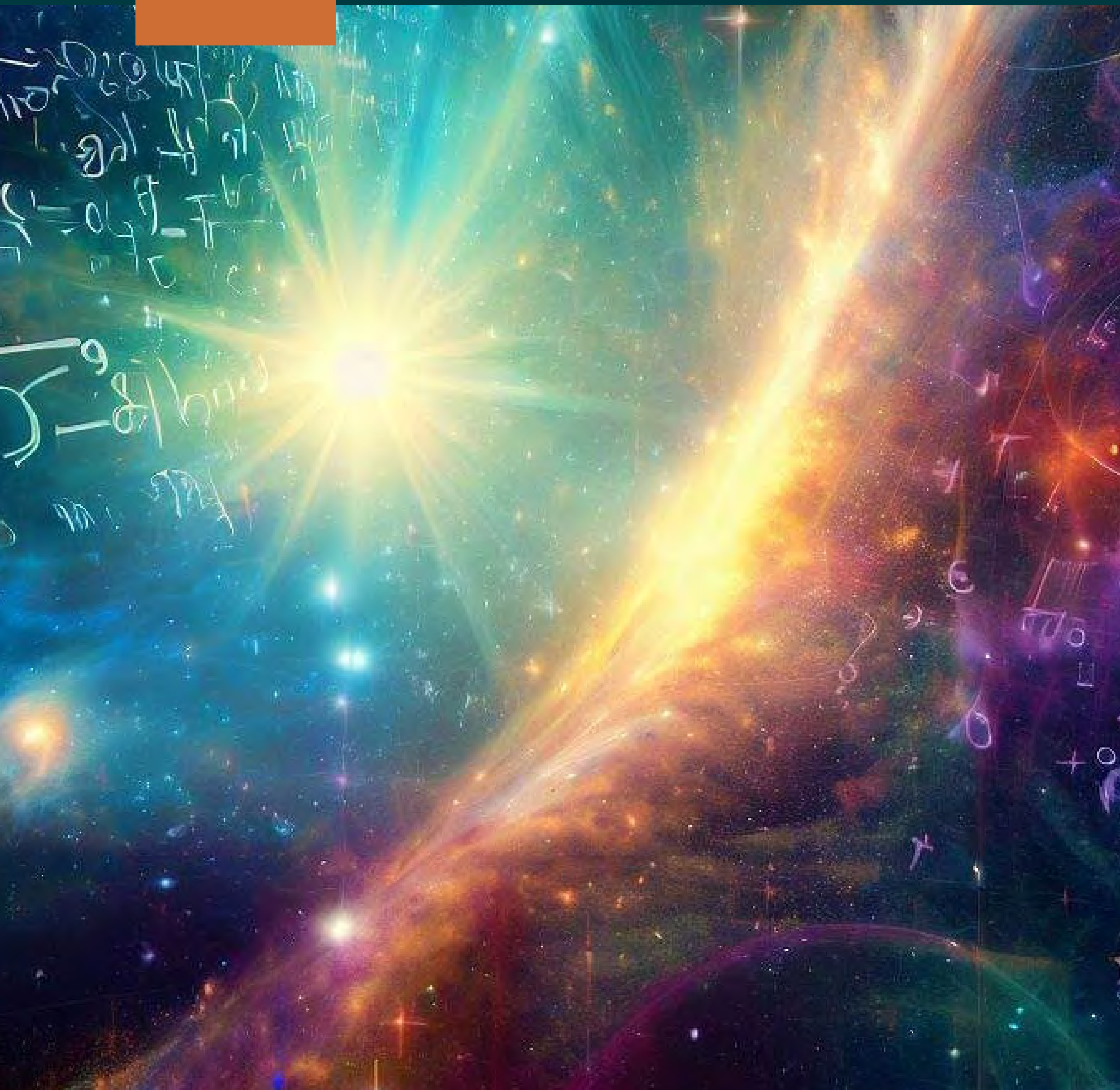
SECTION

8 LIMITS AND DIFFERENTIATION



CALCULUS

Principles of Calculus

INTRODUCTION

A **limit of a function** describes the value that a function approaches as the input gets closer to a particular point. Limits are fundamental in understanding the behavior of functions near specific values. The concept extends to **left-hand limits**(approaching the point from the left) and **right-hand limits** (approaching from the right), which together help determine the overall limit. The properties of limits provide rules for combining limits, such as addition, subtraction, and multiplication, simplifying their computation. Studying the behavior of functions using limits helps identify trends, asymptotes, and potential discontinuities. A function is said to be **continuous** if it has no breaks or gaps in its graph, meaning the limit at any point equals the function's value at that point. Building on limits, the **concept of differentiation** emerges as a way to measure how a function changes. Differentiation begins with finding derivatives from the **first principle**, which involves limits, and extends to using shortcuts like the **power rule** for efficiency, particularly with polynomial functions. Applications of differentiation include determining the **gradient of curves**, which reveals the slope at any point on a graph. This leads to finding the equations of tangents and normals, which are lines closely related to the curve. Additionally, differentiation allows us to calculate the **rate of change** of a function, helping interpret how quantities vary with respect to one another.

At the end of this section, you will be able to:

- Describe and interpret the meaning of limit of a function through graphics and algebraic approaches
- Classify left-hand and right-hand limits algebraically and if possible, with the aid of technology or any creative means
- Distinguish between continuous and discontinuous function near an input value on its domain and investigate them with the use of technology or any other means appropriate
- Use limits of a function to find its derivative

- Use technology or any innovation ways to investigate the rate of change of a function, h(u), with respect to u
- Generalise the behaviour of a moving object along a path or curve
- Use knowledge of differentiation to determine the equation of tangents and normal to curves at a given point
- Apply differentiation to find the rate of change

Key Ideas:

- **Limits**: Limits are the values a function approaches as the input or variable gets close to a given point. It is **denoted** by $\lim_{x \to c} \boldsymbol{f(x) = L}$**.** This means that as x gets closer to c the function $f(x)$ gets closer to L.
- **One-Sided Limits**: Limits can be approached from the left side $f(x) = \lim_{\to C^-} \boldsymbol{f(x)}$ or the right side $f(x) = \lim_{\to C^+} \boldsymbol{f(x)}$
- **Differentiation** is the process of finding the derivative of a function. The derivative measures the rate at which a function's value changes as its input changes. It is fundamentally the slope of the tangent line to the function's graph at any given point.
- **Derivative as a Limit**: The derivative of a function $f(x)$ at a point $x = a$ is defined as: $\frac{dy}{dx} = \lim_{x \to a} \frac{(x + h) - f(x)}{h}$.
- **Rules of Differentiation**:
  - o **Power Rule**: *If* $y = x^n$
  - o $\boldsymbol{\frac{dy}{dx} = nx^{n-1}}$
- **Rate of change** In calculus, the **rate of change** refers to how a quantity changes with respect to another quantity. It is a fundamental concept that helps in understanding how functions behave as their inputs change.
- **Average Rate of Change:** The average rate of change of a function over an interval gives the average amount of change in the function's output (dependent variable) for a unit change in the input (independent variable) over that interval. Mathematically,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- **Instantaneous Rate of Change:** The instantaneous rate of change at a specific point (a) is the rate at which the function is changing at that exact point. This is what the derivative measures.

  Mathematically, $\frac{dy}{dx} = \lim_{x \to a} \frac{f(x+h) - f(x)}{h}$

IDEA OF LIMITS OF A FUNCTION

The concept of limit is the basis for a solid understanding of calculus. For example: frequently, when studying (mostly when trying to sketch) a function, say $f(x)$, we are interested in the function's behavior near a particular point x_0, but not at x_0. For instance, if we seek to evaluate a function at x equals an irrational number, say π (when the value of x can only be approximated), we might evaluate the function at an x-value that is *very close* to our required x i.e., π say 3.14159 instead and thus conclude that the value of $f(\pi)$ is *very close* or *approximately equal to* $f(3.14159)$.

Another situation occurs when trying to evaluate a function at x_0 leads to division by zero, which is undefined. Here is a specific example where we explore numerically how the graph of a function looks near a particular point at which we cannot directly evaluate the function

Example 1

What is the height of the graph of the function $\boldsymbol{g(x) = \frac{x^2 - 2}{x + 2}, at\ x = -2}$

Solution

$g(-2)$ is undefined as the denominator, $x + 2$ will be zero for $x = -2$ so we cannot evaluate the function at $x = -2$ and presume that the result is the height of the graph. However, we can evaluate the function at values of x which are very close to $x = -2$ and predict the height of the graph thus. For example, for some function:

Table 1: The function g(x)

Values of x around $x = -2$	$g(x)$
-2.1	-4.1000
-2.01	-4.0100
-2.001	-4.0010
-2	
-1.999	-3.9990
-1.99	-3.9900
-1.9	-3.9000

From Table 1, it can be observed that the values of $g(x)$ get closer to -4. We can thus assert that the height of the graph of $g(x)$ at $x = -2$ is -4. The graph of $g(x)$ as illustrated in Figure 1 confirms it. There is a hole at $(-2, -4)$ since $g(-2)$ is undefined.

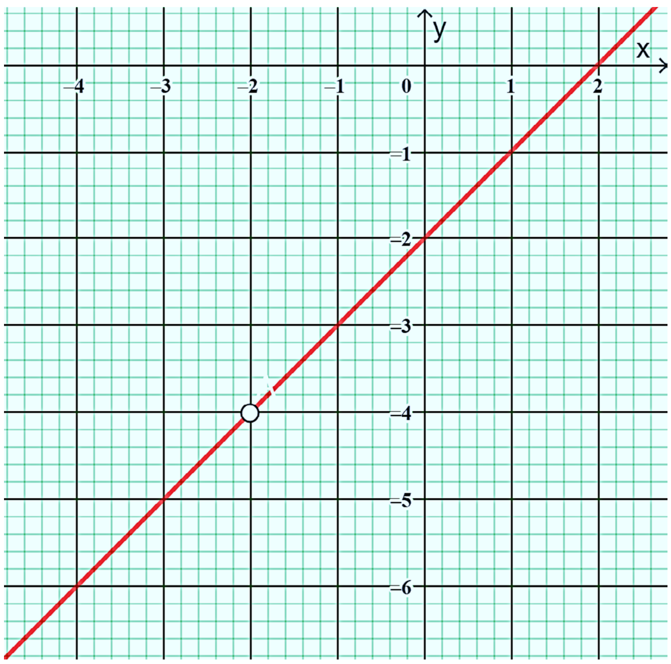


Figure 1: Graph of g(x)

Also, consider a circle C of radius r, its area, A, is πr^2 and circumference, C, is $2\pi r$. One can approximate the area, A and circumference of C, by a region with an area and perimeter respectively that we do know. One approach is to inscribe an equilateral triangle (a regular 3-gon) in C. We add sides, one at a time, to the inscribed figure to create inscribed polygons. The area and circumference of the inscribed figure get closer and closer to the area and circumference of the circle in which it is inscribed respectively as shown in Figure 2: *The area problem*.

