SECTION

2 SURDS, INDICES AND LOGARITHMS

MODELLING WITH ALGEBRA

Number and Algebraic Patterns

INTRODUCTION

Surds will provide a foundation for understanding more complex mathematical concepts. Surds being a square root of Non-perfect Squares, introduce learners to irrational numbers. These numbers cannot be expressed as simple fractions and have decimal expansions that neither terminate nor repeat. Familiarity with surds helps learners to grasp the concept of irrationality. The use of surds enables the engineers to calculate the dimensions and angles for bridges and buildings, ensuring structural stability and load–bearing capacity.

At the end of this section, you will be able to:

- Investigate the properties of surds and perform basic arithmetic operations on surds
- Rationalise surds with binomial denominators
- Recollect the initial laws of indices and establish other laws for negative powers and roots
- Recognise the relationship between surds and indices and apply laws of indices to simplify expressions
- Pose and solve simple equations involving indices
- Establish the relationship between indices and logarithms and use the properties of logarithms to solve related problems in one base

Key Ideas:

- Surds are values expressed in the **square root**, that cannot be further simplified into whole numbers or integers. Examples include $\sqrt{2}, \sqrt{3}, \sqrt{5}$.
- Surds have decimal representations that go on forever without repeating. For example, $\sqrt{2}$ is approximately 1.4142135…………,$\sqrt{3}$ is approximately 1.7320508…………etc. These decimals are considered irrational.

- Values or numbers in the form $\sqrt{16} = 4, \sqrt{4} = 2, \sqrt[3]{8} = 2$ are not surds as their respective answers are whole numbers or exact figures and not repeating decimals.
- In general, expressions of the form $\mathbf{a^m}$ are called indices, where **a** is called the **base** and **m** is called the **index** or **exponent** or **power** and it is read as "a exponent m" or "a to the power of m".
- On the other hand, if we have the equation we say that $\mathbf{y = a^m}$, **the exponent m is the logarithm of y to base a**. it is written $\log_a y = m$.

DEFINITION, PROPERTIES AND SIMPLIFICATION OF SURDS

Surds were discovered and defined by a European mathematician, Gherardo of Cremona, in 1150 BC (Joseph, 2010). He used the Pythagoras' theorem to find the diagonal of a square and the value of the first surd. He termed this value 'voiceless' because the root value had no meaning at that time. Surds are numbers with roots that cannot be simplified to whole numbers. They are square roots, or other roots, that cannot be written as a simple fraction. Surds or radical expressions contain roots (like $\sqrt{2}$) that are not whole numbers. For example, $\sqrt{2} = 1.4142135624$ $\sqrt{3} = 1.7320508076$, $\sqrt{35} = 5.9160797831$ etc. However, $\sqrt{25} = 5$ and $\sqrt{81} = 9$, both have whole number solutions and, thus, are not surds.

The behaviour of numbers can lead to some classifications or groupings.

Activity 2.1

Now, let's go through the following activities either in small groups or individually.

1. Use your calculator to compute the following numbers.

- **i.** $\sqrt{2}, \sqrt{3}, \sqrt{5},$
- **ii.** $\sqrt{4}, \sqrt{36}, \sqrt{49},$

2. What was your observation?

You will observe that the numbers in (1) did not have exact values and so were approximated. However, the numbers in (2) had exact values.

Expressions of the form $\sqrt{2}, 3\sqrt{5}, 2+\sqrt{7}$ and so on that involve or contain square roots of positive integers which are not perfect squares are called **Surds**. When expressed in decimal form, surds are non-terminating and non-repeating.

Though a calculator with a square root key can give values of surds, it must be noted that, these values they provide are only approximations.

A surd is of the form $a \pm \sqrt{b}$ *or* $\sqrt{a} \pm \sqrt{b}$ where a and b are natural numbers, not perfect squares, is often referred to as a binomial surd. The numbers$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, etc, are all surds. On the other hand, the numbers where a certain number multiplied itself to give the number is a perfect square. Examples of **perfect squares** are, 1, 4, 9, 16, 25.

Have a go at writing more perfect squared numbers and verify them with the use of calculator.

Types of Surds

1. Pure Surds: A surd having only a single irrational number is called a pure surd.
   Example, $\sqrt{7}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{2}$
2. Mixed Surds: A surd having a mix of a rational number, and an irrational number is called a mixed surd.
   Example, $5\sqrt{3}$, $12\sqrt{5}$, $7\sqrt{11}$,
3. Compound Surds: A surd composed of two surds, or a surd and a rational number is called a compound surd.
   Example, $\sqrt{3} + \sqrt{10}$, $3 + \sqrt{7}$,
4. Binomial Surd: When two surds give rise to one single surd, the resultant surd is known as a binomial surd.
   Example, $\sqrt{30} = \sqrt{15 \times 2}$

Rules/Properties of surds

Surds have rules which influence the way they behave in algebra. Note the following carefully.

Activity 2.2

In a small group, or individually, use your calculator to verify the following properties of surds using any numbers of your choice.

Multiplication Rules

1. $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = a, \text{ if } a \geq 0$

Examples:

i. $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

ii. $\sqrt{7} \times \sqrt{7} = (\sqrt{7})^2 = 7$

- ***The square of a surd is equal to its radical.***

2. $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$

Examples:

i. $\sqrt{2} \times \sqrt{3} = \sqrt{(2 \times 3)} = \sqrt{6}$

ii. $\sqrt{5} \times \sqrt{11} = \sqrt{(5 \times 11)} = \sqrt{55}$

- ***The product of two surds is equal to the square roots of their products.***

3. $m\sqrt{a} \times \sqrt{a} = m(\sqrt{a})^2 = m \times a = ma$

Examples:

i. $5\sqrt{3} \times \sqrt{3} = 5(\sqrt{3})^2 = 5 \times 3 = 15$

ii. $3\sqrt{7} \times \sqrt{7} = 3(\sqrt{7})^2 = 3 \times 7 = 21$

4. $m\sqrt{a} \times n\sqrt{a} = m \times n(\sqrt{a})^2 = mn \times a = mna$

Examples:

i. $4\sqrt{5} \times 3\sqrt{5} = 4 \times 3(\sqrt{5})^2 = 12 \times 5 = 60$

ii. $2\sqrt{3} \times 5\sqrt{3} = 2 \times 5(\sqrt{3})^2 = 10 \times 3 = 30$

5. $m\sqrt{a} \times \sqrt{b} = m\sqrt{a \times b} = m\sqrt{ab}$

Examples:

i. $5\sqrt{2} \times \sqrt{3} = 5\sqrt{6}$

ii. $3\sqrt{7} \times \sqrt{5} = 3\sqrt{7 \times 5} = 3\sqrt{35}$

6. $m\sqrt{a} \times n\sqrt{b} = (m \times n)\sqrt{a \times b} = mn\sqrt{ab}$

Examples:

i. $5\sqrt{2} \times 4\sqrt{3} = (5 \times 4)\sqrt{2 \times 3} = 20\sqrt{6}$

ii. $-3\sqrt{5} \times 2\sqrt{6} = (-3 \times 2)\sqrt{5 \times 6} = -6\sqrt{30}$

Division Rules

1. $\sqrt{a} \div \sqrt{a} = \frac{\sqrt{a}}{\sqrt{a}} = \sqrt{\frac{a}{a}} = \sqrt{1} = 1$, *where* $a \neq 0$

Examples:

i. $\sqrt{5} \div \sqrt{5} = \frac{\sqrt{5}}{\sqrt{5}} = \sqrt{\frac{5}{5}} = \sqrt{1} = 1$

ii. $\sqrt{3} \div \sqrt{3} = \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{\frac{3}{3}} = \sqrt{1} = 1$

