⇒ Q are referred to as ***Implication***.

Statements 1, 2 and 3 may appear as follows:

- It is a dog ⇒ It is an animal with 4 legs
- Two triangles are congruent ⇒ Two triangles are similar
- a = b, $\Rightarrow a^2 = b^2$

Note that the arrow "⇒" always points to the "then clause".

3.1.2.AS.2

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

Example 1: Which implications are true or false;

- $x^2 > 4$, then $x > 2$
- If a number is a perfect square, then it is positive

Solution

- False, $(-3)^2 > 4$, (But $-3 < 2$)
- True

Converse of an Implication

For all implications, the arrow "⇒" always points at the "then clause", as shown in the following implications:

- It is a dog⇒ It is an animal with 4 legs.
- Two triangles are congruent ⇒ Two triangles are similar.
- $a = b, \Rightarrow a^2 = b^2$

If the arrow is reversed like this "⇐", the implications appear as follows:

It is a dog ⇐ It is an animal with four legs.

Two triangles are congruent ⇐ Two triangles are similar.
$a = b \Leftarrow a^2 = b^2$ Implications a, b and c are the respective converses of the implications 1, 2 and 3.

For converse (1);

- It is a dog ⇐ It has four legs

This means that "if it has four legs, then it is a dog."

The implication is false because not only dogs have four legs. Since this implication is not true, it is written as:

It is a dog ⇐ It has four legs.

Activity 2: The Chain Rule of Implication

Learners in groups discuss the chain rule of implications.

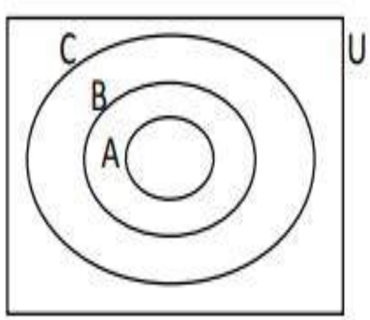
Example 1

Consider the statements that P, Q and R are defined by: P: $x > 7$, Q : $x > 5$, R : $x > 3$ and the implications

P ⇒Q : $x > 7 \Rightarrow x > 5$ Q ⇒R : $x > 5 \Rightarrow x > 3$

This can be represented in a venn diagram as shown below:

∪ = {numbers}
A = {numbers > 7}
B = {numbers > 5}
C = {numbers > 3}



It is observed that $A \subset B \subset C \Rightarrow A \subset C$. Thus, if a number is greater 7, then it is also greater than 3.

Generally, if P ⇒ Q and Q ⇒ R, the conclusively, P ⇒ R. This is called the chain rule of implication, written as P ⇒ Q ⇒ R.

	3.1.2.LI.3	3.1.2.AS.3
	Identify and construct compound statements and form simple statements using the connectives. **Talk for Learning, Group work.** **Learning Experience:** Learners in pairs discuss and construct compound and simple statements. **Activity 1: Compound and simple statements** Learners in groups discuss compound statements with examples. **Example 1** A compound statement is a group of two or more statements connected using words such as 'or', 'and', 'if then', and 'if and only if'. Each statement of a compound statement is a component statement, which can be clearly decided as a true or false statement. The individual statements are represented as p, q, and the compound statements are represented as $p \vee q$, $p \wedge q$, $p \Rightarrow q$, $p \Leftrightarrow q$. **Examples of Compound Statements:** • The grass is green and the sky is blue. • It is cold or it is sunny. • If a person is kind then he is helpful. • The number 12 is an even number if and only if it is divisible by 2. Compound statements are generally formed from simple statements, which are represented as p, q, and the compound statements are represented as $p \vee q$, $p \wedge q$, $p \Rightarrow q$, $p \Leftrightarrow q$. The symbols used to connect the statements p, q are $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ represent the words 'or', 'and', 'if then', 'if and only if', and are referred to as connectives.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

The words 'or', 'and' are useful to form a compound statement, but every statement having these words 'or', 'and' need not be a compound statement.

Example 2: For the compound statement: If it is raining, then it will be very cold", write the converse statement.

Solution: Conditional Statement: P → Q: If it is raining, then it will be very cold.

Converse Statement: Q → P: If it is very cold, then it will be raining.

Example 3: What is the compound statement that can be formed from the statements P: You go regularly to school, and Q: You get good marks?

Solution: The two given statements are:

P: You go regularly to school.

Q: You get good marks.

The four possible connectives that can be used here are ***and***, ***or***, ***if… then***, ***if and only if***. Let us form the four compound statements.

Conjunction Statement: (And connective) You go regularly to school **and** you get good marks.

Disjunction Statement: (Or Connective) You go regularly to school **or** you get good marks.

Conditional Statement: (If then connective) **If** you go regularly to school **then** you get good marks.

Biconditional Statement: (If and only if connective) You go regularly to school **if and only if** you get good marks.
Among the four statements, the conditional statement works well, as the second statement is dependent on the first statement.

	Example 4: From the following pair of statements, form a compound statement using "and" and determine whether the compound statement is true or false. • P: A square has four sides; Q: A triangle has three sides • P: 6 + 7 = 13; Q: 5 + 4 > 8 • P: 4 ∈{primes}, Q: 4 ∈{even numbers}. **Solution** P^Q: A square has four sides, and a triangle has three sides. P^Q is true because P and Q are both true. P^Q: 6 + 7 = 13 and 5 + 4 > 8. P^Q is true because both P and Q are true. P^Q: 4 ∈{primes}∩ {even numbers} P ^Q is false since P is false.	
	3.1.2.LI.4	3.1.2.AS.4
	Use conjunction, disjunctions, implications and negations to construct truth tables of compound statements. **Talk for Learning, and Group work.** **Learning Experience:** Learners in squares discuss and construct truth tables of compound statements. **Activity 1:** Learners in groups discuss various types of compound statements and their connectives. **Example 1** **Disjunction Statement:** The connective used for two simple statements to form a compound statement, which is disjunction, is 'OR.' In a disjunction statement, any one of the statements must be true for the disjunction statement to be true. The two simple statements represented as P and Q can be connected using OR connective and are written as P V Q. Here, any of the two statements should be true for the compound statement to be true.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Conjunction Statement: The compound statement of conjunction uses the connective 'AND' for connecting two simple statements. For this compound statement, both statements must be true for the compound statement to be true. The two simple statements P and Q can be connected using the 'And' connective, and the compound statement can be written as P ^ Q. For a conjunction compound statement, both statements should be true for the compound statement to be true.

Conditional Statement: The connective used for a conditional statement is if then. If Reema does well in the test, then she will be promoted to the next class. Here, the first statement, P, can be taken as the hypothesis, and the second statement, Q, can be taken as the conclusion. We can write condition statements of these two simple statements P, Q as If P then Q. The conditional compound statement does not hold true if the hypothesis is true and the conclusion is false. But in all other situations, the conditional statement is true.

Bi-Conditional Statement: The bi-conditional statement uses the connective 'If and only if,' which is represented by the symbol ⇔. The two statements, P and Q, are represented as a compound statement, P ⇔ Q, and here, the first statement, P, is called the antecedent, and the second statement, Q, is called the consequent. Here, the bi-conditional compound statement is true if both statements are either true or both are false.

Activity 2: Learners in groups construct the truth table of compound statements.

Truth Tables of a Compound Statement: The truth value of a compound statement depends on the truth value of the individual statements and also on the connective used to form the compound statement.

Example 1

Disjunction Truth Table uses the connective 'or' to form the compound statement. Here, even if one of the individual statements is true, then the compound statement also holds true.

P	Q	P V Q
T	F	T
T	T	T
F	T	T
F	F	F

Conjunction Truth Table uses the connective 'and' to form the compound statement. Here, the compound statement is true only if both the individual statements are true. Even if one of the individual statements is false, then the compound statement is considered a false statement.

P	Q	P ^ Q
T	F	F
T	T	T
F	T	F
F	F	F

Conditional Truth Table uses If-then connective, which is represented as ⇒. Here, the statement P is referred to as a ***hypothesis***, the statement Q is referred to as a ***conclusion***. The compound statement is true if the conclusion is true, irrespective of the hypothesis. Also, the compound statement is true if both the hypothesis and the conclusion are false.

P	Q	P ⇒ Q
T	F	F
T	T	T
F	T	T
F	F	T

Bi-conditional Truth Table uses the connective 'if and only if' and is represented as ⇔. Here, the first statement, P, is referred to as antecedent, and the second statement, Q, is referred to as consequent. The bi-conditional compound statement is true if the second statement, the consequent, is false.

P	Q	$P \Leftrightarrow Q$
T	F	T
T	T	F
F	T	F
F	F	T

Activity 3: Valid and invalid arguments using the truth table

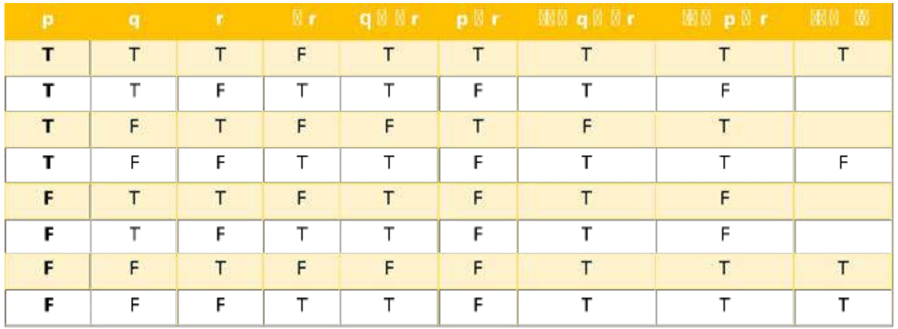
Learners in groups discuss valid and invalid arguments using the truth table.

Notes: A row of the truth table in which all the premises are true is called a critical row. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in every critical row is true, then the argument form is valid.

Example 1

$p \rightarrow q \vee \sim r$
$q \rightarrow p \wedge r$
• $p \rightarrow r$

Solution:



p	q	r	▯ r	q ▯ ▯ r	p ▯ r	▯▯▯ q ▯ ▯ r	▯▯▯ p ▯ r	▯▯▯ ▯
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

	The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false.	
Teaching and Learning Resources	• SHS Curriculum, Graph boards, mathematical set, ICT tools	

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century and GESI	Assessment
3.1.2.CS.2	3.1.2.LI.1	3.1.2.AS.1
Demonstrate the ability to use and apply knowledge of matrices in linear transformations and apply a linear transformation to solve problems in context.	**Find the equation of the image of a line under a linear transformation.** **Think-pair-share, Talk for Learning, and Building on what others say.** **Learning Experience:** Learners in pairs discuss the concept of matrices in transformations. **Activity 1 Concept of matrices in transformation** Learners in convenient groups discuss the following ideas. The image of a point under transformation is generally denoted by A_1, expressed as $A \rightarrow A_1$. The transformation T can also be represented by using the functional notation $T(A) = A_1$. Given the linear transformation, $x_1 = ax + by$ and $y_1 = cx + dy$, the matrix of the linear transformation is expressed as: $$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ **Example 1** Find the matrix of the linear transformation, $x^1 = 3x + y$ and $y^1 = 2x – 3y$ **Solution** $x^1 = 3x + y$ $y^1 = 2x – 3y$ $\begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ The matrix transformation is $M = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$ **Activity 2: Equation of image of a line under a given transformation.** Learners in small groups discuss and determine the equation of the image of a line under a given transformation.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Linear Transformation of a Line:

To find the equation of the image of a line under a translation, reflection, rotation, or enlargement:

- Find the coordinates of any two points on the line.
- Find the coordinates of the images of the two points from step 1.
- Use the two image points from step 2 to find the slope of the image line.
- Use one of the image points, the slope from step 3, and the equation of a line to find the equation of the image line.

Example: Find the equation of the image of the line in part (a) under a 90˚ anticlockwise rotation about the origin.

Solution: To find the equation of the image line, we first start by finding two points that lie in the original line. Let us choose (0,6) and (2,10). Then, we find the image of these two points under the transformation (a 90˚ anticlockwise rotation).

The image point of (0,6) under the rotation is (6,0), and the image point of (2,10) under the rotation is (10,2). We can now use the two image points to find the slope of our image line.

Then, we can use the slope and one point to find the equation of the image line. Let's use (6, 0). The equation of a line is $\boldsymbol{y = mx + b}$, where **m** is the slope and **b** is the y-intercept

$y = mx + b$

$y = \frac{1}{2}x + b$

put in (6,0) and solve for **b**

$0 = \frac{1}{2}(6) + b$

$0 = 3 + b$

$b = 3.$

So, the equation of the image line is $\boldsymbol{y = \frac{1}{2}x + 3}$ **or** $\boldsymbol{y - \frac{1}{2}x - 3 = 0}$

	3.1.2.LI.2	3.1.2.AS.2
	Use linear transformation to determine image and object points. **Think-pair-share, Talk for Learning, and Building on what others say.** **Learning Experience**: Learners in pairs discuss the concept of using linear transformation to determine image and object points. **Activity 1: Image of a point under linear transformation** Learners in convenient groups discuss and determine the Image of a point under a given Linear transformation. To find the image of a point under a given linear transformation; • Identify the given point or the position vector, $\begin{bmatrix} x \\ y \end{bmatrix}$. • Identify the mapping represented by the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. • Multiply the position vector on the left by the mapping to obtain the image of the point. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. **Example 1** Find the image of A(1, 0), B(2, 0) and C(2, 2) under the mapping represented by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. **Solution:** A(1, 0), B(2, 0) and C(2, 2). Mapping $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ Image of A (1, 0);	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

$$\Rightarrow \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\times 1+0\ \times 0\\2\ \times 1+1\ \times 0\end{bmatrix}$$

$$= \begin{bmatrix}1+0\\2+0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$A^{I} = (1, 2)$

Image of $B(2, 0)$;

$$\Rightarrow \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\times 2+0\ \times 0\\2\ \times 2+1\ \times 0\end{bmatrix}$$

$$= \begin{bmatrix}2+0\\4+0\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$B^{I} = (2, 4)$

Image of $C(2, 2)$;

$$\Rightarrow \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1\times 2+0\ \times 2\\2\ \times 2+1\ \times 2\end{bmatrix}$$

$$= \begin{bmatrix}2+0\\4+2\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$$

$C^{I} = (2, 6)$

A^{I}= (1, 2), B^{I} = (2, 4) and C^{I} = (2, 6)

Example 2: Find the image of the matrix where the point is (-2, 3) and the matrix $B(x, y) \to (3x + 5y, 2x + y)$.

Object = (-2, 3)

Matrix × Object = Image

$$\begin{bmatrix}3 & 5\\2 & 1\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}3\times -2+\ 5\ \times 3\\2\ \times -2+1\ \times 3\end{bmatrix}$$

$$= \begin{bmatrix}-6+15\\-4+3\end{bmatrix} = \begin{bmatrix}9\\-1\end{bmatrix}$$

Activity 2: Transformation given a matrix
Learners in small groups find the transformation given the Matrix.

