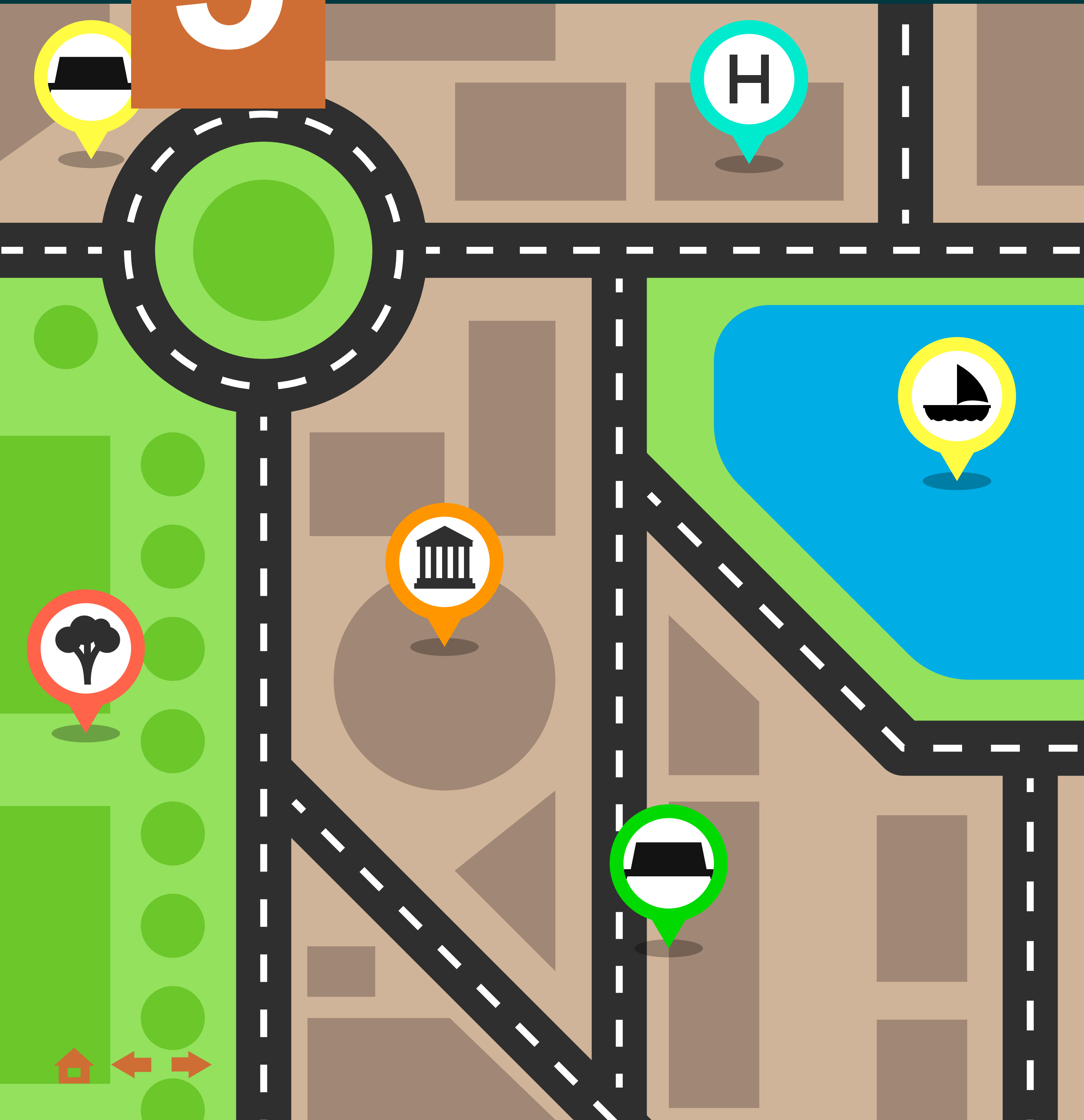
Additional Mathematics **Year 1**

SECTION

5 STRAIGHT LINES



GEOMETRIC REASONING AND MEASUREMENT

Spatial Reasoning

INTRODUCTION

Learning about straight lines is important because it helps us understand direction and route-planning to ensure efficient travel and transportation. We can also apply the concept of straight lines in architecture and different kinds of art forms. This section introduces you to the concept of straight lines, how to determine whether they are parallel or perpendicular and the properties associated with such lines. You also learn how to determine the midpoint of a straight line, divide such lines in a given ratio either internally or by extending the line segment and find the equation of straight lines. You will explore the ways of determining the sizes of acute angles formed from the intersection of two straight lines.

At the end of this section, you will be able to:

- Describe the properties of lines including parallel, perpendicular and midpoints.
- Work out the midpoint of a line segment given two points and find the generalization of the midpoint of a line segment.
- Apply the knowledge of ratio to divide a line segment in a given ratio either internally or externally.
- Recall the formula for finding the gradient of a line and apply it to find the equation of a straight line in various forms
- Use standard algebraic manipulations to find the equation of parallel and perpendicular lines including the equation of perpendicular bisector of a line
- Deduce the shortest distance between a point and a line and use the knowledge of intercepts and right-angled triangles to find the perpendicular distance from an external point to a line.

- Determine the acute angles between two intersecting lines with the aid of technological tools e.g. GeoGebra.

Key Ideas:

- When a line is straight there is no curve on the line.
- *Parallel lines* are straight lines that do not meet.
- *Perpendicular lines* are straight lines that meet at an angle of 90^0.
- A *midpoint* is a point that divides a straight line into two equal parts.
- A straight line is the shortest distance between two points.
- A straight line can be divided internally using a given ratio.
- *Acute angles* are less than 90^0, right angles are exactly 90^0 and obtuse angles are greater than 90^0 but less than 180^0.
- Equations of lines can be written in the *slope-intercept* form and *point-slope* form.

