SECTION 3

SEQUENCES AND FUNCTIONS

MODELLING WITH ALGEBRA

Applications of Algebra

INTRODUCTION

Have you noticed how numbers such as counting, even, or odd numbers are arranged? What have you noticed? What about the regular increments or changes such as counting numbers, days of the week, or workers' monthly salaries? From these scenarios, you can conclude by saying there is a pattern or rule in which things or numbers follow and a relationship between these things. Sequences and functions are essential mathematical concepts that help us identify patterns, model relationships, and solve real-world problems. Sequences reveal ordered patterns, linear functions describe constant rates of change and polynomials allow us to represent and analyse complex scenarios. Learning these concepts equips us with tools to solve practical problems in areas such as science, finance, and engineering, making them highly valuable for both academic and everyday applications. This section comprises the definition of a sequence, number patterns, rules for finding terms of sequence, nth term of pattern, types of sequence, application of sequence, linear functions, quadratic functions, polynomial functions and rational functions.

At the end of this section, you will be able to:

- Recognise sequences in mathematics as an enumerated collection of objects in which repetitions are allowed and classify sequences into linear or exponential
- Find the nth term of linear and exponential sequences
- Identify and describe the relationship of patterns, recognise the difference between a relation and a function and write a function that describes a relationship between two quantities in real-life situations
- Use function notation to evaluate functions for inputs in their domain and outputs in the co-domain
- Determine the composite of two given functions
- Establish, describe and determine bijective functions and composite function

- Find the inverse of simple functions including one-to-one functions
- Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse write an expression for the inverse
- Recognise and construct linear and parabolic functions by hand and with the aid of technology, where appropriate show the intercepts and investigate the turning points.
- Recognise and use appropriate algebraic notions, properties of linear and non-linear functions, linear and non-linear equations and solve linear and non-linear simultaneous systems
- Recognise linear equations in two variables, draw its graph by hand and by using the appropriate technology (e.g., GeoGebra, Demos, PhET Simulations, Geometer's Sketch Pad), find the solution using the graph and determine the area enclosed by the graphs
- Recognise and model statements into linear mathematical equations and solve them simultaneously
- Solve up to three systems of linear equations simultaneously by algebraic manipulations
- Describe polynomial functions and perform the basic arithmetic operations on them
- Use the method of completing the square to transform any quadratic equation in, say x into an equation of the form $(x-p)^2 = q$ that has the same solutions and explains how the quadratic formula is derived from this form
- Use the remainder and factor theorems to find the factors and remainders of a polynomial of degree not greater than 4
- Draw the graph of a polynomial function with degree up to 3 by hand and by using technology (e.g., GeoGebra, Demos, PhET Simulations, and Geometer's Sketch Pad) where appropriate
- Recognise a rational function and determine the domain and range
- Carry out the basic arithmetic operations on rational functions
- Apply partial fraction decomposition up to factors with exponents and irreducible quadratic factors

Key Ideas:

- A **sequence** is any set of objects, often numbers, that can follow a particular pattern infinitely whilst a series refers to the description of the operation that would add all the items in a sequence.
- Each number in the sequence is called a **term**. Sequences can be either **finite (e.g., 2,5,8.)**, meaning they have a specific number of terms, or **infinite (e.g., 2,4.6…)** continuing indefinitely (without ending).
- A ***function*** is a relation between two sets, say X and Y, such that each member of the set X is related to one and only one member of the set Y. In terms of **ordered pairs**, a function is defined as a set of ordered pairs in which each x-element has only one y-element associated with it. There are two main types of functions: **one-to-one function** and **many -to-one function**.
- Algebraic variables where letters of the alphabets are used to represent numbers, things, figures to make problem solving easier will be much needed this week. We must also recall the concept of **slope** which is a measure of the degree of steepness of a line or plane. It must be remembered that points are written as coordinates and can be represented in the x-y plane at this level**.**
- A linear equation can be written in the form $\boldsymbol{y = mx + c}$ where m is the **slope** of the line and c is the **y-intercept**. Here x and y are the variables where y depends on x which is of degree one. The quadratic function can be written in the form $\boldsymbol{y = ax^2 + bx + c}$ where a, b and c are constants, and the degree is two. It must be noted that the nature of the quadratic curve is **parabolic**, that is a maximum turning point or a minimum turning point.
- A ***system of linear equations*** is made up at least two linear equations in which all have to be ***true*** simultaneously. Each equation stands for a ***straight line*** and the solution to the system is where these lines ***intersect.***
- ***Linear Equation*** is an equation in which highest power of variable (usually x or y) is 1. It can be put into form: $ax + by = c$ where a, b, and c are constants while x and y are variables.
- A polynomial function is a mathematical expression that combines numbers and variables using addition, subtraction and multiplication. It involves only whole number exponents of the variables. Examples of polynomials functions are $p(x) = 4x^3 - x - 1$, $p(y) = 7y^5 - 5y^4 + 2 + 3y^2 - y + 1$, $p(a) = 9a^2 - 1$ etc.

- The ***degree*** of a polynomial with one variable is the highest exponent of that variable in any term. The leading term is the term with the highest degree and its coefficient is the leading coefficient. Polynomial functions are typically written with the terms arranged in descending order of degree.
- A rational function, much like a rational number (a fraction of integers), consists of a **ratio between two polynomial expressions**. The denominator cannot be zero.
- A rational function is of the form $Rx = \frac{fx}{gx}$; $g(x) \neq 0$ (That is, a ratio of two polynomials for which the divisor is not zero).

DEFINITION OF SEQUENCES

Sequences and Series are all around us and understanding them helps us make sense of the world. Sequences are seen in our house number system. When we walk down the street, the house numbers typically follow a sequence. Each house number represents a term in that sequence. Also, the page numbers in a book follow a natural order, forming a sequence. Each page number corresponds to a term in that sequence.

Patterns occur all around us. The male honeybee hatches from an unfertilised egg, while the female hatches from a fertilised one. The "family tree" of a male honeybee can be represented by Figure 3.1***:*** **where M represents male, and F represents female**.

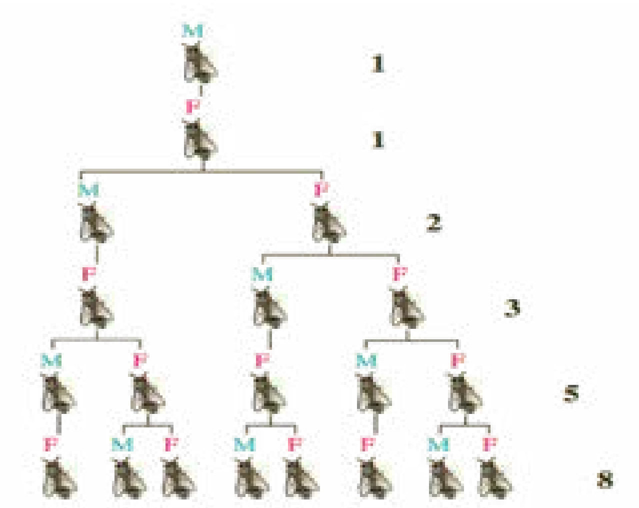


Fig 3.1: "Family tree of a male honeybee" (Lial, Hornsby, McGinnis, 2012)

Starting with the male honeybee at the top, and counting the number of bees in each generation, we obtain the following numbers in the order: 1, 1, 2, 3, 5, 8

Notice the Pattern

After the first two terms (1 and 1), each successive term is obtained by adding the two previous terms. This sequence of numbers is called the *Fibonacci sequence*.

A Sequence is any set of objects, often numbers, that follow a particular pattern infinitely. A series refers to the description of the operation that would add all of the items in a sequence. The objects that make a sequence are called terms and the *nth* term can be represented by S_n, U_n, a_n or similar notations where n is a positive integer indicating the position of the term. For example, U_1 represents the first term and a_4, represents the fourth term in a different sequence.

Problems in various fields and in our daily lives require the application of sequences and series. For example, in trying to predict the value of an investment after a number of years, taking into account the annual interest rate, say, 23%, and a principal of *GH¢* 100.00, the following sequence can model the problem and hence be used for the prediction: 100, 100(1.23), $100(1.23)^2$, $100(1.23)^3$, $100(1.23)^4$, …

The terms of the sequence indicate how much the investment is worth at the beginning of each year i.e., at the beginning of the first year, it is only worth the principal (*GH¢* 100.00) and at the beginning of the fourth year (at the end of the third year), it is worth *GH¢* $100(1.23)^3.00$ = *GH¢* 186.09. The three dots (ellipses) indicates that the sequence continues indefinitely.

Knowledge equips us with skills and strategies to find the sum of things going on even up to infinity which are useful in scientific thesis.

The pattern that is observed in a sequence can be used to classify sequences and also predict other terms. In this section arithmetic (linear) and geometric (exponential) sequences will be discussed.

Let us now, in various small groups, go through the following examples.

Example 1

Given the sequence: 3, 6, 10, 15, 21, …, determine the next term and describe the pattern.

Solution:

The sequence is such that 3 is added to the first number to obtain the second number, 4 added to the second to obtain the third, 5 added to the third to obtain the fourth and 6 added to the fourth to obtain the fifth term hence, the next term should be $21 + 7 = 28$

Mathematically, the sequence can be written thus: $a_n = a_{n-1} + n + 1, n \in \mathbb{Z}, n \geq 2$, $a_1 = 3$ or

$a_n = a_{n-1} + n + 1, n = 2, 3, 4, \ldots, a_1 = 3$

Example 2

Find the next three terms for each of the following sequences and describe the pattern

a. 2, 5, 8, 11…

b. 4, 7, 12, 19…

c. $2, 1, \frac{1}{2}, \frac{1}{4} \ldots$

d. 1, 4, 9, 16…

e. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

Solution

a. The sequence in (a) is such that 3 is added to the first number to obtain the second number, 3 again is added to second number obtain the third number, 3 is again added to the third number to obtain the fourth number and to get next three terms, you have to add 3 to obtain each of the three terms that is $11 + 3 = 14$, $14 + 3 = 17$ and $17 + 3 = 20$. Therefore, the next three terms are **14, 17, 20**

b. Let us look at the sequence in (b) dear learners, where you will find a pattern in the numbers too. In this sequence 3 is added to first number to obtain the second number, 5 is added to the second number to obtain the third number, 7 is added to the third number to obtain the fourth number, 9 will be added to the fourth number to obtain the fifth number, 11 will be added to the fifth number to obtain the sixth number, that is 19+9=28, 28+11=39, 39+13=52. Therefore, the next three terms are **28, 39, 52**.

c. Similarly, in example (c) the sequence is such that $\frac{1}{2}$ is multiplied to the first number to obtain the second number, to get third number the second number is multiplied by $\frac{1}{2}$. To get the subsequent terms we need to multiply each term by $\frac{1}{2}$. That is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, $\frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$, $\frac{1}{16} \times \frac{1}{2} = \frac{1}{32}$. Therefore, the next three terms are $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$

d. The sequence in example (d) is such that we square the natural numbers 1, 2, 3, 4, to obtain the terms that is $1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16$. Hence the next three-term will be $5^2 = \mathbf{25}$, $6^2 = \mathbf{36}$, $7^2 = \mathbf{49}$

e. The sequence in example (e) is such that 1 is added to the numerator and the denominator of first fraction to obtain the second fraction, 1 is added again to the numerator and the denominator of the second fraction to obtain the third. The sequence continues to add 1 to each term of the numerator and the denominator to obtain the other terms, that is $\frac{4+1}{5+1} = \frac{5}{6}$, $\frac{5+1}{6+1} = \frac{6}{7}$, $\frac{6+1}{7+1} = \frac{7}{8}$. Hence the next three terms are $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$.

Example 3

Generate the first six terms of the sequence with the rule: $u_{n+1} = 2u_n$ and $u_1 = 3$

Solution:

$u_{n+1} = 2u_n$, $u_1 = 3$

When $n = 1$,

$$u_{1+1} = 2u_1$$
$$u_2 = 2(3) = 6$$

When $n = 2$,

$$u_{2+1} = 2u_2$$
$$u_3 = 2(6) = 12$$

When $n = 3$,

$$u_{3+1} = 2u_3$$
$$u_4 = 2(12) = 24$$

When $n = 4$,

$$u_{4+1} = 2u_4$$
$$u_5 = 2(24) = 48$$

When $n = 5$,

$$u_{5+1} = 2u_5$$

$$u_6 = 2(48) = 96$$

∴ the first six terms are 3, 6, 12, 24, 48 and 96

NTH TERM OF ARITHMETIC AND GEOMETRIC SEQUENCES

Hello learners, let us consider the following sentences to enable us to understand the components of sequences.

1. Identifying key terms of the sequence such as the first term, the common difference and/or ratio, and the nth term.
2. Establishing the general rule for arithmetic and geometric sequences by using the first terms and relationship between consecutive terms (simple recursions) and the notation of sequences.
3. Determining arithmetic and geometric sequences.
4. Identifying a sequence that is neither arithmetic nor geometric.
5. Insert arithmetic and geometric means for a given sequence.

Arithmetic Progression (or Linear Sequence)

An Arithmetic Progression (AP), also known as Linear Sequence, is a sequence in which the difference between any two consecutive terms remains constant throughout the sequence. The constant difference is called the **common difference** of the AP.

In AP, we come across several important keywords

i. **First term, a**, is the initial term of the progression.

ii. **Common Difference, d,** is the fixed value added to each term to obtain the next term. The Common difference, d, can be calculated as:

$\mathrm{u}_{\mathrm{n}+1} - \mathrm{u}_{\mathrm{n}} = \mathrm{u}_{\mathrm{n}} - \mathrm{u}_{\mathrm{n}-1}$ for all n = 1, 2, 3, …

This means that to find the common difference, we subtract a term from the next term and as subtraction is not commutative, the reverse is inaccurate.

iii. **The nth term, $\mathbf{a}_{\mathbf{n}}$**, is the term at position (n) in the sequence

Example 4

Consider the sequence 1, 4, 7, 10, 13, 16…

This sequence has a **common difference, d**, of 3 (each term is obtained by adding 3) to the previous term i.e., $4 - 1 = 7 - 4 = 3$

The **first term, a**, is 1

The second term is $(a+d) = 1+3 = 4$

The third term is $a+2d = 1 + 2(3) = 7$ and so on.

So, the linear sequence continues, 1, 4, 7, 10, 13, 16, 19, …

The terms of an arithmetic progression are:

$u_1 = a$,

$u_2 = (a + d)$,

$u_3 = (a + 2d)$,

$u_4 = (a + 3d)$, … which is equivalent to $u_1 = (a + (1 - 1) d)$, $u_2 = (a + (2 - 1)d)$, $u_3 = (a + (3 - 1)d)$, $u_4 = (a + (4 - 1)d)$, …

Therefore, the rule for finding the nth term of an arithmetic sequence can thus be written as**:**

$\mathbf{U_n = a + (n - 1)\,d}$ where a is the first term, d the common difference and n is the position of the term.

Now let us perform the follow activities to boost our understanding further.

Example 5

Write the first five terms of the arithmetic sequence with first term -2 and common difference 3

Solution

Let u_n = nth term $= a + (n - 1)\,d$

Where a = first term $= u_1 = -2$,

d = common difference = 3

$u_2 = -2 + 3 = 1$

$u_3 = -2 + 2(3) = 4$

$u_4 = -2 + 3(3) = 7$

$u_5 = -2 + 4(3) = 10.$

$\therefore$ the first five terms are $-2, 1, 4, 7$ and 10

Example 6

Find the 29th term of the arithmetic sequence whose first three terms are -12, $-6, 0, \dots$

Solution

$-12, -6, 0, \dots$

First term $= -12$,

Common difference $= 0 - (-6) = 6$

29^{th} term $= U_n = a + (n - 1)d$

By substitution

$U_{29} = -12 + (29 - 1) \times 6$

$= -12 + 28 \times 6$

$= 156$

Example 7

Three consecutive terms of a linear sequence (or Arithmetic Progression, AP) are $\frac{1}{m}$, 1 and $\frac{6}{m+2}$, where $m \neq 0$ and $m \neq -2$. Find the two possible values for m.

Solution

The Sequence is $\frac{1}{m}$, 1 and $\frac{6}{m+2}$

Common difference $d = 1 - \frac{1}{m} = \frac{6}{m+2} - 1$

Equating the expressions, we have

$1 - \frac{1}{m} = \frac{6}{m+2} - 1$ (finding L C M of the expressions)

$\frac{m-1}{m} = \frac{6-(m+2)}{m+2}$ (Expanding to open bracket)

$\frac{m-1}{m} = 6\frac{-m-2}{m+2}$

$\Rightarrow \frac{\boldsymbol{m}-1}{\boldsymbol{m}} = \frac{6-2-\boldsymbol{m}}{\boldsymbol{m}+2}$

$= \frac{m-1}{m} = \frac{4-m}{m+2}$ (cross multiply the expressions)

$(m-1)(m+2) = m(4-m)$ (Expanding to open the brackets)

$m^2 + 2m - m - 2 = 4m - m^2$

Grouping like terms we have

$= (m^2 + m^2) + (2m - m - 4m) - (2\ )$

$= 2m^2 - 3m - 2 = 0$

Factorising the expression

$=(m-2)\ (2m+1) = 0$

Either $m - 2 = 0$ or $2m + 1 = 0$

Solving for m:

$m = 2\ or\ m = -\ 1\frac{=}{2}$

Example 8

The 13th term of an arithmetic progression is 10 and the 25th term is 20. Calculate:

i) the common difference

ii) the first term.

iii) the 17th term.

Solution

i) Let a be the first term and d be the common difference.

Applying the general rule:

$U_n = a+(n+1)$

$U_{13} = 10$

$\Rightarrow U_{13} = a + (13-1)d$

$= a + 12d = 10$ *equation 1*

$U_{25} = 20$

$\Rightarrow U_{25} = a + (25 - 1)d = 20$

$= a + 24d = 20$ *equation 2*

Solving *equation 1* and *equation 2* simultaneously,

Equation 1 – Equation 2

$(a - a) + (12d - 24d) = (10 - 20)$

$0 + (-12d) = -10$

$-12d = -10$

Dividing both sides by -12

$d = \frac{-10}{-12} = \frac{5}{6}$

$\boldsymbol{d} = \mathbf{\frac{5}{6}}$

ii) To find the value of a

Put $d = \frac{5}{6}$ into *equation 1*

$a + 12\left(\frac{5}{6}\right) = 10$

$a + \frac{60}{6} = 10$

Group like terms

$a + 10 = 10$

$a = 10 - 10$

$\boldsymbol{a} = 0$

iii) the 17th term

$U_n = a + (n - 1)d$

$a = 0, d = \frac{5}{6}, and\ n = 17$

$U_{17} = 0 + (17 - 1) \times \frac{5}{6}$

$U_{17} = 0 + 16 \times \frac{5}{6}$

$U_{17} = \frac{16}{1} \times \frac{5}{6}$

$U_{17} = \frac{80}{6} = \frac{40}{3}$

$\boldsymbol{U}_{17} = \frac{40}{3}$

Geometric / Exponential Sequence

Geometric sequences are sequences in which each term after the first is found by multiplying the preceding term by a non-zero constant. While arithmetic sequences are marked by their common differences, geometric sequences are defined by their common ratios. For example, the terms (except the first) of 3, 6, 12, 24, 48, 96, ... are obtained by multiplying the preceding terms by 2. The common ratio, usually represented by r, is therefore 2. The sequence can hence be rewritten as 3, 3(2), $3(2)^2$, $3(2)^3$, $3(2)^4$, $3(2)^5$, …

By observation, the rule for finding the terms of the sequence is $3(2)^{n-1}$, where 3 is the first term and 2 is the common ratio.

Therefore, the rule for all geometric sequences is:

The nth term, $\boldsymbol{U_n = ar^{n-1}}$ where $\boldsymbol{a}$ is the first term, $\boldsymbol{r}$**,** the common ratio and $\boldsymbol{n}$ is the number of terms in the sequence. Note that $r = \frac{u_n}{u_{n-1}} = \frac{u_{n+1}}{u_n}$

Example 9

Determine the common ratio for the geometric sequence: $\frac{1}{4}, -1, 4, -16, 64, \ldots$

Solution

$\frac{1}{4}, -1, 4, -16, 64, \ldots$

$$Common\ ratio = \frac{64}{-16} = -4$$

Example 10

Find the *nth* term of

i) $5, -10, 20, -40, 80\ldots$

ii) $4, \frac{8}{3}, \frac{16}{9}, \ldots$

Solution

i) $5, -10, 20, -40, 80\ldots$

$First\ term,\ a = 5$

$Common\ ratio,\ r = -\frac{40}{20} = -2$

$$nth\ term,\ u_n = a\,r^{n-1}$$
$$= 5(-2)^{n-1}$$

ii) $4, \frac{8}{3}, \frac{16}{9}, \ldots$

First term, $a = 4$

Common ratio, $r = \frac{8}{3} \div 4 = \frac{2}{3}$

nth term, $u_n = a\,r^{n-1}$

$= 4\left(\frac{2}{3}\right)^{n-1}$

Example 11

The fifth and ninth terms of an exponential sequence are 16 and 256 respectively. Find

i) the first term, a and the common ratio, r if r > 0

ii) the 11th term

Solution

i) $\boldsymbol{U_n = ar^{n-1}}$

Fifth term, $U_5 = a\,r^{5-1} = 16$

$= a\,r^4 = 16$ *equation 1*

Ninth term, $U_9 = a\,r^{9-1}$

$U_9 = a\,r^8 = 256$ *equation 2*

Divide *equation2* by *equation 1*

$\frac{ar^8}{ar^4} = 2\frac{56}{16}$

Applying the quotient law of indices

$r^{8-4} = 2\frac{56}{16}$

$r^4 = 2\frac{56}{16}$

$r^4 = 16$

$r = \sqrt[4]{16} = 2$

Put r =2 into *equation 1* to find a

$a(2^4) = 16$

$16a = 16$

Divide both sides by 16

$a = 1$

ii) $11^{th}\ term\ \ U_{11} = ar^{10}$

$= 1(2)^{10}$

$= 1024$

Example 12

The second and the fifth terms of Geometric Progression (G.P) are 1 and $\frac{1}{8}$ respectively. Find the:

i) Common ratio

ii) First term

iii) Eighth term

Solution

i) $U_2 = 1$

$U_2 = ar^{2-1} = 1$

$U_2 = ar = 1$ *equation 1*

$U_5 = ar^{4-1} = \frac{1}{8}$

$U_5 = ar^4 = \frac{1}{8}$ *equation 2*

Divide *equation 2* by *equation 1*

$\frac{ar^4}{ar} = \frac{1}{8} \div 1$

Using the quotient law of indices

$r^{4-1} = \frac{1}{8}$

$r^3 = \frac{1}{8}$

Taking cube root of both sides

$r = \sqrt[3]{\frac{1}{8}}$

$\boldsymbol{r} = \frac{1}{2}$

a) Substitute $r = \frac{1}{2}$ into *equation 1*

$a\left(\frac{1}{2}\right) = 1$

$\frac{a}{2} = 1$

$2 \times \frac{a}{2} = 1 \times 2$

$\boldsymbol{a} = 2$

ii) 8^{th} term

$U_n = ar^{n-1}$

$U_8 = 2\left(\frac{1}{2}\right)^{8-1}$

$= 2\left(\frac{1}{2}\right)^{7}$

$2 \times \frac{1}{128}$

$= \frac{2}{128}$

$= \frac{1}{64}$

Activity 3.1

Let us perform the following task to add more understanding to the concept of Sequences and their forms (Arithmetic Progressions and Geometric Progressions).

1. Determine the next three terms of the following Sequence

a) 5, 9, 13, 17…

b) $-1, 2, -4, 8\ldots$

Solution

a) To determine the next three terms of sequence in (a) you can perform the following steps

i. First find the common different between the consecutive terms.

ii. The common difference is calculated as: $d = a_2 - a_1 (13 - 9 = 4)$

You then add the common difference (d) to successive terms to obtain the next three terms as: $17 + 4 = 21$, $21 + 4 = 25$, $25 + 4 = 29\ldots.$

Write the next three terms as 21, 25, 29

b) $-1, 2, -4, 8\ldots$

To find the next three terms of the Sequence in (b) follow the guidelines below.

i. Look at the terms in the Sequence and identify the common ratio (r)

ii. Divide the second term by the first term to determine the common ratio (r) that is $r = 2 \div (-1) = -2$

iii. To find the 5^{th} term multiply the 4^{th} term by common ratio that is $8 \times (-2) = -16$

iv. Find the 6^{th} as: $-16 \times (-2) = 32$

v. And the 7^{th} term as: $32 \times (-2) = -64$

vi) So, the next three terms are $-16, 32, -64$

2. In a certain restaurant in Walewale of the North East Region, a square table accommodates four people. Six people visited the restaurant and wanted to sit, two tables were joined, and they sit around perfectly. If three tables are joined horizontally, how many people can sit around it? What about 4 square tables joined horizontally, how many people can sit around it now and so on.

Provide your result on the table that follows

Number of squared table(s)	1	2	3	4	5
Number of people that can sit around it	4				

From the result on the table determine if there a pattern

ii. Deduce if the pattern is Arithmetic Progression (AP) or Geometric Progression (GP).

iii. Suggest a rule for the pattern,

3. Mr. Salifu of Walewale S. H. T. S places in a savings bank Gh¢25.00 on his son's first birthday, Gh¢50.00 on his second, Gh¢75.00 on his third and so on, increasing the amount by Gh¢25 on each birthday. With a partner, or individually, work out how much will be saved up (apart from any accrued interest) when the boy reaches his 16th birthday if the final amount is added on this day?

Present your answer to your teacher or compare your findings with others.

DEFINING FUNCTIONS

The concept of functions has gone through a lot of development and notable among the mathematicians who have worked on its development are René Descartes, Wilhelm Leibniz, Leonhard Euler, Nicolas Bourbáki and John Tate to name but a few. Functions are the building blocks for modelling real-life scenarios and, thereby, designing machines, predicting natural disasters, curing diseases, understanding world economies and keeping airplanes in the air etc.

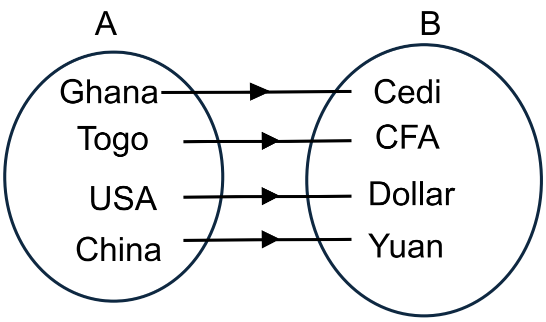
Relations

Hello, welcome to another interesting area of mathematics, relations and functions. How are you related to people around you, your father, mother, siblings, teachers and classmates?

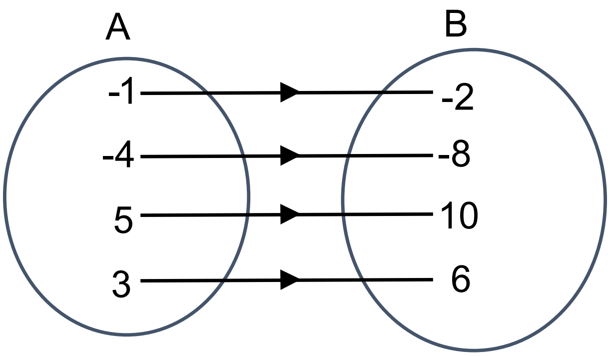
We relate to people around us based on special connections between us. For example, Taninu and Afiba are in one class studying Home Economics, they are related because they are classmates.

Consider the activities (i) to (iii)

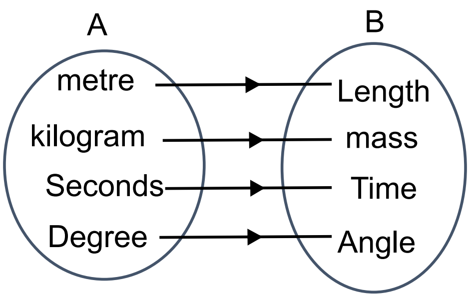
i.



ii.



iii.



What relationship exist between A and B in the activities above?

i. "the currency of countries"

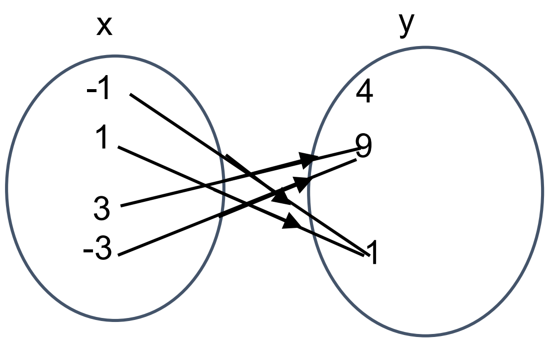
ii. "multiplied by 2"

iii. "SI unit"

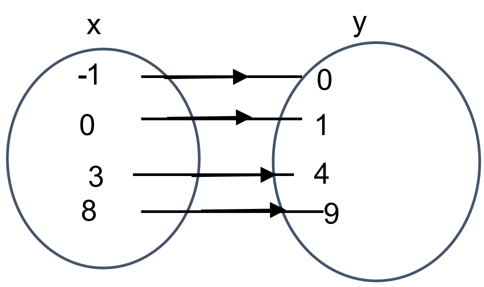
Good job for your answers.

Given the element A and B, on any non-empty sets, then any rule which assigns elements of A to elements of B is referred to as a **relation. Relation** can also be defined as a set of ordered pairs; for example, in activity (ii), the ordered pair are (-1, -2), (-4, -8), (5, 10), (3,6). The first elements in the ordered pairs (the *x*-values), form the domain. The second elements in the ordered pairs (the *y*-values), form the **image or co-domain**. The co-domain is also called the dependent variable**. Range** is the element in the co-domain that are matched to elements in the domain.

1.



2.



Now, look at the diagrams above. Take your pen and write down the domain, co-domain and the range of the relation. Are your answers the same as these?

1. The domain is the set {-1, 1, -3, 3}, the co-domain is the set {1, 4, 9} and the range is the set {1, 9}. Notice that 4 is not part of the range because it is not an image of any of the elements in the domain. Can you write down the ordered pairs? Is your answer the same as this? (-1, 1), (1, 1), (-3, 9), and (3, 9).

2. The domain is the set {-1, 0, 3, 8}, the co-domain is the set {0, 1, 4, 9} and the range is the set {0, 1, 4, 9}. Can you write down the ordered pairs? Is your answer the same as this? (-1, 0), (0, 1), (3, 4), and (8, 9).

Now, investigate the rule or relation between x and y in the activities above.

Congratulations if your findings were

1. $y \rightarrow x^2$
2. $y \rightarrow x + 1$

Several types of relations exist in life.

For example: the relation between the ***head prefect of a school and students at morning assembly***, is ***one–to–many*** because we have only one head prefect as against many students.

Consider the following and guess the kind of relation using either "**one**" or "**many**"

- **i.** Number of people that drive a car at a time.
- **ii.** A teacher and his or her learners
- **iii.** Two football teams
- **iv.** Students in a school and their headteacher

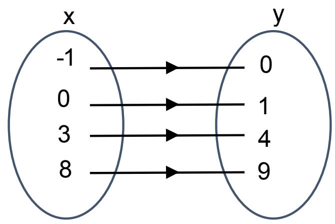
If your guess is or similar to

- **i.** One person → one person
- **ii.** One teacher → many students
- **iii.** Many players → many players
- **iv.** Many students → one headteacher

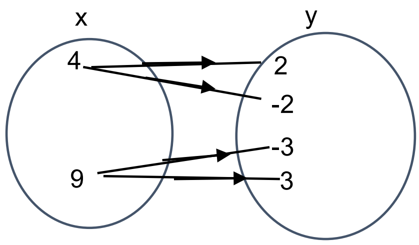
Did you know that similar types of relationship exist in mathematics as well?

Oh yes, there are basically four types of relations in mathematics namely: **one-to-one**, **one-to-many**, **many-to-one**, and **many-to-many**. The arrow diagrams below illustrate the four types of relations.

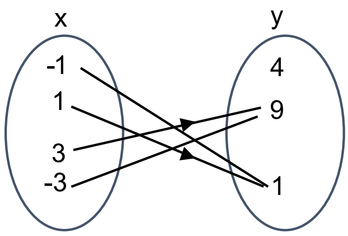
1. **One-to-one relation**: In this relation, each element in the domain has only one image in the co-domain and each element in the co-domain is associated with only one element in the domain. Thus, each element of the domain has a unique image.



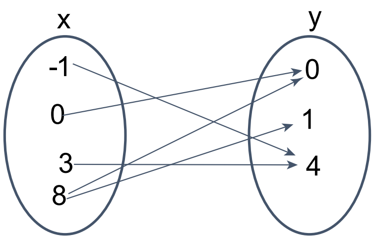
2. **One-to-many relation**: In this relation, one element in the domain is associated with many images in the co-domain.



3. **Many-to-one relation:** In this relation, several elements in the domain have one image in the co-domain.



4. **Many-to-many relation:** In this relation, several elements in the domain have many images in the co-domain and several elements in the co-domain are associated with many elements in the domain.