Note: Given the matrix $\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$, the linear transformation T, represented by the matrix, is $T(x,y) \to (ax+by, cx+dy)$

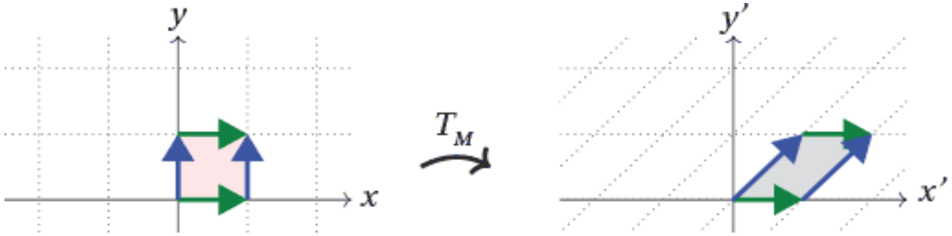
Write down the linear transformation defined by the matrix $\begin{bmatrix}-1 & 2\\3 & -4\end{bmatrix}$

Let the matrix be $M = \begin{bmatrix}-1 & 2\\3 & -4\end{bmatrix}$
The linear transformation, T, defined by M, is $T{:}(x,y) \to (-x+2y, 3x-4y)$.

Activity 3 Geometric transformation
Learners in small groups describe geometrically a plane transformation.

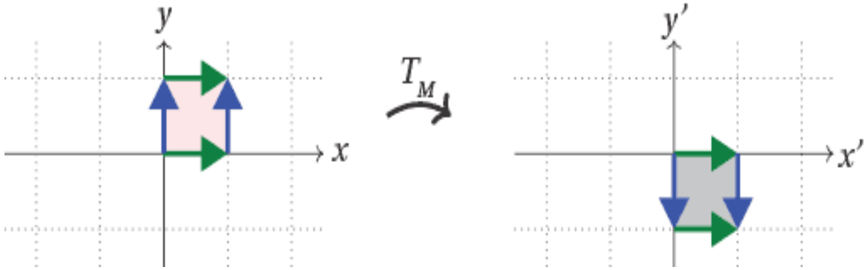
Example 1: Let $M = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$. Describe the linear transformation TM geometrically.

Solution:
Reading the columns of TM tells us that $TM(1,0) = (1,0)$, $TM(1{,}0) = (1{,}0)$, and $TM(0{,}1) = (1{,}1)$; the transformation TM thus turns the unit square into a parallelogram with base 1 and height 1 as shown.



Example 2: Let $M = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. Describe the linear transformation TM geometrically.

Solution:
Reading the columns of M, we have $TM(1,0) = (1,0)$ and $TM(0,1) = (0,-1)$. So TM fixes $(1,0)$ and "flips" $(0,1)$ to its negative $(0,-1)$. Thus, vertical directions are "flipped" so that (x, y) is sent to $(x, -y)$, and TM is reflection in the x-axis.



3.1.2.LI.3

3.1.2.AS.3

Find the composition of linear transformations.

Think-pair-share, Talk for Learning, and Building on what others say.

Learning Experience: Learners in convenient groups discuss and determine the composition of linear transformations.

Activity 1: Composition of linear transformations
This occurs when an object undergoes two or more transformations, one after the other. Thus, $AB(x,\ y\ )$. $BA(x, y\ )$ is defined as the transformation the matrix B, followed by the transformation of matrix A. This is calculated as follows:

- Write the matrix A and B of the linear transformation.
- Find the matrix product AB.
- Write the linear transformation for AB. That is $AB(x,\ y\ )$.
- Notice that the matrix that represents the first transformation, A, is placed on the right, and the second transformation, B, is placed on the left in the matrix product.

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

Example 1: Given the matrices $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$, find the matrix of the transformation A, followed by B.

Solution

$A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$,

A followed by B;

$A \circ B = BA$

$A \circ B = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$

$A \circ B = \begin{bmatrix} 2+15 & -3+21 \\ -2+20 & 3+28 \end{bmatrix} = \begin{bmatrix} 17 & 18 \\ 18 & 31 \end{bmatrix}$

Example 2: Given that $M = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$, find the image of the point $(2, 3)$ under the transformation $M \circ N$.

Solution

$M = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$,

$M \circ N = MN$

$M \circ N = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$,

$M \circ N = \begin{bmatrix} -2+4 & 4+1 \\ -2+12 & 4+3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 10 & 7 \end{bmatrix}$

Image of (2, 3) under transformation $M \circ N$

$= \begin{bmatrix} 2 & 5 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+15 \\ 20+21 \end{bmatrix} = \begin{bmatrix} 19 \\ 41 \end{bmatrix}$

Example 3: Let$S$: $(x, y) \rightarrow (-x, y)$ and $T(x, y) \rightarrow (-y, x)$ be two linear transformations in the $x- y$ plane. Find a single matrix representing the composite transformation TS.

Solution

$$S : (x, y) \rightarrow (-x, y) \ \Rightarrow S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T : (x, y) \rightarrow (- y, x) \ \Rightarrow T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$TS = \Rightarrow S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$TS = \begin{bmatrix} 0+0 & 0-1 \\ -1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Example 4: Two linear transformations are defined by: $A(x, y) \rightarrow (2x- y, 3x + 2y)$ and $B(x, y) \rightarrow (3x + 2y, 4y)$ Find the matrix representing $A \circ B$.

Solution

$$A : (x, y) \rightarrow (2x - y, 3x + 2y)$$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

$$B : (x, y) \rightarrow (3x + 2y, 4y)$$

$$B = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

$$A \circ B = AB$$

$$= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6+0 & 4-4 \\ 9+0 & 6+8 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & 14 \end{bmatrix}$$

	3.1.2.LI.4	3.1.2.AS.4
	Apply linear transformation in finding translation, reflections, rotations and enlargement of points and plane figures. **Think-pair-share, Talk for Learning, and Building on what others say.** **Learning Experience**: Learners in convenient groups discuss the properties and use them to perform translation of points. **Activity 1 Translation and its properties** Learners in groups discuss the concept of translation and its properties. **Note:** Under translation, every point of the plane moves the same distance in the same direction; for all points $P(x\ y\ )$, $OP^1 = OP + T$, where T is the translation vector. If $OP = \begin{pmatrix} x \\ y \end{pmatrix}$ and $T = \begin{pmatrix} a \\ b \end{pmatrix}$, then: $OP^1 = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\ +\ x \\ b + y \end{pmatrix}$ $\boldsymbol{Image = Object + T_v}$ **Properties of Translation** • It requires a translation vector, T. • All points P map unto their images P^1 so that $P - P^1 = T$. • The vector gives the magnitude and direction of the translation. • There is no invariant point. **Example 1:** In a translation in the plane, the image of $P^1(5, 2)$ is $(7, 3)$. Find the image of $Q(-3, 0)$ under the translation.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Solution

Image = Object + Tv

$P'(7, 3) = P(5, 2) + Tv$

$$\begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix} + \text{Tv}$$

$$\text{Tv} = \begin{bmatrix}7-5\\3-2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

Image = Object + Tv

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-3\\0\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-3+2\\0+1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

Activity 1 Reflection and its properties

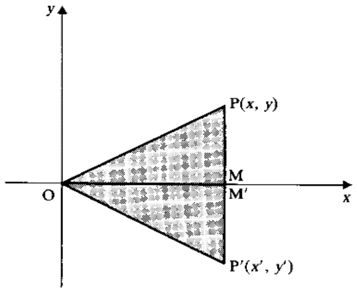
Learners in groups discuss the concept of reflection and its properties.

Notes:

Reflection

Here, an image is produced by reflecting an object or point on either the x–axis or the y–axis. Thus, we have: 1. Reflection in the x–axis (or line y = 0). 2. Reflection in the y–axis (or line x = 0). 3. Reflection in a given line.

Reflection in the x–axis (or line y = 0).



$x' = x$
$y' = -y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is represented by the matrix $[1\ 0\ 0\ -1\ ]$

Image = $[1\ 0\ 0\ -1\ ]$ × object.

Example 1: Find the image of point $A(-2, 5)$ on the x–axis.

Solution:
$A(-2, 5)$
Reflection in the x-axis is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$A' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1(-2) + 0(5) \\ 0(-2) - 1(5) \end{bmatrix} = \begin{bmatrix} -2 \ + \ 0 \\ 0 + -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$A' = (-2, -5)$

Activity 2 Reflection in the y-axis (line $x = 0$).
This is represented by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Image = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times object.$$

Find the image of point $B(-2, 5)$ on the y–axis.

Solution:

B (-2, 5)

Reflection in the y–axis is represented by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$B' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$B' = \begin{bmatrix} -1(-2) + 0(5) \\ 0(-2) + 1(5) \end{bmatrix} = \begin{bmatrix} 2 \ + \ 0 \\ 0 + 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$B' = (2, 5)$

Activity 2 Reflection in the line $y = x$ or $y – x = 0$

This is represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Image = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times object.$

Example: Find the image of point C (5, 7) under reflection in the line y = x.

Solution:

C(5, 7)

Reflection in the y = x is represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$C' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$C' = \begin{bmatrix} 0(5) + 1(7) \\ 1(5) + 0(7) \end{bmatrix} = \begin{bmatrix} 0 + 7 \\ 5 + 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$C' = (7, 5)$

Activity 3

Reflection in the line $y = -x$ or $y + x = 0$

This is represented by the matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$Image = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \times object.$$

Example 1: Find the image of point D (6, 11) under reflection in the line $y = -x$.

Solution:

D(6, 11)

Reflection in the $y = -x$ is represented by the matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

$$D' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$D' = \begin{bmatrix} 0(6) - 1(11) \\ -1(6) + 0(11) \end{bmatrix} = \begin{bmatrix} 0 - 11 \\ -6 + 0 \end{bmatrix} = \begin{bmatrix} -11 \\ -6 \end{bmatrix}$$

$$D' = (-11, -6).$$

Activity 4 Rotation: Learners in groups discuss the properties of rotation and perform some activities in rotation.

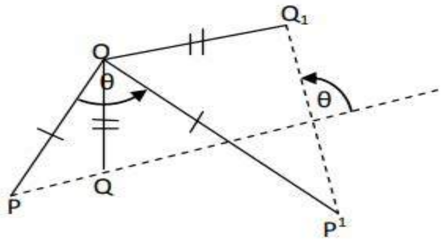
Notes:

Rotation is the turning of an object or point about or around a fixed point (where the fixed point could be the origin). Rotation can be clockwise or anticlockwise through 90^0, 180^0, 270^0 and other angles, represented by θ.

Properties of Rotation

- There is a single invariant point, O, of the rotation.

- All other points P map onto their images P' so that $PO = P'O$, $\angle POP' = \theta$
- The angle θ gives the magnitude of the rotation, the sign convention being positive for clockwise and negative for anticlockwise.
- The angle between any line and its image, for example, PQ and $P'Q'$, is equal to θ.



Example 1 Rotation through an angle θ, anticlockwise about the origin

This is represented by the matrix identity: $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Given that $R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ find $R(45°)$.

Solution

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R(45°) = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix}$$

$$R(45°) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Example 2: Anticlockwise Rotation through an Angle $90°$or Clockwise Rotation through $270°$about the origin. This is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Find the image of $P(-4, 9)$ under anticlockwise rotation through $90°$ about the origin.

Solution: $P(-4, 9)$ under anticlockwise rotation through $90°$about the origin is given by:

$$P' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 9 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0(-4) - 1(9) \\ 1(-4) + 0(9) \end{bmatrix} = \begin{bmatrix} 0 - 9 \\ -4 + 0 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \end{bmatrix}$$

$P' = (-9, -4)$.

Example 3:

Anticlockwise Rotation through $180°$ or Clockwise Rotation through $180°$ about the origin. This is represented by the matrix identity $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Find the image of $Q(7, 10)$ under anticlockwise rotation through $180°$ about the origin.

Solution:

Q(7, 10) under anticlockwise rotation through $180°$ about the origin is given by:

$$Q' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$Q' = \begin{bmatrix} -1(7) + 0(10) \\ 0(7) - 1(10) \end{bmatrix} = \begin{bmatrix} -7 + 0 \\ 0 - 10 \end{bmatrix} = \begin{bmatrix} -7 \\ -10 \end{bmatrix}$$

$Q'' = (-7, -10)$.

Example 4: Anti-clockwise Rotation through $270°$ or Clockwise Rotation through $90°$ about the origin.

This is represented by the matrix identity $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Find the image of $R(-6, 4)$ under anticlockwise rotation through $270°$ about the origin.

Solution:
$R(-6, 4)$ under anticlockwise rotation through $270°$ about the origin:

$$R' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

$$R' = \begin{bmatrix} 0(-6) + 1(4) \\ -1(-6) + 0(4) \end{bmatrix} = \begin{bmatrix} 0 + 4 \\ 6 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

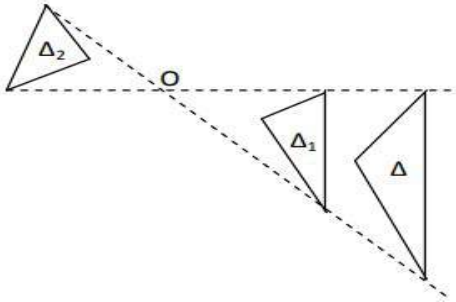
$R' = (4, 6)$.

Activity 5: Enlargement
Learners in groups discuss the properties of rotation and perform some activities in rotation.

Enlargement: Enlargement is the process of making an object bigger than its original size. This is made possible by the use of a scale factor, k. In some cases, the object is made smaller than its original size in a process called reduction or dilatation. Enlargement from the origin, with scale factor k, is represented by the matrix identity.
$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Properties of Enlargement
- There is a single invariant point, the centre of enlargement, O.
- All other points P map onto their images: P' so that $OP' = kOP$, where k is the scale factor of enlargement.
- Under enlargement, any figure is mapped onto a similar figure.
- If $|k| > 1$, the image is larger than the original figure, but if $|k| < 1$, it is smaller. To avoid the concept of an enlargement producing a smaller figure, the term dilatation is often used.

- If $k < 0$, O lies between P and P' and the image figure is inverted.
- A scale factor of -1 is equal to a rotation about the centre O of $180°$.
- With a scale factor, k, similar figures have lengths in the ratio k:1 and areas in the ratio k:1.



Activity 6: Isometrics
Learners in groups perform some activities on isometrics.

Note: Under isometrics, any figure maps onto a congruent figure. Translation and rotation produce directly congruent figures; reflection is inversely congruent.

Activity 7 Inverse of a Linear Transformation
Learners in groups discuss the concept of inverse and solve some problems on it.

The inverse transformation of a linear transformation:
$x' = ax + by$ $y' = cx + dy$ is $x = \frac{1}{k}(dx' - by')$
$y = \frac{1}{k}(-cx' + ay')$
provided $k = ad - bc \neq 0$, the inverse is also linear.