PROPERTIES OF LINES

Like every artifact, some features are specific to it. Dealing with straight lines, certain characteristics determine whether a line is straight or not and these traits are referred to as properties of straight lines. The straight lines can be horizontal, vertical, slanted, parallel and perpendicular.

Activity 5.1

Figure 1: Vertical Line

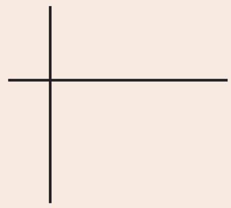


Figure 2: Perpendicular Lines

Figure 3: Parallel lines

Figure 4: Horizontal line

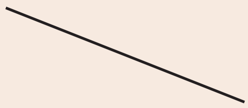


Figure 5: Slanted line

i. Choose a destination (Dining hall, School field, Assembly Hall, etc.).

ii. Walk along a straight path to your chosen destination.

iii. Record the time and distance from your starting point to your destination.

iv. Now, walk along a path with bends and curves to your chosen destination.

v. Record the time and distance from your starting point to your destination.

vi. Compare the distances covered in both scenarios and time taken and share your findings with a colleague.

Some of the properties include:

i. A straight line is formed when two points are joined with the shortest distance, it can be extended forever in both directions.

ii. A straight line has no curves within.

iii. A straight line is one-dimensional and has no width.

Parallel and Perpendicular Lines

Parallel lines are two or more straight lines that are always the same distance (equidistant) apart. They never intersect. Examples include opposite ends of a goal post, railway tracks, edges of a ruler, zebra crossing (parallel white lines).

Perpendicular lines are two lines that intersect, but all the angles at that intersection are the same, that is 90°. For example, “T” junctions on roads, corners of a football pitch etc.

Note:

1. Distances are always positive.
2. Distance can only be zero if the points coincide.
3. The distance from P to Q is the same as the distance from Q to P

Activity 5.2

i. Pick a coordinate grid paper/graph sheet

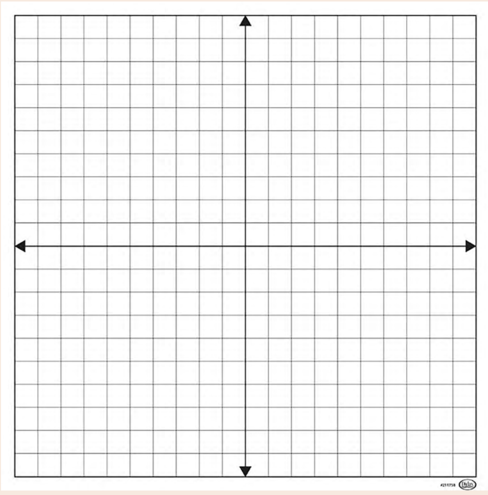


Figure 6: Coordinate grid paper/graph sheet

ii. Plot points B(3, 2) and E(4, 6) on the grid and draw a straight line from one point to the other.

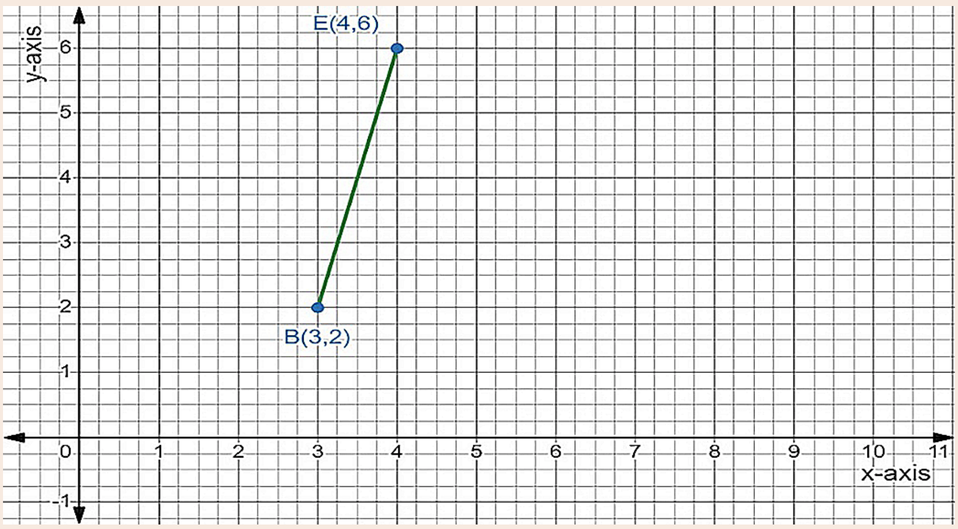


Figure 7: Straight Line

iii. Is line BE slanted, horizonal or vertical?

iv. Measure the length of the line starting from point B and ending at point E and record.

v. Repeat steps 2 and 3 using different self - selected points

vi. Now, use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to calculate the distance of line BE (|BE|) and the other lines created from exploring with other points.

vii. Record your observations and discuss with a classmate.

Generalisation: Distance of a line $\left(d\right) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where x_1, x_2 *and* y_1, y_2 are x and y coordinates of the first and second points respectively.

Ready to consolidate your knowledge? Let's try out some examples!

Example 1

Given that U (-7, 23) and V (6, 18) are the endpoints of a line UV, calculate |UV|.