Activity 3.2

In pairs or individually, investigate which type(s) of relations discussed above have the elements in the domain being matched to only one member in the image.

Your investigation is correct if your findings were: one-to-one, and one–to–many. These two relations (one-to-one, and one–to–many) are called functions in mathematics.

DEFINITION OF FUNCTIONS

A function, f, from set A to set B could be defined as a pairing of elements in A with elements in B in such a way that each element in A is paired with *exactly one* element in B or a rule or relation between A and B that assigns each element $a \in A$ to a *unique* element, $b \in B$.

From the definitions of functions, it can be inferred that "***All functions are relations BUT not all relations are functions***". A function must assign only ONE output to each input. One-to-one and many-to-one relations are therefore the two types of relations that qualify to be functions. If a function is defined by an equation, the variable that represents elements of the domain is the **independent variable.** The variable that represents elements of the range is the **dependent variable.**

If a function describes the relationship between a number and its square, it can be expressed as

a. $\{(-4, 16), (-3, 9), (-2, 4), (-1, 1), (0, 0), (0.5, 0.25), (1,1), (2, 4)\}$ i.e., as ordered pairs,

b. $f(x) = x^2$ read as "*f of x*" or "*f* at *x*" i.e., using a formula,

c. $f{:}x \rightarrow x^2$ read as "*f is such that x maps on to* x^2 i.e., using function notation or

d. graphically.

Functions can be used to calculate electricity bills, supply and demand, and how you are graded. For example, your end-of-semester grade mark is a function in the form of 30% of class work and 70% of exam score which can modelled as say G = 30%(c) +70%(e), where G is the final mark, c is the class score, and e is the exam score.

Activity 3.3

Take your graph books and in small groups plot the points $(-4, 16)$, $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(0.5, 0.25)$, $(1,1)$, $(2, 4)$ on a graph. Did your graph look like the one below, Good! The graph represents a function. If you cannot get the graph, don't worry, seek help from other group members or your teacher.

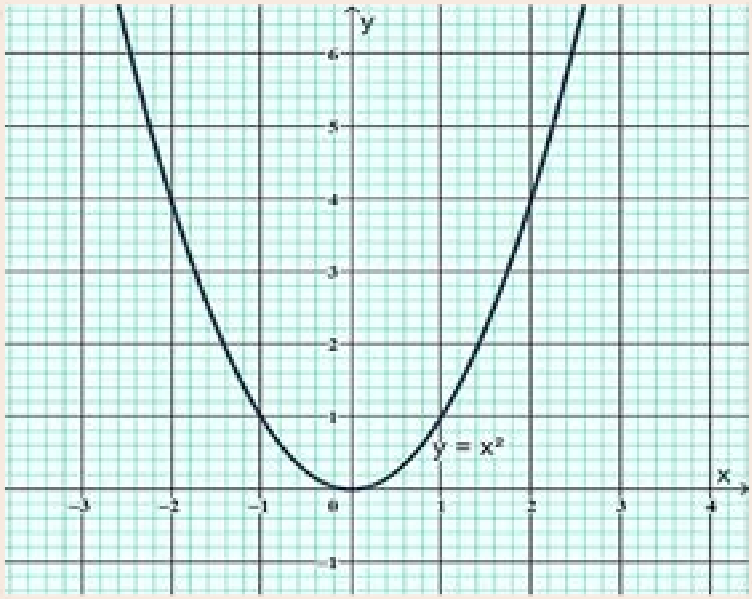


Fig 3.2: Quadratic Graph

Activity 3.4

Draw any vertical line on your graph sheet to meet the curve. How many times did your vertical line meet the curve on your graph sheet? Congratulations on your findings that the vertical line met the curve at only one place. That is how we use **vertical lines** to test for a function.

The Vertical Line Test

This is a test that checks whether or not any vertical line drawn on the same graph as the curve of an equation cuts the curve at more than one point. If the vertical line cuts the curve at more than one point, the curve is said to have failed the test and vice versa. The graph of a function must pass the vertical line test.

Considering 3.3, below, in illustration 1, the vertical line cuts the curve at two points, A and B and thus, the curve does not pass the vertical line test. Automatically, it does not qualify to be the graph of a function. Actually, the curve is the graph of $y^2 = x$. Considering that x represents elements in the domain and y elements in the co-domain, the rule, $y^2 = x$ represents a one-to-many relation and confirms the conclusion that $y^2 = x$ is not a function.

The curve in illustration 2 however, passes the vertical line test as the vertical line cuts the curve at only one point i.e., *A* and hence the equation / rule represented by the curve is a function.

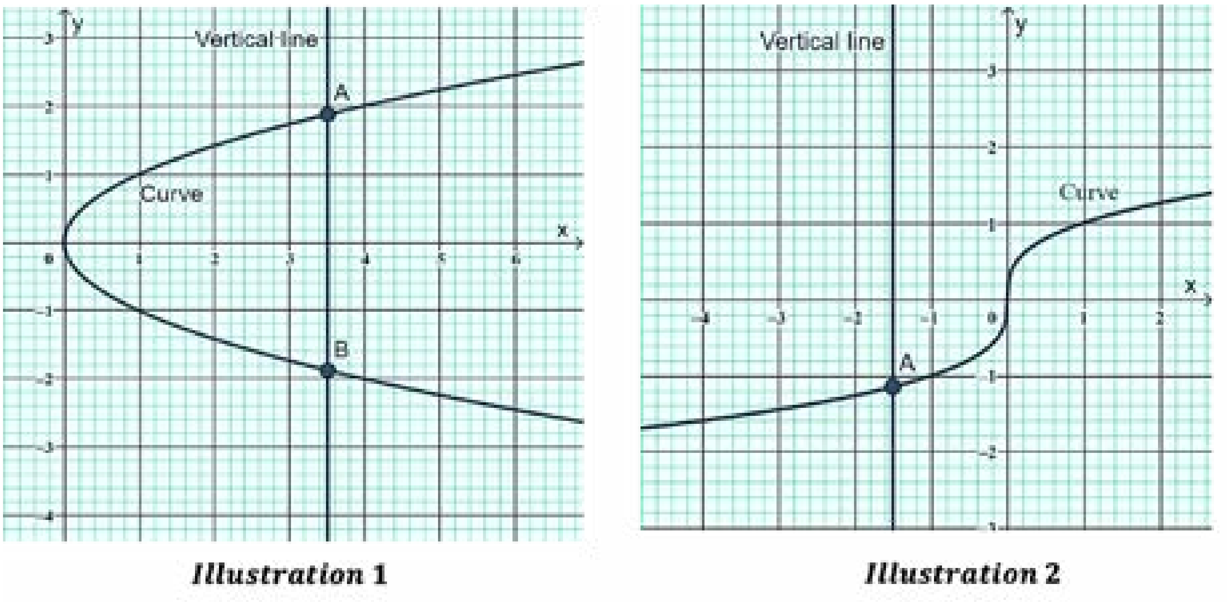


Fig 3.3: Vertical line test

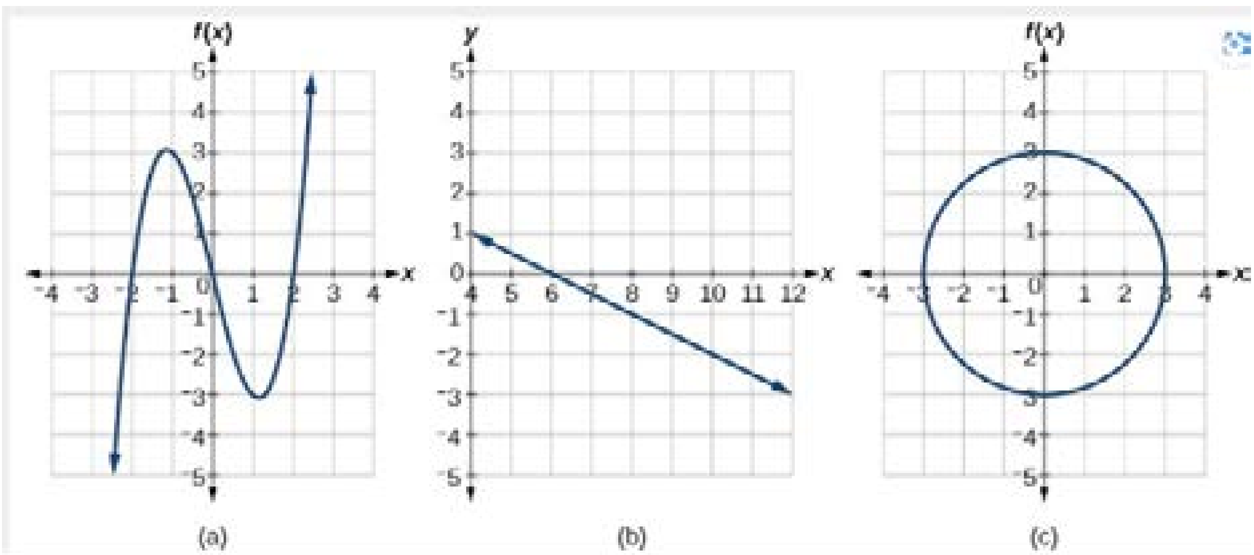


Fig 3.4: Determining functions

Activity 3.5

Investigate which of the diagrams in Figure 3.4 (a, b and c) are functions and discuss your findings with a classmate.

EVALUATING FUNCTIONS

For a function, f, defined as $f(x) = x^2$, $f(x)$ is the functional value of x, it is the image of x and on a two-dimensional plane, it is the corresponding value of x for the point which lies on the graph of x^2. For example, given that $g(x) = -2x$, $g(-3)$ means, apply the rule "*negative of a double of a number*" to -3 and hence, $g(-3) = -2(-3) = 6$

Example 13

If $r(x) = \frac{3x+5}{7x-14}$, evaluate $r(3)$, $x \neq 7$

Solution

$r(x) = \frac{3x+5}{7x-14}$

$r(3) = \frac{3(3)+5}{7(3)-14} = \frac{9+5}{21-14}$

$r(3) = \frac{14}{7}$

$r(3) = 2$

Example 14

If $g(x) = 2x^2 + 3$, simplify

a. $g(3)$

b. $g(-8)$

c. $g(a+1)$

d. $3g(5)$

Solution

a. $g(3) = 2(3)^2 + 3$

$g(3) = 2(9) + 3$

$g(3) = 18 + 3$

$g(3) = 21$

b. $g(-8) = 2(-8)^2 + 3$

$g(-8) = 2(64) + 3$

$g(-8) = 128 + 3$

$g(-8) = 131$

c. $g(a+1) = 2(a+1)^2 + 3$

$g(a+1) = 2[(a+1)(a+1)] + 3$

$g(a+1) = 2\left[a^2 + a + a + 1\right] + 3$

$g(a+1) = 2\left[a^2 + 2a + 1\right] + 3$

$g(a+1) = 2a^2 + 4a + 2 + 3$

$g(a+1) = 2a^2 + 4a + 5$

d. $3g(5) = 3\left[2(5)^2 + 3\right]$

$3g(5) = 3[2(25) + 3]$

$3g(5) = 3[50 + 3]$

$3g(5) = 3[53]$

$3g(5) = 159$

Example 15

Evaluate the following either on your own, or with a classmate, or in a small group.

If $f(x) = 11x - 3x^3$, evaluate:

a) f(1)

b) f (2)

c) $\left(\frac{1}{2}\right)$

d) $f\left(\sqrt{2}\right)$

e) $f(1 + y)$

Excellent work done if your workings gave you the answers below.

a) 8

b) -2

c) $\frac{41}{8}$

d) $5\sqrt{2}$

e) $8 + 2y - 9y^2 - 9y^3$

Example 16

The figure below shows the path, $p(x)$ taken by a particle in the Cartesian plane. Use the graph to predict the vertical distance of the particle from the $x - axis$ when it is 1 unit to the left of the $y - axis$.

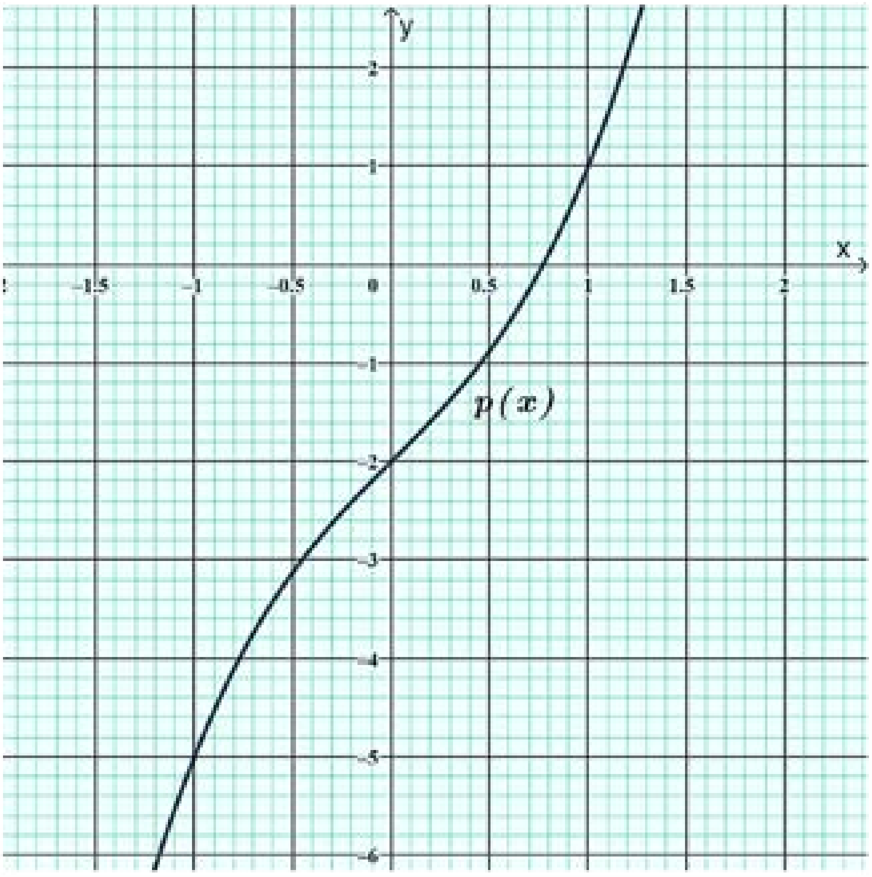


Fig 3.5: Path taken by a particle

Solution

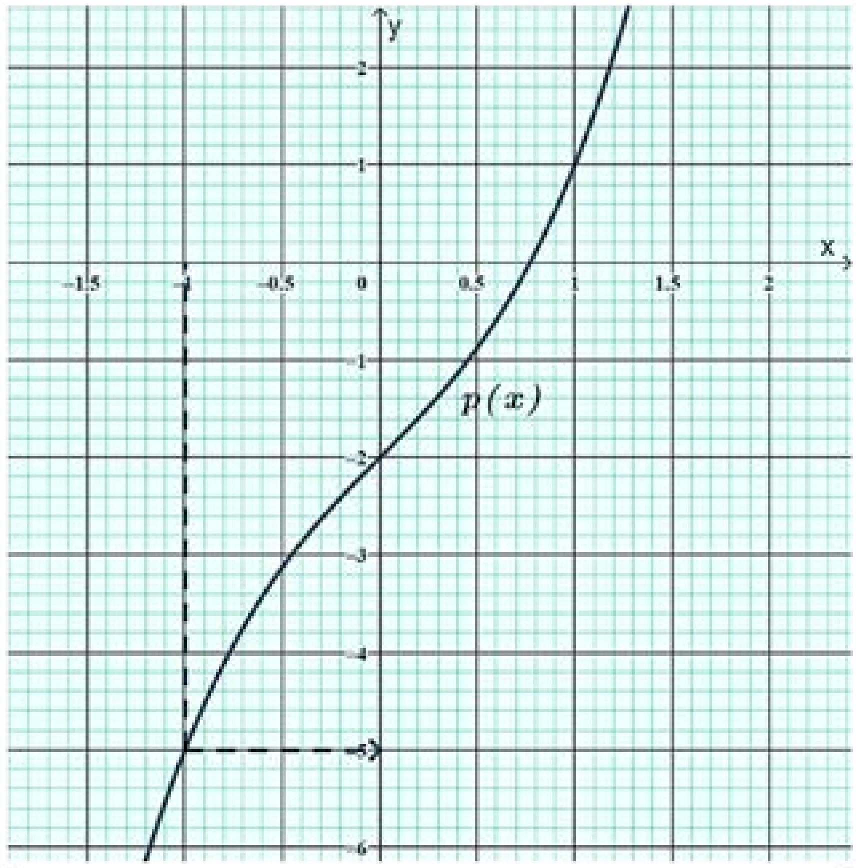


Fig 3.6: Prediction of vertical distance

From the graph, when $x = -1$, $y = -5$ and thus, the vertical distance of the particle from the $x - axis$ when it is 1 unit to the left of the $y - axis$ is 5

Example 17

Temperature conversion is a common task in everyday life, especially when dealing with different temperature scales. Two popular temperature scales are Celsius (°C) and Fahrenheit (°F). To convert temperatures between these scales, we can use a function that takes an input on one scale and produces an output on the other scale.

The function C to F, denoted by $F(C)$ can be used to convert temperatures from Celsius (°C) to Fahrenheit ($°F$). It can be represented as: $F(°F) = \frac{9}{5} \times C(°C) + 32$

In this function, C(°C) represents the input temperature in Celsius, and F(°F) represents the output temperature in Fahrenheit.

For example, $25°C$ in degree Fahrenheit, given by $F(°F) = \frac{9}{5} \times 25 + 32 = 77°F$

Addition and Multiplication Properties of a Function

a. If f_1 and f_2 are two functions from A to B, then $f_1(x) + f_2(x)$ is defined as: $(f_1 + f_2)x = f_1(x) + f_2(x)$.

b. If f_1 and f_2 are any two functions from A to B, then $(f_1 \cdot f_2)x$ is defined as: $(f_1 \times f_2)x = f_1(x) \times f_2(x)$.

c. Function Equality: Two functions are equal only when they have same domain, same co-domain and same mapping elements from domain to co-domain

Example 18

Given that $f(x) = 3x^3 - 2x$ and $g(x) = -2x + 4$, evaluate

a. $(f + g)(-2)$

b. $(f \times g)(5)$

Solution

$f(x) = 3x^3 - 2x$ and $g(x) = -2x + 4$

a. $(f + g)(x) = 3x^3 - 2x + (-2x + 4)$

$= 3x^3 - 2x - 2x + 4$

$= 3x^3 - 4x + 4$

$(f + g)(-2) = 3(-2)^3 - 4(-2) + 4$

$= -12$

b. $(f \times g)(x) = (3x^3 - 2x)(-2x + 4)$

$= (-6)x^4 + 12x^3 + 4x^2 - 8x$

$(f \times g)(5) = (-6)(5)^4 + 12(5)^3 + 4(5)^2 - 8(5)$

$= -2190$

Example 19

Given that $h(x) = x^2 - 2x + 5$, $g(x) = -12x + 16$, and $f(x) = 5x^3 - 7$, evaluate:

a. (f+g) (x)

b. (h+g) (x)

c. (h×g) (x)

d. (f × g) (x)

e. (f + g + h) (3)

Solution

$\boldsymbol{h(x) = x^2 - 2x + 5, g(x) = -12x + 16, \text{and } f(x) = 5x^3 - 7}$

a. $(f+g)(x) = 5x^3 - 7 + (-12x + 16)$

$=5x^3 - 12x - 7 + 16$

$=5x^3 - 12x + 9$

b. $(h+g)(x) = x^2 - 2x + 5 + (-12x + 16)$

$(h+g)(x) = x^2 - 2x - 12x + 16 + 5$

$(h+g)(x) = x^2 - 14x + 21$

c. $(h \times g)(x) = (x^2 - 2x + 5)(-12x + 16)$

$(h \times g)(x) = x^2(-12x + 16) - 2x(-12x + 16) + 5(-12x + 16)$

$(h \times g)(x) = -12x^3 + 16x^2 + 24x^2 - 32x - 60x + 80)$

$(h \times g)(x) = -12x^3 + 40x^2 - 92x + 80$

d. $(f\,x\,g)(x) = (5x^3 - 7)(-12x + 16)$

$(f \times g)(x) = 5x^3(-12x + 16) - 7(-12x + 16)$

$(f \times g)(x) = -60x^4 + 80x^3 + 84x - 112$

e. $(h + g + f)(x) = x^2 - 2x + 5 + (-12x + 16) + (5x^3 - 7)$

$(h + g + f)(x) = 5x^3 + x^2 - 2x - 12x + 5 + 16 - 7$

$(h + g + f)(x) = 5x^3 + x^2 - 14x + 14$

$(h + g + f)(3) = 5(3)^3 + (3)^2 - 14(3) + 14$

$(h + g + f)(3) = 135 + 9 - 42 + 14$

$(h + g + f)(3) = 116$

Activity 3.6

The exchange rate in Ghana in May 2024 was modelled mathematically as $x = \frac{16}{y}$, where x represents Ghana cedis (GHC) and y represents the dollar ($).

Investigate, either in small groups or individually, the value of the Ghana Cedi (x) when

1. $y = 10$

2. $y = 20$

3. $y = 0$

4. $y = 4$

Expected responses for Activity 3.6

1. GHC 1.60

2. GHC 0.80

3. GHC 0.00

4. GHC 4.00

DOMAIN, RANGE AND ZEROES OF FUNCTIONS

Domain of a Function

The domain of a function, say $f(x)$ is the set of all values of x which can be evaluated and hence its image under the function can be obtained. For example, if for $f(x)$, $f(a)$, (where a represents any element in the set, D) can be evaluated, then the domain of $f(x)$ is D.

The range of a function is thus the set of all functional values or images of the elements in the domain.

Mathematically, *Domain*, $D = \{a : \exists b \in Y ; (a, b) \in f\}$ read as *"D is the set of ' a' such that for each 'a', there exists an element, 'b' in set Y, (the range of the function), where* (a, b) *is a pair from the function, f"* and

Range, $Y = \{b : \exists a \in D ; (a, b) \in f\}$ read as *Y is the set of 'b' such that for each 'b', there exists an element, 'a' in set D, (the domain of the function), where* (a, b) *is a pair from the function, f*

For a polynomial function, $y = f(x)$

Domain $= \{x : x \in \mathbb{R}\}$ i.e., the set of all real numbers

Range $= \{y : y \in \mathbb{R}\}$ i.e., the set of all real numbers

For a rational function, $y = R(x) = \dfrac{p(x)}{q(x)}$

Domain $= \{x : x \in \mathbb{R}, q(x) \neq 0\}$ i.e., the set of all real numbers except x values for which $q(x) = 0$

The range of rational functions is the domain of the corresponding inverse function and hence cannot be written in a general form

For a radical function, $y = f(x) = \sqrt{p(x)}$

Domain $= \{x{:}x \in \mathbb{R}, p(x) \geq 0\}$ i.e., the set of all real numbers except values of x for which $q(x)$ is negative ($q(x) < 0$)

The range of radical functions is the domain of the corresponding inverse function and hence cannot be written in a general form

For an exponential function, $y = f(x) = a\,b^x$

Domain $= \{x{:}x \in \mathbb{R}\}$ i.e., the set of all real numbers

Range:

For $b = 1$, the range of $f(x) = a\,b^x$ is simply $\{a\}$.

For b other than 1 and $a > 0$, the *range* $= (0, \infty)$.

For b other than 1 and $a < 0$, the *range* $= (-\infty, 0)$

For a logarithmic function, $y = f(x) = \log(x)$

Domain $= \{x{:}x \in \mathbb{R}, x \geq 0\}$ i.e., the set of all positive real numbers

Range $= \{y{:}y \in \mathbb{R}\}$ i.e., the set of all real numbers

Generally, we determine the domain by looking for those values of the independent variable (usually x) which will make the function defined. We must also avoid 0 in the denominator of a fraction, or negative values under the square root sign. For example, the function $f(x) = x^2 + 2$ is defined for all real values of x, because there are no restrictions on the value of x. Hence, the domain of f(x) is "all real values of x".

Let us consider another function $y = \sqrt{x+3}$, the domain of this function is all real values of $x \geq -3$, since x cannot be less than -3. To see why, try out some numbers less than -3 (like -5 or -9) and some numbers more than -3 (like -2 or 4) in your calculator. The only ones that "work" and give us an answer are the ones greater than or equal to -3. This will make the number under the square root positive.

Example 20

Find the domain for each of the following functions:

1. $f(x) = 2x + 1$

2. $f(x) = \dfrac{3x+1}{x-5}$

3. $f(x) = \sqrt{16 - x}$

4. $f(x) = \sqrt{9 - x^2}$

5. $\dfrac{4x}{\sqrt{13 - x}}$

Solution

1. Domain $= \{x{:}x \in \mathbb{R}\}$, all numbers are defined on the set of real numbers.
2. $f(x) = \frac{3x+1}{x-5}$

   Equating the denominator to zero,

   We have

   $x - 5 = 0$

   $x = 5$

   Domain $= \{x{:}x \in \mathbb{R},\ except\ x = 5\}$ or $\{x{:}x \in \mathbb{R},\ x \neq 5\}$
3. $\mathrm{f}(x) = \sqrt{16 - x}$

   $16 - x \leq 0$

   $x \geq 16$

   Domain $= \{x{:}x \in \mathbb{R}, x \geq 16\}$ or $\{x{:}x \in \mathbb{R}, except\ x \leq 16\}$
4. $f(x) = \sqrt{9 - x^2}$

   $9 - x^2 \leq 0$

   $x^2 \geq 9$

   $x \geq \pm 3$

   Domain $= \{x{:}x \in \mathbb{R}, -3 \leq x \leq 3\}$
5. $\frac{4x}{\sqrt{13 - x}}$

   Domain $= \{x{:}x \in \mathbb{R},\ x < 13\}$ or $\{x{:}x \in \mathbb{R}, except\ x \geq 13\}$

Example 21

Determine the domain of the function $f(m) = \frac{m+5}{m^2 + m - 12}$

Solution

Equating the denominator to zero

$m^2 + m - 12 = 0$

$(m^2 + 4m) - (3m - 12) = 0$

$m(m + 4) - 3(m + 4) = 0$

$(m - 3)(m + 4) = 0$

$(m - 3) = 0\ or\ (m + 4) = 0$

$m = 3$ *or* $m = -4$

Domain $= \{x{:}x \in \mathbb{R}, x \neq -4 \text{ or } 3\}$ or Domain $= \{x{:}x \in \mathbb{R}, \text{except } x = -4 \text{ or } x = 3$
$\}$

Example 22

The domain of the function $f(x) = \frac{1}{x-2}$ is the set of all real numbers except $x = 2$

i.e., Domain of y is $\{x{:}x \in \mathbb{R}, x \neq 2\}$

Example 23

State the largest possible domain of the function defined by $f{:}x \to \frac{2x-1}{2x^2-9x-5}$

Solution

$f(x) = \frac{2x-1}{2x^2-9x-5}$

The function, $f(x)$ is a rational function and hence, is only defined if $2x^2 - 9x - 5$
$\neq 0$

If $2x^2 - 9x - 5 = 0$,

$2x^2 - 10x + x - 5 = 0$

$2x(x-5) + (x-5) = 0$

$(x-5)(2x+1) = 0$

$x - 5 = 0$ or $2x + 1 = 0$

$x = 5$ or $x = -\frac{1}{2}$

$\therefore \text{Domain} = \left\{x{:}x \in \mathbb{R}, x \neq -\frac{1}{2}, x \neq 5\right\}$

Example 24

Find the domain of g if $g{:}x \to \sqrt{3x-4}$

Solution

Since $g(x)$ is a radical function, the expression, $3x - 4$ must be positive for the function to be defined

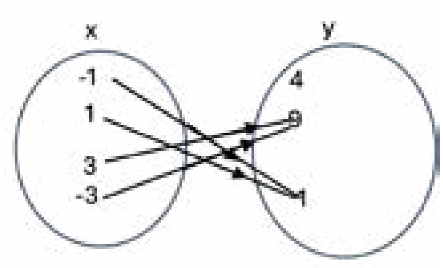
$3x - 4 \geq 0$

$3x \geq 4$

$x \geq \frac{4}{3}$

$\therefore$ *Domain of* $g(x) = \left\{x{:}x \in \mathbb{R}, x \geq \frac{4}{3}\right\}$

Range of a Function



The range is the set of output or image points, or they are the elements in the co-domain that have counterparts in the domain. In the diagram above, which elements in the codomain have a counterpart in the domain? They are {1,9} and this represents the range.

One of the ways to find the range is to first make x the subject and find the values of y which make *x* defined.

Example 25

Find the range of the following functions

1. $f(x) \rightarrow 2x - 3$
2. $f(x) \rightarrow \frac{5 + 2x}{3 - 4x}$
3. $f\left(x\right) \rightarrow \sqrt{25 - x^2}$
4. $g(x) \rightarrow \frac{1}{8x - 13}$

Solution

1. Let $f(x) = y = 2x - 3$ making x the subject

   $y + 3 = 2x$

   $x = \frac{y + 3}{2}$

   Range $= \{y{:}y \in \mathbb{R}, \}$

2. $y = \frac{5+2x}{3-4x}$

$y(3-4x) = 5 + 2x$

$3y - 5 = 2x + 4xy$

$3y - 5 = 2x(1 + 2y)$

$x = \frac{3y-5}{1+2y}$

$1 + 2y = 0$

$y = -\frac{1}{2}$

Range $= \left\{ y{:}y \in \mathbb{R}, y \neq -\frac{1}{2} \right\}$

3. $f\left(x\right) \to \sqrt{25 - x^2}$

$y = \sqrt{25 - x^2}$

$y^2 = 25 - x^2$

$x^2 = 25 - y^2$

$x = \sqrt{25 - y^2}$

Range $= \{y{:}y \in \mathbb{R}, y \geq 5\}$

4. $y = \frac{1}{8x-13}$

$y(8x - 13) = 1$

$8xy - 13y = 1$

$8xy = 1 + 13y$

$x = \frac{1+13y}{8y}$

Range $= \{y{:}y \in \mathbb{R}, y \neq \mathbf{0}\}$

Activity 3.7

Find the range of the following either in pairs or individually and compare your answer with the one provided against the various functions.

Function	Range
1. $\boldsymbol{h}(\boldsymbol{x}) \to 10\boldsymbol{x}$	$\{y{:}y \in \mathbb{R}, y \neq 0\}$
2. $\boldsymbol{h}(\boldsymbol{x}) \to \frac{3}{\boldsymbol{x}+5}$	$\{y{:}y \in \mathbb{R}, y \neq 0\}$
3. $\boldsymbol{g}(\boldsymbol{x}) \to \frac{1}{2}\boldsymbol{x} + 5$	$\{y{:}y \in \mathbb{R}\}$
4. $\boldsymbol{h}\left(\boldsymbol{x}\right) \to \sqrt{2\boldsymbol{x} - 9}$	$\{y{:}y \in \mathbb{R}\}$
5. $\boldsymbol{f}\left(\boldsymbol{x}\right) \to \frac{2x+7}{7x-4}$	$\left\{y{:}y \in \mathbb{R}, y \neq \frac{2}{7}\right\}$

Zeroes of a Function

The zero of a function, say $p(x)$ are the values of x for which $p(x) = 0$

To solve for the zeros, equate the function to zero and find the value(s) of x.