Example 1 Inverse of the linear transformation
Find the inverse of the linear transformation $x' = 2x + 3y$; $y' = 4x + 5y$

	Solution $x' = 2x + 3y \qquad y' = 4x + 5y$ Let $a = 2$, $b = 3$, $c = 4$ and $d = 5$ $K = ad - bc$ $K = (2)\,(5) - (3)\,(4)$ $K = 10 - 12$ $K = -2$ $x = \frac{1}{K}\ (dx' - by')$ $y = \frac{1}{K}\ (-cx' + ay')$, By substitution; $x = \frac{1}{-2}\ (5x' - 3y')$ $y = \frac{1}{-2}\ (-4x' + 2y')$, $x = -\frac{5}{2}x' + \frac{3}{2}y'$ $y = 2x' - 2y'$	
Teaching and Learning Resources	- SHS Curriculum, Graph boards, mathematical set, ICT tools	

Subject ADDITIONAL MATHEMATICS
Strand 2. GEOMETRIC REASONING AND MEASUREMENT
Sub-Strand 1. SPATIAL REASONING

Learning Outcomes	**21st Century Skills and Competences**	**GESI, SEL and Shared National Values**
3.2.1.LO.1		
Construct a parabola of a given quadratic equation ($y = ax^2 + bx + c$) and explain its key features	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups and appreciate everybody's understanding and knowledge of parabola and its related fields. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying innovative strategies of constructing the parabola of a given quadratic equation and deducing the equation of the tangent and normal to a parabola. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse.

	(learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and

		learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they construct a parabola of a given quadratic equation and deduce the equation of the tangent and normal to a parabola.

3.2.1.LO.2		
Sketch a parabola and use it to deduce the relation $y^2 = 4ax$	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to "why, how, when, who, what, and where" questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks,	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups and appreciate everybody's understanding and knowledge of parabola and its related fields. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying innovative strategies of constructing the parabola of a given quadratic equation and deducing the equation of the tangent and normal to a parabola. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies *–Self-Awareness, Self-*

	explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	*Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures

		through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they construct a parabola of a given quadratic equation and deduce the equation of the tangent and normal to a parabola.
3.2.1.LO.3		
Sketch a parabola given the directrix and focus.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups and appreciate everybody's

	actions of others whilst they embark on project works, classroom activities and presentations. **Critical thinking** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts.	understanding and knowledge of parabola and its related fields. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying innovative strategies of constructing the parabola of a given quadratic equation and deducing the equation of the tangent and normal to a parabola. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies *–Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are:

	Digital Literacy Skills: Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application	• Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners

		devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they construct a parabola of a given quadratic equation and deduce the equation of the tangent and normal to a parabola.
3.2.1.LO.4		
Deduce the equation of the tangent and normal to a parabola	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups and appreciate everybody's understanding and knowledge of parabola and its related fields. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying innovative strategies of constructing the parabola of a given quadratic equation and deducing the equation of the tangent and normal to a parabola.

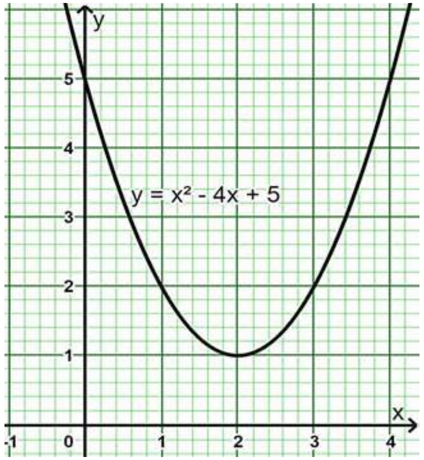
	put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life	• Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the

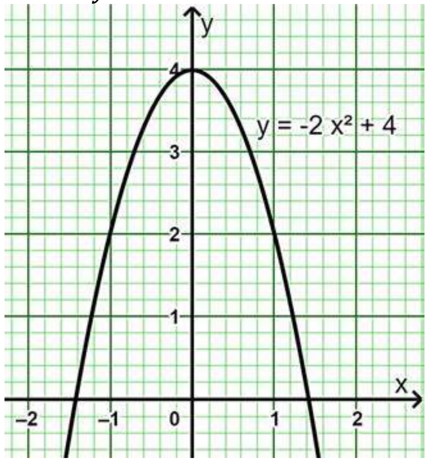
		creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they construct a parabola of a given quadratic equation and deduce the equation of the tangent and normal to a parabola.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI	Assessment
3.2.1.CS.1	3.2.1.LI1.1	3.2.1.AS.1
Demonstrate an understanding of Parabola and its properties.	**Describe the shape of the graphs of quadratic functions.** **Think-pair-share, Talk for Learning, Building on what others say.** **Learning Experience**: Learners work in pairs to describe the shape of the graphs of a quadratic function and provide real life instances and uses. **Activity 1: Shape of Quadratic functions** Learners in pairs and recollect the shape of quadratic functions and state examples in real life objects or situations. **Example** • Kicking a ball upwards • Arc bridges • Satellite dishes	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	3.2.1.LI1.2	3.2.1.AS.2
	Explore the graphs of quadratic functions with different parameters using technological tools such as GeoGebra, Geometer's sketchpad, etc., other available materials, or by hand. **Think-pair-share, Talk for Learning, and Building on what others say.** **Learning Experience:** Learners in groups explore the shapes of a parabola given various quadratic equations. **Activity 1: Sketching graphs of parabola** Learners in groups sketch a variety of parabolas using technological tools such as GeoGebra,	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

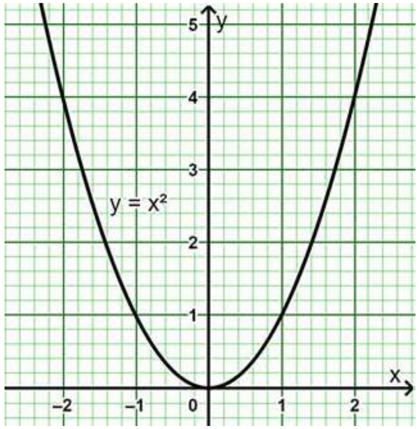
Geometer’s sketchpad, etc., and other available materials or by hand and then investigate how the parabola shape transforms as parameters/coefficients change.

Example 1:
Sketch $y = x^2 - 4x + 5$

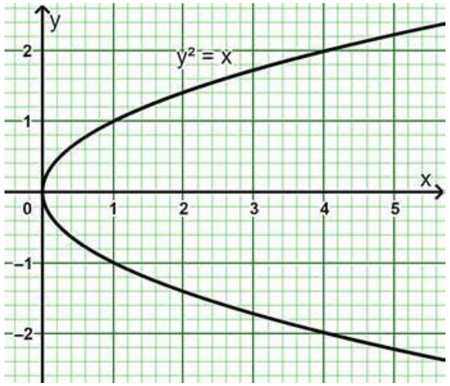


	Example 2: Sketch $y = -2x^2 + 4$  **Activity 2:** Learners in other groups create a quadratics equation for other groups to sketch.	
	3.2.1.LI.3	3.2.1.AS.3
	Explore and describe the features of a parabola (Focus, directrix, axis of symmetry and vertex) using a technological tool and relate it to real life. **Talk for Learning, and Group Work.** **Learning Experience:** Learners in groups sketch parabolas in the form $y = x^2$ and $y^2 = x$ **Activity 1: Parabola in the forms** $y = x^2$ **and** $y^2 = x$ Learners in groups choose an interval to sketch the graph of $y = x^2$, $y^2 = x, y = -x^2$ and $x = -y^2$ using an ITC tool and by hand and discussing the transformation of the parabolic shapes.	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Example 1: Sketch $y = x^2$,



Example 2: Sketch the curve
Sketch $y^2 = x$,



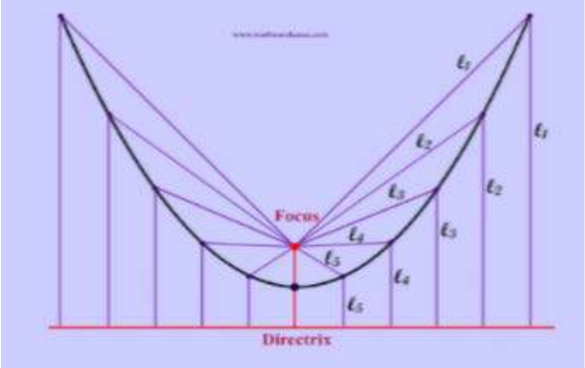
	Activity 2 Transformation of parabola form $y = x^2$ and $y^2 = x$ Learners in groups compare and contrast the graphs of $y = x^2$ and $y^2 = x$ and analyse how the graphs transform when x is positive or negative. **Activity 3 Features of parabola graphs** Learners in groups brainstorm to describe a parabola using visible properties and discuss some keywords in the parabola. **Expected Response** • A parabola graph has a U-shaped curved. • The graph has an extreme point called the **vertex (turning point of the graph).** • If the parabola **opens up**, the vertex is the **lowest point on the graph or the minimum value** of the quadratic function. • If the parabola **opens down**, the vertex is the **highest point on the graph or the maximum value.** The graph is also symmetric with a **vertical line drawn through the vertex**, called the **axis of symmetry**.	
	3.2.1.L1.4	3.2.1.AS.4
	Describe a given locus as a parabola. **Think-pair-share, Talk for Learning, and Building on what others will say.** **Learning Experience:** Learners in mixed-ability groups define a parabola. **Activity 1: Locus describing a parabola** Learners brainstorm to define a parabola using their knowledge in locus and explain some keywords.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

Definition of Terms in a Parabola

- **A parabola** is defined as the **locus** of a point that moves so that it is always the same distance from a fixed point (called the **focus**) and a given line (called the **directrix**).
- **The focus** of a parabola is the fixed point located inside the parabola.
- **Directrix** of a parabola is a line perpendicular to the axis of symmetry.

Activity 2 Identifying vertex, directrix and focus.

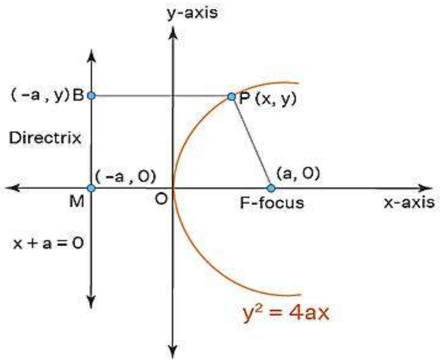
Sketch a parabola and identify the vertex, directrix, and focus.



Activity 3 Relationship between focus and directrix

Learners in groups brainstorm to discover the relationship between the focus and the directrix.

Expected Response: The distance from the focus to the parabola is the same as the distance from the directrix to the parabola.

	3.2.1.LI.5	3.2.1.AS.5
	Apply the formula of distance between two points to find the general equation of a parabola. **Experiential Learning, Talk for Learning, Group work, Building on what others say, and Project-based Learning.** **Learning Experience:** Learners in mixed-ability groups derive the standard equations of a parabola with the vertex at the origin. **Activity 1: Drawing Parabola** Learners in mixed-ability draw a parabola and label the coordinates of the focus and directrix.  **Activity 2 Algebraic expression of a parabola with vertex at the origin.** Learners in mixed-ability groups use the well-labelled diagram and apply their knowledge of the distance of a line to express a parabola algebraically. **Distance formula is given as** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

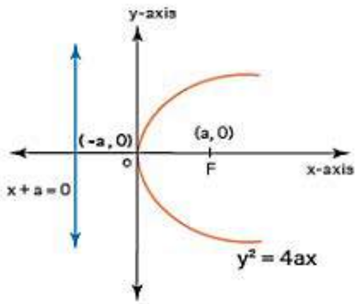
Using the diagram, the relationship between the focus and the directrix implies that $|PF| = |PD|$

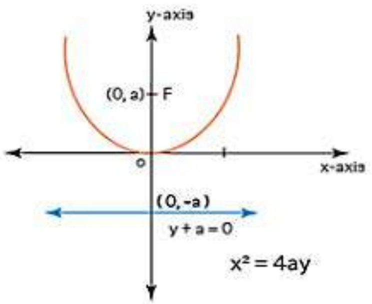
$$\therefore \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x+a)^2 + (y-y)^2}$$

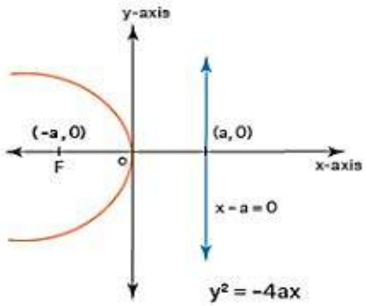
Activity 3 Equation of a parabola ($y^2 = 4ax$)
In mixed-ability groups, learners work to resolve the expression of the relationship between directrix and focus $(a, 0)$ to obtain the equation of a parabola that is $y^2 = 4ax$.

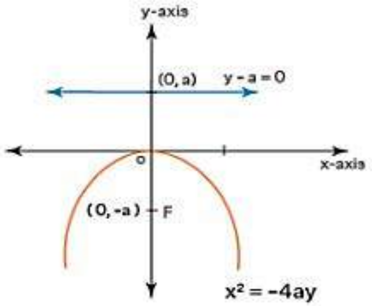
Activity 4 Deriving standard equations of a parabola with vertex at origin.
Learners in their groups brainstorm and share ideas on how to derive the other three standard equations of a parabola and state their observations.
Learners observe that:

- If the equation has the term with y^2, then the axis of symmetry is along the $x - axis,$ and if the equation has the term with x^2, then the axis of symmetry is along the y-axis.
- When the axis of symmetry is along the $x - axis,$ the parabola opens to the right if the coefficient of the x is positive and opens to the left if the coefficient of x is negative.
- When the axis of symmetry is along the $y - axis,$ the parabola opens upwards if the coefficient of y is positive and opens downwards if the coefficient of y is negative.





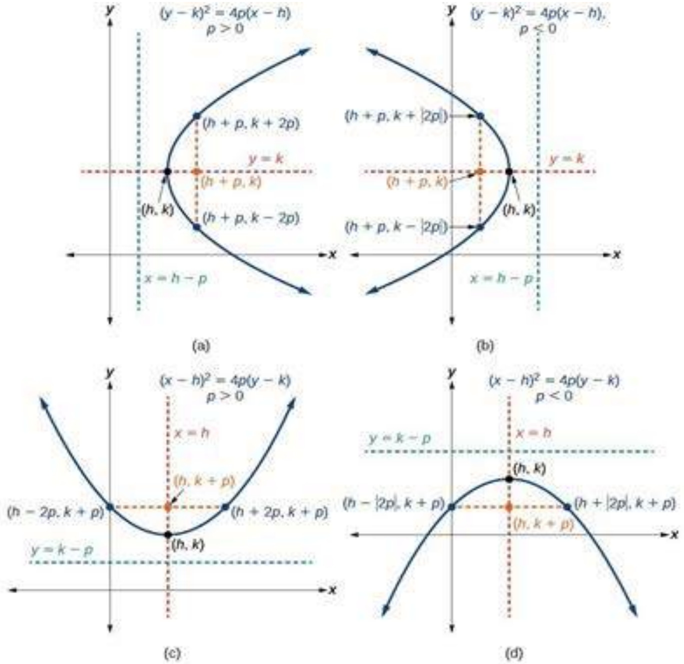




Activity 5 Deriving standard equations of a parabola with vertex not at the origin.