- ***The quotient of two identical surds is equal to 1.***

2. $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, *where* $b \neq 0$

Examples:

i. $\sqrt{2} \div \sqrt{3} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$

ii. $\sqrt{11} \div \sqrt{7} = \frac{\sqrt{11}}{\sqrt{7}} = \sqrt{\frac{11}{7}}$

- ***The quotient of two surds is equal to the square root of the quotient.***

Addition Rule

1. $m\sqrt{a} + n\sqrt{a} = (m + n)\sqrt{a}$

Examples:

i. $4\sqrt{3} + 5\sqrt{3} = (4 + 5)\sqrt{3} = 9\sqrt{3}$

ii. $-3\sqrt{2} + 7\sqrt{2} = (-3 + 7)\sqrt{2} = 4\sqrt{2}$

2. $m\sqrt{a} + n\sqrt{b} = m\sqrt{a} + n\sqrt{b}$

Examples:

i. $6\sqrt{3} + 4\sqrt{5} = 6\sqrt{3} + 4\sqrt{5}$

ii. $3\sqrt{2} + 5\sqrt{3} = 3\sqrt{2} + 5\sqrt{3}$

- ***Like surds can be added together.***
- ***Unlike Surds cannot be added together.***

Subtraction Rule

1. $m\sqrt{a} - n\sqrt{a} = (m - n)\sqrt{a}$

Examples:

i. $7\sqrt{3} - 5\sqrt{3} = (7 - 5)\sqrt{3} = 2\sqrt{3}$

ii. $3\sqrt{2} - 7\sqrt{2} = (3 - 7)\sqrt{2} = -4\sqrt{2}$

2. $m\sqrt{a} - n\sqrt{b} = m\sqrt{a} - n\sqrt{b}$

Examples:

i. $6\sqrt{3} - 4\sqrt{5} = 6\sqrt{3} - 4\sqrt{5}$

ii. $3\sqrt{2} - 5\sqrt{3} = 3\sqrt{2} - 5\sqrt{3}$

- ***Like surds can be subtracted.***
- ***Unlike Surds cannot be subtracted.***

Simplification of surds

In simplifying surds, you find two factors of the number such that one is a perfect square. This can be done by dividing the number by the prime numbers 2, 3, 5, 7, etc in turn as exemplified in the table until you are left with a perfect square. Or you can divide by a perfect square (4, 9, 16 etc) until you have no remainder.

Number	Factors
8	$\mathbf{4} \times 2$
12	$\mathbf{4} \times 3$
32	$\mathbf{16} \times 2$
45	$\mathbf{9} \times 5$
68	$\mathbf{4} \times 17$
567	$\mathbf{81} \times 7$
75	$\mathbf{25} \times 3$

Let us look at $\sqrt{8}$, the number is a product of a perfect square and a prime number. Can you guess the two numbers?

The surd $\sqrt{8} = \sqrt{4 \times 2}$

$\sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2}$

$2 \times \sqrt{2}$

$2\sqrt{2}$

So, to simplify a surd into basic form, look for two products of which one must be a perfect square.

Now let us simplify each of the following surds in small groups or individually.

Example 1

Simplify the following

i. $\sqrt{12}$

ii. $\sqrt{R18}$

iii. $\sqrt{R45}$

iv. $\sqrt{108}$

Solution

i. Finding two numbers that we can multiply to get 12, of which one must be a perfect square, the two numbers will be 4 and 3

$\sqrt{12} = \sqrt{4 \times 3}$

$\sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3}$

$= 2 \times \sqrt{3}$

$= 2\sqrt{3}$

ii. $\sqrt{18} = \sqrt{9 \times 2}$

$\sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$

$= 3 \times \sqrt{2}$

$= 3\sqrt{2}$

iii. $\sqrt{45} = \sqrt{9 \times 5}$

$\sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5}$

$= 3 \times \sqrt{5}$

$= 3\sqrt{5}$

iv. $\sqrt{108} = \sqrt{36 \times 3}$

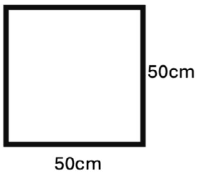
$\sqrt{36 \times 3} = \sqrt{36} \times \sqrt{3}$

$= 6 \times \sqrt{3}$

$= 6\sqrt{3}$

Example 2

A square has an area of 50cm^2. What is the exact length in a simplified form of one side of the square?



Area of a square is (x^2)

For one side $\sqrt{\mathbf{50}} = \sqrt{\mathbf{25 \times 2}} = \mathbf{5}\sqrt{\mathbf{2}}\ cm$

Addition and Subtraction of Surds

Surds are added and subtracted in the same way as algebraic expressions. Just as we add or subtract algebraic expressions by grouping like terms, we do the same for surds i.e. we add and subtract surds by grouping like terms. For example, if we have $2\sqrt{2} + 5\sqrt{2}$, they are alike because they have the same root, we add to get $(2 + 5)\sqrt{2} = 7\sqrt{2}$, likewise, $2\sqrt{2} - 5\sqrt{2}$, $= (2 - 5)\sqrt{2} = -3\sqrt{2}$. We can also simplify surds by writing or breaking some surds down into basic form before adding or subtracting.

Now simplify the following expressions

Example 1

$\sqrt{24} + \sqrt{96} - \sqrt{600}$

Solution

$= \sqrt{4 \times 6} + \sqrt{16 \times 6} - \sqrt{100 \times 6}$ [write surd as product of perfect square and another number]

$= \sqrt{4} \times \sqrt{6} + \sqrt{16} \times \sqrt{6} - \sqrt{100} \times \sqrt{6}$

$= 2\sqrt{6} + 4\sqrt{6} - 10\sqrt{6}$ [simplify perfect squares]

$= (2 + 4 - 10)\sqrt{6}$ [add surds]

$= -4\sqrt{6}$

Example 2

$\sqrt{147} + \sqrt{75} - \sqrt{9}$

Solution

$\sqrt{49 \times 3} + \sqrt{25 \times 3} - 3$

$= \sqrt{49} \times \sqrt{3} + \sqrt{25} \times \sqrt{3} - 3$

$= 7\sqrt{3} + 5\sqrt{3} - 3$

$= (7 + 5)\sqrt{3} - 3 = 12\sqrt{3} - 3$

Example 3

Simplify

$4\sqrt{2} + 5\sqrt{2} - 7\sqrt{2}$

Solution

$(4 + 5 - 7)\sqrt{2}$

$= 2\sqrt{2}$

Example 4

Simplify

$10\sqrt{2} + 12\sqrt{3} - 7\sqrt{2} + 13\sqrt{3}$

Solution

$10\sqrt{2} - 7\sqrt{2} + 12\sqrt{3} + 13\sqrt{3}$

$= 3\sqrt{2} + 25\sqrt{3}$

Example 5

Simplify

$10\sqrt{12} + 12\sqrt{32} - 7\sqrt{27} + 13\sqrt{50}$

Solution

$10\sqrt{4 \times 3} + 12\sqrt{16 \times 2} - 7\sqrt{9 \times 3} + 13\sqrt{25 \times 2}$

$= 10\sqrt{4} \times \sqrt{3} + 12\sqrt{16} \times \sqrt{2} - 7\sqrt{9} \times \sqrt{3} + 13\sqrt{25} \times \sqrt{2}$

$= 10 \times 2 \times \sqrt{3} + 12 \times 4 \times \sqrt{2} - 7 \times 3 \times \sqrt{3} + 13 \times 5 \times \sqrt{2}$

$= 20\sqrt{3} + 48\sqrt{2} - 21\sqrt{3} + 65\sqrt{2}$

$= 20\sqrt{3} - 21\sqrt{3} + 48\sqrt{2} + 65\sqrt{2}$

$= -\sqrt{3} + 113\sqrt{2}$

Multiplication of surds

We multiply surds in same way as algebra including the distributive property.