Solution

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1 in the distance formula

$$\left|UV\right| = \sqrt{(6-(-7))^2 + (18-23)^2}$$

Step 2: Simplify the terms under the square root

$$\left|UV\right| = \sqrt{169 + 25}$$

Step 3: Add the terms under the square root

$$|UV| = \sqrt{194}$$

Step 4: Write out final answer in decimal unless instructed otherwise

$|UV| = 13.93$ (to two decimal places)

Example 2

Calculate the distance between the points N (12, 5) and R (-3, -6).

Solution

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1 in the distance formula

$$\left|NR\right| = \sqrt{(-3-12)^2 + (-6-5)^2}$$

Step 2: Simplify the terms under the square root

$|NR| = \sqrt{225 + 121}$

Step 3: Add the terms under the square root

$|NR| = \sqrt{346}$

Step 4: Write out final answer in decimal unless instructed otherwise

$|NR| = 18.60$ (to two decimal places)

Midpoint of a Line Segment

A midpoint is a point that divides a line segment into two equal parts. The midpoint is equally distant from both ends of the line segment.

Activity 5.3

i. Choose a coordinate grid paper

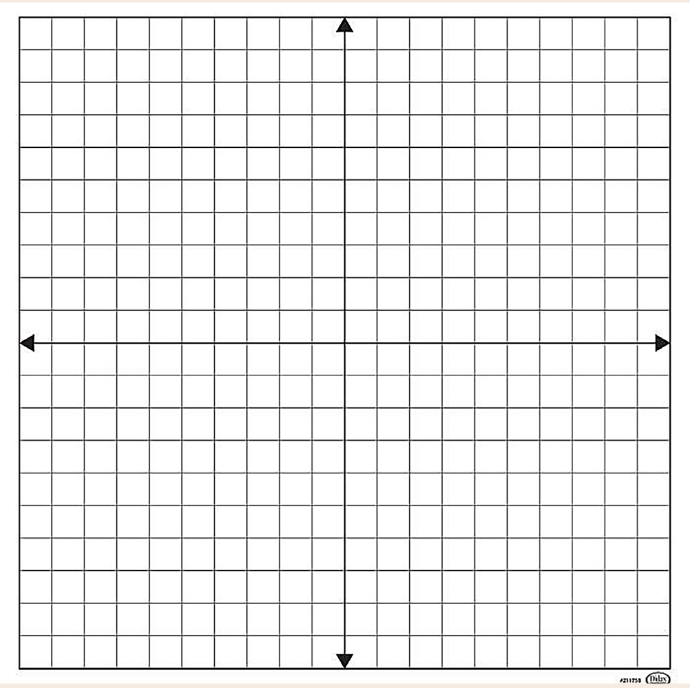


Figure 8: Coordinate grid paper/graph sheet

ii. Plot points A(1, 2) and Y(5, 6) on the grid and draw a straight line from one point to the other.

iii. Fold the grid paper such that the line is divided into two equal parts.

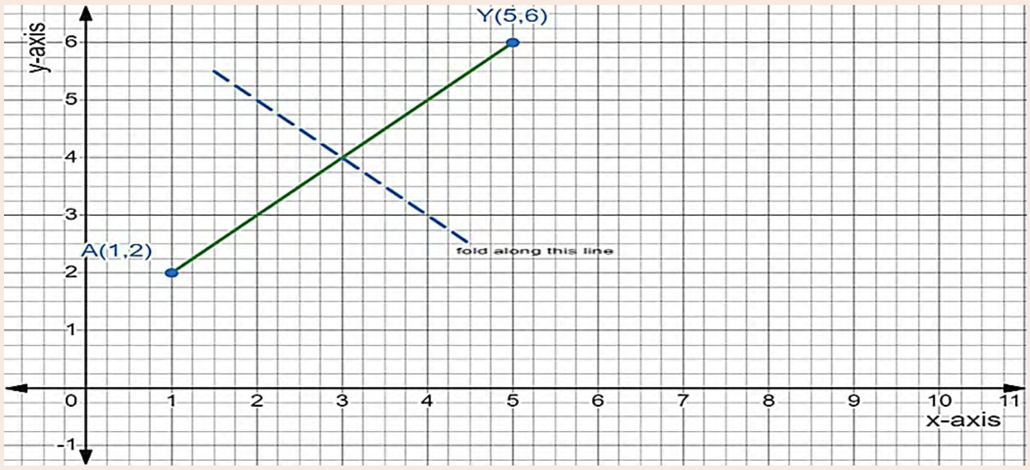


Figure 8: Straight line divided into two equal parts

iv. Indicate the coordinates of the point where the straight line gets divided into two equal parts.

v. Repeat the steps ii to iv using different sets of points to draw different straight lines

vi. Record your observations on the values that make up the coordinates of the points that divide the lines into two equal parts in relation to the end points.

Generalisation: Midpoint (M) = $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$ where x_1, y_1 *and* x_2, y_2 are x and y coordinates of the first and second points respectively.

Are you Ready to consolidate your knowledge? Let's try out some examples!

Example 3

Find the midpoint of the line KG with coordinates K (16, 24) and G (-8, 10).

Solution

Step 1: Let M be the midpoint of line KG

Substitute the corresponding values for x_2, x_1, y_2, y_1 in the midpoint formula

$$M\left(\frac{1}{2}(16 + (-8)), \frac{1}{2}(24 + 10)\right)$$

Step 2: Simplify the terms in the brackets multiplied by $\frac{1}{2}$

$$M\left(\frac{1}{2}(8), \frac{1}{2}(34)\right)$$

Step 3: Divide where necessary

$M(4\ ,\ 17)$

Example 4

Given the coordinates of line RP as R (1, 17) and P (0, 4), determine the midpoint of line RP.

