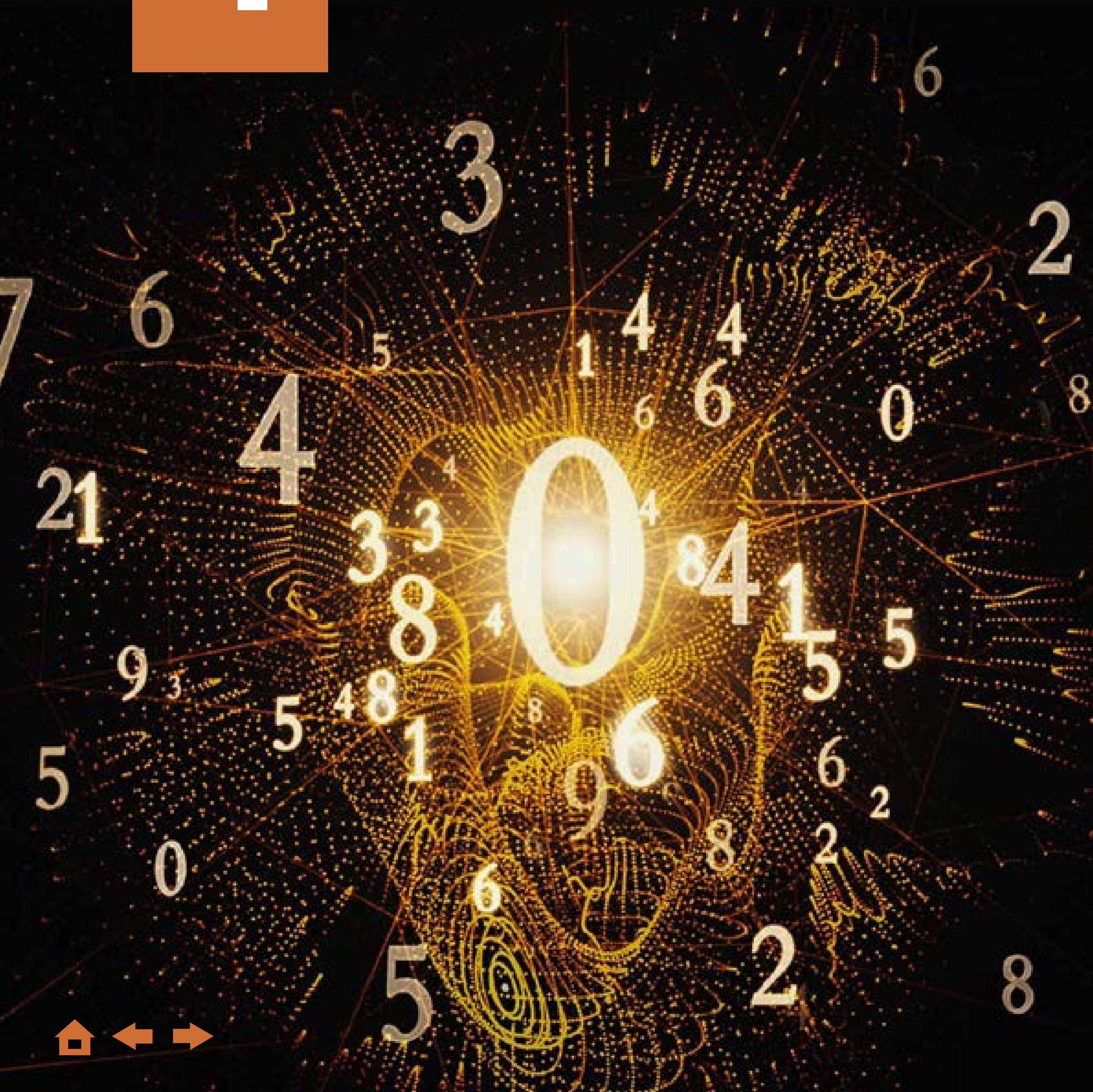
Additional Mathematics Year 1

SECTION 4

MATRICES



MODELLING WITH ALGEBRA

Number and Algebraic Patterns

INTRODUCTION

A matrix is a rectangular arrangement of numbers, symbols, or expressions enclosed within brackets ( ) or [ ]. Each element within the matrix has a specific location identified by its row (horizontal position) and column (vertical position). We describe a matrix by its dimensions, specifying the number of rows and columns. For example, a matrix with 2 rows and 3 columns is called a (2×3) matrix. When a matrix has the same number of rows and columns for example (3×3), it is called a **square matrix**. A square matrix with non-zero entries only on its main diagonal (from top left to bottom right) is called a **diagonal matrix**. A special square matrix with ones (1s) on its main diagonal and zeros (0s) elsewhere is called an identity matrix. It plays a crucial role in solving matrix equations. Matrices have extensive applications in various fields: They are used to organise and analyse large datasets in statistics, health, economics, and social sciences. In computer graphics, matrices are essential for representing 3D objects, transformations, and lighting effects in computer graphics. Matrices are also used to analyse electrical circuits and solve complex problems related to currents and voltages. They are applied in physics to represent physical systems like forces and motion, simplifying calculations and analysis.

At the end of this section, you will be able to:

- Recognise a matrix including types of matrices and state its order
- Find the determinant of a (2×2) matrix
- Add and subtract matrices (2×2) matrix
- Multiply a matrix by a scalar and a matrix by a matrix (2×2) matrices

Key Idea:

A matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions arranged in rows and columns.

DEFINITION, ORDER AND TYPES OF MATRICES

A matrix (plural: matrices) is a rectangular array of numbers, symbols, or expressions, organised in rows and columns. Each entry in a matrix is called an element or an entry, and it is identified by its row and column indices.

$$\begin{array}{cc} & 4\ columns \\ 2\ rows \begin{array}{c} \\ \rightarrow \\ \rightarrow \end{array} & \begin{bmatrix} \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 7 & 3 & 6 \\ -2 & 1 & 3 & 5 \end{bmatrix} \end{array}$$

Everyday situations that exemplify the concept of matrices include