For example, in algebra, $2x \times 4y = 2 \times 4 \times x \times y = 8xy$. If we exchange x and y for surds, this expression can be written in surd form as $2\sqrt{2} \times 4\sqrt{3} = 2 \times 4 \times \sqrt{2} \times \sqrt{3} = 8\sqrt{6}$

Example 1

Simplify the following

i. $5\sqrt{3} \times 13\sqrt{3}$

ii. $10\sqrt{11}\left(3\sqrt{2} - 2\sqrt{11}\right)$

iii. $(2 + 3\sqrt{3})(4 - 5\sqrt{2})$

iv. $(-6 + 2\sqrt{5})(7 - 5\sqrt{5})$

v. $\left(3 - 4\sqrt{3}\right)^2$

vi. $4\sqrt{x}(5\sqrt{x} - 10\sqrt{y})$

vii. $\left(4 + 2\sqrt{3}\right)^2 - \left(4 - 2\sqrt{3}\right)^2$

Solution

i. $5 \times 13\sqrt{3 \times 3}$

$= 5 \times 13 \times 3$

$= 195$

ii. $10\sqrt{11}\left(3\sqrt{2} - 2\sqrt{11}\right)$

$= 10\sqrt{11} \times 3\sqrt{2} - 10\sqrt{11} \times 2\sqrt{11}\big)$

$= 10 \times 3 \times \sqrt{11 \times 2} - 10 \times 2\sqrt{11 \times 11}$

$= 30\sqrt{22} - 10 \times 2 \times 11 = 30\sqrt{22} - 10 \times 2 \times 11$

$= 30\sqrt{22} - 220$

iii. $(2 + 3\sqrt{3})(4 - 5\sqrt{2})$

$= 2(4 - 5\sqrt{2}) + 3\sqrt{3}\left(4 - 5\sqrt{2}\right)$

$= 8 - 10\sqrt{2} + 12\sqrt{3} - 15\sqrt{6}$

iv. $(-6 + 2\sqrt{5})(7 - 5\sqrt{5})$

$= -6(7 - 5\sqrt{5}) + 2\sqrt{5}(7 - 5\sqrt{5})$

$= -42 + 30\sqrt{5} + 14\sqrt{5} - 50$

$= -42 - 50 + 30\sqrt{5} + 14\sqrt{5}$

$= -92 + 44\sqrt{5}$

v. $(3 - 4\sqrt{3})^2$

$= (3 - 4\sqrt{3})(3 - 4\sqrt{3})$

$= 3(3 - 4\sqrt{3}) - 4\sqrt{3}(3 - 4\sqrt{3})$

$= 9 - 12\sqrt{3} - 12\sqrt{3} + 16 \times 3$

$= 9 + 48 - 12\sqrt{3} - 12\sqrt{3}$

$= 57 - 24\sqrt{3}$

vi. $4\sqrt{x}(5\sqrt{x} - 10\sqrt{y})$

$= 4 \times 5 \times x - 4 \times 10 \times \sqrt{x} \times \sqrt{y}$

$= 20\sqrt{x} - 40\sqrt{xy}$

vii. $(4 + 2\sqrt{3})^2 - (4 - 2\sqrt{3})^2$

We can apply the difference between two squares $a^2 - b^2 = (a + b)(a - b)$ to solve this question.

$= (4 + 2\sqrt{3} + 4 - 2\sqrt{3})[(4 + 2\sqrt{3} - (4 - 2\sqrt{3})]$

$= (4 + 4)(4 - 4 + 2\sqrt{3} + 2\sqrt{3})$

$= (8)(4\sqrt{3})$

$= 32\sqrt{3}$

Alternatively, we can expand the brackets and simplify.

$= (4 + 2\sqrt{3})(4 + 2\sqrt{3}) - [(4 - 2\sqrt{3})(4 - 2\sqrt{3})]$

$= (16 + 8\sqrt{3} + 8\sqrt{3} + 12) - [(16 - 8\sqrt{3} - 8\sqrt{3} + 12)]$

$= (28 + 16\sqrt{3}) - [(28 - 16\sqrt{3})]$

$= 28 + 16\sqrt{3} - 28 + 16\sqrt{3}$

$= 32\sqrt{3}$

Conjugate of surds

Conjugate surds are like mirror images in the world of surds. They are formed by simply changing the sign of the radical part in a surd expression. For example, the conjugate of $\sqrt{2} + 3$ is $\sqrt{2} - 3$. When you multiply a surd by its conjugate, the result is a rational expression due to the difference of squares property, making them helpful for simplifying expressions containing surds.

The table below lists various surds and their conjugates.

Surd	**Conjugate**	**Example** **Surd**	**Conjugate**
$\sqrt{a}$	$\sqrt{a}$	$\sqrt{2}$	$\sqrt{2}$
$a\sqrt{b}$	$\sqrt{b}$	$2\sqrt{3}$	$\sqrt{3}$
$a+\sqrt{b}$	$a-\sqrt{b}$	$2+\sqrt{3}$	$2-\sqrt{3}$
$a-\sqrt{b}$	$a+\sqrt{b}$	$2-\sqrt{3}$	$2+\sqrt{3}$
$a+b\sqrt{c}$	$a-b\sqrt{c}$	$2+3\sqrt{5}$	$2-3\sqrt{5}$
$a-b\sqrt{c}$	$a+b\sqrt{c}$	$2-3\sqrt{5}$	$2+3\sqrt{5}$
$\sqrt{a}+\sqrt{b}$	$\sqrt{a}-\sqrt{b}$	$\sqrt{2}+\sqrt{3}$	$\sqrt{2}-\sqrt{3}$
$\sqrt{a}-\sqrt{b}$	$\sqrt{a}+\sqrt{b}$	$\sqrt{2}-\sqrt{3}$	$\sqrt{2}+\sqrt{3}$

Example 1

Write down the conjugate surd of the following

i. $\sqrt{11}$

ii. $2+3\sqrt{3}$

iii. $\sqrt{3}+4\sqrt{2}$

iv. $\sqrt{6}-\sqrt{5}$

Solution

i. $\sqrt{11}$

ii. $2-3\sqrt{3}$

iii. $\sqrt{3}-4\sqrt{2}$

iv. $\sqrt{6}+\sqrt{5}$

It must be noted that the product of a surd (irrational) and its conjugate is a rational number or expression, i.e. the expression is now 'rationalised' as it no longer contains a surd.

RATIONALISATION OF SURDS

Rationalising the denominator of surds means making the denominator of a fraction a rational number by multiplying it by its conjugate surd. If the denominator is a single surd, the conjugate is the same surd. If the denominator is a binomial expression with a surd, the conjugate is the same expression with the opposite sign in the middle. The numerator of the fraction is also multiplied by the same conjugate. The answer is then simplified.

Given the expression $\frac{2}{\sqrt{5}}$, simplifying this expression would make it easier to work with if the denominator was a rational number. Using the idea of equivalent fractions, we can multiply both the numerator and denominator by the conjugate denominator to obtain a 'rationalised denominator' fraction.

$$\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5},$$

We observe from our discussion on conjugate surds that the conjugate of $\sqrt{5}$ is $\sqrt{5}$

To rationalise the denominator of an expression, we multiply both the numerator and the denominator by the conjugate of the surd.

Example 1

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$$

$$= \frac{a\sqrt{b}}{b}$$

Also,

$$\frac{1}{3-\sqrt{5}} = \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$= \frac{3+\sqrt{R5}}{(3-\sqrt{R5})(3+\sqrt{R5})}$$

$$= \frac{3+\sqrt{R5}}{9-5} = \frac{3+\sqrt{R5}}{4}$$

Example 2

Rationalise each of the following surds.

a. $\frac{1}{\sqrt{5}}$ **b.** $\frac{3}{\sqrt{11}}$ **c.** $\frac{\sqrt{5}}{\sqrt{13}}$ **d.** $\frac{3\sqrt{2}}{\sqrt{3}}$

Solution

a. $\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{1}{5}\sqrt{5}$

b. $\frac{3}{\sqrt{11}} = \frac{3}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} = \frac{3\sqrt{11}}{11}$

c. $\frac{\sqrt{5}}{\sqrt{13}} = \frac{\sqrt{5}}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} = \frac{\sqrt{65}}{13}$

d. $\frac{3\sqrt{2}}{\sqrt{3}} = \frac{3\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{6}}{3}$

$= \sqrt{6}$

Example 3

A carpenter was to cut 90m of wood into $2\sqrt{3}$ m, how many pieces will she have? [leave your answer in surd form]

Solution

$\frac{90}{2\sqrt{3}}$

$= \frac{90 \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}}$

$= \frac{90 \times \sqrt{3}}{2 \times 3}$

$= \frac{90 \times \sqrt{3}}{6}$

$= 15\sqrt{3}$

Example 4

Rationalise the following

a. $\frac{1+\sqrt{3}}{2+\sqrt{3}}$ **b.** $\frac{4+\sqrt{2}}{3+2\sqrt{3}}$ **c.** $\frac{3-\sqrt{5}}{6-\sqrt{3}}$ **d.** $\frac{2-3\sqrt{3}}{5-5\sqrt{6}}$

