Additional Mathematics

Year 1

SECTION

7

TRIGONOMETRIC FUNCTIONS AND THEIR APPLICATIONS

GEOMETRIC REASONING AND MEASUREMENT

Measurement of Triangles

INTRODUCTION

Trigonometry is a branch of mathematics that explores the relationships between the angles and sides of triangles. We will focus on the definition of trigonometric ratios, functions and their application, special angles, quadrantal angles and radian measure. On the definition of trigonometry, we will look at the definition of sine, cosine, tangents and their reciprocals. We will use these definitions to solve for the unknown sides and angles in a given right-angled triangle. The application of trigonometry will focus on the angles of elevation and depression and solve related problems. Additionally, we will discuss special angles and how to convert between degrees and radians. The concept of trigonometry is used extensively in other fields and in real life. Examples of such applications are in physics, arts, design, architecture and construction. Understanding and mastering trigonometry will prepare you for more advanced topics in years 2 and 3 as well as advanced studies at the tertiary level.

At the end of this section, you will be able to:

- Recall basic trigonometric ratios and use the knowledge to solve problems relating to triangles
- Use special triangles and the unit circle to determine the geometrical and functional values of trigonometric ratios including special angles.
- Determine radians measure and apply the knowledge to solve practical arc length problems.
- Identify the coordinates of the quadrantal angles in a unit circle and use them to find the trigonometric values of quadrantal angles.

Key Ideas

- The basic trigonometry ratios are sine, cosine, tangent, cosecant, secant and cotangent
- Trigonometry functions relate an angle of a right-angled triangle to ratios of two side lengths.
- 30°, 45°, 60° are examples of special angles. The trigonometry ratios of these special angles can be calculated without using a calculator or a four-figure table.

TRIGONOMETRIC RATIOS

The names of the six trigonometric functions are sine, cosine, tangent, cosecant, secant and cotangent. The names of all these six functions have three-letter abbreviations as shown below:

Table 1: The six trigonometric functions

Name	Abbreviation
Sine	Sin
Cosine	Cos
Tangent	Tan
Cosecant	**Csc**
Secant	Sec
Cotangent	Cot

Except for Cosecant, all abbreviations are the first three letters of the names.

The last three are simply the reciprocals of the first three. Remember that the reciprocal of 2 is $\frac{1}{2}$ and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ etc. This is the same in that the reciprocal of sine is the cosecant etc.

Table 2: The reciprocals of the three trigonometric functions

Name	Reciprocal
Sine	Cosecant
Cosine	Secant
Tangent	Cotangent

This means that:

$$\text{Cosecant}=\frac{1}{Sine} \qquad \text{Secant}=\frac{1}{Cosine} \qquad \text{Cotangent}=\frac{1}{Tangent}$$

All possible ratios of the three sides of a right-angle triangle give the trigonometric ratios.

Let us generate these ratios using the activity below:

1. Draw a right-angle triangle of any reasonable dimensions.
2. Label one of the acute angles, θ
3. Name the sides of the triangles with reference to θ

   The longest side is the side opposite the right angle. It is the hypotenuse. The other two sides are named **opposite** or **adjacent** depending on the acute angle of reference.

   In Figure 1, the names of the sides of the triangle are in reference to the angle θ

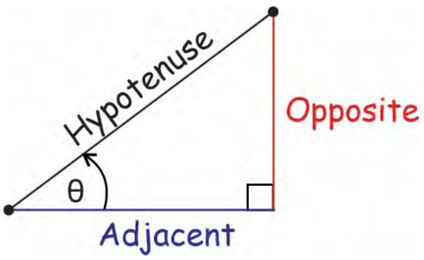


Figure 1: Right-angled triangle

4. Find below the definitions of the trigonometric ratios

   1. $\text{Sin } \theta = \frac{Opposite}{Hypotenuse}$
   2. $\text{Cos } \theta = \frac{Adjacent}{Hypotenuse}$
   3. $\text{Tan } \theta = \frac{Opposite}{Adjacent}$
   4. $\text{Csc } \theta = \frac{Hypotenuse}{Opposite} = \frac{1}{sin\theta}$
   5. $\text{Sec } \theta = \frac{Hypotenuse}{Adjacent} = \frac{1}{cos\theta}$
   6. $\text{Cot } \theta = \frac{Adjacent}{Opposite} = \frac{1}{tan\theta}$

Note the following properties

1. The trigonometry functions are properties of angles, NOT triangles. We use the triangles to help find the ratios.

2. Trigonometry functions have no units.
3. Squared trigonometry functions for example are written as $\cos^2\theta$ and are read as "Cos squared theta"

Let us solve two examples to enhance our understanding!

Example 1

Figure 2 shows a right-angle triangle with vertical, horizontal and diagonal heights measuring 5, 12 and 13 units.

Find:

(i) Sin A

(ii) Cos A

(iii) Tan A

(iv) Sec A

(v) Csc A

(vi) Cot A

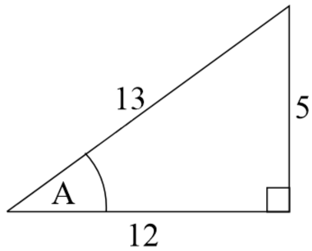


Figure 2

Solution

From the diagram, the side adjacent to angle A is 12, the side opposite to A is 5 and the hypotenuse is 13 (always the longest side which is opposite to the right angle)

(i) Recall that sin of an angle $= \dfrac{\textit{the side Opposite to the angle}}{\textit{Hypotenuse}}$

Therefore, Sin A $= \dfrac{\textit{opposite to A}}{\textit{Hypotenuse}} = \dfrac{5}{13}$

(ii) Cos A $= \dfrac{\textit{Adacent to A}}{\textit{Hypotenuse}} = \dfrac{12}{13}$

(iii) Tan A $= \dfrac{\textit{Opposite to A}}{\textit{Adjacent to A}} = \dfrac{5}{12}$

(iv) Sec A $= \dfrac{\textit{Hypotenuse}}{\textit{Adjacent}} = \dfrac{13}{12} = 1\dfrac{1}{12}$

Alternatively, Sec A $= \dfrac{1}{cosA} = \dfrac{1}{\left(\frac{12}{13}\right)} = 1 \div \dfrac{12}{13} = 1 \times \dfrac{13}{12} = \dfrac{13}{12} = 1\dfrac{1}{12}$

(v) Csc A $= \dfrac{\textit{Hypotenuse}}{\textit{Opposite}} = \dfrac{13}{5} = 2\dfrac{3}{5}$

Alternatively, Cosec A $= \dfrac{1}{sinA} = \dfrac{1}{\left(\frac{5}{13}\right)} = 1 \div \dfrac{5}{13} = 1 \times \dfrac{13}{5} = \dfrac{13}{5} = 2\dfrac{3}{5}$

(vi) Cot A $= \frac{Adjacent}{Opposite} = \frac{12}{5} = 2\frac{2}{5}$

Alternatively, Cot A $= \frac{1}{Tan\ A} = \frac{1}{\left(\frac{5}{12}\right)} = 1 \div \frac{5}{12} = 1 \times \frac{12}{5} = \frac{12}{5} = 2\frac{2}{5}$

Example 2

The diagram below shows a right-angled triangular plate of dimensions 3, 5 and x. Use this information to answer the questions.

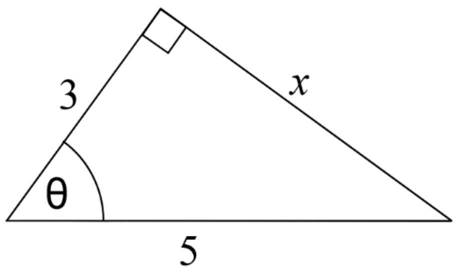


Figure 3: Right-angled triangle

Find the value of

(i) x

(ii) Cotθ

(iii) Sin θ

(iv) θ

Solution

Using θ as our reference, let us first assign names to the sides of the triangle. The side of the triangle *opposite* to θ is x, the side *adjacent* to θ is 3 and the *hypotenuse* is 5

(i) To find x, we will apply the Pythagoras theorem.

This gives $x^2 + 3^2 = 5^2$

$$x^2 = 5^2 - 3^2$$
$$x^2 = 25 - 9$$
$$x^2 = 16$$
$$x = \sqrt{16} = 4$$

(ii) Cot $\theta = \frac{Adjacent}{Opposite} = \frac{3}{x} = \frac{3}{4}$

(iii) Sin $\theta = \frac{x}{hypotenuse} = \frac{4}{5}$

(iv) from (iii), we have Sin $\theta = \frac{4}{5}$. This means, $\theta = \sin^{-1}\left(\frac{4}{5}\right) = 53.13°$

TRIGONOMETRIC FUNCTIONS

The trigonometric function relates an angle in the right-angled triangle to the ratio of lengths of any two sides. Under this sub-topic, we will learn how to find the signs of trigonometry ratios in each of the four quadrants in the cartesian coordinate plane (x-y plane).

The x and the y axis divide the coordinates plane into four parts with each called a quadrant. The diagram below shows an example of a coordinate plane.

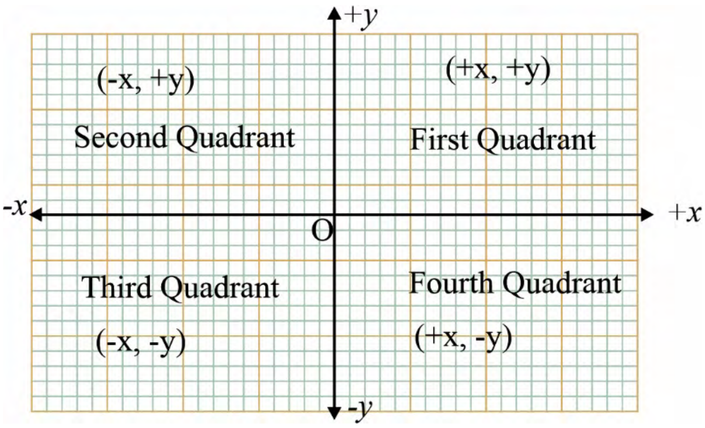


Figure 4: Coordinate plane

1. All points in the first quadrant will have positive values for both x and y coordinates.
2. In the second quadrant, the x coordinates are negative and the y coordinates are positive
3. The third quadrant will have negative values for both x and y coordinates
4. The x coordinates of points in the fourth quadrant are positive and that of their corresponding y coordinates are negative.