Learners in their mixed-ability groups brainstorm and explore the equation of a parabola with the vertex not at the origin. Learners establish that:

- if the vertex of the parabola is not at the origin, then it is ***translated***.
- if a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k).
- this translation will cause x to be replaced with $(x - h)$ and y replaced with $(y - k)$ in the standard form of the equation.



Activity 6: Learners in groups brainstorm and write the equation of a parabola given the vertex and focus.

Example: Find the equation of a parabola with a focus at $(0, 4)$ and vertex at $(0, 0)$.

Solution

- Learners observe that the given vertex at $(0, 0)$ and focus at $(0, 4)$ implies that the parabola opens up on the y – axis.

Therefore, the equation of the parabola is given by
$x^2 = 4ay$

	Distance between the vertex and focus $$\lvert a\rvert = \sqrt{(0-0)^2 + (0-4)^2}$$ $$\lvert a\rvert = 4$$ If the equation of the parabola is $x^2 = 4ay$ Then $x^2 = 4(4)y$ $$x^2 = 16y$$ **Activity 7 Modelling real life problems on parabolas** Learners in groups create and pose problems on the derivation of equations of parabolas given the vertex and focus.	
	3.2.1.LI.6	3.2.1.AS.6
	Deduce the directrix and focus from a parabolic equation and vice versa. **Experiential Learning, Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability groups deduce the directrix and focus of a parabola as well as write the equation of a parabola given its focus and directrix. **Activity 1 Deducing the directrix and focus of a given parabola with vertex at the origin.** Learners in their mixed-ability group brainstorm to deduce the directrix and focus of a given parabola with vertex at the origin. **Example:** Find the focus and directrix of $y^2 = -8x$ **Solution:** Compare the equation given to the standard equation of a parabola, i.e. $y^2 = 4ax$ $$\therefore 4a = -8$$ $$a = -2$$ Therefore, the focus of the parabola is $(0, -2)$, and the directrix is 2.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Activity 2: Learners in their mixed-ability groups create and solve questions on deducing the directrix and focus of a given parabola with vertex at the origin.

Activity 3 Deducing the directrix and focus of a given parabola of a parabola not at the origin.
Learners in their mixed-ability groups to brainstorm to deduce the directrix and focus of a given parabola with the vertex not at the origin.

Example: Find the directrix and focus of $(y-1)^2 = -16(x+3)$
Solution

- Comparing the equation to $(y-k)^2 = 4p(x-h)$, $h = -3 \; and \; k = 1)$
- Also $4p = -16$, and $p = -4$
- Since $p < 0$, the parabola opens left, $\therefore the \; cordinates \; of \; the \; focus \; is \; (h+p, k)$

$$\therefore focus = (-3-4,1)$$
$$(-7,1)$$

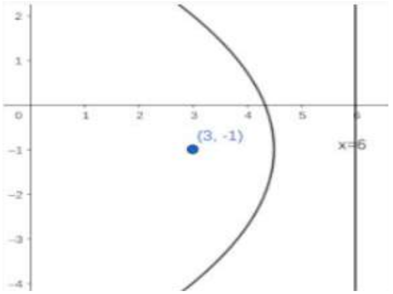
Activity 4: Learners in their mixed-ability groups create and solve questions on deducing the directrix and focus of a given parabola with vertex not at the origin.

Activity 5: Deriving the equation of a parabola given its focus and directrix.
Learners in their mixed-ability groups brainstorm on how to derive the equation of a parabola, given its focus and directrix.

Example: Determine the equation of the parabola with focus $(3,-1)$ and directrix $x = 6$.

Solution

- Learners led by a leader brainstorm to sketch the curve and deduce the general equation of this parabola as $(y-k)^2 = -4a(x-h)$



- Through interaction and building on what others say, learners establish that the vertex (h, k) is the midpoint of $(3, -1)$ and $(6, -1)$

$$\therefore (h, k) = \left(\frac{3+6}{2}, -1\right)$$
$$= \left(\frac{9}{2}, -1\right)$$

- Through interaction and building on what others say, learners establish that a is half the distance between the focus and the directrix, that is $\frac{6-3}{2} = \frac{3}{2}$, since it opens to the left $a < 0$

Therefore, the equation of the parabola is

$$(y+1)^2 = 4 \cdot -\left(\frac{3}{2}\right)\left(x - \frac{9}{2}\right)$$
$$(y+1)^2 = -6\left(x - \frac{9}{2}\right)$$

Example: Determine the equation of the parabola with focus $(-2,13)$ and directrix $y = 5$.

Solution: The equation of the parabola is $(x-h)^2 = 4a(y-k)$
The vertex is the midpoint of the focus and directrix;

$$= \left(-2, \frac{13+5}{2}\right)$$
$$= (-2,9)$$

	a is half the distance between from $(-2,13)$ to $(2,5)$ $\frac{13-5}{2}=4$ $\therefore \mathit{the\ equation\ is}\ (x+2)^2 = 4\cdot 4(y-9)$ $(x+2)^2 = 16(y-9)$ **Activity 5:** Learners in their mixed-ability groups brainstorm to make generalisations on how to derive the equation of a parabola given its focus and directrix. **NB:** In general, the equation of the parabola with focus (a,b) and directrix $y=c$ is $(x-a)^2+b^2-c^2=2(b-c)y$	
	3.2.1.LI.7	3.2.1.AS.7
	Describe the appropriate orientation of the parabola to the given equation. **Experiential Learning, Talk for Learning, Group Work, Building on what others say, and Project-based Learning**. Learners in their mixed-ability groups associate the correct orientation of a parabola with its equation. **Activity 1 Modelling parabolic situations** Learners work in groups to create about 5 parabolas for a project in class. Learners, in groups, play a simple game of mapping given curves to given parabolic equations.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning
	3.2.1.LI.8	3.2.1.AS.8
	Use directrix and focus to sketch a parabola. **Experiential Learning, Talk for Learning, Group work, Building on what others say and Project-based Learning.** **Learning Experience:** Learners in mixed-ability groups sketch parabolas using directrix and focus.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning**

	Activity 1: Learners in the various groups recall the equation of a parabola, its directrix, vertex, and focus. **Activity 2: Sketching a parabola given the directrix and focus** Learners in groups through Talk for Learning apply their knowledge in a parabola to establish how they can use the directrix and focus to draw a parabola. **Activity 3:** Learners create and pose problems using the directrix and focus to sketch a parabola for other groups to solve.	Level 4 Extended critical thinking and reasoning
	3.2.1.LI.9	3.2.1.AS.9
	Describe the equation of lines and derivatives of functions. **Think-pair-share, Talk for Learning, and Building on what others say.** **Learning Experience**: Learners in pairs solve simultaneous equations. **Activity 1:** Learners, in pairs, recall how to solve simultaneous equations and create and pose problems in simultaneous equations for other groups to solve. **Activity 2:** Learners in pairs recall how to find derivatives of functions and create and pose problems in derivates.	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	3.2.1.LI.10	3.2.1.AS.10
	Determine the equations of tangent and normal to a parabola. **Experiential Learning, Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability groups investigate the equation of the tangent and the equation of the normal to the parabola **Activity 1:** Learners in their mixed-ability groups apply their knowledge in finding derivatives and the	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking**

equation of a line to investigate how to derive the equation of a tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1).

- The slope of a tangent is given by $\frac{dy}{dx}$

$\therefore$ differentiating $y^2 = 4ax$ on both sides with respect to x will be

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

- Therefore, the slope (m) of the tangent to the parabola at the point (x_1, y_1) on it is $\frac{2a}{y_1}$

Now, the equation of the tangent to the parabola at the point (x_1, y_1) on it is $y - y_1 = m(x - x_1)$

$$\Rightarrow\ y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Longrightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

But $y_1^2 = 4ax_1$

$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

$\therefore\ yy_1 = 2a(x + x_1)$ is the equation of the tangent at a point (x_1, y_1) on the parabola.

Activity 2: Equation of normal to a parabola.

Learners in mixed-ability groups apply their knowledge in finding derivatives and the equation of a line to explore and investigate how to derive the equation of normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1).

- Learners establish that the slope (m) of the normal is $-\frac{y}{2a}$ since normal and tangent are perpendicular to each other.

$\therefore$ The equation of the normal $y - y_1 = m(x - x_1)$

$$\Rightarrow\ y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$\Longrightarrow y_1x - 2ay_1 - x_1y_1 + 2ay = 0$ is the equation of the normal at a point (x_1, y_1) on the parabola.

Example: A Parabola $y^2 = 8x$ passes through the point $P(1, 2)$. Find:

equation of the tangent

and reasoning

equation of the normal to the parabola at the point P

Solution

i. If $y^2 = 8x$, then comparing it to $y^2 = 4ax$

$$a = 2$$

Equation of the tangent to the parabola is given as

$$yy_1 = 2a(x + x_1)$$
$$\Rightarrow y(2) = 2 \cdot 2(x + 1)$$
$$y - 2x - 2 = 0$$

ii. $y_1x - 2ay_1 - x_1y_1 + 2ay = 0$

$$\Longrightarrow 2x - 2(2)2 - (1)(1) + 2(2)y = 0$$
$$2x + 4y - 9 = 0$$

Activity 3: Learners in groups create and pose problems for other groups to solve and share their solutions.

3.2.1.LI.11

Find the point of intersection for a line and a parabola.

Talk for Learning, Group work, and Building on what others say.

Learning Experience: Learners extend knowledge by finding the point of intersection for a line and a parabola.

Activity 1: The intersection of a line and a parabola

Learners in groups brainstorm to find the intersection for a line and a parabola.

Example: The line $2y + 3x - 6 = 0$ intersects the parabola $y^2 = 6x$ at K and M respectively.

Find the coordinates of K and M.

Find the equation of the normal to the parabola at the point $\left(\frac{2}{3}, 2\right)$.

3.2.1.AS.11

Level 1 Recall

Level 2 Skills of conceptual understanding

Level 3 Strategic reasoning

Level 4 Extended critical thinking and reasoning

Solution:

i)

$$2y + 3x - 6 = 0 \ldots\ldots(1)$$
$$y^2 = 4ax \ldots\ldots\ldots\ldots(2)$$

Equation (1)$\Longrightarrow x = \frac{6-2y}{3}$ and Equation (2)$\Longrightarrow x = \frac{y^2}{6}$

Since the line and parabola intersect, equation (1)=equation (2)

$$\frac{6-2y}{3} = \frac{y^2}{6}$$
$$y^2 + 4y - 12 = 0$$
$$y = 2 \; or \; 6$$

When $y = 2, x = \frac{2}{3}$ and $y = -6, x = 6$

$\therefore K = \left(\frac{2}{3}, 2\right)$ and $M = (6, -6)$

ii) $2ay - 2ay_1 + xy_1 - x_1y_1 = 0$

$$if \; y^2 = 6x$$

Then $a = \frac{2}{3}$ and the point is $\left(\frac{2}{3}, 2\right)$, therefore, the equation of the normal is

$$2\left(\frac{3}{2}\right)y - 2\left(\frac{3}{2}\right) \cdot 2 + (2)x - 2 \cdot \left(\frac{2}{3}\right) = 0$$
$$9y + 6x - 22 = 0$$

Activity 2: Learners in groups create and pose problems on the intersection of a line and a parabola for other groups to solve.

Teaching and Learning Resources	• SHS Curriculum, Graph boards, mathematical set, ICT tools

Subject ADDITIONAL MATHEMATICS

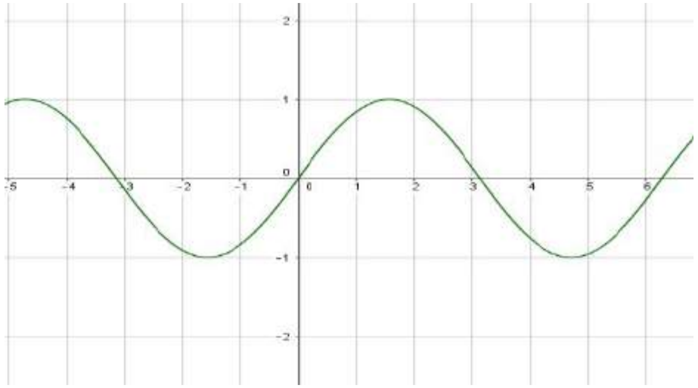
Strand 2. GEOMETRIC REASONING AND MEASUREMENT

Sub-Strand 2. MEASURING TRIANGLES

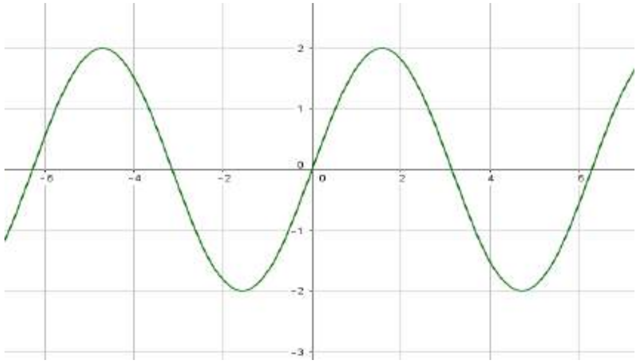
Learning Outcomes	**21st Century Skills and Competences**	**GESI, SEL and Shared National Values**
3.2.2.LO.1		
Draw and analyse basic trigonometric graphs using values of the unit circle, maximum values and minimum values.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical thinking** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners develop skills for drawing and analysing basic trigonometric graphs using values of the unit circle. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying different innovative strategies of drawing and analysing basic trigonometric graphs using values of the unit circle and related mathematical problem fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in

	(learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life.	learning basic trigonometric graphs and other related mathematics discourse problems. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating

		respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they draw and analyse basic trigonometric graphs using values of the unit circle, maximum values and minimum values.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI	Assessment
3.2.2.CS.1	3.2.2.LI1.1	3.2.2.AS.1
Demonstrate understanding of graphs of trigonometric functions.	**Sketch basic trigonometric graphs by hand and/or by technological tools, e.g. GeoGebra and analyse their properties in the unit circle.** **Experiential Learning, Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability groups explore properties of trigonometric graphs using GeoGebra. **Activity 1:** Learners in groups use GeoGebra to sketch trigonometric graphs **Example 1:** Sketch $y = sinx$  **Activity 2: Transformations of graphs of trigonometric function.** Learners in groups discuss how the graphs transform as the trigonometric function transforms.	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Example 2: Sketch $y = 2sinx$



Activity 3: Learners in groups discuss their observations as the graphs transform.

Experiential Learning, Talk for Learning, Group work, and Building on what others say.

Learning Experience: Learners in mixed-ability groups explore sketch trigonometric graphs using the quadrantal angles.

Activity 1: Learners in groups recollect the values of the quadrantal angles for sine and cosine.

θ	$sin\theta$	$cos\theta$
$0°$	0	1
$90°$	1	0
$180°$	0	1
$270°$	1	0
$360°$	0	1

Activity 2: Learners use these values to sketch trigonometric graphs with a specified interval.