Solution

a. $\frac{1+\sqrt{3}}{2+\sqrt{3}}$

$= \frac{(1+\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$

[multiplying both numerator and denominator by conjugate surd]

$= \frac{(2-\sqrt{3}+2\sqrt{3}-3)}{(2)^2-(\sqrt{3})^2}$ [Open the bracket]

$= \frac{-1+\sqrt{3}}{4-3}$ [simplify]

$= -1+\sqrt{3}$

b. $\frac{4+\sqrt{2}}{3+2\sqrt{3}}$

$=\frac{(4+\sqrt{2})(3-2\sqrt{3})}{(3+2\sqrt{3})(3-2\sqrt{3})}$

$=\frac{12+3\sqrt{2}-8\sqrt{3}-2\sqrt{6})}{(3)^2-(2\sqrt{3})^2}$

$=\frac{12+3\sqrt{2}-8\sqrt{3}-2\sqrt{6}}{9-12}$

$=\frac{12+3\sqrt{2}-8\sqrt{3}-2\sqrt{6}}{-3}$

$=\frac{12}{-3}+\frac{3\sqrt{R2}}{-3}-\frac{8\sqrt{R3}}{-3}-\frac{2\sqrt{R6}}{-3}$

$=-4-\sqrt{2}\,\frac{8\sqrt{3}}{3}+\frac{2\sqrt{R6}}{3}$

c. $\frac{3-\sqrt{5}}{6-\sqrt{3}}$

$=\frac{(3-\sqrt{5})(6+\sqrt{3})}{(6-\sqrt{3})(6+\sqrt{3})}$

$=\frac{18+3\sqrt{3}-6\sqrt{5}-\sqrt{15}}{(6)^2-(\sqrt{3})^2}$

$=\frac{18+3\sqrt{3}-6\sqrt{5}-\sqrt{15}}{36-3}$

$=\frac{18+3\sqrt{3}-6\sqrt{5}-\sqrt{15}}{33}$

$=\frac{18}{33}+\frac{3\sqrt{3}}{33}-\frac{\sqrt{15}}{33}$

$=\frac{6}{11}+\frac{\sqrt{3}}{11}-\frac{\sqrt{15}}{33}$

d. $\frac{2-3\sqrt{3}}{5-5\sqrt{6}}$

$=\frac{(2-3\sqrt{R3})(5+5\sqrt{R6})}{(5-5\sqrt{R6})(5+5\sqrt{R6})}$

$=\frac{10+10\sqrt{6}-15\sqrt{3}-15\sqrt{18})}{(5)^2-(5\sqrt{6})^2}$

$=\frac{10+10\sqrt{6}-15\sqrt{3}-15\sqrt{9\times 2}}{25-150}$

$=\frac{10+10\sqrt{6}-15\sqrt{3}-45\sqrt{2}}{-125}$

$=\frac{10}{-125}+\frac{10\sqrt{6}}{125}+\frac{15\sqrt{3}}{125}+\frac{45\sqrt{2}}{125}$

$=-\frac{2}{25}+\frac{2\sqrt{6}}{25}+\frac{3\sqrt{3}}{25}+\frac{9\sqrt{2}}{25}$

Extended Activities

Please have a look at finding the square root of a surd if you have time to spare this week. It is more fun maths!

Finding the square root of a surd

When two or more expressions involving surds are separated by an equal sign, they are said to be equal. Given two surds $a + \sqrt{b}$ *and* $c + \sqrt{d}$, if $a + \sqrt{b} = c + \sqrt{d}$ then $a = c$, $b = d$.

Note: $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$ and $(\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$

We use this concept of equality of two surds to help us determine the square root of surds expressions. Let us with the support of our small group members or teacher go through some examples.

Example 1

Simplify $\sqrt{(16 + 2\sqrt{55})}$

Solution

Let $\sqrt{16 + 2\sqrt{55}} = \pm(\sqrt{a} + \sqrt{b})$

Squaring both sides

$(\sqrt{(16 + 2\sqrt{55})})^2 = (\sqrt{a} + \sqrt{b})^2$

$16 + 2\sqrt{55} = a + 2\sqrt{(ab)} + b$

$16 + 2\sqrt{55} = a + b + 2\sqrt{(ab)}$

Comparing corresponding coefficients

$16 = a + b$ (1)

$2\sqrt{55} = 2\sqrt{(ab)}$

$55 = ab$ (2)

From equation (1)

$a = 16 - b$ (3)

Substitute equation (3) into equation (2)

$(16 - b)\, b = 55$

$16b - b^2 = 55$

$b^2 - 16b + 55 = 0$

$(b-5)(b-11) = 0$

b = 5 or b = 11

when b = 5, a = 16-5 = 11

when b = 11, a = 16-11= 5

$\therefore \sqrt{(16+2\sqrt{55})} = (\sqrt{5}+\sqrt{11})$

It will be necessary to square the results to see if we will get the initial surd.

$(\sqrt{5}+\sqrt{11})^2 = (\sqrt{5})^2 + 2\sqrt{5}\,.\sqrt{11} + (\sqrt{11})^2$

$= 5 + 2\sqrt{55} + 11$

$= 16 + 2\sqrt{55}$

It is now verified.

Example 2

Calculate the square root of $18 - 12\sqrt{2}$

Solution

Let $\sqrt{(18-12\sqrt{2})} = \pm(\sqrt{a}-\sqrt{b})$

Squaring both sides

$(\sqrt{(18-12\sqrt{2})})^2 = (\sqrt{a}-\sqrt{b})^2$

$18 - 12\sqrt{2} = a - 2(\sqrt{ab}) + b$

$18 - 12\sqrt{2} = a + b - 2\sqrt{(ab)}$

Comparing corresponding coefficients

$18 = a + b$ (1)

$-12\sqrt{2} = -2\sqrt{(ab)}$

$6\sqrt{2} = \sqrt{(ab)}$

$72 = ab$ (2)

From equation (1)

$a = 18 - b$ (3)

Substitute equation (3) into equation (2)

$(18 - b)b = 72$

$18b - b^2 = 72$

$b^2 - 18b + 72 = 0$

$b^2 - 12b - 6b + 72 = 0$

$b(b - 12) - 6(b - 12) = 0$

$(b - 6)(b - 12) = 0$

$\therefore b = 6 \text{ or } b = 12$

When b = 6, a = 18 – 6 = 12

When b = 12, a = 18 –12 = 6

$\therefore \sqrt{(18 - 12\sqrt{2})} = \sqrt{12} - \sqrt{6}$

$= 2\sqrt{3} - \sqrt{6}$

Example 3

1. Find the positive square root of
$11 - 4\sqrt{6}$

Solution

The square root of $11 - 4\sqrt{6}$, must be of the form $\sqrt{a} - \sqrt{b}$, where $a \geq b$, as the root must be non-negative.

$\sqrt{11 - 4\sqrt{6}} = \sqrt{a} - \sqrt{b}$

$11 - 4\sqrt{6} = \left(\sqrt{a} - \sqrt{b}\right)^2$

$11 - 4\sqrt{6} = (a + b) - 2\sqrt{ab}$

$11 - 2\sqrt{4 \times 6} = (a + b) - 2\sqrt{ab}$

By comparing the left-hand side and the right-hand side

$11 = (a + b)$.................. ***1***

$24 = ab$...................... ***2***

From equation ***1*** $b = 11 - a$, substituting $b = 11 - a$ into equation **2**, we have

$a(11 - a) = 24$

$11a - a^2 = 24$

$a^2 - 11a + 24 = 0$

$(a - 8)(a - 3) = 0$

$a = 8 \; or \; a = 3$

Hence, the square root of $11 - 4\sqrt{6}$, is $\sqrt{8} - \sqrt{3} = 2\sqrt{2} - \sqrt{3}$.