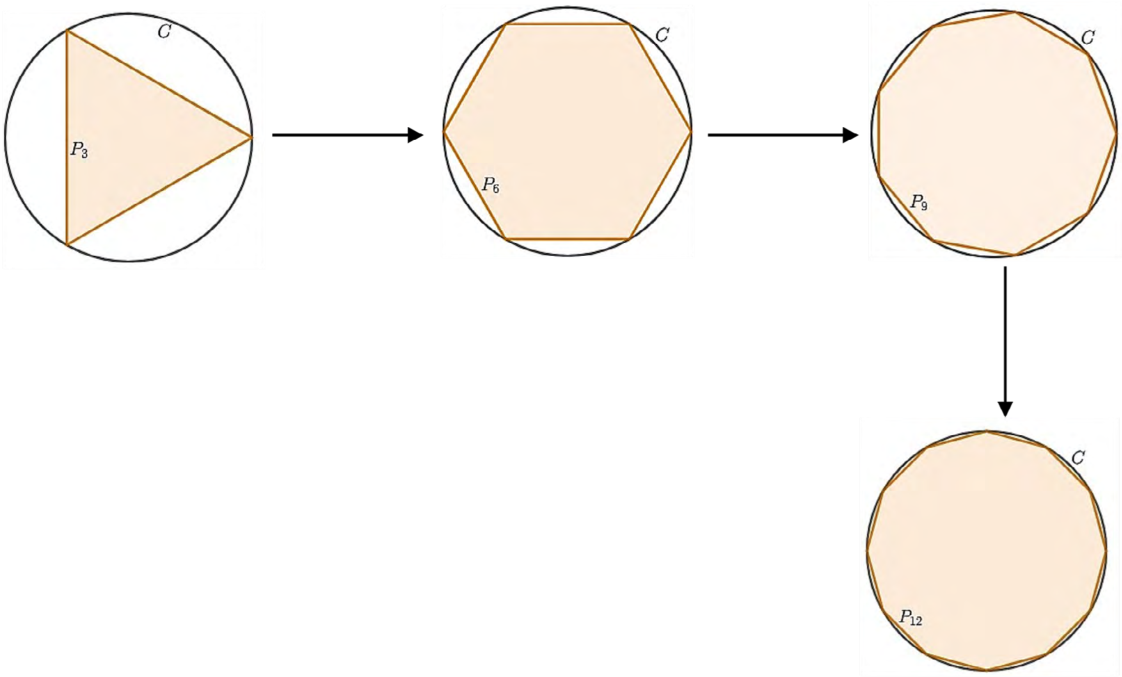


Figure 2: The area problem

The more the number of sides the inscribed polygon, n, becomes, the shorter the sides become. This means as the number of sides, n, gets bigger, the more the inscribed polygon will resemble the circle and the area of the inscribed polygon can be used to approximate the area of the circle just as the perimeter can be used as an approximation for the circumference of the circle. We could also say that as n tends to infinity, the limit of the area, An, inside the polygon, Pn equals the area, A inside the circle and the limit of the perimeter Cn of Pn equals the circumference C of the circle. So, we write this as $A = An$ and $C = Cn$

The statement can also be written as

$\lim_{n \to \infty} A_n = A$ read as *limit of* A_n *as n approaches infinity equals A.*

and $\lim_{n \to \infty} C_n = C$, read as *limit of* C_n *as n approaches infinity equals C.*

Example 2

Given that $f(x) = 2x$, find $\lim_{x \to 2}(f(x))$

Solution

A table of values can be constructed with values of x that are very close to 2 as shown

Table 2: Table of values for f (x) = 2x

x	$f(x)$
2.1	4.2000
2.01	4.0200
2.001	4.0020
↓	
2	
↑	
1.999	3.9980
1.99	3.9800
1.9	3.8000

It can be observed that the values of $f(x)$ approach 4 as the values of x approach 2.

Hence $\lim_{x\to 2}(f(x)) = 4$

LEFT-HAND AND RIGHT-HAND LIMITS

Consider the graph of $f(x) = \frac{x+1}{x-1}$ as shown in Figure 3. The height of $f(x)$ decreases to negative infinity as the graph is traced from the left side of 1 on the $x - axis$ towards $x = 1$ (values of x that approach $x = 1$ from the left) i.e., for values such as 0.9, 0.99, 0.999 and 0.9999 which are less than but very close to 1 as shown in Table 3.

On the other hand, the height of the graph increases infinitely (to positive infinity) as the values of x approach $x = 1$ from the right i.e., for values such as 1.1, 1.01, 1.001 and 1.0001. We say the left-hand limit of $f(x)$ as x approaches 1 is negative infinity and this can be written as $\lim_{x\to 1^-}(f(x)) = -\infty$ and the right-hand limit of $f(x)$ as x approaches $x = 1$ which is written as $\lim_{x\to 1^+}(f(x)) = \infty$

Table 3: Limit of f (x)

x	$f(x)$
0.9	-29.0000
0.99	-299.0000
0.999	-2999.0000
0.9999	-29999.0000
↓	
1	
↑	
1.0001	30001.0000
1.001	3001.0000
1.01	301.0000
1.1	31.00000

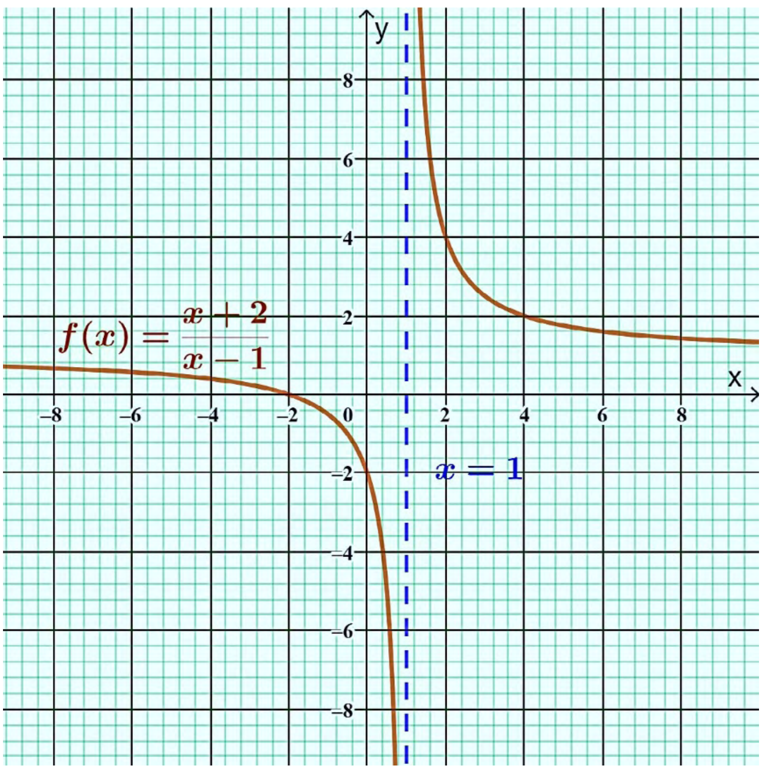


Figure 3: Graph of $f(x) = \frac{x+1}{x-1}$

The left-hand and the right-hand limits are concepts used to describe the behaviour of a function as it approaches a particular point.

Generally, the left-hand limit of a function f (x) as x approaches a point c from the left side (i.e. as x approaches c from the values less than c is denoted as: $\lim_{x \to c^-} (f(x))$

The right-hand limit of a function $f(x)$ as x approaches a point c from the right side (i.e. as x approaches c from the values greater than c is denoted as: $\lim_{x \to c^+} (f(x))$

Existence of a limit

It is important to note that for a limit to exist at a specific point, the left-hand limit and the right-hand limit must be equal.

The limit of a function describes the behaviour of the function as the independent variable approaches a particular value.

The limit of a function, $f(x)$, as x approaches a specific point, c, is the same as the functional value at c and can be obtained when the independent variable x is substituted with the value c in the function $f(x)$.

Example 3

The graph of $k(x)$ is obtained from the combination of the graphs of $f(x)$, $g(x)$, $h(x)$ and $j(x)$ as shown in *Figure 4*.

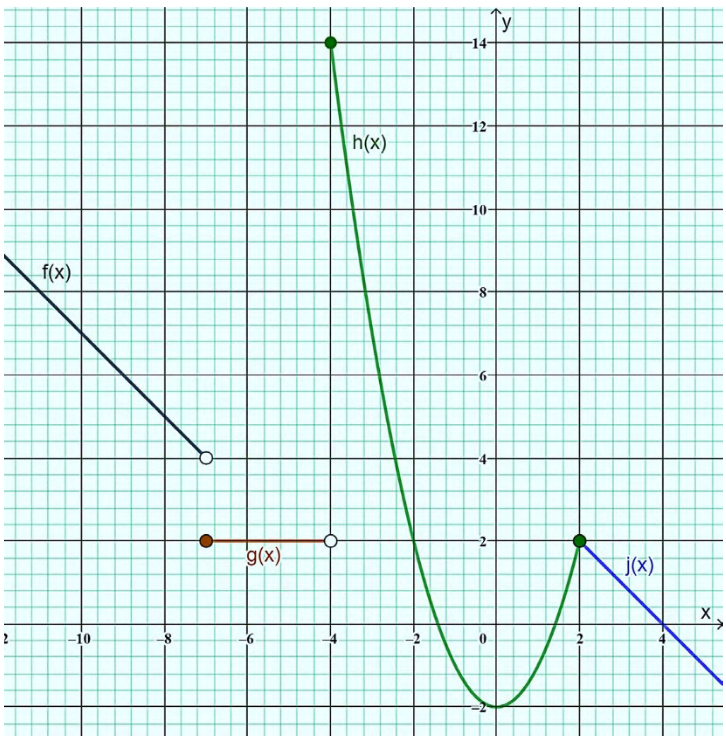


Figure 4: Graph of $k(x)$

The value $k(x)$ approaches when x approaches the number -7 from the left is 4 and from the right is 2. Since the two limits are different, we conclude that the limit does not exist for $k(x)$ as x approaches -7.

Also, $\lim_{x\to-4^-} k(x) = 2$ and $\lim_{x\to-4^+} k(x) = 14$ and since $\lim_{x\to-4^-} k(x) = 2 \neq \lim_{x\to-4^+} k(x) = 14$. $\lim_{x\to-4^-} k(x)$ does not exist.

Contrarily $\lim_{x\to2^-} k(x) = \lim_{x\to2^+} k(x)$ and thus $\lim_{x\to2^-} k(x)$ exist and it has value of 2.

Example 4

What is the meaning of $\lim_{x\to c} f(x) = L$?

Solution

If $f(x)$ is a function, then $\lim_{x\to c} f(x) = L$ means the value of $f(x)$ approaches L as x gets very close to c from the left and right.

Example 5

From Figure 5, what is $\lim_{x\to8} f(x)$

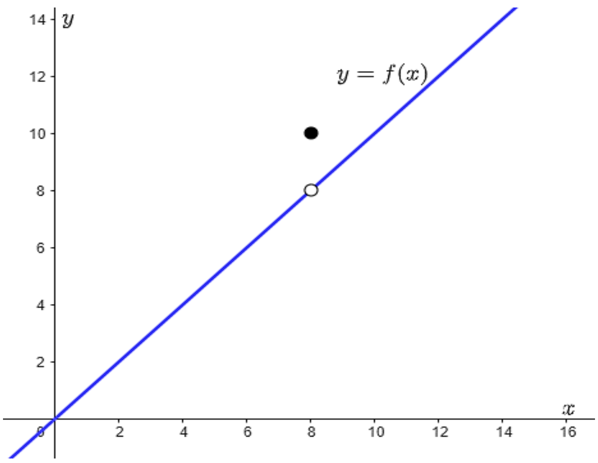


Figure 5: Graph of $\lim_{x\to8} f(x)$

Solution

The function value when $x = 8$, i.e., $f(8) = 10$, however, as x approaches 8 the function approaches 8 and hence $\lim_{x\to8} f(x) = 8$

PROPERTIES OF LIMITS OF A FUNCTION

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g((x))$ both exist, then we have the following results:

1. If k is a constant, then $k \lim_{x\to a} f(x) = \lim_{x\to a}\left(k \times (f(x)\right)$

Example 6

Evaluate $5 \lim_{x\to 2}\left(3x + 8\right) = 5[3(2) + 8] = 5 \times 14 = 70$

2. If r is a positive constant, then $[f(x)]^r = [\lim_{x\to a} f(x)]^r$

3. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Example 7

If $f(x) = x^2$ and $g(x) = 3x + 4$

Evaluate $\lim_{x\to 2}[f(x) + g(x)]$

Solution

$\lim_{x\to 2} x^2 + \lim_{x\to 2} 3x + 4$

$2^2 + 3(2) + 4 = 4 + 6 + 4 = 14 =$

4. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

Example 8

If $f(x) = x^2 + 2$ and $g(x) = 2x - 4$, evaluate $\lim_{x\to 2}[f(x) - g(x)]$

Solution

$\lim_{x\to 2} x^2 + 2 - \lim_{x\to 2} 2x - 4$

$2^2 + 2 - [2(2) - 4] = 4 + 2 - [4 - 4] = 6 - 0 = 6$

5. $\lim_{x\to a}[f(x) \times g(x)] = [\lim_{x\to a} f(x)] \times [\lim_{x\to a} g(x)]$

Example 9

If $f(x) = 2x$ and $g(x) = 2x + 3$, evaluate $\lim_{x\to 3}[f(x) \times g(x)]$

Solution

$\lim_{x\to3}[2x\times(2x+3)] = [\lim_{x\to3}2x\times\left[\lim_{x\to3}(2x+3)\right]$
$2(3)\times([2(3)+3] = 6\times(6+3) = 6\times(6+3) = 6\times9 = 54$

6. If $\lim_{x\to a}g(x)\neq0$, then $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a}f(x)}{\lim_{a\to x}g(x)}$

The following statements about limits should be noted:

1. If $p(x)$ be a polynomial function and a is any real number, then: $\lim_{x\to a}p(x) = p(a)$

2. Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials.