	3.2.2.LI.2	3.2.2.AS.2
	Identify the maximum and minimum values of the trigonometric graphs. **Experiential Learning, Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability identify the maximum and minimum values of the trigonometric graphs. **Activity 1: Turning point of trigonometric graphs** Learners in groups explore the turning point of trigonometric graphs. • Learners observe that when the curve opens upwards, the curve has a minimum value. • Learners observe that when the curve opens downwards, the curve has a maximum value. **Activity 2 Sketch the trigonometric graphs.** Learners in groups create and sketch the trigonometric graphs and identify the minimum and maximum points.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
Teaching and Learning Resources	• Scientific Calculator	

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
3.2.2.CS.2	3.2.2.LI.1	3.2.2.AS.1
Demonstrate understanding of compound angles and multiple angles and solve related problems	**Review compound identities** **Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability recollect compound angles. **Activity 1 Recall of compound angle identity** Learners in pairs recollect compound angle identity as; • $cos\ cos\ (A+B)\ = cosAcosB - sinAsinB$ • $cos\ cos\ (A-B)\ = cosAcosB + sinAsinB$ • $sin\ sin\ (A+B)\ = sinAcosB + cosAsinB$ • $sin\ sin\ (A-B)\ = sinAcosB - cosAsinB$ • $tan\ tan\ (A+B)\ =$ • $\frac{tanA+tanB}{1-tanAtanB}\ tan\ tan\ (A-B)\ = \frac{tanA-tanB}{1+tanAtanB}$	**Level 1 Recall** Level 2 Skills of conceptual understanding Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning
	3.2.2.LI.2	3.2.2.AS.2
	Use knowledge in compound angles to derive harmonic identities and find the maximum and minimum values of a trigonometric function. **Talk for Learning, Group work, and Building on what others say.** **Learning Experience:** Learners in mixed-ability investigate the maximum and minimum values. **Activity 1:** Expression of $acos\theta + bsin\theta$ as $Rcos(\theta - \alpha)$ Learners in mixed-ability groups brainstorm and learn how to express $acos\theta + bsin\theta$ as $Rcos(\theta - \alpha)$	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

Activity 2:

- Learners in groups use the addition formula for cosine to obtain the expansion of $Rcos(\theta - \alpha)$ to obtain

$$(x)\ \ cos\ cos\ (\alpha)\ + R\ sin\ sin\ (x)\ \ sin\ sin\ (\alpha)$$

- Learners brainstorm to find R and α by comparing and equating expressions on both sides of the equations, that is
  - $R\ cos\ cos\ (\alpha)$ and $R\ sin\ sin\ (\alpha)\ = b$
  - $\therefore a^2 + b^2 = R^2(\alpha)\ + R^2(\alpha)$, Factorise R^2 to obtain

$R^2[(\alpha)\ + (\alpha)\ ] = a^2 + b^2$, since

$[(\alpha)\ + (\alpha)\ = 1]$

$R = \sqrt{a^2 + b^2}, \quad R > 0$ and $tan\ tan\ (\alpha)\ = \frac{b}{a}$

Note: $0 \leq \theta \leq 90°$

Example 1: Express $4\ cos\ cos\ (x)\ + 3\ sin\ sin\ (x)$ in the form $R\ cos\ cos\ (x - \alpha)$.

Solution: Let $R\ cos\ cos\ (x + \alpha)\ = 5\ cos\ cos\ (x)\ - 12\ sin\ sin\ (x)$

$\Rightarrow (x)\ \ cos\ cos\ (\alpha)\ - R\ sin\ sin\ (x)\ \ sin\ sin\ (\alpha)\ = 5\ cos\ cos\ (x)\ - 12\ sin\ sin\ (x)$

By comparison,

$R\ cos\ cos\ (\alpha)\ = 5$ and $R\ sin\ sin\ (\alpha)\ = 12$

$R = \sqrt{5^2 + 12^2} = 13\ units$

$tan\ tan\ (\alpha)\ = \frac{12}{5}$

$\Rightarrow \alpha = \left(\frac{12}{5}\right)\ = 67.4°$

$\therefore 5\ cos\ cos\ (x)\ - 12\ sin\ sin\ (x)\ = 13\ cos\ cos\ (x + 67.4°)$

Activity 3: Learners in mixed-ability groups brainstorm how to identify the maximum or minimum value. Using Example 1, learners identify 13 as the maximum value.

Teaching and Learning Resources	• Scientific Calculator

Subject **ADDITIONAL MATHEMATICS**
Strand **3. CALCULUS**
Sub-Strand **1. PRINCIPLE OF CALCULUS**

Learning Outcomes	**21st Century Skills and Competences**	**GESI, SEL and Shared National Values**
3.3.1.LO.1		
Identify and apply the integration rules to evaluate integrals.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical thinking** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena (learners respond to "why, how, when, who, what, and where" questions.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners begin to identify and apply the integration rules to evaluate integrals. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge to identify and apply the integration rules of evaluate integrals and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in the application of integration rules to evaluate integrals. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-*

	Personal and Leadership Skills: Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	*Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment.

		Equity: Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they identify and apply the integration rules to evaluate integrals.

Content Standards	Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI	Assessment
3.3.1.CS.1	3.3.1.LI1.1	3.3.1.AS.1
Demonstrate conceptual understanding of the rules and techniques of integration to select and apply them. appropriately to a function.	**Identify and apply the integration rules to evaluate integrals.** **Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.** **Activity 1:** With reference to a real life instances of derivative functions, learners work in mixed-ability groups to discuss the concept, importance and basic rules of integration. **Rules of Integration** • Power Rule • Sum rule • Difference rule • Multiplication by constant **Example 1:** Which rule of integration is applicable and why? 5. 6. 7. 8. **Example 2** Evaluate the antiderivative of the following; 9. 10. 11. 12. 13.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	Definite Integral: Learners should be aware that from the fundamental theorem of calculus, if is continuous on the interval, and let be an antiderivative of . Then 14. Where, Theorem (Integral Theorems) 15. If are two antiderivatives of the same function, then and differ by a constant. In other words, there is a constant such that 16. If for all, then for some constant . 17. Let be a continuous nonnegative function for. Let be the area of the region under the graph of the function from to the number. The is an antiderivative of **Activity 2:** Evaluating integrals Learners will be working in convenient groups (ability, mixed-ability, mixed gender or pairs etc.) to evaluate integrals. **Example 1:** Evaluate the following integrals 18. 19. 20. 21. **Solution:** d. **Note**: that for questions b and c, the length of the limits of integration for a) is 6-2=4 and that of the b) is 6-0=6. What is the integral value for questions b and c.? Are the same? Why?	

	3.3.1.LI.2	3.3.1.AS.2
	Identify and apply appropriate techniques for integration of a function. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Activity 1: Methods of Substitution** Learners in collaborative groups brainstorm and discuss special features of techniques or methods of integration. **Methods of substitution** If an integral could be written in the form $\int f[g(x)]g'(x)dx$ then we can set $u = g(x)$ and integrate $\int f(u)du$. **Example 1 (Substitution)** For each of the following integrals, learners are to rewrite each integral in the form. $\int f[g(x)]g'(x)dx$ and clearly identify $g(x)$ and $g'(x)$. • $\int \sin(2x)\,dx$ • $\int t\sqrt{t^2+4}\,dt$ • $\int (x^2+1)\cdot 2xdx$ • $\int (x^2+1)^7\cdot 2xdx$ • $\int \frac{5x}{\sqrt{1+x^2}}dx$ **Example 2 (Evaluation of definite integrals)** Evaluate the following: • $\int_4^8 t\sqrt{t^2+4}\,dt$ • $\int_1^5 (x^2+1)\,.\,2xdx$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

- $\int_{-1}^{0}(x^2+1)^7\,.\,2xdx$
- $\int_{2}^{4}\frac{5x}{\sqrt{1+x^2}}dx$

Example

- Find the function $f(x)$ for which

$$f'(x) = x^2 - 2 \; and \; f(1) = \frac{4}{3}$$

Solution:

$$f(x) = \int f'(x)dx = \int(x^2-2)dx = \frac{1}{3}x^3 - 2x + C, C \; is \; a \; constant$$

$$f(1) = \frac{1}{3}(1)^3 - 2(1) + C = \frac{4}{3}$$

$$C = \frac{4}{3} + \frac{5}{3} = 3$$

$$\therefore f(x) = \frac{1}{3}x^3 - 2x + 3$$

- A rocket is fired vertically into the air. Its velocity at t seconds after lift-off is $v(t) = 6t + 0.5$ meter per second. Before launch, the top of the rocket is 8 meters above the launch pad. Find the height of the rocket (measured from the top of the rocket to the launch pad) at time t.

Solution:

$s(t)$ denotes the height of the rocket at time t, then $s'(t) = v(t)$

$$s(t) = \int v(t)dt = \int(6t+0.5)dt$$

$$= 3t^2 + 0.5t + C$$

When $t = 0$, $s(0) = 8$

$$3(0)^2 + 0.5(0) + C = 8$$

$$C = 8$$

$$\therefore s(t) = 3t^2 + 0.5t + 8$$

- A company's marginal cost function is $0.015x^2 - 2x + 80$ dollars, where x denotes the number of units produced in one day. The company has fixed costs of GHS1000 per day.

	• Find the cost of producing x units per day. • Suppose the current production level is $x = 30$. Determine the costs that will rise if the production level is raised to $x = 60$ units. **Solution:** (a) Let $C(x)$ be the cost of producing x units in one day. The derivative $C'(x)$ is the marginal cost. $\therefore C(x) = \int(0.015x^2 - 2x + 80)dx$ $= 0.005x^3 - x^2 + 80x + C$ The GHC1000 fixed costs are the costs incurred when producing 0 units. That is $C(0) = 1000$ $0.005(0)^3 - (0)^2 + 80(0) + C = 1000$ $C = 1000$ $C(x) = 0.005x^3 - x^2 + 80x + 1000$ (b) Increase in cost when production is raised from $x = 30 \; to \; x = 60$ is $C(60) - C(30)$ $C(60) = 0.005(60)^3 - (60)^2 + 80(60) + 1000$ $= 3280$ $C(30) = 0.005(30)^3 - (30)^2 + 80(30) + 1000$ $\therefore C(60) - C(30) = 3280 - 2635 = GHC645$ $= GHC2635$	
	3.3.1.LI.3	3.3.1.AS.3
	Identify and apply appropriate techniques of integration to solve transcendental functions. **Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.** **Activity 1: Integration of exponential and natural logarithm functions.** Learners in their mixed-ability groups distinguish among techniques of integration of exponential and natural logarithm functions.	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning

- $\int e^{kx}dx,\ \ k\ a\ constant\ \neq 0$

  $\int e^{kx}dx = \frac{1}{k}e^{kx} + C,\ \ k \neq 0$
- $\int \frac{1}{x}dx = ln\ ln\ |x|\ + C,\ \ x \neq 0$
- $\int \frac{g'(x)}{g(x)}dx = ln\ ln\ |g(x|\ + C,\ \ x \neq 0$
- $\int e^{h(x)}dx\ \ \ k\ a\ constant\ \neq 0$

  $\int e^{h(x)}dx = \frac{1}{h\prime(x)}e^{h(x)} + C,\ \ k \neq 0$

Example 1: Find the antiderivative of the following;

- $\int \frac{1}{x+8}dx$
- $\int \frac{6}{6x+8}dx$
- $\int \frac{2}{x+8}dx$
- $\int \frac{x}{(x^2+6)}dx$
- $\int \frac{\ln(x)}{x}dx$
- $\int xe^{x^2}dx$

Activity 2: Evaluation of exponential function

Learners in small groups solve problems on techniques of integration and present to the class.

- Compute $\int \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right)dx$

Solution:

$$\int \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right)dx = \int x^{-3}dx + \int 7e^{5x}\,dx + \int \frac{4}{x}dx$$

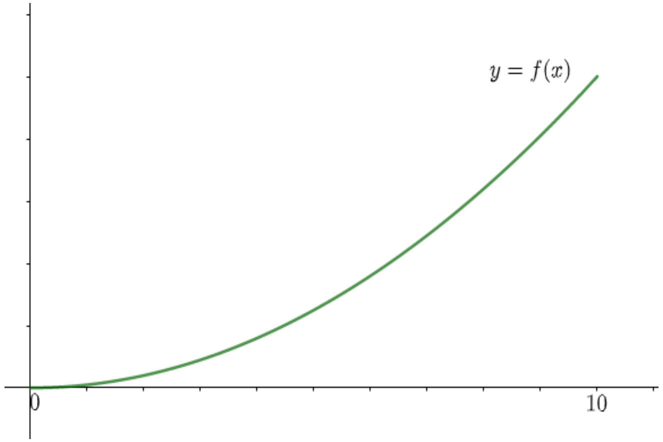
$$= \int x^{-3}dx + 7\int e^{5x}\,dx + 4\int \frac{1}{4}d\,x$$

$$= -\frac{1}{2}x^{-2} + \frac{7}{5}e^{5x} + 4\ ln\ ln\ |x|\ + C$$

Level 4 Extended critical thinking and reasoning

	3.3.1.L1.4	3.3.1.AS.4
	Solve problems involving integrals with no exact antiderivative (Trapezium rule). **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Activity 1: Using the Trapezium rule** Learners brainstorm by way of think-pair-share/square and debate on existence of the trapezium rule and why the need for it. **Theorem** Let $f(x)$ be a continuous function on the interval$[a, b]$. Now divide the intervals $[a, b]$ into n equal subintervals with each of width, $\Delta x = \frac{b-a}{n},\ \Delta x\ is\ the\ step\ size\ and\ 'n'\ is\ the\ number\ of\ partitions$ $T_n \approx \int_a^b f(x)\,dx$ $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_{n-1}) + f(x_n)]$ **Activity 2: Deriving Trapezium rule** Learners work collaboratively on hands-on activity (learning by doing) to establish the trapezium rule for a given definite integral. **Example 1** • Find the area under the curve, $y = x^2$ using the trapezoidal rule where $n = 5$ and x ranges from 0 to 10	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Solution:



$$\Delta x = \frac{10-0}{5} = 2$$

$$T_5 = \frac{2}{2}[(0)^2 + 2(2)^2 + 2(4)^2 + 2(6)^2 + 2(8)^2 + (10)^2]$$
$$= 340\ units\ squared$$

- The table below shows the rate at which water flows through a pipe into a strange tank. This rate is measured every 10 minutes. Estimate the amount of water that will accumulate in the first hour using the trapezoidal rule.

Time (min)	Rate (gal/min)
0	3.8
10	4.5
20	6.2
30	7.0
40	7.5
50	6.9
60	6.2

Solution:

$\Delta x = \frac{60-0}{n} = 10,\ n = 6$

$T_6 = \frac{10}{2}[3.8 + 2(4.5) + 2(6.2) + 2(7.0) + 2(7.5) + 2(6.9) + 6.2]$

$= 371\ gal$

Question: In medical imaging, such as CT (computerized tomography) and MRI (magnetic resonance imaging) processes at Cape Coast hospital, numerous measurements are taken and processed by a computer to construct a three-dimensional image of the tissue the physician wishes to study. Suppose that an MRI scan indicates that the cross-sectional areas of adjacent slices of a tumor are given by the values in the table.

x	$A(x)(cm)^2$
0	0.0
0.0	0.1
0.2	0.4
0.3	0.3
0.4	0.6
0.5	0.9
0.6	1.2
0.7	0.8
0.8	0.6
0.9	0.2
1.0	0.1

Estimate the volume of the tumor using trapezium rule.