Example 4

Find the square root of $23 + 4\sqrt{15}$

Solution

$23 + 4\sqrt{15}$, the square root of $23 + 4\sqrt{15}$ must be of the form $\sqrt{a} + \sqrt{b}$

$\sqrt{23 + 4\sqrt{15}} = \sqrt{a} + \sqrt{b}$

$23 + 4\sqrt{15} = \left(\sqrt{a} + \sqrt{b}\right)^2$

$23 + 2\sqrt{4 \times 15} = a + b + 2\sqrt{ab}$

$23 + 2\sqrt{60} = (a + b) - 2\sqrt{ab}$

Comparing the left-hand side and the right-hand side of the equation

$23 = (a + b)$.................. ***1***

$60 = ab$.......................***2***

From equation ***1*** $b = 23 - a$, substituting $b = 23 - a$ into equation ***2***

$a(23 - a) = 60$

$23a - a\ 2 = 60$

$a^2 - 23a + 60 = 0$

$(a - 3)(a - 20) = 0$

$a = 3 \; or \; a = 20$

$a = 3 \; and \; b = 20$ or $a = 2- \; and \; b = 3$

Hence, the square root of $23 + 4\sqrt{15}$, is $\sqrt{3} + \sqrt{20} = \sqrt{3} + 2\sqrt{5}$.

DEFINITION AND LAWS OF INDICES

We have reached a stage where we must know the difference between the expressions $4a$ *and* a^4.

$$4a = a + a + a + a$$

$$a^4 = a \times a \times a \times a$$

For a^4 which is read as **a** exponent 4 or **a** to the power of 4, **a** is the base and 4 is the power, or exponent.

Index is the power or exponent of a number or a variable. The plural for index is **indices.** For instance, in the expression, 3^4, 3 is called the **base** and 4 is called the **index, power** or **exponent** and it is read as ***"three exponent four"*** or ***"three to the power of 4"***. The concept of indices is widely applied in simplifying mathematical expressions, algebraic manipulation, writing scientific notation, differentiation and integration, geometry and trigonometry, compound interest calculation, etc.

From our previous lessons, we learnt that $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$. Likewise $a \times a \times a \times a = a^4$. In the example $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$, the 2 is called the base, (the number multiplying itself) and the 6 is the index or exponents (the number of times the base is multiplying).

Example 1

Write the following single numbers as exponents in their simplest forms.

i. 16

ii. 27

iii. 100

Solution

i. Least prime factor of 16 = {2}

$16 = 2 \times 2 \times 2 \times 2 = 2^4$

ii. Least prime factor of 27 = {3}

$27 = 3 \times 3 \times 3 = 3^3$

iii. Prime factors of 100 = {2, 5}

$100 = 10 \times 10 = 2 \times 5 \times 2 \times 5 = 2^2 \times 5^2$

Example 2

1. Identify the base and the exponent in following
   - a. 5^8
   - b. $\left(\frac{2}{3}\right)^{-3}$
   - c. x^m
   - d. $16^{\frac{1}{2}}$

Solution

	Base	Exponent
a.	5	8
b.	$\frac{2}{3}$	-3
c.	x	m
d.	16	$\frac{1}{2}$

Example 3

1. Write the following base and exponent form.
   - a. $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$
   - b. $\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
   - c. $y^2 \times y^2 \times y^2 \times y^2 \times y^2 \times y^2 \times y^2$

Solution

- a. 4^8
- b. $\left(\frac{3}{5}\right)^5$
- c. $(y^2)^7$

Verification of rules of indices

Consider $3^4 \times 3^5$, if we expand each of them and simplify, we will have $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$. If we write this in index form, the result will be 3^9. Is there any other way to get the result without the expansion? Yes, we can add the exponent since they are of the same base, thus, $3^4 \times 3^5 = 3^{4+5} = 3^9$.

This helps us to establish a rule that, when multiplying the powers of the same base, we simply add the indices (power or exponent)

$a^m \times a^n = a^{m+n}$ and by extension once the base (a) is equal $a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$

We can apply our knowledge on this rule to verify the other rules that follow.

ii) $a^m \div a^n = a^{m-n}$

Example:

$3^8 \div 3^6 = 3^{8-6} = 3^2$

iii) $(a^m)^n = a^{mn}$, and by extension, $(a^m \times b^m \times \ldots)^n = a^{mn} \times b^{mn} \times \ldots$

$(5^2)^7 = 5^{2\times 7} = 5^{14}$

vi) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$\left(\frac{3}{4}\right)^4 = \frac{3^4}{4^4}$

v) $\left(\frac{a \times b \times \ldots}{c \times d \times \ldots}\right)^m = \frac{a^m \times b^m \times \ldots}{c^m \times d^m \times \ldots}$

$\left(\frac{2 \times 3 \times 4}{5 \times 6 \times 7}\right)^3 = \frac{2^3 \times 3^3 \times 4^3}{5^3 \times 6^3 \times 7^3}$

vi) $(ab)^m = a^m b^m$

$(3 \times 4)^5 = 3^5 \times 4^5$

vii) $\left(\frac{a^m}{b^m}\right)^n = \frac{a^{mn}}{b^{mn}}$

$\left(\frac{2^3}{3^3}\right)^4 = \frac{2^{3\times 4}}{3^{3\times 4}} = \frac{2^{12}}{3^{12}}$

viii) $\left(\frac{a^m \times b^m \times \ldots}{c^m \times d^m \times \ldots}\right)^n = \frac{a^{mn} \times b^{mn} \times \ldots}{c^{mn} \times d^{mn} \times \ldots}$

ix) $a^{\frac{1}{n}} = \sqrt[n]{a}$

$3^{\frac{1}{2}} = \sqrt[2]{3}$

$y^{\frac{1}{2}} = \sqrt{y}$

Other rules are:

1. Negative exponent Rule, that is $a^{-m} = \frac{1}{a^m}$ by extension $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
2. Zero power rule, that is

   $a^0 = 1$, where $a \neq 0$

Example 4

Simplify $8^{\frac{2x}{3}}$

Solution:

$8^{\frac{2x}{3}} = (\sqrt[3]{8})^{2x} = 2^{2x} = 4^{x}$

Or

$2^{3(\frac{2x}{3})} = 2^{2x} = (2^2)^x = 4^x$

Example 5

Simplify $27^{\frac{1}{3}} \times \frac{1}{81}$

Solution

$(3^3)^{\frac{1}{3}} \times 81^{-1}$

$= (3^3)^{\frac{1}{3}} \times (3^4)^{-1}$

$= 3 \times 3^{-4}$

$= 3^{-3}$

Example 6

Simplify the following

a. $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

b. $(0.0125)^{\frac{1}{4}}$

c. $\frac{\sqrt{y} \times \sqrt[3]{y}}{y^3}$

d. $\frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{81^{\frac{1}{6}}}$

e. $\frac{(t+1)^{\frac{1}{3}} \times (t+1)^{\frac{3}{4}}}{(t+1)^{-\frac{1}{2}}}$

Solution

a. $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

$= \left(\frac{27}{64}\right)^{\frac{2}{3}}$

$= \left(\frac{3^3}{4^3}\right)^{\frac{2}{3}}$

$= \frac{3^{3\times\frac{2}{3}}}{4^{3\times\frac{2}{3}}}$

$= \frac{3^2}{4^2}$

$= \frac{9}{16}$

b. $(0.0625)^{\frac{1}{4}}$

$= (625 \times 10^{-4})^{\frac{1}{4}}$

$= (5^4 \times 10^{-4})^{\frac{1}{4}}$

$= 5^{4\times\frac{1}{4}} \times 10^{-4\times\frac{1}{4}}$

$= 5 \times 10^{-1} = \frac{5}{10} = \frac{1}{2}$

c. $\frac{\sqrt{y} \times \sqrt[3]{y}}{y^3}$

$= \frac{y^{\frac{1}{2}} \times y^{\frac{1}{3}}}{y^3}$

$= y^{\frac{1}{2}} \times y^{\frac{1}{3}} \times y^{-3}$

$= y^{\frac{1}{2} + \frac{1}{3} + (-3)}$

$= y^{-\frac{13}{6}}$

d. $\frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{81^{\frac{1}{6}}}$

$= \frac{(3\times 4)^{\frac{1}{3}} \times (2\times 3)^{\frac{1}{3}}}{(3^4)^{\frac{1}{6}}}$