Let us find the signs of the trigonometry ratios starting from the first quadrant.

1. Plot any arbitrary point (x, y) and draw a line to join it to the origin.
2. Label the acute angle formed by the line and the x-axis, θ.
3. Draw a straight line from point (x, y) to intersect the x-axis at right angles.
4. Label the hypotenuse r.
5. Find the trigonometry ratios

Your result should be similar to the one below:

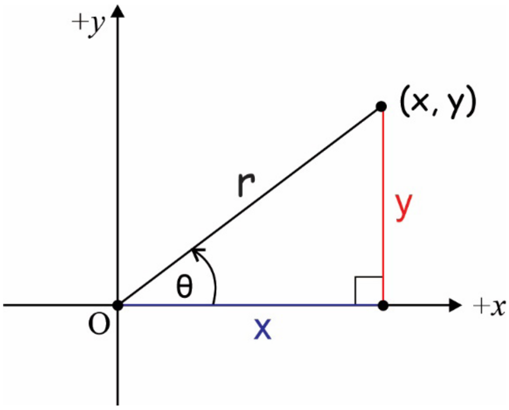


Figure 5: Right-angled triangle

From the right-angle triangle,

$\sin\theta = \frac{+y}{r} = \frac{y}{r}$

$\cos\theta = \frac{+x}{r} = \frac{x}{r}$

$\tan\theta = \frac{+y}{+x} = \frac{y}{x}$

ALL RATIOS ARE POSITIVE

For the second quadrant, we will use $(-x, y)$ and repeat the same steps

You answer should be similar to the one below:

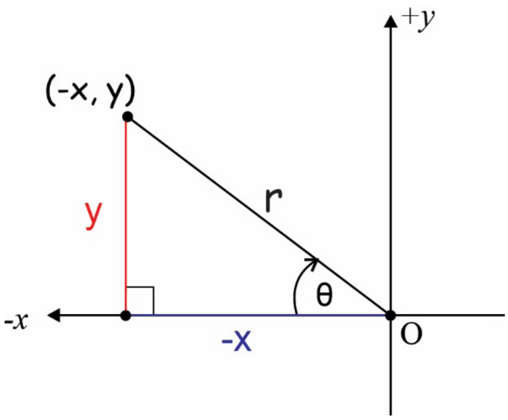


Figure 6: Right-angled triangle

$\sin\theta = \frac{+y}{r} = \frac{y}{r}$

$\cos\theta = \frac{-x}{r} = -\frac{x}{r}$

$\tan\theta = \frac{+y}{-x} = -\frac{y}{x}$

ONLY SINE IS POSITIVE

For the third quadrant, we will use $(-x, -y)$ and repeat the same steps

You answer should be similar to the one below:

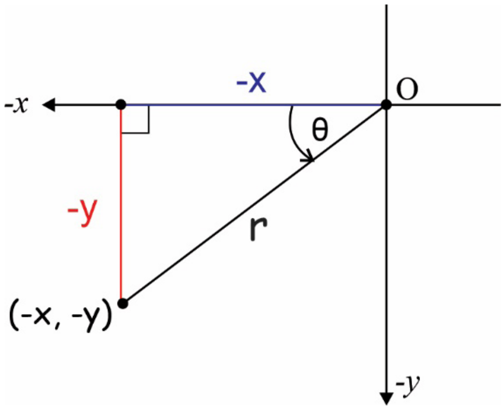


Figure 7: Right-angled triangle

$\sin\theta = \frac{-y}{r} = -\frac{y}{r}$

$\cos\theta = -\frac{x}{r} = -\frac{x}{r}$

$\tan\theta = -\frac{y}{-x} = \frac{y}{x}$

ONLY TAN IS POSITIVE

For the fourth quadrant, we will use $(x, -y)$ and repeat the same steps

You answer should be similar to the one below:

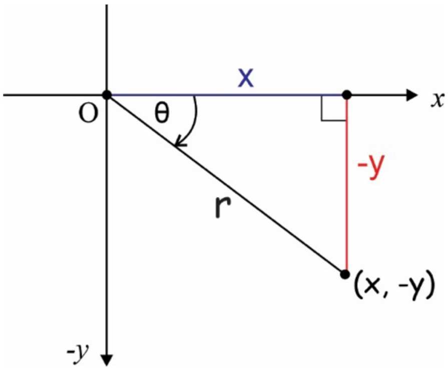


Figure 8: Right-angled triangle

$\sin\theta = \frac{-y}{r} = -\frac{y}{r}$

$\cos\theta = \frac{x}{r} = \frac{x}{r}$

$\tan\theta = \frac{-y}{x} = -\frac{y}{x}$

ONLY COS IS POSITIVE

Figure 9 shows the summarized results:

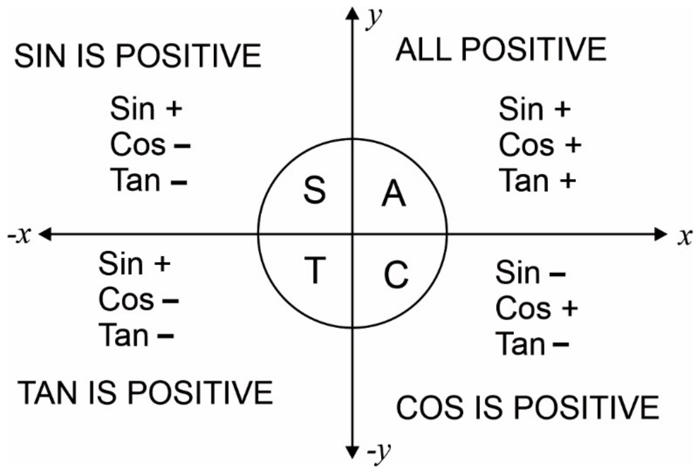


Figure 9: Summary of results

For θ in Q_1 (first quadrant), All sinθ, cosθ and tanθ are positive.

For θ in Q_2 (second quadrant), only sinθ is positive.

For θ in Q_3 (third quadrant), only tanθ is positive.

For θ in Q_4 (fourth quadrant), only cosθ is positive.

The word **CAST** (i.e. the first letters of the ratios beginning from the fourth quadrant) will help you remember the positive ratios.

Table 3: Trigonometric Functions

Quadrant	Trigonometric Function					
	sine	cosine	tangent	cosecant	secant	cotangent
First	+	+	+	+	+	+
Second	+	–	–	+	–	–
Third	–	–	+	–	–	+
Fourth	–	+	–	–	+	–

Let us solve some examples to consolidate our understanding

Example 3

The terminal side of an angle, θ, passes through the point (7, 24). Find the values of the six trigonometric functions of the angle.

Solution

First, plot the point on the x-y axis.

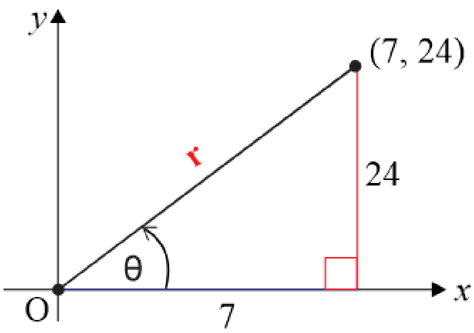


Figure 10: Right-angled triangle

Then, find r.

Using Pythagoras' theorem, we have $r^2 = 7^2 + 24^2$

$$r^2 = 49 + 576$$

$$r^2 = 625$$

$$r = \sqrt{625}$$

$$r = 25$$

Let's use the diagram to find the trig functions

$\text{Sin}\theta = \dfrac{\text{opposite to } \theta}{\text{Hypotenuse}} = \dfrac{24}{25}$

$\text{Cos } \theta = \dfrac{\text{Adjacent to } \theta}{\text{Hypotenuse}} = \dfrac{7}{25}$

$\text{Tan } \theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{24}{7} = 3\dfrac{3}{7}$

$\text{Cosec } \theta = \dfrac{1}{\sin\theta} = \dfrac{25}{24} = 1\dfrac{1}{24}$

$\text{Sec } \theta = \dfrac{1}{\text{Cos}\theta} = \dfrac{25}{7} = 3\dfrac{4}{7}$

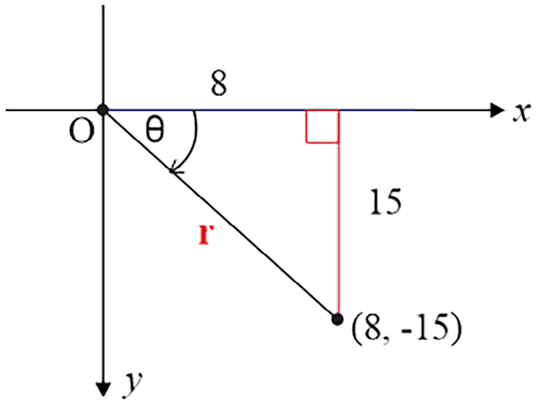
$\text{Cot } \theta = \dfrac{1}{\tan\theta} = \dfrac{7}{24}$

Example 4

The terminal side of an angle passes through the point (8, – 15). Find the values of the six trigonometric functions of angle θ.

Solution

First, plot the point on the x-y axis.



Note: (distance/ length is always positive)

Figure 11: Right-angled triangle

Then, find r.

