Mathematics

Year 1

SECTION

4

LINEAR EQUATIONS, RELATIONS AND FUNCTIONS

ALGEBRAIC REASONING

Applications of Expressions, Equations and Inequalities & Patterns and Relationships

INTRODUCTION

Proficiency in algebraic manipulation and problem-solving lays the foundation for tackling a multitude of mathematical challenges and real-world situations. A bedrock of algebraic proficiency involves the ability to formulate equations that represent real-world situations and to interpret these equations in context. This skill empowers individuals to manipulate formulae to solve problems, including situations necessitating a change of subjects within the formula. Linear equations and inequalities are highly relevant in day-to-day activities, including commerce and industry. For example, they are applied in the following areas: Budgeting and Finance, Supply Chain Management, Production and Manufacturing, Marketing and Sales, Operations Management, Economics and Business Analysis, Risk Assessment and Decision Making, and Optimisation. Relations, mappings, and functions play significant roles in various aspects of daily life, commerce, and industry. These concepts are applied in areas such as: Communication and Relationships, Commerce and Business, Technology and Industry, Data Analysis and Decision Making, Education and Learning, Healthcare and Medicine. In conclusion, relations, mappings, and functions are foundational concepts with wide-ranging applications in daily life, commerce and industry. They help in understanding, analysing, and optimising relationships, processes, and systems across diverse fields, contributing to efficiency, innovation, and informed decision-making. We'll also learn how to find the gradient, equation of a straight line and calculate the distance between two points. Imagine being able to calculate the distance between two cities on a map or the length of a bridge. It's all about understanding the relationships between points and lines.

At the end of this section, you will be able to:

- Construct and interpret formulae for a given task and apply to problems involving a change of subjects.
- Solve linear equations in one and two variable(s); and brackets and fractions for given problems and relate it to real life situations.

- Find solution set or truth set of linear inequalities and illustrate on the number line.
- Identify relations from functions and differentiate between the types of relations and functions using models such as graphs.
- Investigate relationships between two number sets and determine the rules of given mappings or functions.
- Extend the knowledge of coordinates of two points to find the gradient and equation of a straight line
- Recognise and interpret two points on a straight line and use it to find the distance between them.

Key Ideas

- A **formula** is a rule which gives the relationship between things or quantities.
- A **Linear equation** is of the form, $ax + by + c$. where a, b, and c are constants, and x and y are variables. This form is known as the standard form of a linear equation. Or Linear equation can be in slope intercept form as $y = mx + c$ where m is the slope of the line and c is the y-intercept.
- A linear inequality in two variables is typically expressed in the following general form:

  $ax + by \geq c$ or $ax + by \leq c$, where a, b and c are constants, x and y are variables.
- Folding the human right arm resembles the idea of greater than symbol. Folding the human left arm resembles the idea of less than symbol.
- A relation between two sets is a collection of ordered pairs where each element of the first set is related to one or more elements of the second set. A function is a special type of relation where each input (from the domain) is related to exactly one output (from the range).
- There are various types of relations. Some of which are: One-to-One relations, One-to-many relations, many-to-one relations and many-to-many relations. One-to-one and Many-to-one relations are functions.
- A mapping can either be linear, quadratic or exponential. In this section, we shall look at Linear and Exponential mappings as well as determining the rules of mappings. We shall also identify and interpret linear graphs.

CHANGE OF SUBJECTS

This refers to the process of rearranging an equation to isolate a different variable or parameter. This technique is often used in solving equations or formulae where one needs to manipulate the equation to express a different variable as the subject of the formula.

For example, if you have an equation $A = \frac{1}{2}bh$ and you want to solve for the height, you perform a "change of subject" by rearranging the equation to isolate h: $h = \frac{2A}{b}$

In this case, the process involved changing the subject from $A = \frac{1}{2}bh$ to $h = \frac{2A}{b}$, by rearranging the terms of the equation. This method is fundamental in algebraic manipulation and problem-solving in mathematics.

Definition of Key Concepts

A **formula** is a rule which gives the relationship between things or quantities. The letters in the formula always stand for something specific, like cost, speed, number of books. There is always more than one unknown in a formula. For example, the perimeter of a rectangle $P = 2l + 2b$, the area of a triangle, $A = \frac{1}{2}bh$.

Activity 1

Let's break down the process involving a change of subject.

Steps to Construct and Interpret Formulae

Step 1: Identify Variables

First, identify the variables involved in the problem. For example, let's consider a problem involving the formula for the area of a rectangle, which is given by: $A = L \times W$. where A represents the area, L represents the length and W represents the width of the rectangle.

Step 2: Formula Manipulation

Solving for a Specific Variable: Sometimes, you may need to solve the formula for a specific variable. For instance, if you want to solve for **W** in terms of **A** and **L**, you would rearrange the formula:

$W = \frac{A}{L}$

Here, you isolate **W** on one side of the equation.

Step 3: Applying to Problems

Consider a scenario where you are given the area, **A,** and length, **L,** of a rectangle and need to find the width, **W**. Using the formula A = L x W and knowing A and L, you can substitute these values to find W:

$W = \frac{A}{L}$

This formula allows you to calculate **W** directly once you know **A** and **L**.

I trust you can now discuss and manipulate the area, length and breadth of your classroom. Thumbs up!

Example 1

Study the example below.

The area of a rectangular field is 24 square units, and its length is 6 units. Find the width of the field.

Solution

1. Given:

   A = 24 square units

   L = 6 units

2. Use the formula, $A = L \times W$ to find W:

   $W = \frac{24}{6}$

3. Therefore, the width, W of the rectangle is 4 units.

Interpretation

In this example, by constructing and interpreting the formula $A = L \times W$, and applying it correctly to the problem where values of A and L are known, we were able to find the value of W, which represents the width of the rectangle.