Let c be a real number such that $q(c)\neq0$. Then $\lim_{x\to a}r(x) = r(a)$

Example 10

Given $f(x) = 5x$, $g(x) = 2$ and $h(x) = x^2$ use the limit properties to compute each of the following limits. If it is not possible to compute any of the limits, clearly explain why.

a. $\lim_{x\to5}[f(x) - g(x)]$
b. $\lim_{x\to5}[f(x) + g(x)]$
c. $\lim_{\to5}[f(x) - 5g(x)]$
d. $\lim_{x\to5}[f(x)\times g(x)]$
e. $\lim_{x\to5}[f(x) + g(x) + h(x)]$

Solution

a. $\lim_{x\to5}[f(x) - g(x)] = \lim_{x\to5}f(x) - \lim_{x\to5}g(x)$

$= 5(5) - 2 = 25 - 2 = 23$

b. $\lim_{x\to5}[f(x) + g(x)] = \lim_{x\to5}f(x) + \lim_{x\to5}g(x)$

$= 5(5) + 2 = 25 + 2 = 27$

c. $\lim_{x\to5}[f(x) - 5g(x)] = \lim_{x\to5}f(x) - 5\lim_{x\to5}g(x)$

$= 5(5) - 5(2) = 25 - 10 = 15$

d. $\lim_{x\to5}[f(x)\times g(x)] = \lim_{\to5}f(x)\times\lim_{x\to5}g(x)$

$= 5(5)\times2 = 25\times2 = 50$

e. $\lim\limits_{x\to 5}[f(x) + g(x) + h(x)] = \lim\limits_{x\to 5} f(x) + \lim\limits_{\to 5} g(x) + \lim\limits_{\to 5} h(x)$

$= 5(5) + 2 + 5^2 = 25 + 2 + 25 = 52$

Example 11

Evaluate

a. $\lim\limits_{x\to 0}(2x - 5)(3x - 1)(x - 1)$

b. $\lim\limits_{x\to 4}\frac{2x + 1}{3}$

c. $\lim\limits_{x\to -2}\frac{3x + 6}{x + 2}$

d. $\lim\limits_{x\to 3}\frac{x^2 - 9}{x - 3}$

e. $\lim\limits_{x\to -1}\frac{x^2 - 3x + 2}{x - 1}$

f. $\lim\limits_{x\to 4}\sqrt{16 - x^2}$

g. $\lim\limits_{x\to 1}\left(\frac{4x + 2}{3x^3 + 2x^2 + 1}\right)^3$

Solution

a. $\lim\limits_{x\to 0}(2x - 5)(3x - 1)(x - 1) = \lim\limits_{x\to 0}(2x - 5) \times \lim\limits_{x\to 0}(3x - 1) \times \lim\limits_{x\to 0}(x - 1)$

$[(2(0) - 5] \times [3(0) - 1] \times [0 - 1] = (-5)(-1)(-1) = 5(-1) = -5$

b. $\lim\limits_{x\to 4}\frac{2x + 1}{3} = \frac{\lim\limits_{x\to 4} 2x + 1}{\lim\limits_{x\to 4} 3}$

$= \frac{[2(4) + 1]}{3} = \frac{9}{3} = 3$

c. $\lim\limits_{x\to -2}\frac{3x + 6}{x + 2}$

In solving limits involving rational function, you ought to be cautious about direct substitution as seen in this example. For example, evaluating $\lim\limits_{x\to -2}\frac{3x+6}{x+2}$, we obtain $\frac{3(-2)+6}{-2+2} = \frac{0}{0}$, making it undefined. This must be avoided. To solve such question, we must first simplify the expression before the substitution. i.e.

$\lim\limits_{x\to -2}\frac{3(x + 2)}{x + 2} = 3$

d. $\lim\limits_{x\to 3}\frac{x^2 - 9}{x - 3} = \lim\limits_{x\to 3}\frac{(x - 3)(x + 3)}{x - 3} = \lim\limits_{x\to 3} x + 3$

$= 3 + 3 = 6$

e. $\lim_{x\to -1}\frac{x^2-3x+2}{x-1} = \lim_{x\to -1}\frac{(x-2)(x-1)}{x-1} = \lim_{x\to -1}\left(x-2\right)$

$= -1 - 2 = -3$

f. $\lim_{x\to 4}\sqrt{16-x^2} = \sqrt{16-4^2} = 0$

g. $\lim_{x\to 1}\left(\frac{4x+2}{3x^3+2x^2+1}\right)^3 = \left(\frac{4(1)+2}{3(1)^3+2(1)^2+1}\right)^3 = \left(\frac{6}{3+2+1}\right)^3 = \left(\frac{6}{6}\right)^3 = (1)^3 = 1$

Example 12

Given that $f(x) = 3x + 4$ and k is a constant, verify that for $k = 6$, $a = 1$,

$k \times \lim_{x\to a} f((x)) = \lim_{x\to a} k(\times f(x))$

Solution

$\lim_{x\to a}(f(x)) = \lim_{x\to 1}(3x+4)$

$= 7$

$k \times \lim_{x\to a} f((x)) = 6(7) = 42$

$k \bullet f(x) = 6 \bullet (3x+4)$

$= 18x + 24$

$\lim_{x\to a}(k \times f(x)) = \lim_{x\to 1}(18x+24)$

$= 42$

Hence $k \times \lim_{x\to a}(f((x)) = \lim_{x\to a}(k \times f(x))$

Example 13

Find $\lim_{x\to 4}(1+x^2)$, if it exists.

Solution

$\lim_{x\to 4}(1+x^2) = \lim_{x\to 4} 1 + \lim_{x\to 4} x^2$

$= 1 + 4^2 = 17$

Example 14

Find $\lim_{x\to}\frac{x-1}{2x^2+x-2}$, if it exists.

Solution

$$\lim_{x \to 2}\frac{x-1}{2x^2+x-2} = \frac{\lim_{x \to 2}(x-1)}{\lim_{x \to 2}(x^2+x-2)}$$

$$= \frac{2-1}{2^2+2-2} = \frac{1}{4}$$

Determinate Form

An undefined expression involving some operation between two quantities is said to be in a determinate form if it evaluates to a single number value or infinity.

Indeterminate Form

An undefined expression involving some operation between two quantities is in indeterminate form if it does not evaluate to a single number value or infinity.

The indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, ∞^0,$\infty - \infty$, $1^\infty$and $\infty - \infty$

Example 15

Find the following

a. $\lim_{x \to 0}\left(\frac{\sqrt{x+4}-2}{x}\right)$

b. $\lim_{x \to -2}\left(\frac{x^3+8}{x^2-4}\right)$

Solution

Note that applying the properties i.e., evaluating the function at $x = 0$, may yield indeterminate form thus: $\frac{\sqrt{x+4}-2}{x} = \frac{0}{0}$ but rationalising the numerator makes it determinate

a. $$\frac{\sqrt{x+4}-2}{x} \bullet \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \frac{x}{x(\sqrt{x+4+2})}$$

$$= \frac{1}{\sqrt{x+4}+2}$$

$$\lim_{x \to 0}\left(\frac{\sqrt{x+4}-2}{x}\right) = \lim_{x \to 0}\left(\frac{1}{\sqrt{x+4}+2}\right)$$

$$= \frac{1}{\sqrt{0+4+2}} = \frac{1}{4}$$

b. Similarly, evaluating $\frac{x^3+8}{x^2-4}$ at $x = -2$ yields an indeterminate form $\left(\frac{0}{0}\right)$

Since both the numerator and denominator are factorable polynomial expressions, the expression can be simplified thus

$$\frac{x^3+8}{x^2-4} = \frac{(x+2)(x^2-2x+4)}{(x-2)(x+2)}$$

$$= \frac{(x^2-2x+4)}{(x-2)}$$

$$\lim_{x\to-2}\left(\frac{x^3+8}{x^2-4}\right) = \lim_{x\to-2}\left(\frac{(x^2-2x+4)}{(x-2)}\right)$$

$$= -\frac{12}{4} = -3$$

BEHAVIOUR OF FUNCTIONS

Limit at infinity

Consider the graph of $f(x) = \frac{x+1}{x-1}$, $x \neq 1$ as shown in Figure 6 and some points that lie on the graph in Table 4. As the values of x get negatively bigger, i.e., approaches $-\infty$, the value of $f(x)$ increases towards 1. Of course, since the inverse of $f(x)$ is $f^{-1}(x) = \frac{x+1}{x-1}$, $x \neq 1$, the height of $f(x)$ cannot be 1 hence there is a horizontal asymptote at $y = 1$ and thus, no matter how negatively big the value of x will be, $f(x)$ can only get closer 1. We say then, that the limit of $f(x)$ as x approaches negative infinity (our idea of a very big number), written as $\lim_{x\to-\infty}(f(x))$ is 1.