Using Pythagoras' theorem, we have $r^2 = 8^2 + (15)^2$

$r^2 = 64 + 225$

$r^2 = 289$

$r = \sqrt{289}$

$r = 17$

bearing in mind that the triangle is in the fourth quadrant.

Let's use the diagram to find the trig functions :

$\text{Sin}\theta = \frac{opposite\ to\ \theta}{Hypotenuse} = -\frac{15}{17}$

$\text{Cosec}\ \theta = \frac{1}{sin\theta} = -\frac{7}{15} = -1\frac{2}{15}$

$\text{Cos}\ \theta = \frac{Adjacent\ to\ \theta}{Hypotenuse} = \frac{8}{17}$

$\text{Sec}\ \theta = \frac{1}{Cos\theta} = \frac{17}{8} = 2\frac{1}{8}$

$\text{Tan}\ \theta = \frac{Opposite}{Adjacent} = -\frac{15}{8} = -1\frac{7}{8}$

$\text{Cot}\ \theta = \frac{1}{tan\theta} = -\frac{8}{15}$

Recall that in the fourth quadrant, only cosine is positive. Hence all trigonometric functions involving sine and tangent will be negative.

APPLICATIONS OF TRIGONOMETRIC FUNCTIONS (ELEVATION AND DEPRESSION)

Angles of elevation and depression play an important role in solving real-life problems. Understanding this concept will enable you to calculate various distances and heights of objects. In mathematics 'to elevate' is to raise **above** the horizontal. The angle of elevation is the angle an observer must raise their eye through to see an object that is higher than the observer. The diagram below explains this principle.

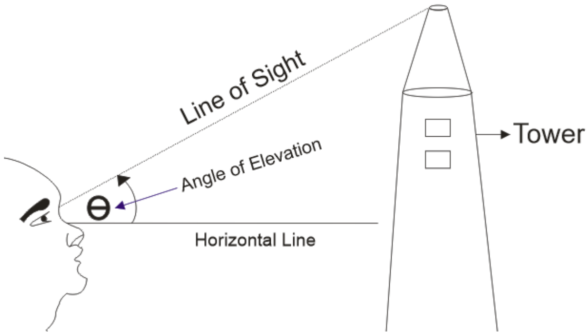


Figure 12: Angle of elevation

The observer is looking at the top of the tower. The distance between his eye and the top of the tower is called the line of sight. It is an imaginary line (straight) that stretches between the observer's eye and the object he is looking at. The line that stretches left to right form the observer's eye is the horizontal line. The angle above the horizontal line and below the line of sight, θ, is the angle of elevation.

Angle of depression on the other hand is the angle formed by a line of sight and the horizontal plane of an observer looking down.

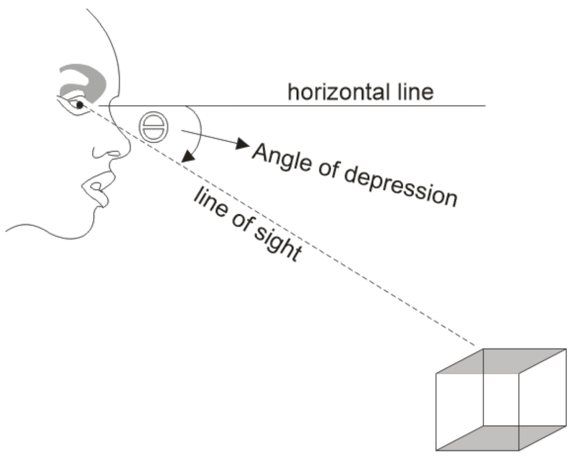


Figure 13: Angle of depression

The observer is looking at an object below their eye level. The angle, θ, is the angle of depression.

To solve problems involving angle of elevation and depression, the following suggestions may prove useful.

- Draw a sketch diagram of the problem if necessary.
- Write in the sketch diagram the given angle of elevation and depression.
- Apply trigonometry and/or Pythagoras' theorem to find missing distances and angles.

Let us now solve some questions.

Example 5

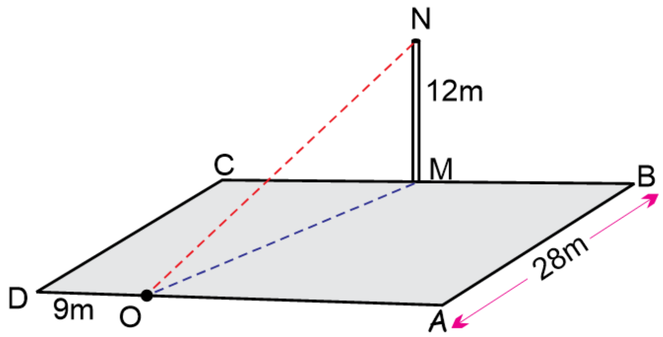


Figure 14: Angle of elevation from a rectangular field

ABCD is a rectangular field of dimension 28m by 60m. Point M, the base of a vertical pole is situated at the mid-point of side BC. The pole, MN, which is 12m, casts a shadow that reaches point O, 9m from D. O is a point on line AD.

Calculate the angle of elevation of the top of the pole, N, from point O.

Solution

We will first complete the missing distances on the diagram.

Since BC= 60m and point M is at the midpoint of BC,

it implies $|BM| = \frac{1}{2} \times 60 = 30\text{m}$

Also since, ABCD is a rectangle, AD = BC = 60m.

This implies $|OA| = 60 - 9 = 51\text{m}$

Draw a perpendicular from point M to intersect AD at E

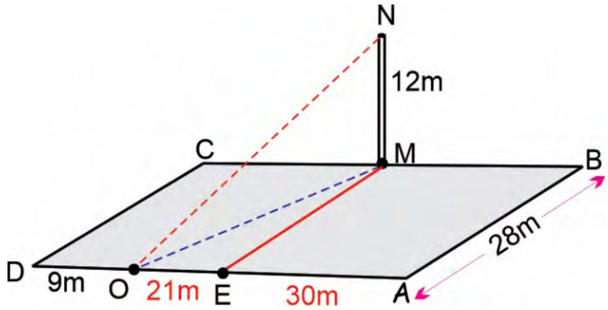


Figure 15: Angle of elevation from a rectangular field

|ME| = $|BA|$ = 28, |AE| =| BM| =30m.

Since |DE| = 30m and |DO| = 9cm, |EO| =30 − 9 = 21m

From triangle OEM, $|OM|^2 = |OE|^2 + |EM|^2$

$|OM|^2 = 21^2 + 28^2$

$|OM|^2 = 441 + 784 = 1225$

$|OM| = \sqrt{1225}$

$|OM| = 35m$

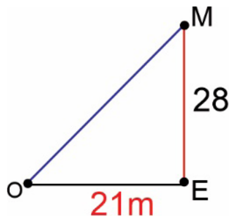


Figure 16: Right-angled triangle

From triangle OMN,

$|ON|^2 = 35^2 + 12^2$

$|ON|^2 = 1225 + 144 = 1369$

$|ON| = \sqrt{1369}$

$|ON| = 37m$

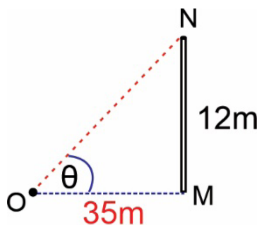


Figure 17: Right-angled triangle

Finally, we can use any of the trigonometry ratios to calculate Angle MON = θ

The side opposite to $\theta = 12$ and side adjacent to $\theta = 35$ and the hypotenuse = 37

$\tan\theta = \frac{12}{35}$

$\theta = \tan^{-1}\left(\frac{12}{35}\right) = 18.9°$

Therefore, the angle of elevation of the top of the pole, N, from point O is 18.9°

Example 6

Two points **A** and **C** are on opposite sides of a vertical pole, BD.

Points A and C are on the same level ground as the foot of the pole, **B**.

The angle of elevation of the top of the pole, **D**, from **A** and **C,** are 30° and 60° respectively. The height of the pole is 15m.

(a) Calculate the shortest distance between points A and B.

(b) Which of the points is closer to the pole? Give a reason.

Solution

First, make a sketch of the problem. It doesn't matter about being perfectly to scale.

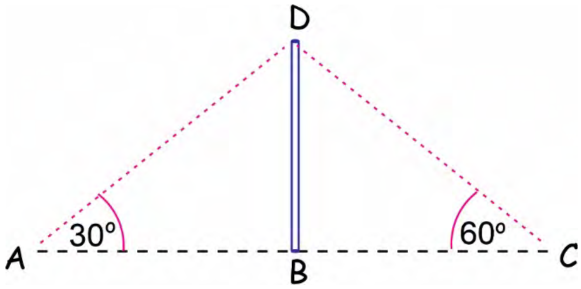


Figure 18: A sketch of the angle of elevation

(a) From triangle ABD, $tan30° = \frac{15}{|AB|}$

Making |AB| the subject, we have $|AB| \times \tan 30° = 15$

Divide through by tan 30

$\frac{|AB| \times Tan30}{Tan30} = \frac{15}{tan30°}$

$|AB| = 15 \div tan30$

$|AB| = 15 \div \frac{1}{\sqrt{3}} = 15 \times \frac{\sqrt{3}}{1} = 15\sqrt{3}$

From triangle BCD, $tan60° = \frac{15}{|BC|}$

Making |BC| the subject, we have $|BC| \times \tan 60° = 15$

Divide through by tan 60

$\frac{|BC| \times Tan60}{Tan60} = \frac{15}{tan60°}$

$|BC| = 15 \div tan60$

$|BC| = 15 \div \sqrt{3} = 15 \div \sqrt{3} = \frac{15}{\sqrt{3}}$

Rationalising, we have:

$|BC| = 1\frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$

The shortest distance from A to B is $15\sqrt{3} + 5\sqrt{3} = 20\sqrt{3}$ metres

(b) Point C is closer to the pole than point A since $|AB| = 15\sqrt{3}$m as compared to $|BC|$ which is $5\sqrt{3}$ m

SPECIAL ANGLES

You will need a calculator or four-figure table to calculate the basic trigonometric values of most angles. However, the trigonometric values of a few angles can be calculated without using a calculator or a four-figure table. Here, we will derive the trigonometric ratios of these special angles. This is important as you will need it when simplifying trig functions and equations. The special angles of interest to us are: 30°, 60° and 45°.