Application of concepts and examples

Hello Learner! Let's now look at the following examples on change of subject.

Example 2

1. Make c the subject of the formula $y = mx + c$

Solution

To make c the subject, subtract mx from both sides of the equation to isolate c.

$$mx - mx + c = y - mx$$

$$\therefore c = y - mx$$

Example 3

From the formula $3c + 2r = md + k$, make r the subject.

Solution

Given $3c + 2r = md + k$,

Make $2r$ the subject, ie isolate the r term on one side of the equation:

$$2r = md + k - 3c$$

Divide both sides by 2 so we only have a single r left.

$$r = \frac{md + k - 3c}{2}$$

Example 4

The relation between energy E, mass m, and velocity of light v, is given by $E = mv^2$.

Find the value v, when $E = 20$ and $m = 5$

Solution

Given that $E = mv^2$, let us rearrange to isolate the v term.

$$v^2 = \frac{E}{m}$$

$$\therefore v = \sqrt{\frac{E}{m}}$$

$$v = \sqrt{\frac{20}{5}}$$

$$\therefore v = 2$$

LINEAR EQUATIONS

An *equation* of the form $ax + b = c$, where a, b and c are real numbers, and $a \neq 0$. A Linear equation in one variable has exactly one solution.

Activity 2

Let's engage in an activity to solve the age problem together.

Steps:

1. Consider the age problem: "John is 5 years older than twice Maria's age. If John is 25 years old, how old is Maria?"

   Define the variable: Let's denote Maria's age to be x.

2. Setting up the Equation

   According to the problem statement, John's age is 25 years, and he is 5 years older than twice Maria's age. So, we can write the equation:

   $25 = 2x + 5$

3. Solving the Equation:

   Now, let's solve the equation to find Maria's age (x).

   $25 = 2x + 5$

   Subtract 5 from both sides to isolate the x term

   $25 - 5 = 2x$

   $20 = 2x$

   Divide both sides by 2 to solve for a single x.

   $x = \frac{20}{2}$

   $x = 10$

Hello Learner! I hope you enjoyed the above activity. Let's now consider the following examples.

Linear Equations Involving one and Two variables

Example 5

Solve for the variable indicated in the following equations.

a) $3x - 12 = 21$

b) $5 - 3y = 3y + 7$

Solution

a) To solve $3x - 12 = 21$,

You need to isolate the x term on one side, so add 12 to both sides of the equation, $3x = 33$

Then divide both sides of the equation by 3, to make a single x the subject

$x = \frac{33}{3} = 11$

b) To solve $5 - 3y = 3y + 7$

First group the like terms (ie isolate the ys onto one side and the constants (the numbers) onto the other)

$5 - 7 = 3y + 3y$

Simplify the terms $-2 = 6y$

Make a single y the subject by dividing both sides by the coefficient of y.

Therefore $y = -\frac{1}{3}$

Example 6

A company produces and sells handmade pottery. The total cost C (in Ghana cedis) to produce ***x*** units of pottery is given by the equation $C = 100x - 500$. The company sells each unit of pottery for ***p*** Ghana cedis. The total revenue R (in Ghana cedis) from selling ***x*** units of pottery is $R = px$

1. Express the company's profit, P (in GH€) as a function of x and p.

2. If the company sells each unit of pottery for GH€20.00 how many units of pottery must they sell to break even (i.e., make zero profit)?

Solution

Step 1: Expressing Profit as a Function

The profit, P is given by subtracting the total cost C from the total revenue, R.

$P = R - C$

Substitute the given equations for C and R:

1. $P = px - (100x - 500)$

   Simplify the expression:

   $P = px - 100x + 500$

   So, the profit, P as a function of x and p is $P(x) = px - 100x + 500$.

Step 2: Finding the Break-even Point

To find the break-even point, we set the profit $P(x)$ equal to zero (break-even condition):

$px - 100x + 500 = 0$

Now, substitute $p = 20$ (since the company sells each unit for GH€20):

$20x - 100x + 500 = 0$

Combine like terms:

$- 80x + 500 = 0$

Subtract 500 from both sides:

$- 80x = - 500$

Divide both sides by –80

$\frac{-80x}{-80} = \frac{500}{-80}$

$x = \frac{50}{8}$

Simplify the fraction:

$x = 6.25$

Since x represents the number of units of pottery, and it must be a whole number, we round up to the nearest whole number because the company cannot sell a fraction of a pottery unit:

$x = 7$

Therefore, the company must sell at least 7 units of pottery to break even (make zero profit) when selling each unit for GH€20.00

Linear Equations Involving Brackets

Here, we shall be applying the concept of distributive property in the expansion of brackets.

Example 7

Solve the equation $7(x - 6) = 3(x + 9)$

Solution

To solve the equation $7(x - 6) = 3(x + 9)$,

First multiply the brackets (expansion) on both sides of the equation,

$7x - 42 = 3x + 27$

Then subtract $3x$ from both sides (so you are isolating your xs on one side of the equation) $4x - 42 = 27$

Then add 42 to both sides of the equation (so the xs are isolated) $4x = 69$.

Divide both sides of the equation by 4 to find the value for a single x.

$x = \frac{69}{4} = 17\frac{1}{4}$

Linear Equations Involving Fractions

This concept was treated in week 10 of section 3. Let's look at the following example to refresh our minds.