$= \frac{3^{\frac{1}{3}} \times 4^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 3^{\frac{1}{3}}}{(3^4)^{\frac{1}{6}}}$

$= \frac{3^{\frac{1}{3}} \times (2^2)^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 3^{\frac{1}{3}}}{(3^4)^{\frac{1}{6}}}$

$= 3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{-\frac{4}{6}} \times 2^{\frac{2}{3}} \times 2^{\frac{1}{3}}$

$= 3^{\frac{1}{3}+\frac{1}{3}-\frac{4}{6}} \times 2^{\frac{2}{3}+\frac{1}{3}}$

$= 3^{\frac{1}{3}+\frac{1}{3}-\frac{4}{6}} \times 2^{\frac{2}{3}+\frac{1}{3}}$

$3^0 \times 2^1$

$= 1 \times 2 = 2$

e. $\frac{(t+1)^{\frac{1}{3}} \times (t+1)^{\frac{3}{4}}}{(t+1)^{-\frac{1}{2}}}$

$= \frac{(t+1)^{\frac{1}{3}+\frac{3}{4}}}{(t+1)^{-\frac{1}{2}}}$

$= (t+1)^{\frac{1}{3}+\frac{3}{4}} \times (t+1)^{\frac{1}{2}}$

$= (t+1)^{\frac{1}{3}+\frac{3}{4}+\frac{1}{2}}$

$= (t+1)^{\frac{1}{3}+\frac{3}{4}+\frac{1}{2}}$

$= (t+1)^{\frac{19}{12}}$

SIMPLIFICATION OF EXPRESSIONS (SURDS AND INDICES)

As already explained, surds are the root values that cannot be written as whole numbers. More so, indices are the exponents of a value. Thus, given 2^5, 5 is the index while 2 is the base. Taking a square root is the inverse process of squaring.

Solving indicial problems involving surds

Example 7

If $a = 3 - \sqrt{3}$, show that $a^2 + \frac{36}{a^2} = 24$

Solution

Substituting $a = 3 - \sqrt{3}$ into $a^2 + \frac{36}{a^2}$

$$\Rightarrow a^2 + \frac{36}{a^2} = (3 - \sqrt{3})^2 + \frac{36}{(3 - \sqrt{3})^2}$$

$$= 9 - 6\sqrt{3} + 3 + \frac{36}{9 - 6\sqrt{3} + 3}$$

$$=12 - 6\sqrt{3} + \frac{36}{12 - 6\sqrt{3}}$$

$$=12 - 6\sqrt{3} + \frac{6}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$=12 - 6\sqrt{3} + 1\frac{12 + 6\sqrt{3}}{4 - 3}$$

$$= 24$$

Example 8

If m= $2 + \sqrt{2}$ and n = $2 - \sqrt{2}$, find the value of $\frac{1}{m^2} + n^2$

Solution

$$\frac{1}{(2 + \sqrt{2})^2} + (2 - \sqrt{2})^2$$

$$= \frac{1}{(2 + \sqrt{2})(2 + \sqrt{2})} + \left(2 - \sqrt{2}\right)\left(2 - \sqrt{2}\right)$$

$$= \frac{1}{(4 + 4\sqrt{2} + 2)} + \left(4 - 4\sqrt{2} + 2\right)$$

$$= \frac{1}{(6 + 4\sqrt{2})} + \left(2 - 4\sqrt{2}\right)$$

$$= \frac{1(6 - 4\sqrt{2})}{(6 + 4\sqrt{2})(6 - 4\sqrt{2})} + \left(2 - 4\sqrt{2}\right)$$

$$= \frac{1(6 - 4\sqrt{2})}{(6 + 4\sqrt{2})(6 - 4\sqrt{2})} + \left(2 - 4\sqrt{2}\right)$$

$$= \frac{1(6 - 4\sqrt{2})}{(36 - 32)} + \left(2 - 4\sqrt{2}\right)$$

$$= \frac{1(6 - 4\sqrt{2})}{4} + \frac{(2 - 4\sqrt{2})}{1}$$

$$= \frac{6 - 4\sqrt{2} + 4(2 - 4\sqrt{2})}{4}$$

$$= \frac{6 - 4\sqrt{2} + 8 - 16\sqrt{2})}{4}$$

$$= \frac{14 - 20\sqrt{2}}{4}$$

$$= \frac{7 - 10\sqrt{2}}{2}$$

INDICIAL EQUATIONS

Areas under indicial equations to be explored include;

1. Solving simple indicial or exponential equations
2. Solving simultaneous equations involving exponents or indices
3. Application of exponential indicial equations (such as growths and/or decays)

Example 9

Solve the equation $2^x = 16$

Solution

We know $2^4 = 16$ [write both sides with the same base]

$2^x = 2^4$ [equate their exponents]

$x = 4$

Example 10

Find the value of $25^{x-1} = 5^{x+2}$

Solution

$(5^2)^{x-1} = 5^{x+2}$

$5^{2(x-1)} = 5^{x+2}$

$2x - 2 = x + 2$

$2x - x = 2 + 2$

$x = 4$

Example 11

If $(5x + 4)^3 = 8$, find the value of x.

Solution

$(5x + 4)^3 = 2^3$ (as exponents are equal we can equate the bases)

$5x + 4 = 2$

$5x = 2 - 4$

$5x = -2$

$x = \frac{-2}{5}$

Example 12

If $5^{\frac{x}{3}} = 0.04$, find the value of x

$5^{\frac{x}{3}} = \frac{4}{100}$

$5^{\frac{x}{3}} = \frac{1}{25}$

$5^{\frac{x}{3}} = \frac{1}{5^2}$

$5^{\frac{x}{3}} = 5^{-2}$

$\frac{x}{3} = -2$

$x = -6$

Example 13

Given that $y = 2x$ and $3^{x+y} = 27$, Find x

Solution

Substitute $y = 2x$ into $3^{x+y} = 27$

Giving us $3^{x+2x} = 3^3$,

equating exponents, $x + 2x = 3$

$3x = 3$

$\therefore x = 1$

Example 14

Find the values of x and y in the equations $2^{x-y} = 8$ and $2^{3x-y} = 128$

Solution

$2^{x-y} = 2^3$

$2^{3x-y} = 2^7$

$\boldsymbol{x} - \boldsymbol{y} = 3$ *equation* 1

$3\boldsymbol{x} - \boldsymbol{y} = 7.$ *equation* 2

$2\boldsymbol{x} = 4.$ *equation* 2 – *equation* 1

$\boldsymbol{x} = 2$

$2 - \boldsymbol{y} = 3.$ ***substitute x*** $= 2$ ***into equation*** 1 ***to find y***

$\boldsymbol{y} = -1$

Example 15

Solve for x and y given that $3^{(x+1)} = 27$ *and* $4^{(y-2)} = 16$

Solution

$3^{(x+1)} = 3^3$

$x + 1 = 3$

$x = 3 - 1$

$x = 2$

$4^{(y-2)} = 16$

$2^{2(y-2)} = 2^4$

$2(y - 2) = 4$

$2y - 4 = 4$

$2y = 8$

$y = 4$

Therefore $x = 2, y = 4$

Example 16

Solve the simultaneous equations

$9^{2a+b} = \frac{1}{729}$

$3^a(9^b) = 27$

Solution

$9^{2a+b} = 1/729$

$3^{2(2a+b)} = \frac{1}{3^6}$

$3^{4a+2b} = 3^{-6}$

$4a + 2b = -6$ ……………..$(eqn1)$

$3^a(9^b) = 27$

$3^a(3^{2(b)}) = 3^3$

$3^a \times 3^{2b} = 3^3$

$3^{a+2b} = 3^3$

$a + 2b = 3$ …………………..$(eqn2)$

From $eqn2$, $a = 3 - 2b$

Substituting $a = 3 - 2b$ into $(eqn1)$

$\Rightarrow 4(3 - 2b) + 2b = -6$

$12 - 8b + 2b = -6$

$-6b = -18$ dividing both sides by -6

$b = 3$

From $a = 3 - 2b$

From $a = 3 - 2(3) = 3 - 6 = -3$

Thus, $a = -3, b = 3$

Example 17

A bacteria culture doubles every hour. If there are initially 100 bacteria, how many bacteria will be present after 5 hours?