a. Classroom seating arrangement

b. Provision items in a shop

c. A pack of bottled water, among others.

Example:

Suppose that we wish to express the information of possession of pens and pencils by Afiba and his two friends Enyonam and Nana, which is as follows:

Afiba has 2 pens and 7 pencils,

Enyonam has 1 pen and 5 pencils,

Nana has 3 pens and 2 pencils.

Now, this could be arranged in tabular form as the individual items in the matrix are called the ***elements*** or ***entries.***

Describing a Matrix

Individuals	Pens	Pencils
Afiba	2	7
Enyonam	1	5
Nana	3	2

Which could be expressed in matrix form as;

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{pmatrix} \text{ or } A = \begin{bmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{bmatrix}$$

The horizontal arrays are called **rows,** and vertical arrays are called **columns**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \textbf{ROWS}$$

$$\downarrow \textbf{COLUMNS}$$

Describing a Matrix (Order or dimension of a matrix)

A matrix is described by stating the dimensions. For example, $[a \quad b]$ is a (1×2) (read one–by–two)n matrix, $\begin{bmatrix} a \\ b \end{bmatrix}$ is a (2×1) (read two–by–one), $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a (2×2) (read two–by–two) and $\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ is (2×3) (read two–by–three) all because of their respective number of rows and columns.

An $(m \times n)$ matrix has m rows (horizontal) and n columns (vertical). Each element of a matrix is denoted by a variable with subscripts. For example, $\boldsymbol{a}_{23,}$ represents the element in the 2nd (second) row and 3rd (third) column of the matrix.

$$\text{For example, the matrix } A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \dots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

is an $(m \times n)$ dimensional matrix having m number of rows and n number of columns.

The order of the matrix $M = \begin{pmatrix} -3 & 4 & -1 \\ 5 & 2 & 0 \end{pmatrix}$ is 2×3

Note: Commas are not used in matrices. Gaps are left between the columns.

Types of Matrices

Square matrix: A matrix that has the same number of rows and the same number of columns, it is called a square matrix.

For example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$$

2×2 matrix $\qquad$ 3×3 matrix

Rectangular matrix: A matrix in which the number of rows is not equal to the number of columns.