Example 26

Find the zeros of the function. $\boldsymbol{f(x) \to 2x - 5}$

Solution

$f(x) = 0$

$2x - 5 = 0$

$2x = 5$

$x = \frac{5}{2}$

Example 27

Find the zeros of the function. $\boldsymbol{f(x) \to \frac{x - 4}{2x + 16}}$

Solution

$\frac{x - 4}{2x + 16} = 0$

$x - 4 = 0(2x + 16)$

$x - 4 = 0$

$x = 4$

Example 28

Find the zeros of the following functions

a. $h\left(x\right) \to \frac{2x^2 + x - 3}{5x - 6}$

b. $h(x) \to \frac{2x - 5}{3x^2 - 17}$

c. $f\left(x\right) \to \frac{16 - x^2}{8}$

d. $g(x) \to \frac{\sqrt{2x - 9}}{3x^2 + 5x + 2}$

e. $m(x) = \log_{81}(3x - 1)$

Solution

a. $\frac{2x^2+x-3}{5x-6}=0$

$2x^2+x-3=0(5x-6)$

$2x^2+x-3=0$

$2x^2-2x+3x-3=0$

$2x(x-1)+3(x-1)=0$

$(2x+3)(x-1)=0$

$x=-\frac{3}{2}\ or\ x=1$

b. $0=\frac{2x-5}{3x^2-17}$

$2x-5=0(3x^2-17)$

$2x-5=0$

$2x=5$

$x=\frac{5}{2}$

c. $0=\frac{16-x^2}{8}$

$16-x^2=8(0)$

$4^2-x^2=0$ applying the difference of two squares

$(2-x)(2+x)=0$

$(2-x)=0\ or\ (2+x)=0$

$x=2\ or\ x=-2$

d. $0=\frac{\sqrt{2x-9}}{3x^2+5x+2}$

$\sqrt{2x-9}=0(3x^2+5x+2)$

$\sqrt{2x-9}=0$

$2x-9=0^2$

$2x-9=0$

$2x=9$

$x=\frac{9}{2}$

e. $0=\log_{81}(3x-1)$

$81^0=3x-1$

$1=3x-1$

$2=3x$

$x=\frac{2}{3}$

Example 29

Find the zeroes of the function, $f\colon x \to \frac{x+3}{x-2}$

Solution

$f(x) = \frac{x+3}{x-2}$

For $f(x) = 0$,

$\frac{x+3}{x-2} = 0$

$x + 3 = 0$

$x = -3$

The zero of f is thus -3

Example 30

State the zeros of the function defined by $f\colon x \to \frac{2x-1}{2x^2 - 9x - 5}$

Solution

For $f(x) = \frac{2x-1}{2x^2 - 9x - 5} = 0$,

$2x - 1 = 0$

$x = \frac{1}{2}$

SURJECTIVE AND INJECTIVE FUNCTIONS

For two sets, A and B the following statements hold

a. A function $f: A \rightarrow B$ is called **surjective** (or is said to map ***A* onto *B***) if $B =$ *range of* f.

b. A function $f: A \rightarrow B$ is called **injective** (or **one-to-one**) if, for all a_1 and a_2 in A, $f(a_1)=f(a_2)$ implies that $a_1 = a_2$.

c. The graph of an injective function must pass the horizontal line test

d. A function $f: A \rightarrow B$ is called bijective if it is both surjective and injective.

e. A function f is strictly increasing if $f(x) > f(y)$ when $x > y$.

f. A function f is strictly decreasing if $f(x) < f(y)$ when $x < y$.

g. A function f is increasing if $f(x) \geq f(y)$ when $x > y$.

h. A function f is decreasing if $f(x) \leq f(y)$ when $x < y$.

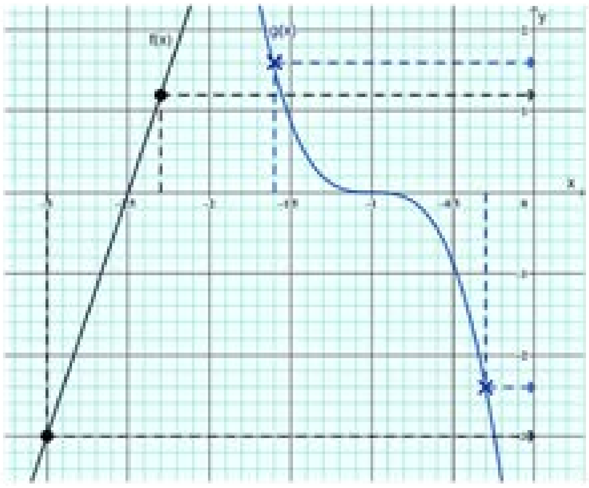


Fig 3.7: Increasing and decreasing functions

From **Fig 3.7: Increasing and decreasing functions. 7,** $f(x)$ is a one-to-one function since no part of the graph of $f(x)$ is horizontal (parallel to the $x - axis$). No more than one value of x has the same value of y. Likewise, $g(x)$ is a one-to-one function as for every position on the $x - axis$ (value of x), there is but one height (corresponding value of y). Both curves pass the horizontal line test (any horizontal line drawn will cut each curve at only one point on each) and the only confirms that they are injective.

Also, $f(x)$ can be described as a strictly increasing function since for $x = -3$, which is less than $x = -2.3$, $y = -3$ which is also less than $y = 1.2$ (corresponding to $x = -2.3$). the reverse can be said of $g(x)$. The height of $g(x)$ for a smaller value of x, i.e., $x = -1.6$ is higher than a bigger value of x, i.e., $x = -0.3$. As the graph of

$g(x)$ is traced from left to right (increasing values of x), the graph goes downwards (decreasing values of y) hence $g(x)$ is a strictly decreasing function.

Conclusions: A function is one-to-one if it is either strictly increasing or strictly decreasing. A one-to-one function never assigns the same value to two different elements of the domain.

For an onto function, range and co-domain are equal.

Example 31

If $f:\mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 3x + 7$, show the function above is one-to-one

Solution

$f(x) = 3x + 7$

$f(y) = 3y + 7$

For one to one function;

$f(x) = f(y) \Rightarrow x = y$

$3x + 7 = 3y + 7$

$3x = 3y$

$x = y$

The function is one to one.

Example 32

Show that the function $f(x) = \frac{5x + 7}{4}$, $x \in \mathbb{R}$ is one to one.

Solution

$f(x) = \frac{5x + 7}{4}$

$f(y) = \frac{5y + 7}{4}$

For one to one function.

$f(x) = f(y) \Rightarrow x = y$

$\frac{5x + 7}{4} = \frac{5y + 7}{4}$

$5x + 7 = 5y + 7$

$5x = 5y + 7 - 7$

$5x = 5y$

$x = y$

The function is one to one.

Example 33

Show that the function $f(x) = \frac{13}{x-8}$, $x \in \mathbb{R}$, $x \neq 8$, is one to one.

Solution

$f(x) = \frac{13}{x-8}$

$f(y) = \frac{13}{y-8}$

For one to one function.

$f(x) = f(y) \Rightarrow x = y$

$\frac{13}{x-8} = \frac{13}{y-8}$

$13(x-8) = 13(y-8)$

$13x - 108 = 13y - 108$

$13x = 13y - 108 + 108$

$13x = 13y$

$x = y$

The function is one to one.

Example 34

Given that $f(x) = x^2$, *for all* $x \geq 0$, show that $f(x)$ is one to one.

Solution

$f(x) = x^2$

$f(y) = y^2$

For one to one function.

$f(x) = f(y) \Rightarrow x = y$

$x^2 = y^2$

$x = \pm y$

$x = y,$

The function is one to one.

In this example, we picked $x = y$ and ignored $x = -y$ because, the condition given was x must be greater zero.

Example 35

Given that $f(x)=\frac{1}{x^2-4}$, $x \neq \pm 2$, determine whether or not $f(x)$ is one to one.

Solution

$f(x) = \frac{1}{x^2-4}$

$f(y) = \frac{1}{y^2-4}$

For one to one function;

$f(x) = f(y) \Rightarrow x = y$

$\frac{1}{x^2-4} = \frac{1}{y^2-4}$

$x^2 - 4 = y^2 - 4$

$x^2 = y^2 - 4 + 4$

$x^2 = y^2,$

$x = \pm y$

$x = -y \; or \; x = y$

In this case, we are not told that $x \neq -y$, therefore there are two options for x.

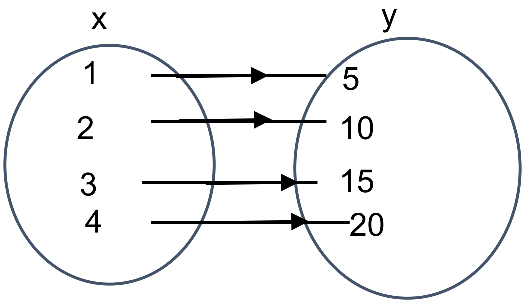
Hence, the function is not one to one.

Example 36

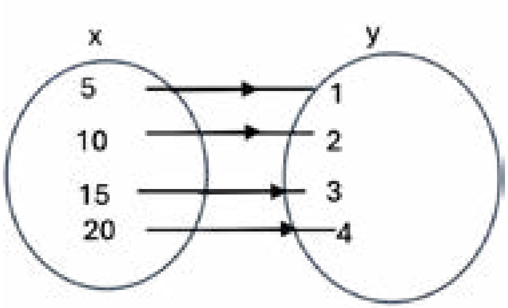
A Bijective function example: Let us consider a real-life scenario of matching students to their unique student IDs. Suppose we have a class of students and a set of unique student identification numbers. A bijective function can be established between the set of students and the set of student IDs, ensuring that each student is assigned a distinct student ID, and no two students have the same ID.

INVERSE OF FUNCTIONS

Inverse functions are functions by which mapping is from the range set to the domain set. Thus, elements of the range set rather map onto elements of the domain set. Let us consider the mapping below.



The above mapping depicts a function which can be denoted by f: $x \rightarrow 5x$. It is the mapping from the domain to the range. Now, consider the following mapping.



This mapping depicts a function that can be denoted by $f^{-1} : x \rightarrow \frac{1}{5}x$. The notation $f^{-1} : x$ is the inverse function of f: x. Hence the inverse of f: $x \rightarrow 5x$ is $f^{-1} : x \rightarrow \frac{1}{5}x$. Note the notation f: x can be expressed as (x) and $f^{-1} : x$ can also be expressed as $f^{-1}(x)$.

The inverse of bijection f denoted as f^{-1} is a function which assigns to b, a unique element a such that $f(a) = b$. hence $f^{-1}(b) = a$

The inverse of a function is visually represented as the reflection of the original function over the line $y = x$ as shown in Fig 3.8: Inverse functions . By definition, functions that have inverses (known as invertible functions) must be bijective and inverse functions are also bijective and hence pass the vertical line and the horizontal line tests

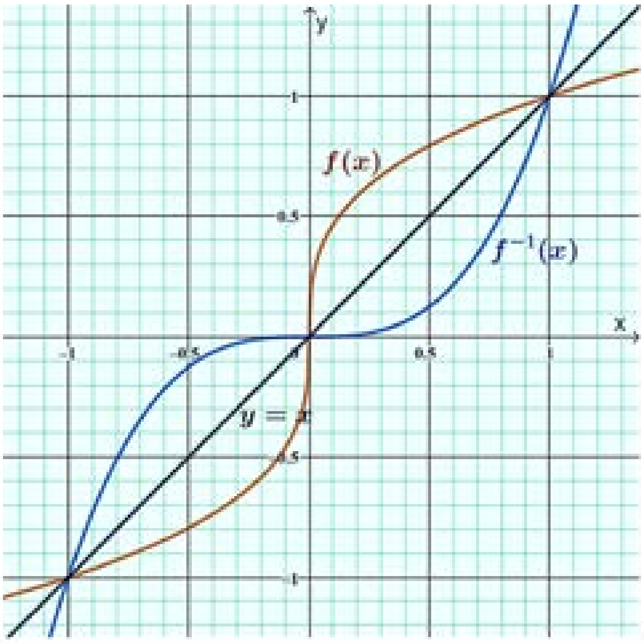


Fig 3.8: Inverse functions

Example 37

You are the manager of a car rental company, and you want to determine at what time during the day the number of available rental cars reaches a specific number (let's say 20 cars). You know that the number of available rental cars (C) at any given time of the day (t) follows a simple linear relationship, represented by the function $f(t) = 40 - 2t$. Now, you want to find the time (t) at which the number of available rental cars is exactly 20 ($C = 20$).

Set up the equation:

We are given the function $f(t) = 40 - 2t$, and we want to find t when $f(t) = 20$.

So, the equation becomes:

$f(t) = 20$

$40 - 2t = 20$

Solve for t:

To solve for t, we need to isolate the variable t on one side of the equation. Let's proceed with the steps:

Subtract 40 from both sides of the equation:

$-2t = 20 - 40$

$-2t = -20$

Now, divide both sides by -2 to solve for t:

$t = \frac{(-20)}{(-2)}$

$t = 10$

Interpretation:

The solution $t = 10$ tells us that at 10 units of time (which could be hours, minutes, etc., depending on the context), the number of available rental cars will be exactly 20.

Write the expression for the inverse function:

Since the function $f(t) = 40 - 2t$ is a one-to-one function (it represents a straight line with a non-zero slope), it has an inverse function.

The inverse function of $f(t)$ is denoted as $f^{-1}(t)$. To find the inverse, interchange the roles of t and $f(t)$ and solve for $f^{-1}(t)$:

$f(t) = 40 - 2t$

Swap t and $f(t)$:

$t = 40 - 2f^{-1}(t)$

Now, solve for $f^{-1}(t)$

$2f^{-1}(t) = 40 - t$

Finally, divide by 2:

$f^{-1}(t) = \frac{(40 - t)}{2}$

So, the expression for the inverse function is $f^{-1}(t) = \frac{(40 - t)}{2}$.

Interpretation: The inverse function tells us that given the number of available rental cars (t), we can determine the time $\left(f^{-1}(t)\right)$at which that specific number of cars will be available. *(NB: Keep in mind that in real-life scenarios, functions and their inverses might not always have straightforward interpretations, and the context of the problem will play a crucial role in understanding their meanings).*

Explore with the help of a classmate or teacher the procedure in finding inverse of a function $f(x) = 2x + 5$ by calculation.

Are your findings the same as Salma whose work was confirmed by her group members and the class teacher as correct which is presented in the table below?

STEPS	WORKINGS
STEP 1 Let your $f(x) = f-1(x)$	$f(x) = f^{-1}(x) = 2x + 5$
STEP 2 Interchange x and $f-1(x)$	$x = 2f^{-1}(x) + 5$
Make $f^{-1}(x)$ subject (write your expression in terms of x)	$x - 5 = 2f^{-1}(x)$ $\frac{x-5}{2} = f^{-1}(x)$
Inverse	$f^{-1}(x) = \frac{x-5}{2}$. $\{x{:}x \in \mathbb{R}\}$

Salma did well by stating the domain of the inverse of her function.

Akasi rather presented his work as follows, study it in pairs or individually and write your observation down.

STEPS	WORKINGS
Step 1 Let your $f(x) = f^{-1}(x)$	$f(x) = f^{-1}(x) = 2x + 5$
Step 2 Make x the subject	$f^{-1}(x) = 2x + 5$ $f^{-1}(x) - 5 = 2x$ $x = \frac{f^{-1}(x) - 5}{2}$
Step 3 Interchange variables	$f^{-1}(x) = \frac{x-5}{2}$
Inverse of the function	$f^{-1}(x) = \frac{x-5}{2}$. $\{x{:}x \in \mathbb{R}\}$

Explore the solutions of both Salamatu's and Akasi's and share your thoughts with your group or the entire class.

Let us now solve more examples, either in small in groups or solve them individually and show them to your teacher or classmates the inverses of the following functions.

1. $f(x) = 9 - 4x$
2. $g(x) = \frac{2x-7}{5}$

3. $h(x) = \sqrt{15x - 2}$

4. $s(t) = u + at$

Solution

1. $f(x) = 9 - 4x$

$f(x) = f^{-1}(x) = 9 - 4x$

$f^{-1}(x) = 9 - 4x$

$x = 9 - 4f^{-1}(x)$

$4f^{-1}(x) = 9 - x$

$f^{-1}(x) = \frac{9 - x}{4}, \{x{:}x \in \mathbb{R}\}$

2. $g(x) = \frac{2x - 7}{5}$

$gx = g^{-1}(x) = \frac{2x - 7}{5}$

$g^{-1}(x) = \frac{2x - 7}{5}$

$x = \frac{2g(x)^{-1} - 7}{5}$

$5x = 2g^{-1}(x) - 7$

$5x + 7 = 2g^{-1}(x)$

$\frac{5x + 7}{2} = g^{-1}(x)$

$\frac{5x + 7}{2} = g^{-1}(x), \{x{:}x \in \mathbb{R}\}$

3. $h(x) = h^{-1}(x) = \sqrt{15x - 2}$

$h^{-1}(x) = \sqrt{15x - 2}$

$x = \sqrt{15h^{-1}(x) - 2}$

$x^2 = 15h^{-1}(x) - 2$

$x^2 + 2 = 15h^{-1}(x)$

$h^{-1}(x) = \frac{x^2 + 2}{15}, \{x{:}x \in \mathbb{R}\}$

4. $s(t) = u + at$

$s(t) = s^{-1}(t) = u + at$

$t = u + as^{-1}(t)$

$t - u = as^{-1}(t)$

$\frac{t - u}{a} = s^{-1}(t), \{t{:}t \in \mathbb{R}\}$

In a mathematics quiz between Saddique SHS and St Paul SHS, the problem of the day was:

Two functions f and g are defined by $f(x) \to 15x - \frac{2x}{3}$ and $g(x) \to 3x + 5$. The schools were asked to find the inverse of these functions. The table below shows the solution of the two schools and the marks awarded by the quiz mistress.

	Sadique SHS	St Paul SHS
1.	$f(x) = f^{-1}(x) = 15x - 2\frac{x}{3}$ $x = 15f^{-1}(x) - \frac{2f^{-1}(x)}{3}$ [1 MARK] $3x = 45f^{-1}(x) - 2f^{-1}(x)$ [1 MARK] $3x = f^{-1}(x)[45 - 2]$ [1 MARK] $3x = 43f^{-1}(x)$ $f^{-1}(x) = \frac{3x}{43}$ [1 MARK $(-\frac{1}{2})$] TOTAL MARK = $3\frac{1}{2}$	$x = 15f^{-1}(x) - \frac{2f^{-1}(x)}{3}$ [1 MARK] $3x = 45f^{-1}(x) - 2f^{-1}(x)$ [1 MARK] [1 MARK] $3x = 43f^{-1}$ $\frac{3x}{43} = f^{-1}(x), \{x : x \in \mathbb{R}\}$ [1 MARK] TOTAL MARK = 4
2.	$g(x) = g^{-1}(x) = 3x + 5$ $g^{-1}(x) = 3x + 5$ $g^{-1}(x) + 5 = 3x$ [0 MARK] $\frac{g^{-1}(x) + 5}{3} = x$ [0 MARK] $\frac{x + 5}{3} = g^{-1}(x), \{x : x \in \mathbb{R}\}$ [0 MARK]	$g(x) = g^{-1}(x) = 3x + 5$ $3x = 5 - g^{-1}(x)$ [0 MARK] $3g^{-1}(x) = 5 - x$ [0 MARK] $g^{-1}(x) = \frac{5 - x}{3}$ [0 MARK] $g^{-1}(x) = \frac{5 - x}{3} = g^{-1}(x), \{x : x \in \mathbb{R}\}$

You were appointed as a consultant to resolve a protest written by the schools for being marked down. Will your model solution be different from what the two schools presented? How will you explain each point where they lost the marks for an amicable settlement?

COMPOSITE FUNCTIONS

Consider a real-life scenario where you are running a food delivery service. You have three functions:

a. The first function calculates the total cost of the food order based on the items selected and their prices.

b. The second function calculates the delivery charge based on the distance between the restaurant and the customer's location.

c. The third function calculates the total time it will take to deliver the order, including preparation and delivery time. Now, to find the total cost for a specific food order, you can create a composite function by combining these three functions:

Total cost = *Third Function* (*Second Function* (*First Function* (*food_items*))).

In this example, the output of the first function (total cost of food items) becomes the input for the second function (delivery charge calculation), and then the output of the second function becomes the input for the third function (total time calculation).

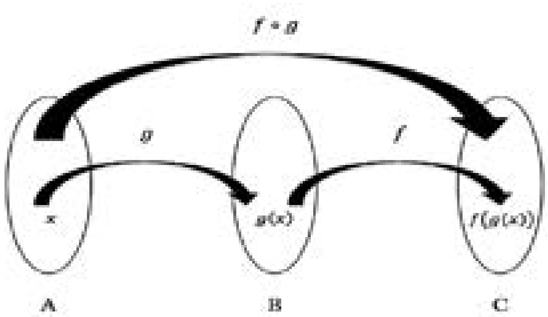


Fig 3.9: Composite functions

Some common properties of composite functions.

Properties include:

1. $fog \neq gof$, except when $g(x) = f(x)$
2. $(f^{-1} \circ f)\,(a) = f^{-1}(f(a) = f^{-1}f(a) = a$
3. $(f \circ f^{-1})(b) = f^{-1}(f(b) = f^{-1}f(b) = b$
4. $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$
5. If f and g both are one to one function, then $f \circ g$ is also one to one.
6. If f and g both are onto function, then $f \circ g$ is also onto.
7. If f and $f \circ g$ both are one to one function, then g is also one to one.

8. If f and fog are onto, then it is not necessary that g is also onto.

Example 38

If $h(x) = -2x^2 + 3x$ and $f(x) = 3x - 2$, evaluate

a. $(h \circ f)(-2)$

b. $(f \circ h)(-2)$

Solution

$h(x) = -2x^2 + 3x$ and $f(x) = 3x - 2$

a. $(h \circ f)(x) = -2(3x - 2)^2 + 3(3x - 2)$

$= -2(9x^2 - 12x + 4) + 9x - 6$

$= -18x^2 + 33x - 14$

$(h \circ f)(-2) = -18(-2)^2 + 33(-2) - 14$

$= -152$

b. $(f \circ h)(x) = 3(-2x^2 + 3x) - 2$

$= -6x^2 + 9x - 2$

$(f \circ h)(-2) = -6(-2)^2 + 9(-2) - 2$

$= -44$

Example 40

If $g(x) = 8x - 3$ and $f(x) = 5 + x$, evaluate

a. $(g \circ f)$

b. $(f \circ g)$

c. $(f \circ f)$

Solution

a. $(g \circ f) = g(5 + x)$

$= g(5 + x) = 8(5 + x) - 3$

$= 40 + 8x - 3$

$(g \circ f) = 8x + 37$

b. $(f \circ g) = f(8x - 3)$

$= f(8x - 3) = 5 + (8x - 3)$

$= 5 + 8x - 3$

$(f \circ g) = 8x + 2$

c. $(f \circ f) = f(5 + x)$

$= f(5 + x) = 5 + (5 + x)$

$= 5 + 5 + x$

$(f \circ f) = 10 + x$

Activity 3.8

In small groups, complete the following table and verify the properties of composite functions for each of the questions.

1. $g(x) = x + 4$ *and* $f(x) = 5x - 16$
2. $f(x) = 1 - x$ and $f(x) = 3x^2$

Property	**Remarks** **Write Verified if the property is true and if not, write Not verified**
$fog \neq gof$, except $g(x) = f(x)$	
$(f^{-1} \circ f)\,(a) = f^{-1}(f(a)$ $= f^{-1}f(a) = a$	
$(f \circ f^{-1})(b) = f^{-1}(f(b)$ $= f^{-1}f(b) = b$	
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$	
If f and g both are one to one function, then $f \circ g$ is also one to one.	
If f and g both are onto function, then $f \circ g$ is also onto	

LINEAR FUNCTIONS

Hello, may I have your attention as we look at the concepts of linear relations and parabola that we see around us and in our daily activities. May I also ask that you pay attention to the behaviour of these relations and be able to make sense out of them in real life situations?

Linear and parabolic functions are among the most essential types of functions. They are part of a wider range of functions called polynomial functions. Whether you are charting the path of a projectile, analysing trends in data, or simply modelling real-world scenarios, these functions play crucial roles.

Linear functions have a constant rate of change, which makes them ideal for representing relationships that evolve uniformly over time or space. They serve as the building blocks for more complex mathematical concepts and are prevalent in various fields, from economics to physics.

Parabolic functions, on the other hand, exhibit a curved, symmetrical shape and are characterized by a squared term. They often describe phenomena such as the trajectory of a thrown object, the shape of a suspension bridge cable, or the arc of a fountain's water.

We will focus on the properties, applications, and graphical representations of both linear and parabolic functions. Through exploration, problem-solving, and real-world examples, you will gain a deeper understanding of these fundamental mathematical concepts and their significance in everyday life.

We discussed linear sequences, and it was established that the general rule for a linear sequence is $u_n = a + (n - 1)d$ with the values of u_n and $(n - 1)$ changing from term to term. This presupposes that there are only two variables (u_n and $(n - 1)$) as the value of 'a' and d would be constant for all the terms of the same linear sequence. One of the features that qualifies $u_n = a + (n - 1)d$ to be a linear equation is the presence of exactly two variables

Linear functions are of the forms:

1. $f(x) = ax + b$: Variables are x and $f(x)$ while 'a' and 'b' are constants
2. $y = mx + c$: Variables are x and y while 'm' and 'c' are constants
3. $y - y_1 = m(x - x_1)$: Variables are x and y while y_1, m and 'x_1' are constants

Note: It must be noted that the degree of linear functions is one whilst that of quadratic functions whose nature is parabolic is two.

GRAPHS OF LINEAR FUNCTIONS

Since linear functions have exactly two variables, any two-dimensional plane / system like the Cartesian coordinate system can be used to represent them. Corresponding values of the two variables are taken as pairs and plotted as locations on the grid. The infinitely many points that satisfy a linear function or equation join to form a straight line.

Some notes:

For a linear equation, $y = mx + c$,

1. The value of the y intercept is the y coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $x = 0$
2. The value of the x intercept is the x coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $y = 0$
3. m is the slope or gradient of the line or a plane. It is the ratio of the increment / decrement of values of the dependent variable to the increment / decrement of values of the independent variable. It often written as $m = \frac{\Delta y}{\Delta x}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$ where (x_1, y_1) and (x_2, y_2) are points on the line. Note that, $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ since x_1 and y_2 are not corresponding coordinates just as x_2 and y_1 are not coordinates of the same point
4. If $m > 0$, the graph strictly increases and when $m < 0$, it strictly decreases

Example 41

To graph the equation $y = 2x - 1$, we can use

A. *the intercepts method:*
   - i. Start by plotting the y-intercept: $(0, -1)$, which can be obtained by finding the value of y when $x = 0$
   - ii. Plot the x-intercept: $(0.5, 0)$, which can be obtained by finding the value of x when $y = 0$
   - iii. Connect the points to get a straight line as demonstrated below.

***Note**: that since linear functions are polynomial functions and hence have the domain and ranges as the set of real numbers and the fact that a straight line has no extremities, it should be emphasised that lines drawn to connect the points should stretch to the ends of the grid.*

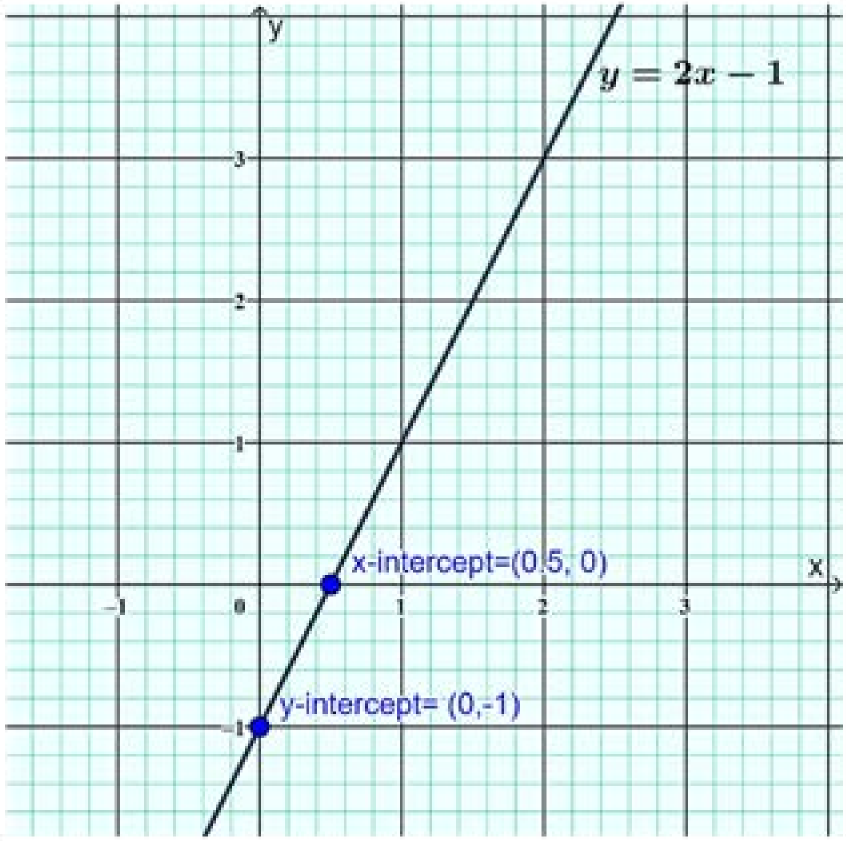


Fig 3.10: Intercept method of graphing linear equations

B. *Using the gradient and y-intercept:*

i. Start by plotting the y-intercept: $(0, -1)$, which can be obtained by finding the value of y when $x = 0$

ii. Use the slope, $m = 2$ to find other points. The slope can be found by comparing the equation to be graphed to the most appropriate general equation of a line, in this example, $y = mx + c$

iii. Since the slope is in the ratio 1:2, we can go up 2 units and right 1 unit from the y-intercept to find the next point $(0 + 1, -1 + 2) = (1, 1)$ and the next, $(1 + 1, 1 + 2) = (2, 3)$ and so on.

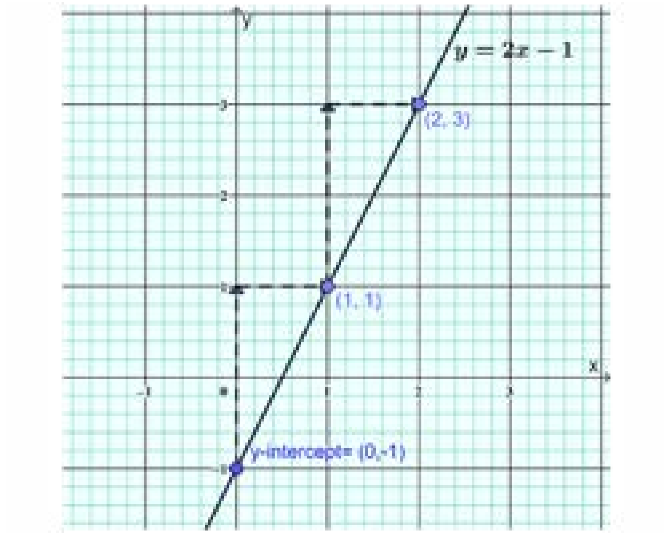


Fig 3.11: Gradient and y-intercept method of graphing linear equations

C. *Using the gradient and x-intercept:*

i. Start by plotting the x-intercept: (0.5, 0), which can be obtained by finding the value of x when $y = 0$

ii. Use the slope, $m = 2$ to find other points.

iii. Since the slope is in the ratio 1:2, we can go up 2 units and right 1 unit from the y-intercept to find the next point $(0.5 + 1, 0 + 2) = (1.5, 2)$ and the next and $(1.5 + 1, 2 + 2) = (2.5, 4)$ and so on.

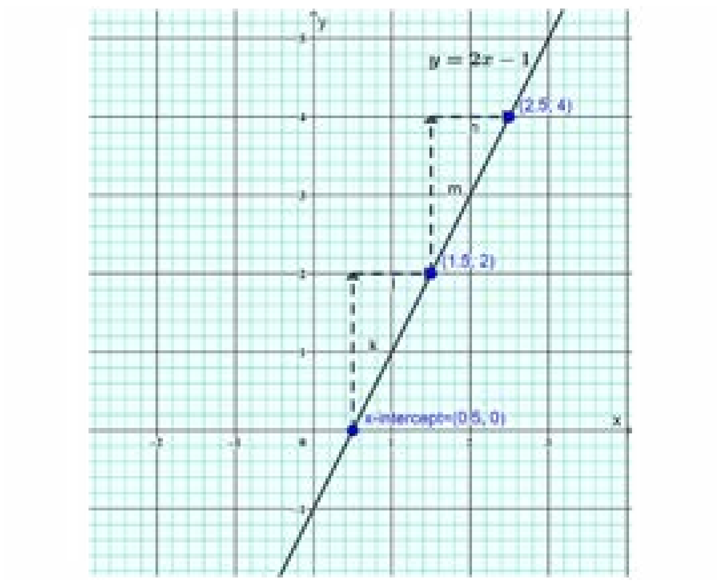


Fig 3.12: Gradient and x-intercept method of graphing linear equations

D. *Plotting points in a suitable range:*

In some cases, it is not convenient to plot the intercepts of the line on the grid. Such situations can occur when multiple graphs have to be constructed on the same grid and a common scale for the axes that allows for all the intercepts to be located and easily plotted cannot be easily obtained. In a case such as this,

i. A convenient value for x may be chosen, say $x = 2$ and its corresponding y value ($y = 3$) found by substituting it into the equation of the line

ii. The process is repeated for a different value of x, say, $x = -2$ to obtain $(-2, -5)$

iii. The two points obtained can then be plotted and a straight line drawn to join the points to the edges of the grid

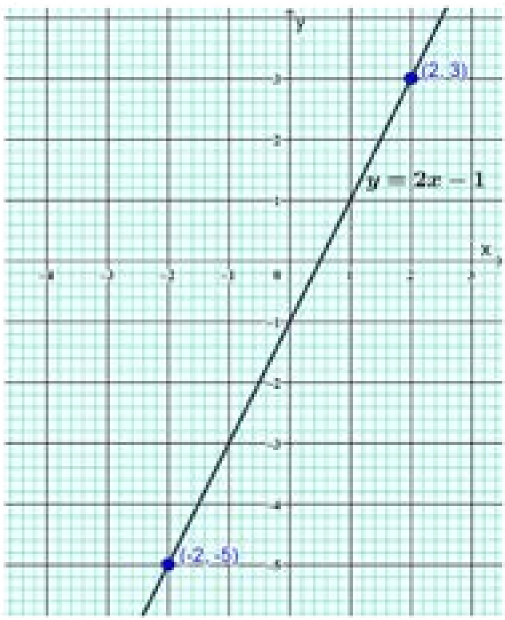
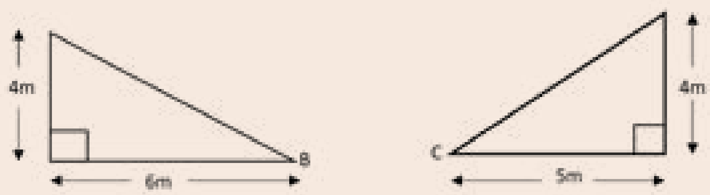


Fig. 3.13: Graphing linear equations by plotting suitable points.