Angle 45°

To derive the trig value of 45°, follow the steps:

1. Draw a square of side 1 unit and bisect it along the diagonal. This will give two congruent right-angle triangles. Do not forget that the diagonal will bisect the angles.

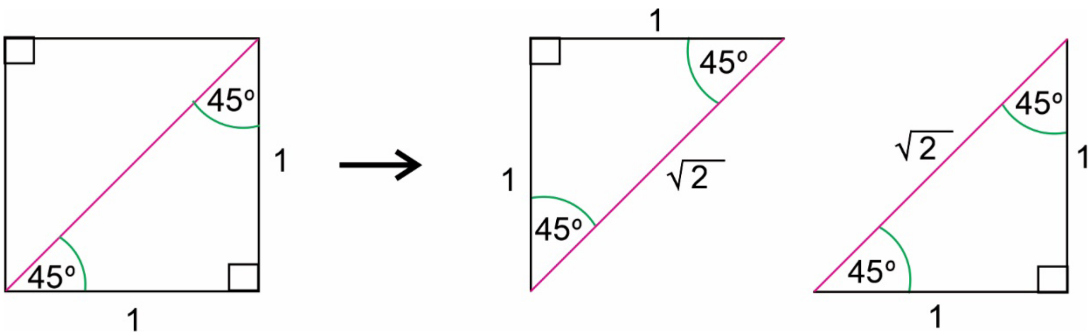


Figure 19: Congruent right-angle triangles

2. Then, find the length of the diagonal of one of the triangles.

   Using Pythagoras Theorem, we have $d^2 = 1^2 + 1^2 = 1 + 1 = 2$

$$d = \sqrt{2}$$

3. Finally, find the trigonometry ratios for the 45°.

$$\text{Sin}45° = \frac{Opposite}{Hypotenuse} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2} \qquad \text{Cosec } 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Cos } 45 = \frac{Adjacent}{Hypotenuse} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \qquad \text{Sec } 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Tan } 45 = \frac{Opposite}{Adjacent} = \frac{1}{1} = 1 \qquad \text{Cot } 45° = \frac{1}{1} = 1$$

Angle 30° and 60°

First, draw an equilateral triangle of side 2 units and bisect it. This will give two congruent 30-60 right-angled triangles.

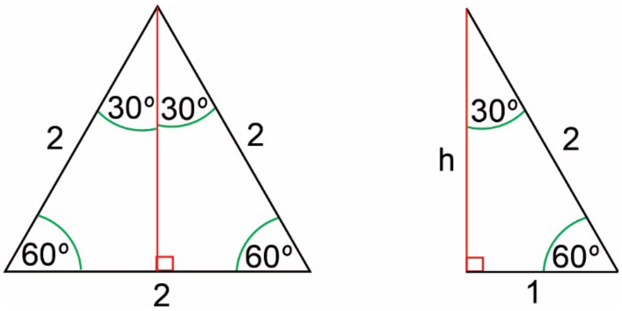


Figure 20: 30-60 right-angled triangles

Then, find the length of the vertical height, h.

Using Pythagoras Theorem, we have $2^2 = 1^2 + h^2$

$$4 = 1 + h^2$$

$$4 - 1 = h^2$$

$$3 = h^2$$

$$h = \sqrt{3}$$

Finally, we find the trigonometry ratios for the 30° and 60°

Let's start with 60°

$$\text{Sin } 60° = \frac{pposite}{Hypotenuse} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3} \qquad \text{Cosec } 60° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$\text{Cos } 60° = \frac{djacent}{Hypotenuse} = \frac{1}{2} \qquad \text{Sec } 60° = \frac{2}{1} = 2$$

$$\text{Tan } 60° = \frac{Opposite}{Adjacent} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \text{Cot } 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

We will use the same diagram for 30°

$\text{Sin } 30° = \frac{Adjacent}{Hypotenuse} = \frac{1}{2}$

$\text{Cos } 30° = \frac{Opposite}{Hypotenuse} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$

$\text{Tan } 30° = \frac{Opposite}{Adjacent} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$

$\text{Cosec } 30° = \frac{2}{1} = 2$

$\text{Sec } 30° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

$\text{Cot } 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$

QUADRANTAL ANGLES

Here we will use the unit circle to find the **quadrantal angles**. Quadrantal angles have special properties. They are angles that terminate on the x and y axis. Therefore, the quadrantal angles between 0° and 360° are 0°, 90°, 180°, 270° and 360° (all multiples of 90). The understanding of quadrantal angles will be used in sketching trigonometric graphs and make calculations easy.

A unit circle is a circle with a radius of 1.

Follow the following activity to generate the trigonometry ratios of the quadrantal angles.

1. Construct a circle with a radius of 1 unit and its centre at the origin of the cartesian coordinate.

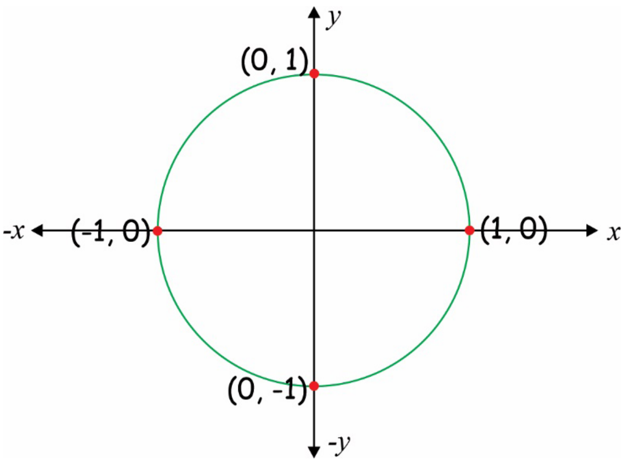


Figure 21: Cartesian coordinate

Since the radius is 1, it means every point on the circumference of the circle is 1 unit away from the centre.

Also, the circle will intersect the positive x-coordinate at (1, 0) and the negative x-coordinate at (-1, 0).

Likewise, it will intersect the positive *y* and the negative *y* coordinates at (0, 1) and (0, -1) respectively.

2. Plot any point (a, b) on the circumference of the circle. This makes an angle of θ with the *x*-axis. Draw a line from this point to intersect the *x*-axis at right angles.

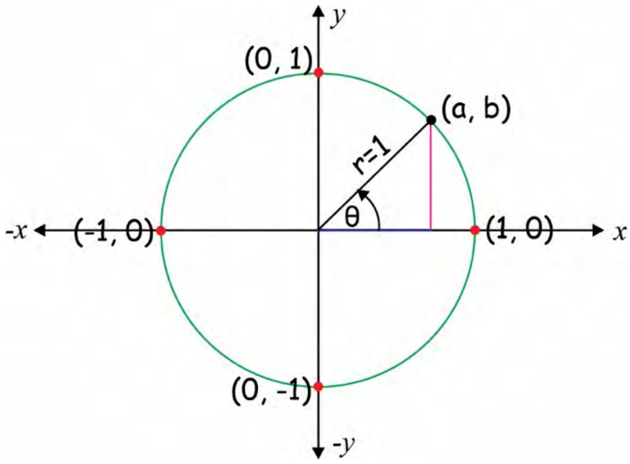


Figure 22: Cartesian coordinate

3. Using the triangle formed, find the value of the side opposite and adjacent to θ

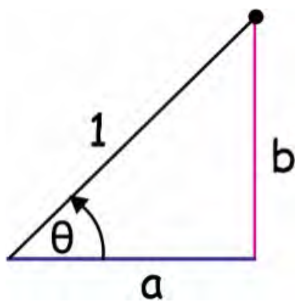


Figure 23: Right-angled triangle

Recall that $\sin\theta = \dfrac{Opposite}{Hypotenuse} = \dfrac{b}{1} = b.$

Also, $\text{Cos}\theta = \dfrac{Adjacent}{Hypotenuse} = \dfrac{a}{1} = a$

Therefore, $(a, b) = (Cos\theta, Sin\theta)$

So, for the unit circle above:

Vertical distance = Opposite = b = Sinθ

Horizontal distance = Adjacent = a = Cosθ

Recall also that $\tan\theta = \dfrac{Opposite}{Adjacent} = \dfrac{Sin\theta}{Cos\theta}$

Note that the signs will depend on the quadrant.

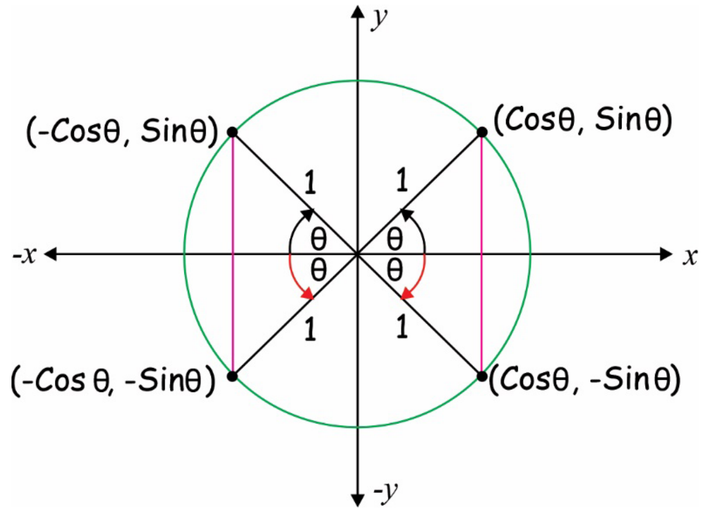


Figure 24: Cartesian coordinate

Let us generate the quadrantal angles using this understanding. In geometry, angles are measured from the positive *x*-axis in an anticlockwise direction. Recall the quadrantal angles are 0°, 90°, 180°, 270°, 360°….