For example, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}$ (2×3) *or* $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ (3×2)

The Zero Matrix

A zero matrix is a matrix whose entries are all zeros or are equivalent to zero.

$$\mathrm{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(0 \quad 0)$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i+i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x-x \end{pmatrix}$ are all zero matrices

Example:

Which of the following matrices is/are zero matrix/matrices?

a. $A = \begin{pmatrix} x-x & 0 & 0 \\ 0 & -b+b & 0 \end{pmatrix}$

b. $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

c. $C = \begin{pmatrix} m-m & 0 \\ 0 & 0 \end{pmatrix}$

Solution:

a. Simplifying the entries in the matrix A gives $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, thus A is a zero matrix

b. Matrix B has two (2) non-zero entries, thus B is not a zero matrix

c. Matrix C can be simplified to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ making it a zero matrix

The Unit or Identity Matrix

A unit matrix or identity matrix is a square matrix having every non-main-diagonal element equal to zero and every main diagonal element equal to one.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Equality of Matrix

Two matrices are said to be equal if their corresponding elements or entries are the same (equivalent).

For example, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $a = e$, $b = f$, $c = g$, $d = h$.

Also, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then $a = 2$, $b = 3$, $c = 4$, $d = 5$

For example, $M = \begin{pmatrix} 3 & 5 & -2 & \frac{3}{4} \\ 4 & \frac{1}{2} & 0 & -\frac{2}{5} \end{pmatrix}$ and N= $\begin{pmatrix} 3 & 5 & -2 & \frac{3}{4} \\ 4 & \frac{1}{2} & 0 & -\frac{2}{5} \end{pmatrix}$ are two equal matrices since they have the same dimension and their corresponding elements are the same.

Example

If $P = \begin{pmatrix} \frac{3}{2} & x \\ y - x & -3 \end{pmatrix}$ and $Q = \begin{pmatrix} x & 1.5 \\ 3.5 & -3 \end{pmatrix}$ are two equal matrices, find the values of x and y.

Solution

Since $P = Q$,

$\frac{3}{2} = x$ and $y - x = 3.5$

$y - \frac{3}{2} = 3.5$

$y = 5$

DETERMINANTS OF (2 × 2) MATRICES

Imagine a woven basket from a village, but instead of holding your favourite fruits, it holds numbers arranged in a neat grid, like two rows of cowrie shells. This grid is what mathematicians call a (2 × 2) matrix. In Ghana, we have a word for unlocking secrets – "Odomankoma" (key). Determinants act as the Odomankoma for these matrices, revealing a unique property based on how the numbers are arranged.

Here's a breakdown for (2 × 2) matrices, like our basket of numbers:

Basic Structure: A (2 × 2) matrix looks like this:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where a represents the element in the first row, first column; b represents the element in the first row, second column, and so on)

Determinant Formula: There's a special formula to calculate the (determinant, *det*) of a (2 × 2) matrix:

$det = (a \times d) - (b \times c)$

where a, b, c and d represent the elements (oduas or twigs) of the matrix as shown above.

Visualizing the Determinant: Imagine drawing a diagonal line across the basket, starting from the top left corner (a) and reaching the bottom right corner (d). Now, draw another diagonal line starting from the top right corner (b) and reaching the bottom left corner (c). The determinant captures the difference between the product of the elements along one diagonal ($a \times d$), $the' main\ diagonal'$, and the product of elements along the other diagonal ($b \times c$).

Example:

Consider this (2 × 2) matrix:

$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

Using the formula, the determinant (det) would be:

$det(A) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

Importance:

The determinant of a (2 x 2) matrix has various applications, including:

a. **Solving systems of linear equations**: Determinants play a crucial role in finding solutions to systems of linear equations with two variables.

b. **Invertibility of matrices:** A non-zero determinant indicates that the matrix is invertible, meaning it has an inverse matrix.

c. **Area calculation:** In specific contexts, the determinant can be used to calculate the area enclosed by a parallelogram defined by the matrix's row vectors. Imagine a farmer needs to calculate the area of a rectangular plot of land represented by a (2×2) matrix, where each element represents the length and width of the plot in meters. The determinant, in this case, can be used to calculate the area (**note**, this application has limitations for general area calculation)