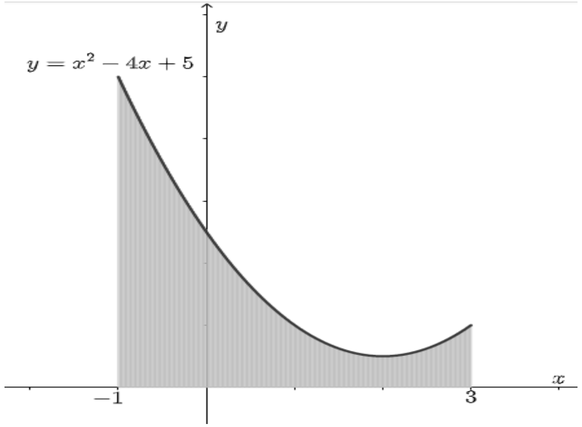
Teaching and Learning Resources	• Reading resource • colour pens • Notebook	• Graph sheets • Mathematical sets, • Technological tools.	• Curriculum • Card boards

Subject **ADDITIONAL MATHEMATICS**
Strand **3. CALCULUS**
Sub-Strand **2. APPLICATION OF CALCULUS**

Learning Outcomes	21st Century Skills and Competences	GESI, SEL and Shared National Values
3.3.2.LO.1		
Determine distance, area under curve and solid of volumes formed under revolution.	**Communication and Collaboration:** As they work in groups, learners learn to share ideas and be sensitive to each other's views on integration, net, sign and total area. **Critical Thinking:** The activities that require learners to reason about to integrate graphs above the horizontal axis and those that have part above and below the horizontal axis with respect to net and total area under a curve.	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups to appreciate and determine distance, area under curve and solid of volumes formed under revolution. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their mathematical knowledge to determine distance, area under curve, solid of voulmes formed under revolution and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in determining distance, area under curve, solid of volumes formed under revolution and related mathematical discourse.

		• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through

		teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they determine distance, area under curve and solid of volumes formed under revolution

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
3.3.2.CS.1	3.3.2.LI.1	3.3.2.AS.1
Demonstrate conceptual understanding of integration to find distances, areas under curve, volumes and other related real life problems.	**Find the area under a constant function, curves and distinguish between total, net and signed area.** **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Activity 1: Area under constant function** Using a real life instances or scenario involving area under curve, Learners in small convenient groups (for instance, mixed gender, mixed-ability, etc.) investigate the total are, net area and signed area under a given function. **Example 1** • Compute the area under the curve $y = x^2 - 4x + 5 \ from \ x = -1 \ to \ x = 3$ **Solution:** 	**Level 1 Recall** **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

$$\int_{-1}^{3}(x^2-4x+5\,)\,dx = \left[\frac{x^3}{3}-2x^2+5x\right]3\ -1$$

$$= \left[\frac{3^3}{3}-2(3)^2+5(3)\right]-\left[\frac{(-1)^3}{3}-2(-1)^2+5(-1)\right]$$

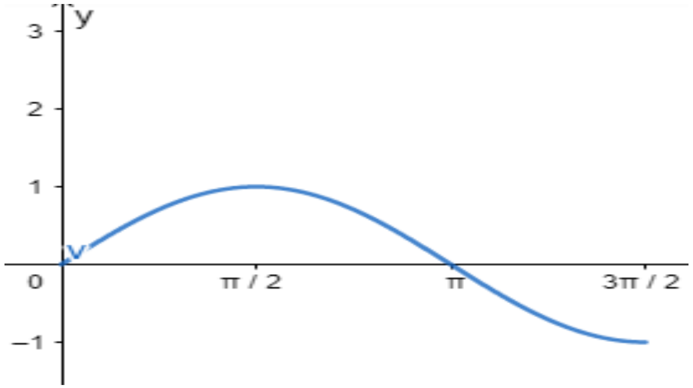
$$= \frac{40}{3}$$

NB: the function, *$f > 0$, hence the integral represent the area.*

determine

- An object moving along a straight line has velocity function $v(t) = sin\ (t)$. If the object starts at position 0, the total distance travelled and the object's position at time $t = \frac{3\pi}{2}$.

Solution:

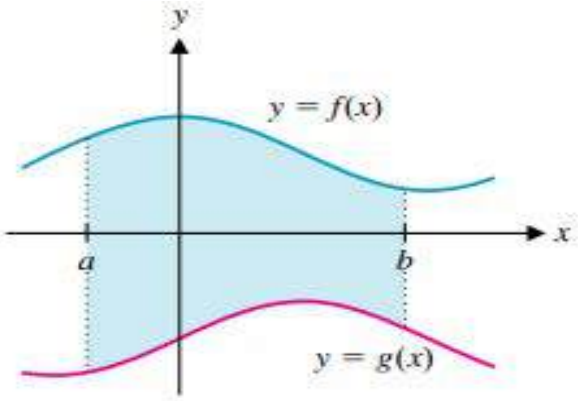


The **total area** bounded by the curve is

$A = \int_0^{\pi} sin\, sin\, t\ \ dt - \int_{\pi}^{\frac{3\pi}{2}} sin\, sin\, t\ \ dt\ \ = 2+1=3$

This means that the total distance travelled is 3 units.

	The overall or total change in position of the object is given by $A = \int_0^{\pi} sin\, sin\, t\ \ dt + \int_{\pi}^{\frac{3\pi}{2}} sin\, sin\, t\ \ dt\ \ = 2 + (-1) = 1$ So, if the object starts at position 0, it ends up at position $0 + 1 = 1$ **Activity 2: Solving problems on Collaborative learning/Group work.** Learners are to work in pairs to debate the expected solution(s) provided to a question and provide reasons for the approaches used. **Question:** Suppose an object moves so that its speed, or more properly velocity, is given by $\mathrm{v(t)} = -\mathrm{t}^2 + 5\mathrm{t}$ in cm/sec. Assume that at time t=0, the position is 0. Find the position at t=6 **Solution:** Let us examine the motion of this object carefully. Let $s(t)$ be the position of the particle at time t. Then, **Approach 1** $s(t) = \int_0^6 v(t)\,dt = \int_0^6 -t^2 + 5t\,dt = (-\frac{t^3}{3} + \frac{5t^2}{2} + C)\|_{t=0}^{t=6}$ =18 **Approach 2** $s(t) = \int_0^6 v(t)\,dt = \int_0^6 -t^2 + 5t\,dt =$ $\int_0^5 -t^2 + 5t\,dx - \int_5^6 -t^2 + 5t\,dx = \frac{71}{3}$ **NB: Learners should be aware of the concepts of "total" and "net."**	
	3.3.2.LI.2	3.3.2.AS.2
	Find the area between two curves. **Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.**	Level 1 Recall Level 2 Skills of conceptual understanding



If $y = f(x)$ lies above $y = g(x)\ from\ x = a\ to\ x = b$, the area of the region between $f(x) and\ g(x)$ from $x = a\ and\ x = b\ is$

$$A = \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

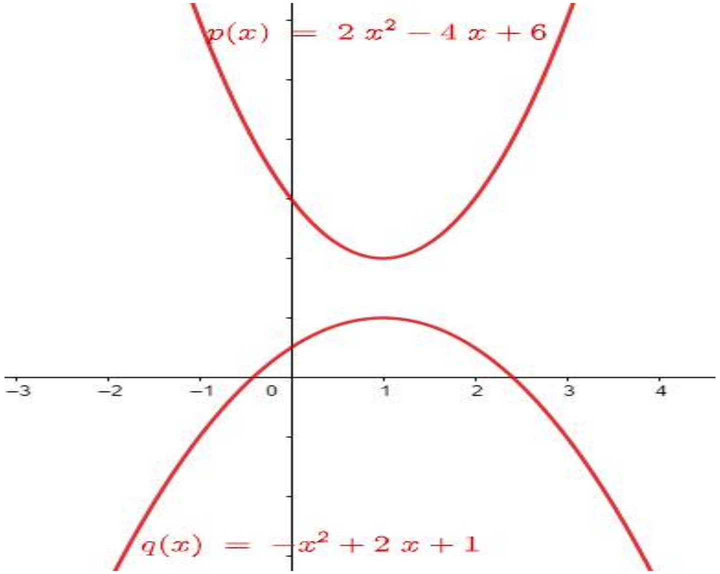
NB: $\int_a^b [\mathrm{f(x) - g(x)}]\,\mathrm{dx}$ is valid only when $\mathrm{f\,(x)} \geq \mathrm{g(x)}$ on the interval $[\mathrm{a, b}]$. In general, the area between $\mathrm{y = f\,(x)}$ and $\mathrm{y = g(x)}$ for $\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}$ is given by $\int_a^b |\,\mathrm{f\,(x) - g(x)}|\,\mathrm{dx}$. Notice that to evaluate this integral, you must evaluate $\int_c^d [\mathrm{f\,(x) - g(x)}]\ \mathrm{dx}$ on all subintervals, where $\mathrm{f\,(x)} \geq \mathrm{g(x)}$, then evaluate $\int_c^d [\mathrm{f\,(x) - g(x)}]\ \mathrm{dx}$ on all subintervals where $\mathrm{g(x)} \geq \mathrm{f\,(x)}$ and finally, add the integrals together.

Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

Example I

- Find the area of the region between $y = 2x^2 - 4x + 6$ and $y = -x^2 + 2x + 1$ from $x = 1$ to $x = 2$

Solution:

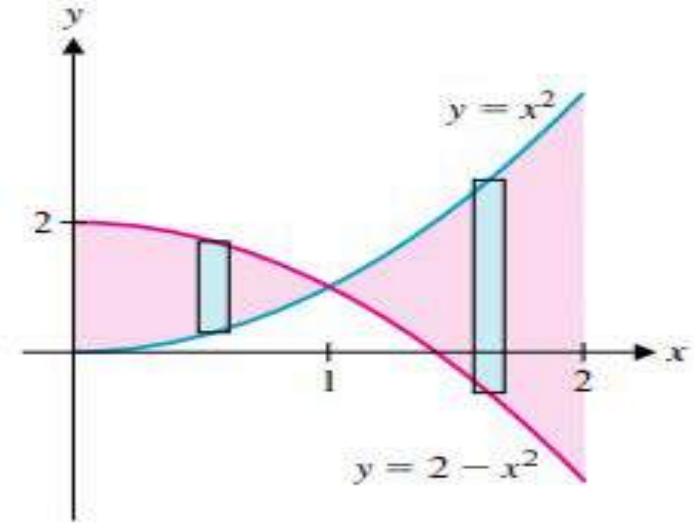


$$\int_1^2 [(2x^2 - 4x + 5) - (-x^2 + 2x + 1)]\, dx = \int_1^2 (3x^2 - 6x + 5)\, dx$$
$$= [x^3 - 3x^2 + 5x]_1^2$$
$$= 3$$

The area enclosed by two graphs.

- Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \le x \le 2$.

Solution:



Graph of $y = x^2$ and $y = 2 - x^2$

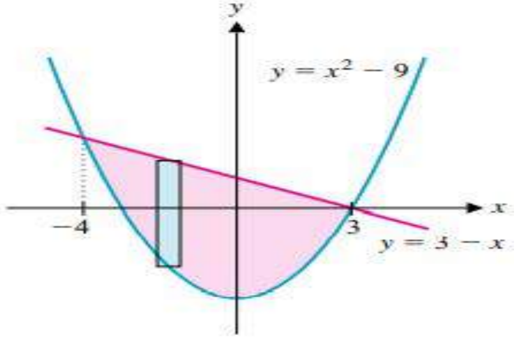
Notice from Figure 2 that since the two curves intersect in the middle of the interval, we will need to compute two integrals, one on the interval where $2 - x^2 > x^2$ and one on the interval where $x^2 \geq 2 - x^2$. To find the point of intersection, we solve $x^2 = 2 - x^2$, so that $2x^2 = 2$ or $x^2 = 1$ or $x = \pm 1$. Since $x = -1$ is outside the interval of interest, the only intersection of note is at $x = 1$. The area is

$$A = \int_0^1 [(2 - x^2) - x^2]\, dx + \int_1^2 [x^2 - (2 - x^2)]\, dx$$

$$= \left[2x - \frac{2x^3}{3}\right] \frac{1}{0} + \left[\frac{2x^3}{3} - 2x\right] \frac{2}{1}$$

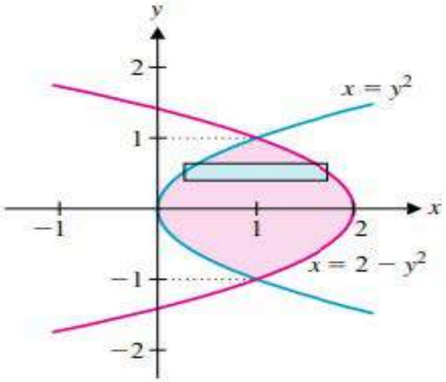
$$= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$$

- Find the area bounded by the graphs of $y = 3 - x$ and $x^2 - 9$.



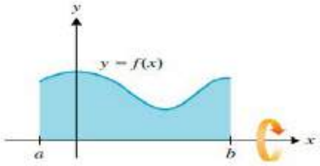
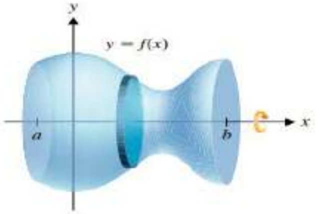
Graph of $y = 3 - x$ and $x^2 - 9$

- Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$.