Solution

Since the culture doubles, the growth factor is 2 (each bacteria becomes 2 after an hour).

We know the initial quantity (100) and want to find the final quantity (let it be Q) after 5 hours (represented by exponent t). The general formula for exponential growth is:

$Q = A\,(growth\ factor)^t$

$Q = 100 \times (2)^5$

$Q = 100 * 32 = 3200$

Therefore, there will be 3200 bacteria after 5 hours.

Example 18

Suppose that a culture initially contains 1000 bacteria and that this number doubles each hour. Write a general formula for the number of bacteria N present after *t* hours

Solution

After one hour, there are 1000×2 bacteria

After two hours, there are $1000 \times 2 \times 2 = 1000 \times 2^2$ bacteria

After three hours, there are $1000 \times 2^2 \times 2 = 1000 \times 2^3$ bacteria and so on.

Following the pattern, if there are bacteria after t hours, then

$N = 1000 \times 2^t$ bacteria

DEFINITION AND RELATIONSHIP BETWEEN INDICES AND LOGARITHMS AND LOGARITHMIC EQUATIONS

Definition: If $N = a^x$, in indices, we observed that **a** is the base, ***x*** is the index or power, and N is the result. For example, $100 = 10^2$

The logarithmic function is defined as $\log_a(N) = x$, ($\log_a N$ is read "logarithm of N to the base a) where N is the number such that $N = a^x$ with N>0 and **a** being positive constant other than **1**

Relationship Between Indices and Logarithms to Solve Problems

To find the logarithm of a number a to the base b, that is, $\log_b(a)$, we ask the question, 'What power do I raise b, to obtain a?

Taking a logarithm is the inverse process of taking a power. Generally, if $a > 0$ and $x > 0$, then

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

Extended Activities

Let's go through the following activities to use the relationship between indices and logarithms applying the laws of logarithm

Example 19

Solve the equation: $\log_3 81$

Solution

The logarithm (log) to base 3 of 81 $((\log_3 81)$ means what is the exponent to which we have to raise 3 to get 81.

$$81 = 3^x$$
$$3^4 = 3^x$$
$$x = 4$$

Example 20

Evaluate $\log_{125}25$

Solution

By writing, $\log_{125}25 = x$; we have

$125^x = 25$

which gives $5^{3x} = 5^2$;

Equating indices, $3x = 2$, so $x = \frac{2}{3}$

Example 21

Solve for y if $\log_2 32 = y + 1$

Solution

$32 = 2^{y+1}$

$2^5 = 2^{y+1}$

$5 = y + 1$

$y = 5 - 1$

$y = 4$

Example 22

Evaluate

a) $\log_2 64$ **b)** $\log_{10}1000$

c) $\log_5 125$ **d)** $\log_{0.1}10$

Solution

a) Let $\log_2 64 = x$

$64 = 2^x$

$2^6 = 2^x$

$\therefore x = 6$

b) Let $\log_{10}1000 = x$

$1000 = 10^x$

$10^3 = 10^x$

$x = 3$

c) Let $\log_5 125 = x$

$125 = 5^x$

$5^3 = 5^x$

$x = 3$

d) Let $\log_{0.1} 10 = x$

$10 = 0.1^x$

$10^1 = 10^{-x}$

$x = -1$

The Laws of Logarithm

1. 1^{st} Law of Logarithm

$\log_a xy = \log_a x + \log_a y$

2. 2^{nd} Law of Logarithm

$\log_a(\frac{x}{y}) = \log_a x - \log_a y$

3. 3^{rd} Law of Logarithm

$\log_a x^n = n\log_a x$

NB: the laws are numbered for convenience, and it is not that they should necessarily be in a certain order. Also, log with no given base can be assumed to be log to the base 10.

Example 23

Let us use the laws of logarithm to express in terms of $\log a, \log b$ *and* $\log c$ each of the following;

i. $\log\frac{a}{c}$ **ii.** $\log\frac{1}{b}$

iii. $\log a^2 b^{\frac{3}{2}}$ **iv.** $\log\frac{1}{100b^2}$

Solution

i. $\log\frac{a}{c} = \log a - \log c$

ii. $\log\frac{1}{b} = \log 1 - \log b$

$= -\log b$

iii. $\log a^2 b^{\frac{3}{2}} = \log a^2 + \log b^{\frac{3}{2}}$

$= 2\log a + \frac{3}{2}\log b$

iv. $\log\frac{1}{100b^2} = \log 1 - \log 100b^2$

$= 0 - \log 100 - \log b^2$

$= -\log 10^2 - 2\log b$

$= -2\log 10 - 2\log b$ Note, log with no given base can be taken to be log to the base 10, ie $log_{10} = log$

$= -2 - 2\log b$

Extended Activities

1. In our small groups, or individually, let us also use the laws of logarithm to express each of the following as a single logarithm;

i. $\log 2 + \log 3$

ii. $\log 18 - \log 9$

iii. $3\log 2 + 2\log 3 - 2\log 6$

iv. $2 + 3\log a$

Solution

i. $\log 2 + \log 3 = \log 2 \times 3$

$= \log 6$

ii. $\log 18 - \log 9 = \log\frac{18}{9}$

$= \log 2$

iii. $3\log 2 + 2\log 3 - 2\log 6 = \log 2^3 + \log 3^2 - \log 6^2$

$= \log 8 + \log 9 - \log 36$

$= \log\frac{8 \times 9}{36}$

$= \log 2$

iv. $2 + 3\log a = 2\log 10 + 3\log a$

$= \log 10^2 + \log a^3$

$= \log 100 + \log a^3$

$= \log 100a^3$

We are now going to consider exponential equations where the bases are different.

2. Solve the following equations;

i. $2^x = 5$

ii. $3^x = 2$

iii. $3^{4x} = 4$

iv. $2^x \times 2^{x+1} = 10$

Solution

i. $2^x = 5$

Taking logarithm to base 10 on both sides

$\log_{10} 2^x = \log_{10} 5$

$x \log_{10} 2 = \log_{10} 5$

$x = \frac{\log_{10} 5}{\log_{10} 2}$

$x = \frac{0.69897}{0.30103}$ using calculator

$x = 2.3219$

ii. $3^x = 2$

Taking logarithm to base 10 on both sides

$\log_{10} 3^x = \log_{10} 2$

$x\log_{10} 3 = \log_{10} 2$

$x = \frac{\log_{10} 2}{\log_{10} 3}$

$x = \frac{0.30103}{0.47712}$ using calculator

$x = 0.6309$

iii. $3^{4x} = 4$

Taking logarithm to base 10 on both sides

$\log_{10} 3^{4x} = \log_{10} 4$

$4x\log_{10} 3 = \log_{10} 4$

$x = \frac{\log_{10} 4}{4\log_{10} 3}$

$x = \frac{0.60206}{1.90849}$ using calculator

$x = 0.3157$

iv. $2^x \times 2^{x+1} = 10$

$2^{x+x+1} = 10$

$2^{2x+1} = 10$

Taking logarithm to base 10 on both sides

$\log_{10} 2^{2x+1} = \log_{10} 10$

$(2x + 1)\log_{10} 2 = 1$

$2x\log_{10} 2 + \log_{10} 2 = 1$

$2x\log_{10} 2 = 1 - \log_{10} 2$

$x = \frac{1 - \log_{10} 2}{2\log_{10} 2}$

$x = \frac{0.69870}{0.60206}$ using full calculator display

$x = 1.1610$

Change of base of logarithm

We can change the base of any logarithm to any base. We cannot directly calculate $log_2(7)$ *or* $log_2(10)$ without a calculator.

In practice, you would use a calculator with a log function and approximate the answer. To help us calculate $log_a(b)$ without using calculator, go through the activity that follow immediately.

Extended Activities

1. With the assistance of your teacher or fellow learner, go through the following work which illustrates how to change base.