Example 1

Evaluate the determinants of the following matrices

a. $\begin{bmatrix} 21 & 4 \\ 17 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -8 \\ -3 & 6 \end{bmatrix}$

c. $\begin{bmatrix} a+3 & 7-a \\ a & 7 \end{bmatrix}$

Solution

Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determinant of $A, det(A) = ad - bc$

a. $det\begin{vmatrix} 21 & 4 \\ 17 & 9 \end{vmatrix} = (21 \times 9) - (4 \times 17) = 121$

b. $det\begin{vmatrix} 2 & -8 \\ -3 & 6 \end{vmatrix} = ((2 \times 6) - (-8 \times -3)) = -12$

c. $det\begin{vmatrix} a+3 & 7-a \\ a & 7 \end{vmatrix} = (7(a+3) - a(7-a)) = 7a + 21 - 7a + a^2 = a^2 + 21$

Example 2

If the matrix A $= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ $and\ B = \begin{pmatrix} 3 & -2 \\ 6 & 5 \end{pmatrix}$, find the determinants of A and B.

Solution

$A = det\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

B$= det\begin{vmatrix} 3 & -2 \\ 6 & 5 \end{vmatrix} = (3 \times 5) - (6 \times -2) = 15 + 12 = 27$

Example 3

If $A = \begin{pmatrix} 4 & -2 \\ 3x & 5 \end{pmatrix}$,find the value of x if the determinant of $A = 32$

Solution

$detA = (4)(5) - (3x)(-2) = 20 + 6x$

$detA = 20 + 6x$, $but\ detA = 32$

$\therefore 32 = 20 + 6x$

$12 = 6x$

$x = 2$

Example 4

Given that $B = \begin{pmatrix} 2 & 4 \\ 6 & 12 \end{pmatrix}$, find the determinant of B

Solution

$detB = det\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 12(2) - 6(4) = 24 - 24 = 0$

Example 5

Evaluate the determinant of:

(a) $A = \begin{pmatrix} -4y & 3 \\ 6 & 5 \end{pmatrix}$, if $y = -2$

(b) $B = \begin{pmatrix} -2 & -2 \\ 2r + 3 & -5 \end{pmatrix}$, if $r = 4$

Solution

$detA = det\begin{vmatrix} -4y & 3 \\ 6 & 5 \end{vmatrix} = 5(-4y) - (6)(3) = -20y - 18$

$but\ y = 2$

$\therefore \text{det}A = -20(2) - 18 = -58$

$detB = det\begin{vmatrix} -2 & -2 \\ 2r+3 & -5 \end{vmatrix} = -2(-5) - (2r+3)(-2\ ) = 10 - (-4r - 6) = 10 + 4r + 6$

$but\ r = 4$

$\therefore \text{det}B = 10 + 4(4) + 6 = 10 + 16 + 6 = 32$

Example 6

Solve the following equations

a. $\begin{vmatrix} x+1 & 2 \\ 2x-1 & 3 \end{vmatrix} = 4$ **b.** $\begin{vmatrix} 5x+2 & 6x-3 \\ 4 & 3 \end{vmatrix} = 0$

c. $\begin{vmatrix} 3x-2 & 2 \\ 1 & 2x+1 \end{vmatrix} = 1$ **d.** $\begin{vmatrix} 2x-1 & 3x+1 \\ x-1 & x+1 \end{vmatrix} = 2$

Solution

a. $\begin{vmatrix} x+1 & 2 \\ 2x-1 & 3 \end{vmatrix} = 4 \Longrightarrow 3(x+1) - 2(2x-1) = 4$

$$3x + 3 - 4x + 2 = 4$$
$$5 - x = 4$$
$$x = 1$$

b. $\begin{vmatrix} 5x+2 & 6x-3 \\ 4 & 3 \end{vmatrix} = 0 \Longrightarrow 3(5x+2) - 4(6x-3) = 0$

$$15x + 6 - 24x + 12 = 0$$
$$18 - 9x = 0$$
$$\mathbf{x = 2}$$

c. $\begin{vmatrix} 3x-2 & 2 \\ 1 & 2x+1 \end{vmatrix} = 1 \Longrightarrow (3x-2)(2x+1) - 2 = 1$

$$\Longrightarrow 6x^2 + 3x - 4x - 2 - 2 = 1$$
$$6x^2 - x - 4 - 1 = 0$$
$$6x^2 - x - 5 = 0$$
$$(x-1)(6x+5) = 1$$
$$x = 1 \text{ } or \text{ } x = -\frac{5}{6}$$

d. $\begin{vmatrix} 2x-1 & 3x+1 \\ x-1 & x+1 \end{vmatrix} = 2 \Longrightarrow (2x-1)(x+1) - (3x+1)(x-1) = 2$

$$2x^2 + x - 1 - \left(3x^2 - 2x - 1\right) = 2$$
$$2x^2 - 3x^2 + x + 2x - 1 + 1 = 2$$
$$-x^2 + 3x = 2$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x = 1 \text{ } or \text{ } x = 2$$

ARITHMETIC OPERATIONS OF MATRICES

Addition of Matrices

Two matrices can be added only if they have the same size. To add two matrices, add the elements in the corresponding positions in each matrix.