Activity 3.9

Consider the figures below:



In small groups or individually, calculate the slope of AB and the slope of CD and by comparing the results determine which one is gentler.

Whichever approach you choose to calculate the slopes, if you arrive at same results as the suggested solution following then you have understood the concept. Attention should be paid on the negative sign for the slope of AB.

$$\text{Slope of AB} = \frac{\textit{Vertical drop}}{\textit{Horizontal distance moved}}$$

$$= \frac{-4}{6}$$

$$= \frac{-2}{3}$$

$$\text{Slope of CD} = \frac{\textit{Vertical rise}}{\textit{Horizontal distance moved}}$$

$$= \frac{4}{5}$$

Since the magnitude of the slope of AB is less than the magnitude of the slope of CD, the slope of AB is gentler.

DETERMINING AREAS ENCLOSED BY GRAPHS

The area enclosed by two or more graphs can be found by finding the area of the plane figures that is bounded by the graphs.

Example 42

Find the area bounded by $y = -3x + 6$, the $x - axis$ and the $y - axis$

Solution

The graph of $y = -3x + 6$ intersects the $x - axis$ at (2, 0) and the $y - axis$ at (0, 6) while the two axes intersect at the origin with coordinates (0, 0)

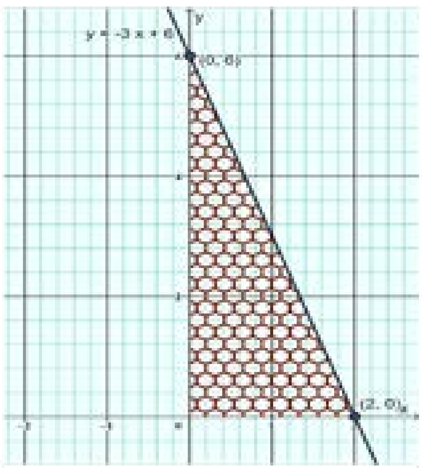


Fig 3.14: Solution region to $y = -3x + 6$, the $x - axis$ and the $y - axis$

The area bounded by $y = -3x + 6$, the $x - axis$ and the $y - axis$ is in the shape of a right triangle with vertices, (0, 0), (0, 6) and (2, 0).

This translates into a base length of 2 *units* and a height of 6 *units*.

The required area is thus: $\frac{1}{2}(2)(6) = 6$ *square units*

Example 43

Consider the linear equation: $2x - 3y = 6$.

i. Draw the graph of this linear equation by hand or using a graphing utility on a coordinate plane.

ii. Find the area bounded by the graph, the $x - axis$, $y - axis$ and $y = 4$

Solution

For $2x - 3y = 6$,

$x - intercept = (3, 0)$

$y - intercept = (0, -2)$

Intersection points between $2x - 3y = 6$ and $y = 4$ is (9, 4)

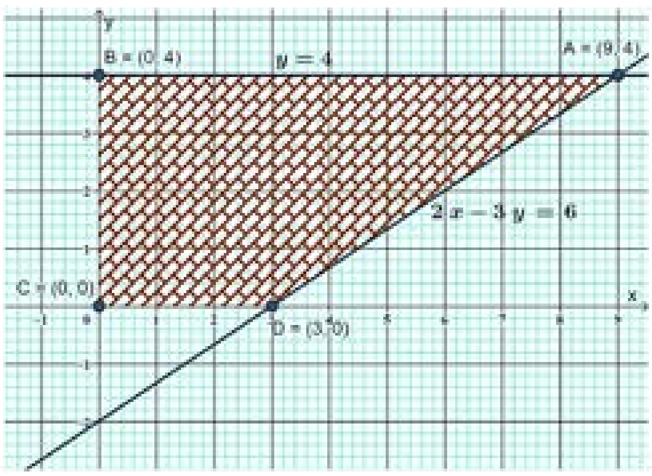


Fig. 3.15: Solution region for $2x - 3y = 6$, $x - axis$, $y - axis$ and $y = 4$

From the graph in Figure 3.15, the bounded area (a trapezium) has vertices at A (9, 4), B(0, 4), C(0, 0) and D(3, 0)

The parallel sides, $|AB|$ and $|DC|$ have lengths, 9 *units* and 3 *units* respectively while the perpendicular height is 4 *units*

$The\ area\ of\ the\ bounded\ region = Area\ of\ ABCD = \frac{1}{2}(|AB| + |DC|)) \times |BC|$

$=\frac{1}{2}(9 + 3)(4)$

$= 24$ *squared units*

MODELLING WITH LINEAR EQUATIONS

Most real-life problems can be translated to the world of mathematics, solved and the solutions interpreted in the real life. Linear functions are among the wide range of mathematical tools for doing just that.

Example 44

A local electronics store is selling a new smartphone model. The store manager has recorded the number of smartphones sold and the corresponding price for each week. The data is as follows:

Week 1: Price *GH¢* 800.00, Number of Smartphones Sold: 30

Week 2: Price *GH¢* 700.00, Number of Smartphones Sold: 40

Week 3: Price *GH¢* 600.00, Number of Smartphones Sold: 50

Week 4: Price *GH¢* 500.00, Number of Smartphones Sold: 60

Construct a linear function that represents the relationship between the price (p) and the number of smartphones sold (n) and predict the number of smartphones that would be sold if the price is reduced to *GH¢* 200.00

Solution

Price (p) / *GH¢*	800	700	600	500
Sales (n)	30	40	50	60

If the required function must be linear, then the pairs from the table must satisfy an equation of the form: $n = mp + c$

From the table,

When $p = 600$, $n = 50$. Thus

$$50 = 600m + c$$

Also, when $p = 500$, $n = 50$. Thus

$$60 = 500m + c$$

A system of two linear equations can be formed and solved to obtain the values of m and c as such:

$$\left.\begin{aligned} 50 &= 600m + c \\ 60 &= 500m + c \end{aligned}\right\}$$

$$-10 = 100m$$

$m = -\frac{10}{100} = -0.1$ and

$$60 = 500\left(-\frac{10}{100}\right) + c$$

$$c = 60 + 50 = 110$$

$\therefore$ the required function is $n(p) = -0.1p + 110$

$\therefore$ When $p = GH¢\ 200.00$, $n(200) = -0.1(200) + 110 = 90$

When the price is reduced to $GH¢$ 200.00, 90 smartphones will be sold

Alternatively,

It can be observed from the table that there is a common difference in the prices i.e., $GH¢$ 100.00 and a common difference in the number of sales too i.e., 10. This suggests that the ratio of difference in the co-domain, range or values of N to the difference in the elements in the domain or values for price will be a constant. That constant is the gradient. The gradient is negative since we expect a decreasing function. While the prices increase, the number of sales decreases, thus:

$$Gradient,\ m = -\frac{10}{100} = -0.1$$

Linear functions are of the form, $y = mx + c$ so we expect the required function to be $n(p) = mp + c$

When $n = 60$, $p = 500$

$$60 = -0.1(500) + c$$

$$c = 60 + 50$$

$$c = 110$$

$\therefore$ the required linear function is $n(p) = -0.1p + 110$

When $p = GH¢\ 200.00$, $n(200) = -0.1(200) + 110 = 90$

$\therefore$ When the price is reduced to $GH¢$ 200.00, 90 smartphones will be sold.

Another solution is to plot the points, (p, n) on the cartesian plane and use the graph to find the gradient and estimate the $y - intercept$ to obtain the values of m and c respectively as illustrated in

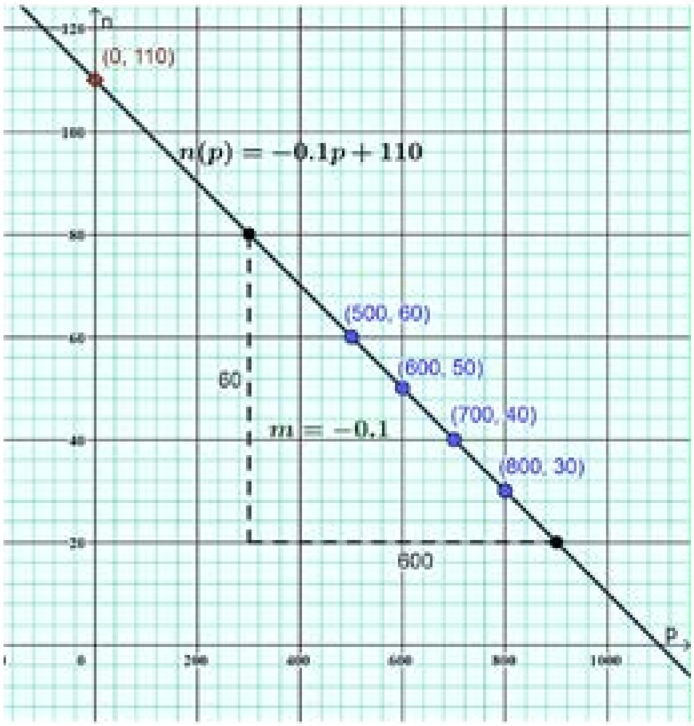


Fig 3.16: Graphical solution to example

From the graph, when $p = GH¢\ 200.00$, $n = 90$

$\therefore$When the price is reduced to $GH¢$ 200.00, 90 smartphones will be sold

PARABOLIC OR QUADRATIC FUNCTIONS

As the name suggests, parabolic functions represent the path taken by points on a parabola.

Parabolic functions are of the forms $f(x) = ax\,2 + bx + c$ where a, b and c are constants and $a \neq 0$. The variables are x and $f(x)$ while 'a' and 'b' are constants. Parabolic functions have a degree of 2.

GRAPHS OF PARABOLIC FUNCTIONS

The graph of a quadratic function in terms of x i.e., $y = ax^2 + bx + c$ is either shaped like the intersection symbol, ∩ thus opening downwards or the union symbol, ∪ opening upward

Some notes:

For a parabolic function, $f(x) = ax^2 + bx + c$ with vertex (turning point) located at (u, v)

1. If $a > 0$,
   a. The graph of $f(x)$ is "$\bigcup$" shaped, same as the turning point
   b. It has a minimum point
   c. It decreases in the interval: $x = (-\infty, u)$ while increasing in the interval: $x = (u, \infty)$
   d. Its range is the interval: $y = [u, \infty)$
2. If $a < 0$
   a. The graph of $f(x)$ is "$\bigcap$" shaped
   b. It has a maximum point, same as the turning point
   c. It increases in the interval: $x = (-\infty, u)$ while decreasing in the interval: $x = (u, \infty)$
   d. Its range is the interval: $y = (-\infty, u]$

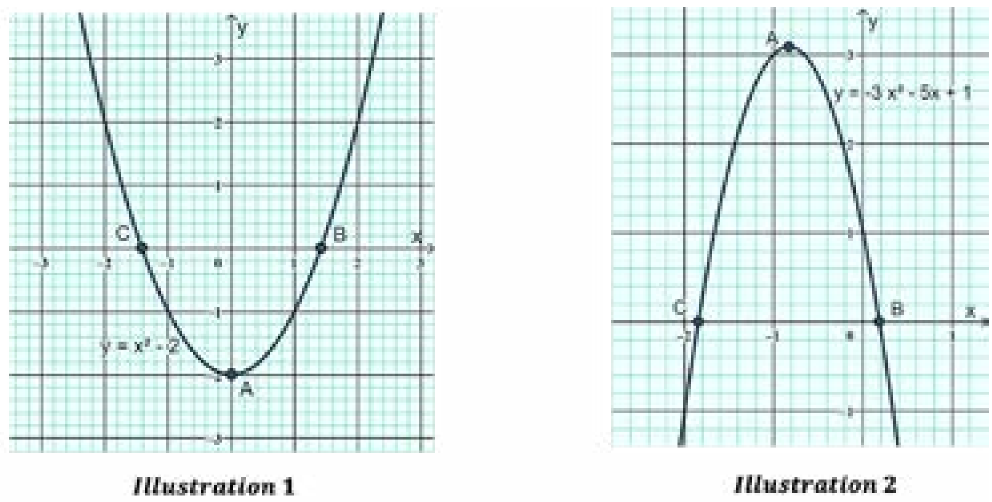


Fig 3.17: Parabolic graphs

Illustration 1 in **Fig 3.17**: Parabolic graphs shows the graph of $y = x^2 - 2$, having a positive number, 1 as the coefficient of the squared term. It has a minimum value at $A(0, -2)$, its turning point. It decreases for values of x in the interval: $(-\infty, 0)$ and increases for $0 < x < \infty$. $y \in [-2, \infty)$ as the values of y would not be less than -2, the y coordinate of the lowest point on the graph.

For $y = -3x^2 - 5x + 1$ (the corresponding graph shown in Illustration 2) however, the coefficient of the squared term, i.e., $-3 < 0$ and thus, is shaped like $\cap$. It has a maximum value at $A(-0.833, 3.083)$, its turning point. It increases for values of x in the interval: $(-\infty, -0.833)$ and increases for $x \in (-0.833, \infty)$ and has a range between negative infinity to 3.0833 (inclusive) as the values of y cannot be greater than 3.0833.

Example 45

Construct a parabolic function whose graph has a minimum value at its vertex at $(1, -2)$ and passes through the point $(2, 1)$.

Solution

Vertex is located at $(1, -2)$

Point on graph $(2, 1)$

From the vertex, a movement of 3 *units* upwards and 1 *unit* to the right results in the location of $(2, 1)$

Since the graph of a parabolic function is bilaterally symmetrical, a movement from the vertex of 3 *units* upwards and 1 *unit* to the left will provide the location of another point $(1 - 1, -2 + 3) = (0, 1)$ which lies on the graph

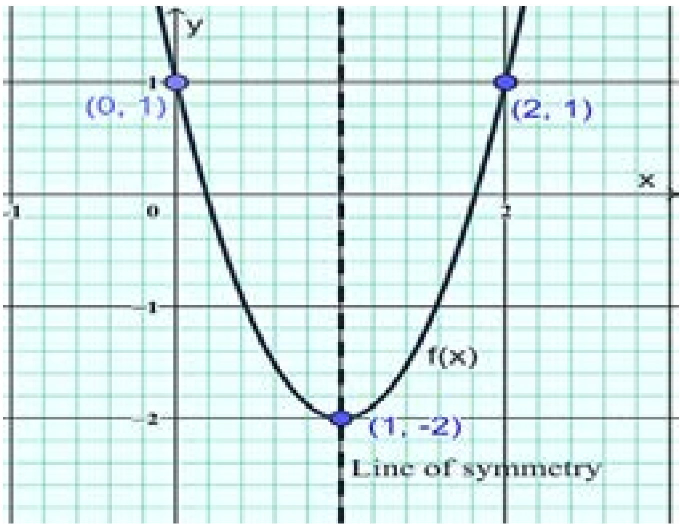


Fig 3.18: Parabolic graph passing through minimum point $(1, -2)$

We expect that any parabolic function to be of the form $f(x) = ax^2 + bx + c$ and since the coordinates of points that lie on its graph must satisfy the equation,

For $(1, -2)$,

$$-2 = a(1)^2 + b(1) + c$$
$$a + b + c = -2$$

For $(2, 1)$,

$$1 = a(2)^2 + b(2) + c$$
$$4a + 2b + c = 1$$

For $(0, 1)$,

$$1 = a(0)^2 + b(0) + c$$
$$c = 1$$

$$\left.\begin{array}{r} a + b + c = -2 \\ 4a + 2b + c = 1 \\ c = 1 \end{array}\right\}$$

The solution to the system of equations results in $a = 3$, $b = -6$ and $c = 1$ which are the constants in the quadratic function hence we obtain

$f(x) = 3x^2 - 6x + 1$

These further examples are to help you appreciate what we have gone through this far. Take your time in small groups or individually and work through them, observing the details and finding out if there are corrections to be made.

Example 46

Plot a graph of $y = -x^2 + x + 2$ for $-3 \leq x \leq 4$.

Solution

Complete a table of values

x	-3	-2	-1	0	1	2	3	4
y	-10	-4	0	2	2	0	-4	-10

Let us go through a few examples to complete the table.

Given $y = -x^2 + x + 2$

For x = -3

$y = -(-3)^2 + (-3) + 2$

$y = -9 - 3 + 2$

$y = -10$

For $x = 0$

$y = -0^2 + 0 + 2$

$y = 2$

For $x = 3$

$y = -3^2 + 3 + 2$

$y = -4$

Then plot the points from the table of values, taking care to join the points with a smooth curve, not straight lines.

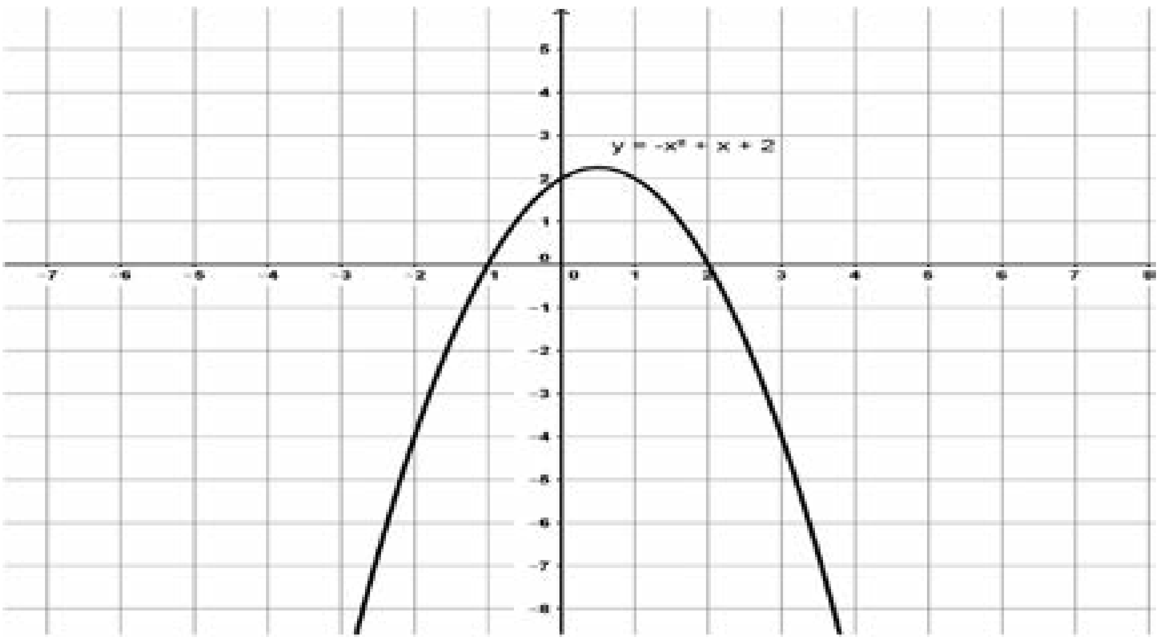


Fig 3.19: Graphical Representation of $y = -x^2 + x + 2$ for $-3 \leq x \leq 4$.

Example 47

Plot a graph of $y = x^2 - 2x - 15$ from the table of values.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
y	9	0	-7	-12	-15	-16	-15	-12	-7	0	9

Solution

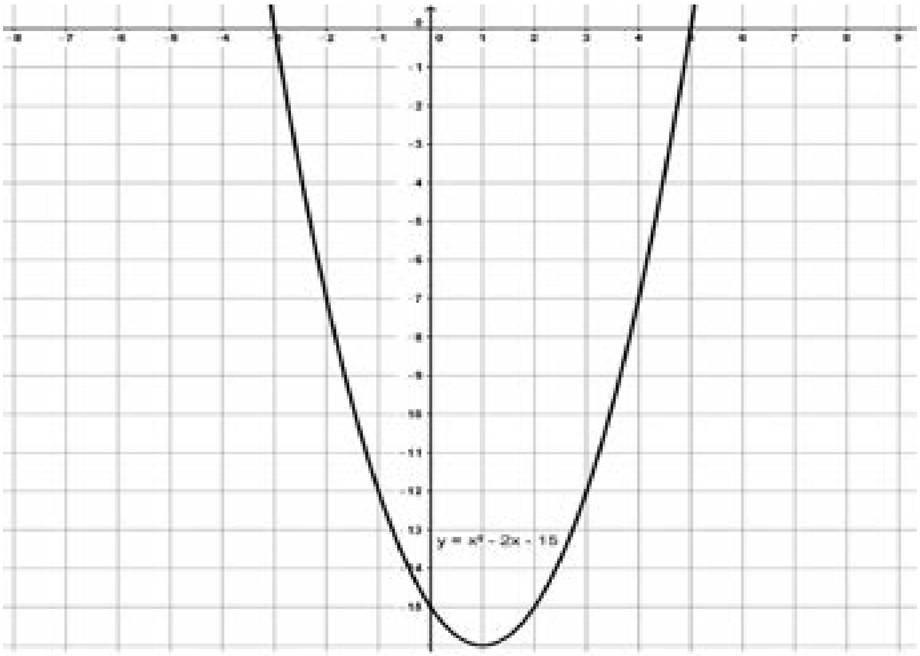


Fig. 3.20: Graphical Representation of $y = x^2 - 2x - 15$

Example 48

Copy and complete the table below for the function.

$y = -x^2 - 7x - 12$ and use it plot the graph.

x	-7	-6	-5	-4	-3	-2	-1	0	1	2
y		-6		0		-2		-12		

Solution

Table of values

x	-7	-6	-5	-4	-3	-2	-1	0	1	2
y	$\mathbf{-12}$	-6	-2	0	$\mathbf{0}$	-2	$\mathbf{6}$	-12	$\mathbf{-20}$	$\mathbf{-30}$

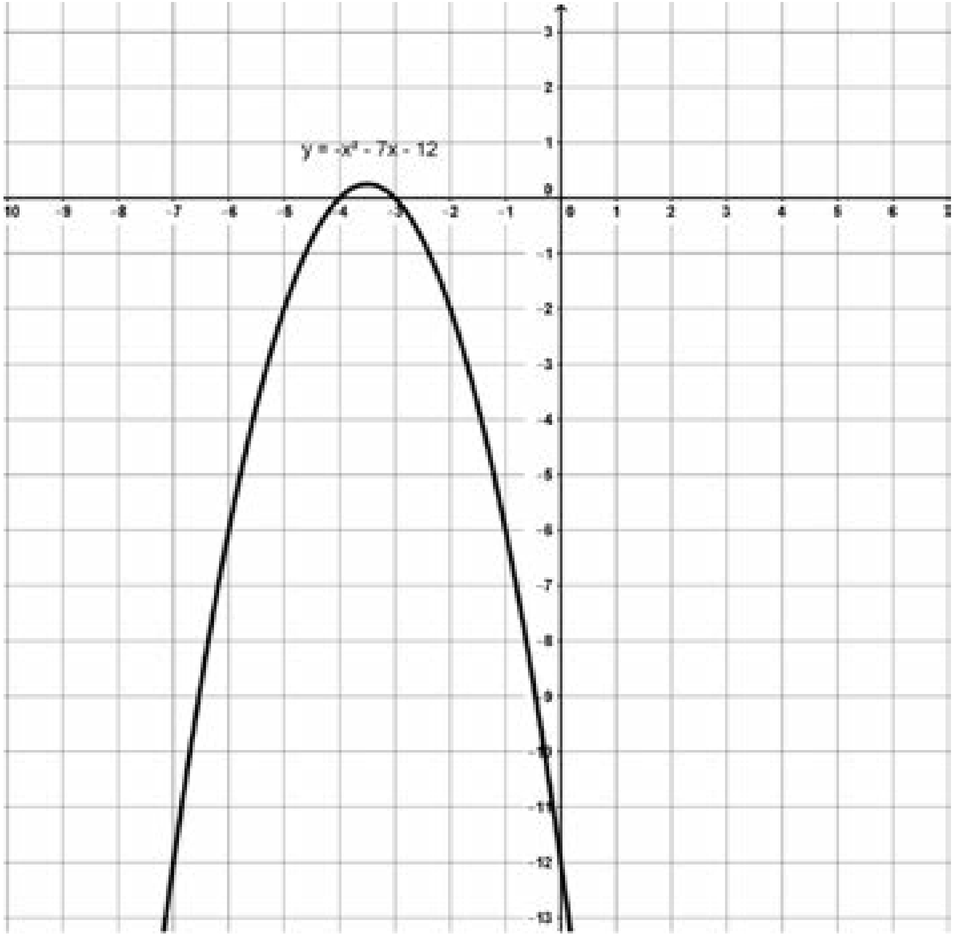


Fig 3.21: Graphical representation of $y = -x^2 - 7x - 12$

Example 49

By using a graphical method, solve the simultaneous equations; $y = x^2 - x - 6$ *and* $y = x + 2$

Solution

By substituting the values of x, the corresponding values y can be obtained.

$y = x^2 - x - 6$

x	-4	-3	-2	-1	0	1	2	3	4	5
y	14	6	0	-4	-6	-6	-4	0	6	14

$y = x + 2$

x	-3	-1	0	1	3	4
y	-1	1	2	3	5	6

Plot both graphs on the same axes.

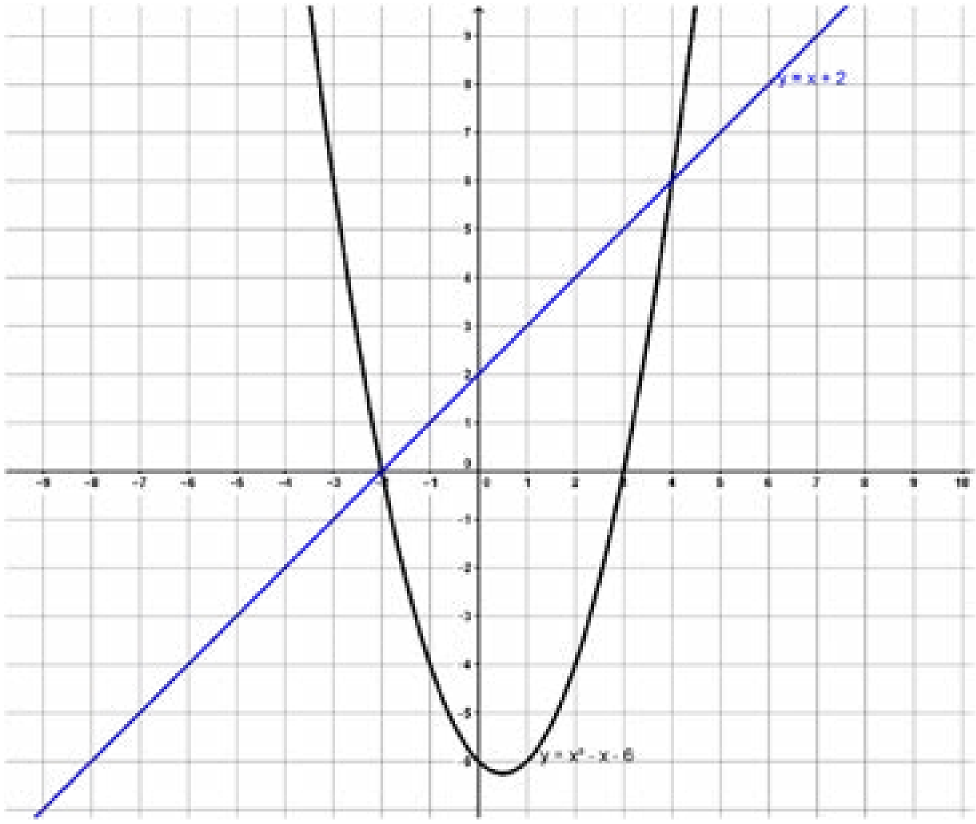


Fig 3.22: Graphical representation of $y = x^2 - x - 6$ *and* $y = x + 2$

From the graph the solutions are where the two graphs intersect, **(-2,0) and (4,6).**

GRAPHICAL SOLUTION TO LINEAR INEQUALITIES INVOLVING ONE VARIABLE

Activity 3.10: Graphing Inequalities Involving *x*

Steps:

1. Understanding the Inequality:

Determine the inequality involving x. For example, let's consider the inequality $x >$ k.

2. Identify the Vertical Line:

The inequality $x >$ k means all values of x that are greater than k. This forms a vertical line on the x-axis at $x =$ k.

3. Graphing the Line:

Draw a vertical line at $x =$ k on your graph paper or graphing tool. Label it as $x =$ k.

4. Determine the Shaded Region:

Based on the inequality:

- **x > k**: Shade the region to the right of the vertical line $x =$ k. This represents all x-values greater than k.
- **x < k**: Shade the region to the left of the vertical line $x =$ k. This represents all x-values less than k.
- $\boldsymbol{x \geq}$ **k**: Include the line $x =$ k itself and shade to the right of it.
- $\boldsymbol{x \leq}$ **k**: Include the line $x =$ k itself and shade to the left of it.

5. Label the Shaded Region:

Clearly label the shaded region on your graph paper or tool to indicate which side of the line satisfies the inequality.

6. Verify:

If using a graphing tool, you can verify by plugging in different values of x to see if they satisfy the inequality and lie in the shaded region.

Example 50

Suppose the inequality is $x > 3$.

a. Draw the vertical line $x = 3$.

b. Shade the region to the right of this line.

c. Your graph should look like this.

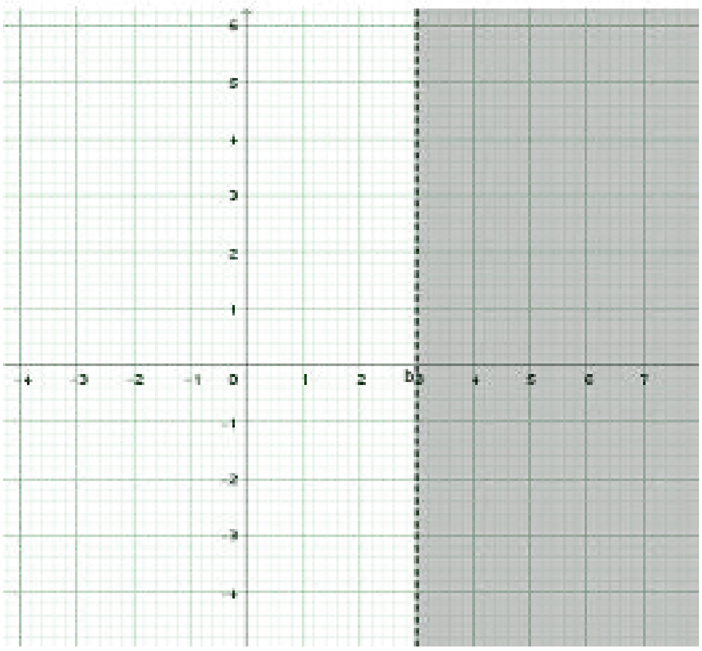


Fig 3.23: graphical illustration of $x>3$.

Note, the line $x = 3$ is dashed as the inequality does not include the line itself.

Example 51

Suppose the inequality is $y \leq 4$.

a. Draw the horizontal line at $y = 4$.

b. Shade the region below this line.

Solution

1. The line $y = 4$ is a horizontal line with y the intercept (0, 4).

Draw a solid line and shade below it as shown below:

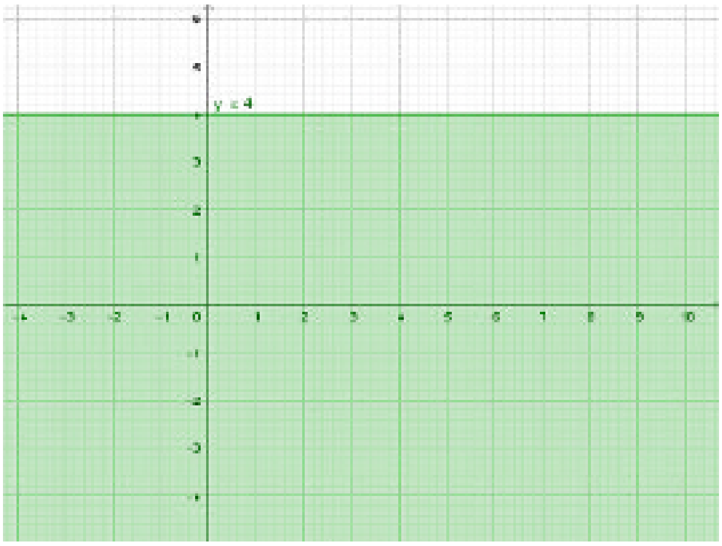


Fig 3.24: Graphical illustration of y = 4

Note, the line here is solid as it is included in the inequality.