Likewise, as the value of x get positively larger, the value of $f(x)$ and in effect the height of its graph decreases to 1 and thus, $\lim_{x\to\infty}(f(x)) = 1$

Table 4: Table of values for the function

x	$f(x)$
$-\infty$	1
↑	↑
-1000	0.998001998
-100	0.980198020
-10	0.818181818
-1	0.000000000
10	1.222222222

x	$f(x)$
100	1.020202020
1000	1.002002002
1000	1.002002002
↓	↓
∞	1

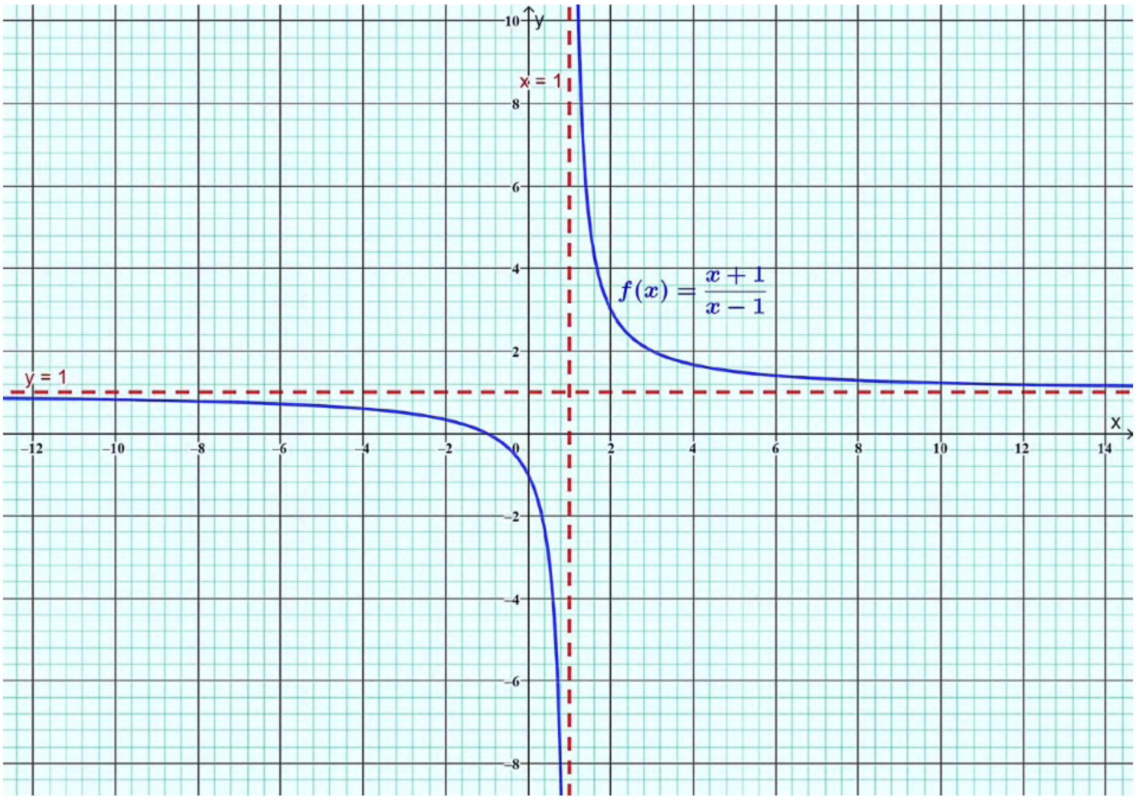


Figure 6: $f(x) = \frac{x+1}{x-1}$,

The idea of limits at infinity can be depended upon to predict the nature or behaviour of functions for extreme values of the independent variables which may be difficult or impractical to evaluate and thus make sketching of such functions possible.

It must be noted that infinity is not a real number. It is only an idea of a certain very huge number. This understanding of infinity helps us to evaluate limits of functions (without having to graph the functions or evaluate functions at very huge values for the independent variable).

Example 16

$\lim_{x \to \infty} (2x^4 - x^2 - 8x)$

Solution

Let $g(x) = 2x^4 - x^2 - 8x$

$g(x)$ is a fourth-degree polynomial function and since the leading term ($2x^4$) has a positive coefficient, the bigger the value of x, the bigger the value of y will be too. This is confirmed by the shape of the graph of the function: an expected ∪ -shape as shown in Figure 7. The height increases as the graph is traced from the turning point towards the right. From these analyses, it can be concluded that $\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \infty$

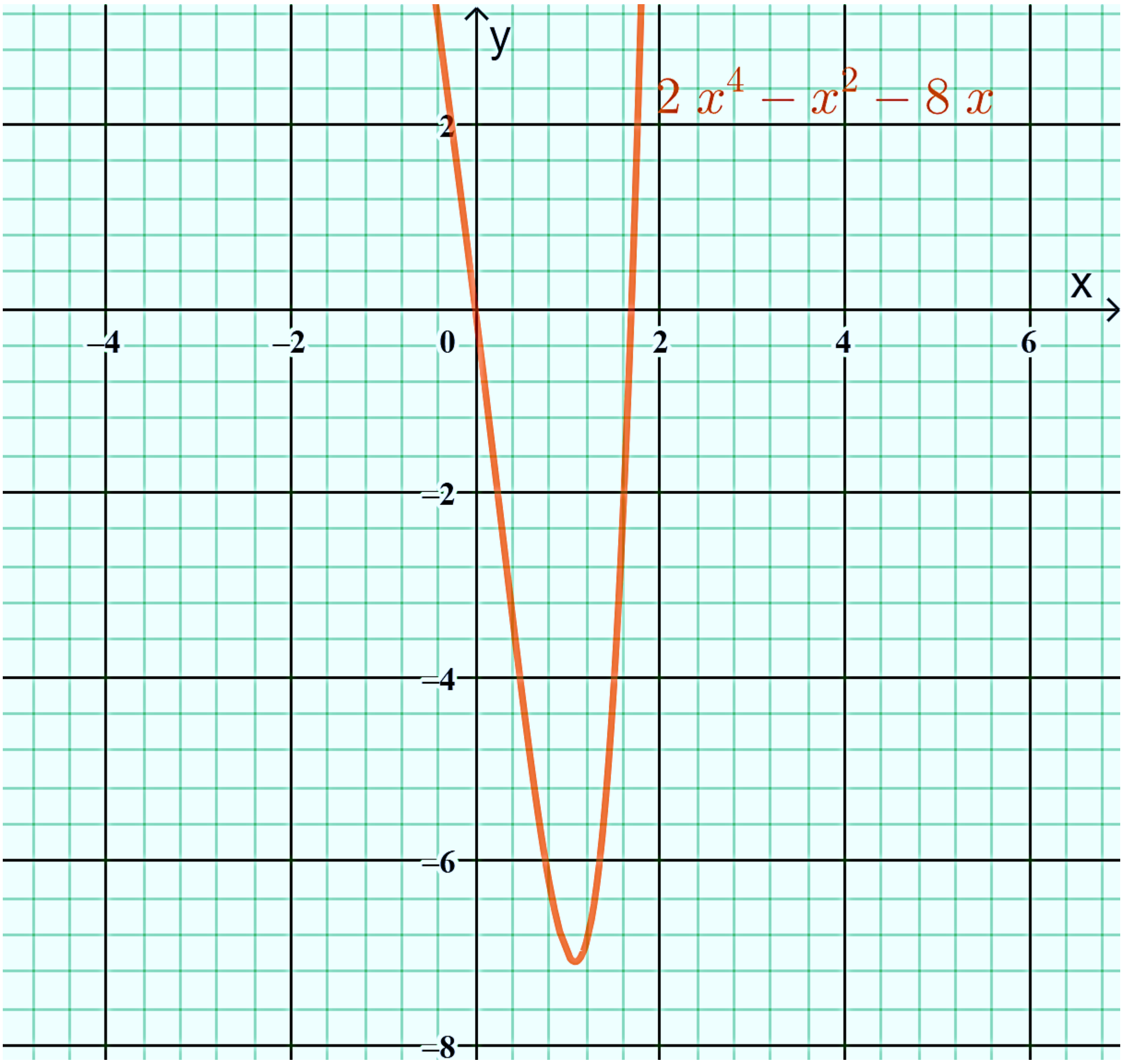


Figure 7: Graph of $\lim_{x \to \infty} (2x^4 - x^2 - 8x)$

Algebraically

"Twice the fourth power of a very big number" i.e., ∞ into $2x^4$ is still a big number, say ∞

"The square of a very big number" i.e., ∞ into x^2 is still a big number (even though, it will be smaller than that for $2x^4$), say ∞

"Eight times a very big number is also big

Consequently, performing the combined operation $2x^4 - x^2 - 8x$ on that very big number still results in a big number, ∞ and hence, it can be concluded that $\lim_{x \to \infty}(2x^4 - x^2 - 8x) = \infty$

Example 17

Evaluate $\lim_{x \to \infty}\left(\frac{x+1}{x-1}\right)^n$

Solution

$\frac{x+1}{x-1}$ is a rational expression and so the algebraic analysis done in example 1 above does not necessarily apply.

$\frac{x+1}{x-1}$ remains the same if it is multiplied by 1 in the form,$\frac{1}{x} \div \frac{1}{x}$ the result of that division is

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \lim_{x \to \infty}\left(\frac{x+1}{x-1}\right) = \lim_{x \to \infty}\left(\frac{1+\frac{1}{x}}{x-\frac{1}{x}}\right) = \left(\frac{\lim_{x \to \infty}\left(1+\frac{1}{x}\right)}{\lim_{x \to \infty}\left(1-\frac{1}{x}\right)}\right)$$

As the value of x becomes bigger,the the value of $\frac{1}{x}$ becomes smaller i.e., as:

$$x \to \infty, \frac{1}{x} \to 0 \textit{ and thus } \lim_{\to \infty}\left(1+\frac{1}{x}\right) = 1 \textit{ and } \lim_{x \to \infty}\left(1-\frac{1}{x}\right) = 1$$

$$\textit{Hence } \lim_{x \to \infty}\left(\frac{x+1}{x-1}\right) = \frac{1}{1} = 1.$$

Example 18

Find the $\lim\limits_{x\to\infty} \dfrac{x^2+1}{3x^3-4x+5}$ if it exists.

Solution

$$\lim_{x\to\infty} \frac{x^2+1}{3x^3-4x+5} = \lim_{\to\infty} \frac{\frac{x^2+1}{x^3}}{3x^3-4x+5}$$

$$= \frac{\lim\limits_{x\to\infty}\left(\frac{1x^3}{x}+\frac{1}{x^3}\right)}{\lim\limits_{x\to\infty}\left(3-\frac{4}{x^2}+\frac{5}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^3}}{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{4}{x^2}+\lim\limits_{x\to\infty}\frac{5}{x^2}}$$

$$= \frac{0+0}{3-0+0} = 0$$

Example 19

Find the $\lim\limits_{x\to\infty} \dfrac{x^3+1}{3x^3-4x+5}$ if it exists.

$$= \lim_{x\to\infty} \frac{x^3+1}{3x^3-4x+5}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3+1}{x^3}}{\frac{3x^3-4x+5}{x^3}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(\frac{x^3}{x^3}+\frac{1}{x^3}\right)}{\lim\limits_{x\to\infty}\left(3-\frac{4}{x^2}+\frac{5}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to\infty}\left(1+\frac{1}{x^3}\right)}{\lim\limits_{x\to\infty}\left(3-\frac{4}{x^2}+\frac{5}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x^3}}{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{4}{x^2}+\lim\limits_{x\to\infty}\frac{5}{x^3}}$$

$$= \frac{1+0}{3-0+0} = \frac{1}{3}$$

Example 20

Evaluate $\lim\limits_{x\to\infty}\dfrac{3x}{x+2}$

$$=\lim_{x\to\infty}\frac{\frac{3x}{x}}{\frac{x}{x}+\frac{2}{x}}$$

$$=\lim_{x\to\infty}\frac{3}{1+\frac{2}{x}}=\frac{3}{1+0}=3$$

Continuity of Functions

A function f is continuous at a point "a" if $\lim\limits_{x\to a}f(x)=f(a)$

Theorem:

If $f(x)$ is continuous at "$x=a$", then the following three conditions hold;

a. $f(a)$ is defined,

b. $\lim\limits_{x\to a}f(x)$ must be defined (thus the left and right limits must be same) and

c. $f(x)=f(a)$

Example 21

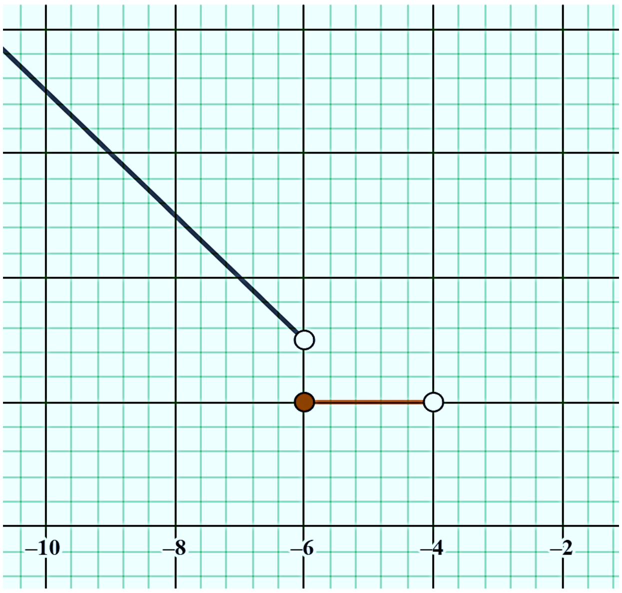


Figure 8: Graph of $h(x)$

Provided that a function, $h(x)$ is defined as

$$h(x)=\begin{cases}-x-3, \; when -\infty < x \le -6 \\ 2, \; when -6 \le x < -4\end{cases}$$

determine, with the aid of the graph in Figure 8, the interval(s)/point(s) for which $h(x)$ is (are) continuous or discontinuous.