For example, given (2 × 2 matrices):

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ resulting in another (2× 2) matrix.

Given that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$,

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Example 1

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$, find

a) $A + B$

b) $B + A$

c) $A + (B + C)$

d) $(A + B) + C$

e) What is the relationship between your answers in a) and b)

f) What is the relationship between your answers in c) and d)

Solution:

a) $A + B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$

$$= \begin{pmatrix} -5+4 & 3-3 \\ 2+7 & -1-5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$$

b) $B + A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4-5 & -3+3 \\ 7+2 & -5-1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$$

c) $B + C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 4-1 & -3+3 \\ 7+5 & -5+4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix}$$

$$A + (B + C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5+3 & 3+0 \\ 2+12 & -1-1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$$

d) $(A + B) + C = \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$$= \begin{pmatrix} -1-1 & 0+3 \\ 9+5 & -6+4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$$

e) $A + B = B + A$ therefore, matrix addition is commutative

f) $A + (B + C) = (A + B) + C$ therefore, matrix addition is associative

Example 2

A shopkeeper has two separate shops which she opens on Mondays and Fridays. In a particular week, she made the following sales in the two shops. Represent the total sales she made in the week in matrix form.

Shop A

	Coke	Sprite
Monday	9	8
Friday	6	7

Shop B

	Coke	Sprite
Monday	5	4
Friday	2	3

Solution

The sales in each shop can be put in (2×2) matrix where the rows are indexed as days of the week and the columns are indexed as types of drinks.

$A = \begin{pmatrix} 9 & 8 \\ 6 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$

The total sales of the two shops is given by the sum of the matrices

$\mathbf{T} = \mathbf{A} + \mathbf{B}$

$$\mathbf{T} = \begin{pmatrix} 9 & 8 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 12 \\ 8 & 10 \end{pmatrix}$$

Example 3

The matrices Q and R are given by $Q = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$.

Find $Q + R$

Solution

$$Q + R = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix} = \begin{pmatrix} 2+3 & 6+(-5) \\ 4+(-7) & 8+9 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -3 & 17 \end{pmatrix}$$

Take care with the signs when adding the numbers.

Example 4

The matrices **A** and **B** are given by; $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -8 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 5 \\ 0 & -4 \end{pmatrix}$.

Find $\mathbf{A} + \mathbf{B}$

Solution

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & -1 \\ -8 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 3+(-2) & -1+5 \\ -8+0 & 6+(-4) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -8 & 2 \end{pmatrix}$$

Subtraction of Matrices

Two matrices can be subtracted only if they have the same size. To subtract two matrices, subtract the elements in the corresponding positions in each matrix. For example, given:

$$A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a - e & b - f \\ c - g & d - h \end{bmatrix}$ resulting in another (2×2) *matrix*

Example 1

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$,

Evaluate the following:

a) $A - B$

b) $B - A$

c) $B - C$

d) $A - (B - C)$

e) $(A - B) - C$

f) What is the relationship between your answers in a) and b)

g) What is the relationship between your answers in c) and d)

Solution

a) $A - B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$

$$= \begin{pmatrix} -5 - 4 & 3 - (-3) \\ 2 - 7 & -1 - (-5) \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix}$$

b) $B - A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-(-5) & -3-3 \\ 7-2 & -5-(-1) \end{pmatrix}$

$= \begin{pmatrix} 9 & -6 \\ 5 & -4 \end{pmatrix}$

c) $B - C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 4-(-1) & -3-3 \\ 7-5 & -5-4 \end{pmatrix}$