Let us start with $\theta = 0°$

When $\theta = 0$, the hypotenuse will line up with the *x*-axis.

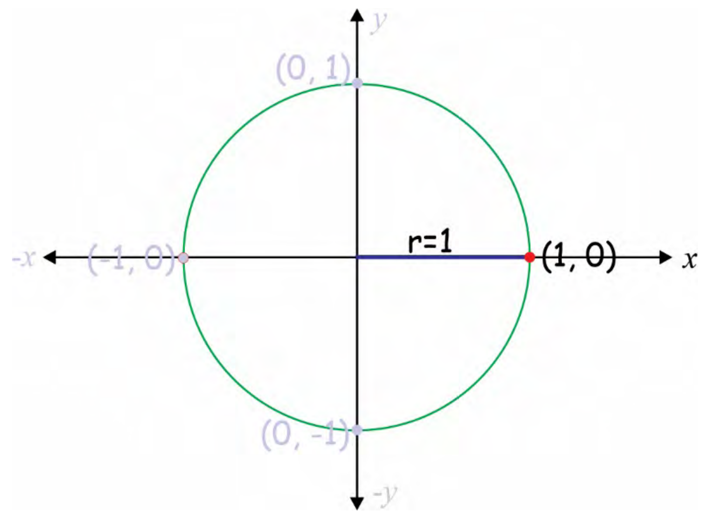


Figure 25: Cartesian coordinate

This means the vertical distance will be 0 units (since there is no vertical movement).

So, sin 0 = 0

Cos 0 =1 (the horizontal movement is 1)

Tan 0°$=\frac{Sin0°}{Cos0°}=\frac{0}{1}=0$

Sec 0°$=\frac{1}{cos0°}=\frac{1}{1}=1$

Csc0°$=\frac{1}{sin0°}=\frac{1}{0}=$ Undefined

Cot0°$=\frac{1}{tan0°}=\frac{1}{0}=$ Undefined

What happens when $\theta = 90°$

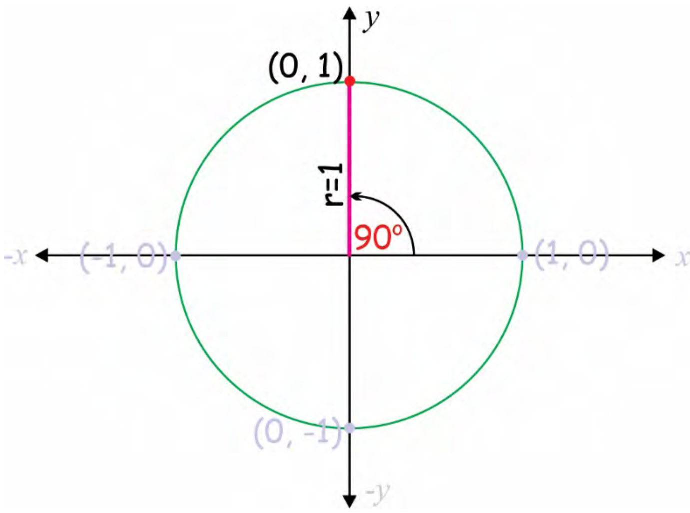


Figure 26: Cartesian coordinate

When $\theta = 90°$, the hypotenuse will line up with the y-axis.

This means the horizontal distance will be 0 units (since there is no horizontal movement).

So, Cos 90° = 0

Sin 90° = 1 (the vertical movement is 1)

Tan 90° $=\frac{Sin90°}{Cos90°}=\frac{1}{0}=$ Undefined

Sec 90°$=\frac{1}{cos90°}=\frac{1}{0}=$ Undefined

Csc 90°$=\frac{1}{sin90°}=\frac{1}{1}=1$

Cot 90°$=\frac{1}{tan90°}=\frac{1}{\infty}=$ Undefined

What happens when $\theta = 180°$

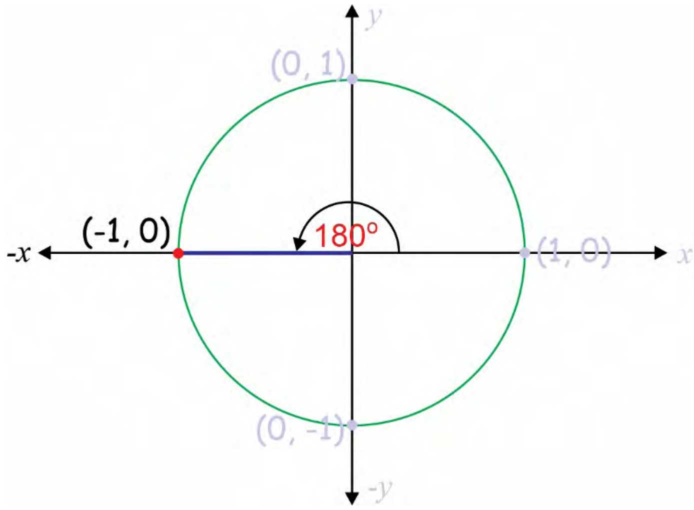


Figure 27: Cartesian coordinate

When $\theta = 180°$, the hypotenuse will line up with the negative x-axis.

This means the vertical distance will be 0 units (since there is no vertical movement).

So, Cos $180° = -1$ (the horizontal movement is 1 in the negative x-axis)

Sin $180° = 0$ (the vertical movement is 0)

Tan $180° = \frac{Sin 180°}{Cos 180°} = \frac{0}{-1} = 0$

Sec $180° = \frac{1}{cos 180°} = \frac{1}{-1} = -1$

Csc $180° = \frac{1}{sin 180°} = \frac{1}{0} =$ Undefined

Cot $180° = \frac{1}{tan 180°} = \frac{1}{0} =$ Undefined

What happens when $\theta = 270°$

When $\theta = 270°$, the hypotenuse will line up with the negative y-axis.

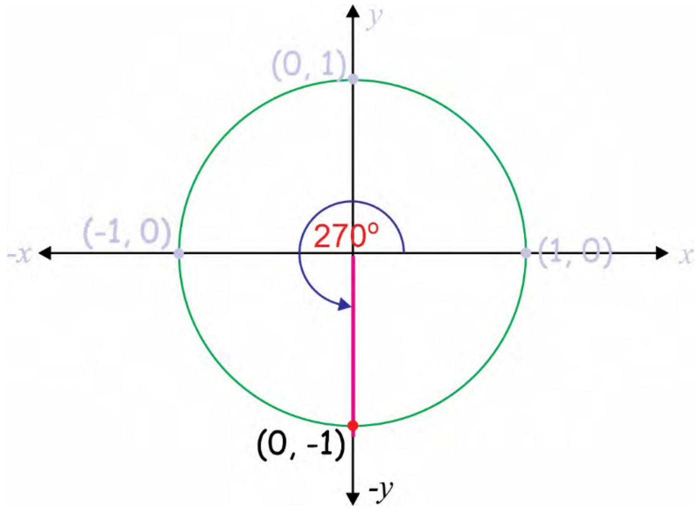


Figure 28: Cartesian coordinate

This means the horizontal distance will be 0 units (since there is no horizontal movement).

So, Cos $270° = 0$

Sin $270° = -1$ (the vertical movement is 1 in the negative y-axis)

Tan $270° = \frac{Sin270}{Cos270°} = \frac{-1}{0} =$ Undefined

Sec $270° = \frac{1}{cos270°} = \frac{1}{0} =$ Undefined

Csc $270° = \frac{1}{sin270°} = \frac{1}{-1} = -1$

Cot $270° = \frac{1}{tan270°} = \frac{1}{\infty} =$ Undefined

The cycle repeats when $\theta = 360°$

Let us take stock of all the trigonometry ratios of the quadrantal angles we have discussed.

Use your calculator to validate these values.

θ	$sin\theta$	$cos\theta$	$tan\theta$	$Csc\theta$	$Sec\theta$	$Cot\theta$
0°	0	1	0	Undefined	1	Undefined
90°	1	0	Undefined	1	Undefined	Undefined
180°	0	– 1	0	Undefined	-1	Undefined
270°	– 1	0	Undefined	-1	Undefined	Undefined
360°	0	1	0	Undefined	1	Undefined

Example 7

Given Sin $x = \frac{1}{2}$, calculate Cos x if $90° < x < 180°$

Solution

Using the given range, make a sketch diagram:

From the given range, angle x will be located in the second quadrant.

We were given Sin $x = \frac{1}{2}$

$$\text{Sin } x = \frac{\textit{Opposite to } x}{\textit{Hypotenuse}} = \frac{1}{2}$$

This means, should we relate this ratio to the sides of a right-angle triangle, the side opposite angle $x = 1$ and the longest side, or the hypotenuse, $= 2$

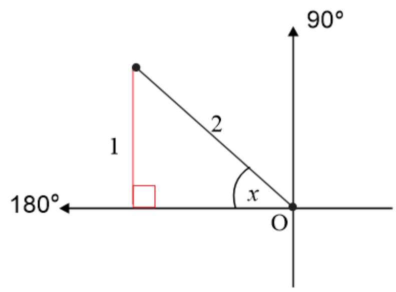


Figure 29: Right-angle triangle

To find the side adjacent to angle x, we will apply Pythagoras Theorem.

This gives us: $1^2 + A^2 = 2^2$, where A=Adjacent side

$A^2 = 4 - 1 = 3$

$A = \sqrt{3}$

On the graph, the adjacent side represents the negative x-coordinate.