Solution

The function $h(x)$ is continuous on the intervals, $-\infty < x < -6$ and $-6 \leq x < -4$ which can also be written as $(-\infty\ , -6)$ and $(-6, -4)$ respectively. $h(x)$ is however discontinuous at $x = -6$ because there is a jump there. In the interval $-4 \leq x < \infty$, the graph discontinues since the graph does not exceed $x = -4$.

Example 22

Determine whether the following functions are continuous at $x = 2$. Why?

a) $f(x) = 3x + 1$

b) $g(x) = \frac{x-4}{x^2-4}$

Solution

a) $f(x)$ is a polynomial function and hence is continuous for all real numbers including 2 and hence, $f(x)$ is continuous at $x = 2$

b) $g(x)$ is a rational function defined for all real numbers except values of x for which $x^2 - 4$ equals zero. When $x = -2$ or $x = 2$, $x^2 - 4 = 0$ and hence $g(2)$ is undefined and $g(x)$ is not continuous at $x = 2$

Example 23

Determine the temperature of a room after 2 hours given the relation,

$f(t) = 20 + 5t - 0.1t^2$

Solution

$\lim_{t\to 2}(20 + 5t - 0.1t^2) = 20 + 5(2) - 0.1(2)^2 = 29.6$

Example 24

A particle moves in a straight line in such a way that its distance s meters from a fixed point O is given by $s(t) = 2t^3 - 5t^2 + 3t + 1$. Find the particles distance travelled when t = 4.

Solution

$\lim_{t\to 4}\left(2t^3 - 5t^2 + 3t + 1\right) = 2(4)^3 - 5(4)^2 + 3(4) + 1 = 61m$

Example 25

Determine whether the following function $h(x) = \dfrac{4}{x^2 + 3x + 2}$ is continuous at $x = -2$

Solution

$h\,(x)$ is a rational function defined for all real numbers except values of x for which

$x^2 + 3x + 2 = 0$. Hence, $h(-1)$ *and* $h(-2)$ is undefined and $h\,(x)$ is not continuous. Therefore, is not continuous when $x = -2$.

THE CONCEPT OF DIFFERENTIATION

Welcome to another interesting area of calculus; ***differentiation***.

Have a look at Dede's salary below in both 2005 and 2023. Has Dede's salary increased over those years? By how much? Can you work out the rate of increase in Dede's salary, i.e., how much her salary has increased year by year, assuming a constant rate of increase?

(1 bundle of GHC20 note)
Salary in the year 2005

(2 bundles of GHC 50 note)
Salary in the year 2023

Figure 9: Dede's salary from the year 2005 to the year 2023.

Changes (increase or decrease) occur normally in our daily lives. These can be anything from changes in population, salary, cost of living, the speed of a car etc. The changes that occur in these variables (population, salary, cost of living, speed) happen because of changes in other variables such as number of years, time, area etc. If the population of a school increases every year, we can say the population (P) depends on the number of years (t). Mathematically, this can be written as $P = f(t)$. P is called the dependent variable and t is the independent variable.

Differentiation is the study of the rate of change of a function with respect to one of its variables. It is a crucial idea in calculus, which is a branch of mathematics that deals with the study of continuous change. It is applied in various fields of study including Economics, Physics, Computer Science, Finance, etc.

Think back to our work on straight lines under section 5. The gradient or slope of a straight line is used to find the ratio of the rise (the change in the height of the graph) to the run (the corresponding change in the horizontal), or the rate of change of the rise with respect to the run; i.e. $m = \frac{y_2 - y_1}{x_2 - x_1}$. This principle will be used to find the gradient of non-linear functions.

Consider the problem of finding the gradient of a non-linear function whose graph is shown below.

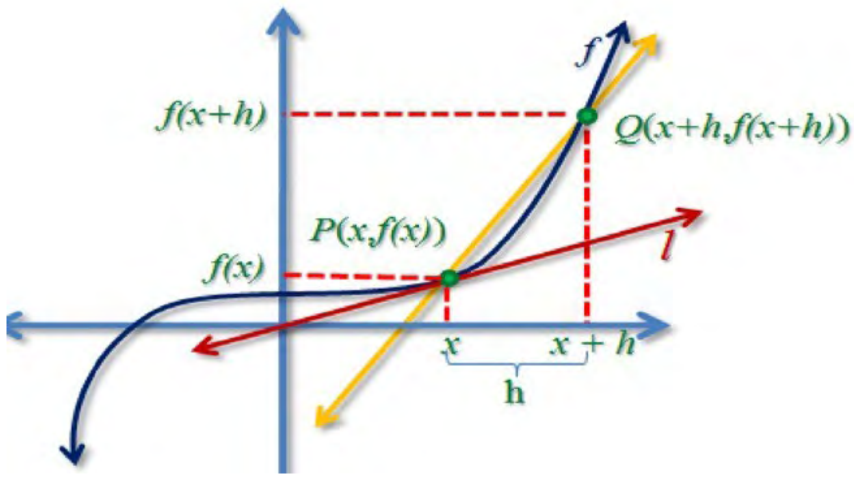


Figure 10: Gradient of a non-linear function

To find the gradient of the curve,

1. Take the coordinate of P to be $P(x,f(x))$.
2. Take the change in the x variable going from P to Q to be h,
3. Take the coordinate of Q to be $Q(x + h, f(x + h))$
4. Find the gradient of the straight line that passes through points P and Q. This is known as a secant line, which is a line which intersects a curve or circle at two different points.

The Gradient of the secant line PQ; $m = \frac{f(x + h) - f(x)}{(x + h) - (x)}$

Unfortunately, as you can see from the graph, the line PQ does not coincide perfectly with the graph of f(x). This means that its gradient will not be a good approximation for the gradient of the curve at that point. We can make a better approximation by decreasing the value of h, which serves as the difference in the two x-coordinates of the points selected for the approximation i.e., P and Q, gradually until we get P and Q to be so close to each other that their coordinates are approximately equal and thus, the gradient obtained will be a sufficient approximation of the gradient of $f(x)$ at $P(x, f(x))$. We are doing this by introducing the limit, $h \to 0$, ie:

$$\text{Gradient} = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{x + h - x} = \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right)$$

It can be inferred that the value for the gradient that will be obtained at different points on the curve will also be different since the tangent line to the curve at different points on the curve will have different slopes. The new function that models the gradient of the curve at points on the curve is called the *derivative* of the function. This new function, as it is derived from $f(x)$, can be written as $f'(x)$ or $\frac{dy}{dx}$ as it is a ratio of the change in y values, representing the change in the height (Δy), to the change in the x values, representing the horizontal change (Δx) or $\frac{d}{dx}(f(x))$ as differentiation can be thought of as an operator. It can also be called the instantaneous rate of change of f at x.

Note: *It is important to note that a function can only be differentiable at a point if it is continuous at that point and the limit of the function as x approaches that point exists*

DIFFERENTIATING POLYNOMIAL FUNCTIONS FROM FIRST PRINCIPLE

Using this method to find gradients (or the derivative of a function) is called differentiating from first principles. Here is a step-by-step guide to doing this:

(1) Write $y = f(x)$

(2) Find $f(x + h)$

(3) Obtain $f(x + h) - f(x)$

(4) Evaluate $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

(5) Find the limit as $h \to 0$

Example 26

Find, from first principles, the derivatives of the following functions

a. $f(x) = 2x$

b. $f(x) = x^2$

c. $g(x) = 3x^3$

d. $t(x) = c$

e. $f(x) = 2x^2 - 4x + 5$

f. $t(x) = \frac{5}{x^2}$

Solution

a. Let $y = f(x) = 2x$

$f(x + h) = 2(x + h) = 2x + 2h$

$f(x + h) - f(x) = 2x + 2h - (2x) = 2h$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2h}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} (2)$$

$$\boldsymbol{\frac{dy}{dx}} = 2$$

b. Let $y = f(x) = x^2$

$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$

$f(x + h) - f(x) = x^2 + 2xh + h^2 - (x^2) = 2xh + h^2$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} 2x + h$$

But $h \to 0$

$$\frac{dy}{dx} = \lim_{h \to 0} 2x + 0$$

$$\boldsymbol{\frac{dy}{dx} = 2x}$$

c. Let $y = g(x) = 3x^3$

$g(x+h) = 3(x+h)^3 = 3(x^3 + 3x^2h + 3xh^2 + h^3)$

$= 3x^3 + 9x^2h + 9xh^2 + 3h^3$

$g(x+h) - g(x) = 3x^3 + 9x^2h + 9xh^2 + 3h^3 - (3x^3)$

$= 9x^2h + 9xh^2 + h^3$

$g(x+h) - g(x) = 9x^2h + 9xh^2 + 3h^3$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{9x^2h + 9xh^2 + 3h^3}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{h(9x^2 + 9xh + 3h^2)}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} (9x^2 + 9xh + 3h^2)$

But $h \to 0$

$\frac{dy}{dx} = \lim_{h\to 0} 9x^2 + 9x(0) + 3(0)^2$

$\mathbf{\frac{dy}{dx} = 9x^2}$

d. Let $y = t(x) = c$

$t(x+h) = c$

$t(x+h) - t(x) = c - c = 0$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{t(x+h - t(x)}{h}$

$\frac{dt}{dx} = \lim_{h\to 0} \frac{0}{h}$

$\mathbf{\frac{dy}{dx} = 0}$

e. Let $y = f(x) = 2x^2 - 4x + 5$

$f(x+h) = 2(x+h)^2 - 4(x+h) + 5 = 2(x^2 + 2xh + h^2) - 4(x+h) + 5$

$= 2x^2 + 4xh + h^2 - 4x - 4h + 5$

$f(x+h) - f(x) = 2x^2 + 4xh + h^2 - 4x - 4h + 5 - (2x^2 - 4x + 5)$

$f(x+h) - f(x) = 2x^2 + 4xh + h^2 - 4x - 4h + 5 - 2x^2 + 4x - 5)$

$f(x+h) - f(x) = 4xh + h^2 - 4h$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{4xh + h^2 - 4h}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{h(4x + h - 4)}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} 4x + h - 4$

But $h \to 0$

$$\frac{dy}{dx} = \lim_{h\to 0} 4x + 0 - 4$$

$$\boldsymbol{\frac{dy}{dx} = 4x - 4}$$

f. Let $y = t(x) = \frac{5}{x^2}$

$$t(x+h) = \frac{5}{(x+h)^2} = \frac{5}{x\,2 + 2xh + h^2}$$

$$t(x+h) - t(x) = \frac{5}{x^2 + 2xh + h^2} - \frac{5}{x^2} = \frac{5x^2 - 5x^2 - 10xh - 10h^2}{x^2\left(x^2 + 2xh + h^2\right)}$$

$$= \frac{-10xh - 10h^2}{x^2\left(x^2 + 2xh + h^2\right)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{t(x+h) - t(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{-10xh - 10h^2}{x^2(x^2 + 2xh + h^2)}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{-10xh - 10h^2}{x^2\left(x^2 + 2xh + h^2\right)} \times \frac{1}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{-h(10x - 10h)}{x^2\left(x^2 + 2xh + h^2\right)} \times \frac{1}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{-(10x - 10h)}{x^2\left(x^2 + 2xh + h^2\right)}$$

But $h \to 0$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{-10x - 10(0)}{x^2\left(x^2 + 2x(0) + 0^2\right)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{-10x}{x^4}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{-10}{x^3}}$$

Example 27

A company manufactures bottled water and the cost C in GHC of producing x bottled water per day is given by the function: $C(x) = 5x + 10$. Help the company to determine the cost changes as production increases.