$= \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix}$

d) $A - (B - C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix}$

$= \begin{pmatrix} -5-5 & 3-(-6) \\ 2-2 & -1-(-9) \end{pmatrix}$

$= \begin{pmatrix} -10 & 9 \\ 0 & 8 \end{pmatrix}$

e) $(A - B) - C = \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} -9-(-1) & 6-3 \\ -5-5 & 4-4 \end{pmatrix}$

$= \begin{pmatrix} -8 & 3 \\ -10 & 0 \end{pmatrix}$

f) $A - B \neq B - A$ therefore, matrix subtraction is not commutative

g) $A - (B - C) \neq (A - B) - C$ therefore, matrix subtraction is not associative

Example 2

A car dealer sells two types of cars Toyota (T) and Opel (O), and two models for each brand Prius (P) and Corsa (C) the inventory of the cars is shown below.

	P	C
T	4	5
O	2	6

If the dealer makes sales the following month as given by the table below, find the new inventory.

	H	C
T	3	3
O	2	1

Solution

The inventory can be put in a (2× 2) matrix where the row of indexed as brands and the columns are indexed as models.

$\mathbf{R} = \begin{pmatrix} 4 & 5 \\ 2 & 6 \end{pmatrix}$, the sales can also be put in a (2×2) matrix as $\mathbf{S} = \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix}$

The new inventory will be $\mathbf{R}-\mathbf{S} = \begin{pmatrix} 4 & 5 \\ 2 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4-3 & 5-3 \\ 2-2 & 6-1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$

Example 3

The matrices **A, B** and **C** are given by:

$\mathbf{A} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}$.

Evaluate

(i) $(A - B)$

(ii) $(B - C)$

Solution

(i) $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2-5 & 4-3 \\ 3-0 & -1-1 \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ 3 & -2 \end{pmatrix}$

(ii) $\mathbf{B} - \mathbf{C} = \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5+1 & 3+2 \\ 0+3 & 1-4 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 3 & -3 \end{pmatrix}$

Multiplication of Matrices

Scalar multiplication of matrices

All the entries of a matrix can be multiplied by a common factor called a ***scale factor***, ***k***, in a process called scalar multiplication to obtain a scalar product. That is, if $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{kA} = \mathrm{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$.

If $\boldsymbol{k} = -1$, then $-\mathbf{A} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is called the negative of **A**

Example 1

Given that $A = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and the matrix $P = -2A$, write out the matrix P

Solution

$$P = -2A = -2\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ -4 \end{pmatrix}$$

Example 2

Given that $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 18 & 9 \end{pmatrix}$, *then* $\mathrm{A} = 3\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ where k is a scale factor (k = 3).

Example 3

If $\mathrm{P} = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$ and $\mathrm{Q} = \begin{pmatrix} 12 & 13 \\ -3 & -9 \end{pmatrix}$, find:

i) $-5P$

ii) $\frac{1}{3}Q$

Solution

i) $\mathrm{P} = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix} \therefore -5p = -5\begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -5\times 4 & -5\times -1 \\ -5\times 2 & -5\times 3 \end{pmatrix} = \begin{pmatrix} -20 & 5 \\ -10 & -15 \end{pmatrix}$

ii) $\mathrm{Q} = \begin{pmatrix} 12 & 13 \\ -3 & -9 \end{pmatrix} \therefore \frac{1}{3}\mathrm{Q} = \frac{1}{3}\begin{pmatrix} 12 & 13 \\ -3 & -9 \end{pmatrix} = \begin{pmatrix} \frac{12}{3} & \frac{13}{3} \\ \frac{-3}{3} & \frac{-9}{3} \end{pmatrix} = \begin{pmatrix} 4 & 4.3 \\ -1 & -3 \end{pmatrix}$

Example 4

If $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$, find $2\mathbf{A} - 3\mathbf{B} + 4\mathbf{C}$.

Solution

$$2A = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix},$$

$$3B = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix}$$

$$4C = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$\therefore \mathbf{2A - 3B + 4C} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix} = \begin{pmatrix} -8 & 3 \\ 27 & -36 \end{pmatrix}$$

Multiplication of matrices

Matrix multiplication, unlike multiplying individual numbers, involves a specific process for combining elements from two matrices to create a new matrix. The two matrices must have compatible dimensions for multiplication. The number of columns in the first matrix $A_{m\times n}$ must equal the number of rows in the second matrix $B_{n\times q}$. The resulting product matrix will have dimensions $m \times q$. We don't directly multiply corresponding elements between the matrices. Instead, to find an element at any row (i) and column (j) of the resulting product matrix, we take the dot product of the row i from the first matrix (A) with column j from the second matrix (B). The dot product involves multiplying corresponding elements between the row and column vectors and summing those products. We repeat this process for each element in the resulting product matrix, considering all possible row-column combinations.