Solution

Step 1: Let M be the midpoint of line RP

Substitute the corresponding values for x_2, x_1, y_2, y_1 in the midpoint formula

$\text{M}\left(\frac{1}{2}(1+0)\ ,\ \frac{1}{2}(17+4)\right)$

Step 2: Simplify the terms in the brackets multiplied by $\frac{1}{2}$

$M\left(\frac{1}{2}(1)\ ,\ \frac{1}{2}(21)\right)$

Step 3: Divide where necessary

$M(0.5\ ,\ 10.5)$

Example 5

W is the midpoint of DV. If the coordinates of D are (−5, 4) and W is (−2, 1), find the co-ordinates of V.

Solution

Step 1: Let point V be (x_v, y_v)

Substitute the corresponding values for x_2, x_1, y_2, y_1 in the midpoint formula

$\text{W} = \left(\frac{1}{2}(-5+x_v)\ ,\ \frac{1}{2}(4+y_v)\right)$

Step 2: Remember W is (−2, 1), now substitute W to get

$(-2, 1) = \left(\frac{1}{2}(-5+x_v)\ ,\ \frac{1}{2}(4+y_v)\right)$

Step 3: Treat the inputs in step 2 as points, pair corresponding x and y coordinates.

For x coordinate;

$-2 = \frac{1}{2}(-5 + x_v)$

For y coordinate;

$1 = \frac{1}{2}(4 + y_v)$

Step 4: Simplify for each of the coordinates and make x_v and y_v the subjects to get;

$x_v = 1,\ y_v = -2$

Step 5: Write out your conclusion

Therefore, the coordinates of V are (1, −2).

DIVISION OF A LINE

Lines can be divided by cutting them into specific parts either equally or unequally. This division can be done with the help of ratios. Take a broomstick for example, you can divide this stick into parts internally (within) or externally (outwards).

Let's talk about internal division of straight lines!

To divide a line segment into two parts internally, place a point somewhere between the two endpoints. This new point creates a ratio between the two resulting segments. For example, dividing the line in half creates a 1:1 ratio which is the same as finding the midpoint discussed earlier. Suppose you are asked to divide a line segment KE in the ratio 2:3, it means that you find a point, say C such that KC:CE = 2:3 which is the same as $\frac{KC}{CE} = \frac{2}{3}$.

Now, let's move on to external divisions of straight lines!

To divide a line segment into two parts externally, place a point somewhere outside the original line segment. This point also creates a ratio but it describes the lengths relative to the original segment **not** parts within it.

Come along as we do this activity for a better understanding

Activity 5.3

Internal Division

i. Draw a line segment KE on a coordinate grid paper.

ii. Choose a ratio m:n (e.g. 2:1).

iii. Divide the line segment into $m + n$ equal parts.

iv. Label the point dividing the line segment according to the ratio as N

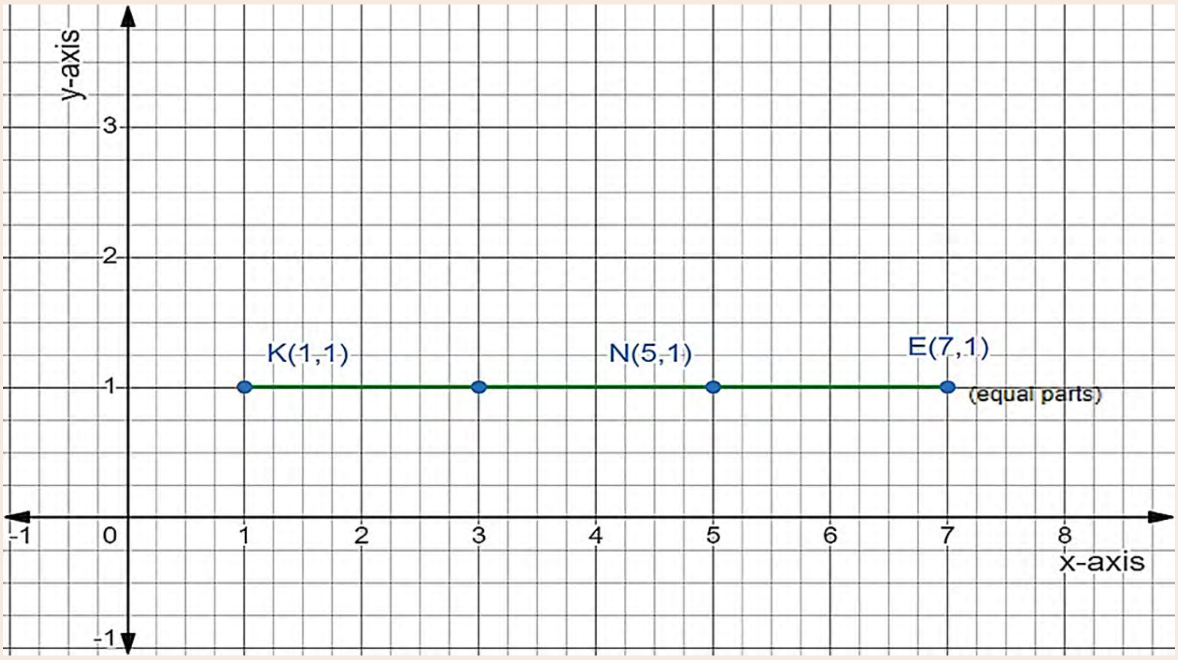


Figure 9: Line Segments

v. Measure the length of KN and NE on the coordinate grid paper.

vi. Verify the length by calculating the length of KN and NE using the ratio

External Division

i. Draw a line segment MQ on a coordinate grid paper. Q is an extension of the line MN, N will lie somewhere along the line MQ.

ii. Choose a ratio m:q (e.g. 5:1), where m is the ratio of the length of MQ and q is the ratio of the length NQ

iii. Divide the extended line segment into m equal parts (*where* $m>q$).