Example 8

Solve $\frac{3x}{2} - 2 = \frac{1}{2}$

Solution

$\frac{3x}{2} - 2 = \frac{1}{2}$

1. Eliminate the fraction by multiplying **each** term on both sides of the equation by the LCM

   $2 \times \frac{3x}{2} - 2 \times 2 = \frac{1}{2} \times 2$

   $3x - 4 = 1$

2. Group the like terms and simplify the equation

   $3x = 1 + 4$

   $3x = 5$

   Divide both sides by 3, $x = \frac{5}{3}$

LINEAR INEQUALITIES

Linear inequality in one variable is of the form $ax + b < c$, $ax + b \leq c$, $ax + b > c$, $ax + b \geq c$.

Folding the human right arm resembles the idea of greater than (>) symbol and that of the left arm also resembles the idea of less than (<) symbol.

$4 < 5$ (4 is less than 5)

$6 > 4$ (6 is greater than 4)

$x \leq 4$ (Depending on the values of x, which is an unknown variable, but it must be less than or equal to 4)

Suppose we are given an inequality of the form $x > -5$, the solution set for an inequality (as it is for an equation) is the set of all values for the variable that make the inequality a true statement.

An appropriate way to picture the solution set is by a graph on a **number line** or the use of **Geodot** to generate conjectures. A sample of the geodot is shown here.

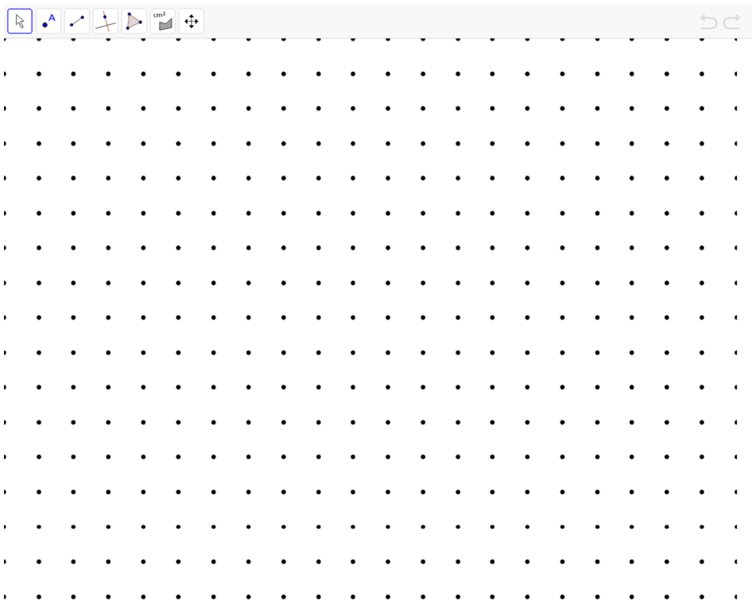


Figure 1: Geodot

Graph the set; $\{x: x < 4\}$

Explanations: We want to include all real numbers less than 4, that is, to the left of 4 on the number line. An open circle is used to indicate that the point corresponding to 4 is not included in the graph. It is called **an open half line**; it extends to the left and not including 4.

Two other symbols as shown in the introduction, $\leq$ and $\geq$, are also used in writing inequalities. In each case, they combine the inequality symbols for less than or greater than with the symbol for equality.

The following explain the use of these symbols. The expression $\boldsymbol{a} \leq \boldsymbol{b}$ is read as "***a*** is less than or equal to ***b***"

Note that this combines the symbol '< 'and '=' and means that either

$\boldsymbol{a} < \boldsymbol{b}$ or $\boldsymbol{a} =$ b. Similarly, $a \geq b$ reads "a is greater than or equal to b". Implying, either $\boldsymbol{a} > \boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{b}$. etc.

Activity 3

Solve the inequality: $2x - 5 > 3$

Steps

1. Isolate the variable:

Start by isolating the x term on one side of the inequality.

$2x - 5 > 3$

Add 5 to both sides to remove the constant term on the left side:

$2x - 5 + 5 > 3 + 5$

Simplify:

$2x > 8$

2. Divide by the coefficient of x

Divide both sides by 2 to solve for a single x

$\frac{2x}{2} > 8=$

Simplify:

$x > 4$

3. Write the solution:

The solution to the inequality $2x - 5 > 3$ is $x > 4$.

This activity shows the basic steps involved in solving a simple linear inequality. Always remember to perform operations on both sides of the inequality to maintain its validity. However, do avoid multiplying or dividing by a negative number as this reverses the inequality sign.

RELATIONS AND FUNCTIONS

What are Relations?

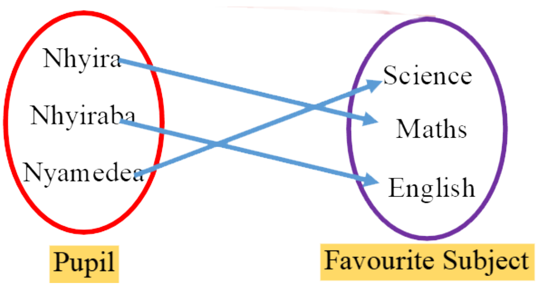
A **relation** is a rule which connects one set to another. We can express relations as an ordered pair (2, 6), in a diagram form, a rule form or graphical form. Relations could be a connection between the first set known as the domain and the second set called the co-domain.

Types of Relations

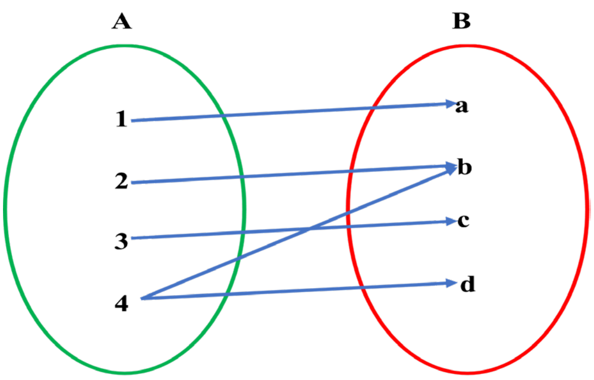
There are **four types of relations** and they are;

1. **One -to-one relation:** This is a relation in which each element in the domain (first set) has exactly one image in the co-domain (second set).