Example 52

$x \geq 0$ is illustrated as follows

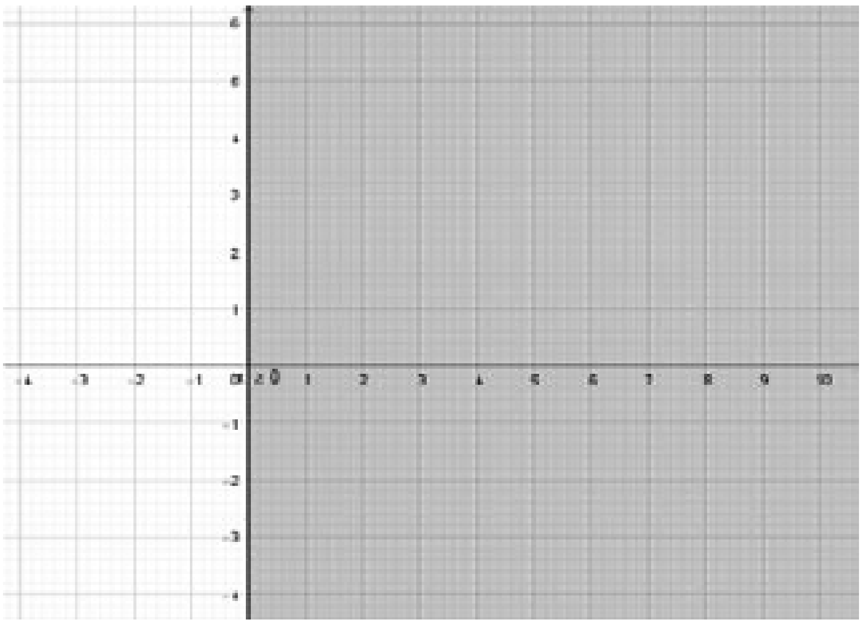


Fig 3.25: Graphical illustration of $x \geq 0$

Example 53

$y \geq 0$ is illustrated as follows

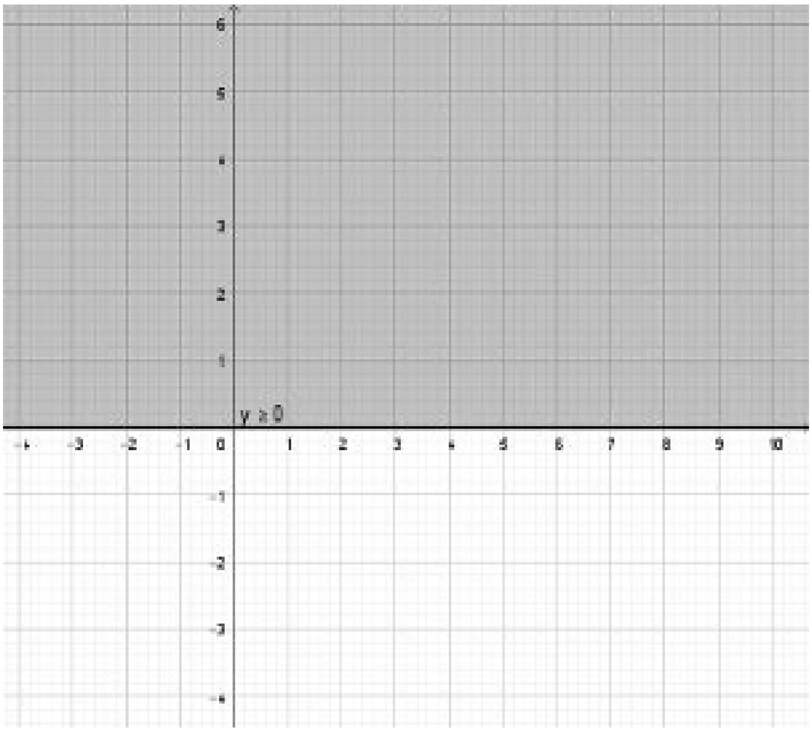


Fig 3.26: Graphical illustration of $y \geq 0$

Example 54

Illustrate $4x > 3$ on a graph.

Solution

The solution to an inequality is the range of values for the variables involved that make the inequality true. For example, given that $4x > 3$:

when $x = 1 \Rightarrow 4x = 4(1) = 4$, the inequality holds true since 4 is greater than 3.

when $x = 3 \Rightarrow 4x = 4(3) = 12$, the inequality also holds true since 12 is greater than 3.

There are infinitely many values for x which will make the inequality hold true hence it is more expedient to write out a range that caters for all those values. For $4x > 3$, all values of x which are greater than 3/4 satisfy. This translates into a *truth set* $= \{x{:}x > 3/4\}$ which can be illustrated by a shaded area on a graph as shown in Figure 9. It can be observed that a broken line is used to indicate the boundary of the solution region (shaded with blue). This is because the solution does not include $x = \frac{3}{4}$

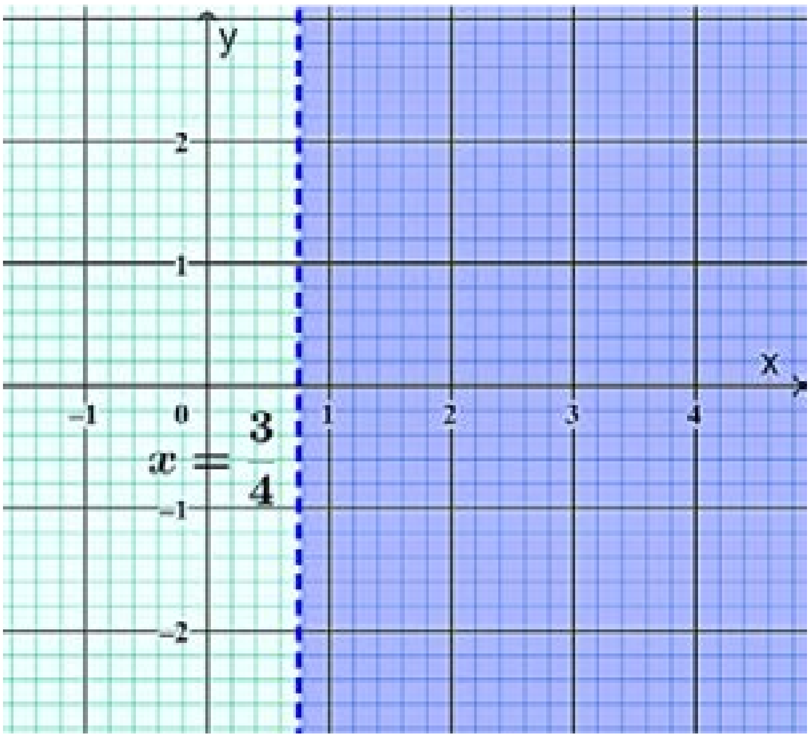


Fig 3.27: Graphical illustration of $4x > 3$

GRAPHICAL SOLUTION TO LINEAR INEQUALITIES INVOLVING TWO VARIABLES

Strategy for Graphing Linear Inequalities in Two Variables

Solve the inequality for y, then graph $y = mx + b$

1. $y > mx + b$ is the region above the line
2. $y = mx + b$ is the line itself
3. $y < mx + b$ is the region below the line.

Steps:

1. Find the intercepts on x and y axes, bearing in mind that at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$.
2. Prepare a table of values.
3. Draw the boundary line according to the table of values, using dashed line for the symbols $<$ and $>$, and solid lines for the symbols $\leq$ and $\geq$.
4. For boundaries with the symbol $>$ or $\geq$ shade the area above the line and for boundaries with symbol $<$ or $\leq$, shade the area below the line.

Example 55

Show the graphical representation of the linear inequality involving two variables, say $2x - 3y < 4$,

1. The inequality is rewritten as an equation, i.e., $2x - 3y = 4$ so that the boundary of the solution can be drawn. It can be easier to write it in the form $y = mx + c$, so in this case, $y = 2/3\ x - 4/3$
2. A test to determine which side of the boundary line forms part of the solution region can then be conducted by determining which side of the line contains points that satisfy the inequality. Thus:
3. select a point, say (0, 0)
4. substitute the coordinates of the point into the inequality to check if it satisfies $2\ (0) - 3\ (0) = 0$

0 is less than 4 hence (0, 0) satisfy the inequality and hence the side of the line, $2x - 3y = 4$ that forms the solution is the side that contains (0, 0) as shown in Figure 10.

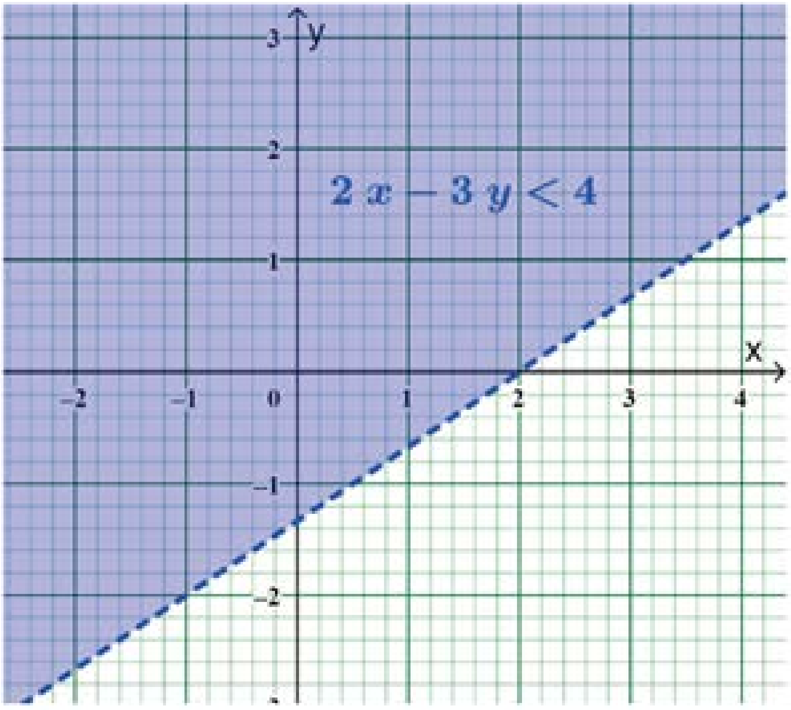


Fig 3.28: Linear inequality in two variables

Example 56

Graph each inequality:

1. $2x - 3y < 6$
2. $y + 2x \geq 3$

Solution

1. If $2x - 3y < 6$, solve for y

$3y > 2x - 6$

$y > \frac{2}{3}x - \frac{6}{3}$

$y > \frac{2}{3}x - 2$

Now, in $y = \frac{2}{3}x - 2$;

Intercept on $x - axis$

When $y = 0$, $0 = \frac{2x}{3} - 2$

$3(0) = 3(\frac{2x}{3}) - 3(2)$

$0 = 2x - 6$

$-2x = -6$

$x = 3$

Intercept on x – axis is (3,0)

Intercept on y-axis,

When $x = 0$, $y = -2$

Intercept on y – axis is (0, -2)

Table of values

x	0	3
y	-2	0

Draw the boundary line according to the table of values, using a dashed line for the boundary because it is not included in the inequality.

Shade the region above the line because of the symbol >.

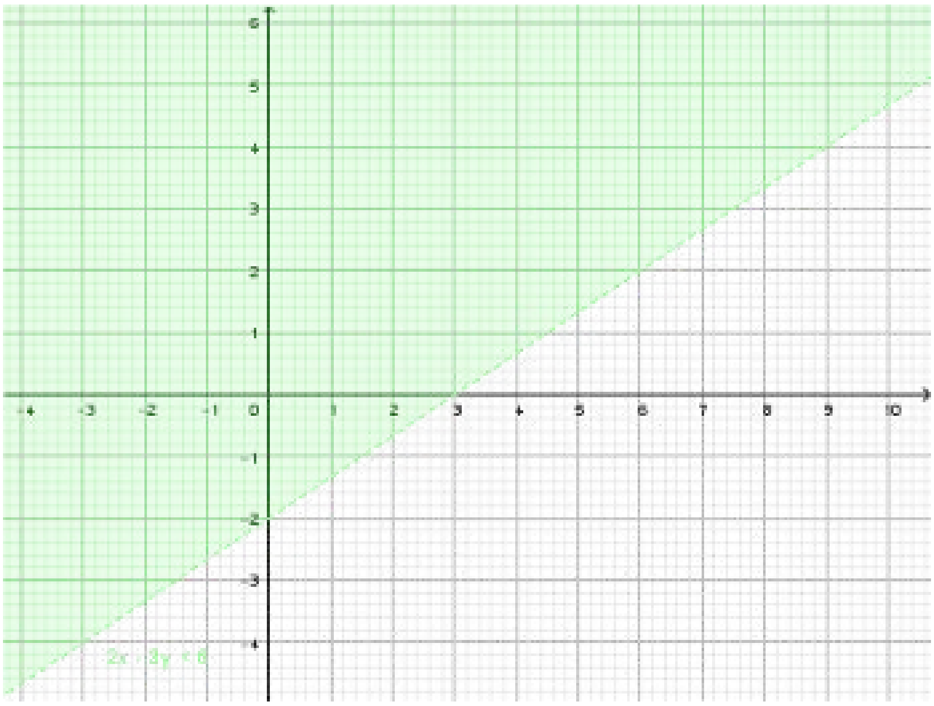


Fig 3.29: Graphical illustration of $2x - 3y < 6$

2. If $y + 2x \geq 3$, solve for y

$y \geq -2x + 3$.

Now in $y = -2x + 3$;

Intercept on x-axis;

When $y = 0$, $0 = -2x + 3$

$2x = 3$

$x = 3/2$

$x = 1.5$

Intercept on x axis is (1.5, 0)

Intercept on y axis;

When x=0, $y = -2(0) + 3$

$y = 3$

Intercept on y-axis is (0, 3)

Table of values

x	1.5	0
y	0	3

Draw the boundary line according to the table of values. Because of the symbol $\geq$, every point on or above the line satisfy the inequality. To show that the line $y = -2x + 3$ is included, draw a solid line and shade the region above.

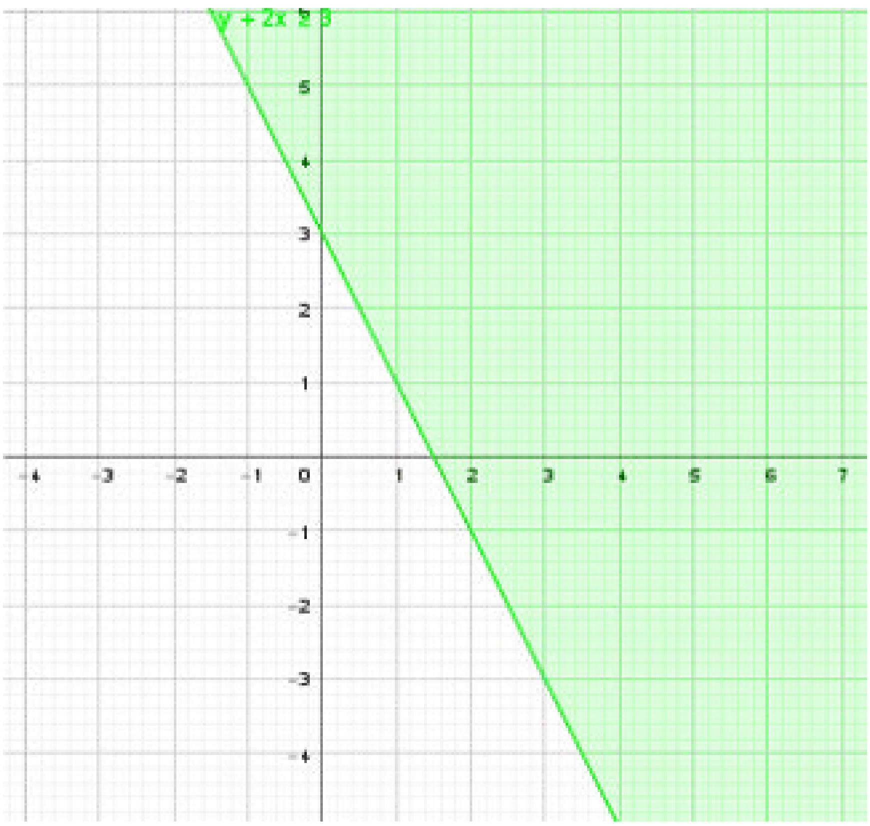


Fig 3.30: Graphical illustration of $y + 2x \geq 3$

GRAPHICAL SOLUTION TO SYSTEMS OF LINEAR INEQUALITIES

The solution region to a system of two or more linear inequalities is the region that is made up of all points that satisfy all the inequalities.

Example 57

1. Graph the solution set of the system

$x + 2y \geq 8$

$x + y \leq 12$

$x \geq 0$

$y \geq 0$

Solution

We will go through the following steps to find the solution region

a. draw the graphs of $x + 2y = 8$, $x + y = 12$, $x = 0$ *and* $y = 0$,

b. identify regions on the grid that contain points that may satisfy all inequalities,

c. verify whether or not test points from those regions satisfy the inequalities

d. make conclusions to determine the solution region

Figure 3.31 shows the graphs of the four equations and possible solution regions namely $A\,1$, $A\,2$ *and* $A\,3$.

We would only consider points in the regions named $A\,1$, $A\,2$ *and* $A\,3$ since points in those regions satisfy both $x \geq 0$ *and* $y \geq 0$

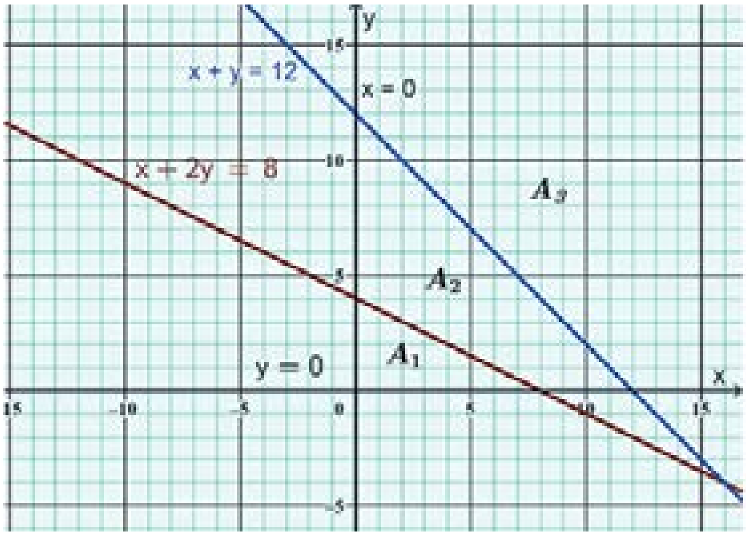


Fig 3.31: Graphical illustration of $x + 2y = 8$, $x + y = 12$, $x = 0$ *and* $y = 0$

Region	Test Point	Inequality	Check	Conclusion
A_1	(1, 1)	$x + 2y \geq 8$	$1 + 2(1) = 3$	Does not satisfy
		$x + y \leq 12$	$1 + 1 = 2$	Satisfies
A_2	(5, 5)	$x + 2y \geq 8$	$5 + 2(5) = 15$	Satisfies
		$x + y \leq 12$	$5 + 5 = 10$	Satisfies
A_3	(10, 10)	$x + 2y \geq 8$	$10 + 2(10) = 30$	Satisfies
		$x + y \leq 12$	$10 + 10 = 20$	Does not satisfy

From the table, A 2, is the solution region since it is the only region which contains points that satisfy all four inequalities

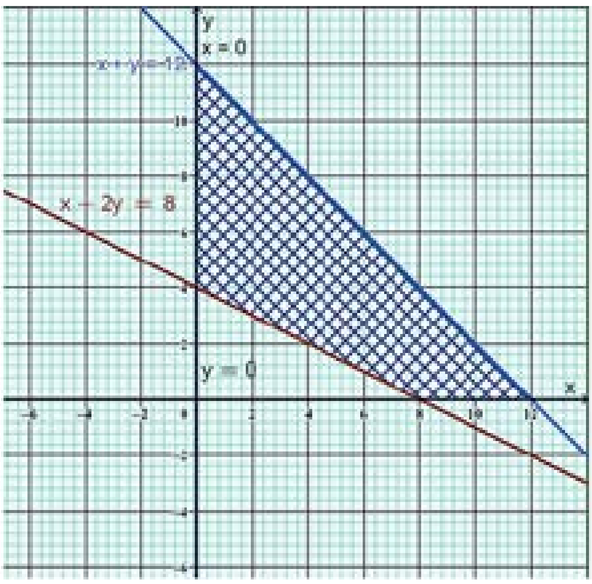


Fig 3.32: Solution Region for example 57

Activity 3.11

In groups or individually, show graphically the region which satisfies simultaneously the inequalities:

$2x + 3y \leq 8$

$x - 2y \geq -3$

$x \geq 0$

$y \geq 0$

Go through the following steps:

- draw the graphs of $2x+ 3y= 8$, $x - 2y = -3$, $x = 0$ and $y =0$,
- identify regions on the grid that contain points that may satisfy all inequalities,
- verify whether or not test points from those regions satisfy the inequalities,
- make conclusions to determine the solution region

After going through the activity above, your graph should be like the one below. The required region is the quadrilateral one which is in the darkest colour as this satisfies all of the inequalities.

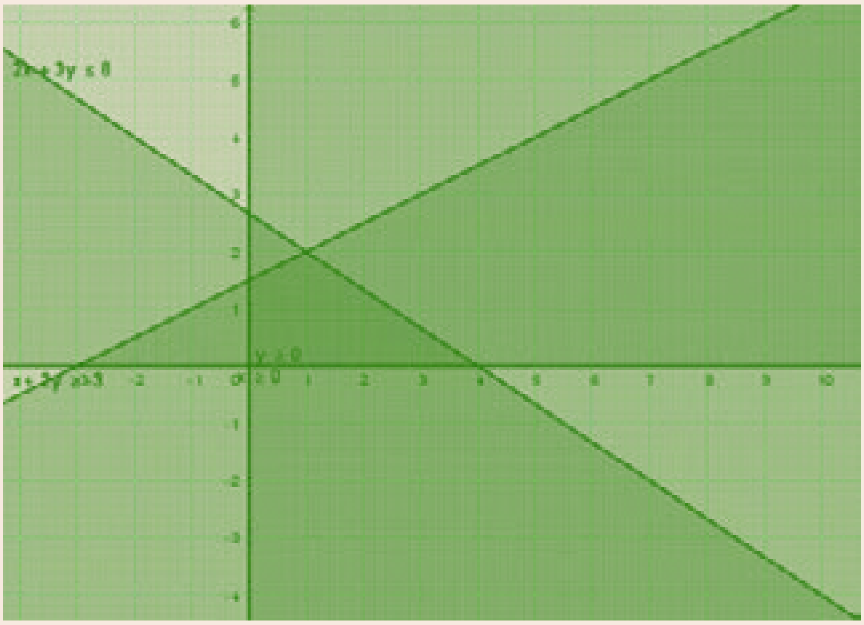


Fig 3.33: Solution region for $2x+ 3y= 8$, $x - 2y = -3$, $x = 0$ and $y =0$

LINEAR PROGRAMMING

We can apply the solution set of the system of linear inequalities to linear programming. Linear programming is a method of finding solutions to a system of linear inequalities that maximises or minimises a function of the form $f(x, y) = ax + by$,where *a and b* are constants. It has a variety of practical applications such as maximizing and minimizing costs, finding the most efficient shipping schedules and so on, to determine the maximum and minimum value of a function, by substituting the coordinates of the vertices of the polygon.

Activity 3.12

A carpenter can make a maximum of 20 tables and 30 chairs per day. Each table requires 3 hours of labour and each chair requires 2 hours of labour. The maximum total number of hours of labour that the carpenter has at his disposal is 96.

(a) Give three inequalities that express the above conditions.

(b) Graph and shade the common region that satisfies these inequalities.

(c) Find the maximum number of chairs and tables that can be made within the time at his disposal.

Expected Solution

(a) Let x be the number of tables made in a day and y the number of chairs made. Hence the inequalities are

$$\mathbf{0} \leq x \leq \mathbf{20}$$
$$\mathbf{0} \leq y \leq \mathbf{30}$$
$$\mathbf{3}x + \mathbf{2}y \leq \mathbf{96}$$

(b) The graph shows the inequalities, the required regions is shaded and labelled P

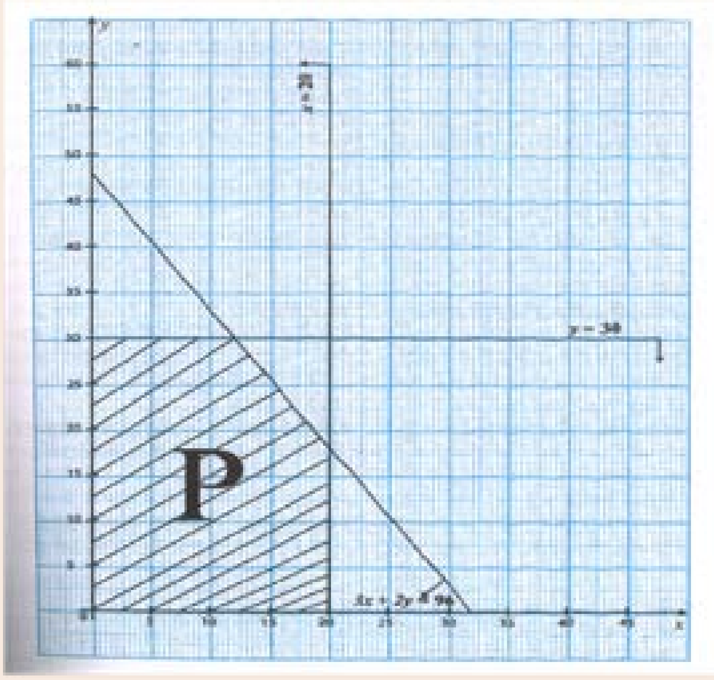


Fig 3.34: Solution region for Activity 3.12

(c) From the graph the coordinates of the vertices of the polygon are:

(0,0), (0,30), (12,30), (20,16) and (20,0).

Hence, we can use any points within area P to decide the optimal number of tables and chairs to be made. For example, 12 tables and 30 chairs = 42 items could be made with 96 hours of labour.

Activity 3.13

In pairs, or individually, work through the following scenario. Then compare your solution to the one below. Which one do you prefer?

1. A factory produces two agricultural pesticides, A and B. For every barrel of A, the factory emits 0.25 *kg* of carbon monoxide (CO) and 0.60 *kg* of sulfur dioxide (SO_2), and for every barrel of B, it emits 0.50 *kg* of CO and 0.20 *kg* of SO_2. Pollution laws restrict the factory's output of CO to a maximum of 75 *kg* and SO_2 to a maximum of 90 *kg* per day.
2. Find a system of inequalities that describes the number of barrels of each pesticide the factory can produce and still satisfy the pollution laws. Graph the feasible region.
3. Would it be legal for the factory to produce 100 barrels of *A* and 80 barrels of *B* per day?

4. Would it be legal for the factory to produce 60 barrels of A and 160 barrels of B per day?

Expected Solution

a. We will first record the data in a table

	CO/kg	SO_2 / kg
A	0.25 kg	0.60 kg
B	0.50 kg	0.20 kg
Maximum	75 kg	90 kg

Let x = *number of barrels of A produced per day* and

y = *number of barrels of B produced per day*

The number of barrels produced cannot be negative hence $x \geq 0$ *and* $y \geq 0$ and since there are maximum masses of CO and SO_2 that are allowed,

Total amount of CO produced $= 0.25x + 0.50y \leq 75$

Total amount of $SO_2 = 0.60x + 0.20y \leq 90$

These bits of data translate into the system:

$$\left.\begin{aligned} x \geq 0 \\ y \geq 0 \\ 0.25x + 0.50y \leq 75 \\ 0.60x + 0.20y \leq 90 \end{aligned}\right\}$$

Test Point	Inequality	Check	Conclusion
(50, 50)	$0.25x + 0.50y \leq 75$	$0.25(50) + 0.5(50) = 37.5$	Satisfies
	$0.60x + 0.20y \leq 90$	$0.60(50) + 0.20(50) = 40$	Satisfies
(50, 200)	$0.25x + 0.50y \leq 75$	$0.25(50) + 0.50(200) = 112.5$	Does not satisfy
	$0.60x + 0.20y \leq 90$	$0.60(50) + 0.20(200) = 70$	Satisfies
(150, 50)	$0.25x + 0.50y \leq 75$	$0.25(150) + 0.50(50) = 62.5$	Satisfies
	$0.60x + 0.20y \leq 90$	$0.60(150) + 0.20(50) = 100$	Does not satisfy

So, we can now draw the inequalities.

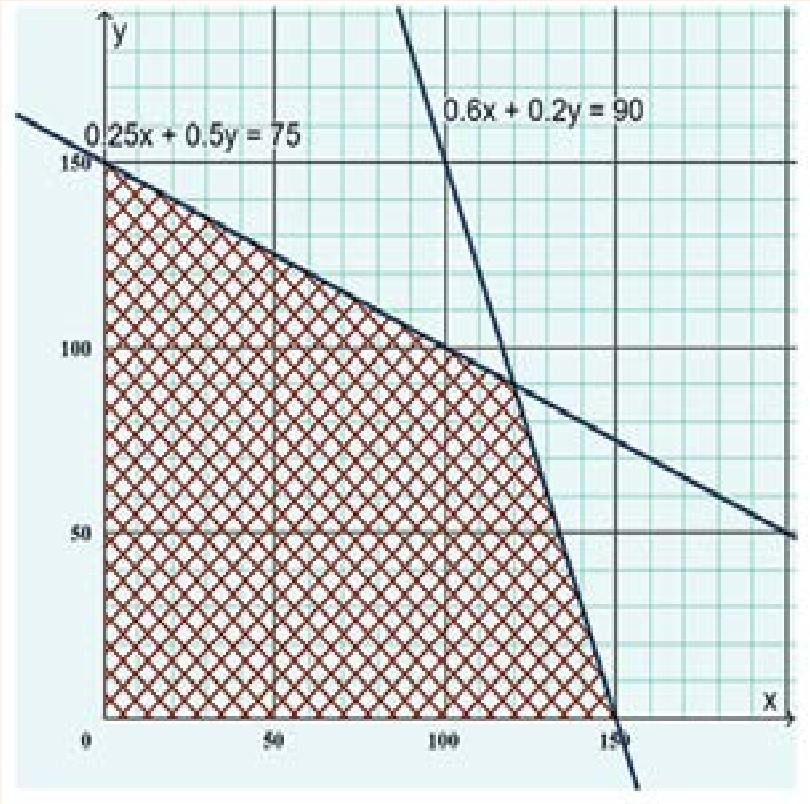


Fig 3.35: Solution region for Activity 3.13

Note that the equations could be multiplied through to a convenient number to obtain an equivalent equation with whole numbers as coefficients. E.g.:

$0.25x + 0.50y \leq 75 \equiv x + 2y \leq 300$

b. (100, 80) falls in the solution region hence it is legal for the factory to produce 100 barrels of *A* and 80 barrels of *B* per day

c. It would be illegal for the factory to produce 60 barrels of A and 160 barrels of B per day since (60, 160) falls outside the shaded region.

Now, let us talk about polynomial functions.

To understand this topic, it is important to revise algebraic expression, graphs of linear and quadratic functions and long division learnt at JHS.

POLYNOMIAL FUNCTIONS

A Polynomial function comprises various combinations on constants, variables, and exponents and is of the form: $P_n(x) = ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots$ where $a, b, c, d, \dots$ are constants and n is a non-negative integer.

Some types of polynomial functions are, linear, quadratic, cubic, quartic and quintic functions. The classification of polynomial functions into these types depends on the degree (the highest exponent of the independent variable) of the polynomial. For example, the highest exponent of x in $-2x^2 + 4x^3 + 2x$ is 3 and thus, the expression is a cubic polynomial expression.

n	Type of polynomial	General Form	Examples
0	Constant	$f(x) = c$	$y = 2, y = -4, y = \frac{2}{3}$
1	Linear	$f(x) = ax + b, , a \neq 0$	$f(x) = -2x + 3,$ $y = 3x - 4, x - 2y = 4$
2	Quadratic	$f(x) = ax^2 + bx + c,$ $a \neq 0$	$f(x) = 2x^2 - 4x + 3,$ $y = -5x^2 - 4,$ $g(x) = x^2 + \frac{3}{2}x$
3	Cubic	$f(x) = ax^3 + bx^2 + cx + d,$ $a \neq 0$	$y = x^3, y = x^3 - 2x,$ $f(x) = 3 + 4x - x^3$
4	Quartic	$f(x) = ax^4 + bx^3 + cx^2 + dx$ $+ g, a \neq 0$	$y = -x^4 + x^3,$ $y = 3x^4 + x^3 - 2x,$ $f(x) = 3 + 4x^2 - x^3 - x^4$
5	Quintic	$f(x) = ax^5 + bx^4 + cx^3 + d$ $x^2 + gx + h, a \neq 0$	$f(x) = -2x^5 + 3,$ $y = x^5 + 3x - 4,$ $x^5 + x^3 - 2y = 4$

PERFORMING ARITHMETIC OPERATIONS ON POLYNOMIAL FUNCTION

Only like terms (terms which have the same variables raised to the same powers) can be added or subtracted.