Graph of $x = y^2$ and $x = 2 - y^2$

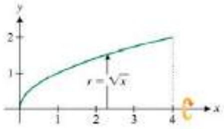
	3.3.2.LI.3	3.3.2.AS.3
	Find the volume of a solid formed after rotation about horizontal or vertical axis. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** • Suppose the downward velocity of a sky diver is given by $v(t) = 30(1 - e^{-t}) ft/s$ for the first 5 seconds of a jump. Compute the distance fallen. **Solution:** $d = \int_0^5 30(1 - e^{-t})\ dt = [30t + 30e^{-t}]5\ 0$ $= (150 + 30e^{-5}) - (0 + 30e^{0})$ $= 120 + 30e^{-5} \approx 120.2 feet$ • Suppose that water can flow in and out of a storage tank. The net rate of change of water is $f(t) = 20(t^2 - 1)$ gallons per minute. • For $0 \leq t \leq 3$, determine when the water level is increasing and when the water level is decreasing. • If the tank has 200 gallons of water at time $t = 0$, determine how many gallons are in the tank at time $t = 3$. **Solution::** Let $w(t)$ be the number of gallons in the tank at time t. • Water level decreases if $w'(t) = f(t) < 0$. $f(t) = 20(t^2 - 1) < 0,\ if\ 0 \leq t < 1$ Water level increases if $w'(t) = f(t) > 0$. $f(t) = 20(t^2 - 1) > 0,\ if\ 1 < t \leq 3$ • $\int_0^3 w'(t)\ dt = \int_0^3 20(t^2 - 1)\,dt$ $w(3) - w(0) = 20\left(\frac{t^3}{3} - t\right)3\ 0$ Since we have $w(0) = 200, then$	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

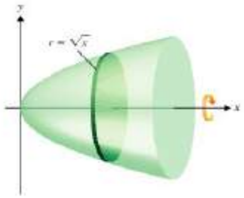
	$w(3) - 200 = 20(9-3) = 120$ Hence, $w(3) = 200 + 120 = 320$ The tank will have 320 gallons at time 3	
	3.3.2.LI.4	3.3.2.AS.4
	Find the volume of a solid formed after rotation about horizontal or vertical axis. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Inter-group competition**: Learners from one group create two binary operations for the other group to investigate volume of solid formed after revolution about and axis. **Volume: Method of Disks** Suppose that $f(x) \geq 0$ and f is continuous on the interval [a, b]. Take the region bounded by the curve $y = f(x)$ and the $x - axis,$ for $a \leq x \leq b$ and revolve it about the $x - axis$ generating a solid (figure 1a and 1b). We can find the volume of this solid by slicing it perpendicular to the $x - axis$ and Recognising that each cross section is a circular disk of radius $r = f(x)$. We then have that the volume of the solid as $V = \sum_{i=1}^{n} [A(c_i)] \Delta x = \int_{a}^{b} \pi [f(x)]^2 \, dx$  $y = f(x) \geq 0$  Solid of revolution	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** **Level 4 Extended critical thinking and reasoning**

Example

- Revolve the region under the curve $y = \sqrt{x}$ on the interval [0, 4] about the $x - axis$ and find the volume of the resulting solid of revolution.

Solution:

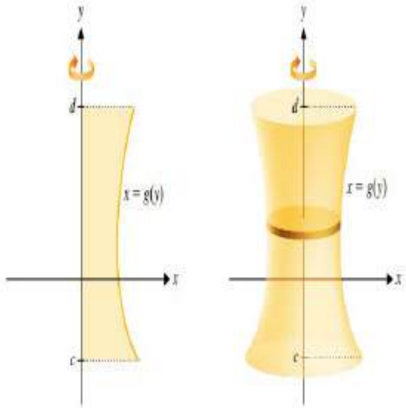




$$V = \int_0^4 \pi[\sqrt{x}]^2\ dx$$
$$= \pi \int_0^4 [\sqrt{x}]^2\ dx$$
$$= \pi \int_0^4 x\ dx$$
$$= \pi \left[\frac{x^2}{2}\right] 4\ 0\ = 8\pi$$

NB: In a similar way, suppose that $g(y)\ \geq\ 0$ and g is continuous on the interval $[c, d]$. Then, revolving the region bounded by the curve $x\ =\ g(y)$ and the $y - \text{axis}$, for $c\ \leq\ y\ \leq\ d$, about the $y - \text{axis}$ generates a solid. (See Figures 3a and 3b.) Once again, notice from Figure 3b that the cross sections of the resulting solid of revolution are circular disks of radius $r\ =\ g(y)$. All that has changed here is that we have interchanged the roles of the variables x and y. The volume of the solid is then given by

$$V = \int_c^d \pi[g(y)]^2\ dy$$



Revolve about the $y-axis$

Example:

Find the volume of the solid resulting from revolving the region bounded by the curve $y=4-x^2$ and $y=1$ from $x=0$ to $x=\sqrt{3}$.

Solution:

By solving for x, we have

$x=\sqrt{4-y},\ y=1$ to $y=4$

$$V=\int_1^4 \pi\left(\sqrt{4-y}\right)^2\,dy=\int_1^4 \pi(4-y)\,dy$$

$$=\pi\left[4y-\frac{y^2}{2}\right]4\ 1\ =\pi\left[(16-8)-\left(4-\frac{1}{2}\right)\right]=\frac{9\pi}{2}$$

$y=4-x^2$

Find the volume of the solid resulting from revolving the portion of the curve $y=2-\frac{x^2}{2}$ from $x=0\ to\ x=2$ about the $y-axis.$

Teaching and Learning Resources	Reading resource colour pens Notebook	Graph sheets Mathematical sets, Technological tools.	Curriculum Cardboard

Subject ADDITIONAL MATHEMATICS
Strand 4. HANDLING DATA
Sub-Strand 1. ORGANISING AND REPRESENTING AND INTERPRETING DATA

Learning Outcomes	21st Century Skills and Competences	GESI, SEL and Shared National Values
3.4.1.LO.1		
Describe the nature and strength of relationship between two given variables using scatter diagram and correlation coefficient.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others, and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners describe the nature and strength or relationship between two given variables using scatter diagram and correlation coefficient. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of scatter diagram and correlation coefficient in mathematics and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in a mathematics discourse.

	phenomena (learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Conscious efforts will enable learners to collectively develop and implement innovative actions that further sustainability at the local level for application to life	• Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and

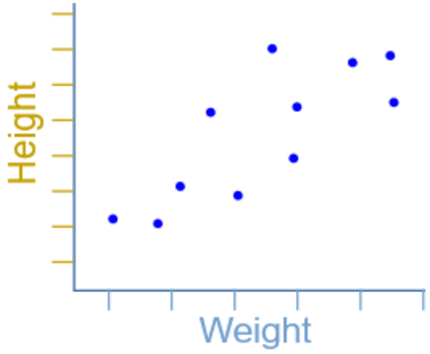
		learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote respect for divergent views to ensure inclusivity in the mathematics-learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they describe the nature and strength or relationship between two given variables using scatter diagram and correlation coefficient.

3.4.1.LO.2		
Model and solve problems using regression analysis.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups to model and solve problems using regression analysis. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying mathematics and related fields. • Examine and dispel misconceptions/ myths about gender as they relate to each other in a mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving

		each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work.

		Diversity: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning through mixed-ability grouping within a differentiated mathematics classroom instruction and assessment as they Model and solve problems using regression analysis.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
3.4.1.CS.1	3.4.1.LI.1	3.4.1.AS.1
Demonstrate understanding of the nature and strength of relationship between two given variables.	**Distinguish between univariate and bivariate data and give examples and explain the concept of correlation.** **Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Learning Experience:** Learners in groups discuss correlation. **Activity 1: The Idea of Correlation** Learners provide examples of univariate data they are familiar with and brainstorm on bivariate data. Learners share their ideas on the concept and the use of correlation. • Univariate means one variable (one type of data) and bivariate means two variables (two types of data). • With bivariate data we have **two** sets of related data we want to **compare**. **Examples of Univariate data:** Age of students, height of students, travel time etc. • With univariate data, you can find the central value using mean, median and mode or find how spread out the data is using range, quartiles and standard deviation. However, this form of analysis gives only just simple explanation because only one quantity is used, unlike bivariate data. • With univariate data you can make plots like bar graphs, pie charts and histogram. **Examples of bivariate data**: Age of students and their height, price of goods and quantities of goods, hours students spend studying and their test scores. Bivariate data can be used to more comparison.	**Level 1 Recall** **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	3.4.1.LI.2	3.4.1.AS.2
	Construct a scatter plot using given data sets and use it to describe the relationship between two variables. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Learning Experience:** Learners in groups discuss correlation. Learners establish that: • Correlation is statistical measurement that is used to describe the nature and strength of relationship between two variables. • Correlation can be used to determine how performance relates to number of hours used to study and also to make decision on the likelihood of rains in the dry season, likelihood of disease outbreak in a community, number of people in the house and amount of water used. **Activity 1: Representation of bivariate Data in a scattered plot diagram** Learners in pairs recollect how to plot coordinates in the x-y plane. Learners in groups create a bivariate data and construct a scatter plot to illustrate the relationship between the data. **Example of Scatter plot** 	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended critical thinking and reasoning

	A scatter plot showing weight versus height **Activity 3:** Learners in groups discuss and describe the nature of relationship between weight and height using the scatter plots. E.g. the scatter plot above shows positive slope indicating a positive correlation between weight and height, i.e. as the weight increases so is the height also increases. NB: this does not however mean that height is causing the weight to increase or vice versa.	
	3.4.1.LI.3	3.4.1.AS.3
	Analyse and describe visual data in a scatter plot by interpreting the relationship between given bivariate data sets. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Learning Experience:** Learners in groups analyse the data in scatter plot **Activity 1: Analysis of bivariate data** Learners in groups collect on two related variables (E.g. price of goods and quantities of goods, hours students spend studying and their test scores) and construct a scatter plot to illustrate the relationship between the variables. **Activity 2: Kinds of relationships between two variables** Learners in groups discuss how to describe the scatter plot to reflect the relationship between the data and to ascertain whether the relationship is linear or not. Learners in groups make generalisation of the relationship between data on a scatter diagram. The relationship between the data can be positive linear, negative linear, non-linear and no association.	Level 1 Recall Level 2 Skills of conceptual understanding **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	 Scatter diagram showing positive linear relationship between salary earned by government workers in Ghana and the price of car they own. That is as the salary increases there is likelihood to buy expensive car.	
	3.4.1.LI.4	3.4.1.AS.4
	Describe the Spearman's Rank correlation coefficient and interpret the results within a given situation. **Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Learning Experience:** Learners in groups calculate and interpret Spearman's Rank correlation**.**	Level 1 Recall **Level 2 Skills of conceptual understanding** Level 3 Strategic reasoning Level 4 Extended

Activity 1: Spearman's Rank correlation

Learners in groups discuss the type of data suitable for the use of Spearman's rank correlation and, hence, apply the formula $r_s = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$ to find the Spearman's rank correlation coefficient.

Where n is the sample size and Σd^2 is the sum of the squares of the difference in ranks.

Condition for use of Spearman's rank correlation: When both variables are in ordinal scale or rank order form or when the data set can be converted to ordinal form. The data must be at least ordinal and the scores on one variable must be monotonically related to the other variable.

Example 1: The following are scores obtained by 5 students in Physics and Biology

Physics	**98**	**67**	**93**	**90**	**83**
Biology	**91**	**65**	**92**	**93**	**85**

Calculate the Spearman's rank of correlation

Solution

Physics	**Biology**	**Physics Rank**	**Biology Rank**	**difference**	**Squared difference**
98	91	1	3	-2	4
67	65	5	5	0	0
93	92	2	2	0	0
90	93	3	1	2	4
83	85	4	4	0	0

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$
$$= 1 - \frac{6(8)}{5(5^2-1)}$$
$$= 0.6$$

Conclusion: Since this value is positive, it indicates a positive correlation.

Therefore, there is a positive correlation between the rankings of students in Physics and biology.

critical thinking and reasoning

	Activity 2 Solving daily problems using Spearman's Rank correlation Learners solve other examples and draw conclusions based on the value of Spearman's rank coefficient. **Conclusion**: If the value of the coefficient is 1, it means there is perfect positive correlation between the two sets of data. If the coefficient is -1, then there is perfect negative correlation. If the coefficient is zero, it means there is no correlation between the two variables.	
Teaching and Learning Resources	• ICT tools • Calculators • SHS curriculum	

<table>
<tr><th>Content Standards</th><th>Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI</th><th>Assessment</th></tr>
<tr><td>3.4.1.CS.2</td><td>3.4.1.LI.1</td><td>3.4.1.AS.1</td></tr>
<tr><td>Demonstrate an understanding of relationship arising from cause and effect or contribution of one or more variables to another.</td><td>

Describe the concept of regression and distinguish between correlation and regression, and describe situations that call for the use of regression.

Project-based Learning; Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.

Learning Experience: Learners in groups discuss regression and distinguish between correlation and regression.

Activity 1: Simple linear regression
Learners brainstorm and share ideas on the situations that require the use of simple linear regression analysis.

Definition: Simple linear regression is a statistical measurement used to describe the relationship between independent variable and dependent variables where change in the independent variable affect the dependent variable.

Activity 2: Differences between correlation and regression
Learners work in pairs to describe events where correlation and regression occur and distinguish between them in tabular form, as shown.

<table>
<tr><th>Correlation</th><th>Regression</th></tr>
<tr><td>Correlation is used to determine whether two variables relate in any manner or not.</td><td>Regression is used to numerically describe how a dependent variable changes with a change in an independent variable.</td></tr>
</table>

</td><td>

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

</td></tr>
</table>

Correlation is used to tell the strength (strong, weak) and direction ((positive, negative) of the relationship between the two variables.	Linear regression uses an equation to express the amount of effect (contribution) that one variable has on another in a relationship.
The variables can be used interchangeably.	The variables cannot be interchanged.

Activity 4: Independent variable versus dependent variable

Learners brainstorm to identify independent and dependent variables and describe examples or situations where both variables are used for regression analysis.

- The value of an independent variable does not change due to the impact of any other variable.
- Dependent variable is the variable that changes due to change in another variable.

Example: Consider a situation where we want determine what happens to students' test scores when students increase the number hours they spend studying.

Independent variable: Hours students spend sleeping

Dependent variable: Test scores

3.4.1.LI.2

Fit a linear function to a given data and use the line of best fit to solve problems in the context of the data.

Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.

Learning Experience: Learners in groups brainstorm to find the line of best fit for better accuracy using least square method.

3.4.1.AS.2

Level 1 Recall

Level 2 Skills of conceptual understanding

Level 3 Strategic reasoning

Level 4 Extended

Activity 1: Line of best fit
Learners in groups draw a tabular representation for a given data set. Learners in groups use the formula $m = \frac{\sum_{i=1}^{n} (x_i - \underline{x})(y_i - \underline{y})}{\sum_{i=1}^{n} (x_i - \underline{x})^2}$ to estimate slope the line of best fit and use these to construct a table with the details in the formula. Learners work to obtain the $y - intercept$ using the formula of equation of a line.