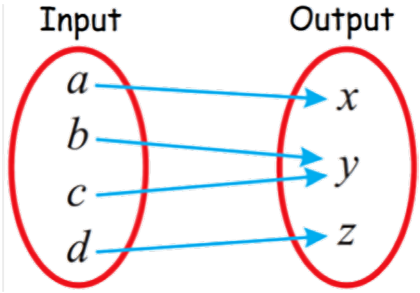
   Example:



2. **One-to-many relation**: This is a relation in which one element in the domain has more than one (many) in the co-domain.



3. **Many- to-one relation:** This is a relation in which more than one element in the domain has only one element in the co-domain.

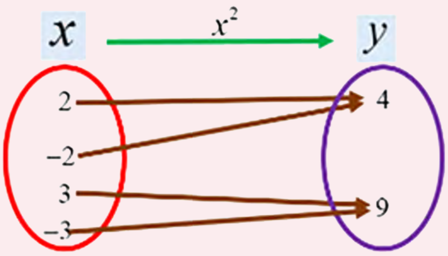


4. **Many-to-many relation**: This is a relation in which many elements in the domain have many images in the co-domain.

Activity 4

Can you draw a diagram to show many-to-many relations?

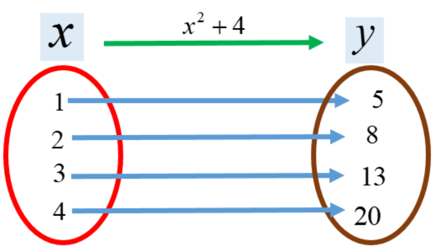
It could look something like this, but you could have a better one.



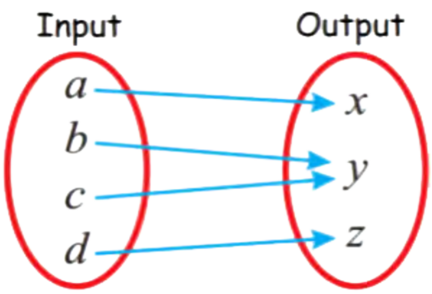
Many-to-many function

Functions

Functions are relations where each element in the domain has only one image in the co-domain. **One-to-one and many-to-one relations are functions** because each element in the domain has exactly one image in the co-domain. Thus, every member of the domain has only one image in the co-domain.



One-to-one function

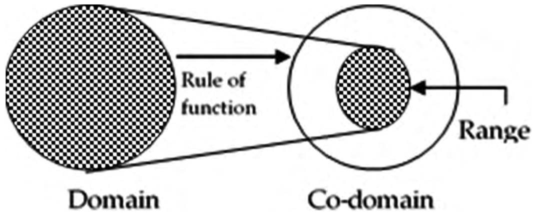


Mapping

Mapping is a relation such that each element in the domain is associated with an element in the co-domain**.**

A subset of the ***co-domain*** which is actually used by the function is called the ***range*** of the function. This is illustrated in figure below.

A subset of the ***co-domain***, the ***range*** of the function as shown



Rules of Mappings

There are two rules for mappings

1. **Linear mapping**

   A **mapping is said to be linear** if the difference between the consecutive elements in both the domain and the co-domain is constant.

x	1	2	3	4	5 ...
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
y	1	3	5	7	9 ...

x	0	2	4	6	8 ...
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
y	4	8	12	16	20 ...

The rule for linear mapping is of the form $y = mx + c$

Where,

$$m = \frac{\text{constant difference of the co} - \text{domain}}{\text{constant difference of the domain}}$$

Example 9

What is the rule of the mapping?

x	0	2	4	6	8 ...
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
y	4	8	12	16	20 ...

Solution

The rule of the mapping is of the form $y = mx + c$

$$m = \frac{8-4}{2-0}$$

$$m = 2$$

Put the value of m into the equation $y = 2x + c\ldots(1)$

Now take any coordinate say (0, 4) and put it into equation 1(i.e. $x = 0$ and $y = 4$),

$4 = 2(0) + c$

$c = 4$

Put c back into equation 1 to give the rule for the mapping above.

$\therefore y = 2x + 4$

2. Exponential mapping

A **mapping is said to be an exponential mapping** if the ratio between the consecutive elements in the co-domain is constant. The rule for exponential mapping is given as; $\boldsymbol{y = br^{x-a}}$

Example 10

i.

$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 4 & 8 & 16 \end{array}$$

Common ratio (r) $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$

The first term of $x = a = 1$

The first term of $y = b = 2$

Y can be expressed in terms of x by $y = br^{x-a}$

Hence the rule for the mapping is: $y = 2 \times 2^{x-1}$

ii.

$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \end{array}$$

common ratio (r) $\frac{1/9}{1/3} = \frac{1/27}{1/9}$

$= \frac{1/81}{1/27} = \frac{1}{3}$

the first term of $x = a = 1$

the first term of $y = b = \frac{1}{3}$

y can be expressed in terms of x by $y = br^{x-a}$

Hence the rule for the mapping is:y

$= \ y = \frac{1}{3} \times \left(\frac{1}{3}\right)^{x-1}$

Application of Concepts in Real-world

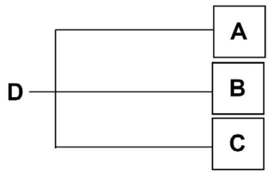
Example 11

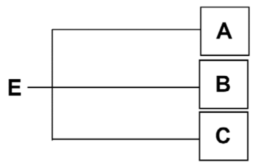
Adowa has three jumpers and two skirts, combine these in six possible ways using arrow diagram.