Let us consider $y = \log_a b$

$\Rightarrow b = a^y$ [converting logarithm to indices]

Taking logarithm to base c on both sides

$\log_c b = \log_c a^y$

$\log_c b = y\log_c a$

$\therefore y = \frac{\log_c b}{\log_c a}$

Hence $\log_a = \frac{\log_c b}{\log_c a}$

2. Let us use this relationship to change the base of the following to $\log_{10}$;

i. $\log_4 15$

ii. $\log_{2.5} 8$

iii. $\log_2 25 \times \log_5 8$

Solution

i. $\log_4 15 = \frac{\log_{10} 15}{\log_{10} 4}$

$= \frac{1.17609}{0.60206}$ using calculator

$= 1.9534$

ii. $\log_{2.5} 8 = \frac{\log_{10} 8}{\log_{10} 2.5}$

$= \frac{0.90309}{0.39794}$ using calculator

$= 2.2694$

iii. $\log_2 25 \times \log_5 8 = \frac{\log_{10} 25}{\log_{10} 2} \times \frac{\log_{10} 8}{\log_{10} 5}$

$= \frac{\log_{10} 5^2}{\log_{10} 2} \times \frac{\log_{10} 2^3}{\log_{10} 5}$

$= \frac{2\log_{10} 5}{\log_{10} 2} \times \frac{3\log_{10} 2}{\log_{10} 5}$

$= 2 \times 3$

$= 6$

REVIEW QUESTIONS

Review Questions 2.1

1. Put the following surds in their simplest form where possible. For those that cannot be further simplified, state the reasons why.
   i. $\sqrt{5} + \sqrt{7}$
   ii. $3\sqrt{2} + 5\sqrt{2}$
   iii. $\sqrt{7} - \sqrt{5}$
   iv. $3\sqrt{2} - 5\sqrt{2}$
   v. $3\sqrt{2} \times 5\sqrt{2}$
   vi. $\sqrt{15} \div \sqrt{5}$
2. Simplify the following surd expression:
   $\sqrt{12} + \sqrt{27}$
3. Solve for x:
   $2\sqrt{3x+5} = 4\sqrt{x+1}$
4. Given that $\sqrt{a} + \sqrt{b} = 7$ and $\sqrt{a} - \sqrt{b} = 1$, find the value of a and b.
5. Rationalise the denominator of the fraction: $\frac{1}{\sqrt{5}}$
6. Simplify and rationalise the expression: $\frac{2+\sqrt{6}}{\sqrt{2}}$
7. Rationalise the denominator of the expression: $\frac{(\sqrt{3}+\sqrt{7})}{(\sqrt{7}+3)}$
8. Calculate $\sqrt{0.9}$
9. Which of the following statements is/are true?
   i. $2\sqrt{3} > 3\sqrt{2}$,
   ii. $4\sqrt{2} > 2\sqrt{8}$
10. What is the conjugate of
   i. $1 + \sqrt{3}$
   ii. $8\sqrt{5} + 6$

11. What is the square root of $(10+\sqrt{25})(12-\sqrt{49})$?
12. If $(3+2\sqrt{5})^2 = 29 + k\sqrt{5}$, find the value of k
13. Find the value of y, if $\sqrt{64} - 3\sqrt{64} = -4\sqrt{y}$, where $y > 0$,
14. Simplify $\frac{\sqrt{48}+2\sqrt{27}}{\sqrt{12}}$, given your answer as an integer
15. Express $\frac{1-5\sqrt{5}}{3+\sqrt{5}}$ in the form $m - n\sqrt{5}$; $m, n \in Z$
16. Show that $\frac{\sqrt{75}+\sqrt{27}}{\sqrt{3}}$ is an integer and find its value
17. Show that $\frac{x-25}{\sqrt{x}+5} = \sqrt{x} - 5$
18. Rationalise the denominator of $\frac{8}{1+2\sqrt{3}}$
19. Simplify $\frac{8}{1+2\sqrt{3}} \times \frac{8}{1-2\sqrt{3}}$

Review Questions 2.2

1. Simplify and write the answer with positive indices: $\frac{(x^3)^4}{(x^5)^2}$
2. Simplify and write the answer with positive indices: $\frac{(a^2b^4)}{(a^5b^3)}$
3. Simplify $2^2 \times 4^{-4} \div 16^{-3}$
4. Find the value of x in the equation $2^3 \times 3^4 \times 72 = 6^x$
5. If $2^n = 32$ find the value of n
6. Solve $3^{3-x} = 27^{x-1.}$
7. Show that
   i. $32^{-\frac{2}{5}} = \frac{1}{4}$
   ii. $(2x^{-\frac{2}{5}})^5 = \frac{32}{x^2}$
8. Find the value of x given $625^{0.17} \times 625^{0.08} = 25^x \times 25^{-\frac{3}{2}}$
9. If $\left(\frac{3}{5}\right)^x = \left(\frac{81}{625}\right)$, then what is the value of x^x
10. Given that $\left(\frac{7}{5}\right)^{4x} \times \left(\frac{7}{5}\right)^{3x-1} = \left(\frac{7}{5}\right)^8$, find the value of x that satisfies this equation.
11. Find the value of a if $5^{3a-1} \times 125 = 25^{2a-1}$

12. Given that $y = 5x$ and $3^{x+y} = 81$, Find x

13. Solve the simultaneous equations $9^{2a+b} = 2187$ and $3^a \times 9^b = 3$

14. Simplify $8^{\frac{2}{3}}$

15. Simplify $\log_b x^2 + \log_b x^3 - \log_b x^4$

16. Calculate $\log_7 8$ to four decimal places

17. If $\sqrt{5^x} = 25$, find the value of x

18. It is given that $x = \sqrt{3}$ and $y = \sqrt{12}$.

Find in the simplest form, the value of

i. xy

ii. $\frac{y}{x}$

iii. $(x + y)^2$

19. Given $\log_7 2 = \alpha$, $\log_7 3 = \beta$ and $\log_7 5 = \gamma$, express in terms of α, β, and γ;

i. $\log_7 6$

ii. $\log_7 \frac{15}{2}$

ANSWERS TO REVIEW QUESTIONS

Answers to Review Questions 2.1

1. i. Cannot be simplified further as they are unlike terms.

   ii. $8\sqrt{2}$

   iii. Cannot be simplified further as they are unlike terms.

   iv. $-2\sqrt{2}$

   v. 30

   vi. $\sqrt{3}$

2. $5\sqrt{3}$

3. $x=1$

4. $a = 16, b = 9$

5. $\frac{\sqrt{5}}{5}$

6. $\sqrt{3} + \sqrt{2}$

7. $\frac{\sqrt{21} - 3\sqrt{3} + 7 - 3\sqrt{7}}{-2}$

8. $\frac{3\sqrt{10}}{10}$

9. i. false

   ii. false

10.

    i. $1 - \sqrt{3}$

    ii. $8\sqrt{5} - 6$

11. $5\sqrt{3}$

12. k=12

13. y = 16

    - 5
    - $7 - 4\sqrt{5}$
    - 8

- Work it to show that the expression is $\sqrt{x} - 5$
- $\frac{8 - 16\sqrt{3})}{-11}$
- $-\frac{64}{11}$

Answers to Review Questions 2.2

1. x^2

2. $\frac{b}{a^3}$

3. 2^6

4. $x = 6$

5. $n = 5$

6. $x = \frac{3}{2}$

7. **i.** Show that this is true

ii. Show that this is true

8. $x = 2$

9. 256

10. $\frac{9}{7}$

11. a = 4

12. $x = \frac{2}{3}$

13. $a = \frac{4}{3}$ $b = \frac{5}{6}$

14. 4

15. $\log_b x$

16. 1.0686

17. $x = 4$

18. **i.** 6

ii. 2

iii. 27

19. **i.** $\alpha + \beta$

ii. $\alpha + \beta - \gamma$

EXTENDED READING

Cambridge Additional Mathematics by Michael Haese, Sandra Haese, Mark Humphries, Chris Sangwin. Haese Mathematics (2014). Page(s) 101 – 128.

REFERENCES

1. Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952. Page(s) 34 - 50, 473 – 154.
2. Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740. Page(s) 17 – 20, 92 – 99.
3. Tuttuh-Adegun, M.R & Adegoke, D. G. New (2011) *Further Mathematics Project:* Bounty Press Limited, Ibadan. Page(s) 22 - 60, 32 – 47.

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