Example 1: Use the least square method to determine the equation of best fit of the line for the given data and plot the graph.

x	8	2	11	6	5	4	12	9	6	1
y	3	10	3	6	8	12	1	4	9	14

Solution
NB: $\underline{x} = 6.4$ **and** $\underline{y} = 7$

x	y	$(x_i - \underline{x})$	$(y_i - \underline{y})$	$(x_i - \underline{x})(y_i - \underline{y})$	$(x_i - \underline{x})^2$
8	3	1.6	-4	-6.4	2.56
2	10	-4.4	3	-13.2	19.36
11	3	4.6	-4	-18.4	21.16
6	6	-0.4	-1	-0.4	0.16
5	8	-1.4	1	-1.4	1.97
4	12	-2.4	5	-12	5.76
12	1	5.6	-6	-33.6	31.36
9	4	2.6	-3	-7.8	6.76
6	9	-0.4	2	-0.8	0.16
1	14	-5.4	7	-37.8	29.16
$\Sigma =$				-131	118.4

critical thinking and reasoning

	$\therefore m = \frac{\sum_{i=1}^{n} \quad (x_i - \underline{x})\left(y_i - \underline{y}\right)}{\sum_{i=1}^{n} \quad (x_i - \underline{x})^2} = \frac{131}{118.4}$ $= -1.1$ To find the $y - intercept$, the equation of the line is given as $\underline{y} = m\underline{x} + b$ $\therefore 7 = (-1.1 \times 6) + b$ $b \approx 14$ Therefore, the equation of the best-fit line is $y = -1.1x + 14$ **Activity 4:** Learners work in groups sketch of $y = -1.1 + 14$ in graph sheets. **Activity 5:** Learners work in groups to solve given examples and use the equation of best fit to estimate values within the domain and the range of the data. Learners to model their own data and use the idea of line of best fit to arrive at estimated values.	
Teaching and Learning Resources	• ICT tools • Calculators • SHS curriculum	

Subject **ADDITIONAL MATHEMATICS**
Strand **4. HANDLING DATA**
Sub-Strand **2. MAKING PREDICTIONS WITH DATA**

Learning Outcomes	**21st Century Skills and Competences**	**GESI, SEL and Shared National Values**
3.4.2.LO.1		
Solve problems involving conditional probability using permutations and combinations.	**Communication**: Provide learners the opportunity to engage and participate in the mathematical talk, ensuring that learners are tolerant to listen to the views and perspectives of others and use appropriate vocabulary confidently and effectively to present their ideas. **Collaboration**: Create an atmosphere, environment and opportunity that fosters the spirit of team success where learners understand and respect the needs, contributions, perspectives, and actions of others whilst they embark on project works, classroom activities and presentations. **Critical Thinking:** Provide the atmosphere, environment and opportunity for learners to observe patterns and make predictions, gather and interpret data, draw conclusions based on relevant data or personal knowledge and experience, present and receive information with others verbally, nonverbally and in writing, and put together relevant sources of information (making connections) to solve problems. Learners should compare and contrast, evaluate, analyse, generate possibilities, make inferences and interpret phenomena	**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners solve problems involving conditional probability using permutations and combinations. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of permutation and combination in solving problems involving conditional probability and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in a mathematics discourse.

	(learners respond to “why, how, when, who, what, and where” questions. **Personal and Leadership Skills:** Provide the incentives for all learners to feel safe, confident, and happy to participate in all activities; have all learners taste leadership roles and responsibilities whilst they work in groups; demystify the mathematics classroom and get all learners to overcome fear in the mathematics classroom. Encourage learners to keep a mathematics journal, take risks, explore and observe others they admire (mentors) to learn from them and their actions. **Creativity and Innovation:** • Provide a classroom atmosphere and environment that is flexible and democratic; ensure that learners reflect and visualise ideas and goals; encourage journaling and set aside a dedicated time of mindfulness each school day. • Ask open-ended questions and set problem-finding contexts. **Digital Literacy Skills:** Provide learners the opportunity to effectively and independently use technological tools to research for information and solve problems, including drawing graphs and arithmetic computations. **Strategic Competency:** Make conscious efforts to enable learners to collectively develop and implement innovative actions that further sustainability at the local level life application.	• Value and promote justice in the mathematics classroom and beyond. **SEL**: Creating opportunities for learners to build their Social Emotional Learning Competencies –*Self-Awareness, Self-Management, Social Awareness, Relationship Skills and Responsible Decisions* are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the SHS Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and

		learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work. **Diversity**: Promote the respect for divergent views to ensure inclusivity in the mathematics-learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they solve problems involving conditional probability using permutations and combinations.
3.4.2.LO.2		
Use the concepts of permutation and combination to solve real life problems.		**GESI:** Learners having experienced a teaching approach that ensures gender equality and social inclusion, where they

		work with each other in an inclusive way; cross-sharing knowledge and understanding among groups and individuals lead them to: • Respect individuals of different backgrounds in their groups as learners demonstrate the usefulness of permutation and combination to solve real life problems. • Interrogate their stereotypes and biases about the roles and abilities of different groups in learning and applying their knowledge of the mathematical concepts of permutation and combination to solve real life problem and related fields. • Examine and dispel misconceptions/ myths about GESI as they relate to each other in the concept of permutation and combination in mathematics discourse. • Value and promote justice in the mathematics classroom and beyond. **SEL:** Creating opportunities for learners to build their Social Emotional Learning Competencies – *Self-Awareness, Self-Management, Social Awareness, Relationship*

		Skills and Responsible Decisions are integrated throughout all lessons to encourage inclusion. As part of achieving each learning outcome in the Additional Mathematics curriculum, the teacher should apply the Social Emotional Learning strategies to ensure that learners are: • Self-reflecting and developing confidence • Exhibiting motivation and SMART goal-setting • Managing emotions and conflicts • Showing empathy and cooperation These may be done by the teacher through modelling emotional self-regulation and decision-making, the promotion of positive self-talk with self-made portraits, the creation of a vision board, creating respectful icebreakers for healthy debates, encouraging diversity presentations, and learners writing on the sequence of their activities. **National Core Values:** **Leadership and Respect for others' views:** Inculcate the habit of leadership through teamwork; and respect for individuals' views, beliefs, religions, and cultures through interactive and collaborative/group work.

		Diversity: Promote respect for divergent views to ensure inclusivity in the mathematics learning environment. **Equity:** Develop fair and impartial opportunities or resources for learners devoid of unwanted segregation or discrimination among learners. **Tolerance:** Model tolerance among learners by creating opportunities for collaborative learning in their mixed-ability groups, emphasising differentiated instruction and assessment as they use the concepts of permutation and combination to solve real life problems.

Content Standards	**Learning Indicators and Pedagogical Exemplars with 21st Century Skills and Competencies, and GESI**	**Assessment**
3.4.2.CS.1	3.4.2.LI.1	3.4.2.AS.1
Apply and extend the knowledge of the laws of probability and the concept of combination and permutation to solve real life problems involving conditional and binomial probability.	**Explain the concepts that give rise to conditional probabilities.** **Project-based Learning, Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.** **Learning Experience:** Work individually/in pairs in mixed-ability gender groups to identify the set of real numbers with the aid of models, including the number line. **Activity 1:** Learners use Talk for Learning to discuss two events from the same experiment and also from two different experiments. Find the probability of two events both from the same experiment and also from two different experiments using the idea of combination or selection. **Example 1:** A bag contains 7 black and 2 white identical marbles. Another bag contains 1 white and 5 black of the same marbles. A boy selects one marble from the selected bag. Find the probability that a black marble is selected. **Possible solution:** $P(\ Black\ marble\ ) =$ $= P(either\ bag) \times [\ P(black\ from\ 1st\ bag) \cup P(black\ from\ 2nd\ bag)$ $= \frac{1}{2} \times \frac{7C_1}{9C_1} + \frac{5C_1}{6C_1}$ $= \frac{1}{2} \times [\frac{7}{9} + \frac{5}{6}]$ $= \frac{1}{2} \times \frac{29}{18}$ $= \frac{29}{36}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	Activity 2: Learners in groups model similar examples and solve them.	
	3.4.2.LI.2	3.4.2.AS.2
	Describe the probability of two equally likely events occurring from given experiments. **Project-based learning; Talk for Learning, Think-pair-share, Experiential Learning, and Group Work/Collaborative Learning.** **Learning Experience:** Learners work in mixed-ability gender groups to establish that, for any two equally likely events, A and B, the probability that B will occur given that A has occurred is given by $P(B/A) = \frac{P(B \cap A)}{P(A)}$ **Activity 1: Conditional Probability** Learners work in groups to perform a simple activity or experiment and describe the probability of two likely events occurring. An example is the probability of a Aduana Star scoring better in the next match as they have a former Olympian for a coach. **Definition**: Given two events A and B, the probability of event B occurring, given that event A has occurred, is a conditional probability $P(B/A)$, which is given by; $P(B/A) = \frac{P(B \cap A)}{P(A)}$ **Example 1:** There are a total 40 students in a science class, out of which 15 are girls and 20% of the boys wear spectacles. Find the probability that a student selected at random from the class is a boy who wears spectacles. **Solution:** P(selecting a boy) $= P(A) = \frac{25}{40}$ P(spectacle/boy) =$P(B/A) = \frac{20}{100}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	We want to find the probability of selecting a boy who wears spectacles, which is given by $P(B \cap A) = P(A) \times P(B/A)$ $P(B \cap A) = \frac{25}{40} \times \frac{20}{100} = \frac{1}{8}$	
	3.4.2.LI.3	3.4.2.AS.3
	Model and solve real life problems involving conditional probabilities. **Project-based Learning, Talk for Learning, Think-pair-share, Experiential Learning and Group Work/Collaborative Learning.** **Learning Experience:** Learners work in mixed-ability groups to model conditional situations and solve examples of conditional probability. **Activity 1: Solving problems on conditional probability** Learners discuss situations where conditional probabilities could arise, model some likely probabilities and solve problems relating to conditional probability. **Example:** Two bags P and Q contain 3 white, 4 green and 5 white and 2 green identical balls respectively. A ball is picked at random from bag P and put into bag Q. A ball is then picked at random from bag Q. Find the probability that the ball drawn from bag Q is white. **Solution:** $= P(W_Q/W_P)$ **or** $P(W_Q/G_P)$ $= \frac{3}{7} \times \frac{6}{8} + \frac{4}{7} \times \frac{5}{8} = \frac{9}{28} + \frac{10}{28} = \frac{19}{28}$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

	3.4.2.LI.4	3.4.2.AS.4
	Model and solve real life problems involving binomial probability. **Project-based Learning; Talk for Learning, Think-pair-share, Experiential Learning; and Group Work/Collaborative Learning.** **Learning Experience:** Learners work in mixed-ability gender groups, to solve binomial probability problems. **Activity 1: Binomial probability** Learners think pair and share ideas on situations or events involving likelihood of combining or selecting or succeeding in one out of two outcomes in a certain number of attempts and use the idea to solve binomial probability problems. E.g. The probability of passing 2 times in a semester examination in Add Mathematics after attempting 5 times. **Definition**: ***Binomial probability refers to the probability of exactly x successes on n repeated trials in an experiment which has two possible outcomes.*** **Activity 2: The idea of combination in binomial probability** Learners in mixed-ability groups solve real life problems on combination and discuss how combination relates to the binomial probability. **Example 1:** A group of 4 SRC executives is to be selected from 5 males and 6 females of the SRC. Find the number of ways of selecting the group if: • There are no restrictions on its composition • There is an equal number of males and females • It consists of all males or all females. **Solution:** $11C_4 = \frac{11!}{4!7!} = \frac{11\times10\times9\times8}{4\times3\times2} = 330\ ways$	Level 1 Recall **Level 2 Skills of conceptual understanding** **Level 3 Strategic reasoning** Level 4 Extended critical thinking and reasoning

$$5C_2 \times 6C_2 = 10 \times 15 = 150\, ways$$

$$5C_2 + 6C_2 = 10 \times 15 = 25\, ways$$

Activity 3: Solving real life problems on binomial probability
Learners use Talk for Learning strategy to describe the probability P that an event will happen in any single trial (success) and the probability that the event will fail to happen (failure) after a given number of attempts. Mathematically, the probability that an event will happen "x" times in 'n' trials is given by
$P(X = x) = nC_x\ p^x q^{n-x}$ where

$$n = number\ of\ trials$$
$$p = probabilty\ of\ success$$
$$q = probabilty\ of\ failure$$

Note:

- p + q = 1
- Success is what we are interested in
- Failure is what we are not interested in.

Solution:

$$n = 6$$
$$p = p(defective) = \frac{3}{12} = \frac{1}{4}$$

$$q = q(non - defective) = 1 - \frac{1}{4}$$

- $P(x = 1) = \binom{6}{0}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^5$

$$= 6 \times \frac{1}{4} \times \frac{243}{1024} = 0.355957 = 0.356$$

- $P(x \leq 1) = P\ (x = 0) + \ P(x = 1)$

$$= \binom{6}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^6 + P(x = 1)$$

$$= \frac{729}{4096} + 0.355957$$

$$= 0.1779785\ + 0.355957$$

$$= 0.534$$

Example 2: A student tosses a fair coin 5 times. Find the probability that there will be:

- Exactly 3 tails
- At least 1 head.

Solution:

$$n = 5$$

$$p = \frac{1}{2}, q = \frac{1}{2}$$

$P(x = 3) = 5C_3\ \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \times \frac{1}{8} \times \frac{1}{4} = \frac{10}{32} = \frac{5}{16}$

ii) $P(at\ least\ one\ head) = P(X \geq 1)$

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - (5\ 0\ )\left(\frac{1}{2}\right)^0\left(\frac{1}{5}\right)^5$$

$$= 1 - \frac{1}{32}$$

$$= \frac{31}{32}$$

Activity 3: Modelling real life problems on binomial probability

Learners in groups create more real life examples of binomial distribution and calculate the probabilities.

3.4.2.LI.5

Model and solve real life problems involving permutation and combination.

Group work, Talk for Learning, and Building on what others say.

Learning Experience: Learners in groups model and solve problems involving permutation and combination.

Activity 1: **Fundamental counting principles**
Learners in pairs describe the fundamental counting techniques and principles for the outcomes of some experiments and state the number of ways each can be performed.

Example 1: If a sports club decides to elect a chairman and a secretary from 5 members, assuming the same person cannot hold both positions, find the number of ways this can be done?

Solution: Thus, number of ways of choosing a chairman $= 5$
And the number of ways for choosing a secretary $= 4$

$$= 5 \times 4$$

$$= 20$$

Example 2: The Department of Vehicle and Licensing Authority designs the registration of each car with 2 letters from E, R, G, T and 3 digits from $1,4,5,6$ on a given day. From the prescribed solution, how many cars can be registered on that day under the required arrangement?

Solution: Firstly, how many 2-letter arrangements can be formed from the 4 letters, E, R, G, T?
This is $4P_2 = \frac{4!}{(4-2)!}$

$$= \frac{4!}{2!} = 12$$

Similarly, arranging 3 digits from the digits $1,4,5,6$ is $4P_3$

$$4P_3 = \frac{4!}{(4-3)!}$$

3.4.2.AS.5

Level 1 Recall
Level 2 Skills of conceptual understanding
Level 3 Strategic reasoning
Level 4 Extended critical thinking and reasoning

	$= \frac{4!}{1!}$ $= 24$	
	3.4.2.LI.6	3.4.2.AS.6
	Determine areas in business, commerce, industry, etc. where permutations and combinations can be applied to improve the systems. **Group work and Project-Based Learning** **Learning Experience:** Learners in groups conduct research on areas where permutation and combination can be applied. **Activity 1 Application of Permutations and Combinations** Learners in groups investigate the application of permutation and combination in business, commerce, industry, factory and games and present the findings from the research in class**.** **Example:** Arranging digits in passwords, alphabets in concatenation and colour schemes in clothing in factories are examples of permutations. The selection of menu, food, clothes, subjects and footballers for a match are examples of combinations. Other applications include communication networks, cryptography and network security, computer architecture, electrical engineering, computational molecular biology, languages, both natural and computer and pattern analysis.	Level 1 Recall Level 2 Skills of conceptual understanding Level 3 Strategic reasoning **Level 4 Extended critical thinking and reasoning**
Teaching and Learning Resources	• SHS curriculum • calculators	