For example, given a matrix, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ Multiplying } N \text{ by the 1}st \text{ row of } M$$

Which gives the entries for first-row elements of the product as $(ae + bg \quad af + bh)$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ Multiplying } N \text{ by the 2}nd \text{ row of } M$$

Giving the entries for first-row elements of the product MN as $\begin{pmatrix} ce+dg & cf+dh \end{pmatrix}$

The result will be $MN = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

In summary,

$\begin{bmatrix} \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ \begin{pmatrix} c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & g \end{pmatrix} \end{bmatrix}$ multiply the first row by the second matrix and the second row by the second matrix

$$\begin{pmatrix} \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} e \\ g \end{pmatrix} & \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} f \\ h \end{pmatrix} \\ \begin{pmatrix} c & d \end{pmatrix}\begin{pmatrix} e \\ g \end{pmatrix} & \begin{pmatrix} c & d \end{pmatrix}\begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}$$

Multiply:

- First row in first matrix by first column in second matrix,
- Second row in first matrix by the first column in the second matrix
- First row in first matrix by first second column in second matrix
- Second row in first matrix by the second column in the second matrix

You will obtain

$$\begin{pmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{pmatrix}$$

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

NOTE: Matrix multiplication is not commutative ($AB \neq BA$) in general.

The order of the matrices matters when multiplying them.

Example 1

Given that $A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}$

Evaluate the following:

a) AB

b) BA

c) what is the relationship between your answers in a) and b)

Solution

a) $AB = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix} = \begin{pmatrix} 4(-2)+(-2)(-2) & 4(3)+(-2)(-7) \\ 1(-2)+3(-2) & 1(3)+3(-7) \end{pmatrix}$

$= \begin{pmatrix} -4 & 26 \\ -8 & -18 \end{pmatrix}$

b) $BA = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -2(4)+3(1) & (-2)(-2)+3(3) \\ (-2)(4)+(-7)(1) & (-2)(-2)+(-7)(3) \end{pmatrix}$

$= \begin{pmatrix} -5 & 13 \\ -15 & -17 \end{pmatrix}$

c) $AB \neq BA$ therefore, matrix multiplication is not commutative

Example 2

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$, *find p and q* if $AB = \begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + 3\begin{pmatrix} 4 & 2 \\ -3 & q \end{pmatrix}$

Solution

$\text{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix}$

$\text{AB} = \begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + 3\begin{pmatrix} 4 & 2 \\ -3 & q \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix}$

$\begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + \begin{pmatrix} 12 & 6 \\ -9 & 3q \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix}$

$\begin{pmatrix} p+12 & 4 \\ -15 & 4+3q \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix}$

Equating corresponding entries, it follows that

$-7 = p + 12 \therefore p = -19$

$10 = 4 + 3q \Longrightarrow 3q = 6 \therefore q = 2$

Example 4

Find AV where $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $V = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Solution

$AV = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\times 5 + 3\times 4 \\ 1\times 5 + 2\times 4 \end{pmatrix} = \begin{pmatrix} 10+12 \\ 5+8 \end{pmatrix} = \begin{pmatrix} 22 \\ 13 \end{pmatrix}$

Example 5

If $P = (2 \quad 3)$ and $A = \begin{pmatrix} 4 & 6 \\ 5 & 2 \end{pmatrix}$, find PA

Solution

$PA = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 5 & 2 \end{pmatrix} = (2 \times 4 + 3 \times 5 \quad 2 \times 6 + 3 \times 2) = \begin{pmatrix} 23 & 18 \end{pmatrix}$

Example 6

Given the matrices $A = \begin{pmatrix} 5 & 9 \\ -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$, evaluate

(i) AB

(ii) BA

(iii) A(AB)