Solution

Let the jumpers be A, B, and C

Let the skirts be D and E.





Graphs of Linear Functions

Let's look at graphs of linear functions and their interpretations.

A **straight line graph** is a visual representation of a linear function.

A straight line has a general equation of

$$y = mx + c$$

gradient y-intercept

Example

$y = 2x + 1$

$m = 2$, and $c = 1$

The graph of this equation looks like this:

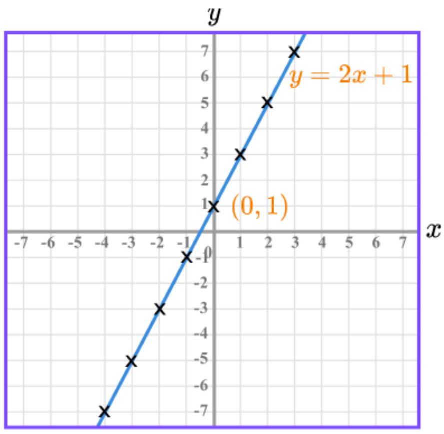


Figure 2: Straight line graph

Identifying Linear Graphs

Example 12

Mr. Benyah asks Yakubu to identify whether the given equation $3x - 7y = 16$ forms a linear graph or not without plotting its values.

Solution

First, Yakubu needs to identify the type of equation. Next, he needs to remember that any linear equation in two variables always represent a straight line if it can be rearranged to be in the form $y = \mathrm{m}x + \mathrm{c}$.

$3x - 7y = 16$ can be rearranged to $7y = 3x - 16$ and then further rearranged to $y = \frac{3}{7}x - \frac{16}{7}$. Therefore, the above equation represents a straight line.

Example 13

Draw a graph of a straight line with a **gradient of 1** and explain your answer.

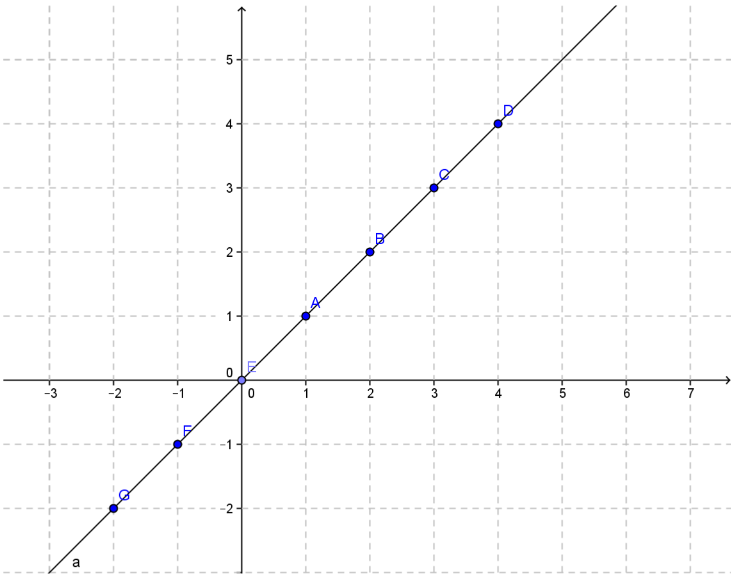


Figure 3: Straight line graph

This graph with a gradient 1 pass through the origin and slopes from left to right. For every one unit we go to the right, we go up one unit.

Now draw a straight-line graph with a **gradient of -1** and explain your answer.

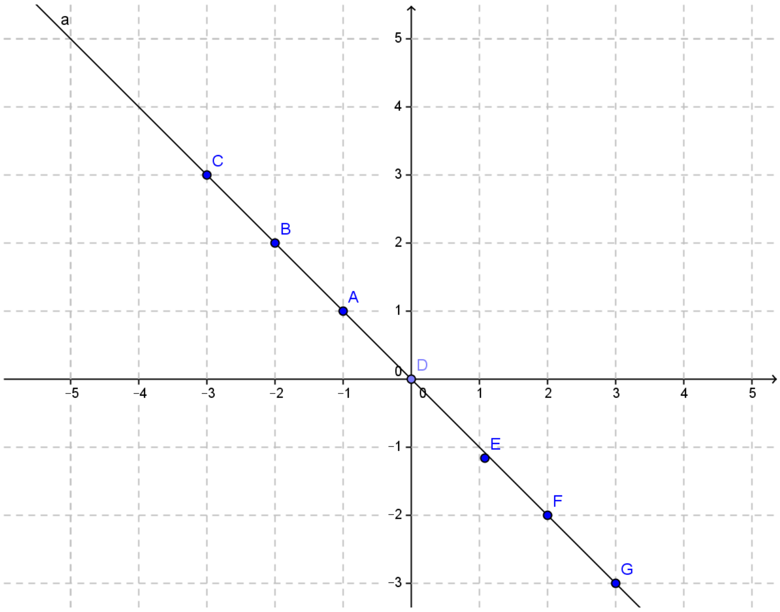


Figure 4: Straight line graph

This graph with a gradient of -1 slopes downwards and it passes through the origin. For every one unit we go to the right, we go down one unit.

Example 14

Draw the graph of $y = 4x + 6$ and explain what happens if the constant 6 is changed to 1.