So, A $= -\sqrt{3}$

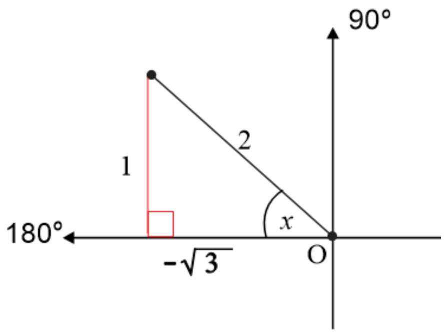


Figure 30: Right-angle triangle

Finally, we calculate the value of cos$x$

$$\text{Cos}x = \frac{Adjacent}{Hypotenuse} = -\frac{\sqrt{3}}{2}$$

Therefore, if Sin$x = \frac{1}{2}$ for $90° < x < 180°$, Cos$x = -\frac{\sqrt{3}}{2}$

Example 8

Given $tan\theta = \sqrt{3}$,

Calculate $\frac{Sec\theta - Cosec\theta}{Cot\theta}$ for $180° < x < 270°$

Solution

$$tan\theta = \frac{\sqrt{3}}{1} = \frac{Opposite}{Adjacent}$$

So, relating this to the right-angle triangle from which the ratio was derived, we have Opposite $= \sqrt{3}$ and Adjacent $= 1$

This means we have to find the hypotenuse.

Let $r =$ hypotenuse

$$r^2 = 1^2 + (\sqrt{3})^2 = 4$$

$$r = \sqrt{4} = 2$$

The range given falls in the third quadrant. Recall that the third quadrant has both x and y-axis being negative. A sketch of the result is shown below

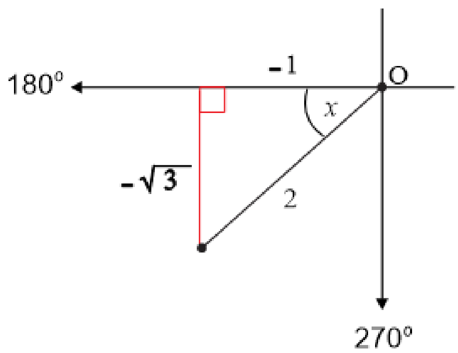


Figure 31: Right-angle triangle

Let us now calculate $\frac{Sec\theta - Cosec\theta}{Cot\theta}$

$$\text{Sec}\theta = \frac{hypotenuse}{Adjacent} = \frac{2}{-1} = -2$$

$$\text{Cosec}\theta = \frac{hypotenuse}{Opposite} = \frac{2}{-\sqrt{3}} = -\frac{2}{3}\sqrt{3}$$

$$\text{Cot }\theta = \frac{Adjacent}{Opposite} = -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

This implies $\frac{Sec\theta - Cosec\theta}{Cot\theta} = \frac{-2-\frac{2}{3}\sqrt{3}}{\frac{\sqrt{3}}{3}} = \left(-2 - \frac{2}{3}\sqrt{3}\right) \times \frac{3}{\sqrt{3}}$

$$= \left(\frac{-6 - 2\sqrt{3}}{3}\right) \times \frac{3}{\sqrt{3}}$$

$$= \left(\frac{-6 - 2\sqrt{3}}{\sqrt{3}}\right)$$

Rationalise the resulting expression

$$\left(\frac{-6 - 2\sqrt{3}}{\sqrt{3}}\right) = \left(\frac{-6 - 2\sqrt{3}}{\sqrt{3}}\right) \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{-6\sqrt{3} - 2(3)}{3}$$

$$= \frac{-6(\sqrt{3} + 1)}{3}$$

$$= -2(1 + \sqrt{3})$$

Therefore, $\frac{Sec\theta - Cosec\theta}{Cot\theta} = -2\left(1 + \sqrt{3}\right)$

Example 9

Without using a calculator, find the value of Sin 315°

Solution

First make a sketch of the angle 315° in the x-y axis. This angle falls on the fourth quadrant. This will result in a right-angle triangle with acute angle 45°

Recall that 45 is a special angle which is derived from a right-angle triangle of sides 1, 1 and $\sqrt{2}$. Also, in the fourth quadrant, the x-axis is positive and the y-axis is negative.

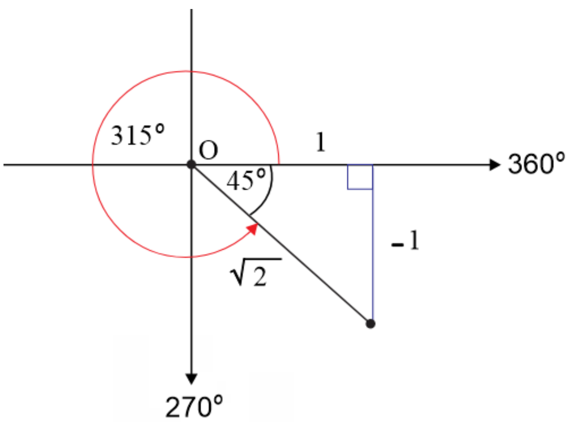


Figure 32: Right-angle triangle

From the graph, Sin 315° is the same as the value of Sin 45° in the fourth quadrant.

Therefore, Sin 315°$= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

RADIAN MEASUREMENT

You might be more used to measuring angles in degrees. Radians are another way of measuring angles. Aside from using degrees in practical geometry, angles are mostly measured in radians. In addition to being widely used in other branches of mathematics and other fields, it makes some calculations much easier.

Our focus here is to find the relationship between degree measurement and radians.

To find the relationship, we ask you to do the following activity:

1. Construct a circle whose radius is equal to a piece of string. Draw the radius and label it |OA| where O is the centre.

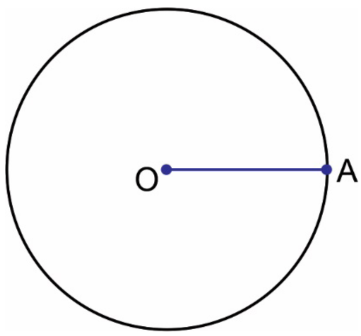


Figure 33: A circle with radius OA

2. Place one end of the string at A and lay it out on the circumference. Mark the end of the string B on the circumference of the circle.

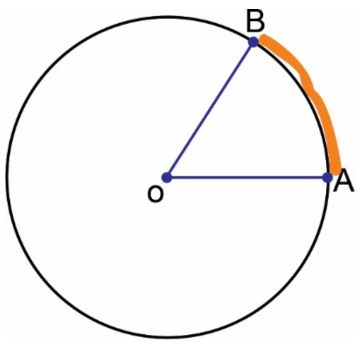


Figure 34: A circle

3. Join points O and B with a straight line. Acute angle $A\hat{O}B$ is **1 Radian**.

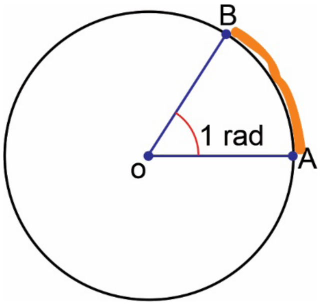


Figure 35: A circle

4. Using a protractor measure the approximate size of this radian in degrees.

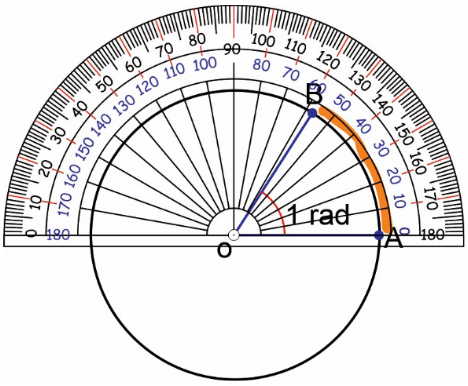


Figure 36: A circle

You should get a value of approximately 57°.

The more accurate figure, corrected to 4 decimal places, is 57.2958°

5. Repeat step 2 until the entire circumference has been covered.

6. Use your findings to estimate the number of radians in a full circle.

Compare your result to $2\pi = 6.283$.

7. You know that the total degrees in a circle is 360°. This value is equivalent to 2π radians:

Therefore, we have 360°$= 2\pi$ radians

Dividing both sides by 2 we have:

180°$= \pi$ radian

To find the degree measure of 1 radian, divide by π

$\frac{180°}{\pi} = \frac{\pi}{\pi}$ *radians*

$\frac{180°}{\pi} = 1$ ***radian***

To find the value of 1 degree, divide through $180° = \pi$ radian by $180°$

$\frac{180°}{180°} = \frac{\pi}{80°}$ *radians*

$1° = \mathbf{\frac{\pi}{180}}$

8. Write, in your own words, what you understand by the phrase "*An angle of size one radian*."

One radian is the angle made by an arc length equal to the radius.

9. Discuss and compare your findings with your classmates.

How to Convert Degree Measure to Radian Measure

To convert the degree measure to radian measure, we multiply the degree measure by the factor $\frac{\pi}{180°}$. We shall consider some examples of how to convert from degrees to radians.

Example 10

Convert 60° to radian measure.

Solution

$60° = 60 \times \frac{\pi}{180}$

$= \frac{1}{3}\pi\ rad$

Example 11

Convert 135° to radian measure.

Solution

$135° = 135 \times \frac{\pi}{180}$

$= \frac{3}{4}\pi\ rad$

Example 12

Convert 210° to radian measure.

Solution

$210° = 210 \times \frac{\pi}{180}$

$= \frac{6}{7}\pi\ rad$

Example 13

Convert 65.5° to radian measure correct to 4 significant figures.

Solution

$$65.5° = 65.5 \times \frac{\pi}{180}$$
$$= 1.143 \; rad \; (to \; 4 \; significant \; figures)$$

How to Convert Radian Measure to Degree Measure

Conversely, to convert the radian measure to degree measure, we multiply the radian measure by the factor $\frac{180°}{\pi}$.

Example 14

Convert $\frac{2\pi}{3}$ rad to degree measure.