Solution

(i) $AB = \begin{pmatrix} 5 & 9 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 5(2) + 9(4) & 5(3) + 9(1) \\ -2(2) + 4(4) & -2(3) + 4(1) \end{pmatrix}$$

$$= \begin{pmatrix} 10 + 36 & 15 + 9 \\ -4 + 16 & -6 + 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 46 & 24 \\ 12 & -2 \end{pmatrix}$$

(ii) $BA = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5 & 9 \\ -2 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 2(5) + 3(-2) & 2(9) + 3(4) \\ 4(5) + 1(-2) & 4(9) + 1(4) \end{pmatrix}$$

$$= \begin{pmatrix} 10 - 6 & 18 + 12 \\ 20 - 2 & 36 + 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} -4 & 30 \\ 18 & 40 \end{pmatrix}$$

(iii) A(BA)

We obtained $BA = \begin{pmatrix} -4 & 30 \\ 18 & 40 \end{pmatrix}$

$A(BA) = \begin{pmatrix} 5 & 9 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -4 & 30 \\ 18 & 40 \end{pmatrix}$

$= \begin{pmatrix} 5(-4) + 9(18) & 5(30) + 9(40) \\ -2(-4) + 4(18) & -2(30) + 4(40) \end{pmatrix}$

$= \begin{pmatrix} -20 + 162 & 150 + 360 \\ 8 + 72 & -60 + 160 \end{pmatrix}$

$A(BA) = \begin{pmatrix} 142 & 510 \\ 80 & 100 \end{pmatrix}$

Activity 4.1

Use the appropriate steps to solve the following question in pairs or individually and cross-check your answer with the ones provided.

Given that $M = \begin{pmatrix} y+1 & 7 \\ \frac{1}{3} + x & q \end{pmatrix}$ and $N = \begin{pmatrix} -2 & m + 11 \\ -2 & -8 - q \end{pmatrix}$ and MN are equal vectors.

(i) Find the values of x, y, m and q

(ii) Hence find the determinant of M^2

Expected Answers

(i) $y = -3$, $m = -4$, $x = -\frac{7}{3}$, $q = -4$

(ii) the determinant of $M^2 = 484$

REVIEW QUESTIONS

Review Questions 4.1

1. The matrices A = (3 0 0 4) and B = (a b 0 c) are such that AB = A + B. Find the values of a, b *and* c

2. Given that the following matrices are equal, find the values of x, y *and* z :
($x + 3$ -1 4 5) = (6 y $z - 3$ 5)

3. If ($a - b$ 3 2 $a + b$) = (2 3 2 6) , find the values of a *and* b

4. Find a (2 × 2) matrix A and B such that $A + 2B$ = (1 −2 0 1) *and*

   $$2A + 3B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

5. If A = (1 2 3 4) *and* B = (−1 2 −3 1) , find p and q,
if AB = (p −2 −6 4) + 3(4 2 −3 q).

6. If T = (4 −2 3x 5), find the value of x, if the determinant of $T = 32$

7. A = (1 2 3 4) and B = (5 1 2 0)
   - (i) Find the sum $(A + B)$.
   - (ii) Find the difference $(A - B)$.

8. Given that A = [2m + 1 −1 mn + 4 8 −m + 7 −4] and

   $$B = \begin{bmatrix} -5 & 5 - 3n & -2 \\ n - 2m & 10 & -n + \frac{2}{3}m \end{bmatrix}$$ and $A = B$, find the values of m and n

9. Evaluate the determinant (−6 2 5p + 5 −4), *if* $p = 2$

10. Evaluate the determinant (−4 −3 −4 2n − 4), *if* $n = -4$

11. For the matrices A = (2 3 1 4), B = (3 0 2 −1) and C = (1 1 −2 5), verify that:
    - (i) $(AB)C = A(BC)$
    - (ii) $A(B + C) = AB + AC$
    - (iii) $(A + B)C = AC + BC$

12. Find the (2 × 2) matrices **A** and **B** such that $2A + B$ = (3 1 −8 6),

    $$3A + 2B = \begin{pmatrix} 5 & 0 \\ -13 & 10 \end{pmatrix}.$$

13. Consider two matrices:

$C = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ Multiply matrix C by Matrix D

EXTENDED READING

Cambridge Additional Mathematics by Michael Haese, Sandra Haese, Mark Humphries, Chris Sangwin. Haese Mathematics (2014). Page(s) 305 – 332.

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