Solution

The graph's gradient is 4, so for every unit we move to the right, we move up 4 units. The graph intersects the y axis at (0,6). This means the graph looks like this:

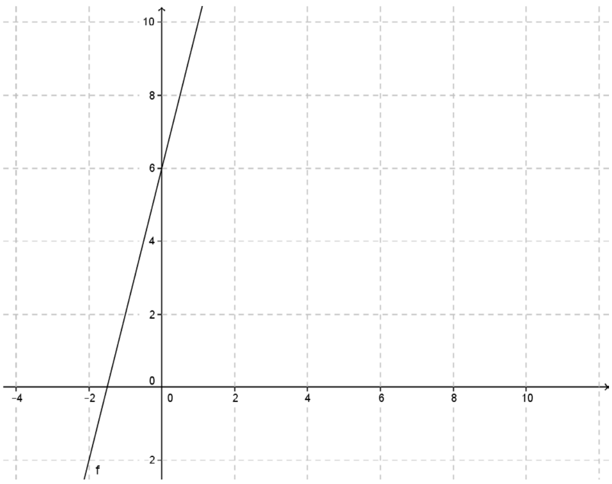


Figure 5: Straight line graph

If the equation is changed to $y = 4x + 1$ the gradient remains the same, but the intersection with the y axis is now at (0,1), so the moves "down" in order to cut y-axis at (0, 1).

Example 15

Draw the graph of $y = 4x + 6$ and explain what happens if the coefficient 4 is changed to 2, 1, 0 and -1 respectively.

Solution

As the co-efficient decreases, the gradient changes as the lines are rotated clockwise about (0, 6)

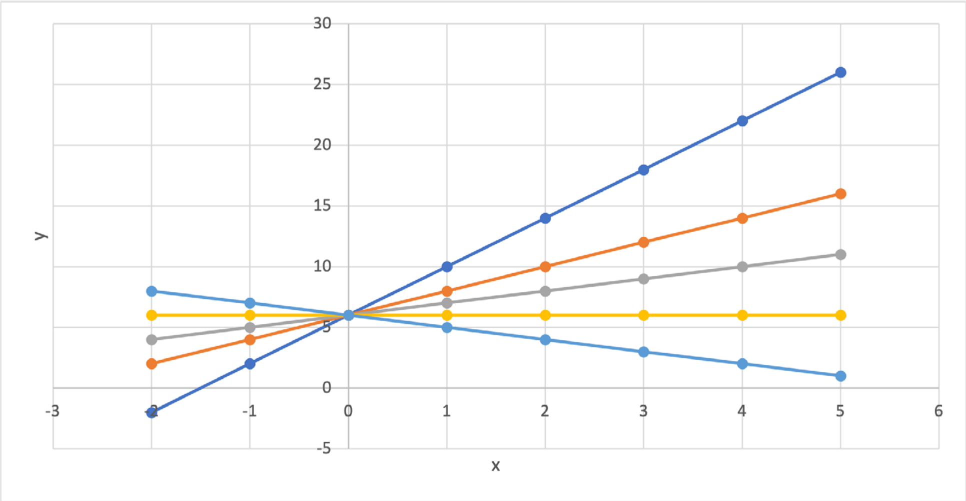


Figure 6: Straight line graph

Activity 5

Family Relations

Relation: Consider a family where each person is related to others through various roles (parent, child, sibling).

Function: If we focus on the relation between parents and children, each parent (father or mother) typically has a set of children. In this case, the relation from parent to child is a function because each parent (input) is associated with a unique child or set of children (output). For example, Mr. Kwame has children Ama, Kofi, and Kwesi; Mrs. Akua has children Yaa and Kojo. This demonstrates a function where each parent has a specific set of children.

Hello Learner! Now that you have studied the above scenario, discuss with your friend (either at home or in school) how functions can be identified from the following relations:

i) Academic adviser to student

ii) Employer to employees

iii) Market vendor to products

Activity 6

Ama has three Defenders and two Strikers, combine these in six possible ways using arrow diagram.

Solution

Step-by-step solution:

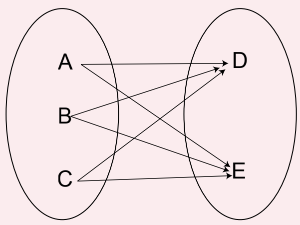
1. List all Defenders and Strikers:

- Defenders: A, B, C
- Strikers: D, E

2. Pair each Defender with each Striker:

- A with D
- A with E
- B with D
- B with E
- C with D
- C with E

3. An arrow diagram below is a visual representation that combines all. Each arrow from a Defender to a Striker represents a specific outfit combination.



You can see from the arrow diagram that the solution is complete with six possible combinations shown by the six arrows.

Activity 7

Example 16

Consider a linear mapping having the differences between consecutive images being constant. E.g. Given two sets

$X = \{0 ,1 ,2 ,3 ,4\}$ and $Y = \{2, 5, 8, 11, 14\}$

- Let the first term of set X be a
- Let the first term of set Y be b
- Denote each element in set X be x and that of set Y be y as shown below.

$$\begin{array}{cccccc} X & 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Y & 2 & 5 & 8 & 11 & 14 \end{array}$$

- Let the common difference between consecutive image in Y be d
- Let the common difference between consecutive image in X be k
- Calculate the common difference between consecutive image in Y

  $d = 5 - 2 = 8 - 5 = 11 - 8 = 14 - 11 = 3$
- Calculate the common difference between consecutive image in X

  $k = 5 - 4 = 6 - 5 = 7 - 6 = 8 - 7 = 1$
- The first term of the set $x = a = 0$
- The first term of the set $y = b = 2$
- Express y in terms of x by $y = b + \frac{d}{k}(x - a)$
- Hence you arrived at the rule for the above mapping as