Solution

$$\frac{2\pi}{3} rad = \frac{2\pi}{3} \times \frac{180}{\pi}$$
$$= 120°$$

Example 15

Convert 1.25 rad to degree measure correct to 1 decimal place.

Solution

$$1.25 \; rad = 1.25 \times \frac{180}{\pi}$$
$$= 71.6°$$

Example 16

Convert $1\frac{5}{6}$ rad to degree measure correct to the nearest degrees.

Solution

$$1\frac{5}{6} rad = 1\frac{5}{6} \times \frac{180}{\pi}$$
$$= 105°$$

Let us use our understanding to find the length of an arc of a circle.

From elementary geometry, we know that circumference of a circle is $2\pi r = \pi d$

Where r = radius and d = diameter. Remember that $2r = d$

Also, the length of an arc using **degree** measure $= \frac{\theta}{360} \times 2\pi r = \frac{\theta}{360} \times \pi d$

How do you find the arc length given the angle in radians?

Well, we know $360° = 2\pi$ radians. So, we just replace $360°$ with 2π.

Therefore, the length of an arc subtended at θ radians $= \frac{\theta}{2\pi} \times 2\pi r = \theta \mathrm{r} = \frac{1}{2}\boldsymbol{\theta d}$

Example 17

The diagram shows a sector of radius 7cm subtended at angle θ.

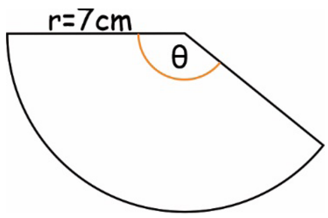


Figure 37: A Sector

If $\theta = \frac{5}{6}\pi$, find:

(a) Length of arc of the sector

(b) Perimeter of the sector.

(c) Circumference of the circle from which the sector was obtained.

Solution

(a) The length of an arc given θ in radians $= \theta r = \frac{5}{6}\pi \times 7cm$

$$= \frac{35}{6}\pi = 18.33cm$$

Alternatively, we can also convert the radian measure to degree measure and use the formula $\frac{\theta}{360} \times 2\pi r$

$\frac{5}{6}\pi = \frac{5}{6} \times 180 = 150°$

Length of an arc $= \frac{150}{360} \times 2 \times \pi \times 7 = \frac{35}{6} \times \pi = 18.33\text{cm}$

(b) Perimeter = distance round an object = sum of the sides of the object.

The sector is enclosed by 3 sides; an arc and two radii.

Perimeter of the sector $= 18.33 + 7 + 7 = 32.33\text{cm}$

(c) Circumference of a circle $= 2\pi r = 2 \times \pi \times 7 = 44$cm.

Example 18

The diagram below shows a large circular plate in the form of a sector. The radius of the sector is 2.1m and arc of the sector is 4.4m.

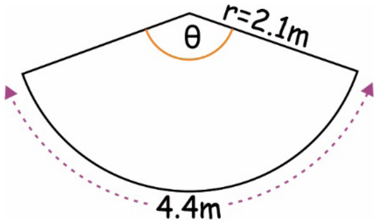


Figure 38: A Sector

Find in π radians the angle subtended by the arc.

Solution

Arc of a circle = 4.4m

since we need the angle in radians, we will use the formula $\theta r = 4.4m$. From the question, $r = 2.1m$

$$\Longrightarrow \theta \times 2.1 = 4.4$$

$$\frac{\theta \times 2.1}{2.1} = \frac{4.4}{2.1}$$

$$\theta = \frac{44}{21}$$

We have to leave our answer in π radians. So, we divide the figure by π to see how many times πgoes into it

Therefore, we have $\theta = \frac{44}{21}\pi \div \pi = \frac{2}{3}\pi$

Alternatively, you can also find the degree measure and convert the result to radians.

$$\frac{\theta}{360} \times 2\pi r = 4.4$$

$$\frac{\theta}{360} \times 2 \times \pi \times 2.1 = 4.4$$

$$\frac{\theta}{360} \times 4.2\pi = 4.4$$

$$\theta = \frac{4.4 \times 360}{4.2\pi} = 120°$$

Since we need the answer in radians, we will convert 120° to radians

$120° = 120° \times \frac{\pi}{180°} = \frac{2}{3}\pi \; radians$

Example 19

A windshield wiper is 0.49m long. In one sweep, it turns through an angle of $\frac{5}{6}\pi$. Calculate the distance covered by the tip of the wiper in one sweep.

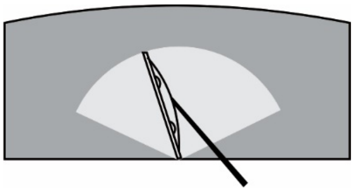


Figure 39: A windshield wiper

Solution

Distance covered by the tip of the wiper in one sweep= $\frac{5}{6}\pi \times 0.49 = 1.283m$

Example 20

A conference table is in the shape of a rectangle with a circle segment at both ends, as shown in the diagram below. The rectangle at the centre measures 5.5m by 2.1 metres. PO and QO are radii of the circle and POQ $= \theta = \frac{2}{3}\pi$ radians

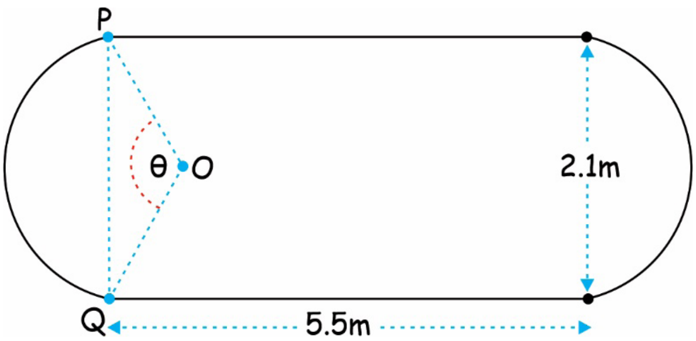


Figure 40: A rectangle with a circle segment at both ends

To sit comfortably around this table, it is estimated that an average person needs 80cm of table edge. How many people can sit comfortably at the table?

Solution

We first have to find the radius of the sector from which the segment was obtained.

To do this, we will draw a line from point O to intersect PQ at right angles. This will bisect angle POQ and line PQ

$\theta = \frac{2}{3}\pi$. Half of $\theta = \frac{1}{2} \times \frac{2}{3}\pi = \frac{1}{3}\pi = 60°$

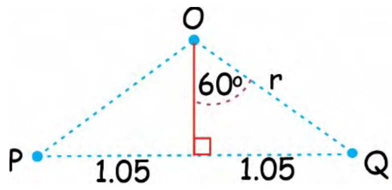


Figure 41: Triangle POQ

Using trig ratios, we have $Sin60° = \frac{1.05}{r}$

$r = \frac{1.05}{Sin60°} = 1.2124m$

Arc PQ= $\frac{2}{3}\pi \times 1.2124 = 2.5393m$

This implies the perimeter of the top of the table is $2.5393 + 5.5 + 2.5393 + 5.5 = 16.079m$

To sit comfortably this table will require $= \frac{16.079m}{80cm}$ people

The distances should be in the same units. We will convert 80cm to m

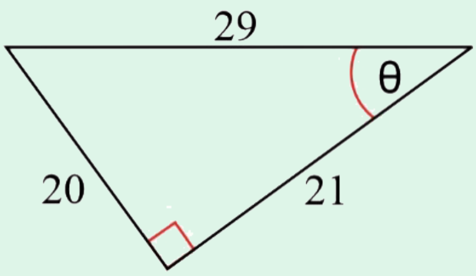
80cm $= 0.8m$

To sit comfortably this table will require $= \frac{16.0786m}{0.8m} = 20.09$ people

Since we cannot have decimal number of people, we will round down our answer and conclude that 20 people can sit comfortably around the table.

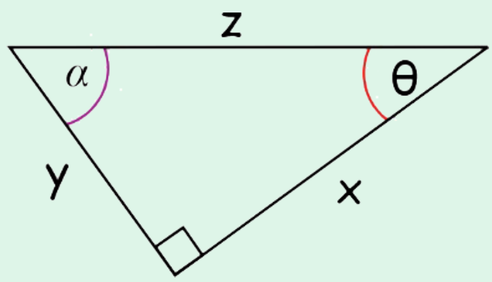
REVIEW QUESTIONS

1. The figure below shows a right-angled triangular board of dimensions 20, 21 and 29. Use it to find the trigonometric ratios for θ.



2. (a) Find the six trigonometric ratios for α in the figure below.

   (b) Find the six trigonometric ratios for θ in the figure below.



3. The terminal side of an angle, θ, passes through the point (-9, -12).

   Find the values of the six trigonometric functions of the angle.

4. Use the following clues to help Madam Adom find the height of a building.

   From a given point on the ground, the angle of elevation of the top of the building is 42°.

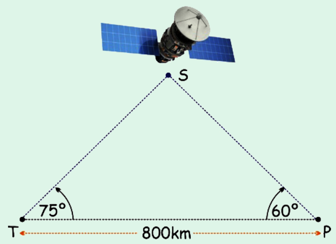
   The angle of elevation of another point on the same level ground which is 70ft farther from the first point is 30°.

5. A Pole is 10 metres tall. The angle of elevation of the top of the pole from the base of the building is 52°. The bases of the building and the pole are on the same level ground. A plumber wants to connect the bases of the pole and the building with a pipe.

   What should be the least length of the pipe?

6. GTV wants to launch a satellite which will receive signals from a transmitter (T) at an angle of 75°. The satellite when launched will send signals at

an angle of 60 to Pulmakom, a town 800km away from the transmitter as shown in the diagram below.