iv. Label the point dividing the line segment as N

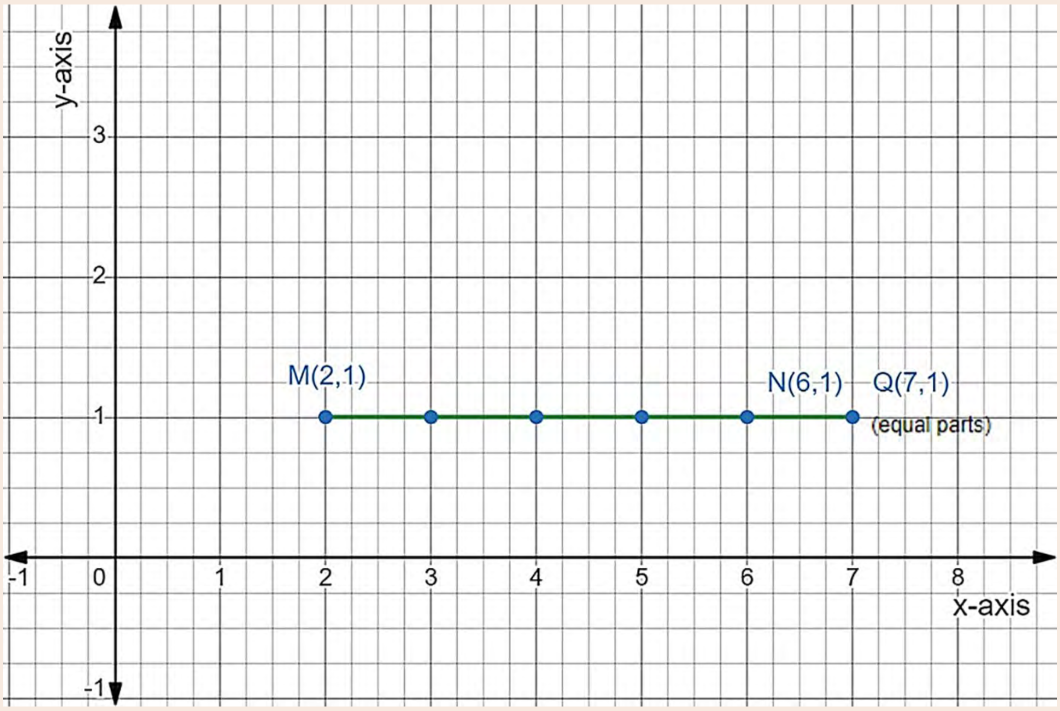


Figure 10: Line Segment

v. Measure the length of MQ and QN on the coordinate grid paper.

vi. Verify the length by calculating MQ and QN using the ratio.

Let's tap into your creativity, critical thinking and collaborative skills!

1. Pick out your own sets of points for different line segments and choose different ratios.
2. Now, apply the steps in the activities on internal and external division of lines and document the different results you get.
3. How about using the generalizations below on the points and ratios you picked?
4. Did you get the same results as that generated from the activity?
5. Discuss your observations with a friend or classmate.

Generalisations:

1. The coordinates P, which divides the line segment K (x_1, y_1) and E (x_2, y_2) internally in the ratio $m{:}n$ is given by:

$$P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right).$$

2. The coordinates Q, which divides the line segment A(x_1, y_1) and B(x_2, y_2) externally in the ratio $m{:}n$ is given by:
   $Q = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right).$

Note: You do not always negate '*n*' but you negate the smaller number in the ratio for external division of lines.

Understanding both internal and external division of lines is important in various applications like scaling distances on maps or solving geometric problems.

Are you up for a challenge? Let's try out some examples!

Example 6

G and K divide the line FH, F(13, 9) and H(6, -17) internally and externally respectively in the ratio 11:3. Find the coordinates of G and K.

Solution

Focusing the internal division, Let m = 11 and n = 3.

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1, m and n in the internal division formula.

$$G\left(\frac{11(6)+3(13)}{11+3}, \frac{11(-17)+3(9)}{11+3}\right),$$

Step 2: Simplify the terms substituted

G(7.5, −11.43).

Focusing the external division, Let m = 11 and n = 3.

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1, m and n in the external division formula.

$$K\left(\frac{11(6)-3(13)}{11-3}, \frac{11(-17)-3(9)}{11-3}\right),$$

Step 2: Simplify the terms substituted

K(3.375, −26.75).

Example 7

Find the coordinates of the point that divides the lines segment (−4, 3) and (6, -12) in the ratio 3: 2, internally and externally.

Solution

Focusing the internal division, Let m = 3 and n = 2.

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1, m and n in the internal division formula.

$$\left(\frac{3(6)+2(-4)}{3+2}, \frac{3(-12)+2(3)}{3+2}\right),$$

Step 2: Simplify the terms substituted

(2, −6).

Focusing the external division, Let m = 3 and n = 2.

Step 1: Substitute the corresponding values for x_2, x_1, y_2, y_1, m and n in the external division formula.

$$\left(\frac{3(6)-2(-4)}{3-2}, \frac{3(-12)-2(3)}{3-2}\right),$$

Step 2: Simplify the terms substituted

(26, −42).

EQUATION OF STRAIGHT LINES

If $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ then given any arbitrary point $P(x, y)$ on the line AB we can generate the equation of a line.

Let's derive the formula for the equation of a line using standard algebraic manipulations.