  $y = 2 + \frac{3}{1}(x - 0)$

  $\boldsymbol{y} = 2 + 3\boldsymbol{x}$

Example 17

Find the rule for the mapping

X	0	2	4	6	8
↓	↓	↓	↓	↓	↓
Y	2	6	10	14	18

- Cross check both the substitutions and the answer if they are correct or not
- Check the substitutions $a = 0$, $b = 2$, $d = 4$ *and* $k = 2$
- Check the rule for the mapping as given by $y = 2 + \frac{4}{2}(x - 0)$
- Check the final answer $\mathbf{y} = 2 + 2\boldsymbol{x}$

Activity 8

Given an Exponential mapping having a common ratio between consecutive images being constant. E.g. Consider the two sets

$X = \{0, 1, 2, 3, \}$ and $Y = \{3, 6, 12, 24,\}$

- Let the first term in the set $X = a$ and that of set $Y = b$
- Represent each element of set X be x and that of Y be y

  Using the diagram

X	0	1	2	3
↓	↓	↓	↓	↓
Y	3	6	12	24

- The common ratio of the element $Y = r = \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = 2$
- Let the first term of the set $X = a = 0$
- let the first term of the set $Y = b = 3$
- Express y in terms of x by the standard relation $y = b\,r^{x-a}$ and substitute all the values as mentioned above and we have $y = 3 \times 2^x$

GRADIENT AND EQUATION OF A STRAIGHT LINE

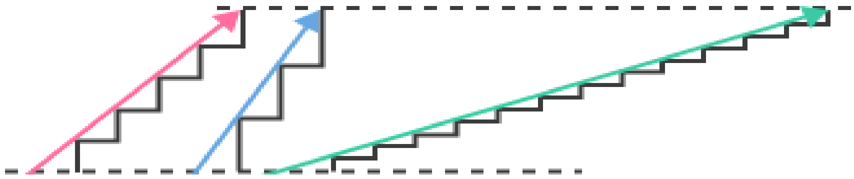
Facts about gradient

1. The gradient of a line is the measure of the steepness of a straight line.
2. The gradient of a line can be either positive (uphill) or negative (downhill) or 0 (horizontal) and does not need to be a whole number.
3. The gradient of a line is the measure of the steepness of a straight line.

How to Understand the Gradient of a Line

Imagine walking up a set of stairs. Each step has the same height and you can only take one step forward each time you move. If the steps are taller, you will reach the top of the stairs quicker, if each step is shorter, you will reach the top of the stairs more slowly.

Let's look at sets of stairs,



The blue steps are taller than the red steps and so the gradient is steeper (notice the blue arrow is steeper than the red arrow).

The green steps are not as tall as the red steps so the gradient is shallower (the green arrow is shallower than the red arrow).

Gradients can be positive or negative but are always observed from left to right.

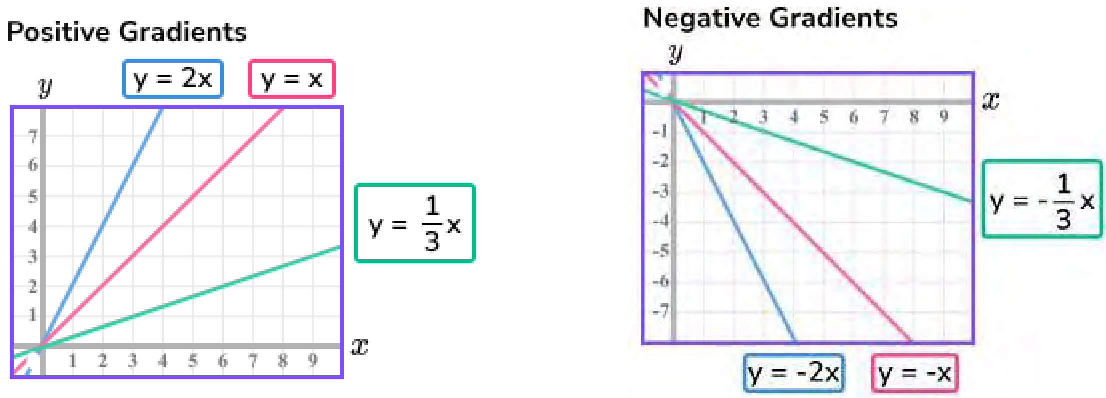


Figure 7: Straight line graph

Finding gradients: In placing a ladder against a wall or tree, a change in the position of the top of the ladder will be because of a change in position of the foot of the same ladder.

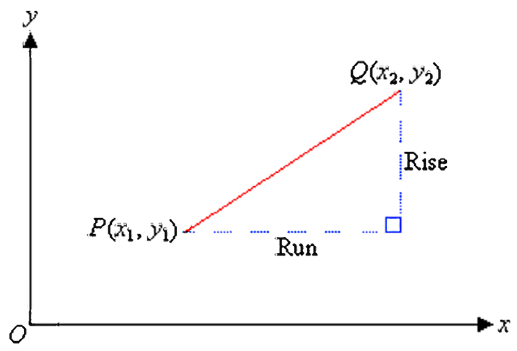


Figure 8: Gradient of a line

$$\text{Gradient} = \frac{\text{Rise}}{\text{Run}}$$
$$= \frac{\text{Change in } y}{\text{Change in } x}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient of a straight line is denoted by m where: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Now that we have understood the concept of gradient and learned the formula to find the gradient of a straight line joining two points, it's time to put our knowledge into practice. Let's undertake the two activities below on finding gradient between two points using the formulae.

Activity 9

Find the gradient of the straight line joining the points P(– 4, 5) and Q(4, 17).