Solution

Let $y = f(x) = 5x + 10$

$f(x+h) = 5(x+h) = 5x + 5h + 10$

$f(x+h) - f(x) = 5x + 5h + 10 - (5x + 10) = 5h$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{5h}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} 5$$

$$\frac{dy}{dx} = 5$$

This means that for every additional bottled water produced, the cost increases by GH¢ 5.

DIFFERENTIATING FUNCTIONS USING THE POWER RULE

There is a way to determine the derivative (gradient of the curve) other than finding it from first principles. Let us go through the steps below to enable the use of the power rule to find the derivative of functions.

Given that $f(x)$ is a polynomial function of degree n say $f(x) = x^n$, differentiating from first principles requires that $f(x + h)$ be found.

$f(x+h) = (x+h)^n$

From the binomial expansion, we have:

$$f(x+h) = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + h^n$$

$$f(x+h) - f(x) = nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + h^n$$

$$\frac{f(x+h) - f(x)}{h} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots + h$$

$$\frac{dy}{dx} = \lim_{h \to 0}\left(\frac{fx + h - fx}{h}\right)$$

$$= \lim_{h \to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots + h\right)$$

$$= nx^{n-1}$$

Generally, given a polynomial function of degree n, $f(x) = x^n$, $\frac{dy}{dx} = f'(x) = nx^{n-1}$.

Note, that the derivative of a constant is 0. i.e. $\frac{dy}{dx}(5) = 0$.

In summary, given a polynomial function of degree n, for example, $f(x) = x^n$, the derivative of such functions is obtained as follows:

1. Bring the exponent of the independent variable in a term down
2. Multiply it by the original term
3. Reduce the original exponent by one.
4. Repeat the process for all the terms if there is more than one term.
5. Combine all terms, and that represents the derivative of the function

Go through the following example using the steps above.

Example 28

Find the derivative of the following functions:

a. $y = 4x^3$

b. $y = 20x^{-3}$

c. $y = 2x^2 + 5x - 9$

d. $y = x^4 + 3x^3 + 5x^2$

e. $y = \frac{7}{x^3}$

f. $y = 6x^{\frac{3}{4}}$

g. $y = -5 + 3x - \frac{7}{2}x^2 - 4x^3$

h. $y = 2\sqrt{x}$

Solution

a. $y = 4x^3$

Bring the exponent of x down and multiply it by the given term.

$\frac{dy}{dx} = 3 \times 4x^3$

Subtract or reduce the power by 1

$\frac{dy}{dx} = 3 \times 4x^{3-1}$

$\frac{dy}{dx} = 3 \times 4x^2$

Simplify to obtain your derivative, $\frac{dy}{dx}$

$\boldsymbol{\frac{dy}{dx} = 12x^2}$

b. $y = 20x^{-3}$

$\frac{dy}{dx} = -3 \times 20x^{-3-1}$

$\mathbf{\frac{dy}{dx} = -60x^{-4}}$

c. $y = 2x^2 + 5x - 9$

Take note that the constant can be written as $-9x^0$

$y = 2x^2 + 5x - 9x^0$

$\frac{dy}{dx} = 2 \times 2x^{2-1} + 5 \times x^{1-1} - 9 \times 0 \times x^0$

$\frac{dy}{dx} = 2 \times 2x^1 + 5 \times x^0 - 9 \times 0 \times x^0$

$\mathbf{\frac{dy}{dx} = 4x + 5}$

d. $y = x^4 + 3x^3 + 5x^2$

$\frac{dy}{dx} = 4 \times x^{4-1} + 3 \times 3x^{3-1} + 2 \times 5x^{2-1}$

$\mathbf{\frac{dy}{dx} = 4x^3 + 9x^2 + 10x}$

e. $y = \frac{7}{x^3}$

From our lesson on indices, we can write:

$y = \frac{7}{x^3} = 7x^{-3}$

$y = 7x^{-3}$

$\frac{dy}{dx} = -3 \times 7x^{-3-1}$

$\frac{dy}{dx} = -3 \times 7x^{-3-1}$

$\mathbf{\frac{dy}{dx} = -21x^{-4}}$

f. $y = 6x^{\frac{3}{4}}$

$\frac{dy}{dx} = \frac{3}{4} \times 6x^{\frac{3}{4}-1}$

$\frac{dy}{dx} = \frac{3}{2} \times 3x^{-\frac{1}{4}}$

$\mathbf{\frac{dy}{dx} = \frac{9}{2}x^{-\frac{1}{4}}}$

g. $y = -5 + 3x - \frac{7}{2}x^2 - 4x^3$

$\mathbf{\frac{dy}{dx} = 3 - 7x - 12x^2}$

h. $y = 2\sqrt{x}$

$\frac{dy}{dx} = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 2x^{\frac{1}{2}-1}$

$\frac{dy}{dx} = x^{-\frac{1}{2}}$

$\boldsymbol{\frac{dy}{dx} = \frac{1}{\sqrt{x}}}$

GRADIENT OF CURVES

We have learnt that the derivative of a function which can be graphed on a plane represents the slope or gradient of the tangent line to the curve at points that lie on the curve.

In addition, the value of the gradient of a linear graph (straight line) gives information on the behaviour of the line (whether it is increasing or decreasing). We will now apply that knowledge to predict the behaviour of the graph of curves for functions, for some intervals of the independent variable.

To predict the behaviour of the graph of curves for functions:

1. Find the derivative of the given function $f(x)$ to obtain $f'(x)$
2. Take notice of the value of $f'(x)$, if
   - **a)** $f'(x) > 0$ (positive value), then $f(x)$ increases for all values of x
   - **b)** $f'(x) < 0$ (negative), then $f(x)$ decreases for all values of x
   - **c)** $f'(x) = 0$, then $f(x)$ is momentarily at rest for all values of x

Example 29

Determine the nature of the following functions as either increasing, decreasing or momentarily at rest

a. $f(x) = -3x + 14$

b. $f(x) = 10x - 5$

c. $f(x) = 20$

Solution

a. $f(x) = -3x + 14$

$f'(x) = -3$

$f'(x) = -3 < 0$

$f'(x) < 0$ the function $f(x)$ decreases

b. $f(x) = 10x - 5$

$f'(x) = 10$

$f'(x) = 10 < 0$

$f'(x) > 0$, the function $f(x)$ increases

c. $f(x) = 20$

$f'(x) = 0$

$f'(x) = 0$, then $f(x)$ is momentarily at rest. We can infer that the graph of $f(x)$ is a horizontal line as horizontal lines have their gradients to be 0. As the value of x changes, there is no change in the y values

To determine the nature of functions other than linear functions;

1. Find the derivative, $f'(x)$
2. Solve for $f'(x) = 0$
3. Substitute the value(s) for $f'(x) = 0$ into the given function $f(x)$
4. Use the following to determine whether the function $f(x)$ increases, decreases or momentarily at rest
   - $f'(x) > 0$ (positive value), then $f(x)$ increases for all values of x
   - $f'(x) < 0$ (negative), then $f(x)$ decreases for all values of x
   - $f'(x) = 0$, then $f(x)$ is momentarily at rest for all values of x

Example 30

Find the value(s) of x for which $f(x) = 3x^2 - 12x$ increases or decreases.

Solution

Find the derivative

$f'(x) = 6x - 12$

Solve $f'(x) = 0$

$6x - 12 = 0$

$x = 2$

Substitute $x = 2$ into $f(x)$:

$f(2) = -12$

Therefore, the graph is momentarily at rest at (2, -12) and the graph is decreasing $(-\infty, 2)$ and increasing at $(2, +\infty)$

Example 31

Find the value(s) of x for which $f(x) = x^2 + 5x - 3$ increases or decreases

Solution

Step 1. Differentiate the given function.

$f(x) = x^2 + 5x - 3$

$f'(x) = 2x + 5$

Step 2. Find the values of x for which $f(x)$ is momentarily at rest,

$f'(x) = 0$

$2x + 5 = 0$

$x = -\frac{5}{2}$

Hence at $x = -\frac{5}{2}$, $f(x)$ is momentarily at rest.

Step 3. Use the values of x in step 2 to partition the real number line to obtain a real interval.

Partitioned intervals: $(-\infty, -\frac{5}{2})$, $(-\frac{5}{2}, +\infty)$

Decreases at $(-\infty, -\frac{5}{2})$, and increases at $(-\frac{5}{2}, +\infty)$

Example 32

Find the value(s) of x for which $f(x) = 2x^3 - 3x^2 - 36x + 7$ increases, decreases or is momentarily at rest

Solution

Step 1. Differentiate the given function.

$f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36$

Step 2. Find the values of x for which f is momentarily at rest,

$f'(x) = 0$

$6(x^2 - x - 6) = 0$

Factorising:

$(x - 3)(x + 2) = 0$

$(x - 3) = 0 \text{ or } (x + 2) = 0$

$x = 3 \text{ or } x = -2$

Hence at $x = -2$ or $x = 3$, $f(x)$ is momentarily at rest.

Step 3. Use the values of x from step 2 to partition the real number line to obtain a real interval. Note that values that are momentarily at rest should not be part of the intervals constructed.

Partitioned intervals: $(-\infty\ , -2)$, $(-2, 3)$ $(3, +\infty)$

Table 5: Table of values of $f(x)$

Interval	Test value	Derivative	Conclusion
$(-\infty\ , -2)$,	-3	$6(3)^2 - 6(-3) - 36 = 36 > 0$	Increasing
$(-2, 3)$	1	$6(1)^2 - 6(1) - 36 = -36 < 0$	Decreasing
$(3, +\infty)$	4	$6(4)^2 - 6(4) - 36 = 36 < 0$	Increasing

Hence, $f(x)$ is strictly increasing at $(-\infty, -2)$, $(3, +\infty)$ and strictly decreasing at $(-2, 3)$ as can be seen from the graph of $f(x) = 2x^3 - 3x^2 - 36x + 7$ below.

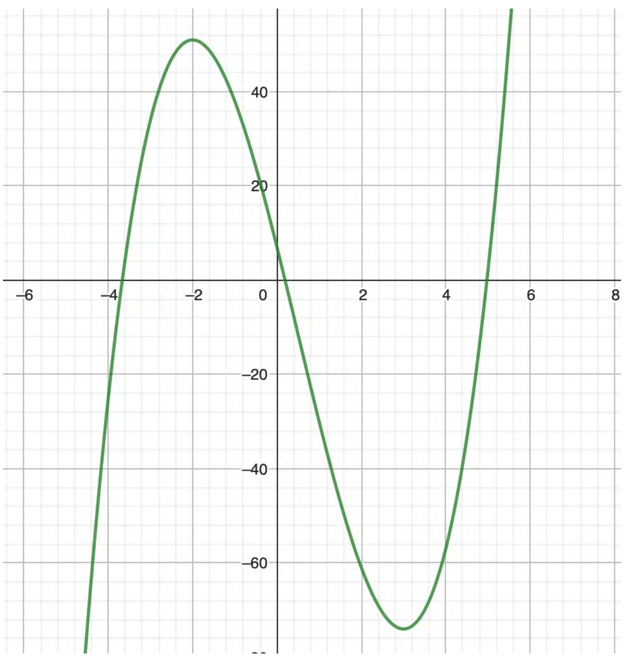


Figure 11: Graph of $f(x) = 2x^3 - 3x^2 - 36x + 7$

To find the coordinates at which the function is momentarily at rest, also called a turning point:

Substitute the x value(s) obtained at $f'(x) = 0$ into $f(x)$, ie substitute $x = -2$ *or* $x = 3$ into $f(x) = 2x^3 - 3x^2 - 36x + 7$

$f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 7 = 51$

$f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 7 = -74$

Hence the coordinates when the function is momentarily at rest are $(-2, 51)$, and $(3, -74)$

Example 33

Find the value(s) of x for which $f(x) = x^3 - x^2 - 8x - 5$ increases, decreases or is momentarily at rest

Solution

Step 1. Differentiate the given function.