Recall that the formula for finding the gradient of a line is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}.$$

Well, if there is another point $P(x, y)$ on the line segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$, then the gradient of line AP becomes $m = \frac{y - y_1}{x - x_1}$ which is equal to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Activity 5.4

i. Equate the gradient for lines AP and AB

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

ii. Multiply both sides by $(x - x_1)$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \text{ ....}(\textit{equation 1})$$

iii. Remember $m = \frac{y_2 - y_1}{x_2 - x_1}$, now substitute $m = \frac{y_2 - y_1}{x_2 - x_1}$ into *equation 1*

$$\boldsymbol{y - y_1 = m\,(x - x_1)}$$

iv. Expand terms on the RHS of the equation

$$y - y_1 = mx - mx_1$$

v. Add y_1 to both sides:

$$y = mx - mx_1 + y_1$$

vi. Represent $-mx_1 + y_1$ by a variable (c)

Now, $\boldsymbol{y = mx + c}$

Congratulations!! You have used algebra to generate the equation of a straight line in two forms. First the point-slope form ($\boldsymbol{y - y_1 = m\,(x - x_1)}$) *and the slope-intercept form* ($\boldsymbol{y = mx + c}$).

Note, horizontal lines have zero gradient, so these are written in the form $y = c$. Vertical lines have infinite gradient, so these are written in the form $x = c$.

EQUATION OF PARALLEL AND PERPENDICULAR LINES

Two lines are perpendicular when they meet (intersect) at a right angle (90^0) and two lines are said to be parallel when they never intersect no matter how far you extend the lines.

Whether perpendicular or parallel, they are lines and thus their equations can be determined.

Let us explore the relationship between parallel and perpendicular lines!

Activity 5.5

i. Pick a coordinate grid paper

ii. Choose two points and draw a line segment

iii. Choose another set of points and draw a different line segment (**Note:** this line should not intersect the first no matter how far it is extended).

iv. Find the gradients of the two lines

v. Record your observation

vi. Pick a different set of points and draw a line (Select your points in such a way that the line intersects the first line in step iii at 90^0).

vii. Find the gradient of the third line.

viii. What conclusion can you draw on the gradients of the first and the third lines; second and third lines?

Generalisations:

1. Parallel lines have the same gradient

2. When the slope of one line is m, then the slope of the line perpendicular to it will be $\frac{-1}{m}$.

3. If two lines are perpendicular then the product of their gradients, $m \times \frac{-1}{m}$ is -1

Let's try out some examples!

Example 8

a) Find the equation of the line that passes through the points S (4, −2) and T (8, 4), leaving your final answer in the form $ax + by + c = 0$ where a, b and c are integer values.

b) If line JK is perpendicular to line ST and passes through the point W (-7, 5) find the equation of line that passes through JK.

c) Suppose line ED is parallel to line JK, and passes through L (2, 9), find the equation of line that passes through ED.

Solution

a)

Step 1: Find the gradient of line ST.

$m_1 = \frac{4 - (-2)}{8 - 4}$

$m_1 = \frac{3}{2}$

Step 2: Use the point-slope form for equation of line and substitute values accordingly.

$y - y_1 = m_1 (x - x_1)$

$y - (-2) = \frac{3}{2}(x - 4)$

Step 3: Expand the brackets and multiply both sides of the equation by 2.

$2y + 4 = 3x - 12$

Step 4: Group all terms on one side of the equation and simplify.

$2y + 4 - 3x + 12 = 0$

$-3x + 2y + 16 = 0$

The equation of the line that passes through ST is $-3x + 2y + 16 = 0$ or $3x - 2y - 16 = 0$.

b)

Step 1: Determine gradient of JK based on relations between perpendicular lines.

$m_2 = \frac{-1}{m_1}$

$m_2 = \frac{-2}{3}$

Step 2: Use the point-slope form for equation of line and substitute values accordingly.

$y - y_1 = m_2 (x - x_1)$

$y - 5 = \frac{-2}{3}(x - (-7))$

Step 3: Expand the bracket and multiply both sides of the equation by 3.

$3y - 15 = -2x - 14$

Step 4: Group all terms on one side of the equation and simplify.

$3y - 15 + 2x + 14 = 0$

$2x + 3y - 1 = 0$

The equation of the line that passes through JK is $2x + 3y - 1 = 0$.

c)

Step 1: Determine gradient of ED based on relations between parallel lines.

$m_3 = m_2$

$m_3 = \frac{-2}{3}$

Step 2: Use the point-slope form for equation of line and substitute values accordingly.

$y - y_1 = m_3\,(x - x_1)$

$y - 9 = \frac{-2}{3}\,(x - 2)$

Step 3: Expand the bracket and multiply both sides of the equation by 3.

$3y - 27 = -2x + 4$

Step 4: Group all terms on one side of the equation and simplify.

$3y - 27 + 2x - 4 = 0$

$2x + 3y - 31 = 0$

The equation of the line that passes through ED is $2x + 3y - 31 = 0$.

DISTANCE BETWEEN A POINT AND A LINE

Investigate the shortest distance between a point and a line, as well as the shortest distance between two lines. Establish that the shortest distance between a line and a point is the perpendicular distance, D.

Activity 5.6

Perpendicular Distance

i. Graph a line on the coordinate grid, $y = mx + \mathrm{c}$.

ii. Plot a point **not** on the line: (x_1, y_1).

iii. Draw a perpendicular line from the point (x_1, y_1) to the original line

iv. Label the point of intersection: (x_2, y_2)

v. Calculate the distance between $(x_1, y_1)$and (x_2, y_2)) using the distance formula:

$$\mathrm{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

vi. Use the formula to find the distance from a point to a line $D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

vii. Plug in values for a, x_1, y_1, b and c.

viii. Record your observations and discuss with a classmate.

Example 9

Find the shortest distance between the perpendicular line drawn from the point J (1, 5) to the straight line $5x + 12y + 7 = 0$.