Example 58

$f(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$ and $h(x) = x^3 + 3x^2 - 2x + 2$ then,

1. $f(x) + h(x) = 4x^4 - x^3 - 9x^2 + 2x - 5 + (x^3 + 3x^2 - 2x + 2)$

$= 4x^4 - 6x^2 - 3$

2. $f(x) - h(x) = 4x^4 - x^3 - 9x^2 + 2x - 5 - (x^3 + 3x^2 - 2x + 2)$

$= 4x^4 - 2x^3 - 12x^2 + 4x - 7$

Example 59

If $h(x) = 5x^3 + 2x^2 + x + 5$ *and* $g(x) = 2x^3 + 5x + 7$, find:

1. $h(x) - g(x)$

2. $h(x) + g(x)$

Solution

1. $h(x) - g(x) = (5x^3 + 2x^2 + x + 5) - (2x^3 + 5x + 7)$

Group terms with the same degree or power

$= (5x^3 - 2x^3) + (2x^2) + (x - 5x) + (5 - 7)$

Then simplify

$= 3x^3 + 2x^2 — 4x - 2$

2. $hx + gx = 5x^3 + 2x^2 + x + 5 + (2x^3 + 5x + 7)$

Again, group the terms with the same degree or power

$= (5x^3 + 2x^3) + (2x^2) + (x + 5x) + (5 + 7)$

And then simplify:

$= +2 + 6x + 12$

Go through the following steps with a classmate or in a small group to learn how to divide two polynomial functions.

Example 60

Divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$

Steps to follow when dividing polynomials

1. Divide x^3 by x to get the first term of the quotient
2. Multiply $x - 2$ by x^2, make up the rest of the terms by simply adding $0x$ *and* 0 *respectively*
3. Subtract and bring down $4x - 3$
4. Divide $-3x^2$ by x to get $-3x$ that is the second term of the quotient.
5. Repeat the process in step 2 and subtract
6. Bring down -3
7. Divide $2x$ by x and multiply $x - 2$ by -2
8. Finally subtract to get-7 that is the Remainder

Solution

Here it is in action with the Long Division Method:

$$\begin{array}{r}
x^2 - 3x - 2 \\
x - 2\,\overline{)\,x^3 - 5x^2 + 4x - 3} \\
-\underline{(x^3 - 2x^2 + 0x + 0)} \\
-3x^2 + 4x - 3 \\
-\underline{(-3x^2 + 6x - 0)} \\
-2x - 3 \\
-\underline{(-2x + 4)} \\
-7
\end{array}$$

$\frac{x^3 - 5x^2 + 4x - 3}{x - 2} = x^2 - 3x - 2 - \frac{7}{x - 2}$ where $x^2 - 3x - 2$ is the quotient and -7 is the remainder.

Example 61

With the steps of dividing a polynomial by a divisor using the long division method, find the quotient and the remainder of $P(x) = \frac{x^4 + x^3 - 11x^2 - 5x + 30}{x - 2}$.

Solution

$$\begin{array}{r l}
 & x^3 + 3x^2 - 5x - 15 \\
x\text{-}2 & \overline{)\, x^4 + x^3 - 11x^2 - 5x + 30} \\
 & \underline{x^4 - 2x^3} \\
 & \quad 3x^3 - 11x^2 \\
 & \quad \underline{3x^3 - 6x^2} \\
 & \qquad 5x^2 - 5x \\
 & \qquad \underline{5x^2 + 10x} \\
 & \qquad\quad \text{-}15x + 30 \\
 & \qquad\quad \underline{15x + 30} \\
 & \qquad\qquad 0
\end{array}$$

Therefore, the quotient is $x^3 + 3x^2 - 5x - 15$ and the remainder is zero.

Example 62

With the steps of dividing a polynomial by a divisor using the long division method, find the quotient and the remainder of P(x)= $\frac{2x^3 - 13x^2 + 26x - 24}{x - 1}$.

Solution

$$\begin{array}{r l}
 & 2x^2 - 11x + 15 \\
x - 1 & \overline{)\, 2x^3 - 13x^2 + 26x - 24} \\
 & \underline{2x^3 - 2x^2} \\
 & \quad -11x^2 + 26x \\
 & \quad \underline{-11x^2 + 11x} \\
 & \qquad 15x - 24 \\
 & \qquad \underline{15x - 15} \\
 & \qquad\quad -9
\end{array}$$

Therefore, the quotient is $2x^2 - 11x + 15$ and the remainder is -9.

Dear learners, let us explore an **alternative method** to carry out the division of polynomial functions which is **the Synthetic Division**. To effectively do this and understand, you have to go through the following steps below:

1. Write down the coefficients of the polynomial functions (*i.e.* $\mathbf{1}x^2 + \mathbf{1}x - \mathbf{2}$)
2. Use the zero of the linear factor (divisor), as your divisor ($i.e.$ *if* $x + 2 = 0$, $\boldsymbol{x = -2}$
3. Perform the synthetic division

a. Bring down 1 (the leading coefficient), multiply -2 *by* 1 and get -2

b. Add 1 *and* -2 to get -1

c. Multiply -2 *by* -1 to get 2

d. Add -2 *and* 2 to get 0

4. Finally write down the quotient $(x - 1)$

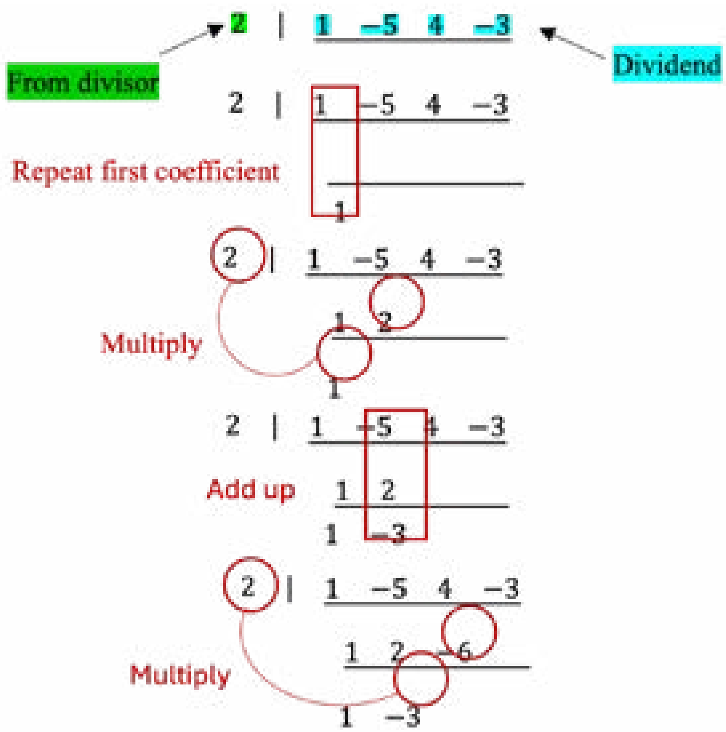
Synthetic Method

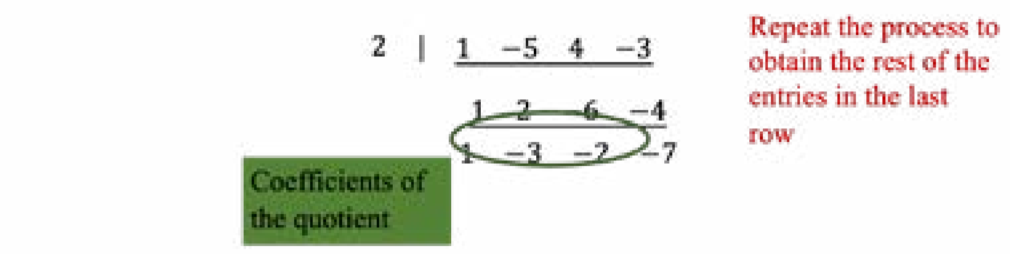
Here it is in action, divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$

We begin by writing the coefficients to represent the divisor and the dividend.

$Divisor = x - 2$

$x - 2 = 0 \Rightarrow \boldsymbol{x = 2}$





Since the coefficients of the quotients are only three, it suggests that there are only three terms which are x^2, $-3x$ and -2. -7 is the remainder of the division.

$$\therefore \frac{x^3 - 5x^2 + 4x - 3}{x - 2} = x^2 - 3x - 2 - \frac{7}{x-2}$$

Example 63

With the steps of dividing a polynomial by a divisor using the synthetic division method, find the quotient and the remainder of $P(x)= \frac{2x^3 - 13x^2 + 26x - 24}{x - 1}$

Solution

Consider the div isor $x - 1 = 0$, $\therefore x = 1$

1	2	–13	26	–24
		2	–11	15
	2	*–11*	*15*	**– 9**

Therefore, the quotient is $2x^2 - 11x + 15$ and the remainder is -9.

Example 64

With the steps of dividing a polynomial by a divisor using the synthetic division method, find the quotient and the remainder of $P(x) = \frac{x^4 + x^3 - 11x^2 - 5x + 30}{x - 2}$

Solution

Consider the divisor $x - 2 = 0$, $\therefore x = 2$

2	1	1	-11	-5	30
		2	6	-10	-30
	1	3	-5	-15	0

Therefore, the quotient is $x^3 + 3x^2 - 5x - 15$ and the remainder is zero.

POLYNOMIAL GRAPHS

Constant Functions

Constant functions of the form $y = c$ or $x = c$ represent horizontal or vertical lines.

For $y = c$, the value of y (which indicates the height of the graph) remains constant regardless of the value of x. c indicates the value of y where the line cuts the $y - axis$

The graph of $x = c$ is a vertical line that cuts the $x - axis$ when $x = c$. Figure 3.36 shows the graphs of $x = -3$, $x = 2$, $y = -2$ and $y = 5$.

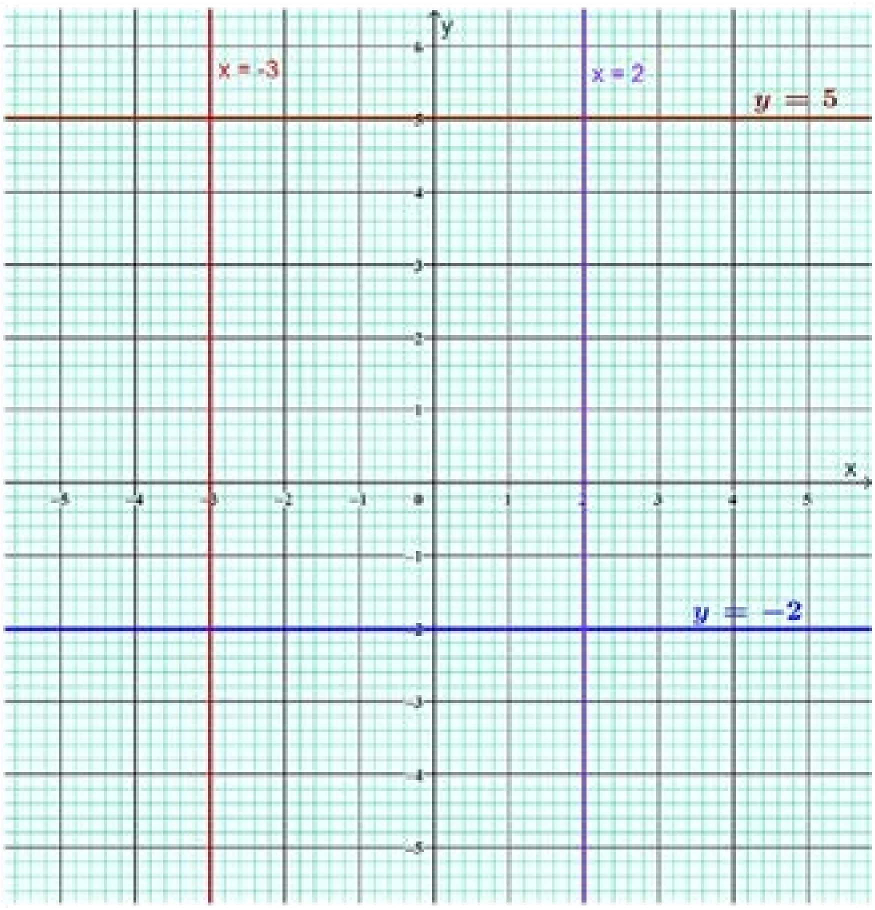


Fig 3.36: Graphs of constant functions

The lines $x = 0$ and $y = 0$ are special lines i.e., they coincide with the $y - axis$ and the $x - axis$ respectively

Linear Functions

$f(x) = ax + b$ represents a straight line which cuts the $y - axis$ at 'b' and has slope, 'a'. The value of 'a', i.e., negative of positive, determines the nature of the straight line. If $a < 0$, the straight line strictly decreases and if $a > 0$, the line strictly increases.

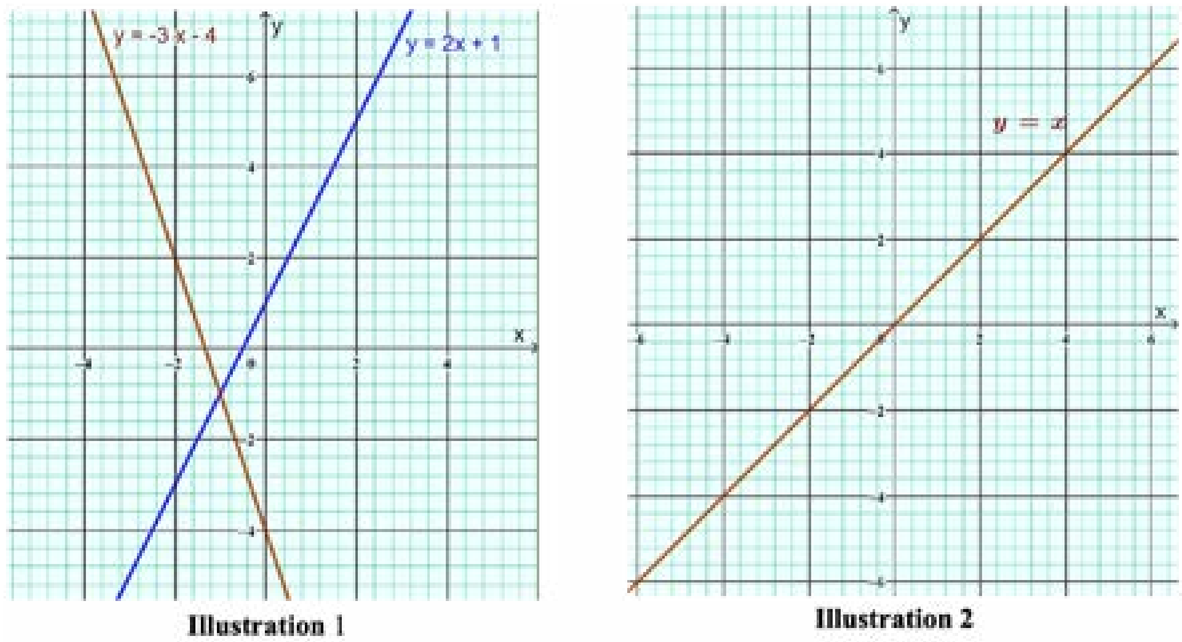


Fig 3.37: Graphs of linear functions

The basic linear function with equation $y = x$ is the straight line that bisects the Cartesian plane as shown in illustration 2 in Figure 3.37. It passes through all points that have the same x and y coordinates including the origin.

Quadratic Functions

Quadratic functions are also called parabolic functions because their graphs are parabolas. Graphs of parabolic functions have been discussed extensively in week 8.

Cubic Functions

Parabolic graphs can be used as a model to graph cubic functions.

Example 65

$f(x) = \frac{x^3}{4} + \frac{3x^2}{4} - \frac{3}{2}x - 2$

$$= \frac{1}{4}(x^3 + 3x^2 - 6x - 8)$$

$$= \frac{1}{4}(x + 4)(x + 1)(x - 2)$$

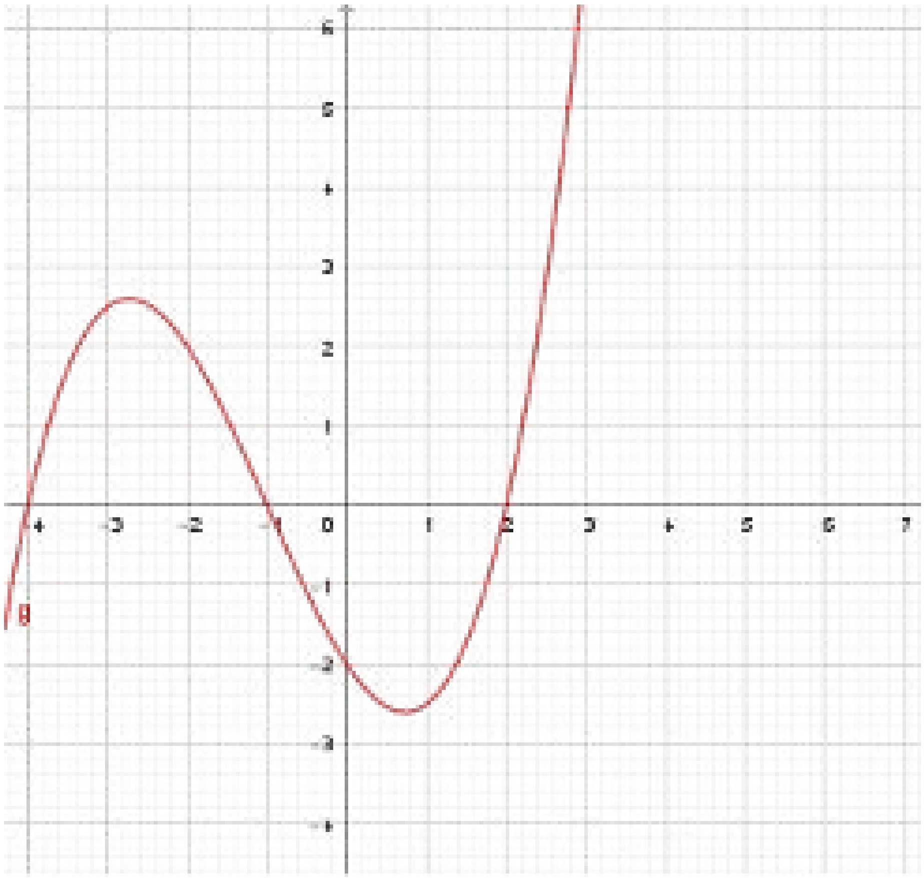


Fig 3.38: Graphical illustration of $f(x) = \frac{x^3}{4} + \frac{3x^2}{4} - \frac{3}{2}x - 2$

The roots of the graph (where it crosses the x-axis) are $x = -4$, $x = -1$ and $x = 2$

The graph increases in the interval $(-\infty, -2.7)$ and $(0.73, \infty)$ and decreases in the interval (-2.7, 0.73).

COMPLETING THE SQUARE TO FIND THE QUADRATIC FORMULA

From the general form of the quadratic equation $a\,x^2 + bx + c = 0$

Follow the steps below to ‘complete the square’.

1. Divide through by the coefficient of the quadratic term ie a
   $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$
2. Transfer (carry) the constant term to the Right Hand Side (RHS)
   $x^2 + \frac{bx}{a} = -\frac{c}{a}$
3. Add the square of half the coefficient of the $x - term$ to both sides of the equation
   $x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
4. Write the Left Hand Side (LHS) as a perfect square and simplify the RHS
   $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$
   $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
5. Take the square root of both sides
   $x + \frac{-b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
6. Proceed to solve for x
   $x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$
7. Further simplifying the expression we have:
   $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
   $\boldsymbol{x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}}$

This is the general formula for solving quadratic equations.

Activity 3.14

Either working with a classmate or individually work through these examples of completing the square.

Example 66

Using completing the square, find the roots of the equation $x^2 - 6x + 7 = 0$

Solution

The coefficient of is 1

$x^2 - 6x = -7$

Next thing is to add the square of half the coefficient of x on both sides of the equation

The coefficient of x is -6 and half of it is -3 and square of it is $(-3)^2$

$x^2 - 6x + (-3)^2 = -7 + (-3)^2$

$(x - 3)^2 = -7 + 9$

Simplifying the equation we have:

$(x - 3)^2 = 2$

Taking the square root of both side

$x - 3 = \pm \sqrt{2}$

Solve for x :

$x = 3 \pm 1.414$

$x = 3 + 1.414 = 4.414$ *or*

$x = 3 - 1.414 = 1.586$

Example 67

Solve the equation $2x^2 - 6x + 4 = 0$ by completing the square.

Solution

First, always ensure the coefficient of the polynomial function is 1 (i.e. the coefficient of x^2 is 1), before completing the square. In this case, it is not, so we need to make it so, by dividing through by 2.

$x^2 - \frac{6}{2}x = -\frac{4}{2}$

To obtain

$x^2 - 3x = -2$

Then add the square of half the coefficient of x to both sides of the equation

The coefficient of x is -3 and half of it is $\left(\frac{-3}{2}\right)^2$

Now

$x^2 - 3x + (-\frac{3}{2})^2 = -2 + (-\frac{3}{2})^2$

$(x - \frac{3}{2})^2 = -2 + \frac{9}{4}$

$(x - \frac{3}{2})^2 = \frac{-8+9}{4}$

$(x - \frac{3}{2})^2 = \frac{1}{4}$

Taking square root of both sides:

$x - \frac{3}{2} = \pm\sqrt{\frac{1}{4}}$

Solve for x :

$x = \frac{3}{2} \pm \frac{1}{2}$

$x = \frac{3}{2} + \frac{1}{2}$ or $x = \frac{3}{2} - \frac{1}{2}$

$x = \frac{4}{2}$ *or* $x = \frac{1}{2}$

$x = 2$ *or* $x = \frac{1}{2}$

Exampel 68

Solve, by completing the square, $x^2 + 6x + 2 = 0$

Solution

Since the coefficient of x^2 is already 1, you go ahead to complete the square.

$x^2 + 6x + 2 = 0$

$x^2 + 6x + 2 = -2$

Then add the square of half the coefficient of x to both sides of the equation. The coefficient of x is 6 and half of it is 3.

$x^2 + 6x + +(3)^2 = -2 + (3)^2$

Simplify by grouping the perfect square:

$(x + 3)^2 = -2 + 9$

$(x + 3)^2 = 7$

Taking the square root of both sides:

$x + 3 = \pm\sqrt{7}$

Make x the subject

$x = -3 + \sqrt{7}$ *or* $x = -3 - \sqrt{7}$

Use your calculator to simplify the value of x

$x = -0.35$ or $x = -5.65$

Example 69

Evaluate by completing the squares $2x^2 — 5x + 1 = 0$

Solution

First, always ensure the coefficient of the polynomial function 1 (i.e. x^2 is 1), before completing the square. To make it so in this case, divide through by 2

$2\,x^2 — 5x = -1$

$x^2 - \frac{5}{2}x = -\frac{1}{2}$

Then add the square of half the coefficient of x to both sides of the equation. The coefficient of x is $-\frac{5}{2}$ and half of it is $-\frac{5}{4}$

$x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = -\frac{1}{2} + \left(\frac{5}{4}\right)^2$

Grouping perfect squares:

$\left(x - \frac{5}{4}\right)^2 = -\frac{1}{2} + \frac{25}{16}$

$\left(x - \frac{5}{4}\right)^2 = \frac{-8 + 25}{16}$

$\left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$

$x - \frac{5}{4} = \pm\sqrt{\frac{17}{16}}$

Make x the subject:

$x = \frac{5}{4} \pm \sqrt{\frac{17}{16}}$

$x = \frac{5}{4} + \sqrt{\frac{17}{16}}$ or $x = \frac{5}{4} - \sqrt{\frac{17}{16}}$

Use your calculator to get the final answer

$x = 2.28$ *or* $x = 0.22$

The quadratic formula and its usage.

Use the method of completing the square to write the general quadratic functions $ax^2 + bx + c = 0$ in the form $A\ (x \pm B)^2 \pm C = 0\ or\ A\ (x \pm B)^2 = C$, where A, B and C are constants, and C is the maximum or minimum value of the function, and the maximum and minimum occurs at $x \pm B = 0$.

From the general form of quadratic expression $y = ax^2 + bx + c$ we can derive the function in the form $A(x \pm B)^2 \pm C = 0$,where A, B and C are constants and determine the maximum and minimum value of the function and where they occur.

$y = a\,x^2 + bx + c$

Factorising out a we have:

$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

Completing square:

$a\left[x^2 + \frac{b}{a} + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$

$a\left[\left(x + \frac{b}{2}a\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$

$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$

$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$

$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]$

$a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right) = 0$

Activity 3.15

Either working with a classmate or individually work through these examples of completing the square.

Example 70

Express $y = 5 - 2x - 4\,x^2$ in the form $y = A - B(x + C)$,where A, B and C are constants. Hence state the maximum value of the function $f{:}x \to 5 - 2x - 4\,x^2$.

Solution

Start by completing the square:

$y = 5 - 2x - 4\,x^2$

Re arrange the expression:

$y = -4\,x^2 - 2x + 5$

Factorise out the x^2 *coefficient*, -4

$$y = -4\left[x^2 + \tfrac{1}{2}x - \tfrac{5}{4}\right]$$

$$y = -4\left[x^2 + \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 - \tfrac{5}{4}\right]$$

Write it in perfect square and simplify:

$$y = -4\left[\left(x + \tfrac{1}{4}\right)^2 - \tfrac{1}{16} - \tfrac{5}{4}\right]$$

$$y = -4\left[\left(x + \tfrac{1}{4}\right)^2 + \frac{-1 - 20}{16}\right]$$

$$y = -4\left(x + \tfrac{1}{4}\right)^2 + \tfrac{21}{4}$$

Re write it in form $A - B(\,x + C)^2$:

$$y = \tfrac{21}{4} - 4\left(x + \tfrac{1}{4}\right)^2$$

Compare the expression to that of $A - B(\,x + C)^2$:

$$y = \tfrac{21}{4} - 4\left(x + \tfrac{1}{4}\right)^2$$

Hence $A = \tfrac{21}{4}$, $B = -4$, $C = \tfrac{1}{4}$

$\therefore$ The maximum value of $f\!:x \rightarrow 5 - 2x - 4\,x^2$ is $\tfrac{21}{4}$

Example 71

Express the following equation in the form $A(x + B)^2 + C$. Hence find the least or greatest value of the function and the value of x at which it occurs.

$f(x) = 3x^2 - x - 6$

Solution

First method – completing the square

$$f(x) = 3\left(x^2 - \tfrac{1}{3}x - 2\right)$$

$f(x) = 3\left(x^2 - \frac{1}{3}x + \left(-\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right)^2 - 2\right)$

$f(x) = 3\left(\left(x - \frac{1}{6}\right)^2 - \left(\frac{1}{36} - 2\right)\right)$

$f(x) = 3\left(\left(x - \frac{1}{6}\right)^2 - \left(\frac{1-72}{36}\right)\right)$

$f(x) = 3\left(\left(x - \frac{1}{6}\right)^2 - \left(\frac{73}{36}\right)\right)$

$f(x) = 3\left(x - \frac{1}{6}\right)^2 - 3\left(\frac{73}{36}\right)$

$f(x) = 3\left(x - \frac{1}{6}\right)^2 - \frac{73}{12}$

Equating:

$3(x - \frac{1}{6})^2 - \frac{73}{12}$ to $A(x + B)^2 + C$ and comparing terms we have:

$A = 3, B = -\frac{1}{6}, C = \frac{-73}{12}$

The least value of $f(x) = \frac{-73}{12}$ and it occurs at $x = \frac{1}{6}$

Second Method - Equating

Equate $f(x) = 3x^2 - x - 6$ to $A(x + B)^2 + C$

Expand the RHS:

$RHS = A(x^2 + 2Bx + B^2) + C$

$RHS = Ax^2 + 2ABx + AB^2 + C$

Equate $f(x) = 3x^2 - x - 6$ to $Ax^2 + 2ABx + AB^2$+C and compare terms:

$A = 3, 2AB = -1, AB^2 + C = -6$

We know $A = 3$ and $2AB = -1$, therefore $2(3)B = -1, 6B = -1, therefore, B = \frac{-1}{6}$

We also know that $AB^2 + C = -6$, but $A = 3$ *and* $B = \frac{-1}{6}$

$3\left(-\frac{1}{6}\right)^2 + C = -6$

$3\left(\frac{1}{36}\right) + C = -\frac{1}{6}$

$\frac{1}{12} + C = -\frac{1}{6}$

$C = -\frac{1}{12} - \frac{1}{6} = \frac{-1-72}{12} = -\frac{73}{12}$

The least value of $f(x) = \frac{-73}{12}$ and it occurs at $x = \frac{1}{6}$

Recognise the importance of the expression $b^2 - 4ac$ from the general quadratic formula:

We know that the quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ The part inside the square root,

$b^2 - 4ac$, is called the discriminant and it determines the nature of the roots of the function and the following properties holds:

a. If $b^2 - 4ac \geq 0$, the roots are real

b. If $b^2 - 4ac > 0$, the roots are real and different/discrete, ie the quadratic has two distinct intersections on the x-axis.

c. If $b^2 - 4ac < 0$, the roots are complex and imaginary, ie the graph has no real roots, so it does not intersect the x-axis.

d. If $b^2 - 4ac = 0$, the roots are real and equal or the equation is a perfect square, ie the graph just touches the x-axis at the maxima or minima.

Example 72

Describe the nature of the roots of the following equations:

1. $x^2 - 6x + 9 = 0$

Solution

Using the discriminant formula $b^2 - 4ac$,

$a = 1$ (*ie coefficient of the x* 2 *term*), $b = -6$ (*coefficient of x term*)

$c = 9$ (*the constant*)

Substituting into the formula $(-6)^2 - 4(1)(9) = 36 - 36 = 0$

The equation has equal roots since the discriminant is equal to zero

2. $2x^2 - 5x + 3 = 0$

Solution

Using the discriminant formula $b^2 - 4ac$.

$a = 2, b = -5, c = 3$

Substituting into the formula $(-5)^2 - 4(2)(3) = 25 - 24 = 1 > 0$

As the discriminant is greater than 0, the equation has two real and different roots

3. $3\,x^2 - 2x + 4 = 0$

Solution

Using the discriminant formula $\boldsymbol{b^2 - 4ac}$.

$a = 3,\ b = -2,\ c = 4$

Substituting into the formula= $(-2)^2 - 4(3)(4) = 4 - 48 = -44 < 0$

As the discriminant is less than zero, the equation has no real roots, ie, it has complex roots

Use of Quadratic Equations in Real-Life Situations

Quadratic equations are important and widely used in various fields, including mathematics, physics, engineering, economics, and computer science. Some common applications and uses of quadratic equations include:

1. Solving for unknowns: The primary use of quadratic equations is to find the values of the variable x that satisfy the equation. This is especially useful when dealing with problems that involve motion, distance, area, or quantities that vary quadratically.
2. Geometry: Quadratic equations are often used in geometry to find the coordinates of points, lengths of sides, or angles of shapes.
3. Projectile motion: When an object is thrown or launched into the air, its path can often be modelled by a quadratic equation, helping to determine its maximum heigh time of flight, and range

Example 73

A company manufactures and sells a product. The revenue R (in dollars) from selling x units of the product is given by the polynomial function $\mathrm{R}(x) = 50x - 0.5x^2$. The cost C (in dollars) to produce x units is given by $\mathrm{C}(x) = 10x + 100$. How many units should the company produce and sell to maximise its profit?

Solution

The profit P is given by the difference between revenue and cost: $P(x) = C(x) - P(x)$

$P(x) = (50x - 0.5\,x^2\,) - (10x + 100)$

$P(x) = 50x - 0.5\,x^2 - 10x - 100$

$P(x) = -0.5x^2 + 40x - 100$

The maximum profit, is found at the maxima of the quadratic function:
$P(x) = -0.5x^2 + 40x - 100$

The vertex of a parabola $a\,x^2 + bx + c$ is given by $x = -\frac{b}{2a}$.
By comparison, $a = -0.5$ and $b = 40$:

$$x = -\frac{40}{2(-0.5)}$$

$x = 40$

$\therefore$ The Company should produce and sell **40 units** to maximise its profit.

Example 74

A car's fuel efficiency (in miles per gallon) can be modeled by the polynomial function $(v) = -0.02v^2 + 1.2v + 20$, where v is the speed of the car in miles per hour.

At what speed should the car be driven to achieve the maximum fuel efficiency?

Solution

To find the speed that maximises fuel efficiency, we need to find the maxima of the quadratic function $(v) = -0.02v^2 + 1.2v + 20$.

The vertex of a parabola $a\,x^2 + bx + c$ is given by $v = -\frac{b}{2a}$.

By comparison, $a = -0.02$ and $b = 1.2$:

$$v = -\frac{1.2}{2(-0.02)^2}$$

$$v = \frac{1.2}{0.04}$$

$v = 30$

$\therefore$ The car should be driven at 30 miles per hour to achieve the maximum fuel efficiency.

Use the roots (sum and products) to find other quadratic equations.

Explore the relationship between the constants a, b *and* c of the general quadratic function: $ax^2 + bx + c = 0$ and α *and* β, and use these relations to write other quadratic equations with given roots.

Relationships that exist between the roots and the coefficients of a quadratic equation.