$f(x) = x^3 - x^2 - 8x - 5$

$f'(x) = 3x^2 - 2x - 8$

Step 2. Find the values of x for which f is momentarily at rest,

$f'(x) = 0$

$3x^2 - 2x - 8 = 0$

Factorising:

$(x - 2)(3x + 4)$

$(x - 2) = 0$ *or* $(3x + 4) = 0$

$x = 2$ *or* $x = - 4=$

Hence at $x = 2$ *or* $x = -\frac{4}{3}$, $f(x)$ is momentarily at rest.

Step 3. Use the values of x in step 2 to partition the real number line to obtain a real interval. Note that values that are momentarily at rest should not be part of the intervals constructed.

Partitioned intervals: $(-\infty, -\frac{4}{3})$, $(-\frac{4}{4}, 2)$ $(2, +\infty)$

Table 6: Table of values of $f(x)$

Interval	Test value	Derivative	Conclusion
$(-\infty, -\frac{4}{3})$,	-2	$3(-2)^2 - 2(-1) - 8 = 6 > 0$	Increasing
$(-\frac{4}{3}, 2)$	1	$3(1)^2 - 2(1) - 8 = -9 > 0$	Decreasing
$(2, +\infty)$	3	$3(3)^2 - 2(3) - 8 = 13 > 0$	Increasing

Hence, $f(x)$ is strictly increasing at $(-\infty, -\frac{4}{3})$, $(2, +\infty)$ and strictly decreasing at $(-\frac{4}{3}, 2)$

To find the coordinates at which the function is momentarily at rest (or the function's turning point):

Substitute the x value(s) obtained at $f'(x) = 0$ into the $f(x) = 0$

So, substitute $x = -\frac{4}{3}$ *or* $x = 2$ into $f(x) = x^3 - x^2 - 8x - 5$

$f\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 - \left(-\frac{4}{3}\right)^2 - 8\left(-\frac{4}{3}\right) - 5 = \frac{41}{27}$

$f(2) = (2)^3 - (2)^2 - 8(2) - 5 = -17$

Hence the points of momentarily at rest are $\left(-\frac{4}{3}, \frac{41}{27}\right)$, and $(2, -17)$. All of this information can be seen in the graph of $f(x)$ below.

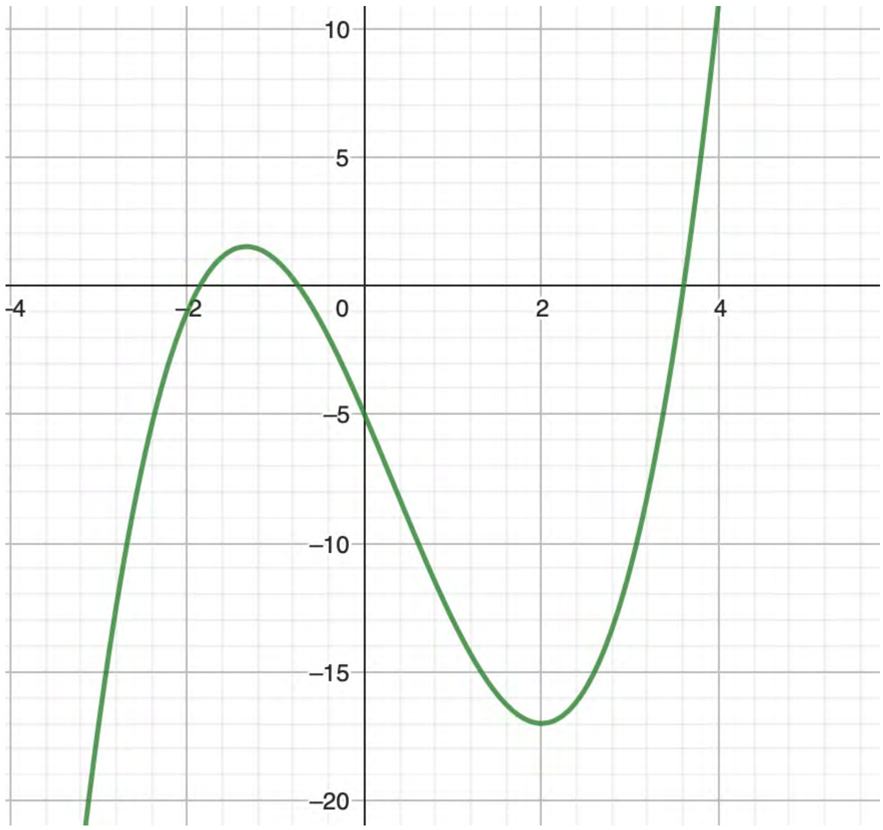


Figure 12: Graph of $f(x)$

EQUATION OF TANGENTS AND NORMALS TO CURVES

We will now apply our knowledge to find the equation of tangents and normal to curves through some given points on the curve. It is important to remember the following statements:

1. The equation of any straight line is of the form $y - y_1 = m(x - x_1)$ where (x_1, y_1) is a point that lies on the line and the gradient of the line is m
2. If m_1 is the gradient of a line, then the gradient of any straight line perpendicular to it, $m_2 = -\frac{1}{m_1}$

Equation of tangent to a Curve

A tangent is a line that passes at a given point on a curve without passing through it.

To find the equation of tangent to a curve at a given point P(x, y);

1. Find the derivative of the function $f(x)$.
2. Substitute the value of the x component in the given point P(x, y) into your gradient function.
3. Use the formula for finding a straight line to find the equation.

Example 34

Find the equation of the tangent line to the curve $y = 2x^2 + 5x$ at the point (2, 18)

Solution

Step 1: Find the gradient of the tangent

$y = 2x^2 + 5x$

$m = \frac{dy}{dx} = 4x + 5$

Step 2: Substitute the x coordinate of the given point into the gradient obtained

$m = \frac{dy}{dx} = 4(2) + 5 = 13$

Step 3: Using $y - y_1 = m(x - x_1)$ and the point (2, 18),

$y - 18 = 13(x - 2)$

$y - 18 = 13x - 26$

$\boldsymbol{y} = 13\boldsymbol{x} - 8$

Example 35

Determine the equation of tangents to the given functions against the points indicated.

a. $y = 3x^2 + 5x$ at $(-4, 28)$

b. $y = -5x^3 - 12x + 4$ at (0, 4)

c. $y = \frac{x^3 + 3x^2 - 5x}{x}$ at (2, 5)

d. $y = \frac{8}{5}x^3$ at (5, 200)

Solution

a. $y = 3x^2 + 5x$ at $(-4, 28)$

$m = \frac{dy}{dx} = 6x + 5$

$m = \frac{dy}{dx} = 6(-4) + 5 = -19$

Using $y - y1 = m(x - x1)$ and the point $(-4, 28)$,

$y - 28 = -19(x + 4)$

$y - 28 = -19x - 76$

$\mathbf{y = -19x - 48}$

b. $y = -5x^3 - 12x + 4$ at $(0, 4)$

$m = \frac{dy}{dx} = -15x^2 - 12$

$m = \frac{dy}{dx} = -15(0)^2 - 12 = -12$

Using $y - y_1 = m(x - x_1)$ at the point (0,4),

$y - 4 = -12(x - 0)$

$y - 4 = -12x + 0$

$\mathbf{y = -12x + 4}$

c. $y = \frac{x^3 + 3x^2 - 5x}{x}$ at $(2, 5)$

To be able to work with the function, we have to simplify the expression:

$y = \frac{x(x^2 + 3x - 5)}{x} = x^2 + 3x - 5$

$m = \frac{dy}{dx} = 2x + 3$

$m = \frac{dy}{dx} = 2(2) + 3 = 7$

Using $y - y_1 = m(x - x_1)$ at the point (2,5),

$y - (5) = 7(x - 2)$

$y - 5 = 7x - 14$

$\mathbf{y = 7x - 9}$

d. $y = \frac{8}{5}x^3$ at $(5, 200)$

$m = \frac{dy}{dx} = \frac{24}{5}x^2$

$m = \frac{dy}{dx} = \frac{24}{5}(5)^2 = 120$

Using $y - y_1 = m(x - x_1)$ at the point (5, 200),

$y - 200 = 120(x - 5)$

$y - 200 = 120x - 600$

$y = 120x - 400$

Equation of Normal to a Curve

A **normal** is a straight line that intersects a curve at a point and is **perpendicular** to the tangent line at that same point.

In other words, the tangent and normal lines share a common point of contact on the curve, as illustrated in the diagram. This means that the tangent and normal lines meet at a single point on the curve, with the normal line being perpendicular to the tangent line at that point.

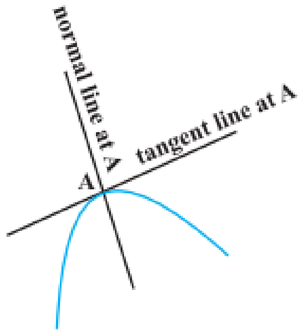


Figure 13: A normal to a curve

We have already learnt that, if two lines are perpendicular, then a relationship exists between their gradients. The relationship is the product of their gradients is -1.

$$\boldsymbol{m_1 m_2 = -1}$$

Since the normal is perpendicular to the tangent, the gradient of the normal, $\boldsymbol{m_1 = -\frac{1}{m_2}}$

To find the normal to a curve;

1. Find the derivative of the function.
2. Substitute the value of x of the given point into step 1 to find the gradient at that point
3. Find the gradient of the normal to the curve using $m_1 = -\frac{1}{m_2}$
4. Use $y - y_1 = -\frac{1}{m_2}(x - x_1)$ to find the equation of the line

Example 36

Find the equation of the normal to the curve $y = 3x^2 - 2x$ at a point Q(2, 8)

Solution

Step 1 Find the derivative

$y = 3x^2 - 2x$

$m = \frac{dy}{dx} = 6x - 2$

Step 2: Substitute the x component in the given point Q(2, 8) into step 1

$m = \frac{dy}{dx} = 6(2) - 2 = 10$

Step 3: Find the gradient of the normal to the curve using $m_1 = -\frac{1}{10}$

Using $y - y_1 = m_1(x - x_1)$ at the point (2, 8),

$y - 8 = -\frac{1}{10}(x - 2)$

$10y - 80 = -x + 2$

$\boldsymbol{x + 10y - 82 = 0}$

Example 37

Find the equation of the normal to the curve $y = 4x^3 - 6x^2 + 4x + 1$ at (1, 3)

Solution

Find the derivative

$y = 4x^3 - 6x^2 + 4x + 1$

$m = \frac{dy}{dx} = 12x^2 - 12x + 4$

Substitute the x component in the given point (1, 3) into $\frac{dy}{dx} = 12x^2 - 12x + 4$

$m = \frac{dy}{dx} = 12(1)^2 - 12(1) + 4 = 4$

Find the gradient of the normal to the curve using $m_1 = -\frac{1}{4}$

Using $y - y_1 = m_1(x - x_1)$ at the point (1, 3),

$y - 3 = -\frac{1}{4}(x - 1)$

$4y - 12 = -x + 1$

$\boldsymbol{x + 4y - 13 = 0}$

Example 38

Find the equation of the normal to the curve $y = x^2 + 4x - 2$ at $(-1, -5)$

Solution

Find the derivative

$y = x^2 + 4x - 2$

$m = \frac{dy}{dx} = 2x + 4$

Substitute the x component in the given point $(- 1, -5)$ into $\frac{dy}{dx}$.