Solution

Step 1: Substitute the corresponding values for a, b, c, x_1, y_1 in the shortest distance formula

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|5(1) + 12(5) + 7|}{\sqrt{5^2 + 12^2}}$$

Step 2: Simplify the terms in absolute and those under the square root

$$D = \frac{|72|}{\sqrt{169}}$$

Step 3: Simplify the result

$D = 5.54$ units to two decimal places

Example 10

Telecel is setting up a new cell tower at a location C (6, −2) on the map of Accra. The main road, which serves as a reference line, is represented by the equation $3x - 4y = 12$. Determine the perpendicular distance from point C (6, −2) to the main road.

Solution

Step 1: Substitute the corresponding values for a, b, c, x_1, y_1 in the shortest distance formula

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|3(6) - 4(-2) - 12|}{\sqrt{3^2 + 4^2}}$$

Step 2: Simplify the terms in absolute and those under the square root

$$D = \frac{|14|}{\sqrt{25}}$$

Step 3: Simplify the result

$$D = 2.8$$

ACUTE ANGLES BETWEEN TWO INTERSECTING LINES

In geometry, a triangle is a closed, three-sided polygon. Each of the three sides meet at a point called a vertex and the angles formed between these sides within the triangle are called interior angles. Every triangle has exactly three interior angles. The measures of the three interior angles of any triangle sums up to 180^0.

Triangles can be classified into categories based on the measure of their interior angles, namely:

- Acute Triangle: All three interior angles are less than 90 degrees (acute).
- Right Triangle: One interior angle is exactly 90 degrees (right angle).
- Obtuse Triangle: One interior angle is greater than 90 degrees (obtuse).

Let's focus on measuring acute angles through this activity!

Activity 5.7

i. Plot the points (1, -1) and (3, 3) and join them with a straight line on a coordinate grid paper.

ii. Plot the different sets of point (1, -1) and (-1, 5) and connect with a straight line on the same coordinate grid paper.

iii. Measure the interior angle formed by the lines at the point of intersection using a protractor.

iv. Label the angle as θ (theta).

v. Calculate the difference between the slopes (gradient) of the two lines.

vi. Multiply the gradients of the two lines.

vii. Compute $\frac{m_1 - m_2}{(1 + m_1 \times m_2)}$ where m_1, m_2 are the gradients of the two lines

viii. Calculate $tan^{-1}\left(\frac{m_1 - m_2}{(1 + m_1 \times m_2)}\right)$ on your calculator.

ix. Compare the values recorded in steps 3 and 8.

x. Write your observations

Do you have the GeoGebra Software? If yes, let's explore how to measure angles!

Activity 5.8

i. Open GeoGebra and create a new worksheet

ii. Construct two lines using the "Line" tool, $y = 2x - 3$ and $y = -3x + 2$

iii. Use the "Slider" tool to adjust the slopes (m_1 and m_2) and y-intercepts (c_1 and c_2)

iv. Now, use the "Angle" tool to measure the angle between the two lines

v. Label the angle as θ (theta)

vi. Use the "Slider" tool to adjust the slopes and y-intercepts and observe how the angle changes.

vii. Use the "Calculate" tool to calculate the difference between the slopes , m_1 and m_2.

viii. Calculate the product of the slopes, m_1 and m_2.

ix. Explore the relationship between the angle θ and the slope calculations

x. Now find the value of θ using $\tan(\theta) = \left(\frac{m_1 - m_2}{(1 + m_1 \times m_2)}\right)$, with the help of a calculator.

xi. What are your observations? Record and discuss with a classmate.

Note: For acute angles $\tan\theta = \left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

Intersection of Straight Lines

The point of intersection of straight lines refer to the exact points where straight lines meet.

When dealing with two straight lines, given by equations $y = m_1x + c_1$and y $= m_2 x + c_2$, to find their intersection point, we equate the two equations: $m_1x + c_1 = m_2 x + c_2$.

Then, solve for x to determine the x-coordinate of the intersection point. Once x is found, substitute it into either of the original equations to obtain the corresponding y-coordinate.

Example 11

Given two lines $y = 7x - 13$ and $y = -2x + 6$ find their point of intersection.

Solution

Step 1: Equate the two equations

$7x - 13 = -2x + 6$

Step 2: Make x the subject

$7x + 2x = 6 + 13$

$9x = 19$

$x = \frac{19}{9}$

Step 3: Substitute x in any of the original equations to find y

$y = 7\left(\frac{19}{9}\right) - 13$

Step 4: simplify the equation

$y = \frac{16}{9}$

The lines intersect at $\left(\frac{19}{9}, \frac{16}{9}\right)$

Example 12

A transportation planner is designing two new bus routes in Kasoa. The routes will be represented by straight lines on the map of Kasoa. The goal is to identify the most favourable point of intersection between the two routes that can serve as a major bus stop. Route A is designed to pass through the points (2, 5) and (8, 17). Route B is designed to pass through the points (3, 20) and (9, 2). Calculate the point of intersection between Route A and Route B.

Solution

Step 1: Find the equation of the line passing through Route A

$m_A = \frac{17 - 5}{8 - 2}$

$m_A = \frac{12}{6}$

$m_A = 2$

$y - 5 = 2\,(x - 2)$

$y - 5 = 2x - 4$

$y = 2x + 1$

Step 2: Find the equation of the line passing through Route B

$m_B = \frac{2-20}{9 - 3}$

$m_B = \frac{-18}{6}$

$m_B = -3$

$y - 20 = -3\,(x - 3)$

$y - 20 = -3x + 9$

$y = -3x + 29$

Step 3: Equate the two equations

$2x + 1 = -3x + 29$

Step 4: Make x the subject

$5x = 28$

$x = \frac{28}{5}$

Step 5: Substitute x in any of the original equations to find y

$y = 2\left(\frac{28}{5}\right) + 1$

Step 6: Simplify the equation

$y = \frac{61}{5}$

The lines intersect at $\left(\frac{28}{5}, \frac{61}{5}\right)$.