Assuming α *and* β are now the roots of the general quadratic equation $ax^2+bx+c=0$ instead then $(x-\alpha)(x-\beta)=0$ (Now expand your expression)

$x(x-\beta)-a(x-\beta)=0$

$x^2-x\beta-x\alpha+\alpha\beta=0$

Factorise the x terms:

$x^2-(\alpha+\beta)x+\alpha\beta=0 \ldots\ldots\ldots equation\ 1$

Also consider:

$ax^2+bx+c=0.$

Divide through by a

You will have:

$x^2+\frac{b}{a}x+\frac{c}{a}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots equation\ 2$

Comparing the two equations, you can see and write the relationship that connects the coefficients of the equation and the roots ie

Sum of roots $\quad \boldsymbol{\alpha+\beta=-\frac{b}{a}}$

Product of roots $\quad \boldsymbol{\alpha\beta=\frac{c}{a}}$

You then write the equation as: $x^2-(\text{sum of roots})x+$ ***product of roots*** $=0$

Example 75

If $2x^2-3x+6=0$ has roots α *and* β, find the sum and product of the roots.

Divide through by 2

$x^2-\frac{3}{2}x+\frac{6}{2}=0$

The sum of roots

$\alpha+\beta=-(-\frac{3}{2})=\frac{3}{2}$

The product of roots:

$\alpha\beta=\frac{6}{2}=3$

Example 76

Find the quadratic equation whose roots are -2 and 3 and has a minimum value.

Solution

$x^2 - (sum\ of\ roots)x + (products\ of\ roots) = 0$

$x^2 - (-2 + 3)x + (-2)(3) = 0$

$x^2 - x - 6 = 0$

Example 77

If the roots of a quadratic equation are $\frac{2}{3}$ and -4, find the equation if it has a minima.

Solution

$x^2 - (sum\ of\ roots)x + productsof\ roots) = 0$

$x^2 - \left(\frac{2}{3} + (-4x)\right) + (\frac{2}{3} \times (-4) = 0$

$x^2 - \left(\frac{2 + -12}{3}\right)x + \left(\frac{2 \times -4}{3}\right) = 0$

$x^2 - \frac{-10}{3}x + \frac{-8}{3} = 0$

Multiply through by 3

$3\,x^2 + 10x - 8 = 0$

Example 78

If the roots of a quadratic equation are 2 and 5, find the equation if it has a minima.

Solution

Sum of roots $= 2 + 5 = 7$

Product of roots $= 2 \times 5 = 10$

Put this in the formula of the equation:

$x^2 - (7)x + (10) = 0$

$x^2 - 7x + 10 = 0$

Example 79

If the roots of a quadratic equation are 5 and –2, find the equation if it has a minima.

Solution

Using the formula, you will have:

$x^2 - (5 + (-2))x + (5 \times (-2)) = 0$

Simplifying you will obtain

$x^2 - 3x - 10 = 0$

Example 80

Find the equation whose roots are 7 *and* $\frac{5}{2}$ and it has a minima.

Solution

Sum of roots $= 7 + \frac{5}{2} = \frac{14 + 5}{2} = \frac{19}{2}$

Product of roots $= 7 \times \frac{5}{2} = \frac{35}{2}$

Using the formula, you will have:

$x^2 - \left(\frac{19}{2}\right)x + \frac{35}{2} = 0$

Tidy it up by multiplying through by 2:

$2\,x^2 - 19x + 35 = 0$

FACTOR AND REMAINDER THEOREMS

Use the factor and remainder theorems to solve problems of polynomial functions in context:

We must now explore and recognise that generally for any polynomial function $P(x)$ division by another polynomial result in:

$$\frac{Dividend}{Divisor} = quotient + \frac{Remainder}{Divisor}$$

and so;

$$Dividend = Quotient \times Divisor + Remainder$$

It is important to recognise that for any polynomial functions $P(x)$, if $(x - a)$ is a divisor then;

$P(x) = q(x) \bullet (x - a) + r$; where $q(x)$ is the quotient and r is the remainder.

We can now establish the **Remainder theorem** and use it to solve related problems:

***Remainder Theorem*:** Given any polynomial functions $P(x)$, if $(x - a)$ is a divisor then;

$P(a) = q(a) \bullet (a - a) + r$; and $P(a) = r$

Example 81

Find the remainder when the polynomials $f(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$ is divided by $(x - 2)$

Solution

$x - 2 = 0$

$\Rightarrow x = 2$

$f(2) = 4(2)^4 - 2^3 - 9(2)^2 + 2(2) - 5 = 19$

Example 82

If $f(x) = x^3 - 3x^2 + x - 5$, what is the remainder when $f(x)$ *is divided by* $x - 1$?

Solution

In this question the divisor is $(x - 1)$ hence from the remainder theorem, $f(1)$ is the remainder

$$f(1) = (1)^3 - 3(1)^2 + (1) - 5$$
$$= 1 - 3(1) + 1 - 5 = -6$$

Example 83

Find the remainder when $f(x) = 8x^3 - 3x^2 - 5x + 2$ is divided by

a) $x - 1$

b) $x + 1$

c) $3x + 1$

Solution

a) $x - 1 = 0$

$x = 1$

$f(1) = 8(1)^3 - 3(1)^2 - 5(1) + 2 = 8 - 3 - 5 + 2 = 2$

b) x **+ 1 = 0**

$x = -1$

$f(-1) = 8(-1)^3 - 3(-1)^2 - 5(-1) + 2$

$-8 - 3 + 5 + 2 = -4$

c) $3x + 1 = 0$

$3x = -1$

$x = -\frac{1}{3}$

$$f\left(-\frac{1}{3}\right) = 8\left(-\frac{1}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^2 - 5\left(-\frac{1}{3}\right) + 2$$
$$= 8\left(-\frac{1}{27}\right) - 3\left(\frac{1}{9}\right) + 5 \times \frac{1}{3} + 2$$
$$= -\frac{8}{27} - \frac{3}{9} + \frac{5}{3} + 2$$
$$= \frac{-8 - 9 + 45 + 54}{27}$$
$$= \frac{82}{27}$$

FACTOR AND REMAINDER THEOREMS

If a polynomial $f(x)$ is divided by $x-2$ and $f(a)=0$ then the remainder is zero (0). If the remainder is zero, then the divisor $(x-a)$ is a factor of $f(x)$. This result which is the factor theorem, states that:

If $f(a)=0$, then $(x-a)$ is the factor of $f(x)$.

Similarly, if $f(-a)=0$ then $(x+a)$ is also a factor of $f(x)$

Example 84

Determine whether or not $(x+2)$ is a factor of $f(x)=x^3+2x^2-x-2$

Solution

$$f(-2)=(-2)^3+2(-2)^2-(-2)-2$$

$$=-8+8+2-2=0$$

Therefore, $(x+2)$ is a factor of f(x).

Example 85

Determine whether or not $(x-1)$ is a factor of $f(x)=x^3+2x^2-x-2$

Solution

$$f(x)=(1)^3+2(1)^2-(1)-2=1+2-1-2=0$$

Therefore, $(x-1)$ is a factor of f(x).

Example 86

Determine whether or not $(x-2)$ is a factor of $f(x)=x^3+2x^3-x-2$

Solution

$$f(2)=(2)^3+2(2)^2-2-2=8+8-2-2=1$$

Since$f(2)\neq 0$, it means that $(x-2)$ is not a factor of $f(x)$

Example 87

Show that $f(x)=6x^3+7x^2-x-2$ is divisible by $(2x-1)$

Solution

$2x - 1 = 0$

$x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2$$

$$= 6\left(\frac{1}{8}\right) + 7\left(\frac{1}{4}\right) - \frac{1}{2} - 2$$

$$= \frac{6}{8} + \frac{7}{4} - \frac{1}{2} - \frac{2}{1}$$

$$= \frac{6 + 14 - 4 - 16}{8}$$

$$= \frac{0}{8} = 0$$

Therefore, $6x^3 + 7x^2 - x - 2$ is divisible by $2x - 1$

Example 88

Show that $(x + 2)$ is a factor of $f(x) = x^3 - 2x^2 - 5x + 6$

Solution

Let $x + 2 = 0$

$x = -2$

$$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6$$

$$= -8 - 8 + 10 + 6 = 0$$

$(x + 2)$ is a factor $x^3 - 2x^2 - 5x + 6$

GRAPHS OF POLYNOMIAL FUNCTIONS

We need to be able to use graphs of polynomial functions to solve problems.

To do this we need to draw quadratic and cubic polynomial functions by hand and with the aid of appropriate technology and discuss what happens to the curve as $x - values$ decreases or increases, the point where the curve touches the $y - axis$, axis of symmetry and turning points.

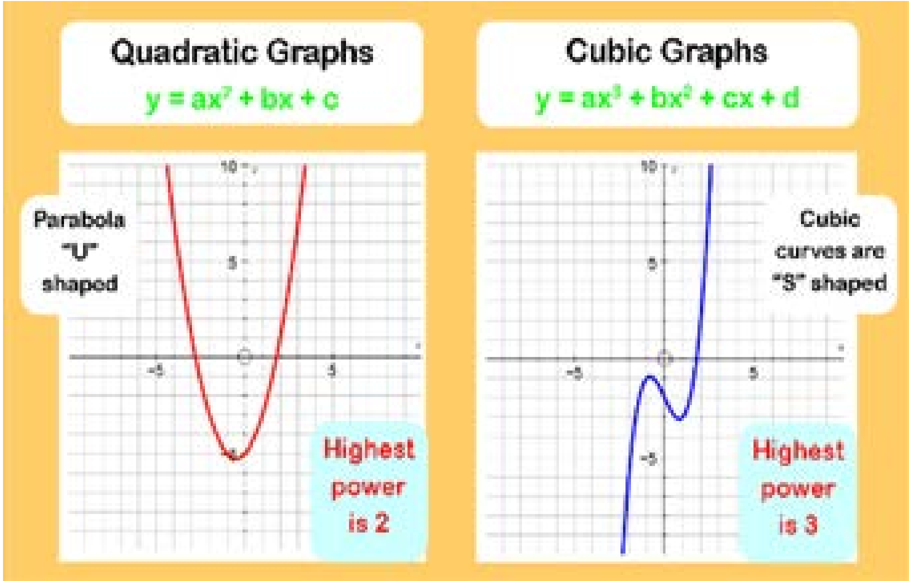


Fig 3.39: Quadratic and Cubic graphs

We are going to study the behaviour of some graphs of polynomials in the following examples;

Example 89

Let's look at the cubic graph of $f(x) = x^3 - x^2 - 5x - 3$ from $x = -4$ to $x = 4$

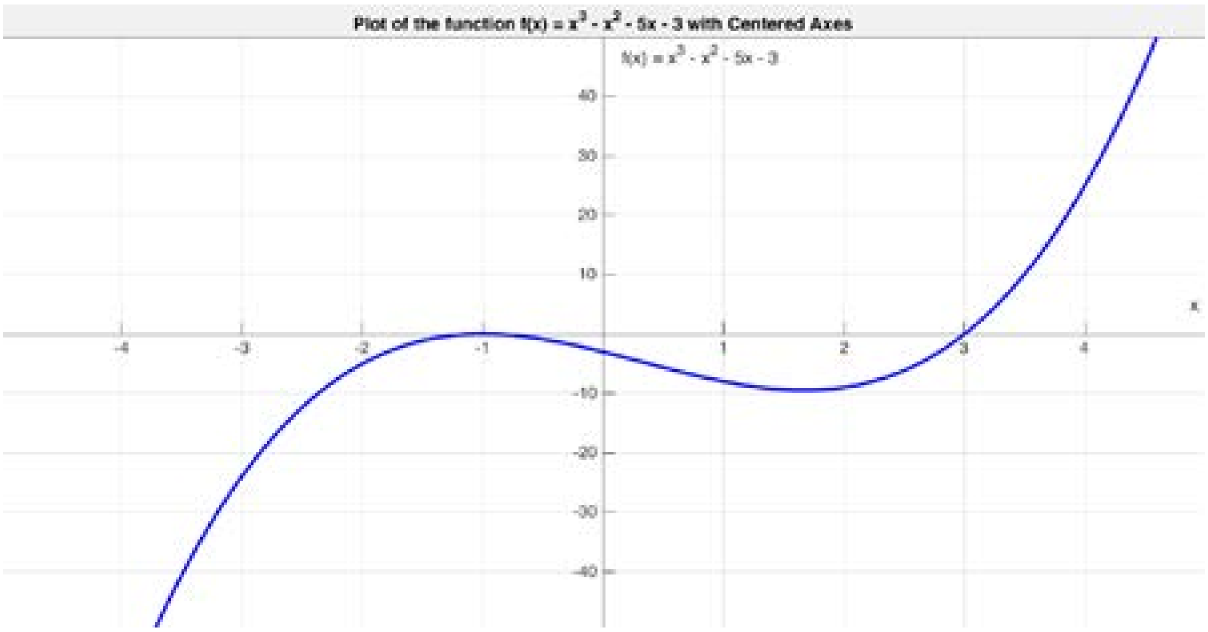


Fig 3.40: Plot of $f(x) = x^3 - x^2 - 5x - 3$ with centred axes

It can be observed that as x increases from -4 to -1, y is increasing from about -45 to 0. At the point (-1,0) is where a local maximum turning point occurs. Beyond this point as x continues to increase from -1 to about 1.67, y then decreases gently to a local minimum turning point of about (1.67, -13.19) after which as x continue to increase, y also increases. Leaners, please do pay attention to the quoted estimates of the points and based on the scale of the graph make sense of them.

Example 90

If we now consider the cubic graph of $f(x) = x^3 + 2x^2 - 8x$ from $x = -4$ to $x = 4$

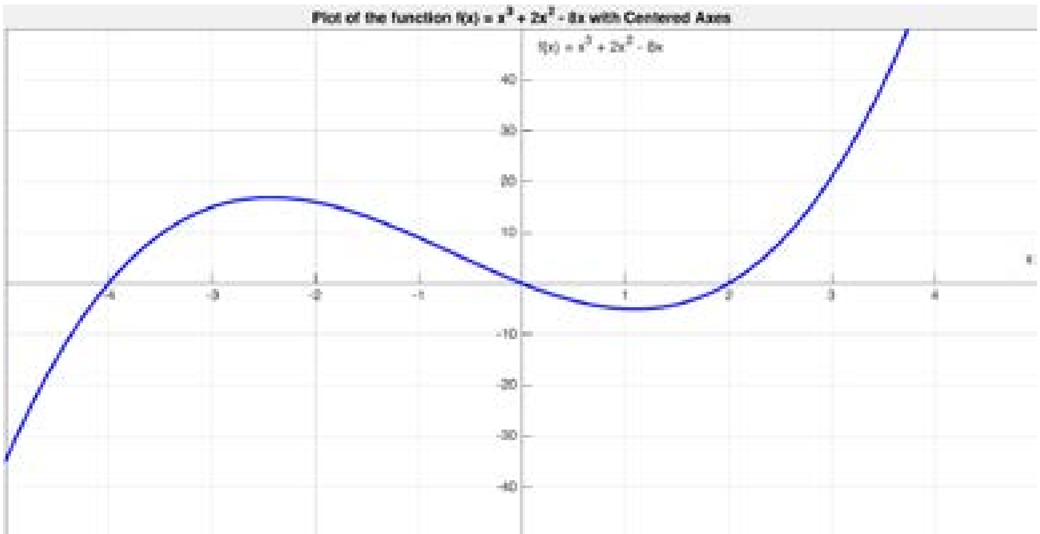


Fig 3.41: Plot of $f(x) = x^3 + 2x^2 - 8x$ with centred axes

Here, we see a similar trend where, as x increases from -4 to -1, y is increasing from about -35 to about 16.9. At the point (-2.43,16.9) is where a local maximum turning point occurs. Beyond this point as x continues to increase from -2.43 to about 1.10, y is decreasing to a local minimum turning point of about (1.10, -5.26) after which as x continue to increase, y also increases. Again, please do pay attention to the quoted estimates of the points and based on the scale of the graph make sense of them.

Example 91

The next example is a quadratic function of the form $f(x) = x^2 - 2x - 15$

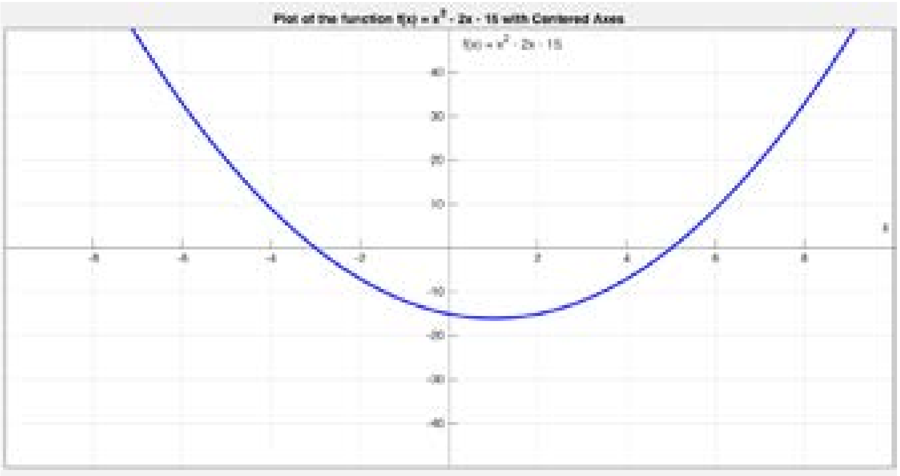


Fig 3.42: Plot of $f(x) = x^2 - 2x - 15$ with centred axes

The nature of this curve is a minimum parabola with the minimum point occurring at the point (1, -16). It must be noted that there is a line of symmetry at x=1.

Example 92

The final example is a cubic of the form $f(x) = (x - 2)^3$

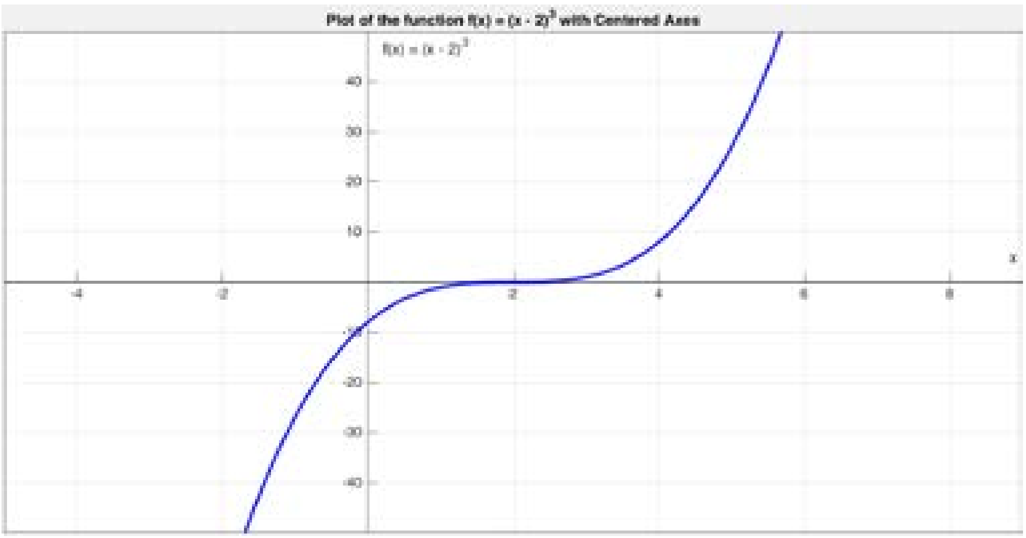


Fig 3.43: Plot of $f(x) = (x - 2)^3$with centred axes

This last example depicts a cubic function with a point of inflexion at x=2. Note that other than a momentary stationary point at (2,0), the curve is always increasing.

DEFINITION, DOMAIN, RANGE AND ZEROS OF RATIONAL FUNCTIONS

Congratulations for your hard work, and commitment in your studies so far. Are you conversant with the factorisation of binomial terms, factorisation of quadratic expressions? For example, $mx + my$ can be factorised as $m(x + y)$ and the quadratic expression $x^2 - 2x - 15$ can be factorised as $(x + 3)$ *and* $(x - 5)$. There is also the need to be able to distinguish between a perfect square and difference of two squares. A perfect square $(a + b)^2$ can be expanded to get $a^2 + 2ab + b^2$ or $(a + b)(a + b)$ whereas difference of two squares which is $a^2 - b^2$ *which can be expressed as* $(a - b)(a + b)$.

A rational function, much like a rational number (a fraction of integers), consists of a **ratio between two polynomial expressions**. The denominator cannot be zero.

A rational function is of the form

$Rx = \frac{fx}{gx}$; $g(x) \neq 0$ (That is, a ratio of two polynomials for which the divisor is not zero).

For example, $f(x) = \frac{x^2 + 1}{x - 2}$ is a rational function.

The concept of rational functions finds their use in various fields like modelling rates of change, describing physical phenomena, solving optimisation problems, Music and Sound Engineering and in Biology and Medicine.

Domain of rational functions

The domain of a rational function, $R(x) = \frac{f(x)}{gx}$ is the set of all real numbers except numbers for which $g(x)$ is zero.

Example 93

What is the domain of the function $y = \frac{1}{x - 2}$

Solution

As per our definition of domain of a rational function, which expression in $y = \frac{1}{x - 2}$ should not be zero.

$x - 2 = 0$

Solving, we obtain:

$x = -2$

The domain of y is the set of all real numbers except $x = 2$.

That is, domain of y is $\{x{:}x \in R, x \neq 2\}$

Example 94

Determine the domain of the following rational functions;

a. $f(x) = \dfrac{5x - 3}{2x - 14}$

b. $p(x) = \dfrac{3x + 1}{x^2 - 9}$

c. $q(x) = \dfrac{3x + 8}{(x - 4)(2x + 4)}$

Solution

a. Given that $f(x) = \dfrac{5x - 3}{2x - 14}$, let the denominator $2x - 14 = 0$

$\Rightarrow 2x = 14$

$x = 7$

Thus, the domain of $f(x)$ is real except $x = 7$. That is $\{x{:}x \in R, x \neq 7\}$

b. Given $p(x) = \dfrac{3x + 1}{x^2 - 9}$, let $x^2 - 9 = 0 \Rightarrow (x - 3)(x + 3) = 0 \Rightarrow x = 3$ *or* $x = -3$

Domain: $\{x{:}x \in R, x \neq 3 \text{ } or \neq -3\}$

c. Given $q(x) = \dfrac{3x + 8}{(x - 4)(2x + 4)}$,

Let $(x - 4)(2x + 4) = 0 \Rightarrow (x - 4) = 0 \text{ } or \text{ } (2x + 4) = 0$

$\Rightarrow x = 4 \text{ } orx = -2$

Domain: $\{x{:}x \in R, x \neq 4 \text{ } or - 2\}$

Range of rational functions

The range of a rational function refers to the set of all possible output values (y-values) it can produce. That is the range is the same as the domain of the inverse function. Thus, in finding the range, we first find the inverse function, then its domain, which is the range.

Example 95

Determine the range of the following rational functions;

a. $f(x) = \frac{5x-3}{2x-14}$

b. $p(x) = \frac{3x+8}{2x-7}$

Solution:

a. Given $f(x) = \frac{5x-3}{2x-14}$

let $f(x) = y$

$\Rightarrow y = \frac{5x-3}{2x-14}$ making x the subject

$y(2x-14) = 5x-3$

$2xy - 14y = 5x - 3$

$2xy - 5x = 14y - 3$

$x(2y-5) = 14y - 3$

Thus, the inverse function is;

$$x = \frac{14y-3}{2y-5}$$

Finding the domain of the inverse function, let $2y - 5 = 0 \Longrightarrow y = \frac{5}{2}$

Range: $\left\{y{:}y \in R, x \neq \frac{5}{2}\right\}$

b. Given $p(x) = \frac{3x+8}{2x-7}$

Let $p(x) = y$

$\Rightarrow y = \frac{3x+8}{2x-7}$

$y(2x-7) = 3x+8$

$2xy - 7y = 3x + 8$

$2xy - 3x = 7y + 8$

$x(2y-3) = 7y + 8$

$x = \frac{7y+8}{2y-3}$ which is the inverse function

For the domain of the inverse, let $2y - 3 = 0 \Longrightarrow y = \frac{3}{2}$

Range: $\left\{y{:}y \in R, x \neq \frac{3}{2}\right\}$

Example 96

Determine the range of each of the following rational functions;

a. $f(x) = \frac{x^2 + 2}{2x^2 - 4}$

b. $f(x) = \frac{10}{x + 5}$

Solution

a. For the range of $f(x) = x^2 + \frac{2}{2}x^2 - 4$, let $y = f(x)$

$y = \frac{x^2 + 2}{2x^2 - 4}$

Making x the subject of the equation

$y(2x^2 - 4) = x^2 + 2$

$2x^2y - 4y = x^2 + 2$

$2x^2y - x^2 = 4y + 2$

$x^2(2y - 1) = 4y + 2$

$x^2 = \frac{4y + 2}{2y - 1}$

$x = \pm\sqrt{\left(\frac{4y + 2}{2y - 1}\right)}$

For the range $\frac{4y + 2}{2y - 1} \geq 0$

$(4y + 2)(2y - 1) \geq 0$

$y \leq -\frac{1}{2}$ *and* $y \geq \frac{1}{2}$

Therefore the range is $\left\{ \text{y: y} \in R, \frac{1}{2} \leq y \textit{ and } y \leq \frac{-1}{2} \right\}$

b. For the range of $f(x) = \frac{10}{x + 5}$, let $y = f(x)$

$y = \frac{10}{x + 5}$

Make x the subject of the equation,

$y(x + 5) = 10$

$xy + 5y = 10$

$xy = 10 - 5y$

$x = \frac{10 - 5y}{y}$

For the range $y \neq 0$

Therefore the range is $\{\text{y: y} \in R, y \neq 0\}$

Zeroes of rational functions

The zeroes, or roots, of rational functions, are the specific input values ($x - values$) that cause the entire function's output ($y - value$) to become zero. They basically "cancel out" the numerator when the denominator doesn't equal zero. Finding the zeroes of rational functions involves solving the equation formed by setting the function equal to zero and manipulating the expression to isolate x.

Example 97

What is/are the zeros of the function $f{:}x \to \frac{x+3}{x-2}$

Solution

The zeros are computed as $x + 3 = 0 \Longrightarrow x = -3$

Example 98

Find the zeros of the function defined by $f{:}x \to \frac{2x-1}{2\,x^2 - 9x - 5}$

Solution

To find the zeros of $f(x) = \frac{2x-1}{2x^2 - 9x - 5}$

$$\frac{2x-1}{2x^2-9x-5} = 0$$

$$\Longrightarrow 2x - 1 = 0$$

$$\Longrightarrow x = \frac{1}{2}$$

Thus, the zeros of $f(x)$ *are* $x = \frac{1}{2}$

Example 99

Determine the zeros of the following functions;

a. $f(x) = \frac{(x-3)(5x^2+7x+2)}{(x^2+2)^2}$

b. $f(x) = \frac{2x^2-8}{17}$

Solution

a. For the zeros of $f(x) = \frac{(x-3)(5x^2+7x+2)}{(x^2+2)^2}$

$$(x-3)(5x^2+7x+2) = 0$$

$$\Longrightarrow x - 3 = 0 \text{ or } 5x^2 + 7x + 2 = 0$$

$$(5x+2)(x+1) = 0$$

$$x = 3 \text{ or } x = \frac{-2}{5} \text{ or } x = -1$$

Therefore the required zeros are $x = 3$ *or* $x = \frac{-2}{5}$ or $x = -1$

b. For the zeros of $f(x) = \frac{2x^2-8}{17}$

$$\frac{2x^2-8}{17} = 0$$

$$\Longrightarrow 2x^2 - 8 = 0$$

$$\Longrightarrow 2(x^2-4) = 0$$

$$\Longrightarrow (x-2)(x+2) = 0$$

$$\Longrightarrow x = 2 \text{ or } x = -2$$

Therefore the required zeros are $x = \mathbf{2}$ *or* $x = \mathbf{-2}$

BASIC OPERATIONS ON RATIONAL FUNCTIONS

Basic arithmetic operations involving rational functions may be performed by factorising both the numerator and denominator where applicable and cancelling out any common factors.

Example 100

By factorising out the numerator and denominator simplify each of the following;

i) $\dfrac{3x-6}{x^2-4}$

ii) $\dfrac{5\,x^2-5}{x^2-4x-5}$

iii) $\dfrac{2\,x^2+x-6}{2\,x^2-x-3}$

Solution

i)
$$\frac{3x-6}{x^2-4}=\frac{3(x-2)}{x+2x-2}$$
$$=\frac{3}{(x+2)}$$

ii)
$$\frac{5\,x^2-5}{x^2-4x-5}=\frac{5(x^2-1)}{(x-5)(x+1)}$$
$$=\frac{5(x-1)(x+1)}{(x-5)(x+1)}$$
$$=\frac{5(x-1)}{(x-5)}$$

iii)
$$\frac{2x^2+x-6}{2x^2-x-3}=\frac{(x+2)(2x-3)}{(x+1)(2x-3)}$$
$$=\frac{x+2}{x+1}$$

Example 101

Given that $f(x)=\dfrac{2x-1}{x-2}$ and $g(x)=\dfrac{x-3}{x+3}$. Simplify:

a. $f(x)-g(x)$

b. $f(x)+g(x)$

Solution

a. $f(x) - g(x) = \frac{2x-1}{x-2} - \frac{x-3}{x+3}$

$$= \frac{(2x-1)(x+3) - (x-3)(x-2)}{(x-2)(x+3)}$$

$$= \frac{(2\,x^2 + 6x - x - 3) - (x^2 - 2x - 3x + 6)}{(x-2)(x+3)}$$

$$= \frac{(2\,x^2 + 5x - 3) - (x^2 - 5x + 6)}{(x-2)(x+3)}$$

$$= \frac{x^2 + 10x - 9}{(x-2)(x+3)}$$

b. $f(x) + g(x) = \frac{2x-1}{x-2} + \frac{x-3}{x+3}$

$$= \frac{(2x-1)(x+3) + (x-3)(x-2)}{(x-2)(x+3)}$$

$$= \frac{(2\,x^2 + 6x - x - 3) + (x^2 - 2x - 3x + 6)}{(x-2)(x+3)}$$

$$= \frac{(2\,x^2 + 5x - 3) + (x^2 - 5x + 6)}{(x-2)(x+3)}$$

$$= \frac{3x^2 + 3}{(x-2)(x+3)}$$

Example 102

Simplify the expression $\frac{2w-6}{3-w}$

Solution

Solve this by factorising both the numerator and denominator,

$\frac{2(w-3)}{(3-w)}$

It appears that there are no common factors for both the numerator and denominator.

What do think can be done to get common factors for them?

You are right by your response that we should multiply for both numerator and denominator by -1

$\frac{-1[2(w-3)]}{-1(3-w)}$

$\frac{-2w+6}{-3+w}$, then factorise this:

$\frac{-2(w-3)}{(w-3)}$, then simplify:

$\frac{-2(w-3)}{(w-3)} = -2$

Example 103

Simplify $\frac{2x-4}{4-x^2}$

Solution

Solve this by factorising both the numerator and denominator,

$\frac{2(x-2)}{(2-x)(2+x)}$

Once again, it appears that there are no common factors for both the numerator and denominator.

Once again, we need to multiply both the numerator and denominator by -1.

$\frac{-1(2x-4)}{-1(4-x^2)}$

$\frac{-2x+4}{-4+x^2}$

$\frac{-2x+4}{x^2-4}$, then factorise this:

$\frac{-2(x-2)}{(x-2)(x+2)}$, then simplify:

$\frac{-2(x-2)}{(x+2)(x-2)}$

$\frac{-2}{(x+2)}$

Activity 3.17

In small groups or individually, simplify each of the following rational expressions:

a. $\frac{9-x^2}{x^2+2x-15}$ **b.** $\frac{5x-20}{16-x^2}$ **c.** $\frac{x^2-6x-27}{81-x^2}$

Congratulations on your effort. If your answers are similar to the following, you are on the right path, if not, consult a teacher or a friend for assistance.

a. $\frac{-(x+3)}{x+5}$ **b.** $\frac{-5}{x+4}$ **c.** $\frac{-(x+3)}{x+9}$

DECOMPOSITION INTO RATIONAL FUNCTIONS

Decomposition into partial fractions is a technique used to simplify and decompose a rational function into a sum of simpler fractions. It is particularly useful when dealing with complex rational functions, making integration, differentiation and solving equations more manageable.