$m = \frac{dy}{dx} = 2(-1) + 4 = 2$

Find the gradient of the normal to the curve using $m_1 = -\frac{1}{2}$

Using $y - y_1 = m_1(x - x_1)$ at the point $(-1, -5)$

$y - (-5) = -\frac{1}{2}(x - (-1))$

$y + 5 = -\frac{1}{2}(x + 1)$

$2y + 10 = -x - 1$

$\mathbf{x + 2y + 11 = 0}$

Example 39

Find both the tangent and the normal to the curve $y = \frac{1}{x^2}$ at $\left(-1, \frac{1}{4}\right)$

Solution

For the tangent:

$\frac{dy}{dx} = \frac{-2}{x^3}$

$\frac{dy}{dx} = \frac{-2}{(-2)^3} = \frac{-2}{-8} = \frac{1}{4}$

$y - y_1 = m(x - x_1)$ at the point $(-1, \frac{1}{4})$

$y - \left(\frac{1}{4}\right) = \frac{1}{4}(x - (-1)$

$y - \left(\frac{1}{4}\right) = \frac{1}{4}(x + 1)$

$4y + 1 = x + 1$

$\mathbf{-x + 4y = 0}$

For the normal:

$\frac{dy}{dx} = -\frac{2}{x^3}$

$\frac{dy}{dx} = \frac{-2}{(-2)^3} = \frac{-2}{-8} = \frac{1}{4}$

Gradient of the normal $= -4$

$y - y_1 = m(x - x_1)$ at the point $\left(-1, \frac{1}{4}\right)$

$y - \left(\frac{1}{4}\right) = -4\left(x - (-1)\right)$

$y - \frac{1}{4} = -4(x + 1)$

$y - \frac{1}{4} = -4x - 4$

$4y - 1 = -16x - 16$

$\mathbf{16x + 4y + 15 = 0}$

Example 40

If $y = 5x^2 - 2x + 5$, find the coordinates of the point on the curve at which the tangent is parallel to the line $y = 8x + 5$

Solution

Here the coordinate of the tangent is not given. However, we have already learnt that the gradients of parallel lines are equal, i.e $m_1 = m_2$.

The gradient function of $5x^2 - 2x + 5$ is $\frac{dy}{dx} = 10x - 2$ and that of $y = 8x + 5$ is $\frac{dy}{dx} = 8$

Equating the two gradients:

$10x - 2 = 8$

$10x = 10$

$x = 1$

Substitute $x = 1$ into $y = 5x^2 - 2x + 5$ to obtain the y coordinate

$y = 5(1)^2 - 2(1) + 5$

$y = 8$

Hence the coordinates of the point on the curve is $(\mathbf{1, 8})$

RATE OF CHANGE OF A FUNCTION

Did you know that the gradient of a line is the rate of change of y with respect to x? Likewise, the gradient of a curve $y = f(x)$ at a point P is equal to the derivative of the function $f(x)$ evaluated at the x-coordinate of $P\ (x, y)$, denoted as $f'(x)$. This represents the rate of change of y with respect to x at point $P(x, y)$.

To determine the rate of change of a function at an instant:

1. Find the derivative of the function
2. Evaluate at x.

Example 41

Consider the velocity of an object at time t = 2, where the displacement function is $s(t) = t^2$ Using differentiation, find the rate of change of displacement which represents the velocity at t = 2.

Step 1: Find the derivative

$s(t) = t^2$

$\frac{ds}{dt} = 2t$

Step 2: Substitute in $t = 2$ to find the rate of change at that instant.

$\frac{ds}{dt} = 2(2) = 4$

Therefore, the rate of change of displacement, or velocity, when t = 2, is 4m/s

Example 42

The height of a ball being thrown is modelled by the function $h(t) = 6t^2 + 5t - 10$ where t represents the time in seconds and h represents the height in metres.

Determine the rate of change in height when t = 2.

Solution

$\frac{dh}{dt} = 12t + 5$

When t = 2

$\frac{dh}{dt} = 12(2) + 5 = 29$

Rate of change of height when t = 2 is $29m/s$

Example 43

Find the rate of change in the area of a circle with respect to its radius, r, when r = 3cm.

Solution

Let A be the area of the circle

Area of circle $= \pi r^2$

The rate of change of area with respect to radius $= \frac{dA}{dr}$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dr} = 2 \times \pi \times 3$

Therefore, the rate of change of the area with respect to the radius is 6 πcm

Note: where the change occurred between two points, where Δx is very small, use the formula:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 44

(a) Suppose that $f(x) = x^2$. Calculate the average rate of change of $f(x)$ over the intervals 1 to 2, 1 to 1.1, and 1 to 1.01

(b) Determine the (instantaneous) rate of change of $f(x)$ when $x = 1$

Solution

(a) The intervals are of the form 1 to $1 + \Delta x$ for $\Delta x = 1$, 0.1 and 0.01. the average rate of change is given by the ratio:

$$\frac{\Delta y}{\Delta x} = \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{(1 + \Delta x)^2 - 1}{\Delta x}$$

For the three given values of Δx, this expression has the following respective values:

$\Delta x = 1$ for which

$$\frac{\Delta y}{\Delta x} = \frac{2^2 - 1}{1} = \frac{4 - 1}{1} = 3$$

$\Delta x = 0.1$ for which

$$\frac{\Delta y}{\Delta x} = \frac{1.1^2 - 1}{0.1} = \frac{1.21 - 1}{0.1} = 2.1$$

$\Delta x = 0.01$ for which

$$\frac{\Delta y}{\Delta x} = \frac{1.01^2 - 1}{0.01} = \frac{1.0201 - 1}{0.01} = 2.01$$

Thus, the average rate of change for $\Delta x = 1$, 0.1 and 0.01 is 3, 2.1 and 2.01 units per unit change in x respectively

(b) The instantaneous rate of change of $f(x)$ at $x = 1$ is equal to $f'(1)$. We have:

$f'(x) = 2x$

$f'(1) = 2(1) = 2$

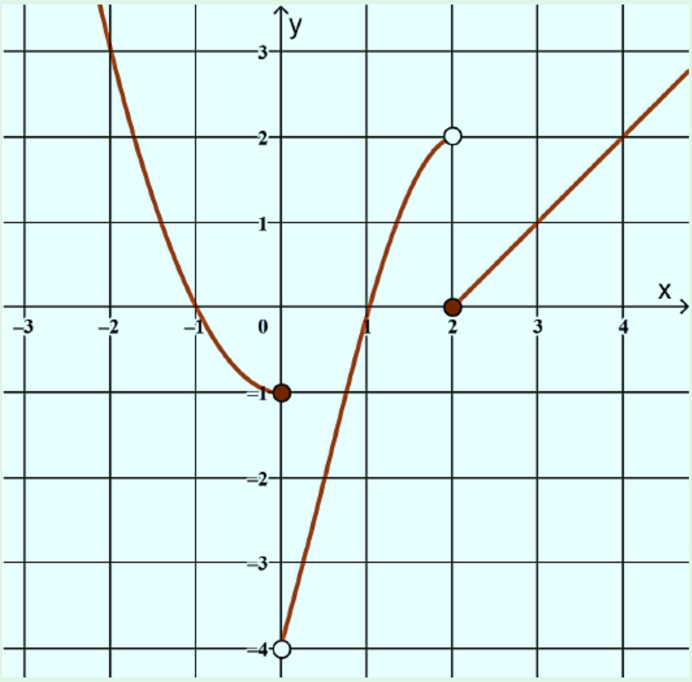
That is, the instantaneous rate of change is 2 units per unit change in x.

REVIEW QUESTIONS

Review Questions 8.1

1. What are the conditions that should be met for a function to be continuous?

2. For the function $g(x)$ whose graph is given below, state the value of each quantity, if it exists. If it does not exist, explain why



a) $\lim_{x \to 0-} g(x)$

b) $\lim_{x \to 0+} g(x)$

c) $\lim_{x \to 2-} g(x)$

d) $\lim_{x \to 2+} g(x)$

e) $g(2)$

f) $\lim_{x \to 4} g(x)$

3 Find the limiting value of:

(a) $\lim_{x \to 2} \left\{ \frac{2x - 2}{x - 1} \right\}$

(b) $\lim_{x \to 1} \left\{ \frac{x^2 + 1}{2x^3 - 1} \right\}$

4 Evaluate the following limits:

(a) $\lim_{x \to 2}\left\{\frac{x^2 - 4}{x - 2}\right\}$

(b) $\lim_{x \to 3}\left\{\frac{x - 3}{x^2 - 5x + 6}\right\}$

5 Evaluate the following limits

(a) $\lim_{x \to \frac{2}{3}} \frac{6x^2 - x - 2}{3x - 2}$

(b) $\lim_{x \to \frac{3}{2}} \frac{2x - 3}{8x^2 - 18x + 9}$

6 Evaluate the following limits

(a) $\lim_{x \to \infty} \frac{2x^2 - 3x}{3x^2 - 2}$

(b) $\lim_{x \to \infty} \frac{4x^3 - 5x^2}{2x^2 + x}$

7 Determine whether the following function are continuous at $x = 3$ and why?

(a) $\frac{3x^2 - 4}{x^2 - x - 6}$

(b) $\frac{x + 1}{x^{2+}3}$

Review Questions 8.2

1. A school population has been modelled since the year 2017 as

$P(t) = 2t^2 + 20t + 1115$ students. Where t is the number in years.

Find the rate of change in the school in the year 2022.

2. A train moves along a railway track in such a way that the distance it covers, starting from Kojokrom to Effiekuma is given by $x = 5t^2 + t$ where x is in metres and t is in seconds. What will be the velocity of the train at 25 seconds?

3. Find the equation of the tangent to the curve $y = 5 - x^3 - 4x^2 + 13x$ at P(2, -4)

4. Given $f'(-2) = -11$ and $f(x) = 3x^2 + 5mx + 8$, find the value of m.

5. Show that the equation of tangent to the curve $y = \frac{1}{x^2}$ at $x =$ a is $2x + a^3y - 3a = 0$

6. Find, from first principles, the derivative of the following functions

   a. $f(t) = t^3 - 6t$

   b. $y = ax^2 + bx + c$ where a, b and c are constants

7. The bridge below has the model $y = -3x^2 + 5x + 2$, find the point(s) where the curve on the bridge is decreasing.

8. Differentiate the following functions

   a. $y = \frac{1}{3}t^9 - 5t^5 + 8t^4 + 9t + 5$

   b. $y = \frac{1}{4}(x^3 + 8x)$

   c. $y = \frac{3x^5 - 5x^4 + 12x^2}{x^3}$

   d. $y = 2(3x^3 + 4x)$

9. Assuming the population $P(t)$ of first-year students in SHS grows according to the function $P(t) = 1000 + 100\,000t + 2t^2$, where $P(t)$ is the population at time t years. The Ministry of Education wants to know how the rate of change of the population will alter in five years.

10. Find the equation of the tangent and normal to curve $y = 10 - 3x^2$ at $(2, -2)$

11. Find the equation of tangent and normal to the following curves at the given point.

   a. $y = 3x^2 - 2x + 4$ at (2, 12)

   b. $y = x^3 - 2x^2 - x - 1$ at $x = 2$

GLOSSARY

Limits are the values a function approaches as the input or variable gets close to a given point.

Determinate Form is an undefined expression involving some operation between two quantities. It is said to be in a determinate form if it evaluates to a single number value or infinity.

Indeterminate Form is an undefined expression involving some operation between two quantities. It is said be in indeterminate form if it does not evaluate to a single number value or infinity.

Differentiation is the process of finding the derivative of a function. The derivative measures the rate at which a function's value changes as its input changes.

Rate of change refers to how a quantity changes with respect to another quantity. It is a fundamental concept that helps in understanding how functions behave as their inputs change.

A **tangent** to a curve at a given point is a straight line that touches the curve at that point and has the same gradient as the curve at that point.

A **normal** to a curve at a given point is a straight line that is perpendicular to the tangent at that point.

A **secant** to a curve is a straight line that intersects the curve at two or more points.

ACKNOWLEDGEMENTS

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