Example 13

Determine the acute angle between two straight lines having slopes of 4 and $\frac{2}{7}$. Give your answer to two decimal places.

Solution

Step 1: Substitute the gradients in the formula for finding acute angles

$\tan(\theta) = \left|\frac{4 - \frac{2}{7}}{1 + 4\left(\frac{2}{7}\right)}\right|,$

Step 2: Simplify terms on RHS of the equation

$tan\ (\theta) = \frac{26}{15}$

Step 3: Take tan^{-1} of the terms

$\theta = tan^{-1}\left(\frac{26}{15}\right)$

$\theta = 60.02^0$

The acute angle between the two straight lines is 60.02^0.

Example 14

Find the acute angle formed between the lines $y = 19x\ - 5$ and $y = \ -6x + 4$

Solution

Step 1: Determine the gradients of the two lines.

$m_1 = 19, m_2 = -6$

Step 2: Substitute the gradients into the acute angle formwula

$\tan(\theta) = \left|\frac{|19 - (-6)|}{(1 + 19(-6))}\right|,$

Step 3: Simplify terms on RHS of the equation

$tan\ (\theta) = \frac{25}{113}$

Step 4: Take tan^{-1} of the terms

$\theta = tan^{-1}\left(\frac{25}{113}\right)$

$\theta = 12.48^0$

The acute angle between the two straight lines is 12.48^0.

Now, pick up some challenging tasks in the Review Questions. You've got this!

REVIEW QUESTIONS

1. The coordinates of two points are F (3, 7) and Y (-2, 8).

   Determine the distance between F and Y, $|FY|$.

2. An engineer is designing a suspension bridge that spans two river banks. The coordinates of the feet of the bridge on the river banks are R (3,5) and G (11,7). Find the coordinates of the midpoint of the river where the main support pillar will be anchored.

3. Given three points A (3, 2), B (8, 11) and C (14, 5) on a coordinate plane:

   i. Determine a point P on segment AB such that the line segment AB is divided internally in the ratio 3:2.

   ii. Determine a point Q on segment BC such that the line segment BC is divided internally in the ratio 4:1.

   iii. Find the equation of the line passing through P and Q.

   iv. Find a point R on segment AC such that AR is divided internally in the ratio 5:3.

   v. Analyse the position of this line relative to the original triangle $\triangle$ABC.

      Does this line bisect any sides of the triangle?

      Provide a reasoned argument based on the coordinates and the properties of line division.

4. Given three points A (-6, 4), B (2, 0) and M (12, -5) on a coordinate plane. If M divides line AB externally in the ratio 9:n, find the value of 'n'.

5. Find the equation of the line that passes through the points D (25, 6) and C (29, -14).

6. The equation of a line passing through the points M (4, 2) and T (-8, -2) is: $3y = ax + b$, where a and b are constants. Find the values of a and b.

7. Find the equation of the line perpendicular to $5x - 3y - 18 = 0$ that passes through the points (-5, 6).

8. Determine the equation of the line parallel to $3x + 5y = 108$ and passes through the point (5, 10).

9. Find the measure of acute angles between the lines $y = 2x + 10$ and $y = -5x + 4$.

10. Find the length of the perpendicular line drawn from the point B$(-1, -7)$ to the straight line passing through the points E$(6, -4)$ and Y$(9, -5)$.

11. Determine the equation of the line passing through points (3, 6) and (1, 2) using the point-slope form.

12. The Agona West Municipal Assembly wants to ensure that specific streets are parallel or perpendicular to each other to create an organised layout in its residential area. Main Street runs through the city and follows the equation $y = 2x + 3$. Nana Botwe Street runs parallel to the Main Street and pass through the point (4, 1). Bebianiha Street will be perpendicular to the Main Street and intersect it at the point (1, 5).

i. Determine the equation of Nana Botwe Street.

ii. Determine the equation of Bebianiha Street.

iii. The Assembly also wants to build a new playground (Children's Park), at a point that intersects with Bebianiha Street and a new road called Park Road. Park Road should be parallel to the Nana Botwe Street and pass through the point (2, 3).

iv. Determine the coordinates of the playground.

13. Discuss the geometric relationship that exists between the four Roads and their equations.

i. Road A, $y = -12x + 10$

ii. Road B, $-x + 12y = 56$

iii. Road C, $12x = 76 - y$

iv. Road D, $12y = x - 52$

GLOSSARY

- Straight line - this line is the shortest distance between any two points. It extends infinitely in both directions without curving.
- Perpendicular lines – these are lines that intersect at a right angle (90^0). If two lines are perpendicular, the product of their slopes (gradients) is -1.
- Parallel lines – these are lines in the same plane that never intersect, no matter how far they are extended. These lines have the same slope (gradient) but different y-intercepts.
- Gradient of a line - The gradient (or slope) of a line measures its steepness and direction. It is calculated as the ratio of the vertical change (rise) to the horizontal change (run) between two points on the line. Mathematically, it is expressed as:

  Gradient $= \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x}$.
- Acute angle - this is an angle that measures greater than 0^0 but less than 90^0. It is smaller than a right angle.
- Obtuse angle - it is an angle that measures greater than 90^0 but less than 180^0. It is larger than a right angle but smaller than a straight angle.
- Right angle – this is an angle that measures exactly 90^0. It is the angle formed when two perpendicular lines intersect.

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