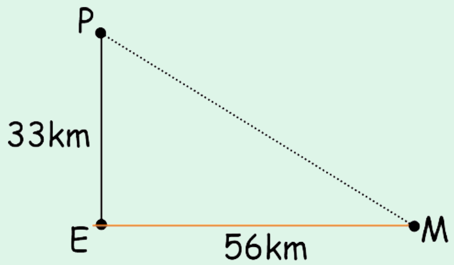


7. At what height should the satellite be launched above the ground?

8. Three branches of a Bank, Premium(P), Ebony(E) and Makola(M) are situated in Ghana.

Ebony branch is 33km due south of Premium and 56km east of the Mokola branch as shown in the diagram below.

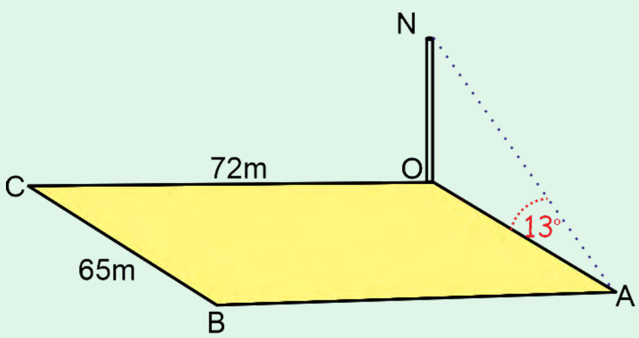
Find the shortest distance between Makola and the Premium branch.



9. The diagram below shows a vertical pole, |ON|, at a corner of a rectangular park, OABC. |OC| = 72m, |OA| = 65m and the angle of elevation of N from A is 13°.

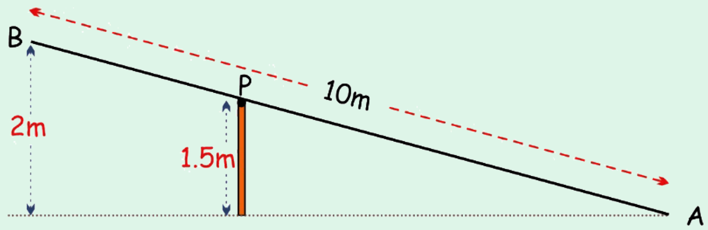
(a) calculate the height of the pole.

(b) A sprinkler is to be situated at the centre of the park, how far will it be from C? Find the elevation of the top of the pole from the sprinkler.

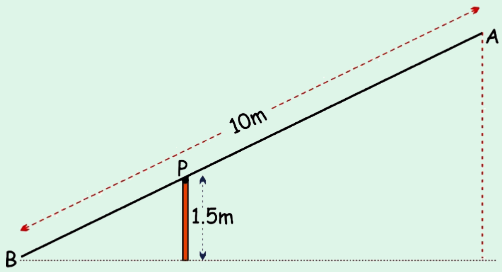


10. A metal beam AB is 10 metres long. It is hinged at the top, P, of a vertical post 1.5 meters high.

When A touches the ground, B is 2 metres above the ground, as shown in Fig1



When B comes down to the ground, A rises as shown in the Fig2

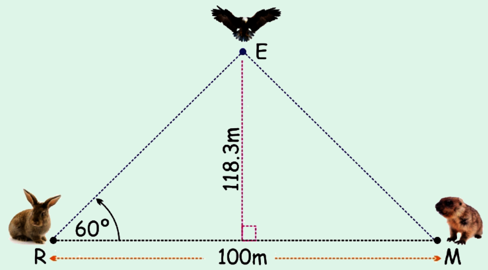


(a) Find the height of A above the ground in Fig 2.

(b) Find the elevation of the beam from the ground in Fig 2

(c) Find |AP|

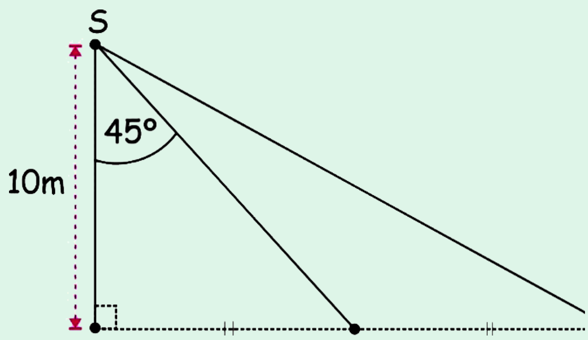
11. An Eagle, E, hovers between two preys, Rabbit(R) and Marmot(M), 100 metres apart. From the Eagle, the angle of depression of the Rabbit is 60° . The eagle is 118.3m above the ground.



(a) If the Eagle decided to attack the nearer prey, which of the animals would it attack? Justify your response.

(b) From the eagle, what is the angle of depression of the Marmot?

12. The diagram below shows a spotlight at point S, mounted 10m directly above point P at the front edge of the stage. The spotlight swings 45° from the vertical to illuminate another point Q, also at the front edge of the stage.



Through how many more degrees must the spotlight swing to illuminate point B, where Q is the mid-point of PB

13. A 14-foot ladder is used to scale a 13-foot wall. At what angle of elevation must the ladder be situated in order to reach the top of the wall?

14. Standing on a cliff 380 meters above the sea, Pat sees an approaching ship and measures its angle of depression, obtaining 9 degrees.

(a) How far from shore is the ship?

(b) Now Pat sights a second ship beyond the first. The angle of depression of the second ship is 5 degrees. How far apart are the ships?

15. The diagonal of a rectangle is 15 cm, and the perimeter is 42 cm. Find the acute angles between the diagonals.

16. A ladder is leaning against an outside wall of a building. If the angle of elevation at the base of the ladder to the wall is 55° and the ladder is 5 metres long, how far up the wall does the ladder reach?

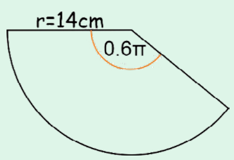
17. Without using a calculator, find Cot150°

18. Convert $\frac{5}{6}\pi$ radians to degrees.

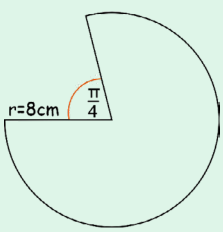
19. Convert 60^0 to radians.

20. Calculate the length of arcs of the following sectors. (Leave your answer in π)

(a)

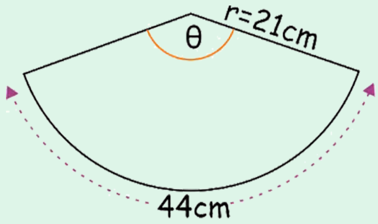


(b)

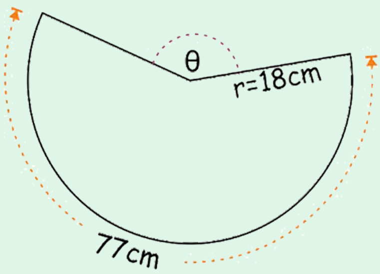


21. Calculate angle θ in each of the following sectors. Leave your answer in π radians

(a)



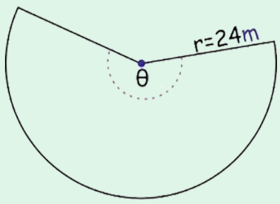
(b)



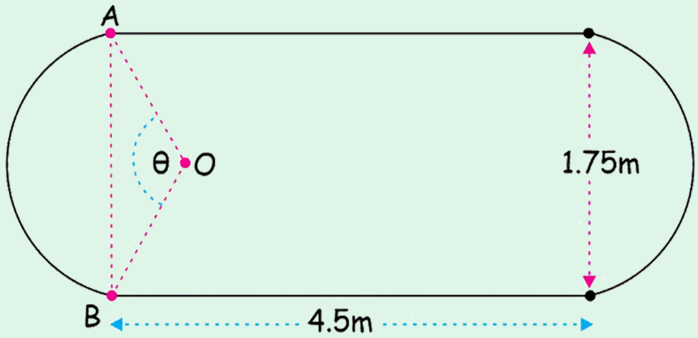
22. If the minute hand of a clock is 6 inches. How far does the tip of travel in 40 minutes?

23. The pendulum of a clock is 35 cm long. If it swings through an angle of $\frac{2}{7}\pi$ find the total distance travelled in one complete swing.

24. A sensor in a security system covers an area in the form of a sector as shown below. If the perimeter of the sector is $3(7\pi + 16)$, find in π radians, the angle subtended by the sector.



25. A large dining table is in the shape of a rectangle with a circle segment at both ends, as shown in the diagram below. The rectangle at the centre measures 4.5m by 1.75 metres. OA and BO are radii of the circle and $A\widehat{O}B = \theta = \frac{3}{5}\pi$ radians



To sit comfortably around this table, it is estimated that an average person needs 95cm of table edge. How many people can sit comfortably at the table?

GLOSSARY

1. **Trigonometry** is a branch of mathematics that explores the relationships between the angles and sides of triangles.
2. **Trigonometry functions** are the six trigonometric functions, namely, sine, cosine, tangent, cosecant, secant and cotangent. They relate an angle in the right-angled triangle to the ratio of lengths of any two sides.
3. The **angle of elevation** is the angle an observer must raise his eye through to see an object that is higher than the observer. '*To elevate*' is to raise **above** the horizontal.
5. The **angle of depression** is the angle formed by a line of sight and the horizontal plane of an observer looking down.
6. **Quadrantal Angles** are angles that terminate on the x and y axis. Among these angles are 0°, 90°, 180°, 270°, 360°,……. These angles are in multiples of 90°.
7. **One radian** is the angle made by an arc length equal to the radius.
8. An **angle** is formed by two rays (or line segments) that share a common endpoint, called the vertex. In trigonometry, angles are typically measured in degrees or radians.
9. The **sine** of an angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse.
10. The **cosine** of an angle in a right triangle is the ratio of the length of the adjacent side to the length of the hypotenuse.
11. The **tangent** of an angle in a right triangle is the ratio of the length of the opposite side to the length of the adjacent side.

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