Equivalence of expressions

Example 104

Find the values of the constants A, B and C such that
$x^2 - 4x + 5 = A(x-1)(x+2) + B(x+2)(x-4) + C(x-4)(x-1)$

Solution

$x^2 - 4x + 5 = A(x-1)(x+2) + B(x+2)(x-4) + C(x-4)(x-1)$

$x^2 - 4x + 5 = A\,x^2 + Ax - 2A + B\,x^2 - 2Bx - 8B + C\,x^2 - 5Cx + 4C$

$x^2 - 4x + 5 = A\,x^2 + B\,x^2 + C\,x^2 + Ax - 2Bx - 5Cx - 2A - 8B + 4C$

$x^2 - 4x + 5 = (A + B + C)\,x^2 + (A - 2B - 5C)x - 2A - 8B + 4C$

By comparison, $A + B + C = 1$,

$A - 2B - 5C = -4$ and

$-2A - 8B + 4C = 5$

Solving the three equations simultaneously,

$A = \frac{5}{18}$, $B = -\frac{2}{9}$ and $C = \frac{17}{18}$

Decomposition into Partial Fractions

Rational expressions with linear factors as denominators can be decomposed as follows:

$$\frac{(ax-b)}{((x-p)(x+q)(x-r)\ldots)} = \frac{A}{(x-p)} + \frac{B}{(x+q)} + \frac{C}{(x-r)} \cdots$$

Rational expressions with repeated factors as denominators can be decomposed as follows:

$$\frac{ax-b}{(x-p)^3} = \frac{A}{(x-p)} + \frac{B}{(x-p)^2} + \frac{C}{(x-p)^3}$$

Rational expressions with irreducible quadratic factors as denominators can be decomposed as follows:

$$\frac{ax-b}{a\,x^2+bx+c}=\frac{Ax+B}{a\,x^2+bx+c}$$

Rational expressions with numerators having degrees greater or equal to the degree of the denominator are deemed improper. Such expressions need to be reduced by long division before being decomposed into partial fractions

Example 105

Resolve $\frac{2x^2+1}{(x-1)(x+2)}$ in partial fractions.

Solution

$$\frac{2x^2+1}{(x-1)(x+2)}=2+\frac{-2x+5}{(x-1)(x+2)} \text{ Using long division}$$

$$=2+\frac{A}{x-1}+\frac{B}{x+2}$$

$$=\frac{2(x-1)(x+2)}{(x-1)(x+2)}+\frac{A}{x-1}+\frac{B}{x+2}$$

$$=\frac{2(x-1)(x+2)+A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$2x^2+1=2(x-1)(x+2)+A(x+2)+B(x-1)$$

$$2x^2+1=2\,x^2+2x-4+Ax+2A+Bx-B$$

$$2x^2+1=2\,x^2+(A+B+2)x+2A-B-4$$

By comparison, $A+B+2=0$ and

$2A-B-4=1$

Solving the equations simultaneously yields $A=1$ and $B=-3$

$$\therefore \frac{2x^2+1}{(x-1)(x+2)}=2+\frac{1}{(x-1)}-\frac{-3}{(x+2)}$$

Now let us work through the following examples together paying attention to the details in each one of them.

Example 106

Resolve $\frac{3x+2}{(x+3)(x+2)}$ in partial fractions.

Solution

$$\frac{3x+2}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x+3)}{(x+3)(x+2)}$$

Since the denominators are the same, then the numerators are also the same.

$3x + 2 = A(x + 2) + B(x + 3)$

When $x = -2$

$3(-2) + 2 = A(-2 + 2) + B(-2 + 3)$

$-6 + 2 = \mathrm{B}$

$\Rightarrow B = -4$

When $x = -3$

$3(-3) + 2 = A(-3 + 2) + B(-3 + 3)$

$-9 + 2 = -A$

$\Rightarrow A = 7$

$$\frac{3x+2}{(x+3)(x+2)} = \frac{7}{x+3} - \frac{4}{x+2}$$

Example 107

Express $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fractions.

Solution

$$\frac{3x^2+x+1}{(x-2)(x+1)^2} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2+B(x-2)(x+1)+C(x-2)}{(x-2)(x+1)^2}$$

Since the denominators are the same

$3x^2 + x + 1 = A(x + 1)^2 + B(x - 2)(x + 1) + C(x - 2)$

When $x = 2$

$3(2)^2 + 2 + 1 = A(2 + 1)^2 + B(2 - 2)(2 + 1) + (2 - 2)$

$12 + 3 = 9\mathrm{A}$

$15 = 9\mathrm{A}$

$\Rightarrow A = \frac{5}{3}$

When $x = -1$

$$3(-1)^2 - 1 + 1 = A(-1+1)^2 + B(-1-2)(-1+1) + C(-1-2)$$
$$3 = -3C$$
$$\Rightarrow C = -1$$

When $x = 1$ (or any value of x)

$$3(1)^2 + 1 + 1 = A(1+1)^2 + B(1-2)(1+1) + C(1-2)$$
$$3 + 2 = 4\left(\frac{5}{3}\right) - 2B - (-1)$$
$$5 - \frac{20}{3} - 1 = -2B$$
$$\Rightarrow -\frac{8}{3} = -2B$$
$$\Rightarrow B = \frac{4}{3}$$
$$\therefore \frac{3x^2 + x + 1}{(x-2)(x+1)^2} = \frac{5}{3(x-2)} + \frac{4}{3(x+1)} - \frac{1}{(x+1)^2}$$

Example 108

Resolve $\frac{3x+1}{(x-1)(x^2+1)}$ in partial fractions.

Solution

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$$
$$= \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

Since the denominators are the same

$$3x + 1 = A(x^2+1) + (Bx+C)(x-1)$$

When $x = 1$

$$3(1) + 1 = A\{(1)^2 + 1\} + \{B(1) + C\}(1-1)$$
$$3 + 1 = 2A$$
$$4 = 2A$$
$$\Rightarrow A = 2$$

When $x = 0$

$$3(0) + 1 = A\{(0)^2 + 1\} + \{B(0) + C\}(0-1)$$
$$1 = A - C$$
$$1 = 2 - C$$
$$\Rightarrow C = 1$$

When $x = 2$ (or any value of x)

$3(2) + 1 = A\{(2)^2 + 1\} + \{B(2) + C\}(2 - 1)$

$6 + 1 = 5A + 2B + C$

$7 = 5(2) + 2B + 1$ substituting $A = 2$ and $C = 1$

$7 - 10 - 1 = 2B$

$-4 = 2B$

$\Rightarrow B = -2$

$$\therefore \frac{3x+1}{(x-1)(x^2+1)} = \frac{2}{(x-1)} + \frac{-2x+1}{(x^2+1)}$$

Example 109

Resolve $\frac{2x^2+1}{((x-1)(x+2))}$ in partial fractions.

An improper fraction is one whose numerator is of equal degree or greater than that of the denominator. To deal with this situation, we first obtain a quotient and proper fraction and split the latter into partial fractions.

Solution

Consider the expansion of the denominator

$(x - 1)(x + 2) = x^2 + x - 2$

$$\begin{array}{r|l} & \quad\quad\quad\quad 2 \\ \hline x^2 + x - 2 & 2x^2 \quad\quad +1 \\ & -(2x^2 + 2x - 4) \\ \hline & \quad -2x + 5 \end{array}$$

$$\frac{2x^2+1}{((x-1)(x+2))} = 2 + \frac{(-2x+5)}{(x-1)(x+2)}$$

Consider:

$$\frac{(-2x+5)}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$= \frac{(A(x+2) + B(x-1)}{(x-1)(x+2)}$$

Since the denominators are the same:

$-2x + 5 = A(x + 2) + B(x - 1)$

When $x = 1$

$-2(1) + 5 = A(1 + 2) + B(1 - 1)$

$-2 + 5 = 3A$

$3 = 3A$

$A = 1$

When $x = -2$

$-2(-2) + 5 = A(-2 + 2) + B(-2-1)$

$4 + 5 = -3B$

$9 = -3B$

$B = -3$

Therefore $\frac{2x^2 + 1}{((x-1)(x+2))} = 2 + \frac{1}{(x-1)} - \frac{3}{(x+2)}$

Activity 3.18

Now in groups of three or individually, solve each of the following problems to confirm the answers given.

1. Find the values of A and B such that $31x - 8 = A(x-5) + B(4x+1)$

Ans. $A = 3$ *and* $B = 7$

2. Find the values of A, B and C such that

$4x^2 + 4x - 26 = A(x+2)(x-4) + B(x-4)(x-1) + C(x-1)(x+2)$

Ans. $A = 2$, $B = -1$ *and* $C = 3$

3. Express $h(x) = \frac{2x^3 + x^2 - 15x - 5}{(x+3)(x-2)}$ in partial fractions.

Ans. $\frac{2x^3 + x^2 - 15x - 5}{(x+3)(x-2)} = (2x - 1) + \frac{1}{x+3} - \frac{3}{x-2}$

4. A factory produces a certain product, and the average cost per unit is given by the function

$f(x) = \frac{5000 + 2x}{x}$ where x is the number of units produced. What is the average cost in cedis per unit if 100 units are produced?

Ans. The average cost per unit is GHC 52 if 100 units are produced.

EXTENDED READING

The simplest example of a rational function is $f(x) = \frac{1}{x}$ whose nature is a rectangular hyperbola which can be illustrated graphically as follows;

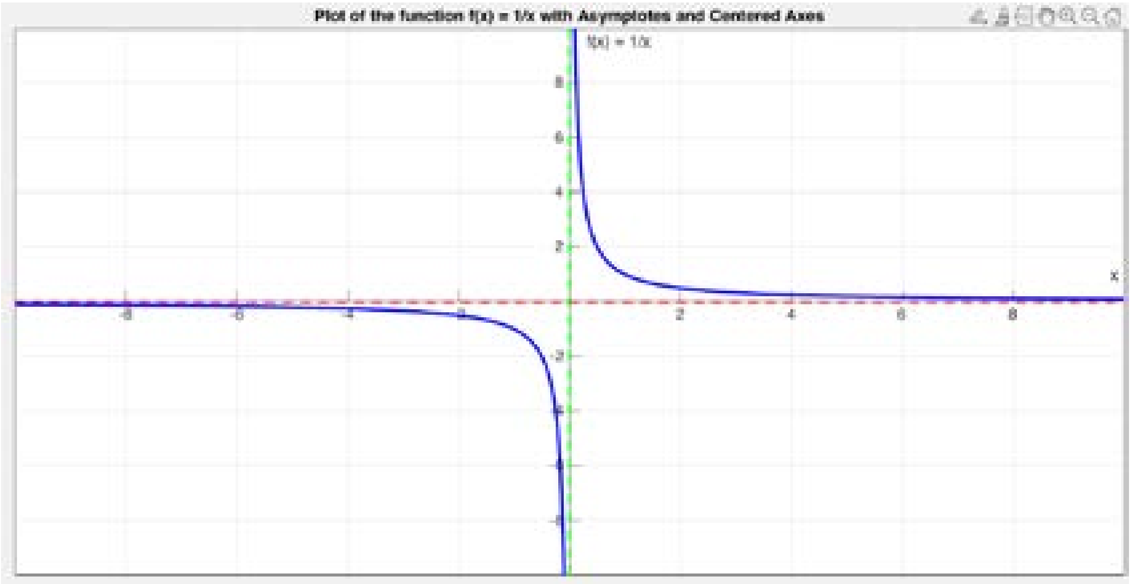


Fig 3.44: Plot of $f(x) = \frac{1}{x}$ with asymptote and centred axes

Here, the x and y axes are asymptotes. An asymptote is line horizontal, vertical or oblique which the graph does not touch or cross.

Graphical representation of rational functions

There are certain rational functions that after simplifying will result in a linear function. In such situations, it is important that we note the x values that need to be excluded, particularly when we are trying to draw the graph of the function. The set of ordered pairs of the simplified function will be exactly the same as the set of ordered pairs of the original function. If we plug the excluded value(s) for x into the simplified expression, we get a set of ordered pairs that represent "holes" in the graph. These holes are breaks in the curve. We use an open circle to designate them on the graph.

For example, considering the function $f(x) = \frac{x^2 + 2x + 1}{x + 1}$

Simplifying we get

$$f(x) = \frac{(x + 1)(x + 1)}{x + 1}$$

$$= x + 1, \text{ for } x \neq -1$$

For the hole in the graph put $x = -1$ into the simplified $f(x)$

$f(-1) = -1 + 1 = 0$

This occurs at (–1, 0)

Therefore, the graph of the simplified $f(x) = x + 1$ can be drawn as follows:

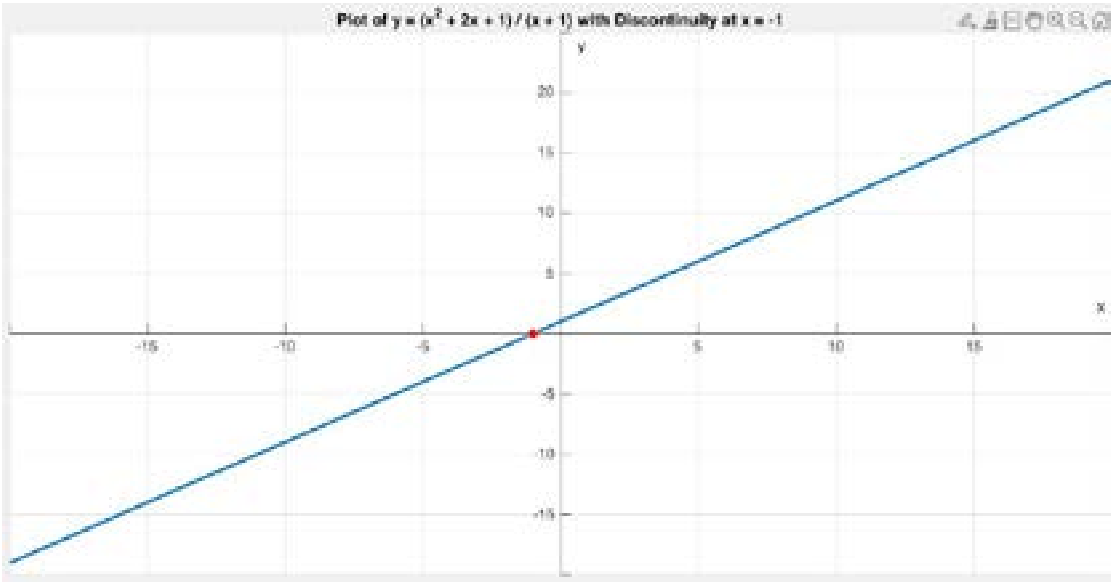


Fig 4.45: Plot of $f(x) = \frac{x^2 + 2x + 1}{x + 1}$ with discontinuity at $x = -1$

It must be observed that the red spot is the hole which occurs at (-1,0)

Similarly, the graph of $f(x) = \frac{5x^2 - 10x}{5x}$

This can be simplified as;

$f(x) = \frac{5x(x-2)}{5x}$

$= x - 2$ for $x \neq 0$

Therefore the hole is (0,–2)

The graph is as shown:

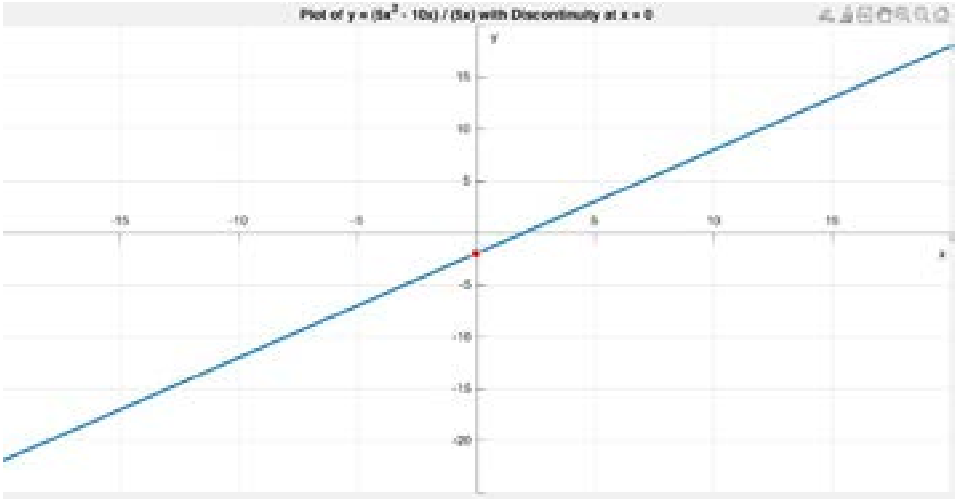


Fig 3.46: Plot of $f(x) = \frac{5x(x-2)}{5x}$ with discontinuity at x = 0

Once again, the hole is the red spot.

Let us now study graphically the behaviour of the rational function $f(x) = \frac{x-3}{x+2}$ as shown;

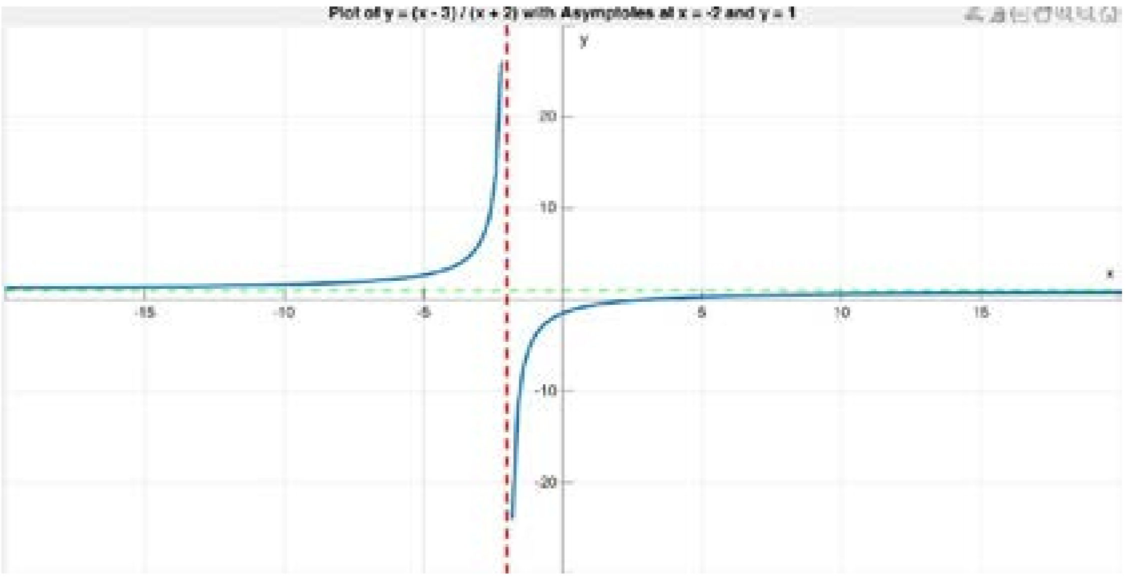


Fig 3.47: Plot of $f(x) = \frac{x-3}{x+2}$ with asymptote at x = -2 and y = 1

The broken red line is the vertical asymptote and broken green line the horizontal asymptote.

Again, for the function $f(x) = \frac{x+2}{x-2}$, the graph is;

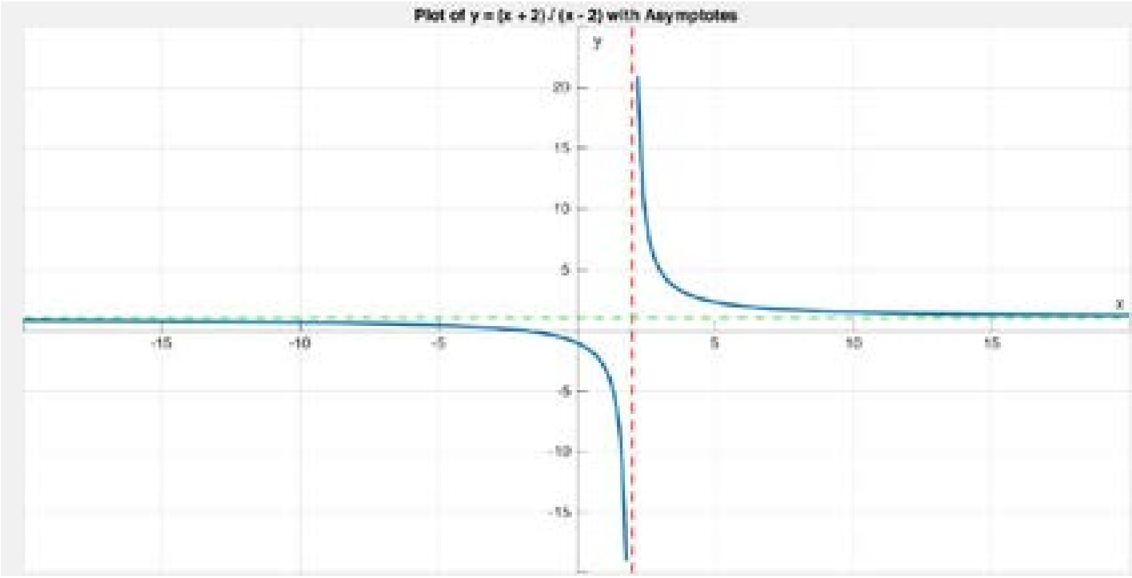


Fig 3.48: Plot of $f(x) = \frac{x+2}{x-2}$ with asymptotes

The broken red line is the vertical asymptote and broken green line the horizontal asymptote

Identifying rational function

Which of the following is/are rational function(s);

a. $f(x) = 3\,x^2 - 5x + 2$

This is a rational function; it could be written over the denominator 1 and 1 is a polynomial.

b. $f(x) = \dfrac{3x^2 - 5x + 2}{2x - 1}$

This is a rational function; it is a ratio of two polynomials.

c. $f(x) = 3x^3 + 3\sqrt{x}$

This is not a rational function; it is not the ratio of two polynomials.

Activity 3.18

In small groups or individually, solve the following problem:

The total revenue from the sale of a popular video of a Ghanaian film is approximated by the rational function;

$$\boldsymbol{R(x)=\frac{300\,x^2}{x^2+9}}$$

Where x is the number of months since the video was released and $R(x)$ is the total revenue in hundreds of Ghana cedis.

i. Find the total revenue generated by the end of the first month.

ii. Find the total revenue generated by the end of the second month.

iii. Find the total revenue generated by the end of the third month.

iv. Find the revenue in of the second month only.

Congratulations if your answers are;

i. GH¢ 3000

ii. GH¢ 9231

iii. GH¢ 15000

iv. GH¢ 6231

REVIEW QUESTIONS

Review Questions 3.1

1. Determine the next three terms of the following sequence
   - a. 5, 9, 13, 17, …
   - b. 1, 3, 9, 27, …
   - c. 2, 6, 18, 54, …
   - d. 1, 3, 5, 7 …
   - e. 2, 4, 8, 16, …
2. What kind of sequence is the following: $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$?
   - a. Neither geometric nor arithmetic
   - b. Geometric and arithmetic
   - c. Geometric only
   - d. Arithmetic only
3. Which of the following sequences is not classified as arithmetic or geometric?
   - a. $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
   - b. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
   - c. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$
   - d. $\frac{1}{3}, \frac{-1}{3}, -1, \frac{-5}{3}, \ldots$
   - e. $1, \frac{1}{2}, 0, -\frac{1}{2} \ldots$
4. Find the 10th term of the arithmetic sequence: 2, 5, 8, 11, ...
5. Given the arithmetic sequence: 3, 7, 11, 15, ... Find an expression for the nth term of this sequence.
6. For the geometric sequence: 4, 12, 36, 108, ... Find the formula for the nth term of the sequence.

7. Complete the table for the sequence given by the formula $U_n = 13(2)^{n-1}$, n> 0

N	1	2	3	4	5	6
$U_n = 13(2)^{n-1}$						

8. The second and fourth terms of exponential sequence (GP) Of positive terms are 4 and 9 respectively. Find:
   a) the common ratio
   b) the first term
9. Ali bought a new car for Gh¢50,000.00. It is believed that the value of depreciation per annum is 20% of the value at the beginning of the year.

   Calculate the number of years after which the car is worth GH¢13,110.00.
10. Three consecutive terms of an AP. Have a sum of 21 and a product of 315. Find the numbers.
11. The first term of an AP is 11 and the fifteenth term is 21.

    Find:
    a) the first term
    b) the common difference
    c) the nth term

Review Questions 3.2

1. Evaluate $g(-3)$ if $g(x) = 2\ x\ 2 - 4x + 7$
2. $Let\ f\ (x) = 3x + 2\ and\ g(x) = 2x\ 2 - 1$. Determine the composite function $(g \circ f)(x)$
3. $Given\ f(x) = \frac{1}{2}\ and\ g(x) = log\ 10(x)$, find the composite function, $(g \circ f)(x)$. , giving your answer to 4 decimal places.
4. Given the functions $f(x) = 3x$ and $g(x) = x - 2$, find the composite function $(f \circ g)(x)$
5. Given the function $f(x) = 2x + 3$, evaluate $f(4)$
6. Consider the function $h(t) = 3t^3 - 2t^2 + 5$. Calculate $h(-1)$.
7. Consider the function $v(t) = log_{10}(t^2 + 1) - \sqrt{(t)}$. Calculate $v(4)$.
8. State the domain of $P(x) = -3x^3 + 6x - 7$

9. State the domain of $f(x) = \sqrt{2x + 3}$,

10. Find the domain of function $g(t) = \dfrac{3 + x}{x^2 - 4x + 3}$

11. Find the inverse of the function $f(x) = 2x + 5$

12. Find the inverse of the function $h(x) = 4 - 3x$

13. Find the inverse of the one-to-one function $p(x) = 5x - 7$

14. Determine whether the function $f(x) = 2x + 3$ is bijective

15. If an astronaut weighs 65 *kg* on the surface of the earth, then her weight when she if d miles from the earth is given by
$w(d) = 65\left(\dfrac{3960}{3960 + d}\right)^2$
    a. Determine her weight when she is 150 miles above the earth.
    b. Construct a table of values for the function w that gives her weight at heights in the interval: [0, 300] using 50 miles for the increment.
    c. From your table, describe the nature of the function, $w(d)$, (increasing or decreasing, injective, surjective and bijective).

16. Establish whether the function $g(x) = x^2$ is bijective on the domain of real numbers.

17. Consider the functions $p(x) = x^3$ and $q(x) = \sqrt{x}$. Find the composite function $(q \circ p)(x)$

Review Questions 3.3

1. Construct a linear function passing through the points (1, 4) and (3, 10)
2. Given the linear function $y = 2x - 5$, find its x–intercept and y-intercept.
3. Determine the turning point of the parabolic function $y = -2x^2 + 4x + 1$, state whether it is a maximum or minimum and describe the nature of its graph
4. Kojo and Afi went to the shop to buy stationery for school. Kojo bought 5 exercise books and 3 pens for *GH¢*18. Afi bought 3 exercise books and 4 pens for *GH¢*13. Find the cost of an exercise book and a pen.

5. A manufacturing company produces two types of products: Product A and Product B. The company's production costs and sales revenue for each product are as follows:

   Product *A*:

   Production Cost per unit: $GH¢10$

   Selling Price per unit: $GH¢25$

   Product B:

   Production Cost per unit: $GH¢15$

   Selling Price per unit: $GH¢30$

   In a given month, the company produced a total of 1,000 units of **Product *A*** and **Product *B*** combined. The total production cost was $GH¢12{,}000$, and the total sales revenue was $GH¢27{,}000$.

   a. Use appropriate algebraic notations and properties to represent the situation algebraically.

   b. Solve the simultaneous equations to find the number of units produced for each product

6. On the same graph sheet, plot the straight lines $y = 4x - 2$ *and* $y = 2x + 4$ and state the point of intersection.

7. Plot the graphs of $y = x^2 - 4x - 21$ *and* $y = x + 3$ for $-4 \leq x \leq 10$ using scale of 2cm to 2 units on the x-axis and 2 cm to 5 units on the y-axis . Hence from graph state the point of intersection.

8. i) Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes ox and oy.

   ii) Draw on the graph sheet, the equation $y = 2 - 2x$ for values of x from -1 to 3.

   iii) On the same graph sheet, draw a graph for the equation $y = \frac{1}{2}(x + 1)$ for values of x from -1 to 3

   iv) Using the graphs, find the values of x and y at the point where the two lines meet.

9. Plot the graphs of $y = 4x - x^2$ *and* $y = x^2 - 16$ on the same graph sheet for values of x from –5 to 5, using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 4 units on the y-axis.

   Hence, state from the graph.

   a. The minimum point of the curve $y = x^2 - 16$

   b. The maximum point of the curve $y = 4x - x^2$

   c. The points of intersection of the two curves.

10. Find the area bounded by $y = -2x + 8$, the x-axis and the y-axis

11. Find the area bounded by $3x - 2y = 6$, y = 4, the x-axis and the y-axis

Review Questions 3.4

1. Graph the solution set to the system of inequalities and identify each vertex of the region: $x \geq 0$, $y \geq 0$, $3x + 2y \leq 12$ *and* $x + 2y \leq 8$

2. Graph the following system of inequalities: $y > -3x + 4$ *and* $2y—x > 2$

3. One kind of cake requires 200 g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Suppose we want to make as many cakes as possible but have only 4000g of flour and 1200g of fat available, although there is no shortage of the various other ingredients. How many cakes of each kind should we make?

4. Graph the system of inequalities; $x > 4$ *and* $y < 3$

5. Illustrate the solution set of the linear inequalities: $3y - 2x \leq 5$ *and* $y + 2x \geq 4$

Review Questions 3.5

1. Given $g(x) = 3x^2 - 5x + 2$ and $h(x) = 2x^2 + 4x - 1$

   a. Determine the sum of $g(x)$and $h(x)$

   b. Find the difference between $g(x)$and $h(x)$

2. Consider the polynomial function $p(x) = 6x^4 - 3x^3 + 2x^2 - 8x + 5$

   a. What is the degree of the polynomial function?

   b. What is the leading coefficient of the polynomial?

   c. Evaluate $p(1)$

3. Given the polynomial $p(x) = 3x^4 - 4x^3 + 2x^2 - 7x - 10$, use the Remainder Theorem to determine if $p(x)$is divisible *by* $x - 2$
4. A local farmer wants to build a rectangular pen for his animals. He plans to use one side of his barn as one side of the pen, and he has 200 *feet* of fencing material to complete the other three sides. What dimensions should the farmer use to maximise the area of the pen?
5. The number of tickets sold during the Senior High School Football season can be modeled by $t(x) = x^3 - 12x^2 + 48x + 74$. Use the remainder theorem to find the number of tickets sold during the 12 game of the school football season
6. Given the polynomial $p(x) = 5x^4 - 3x^3 + x^2 - 10x + 10$. Find the remainder when it is divided by $x - 2$
7. Write down the equations whose sums and products of roots are respectively
   - a) 5, 9
   - b) $\frac{2}{3}, \frac{1}{6}$
   - c) $-\frac{4}{7}, -\frac{1}{2}$
   - d) $-1, \frac{3}{5}$
8. From each equation, calculate the discriminant and hence describe the nature of its roots.
   - i $4x^2 + 3x + 1 = 0$
   - ii. $2a^2 - 5a - 8 = 0$
   - iii. $9x^2 + 12x + 4 = 0$
9. Find the minimum or the maximum values of :
   - i. $4 + 3x - x^2$
   - ii. $x^2 - 5x + 4$ and where they occur
10. Using completing squares solve the following
    - a. $x^2 + 6x + 9 = 0$
    - b. $2x^2 - 8x - 4 = 0$
    - c. $x^2 + 3x + 1 = 0$
    - d. $2x^2 + 2x - 1 = 0$
    - e. $x^2 + 4x - 8 = 0$

Review Questions 3.6

1. What is the restriction on the domain of the rational function $f(x) = \frac{2x+1}{x-3}$?
2. The range of the function $g(x) = \frac{x^2}{x-1}$ excludes what value/s?
3. What are the zeros of the function $h(x) = (x+2)(x-5)$?
4. If $f(x) = \frac{x+1}{x-2}$ and $g(x) = \frac{2x-3}{x+1}$ what is the numerator of $\frac{f(x)}{g(x)}$?
5. What is the decomposed form of the function $j(x) = \frac{1}{x^2-4}$
6. Why is the function $k(x) = \frac{x^2+1}{x-2}$ defined for all real numbers except $x = 2$.
7. Given $f(x) = \frac{2x+1}{x}$ and $g(x) = \frac{x-3}{x+2}$, find the product $f(x) \times g(x)$.
8. Given that $f(x) = \frac{2x+3}{x^2-9}$, find the constants A and B in the following form: $\frac{A}{x+3} + \frac{B}{x-3}$
9. Resolve the following into partial fractions; $\frac{3x^3+5x^2-7x+3}{(x^2-1)^2}$
10. Find the values of the constants A, B and C such that $x^2 - 4x + 5 = A(x-1)(x+2) + B(x+2)(x-4) + C(x-4)(x-1)$
11. A mining company in Tarkwa, a town in the Western Region of Ghana extracts ore from a mine, and the cost of extraction per ton is given by the function $C(t) = \frac{1000 + 20t}{t}$, where t is the number of tons of ore extracted. What is the cost per ton if the company extracts 50 tons in cedis?
12. A pharmaceutical company in Cape Coast produces a certain medication, and the average cost per dose is given by the function $(d) = \frac{500+10d}{d}$ where d is the number of doses produced.

    What is the average cost per dose if the company produces 200 doses in cedis?
13. A water tank is being filled and the amount of water in the tank is represented by the function $(t) = \frac{500t}{t+2}$, where (t) is the amount of water in litres after t hours. I How much water will be in the tank after 4 hours?

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