Step 1: Identify the coordinates of the points

P(–4, 5) and Q(4, 17)

Step 2: Write the formula for gradient (m)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 3: Plug in the coordinates into the formula

$$m = \frac{17 - 5}{4 - (-4)}$$

Step 4: Simplify the expression

$m = \frac{12}{8} = \frac{3}{2}$

Step 5: Calculate the gradient

$m = 1.5$

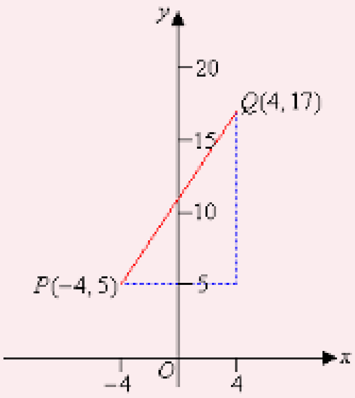


Figure 9: Gradient of a line

The gradient of the straight line joining points P(–4, 5) and Q(4, 17) is 1.5.

Note: If the gradient of a line is positive, then the line slopes upward as the value of x increases.

Activity 10

Find the gradient of the straight line joining the points A(6, 0) and B(0, 3).

Step 1: Identify the coordinates of the points

A(6, 0) and B(0, 3)

Step 2: Write the formula for gradient (m)

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 3: Plug in the coordinates into the formula

$m = \frac{3 - 0}{0 - 6}$

Step 4: Simplify the numerator

$3 - 0 = 3$

Step 5: Simplify the denominator

$0 - 6 = -6$

Step 6: Write the simplified formula

$m = \frac{3}{-6}$

Step 7: Calculate the gradient

$m = -\frac{1}{2}$

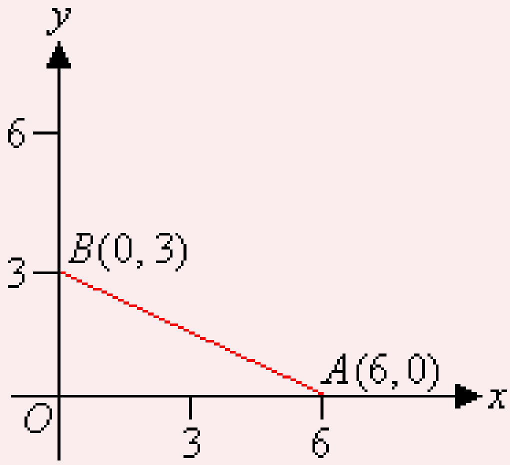


Figure 10: Gradient of a line

The gradient of the straight line joining points A(6, 0) and B(0, 3) is $-\frac{1}{2}$

Note: When the gradient of the line is negative. It indicates that the line slopes downward from the left to right. As the value of (x) increases, the corresponding (y) values decreases, resulting in a reduction along the line.

Now that you have mastered the skill of finding the gradient of a line when given two points, let's explore how gradients are applied in real-world scenarios

Applications of Gradients

Gradients are an important part of life. The roof of a house is built with a gradient to enable rain water to run down the roof. An aeroplane ascends at a particular gradient after take-off, flies at a different gradient and descends at another gradient to safely land. Tennis courts, roads, football and cricket grounds are made with a gradient to assist drainage.

Activity 11

Example 18

A horse gallops for 20 minutes and covers a distance of 15 km, as shown in the diagram. Find the gradient of the line and describe its meaning

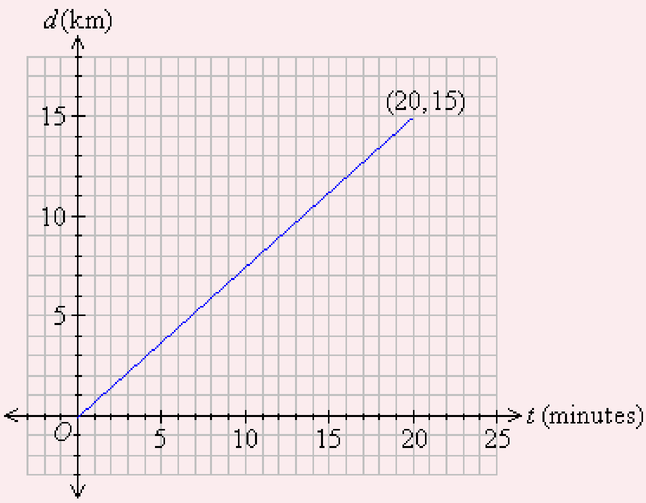


Figure 11: Gradient of a line

Finding the Gradient of a Horse's Gallop

Solution:

Let $(t_1, d_1) = (0, 0)$ and $(t_2, d_2) = (20, 15)$

Step 1: Identify the coordinates

- Time (x-axis): 0 minutes to 20 minutes
- Distance (y-axis): 0 km to 15 km

Step 2: Write the formula

$$m = \frac{d_2 - t_2}{d_1 - t_1}$$

Step 3: Plug in the values

$$m = \frac{(15\ km - 0\ km)}{(20\ minutes - 0\ minutes)}$$

Step 4: Simplify and calculate

$$m = \frac{15\ km}{20\ minutes}$$

$$m = \frac{3}{4} km/minute$$

The gradient of the horse's gallop is $m = \frac{3}{4} km/minute$

Meaning: So, the gradient of the line is 3/4 km/min. In the above example, we notice that the gradient of the distance-time graph gives the speed (in kilometres per minute); and the distance covered by the horse can be represented by the equation: $d = \frac{3}{4}t$ ($\therefore$ Distance = Speed × Time)

Interpretation: The gradient is a measure of the horse's speed. A steeper gradient would indicate a faster speed, while a shallower gradient would indicate a slower speed.

Hello learner, now that you have completed the task of determining the gradient of a horse's gallop, let's move on to another example.

Example 19

The cost of transporting documents by courier is given by the line segment drawn in the diagram. Find the gradient of the line segment; and describe its meaning.

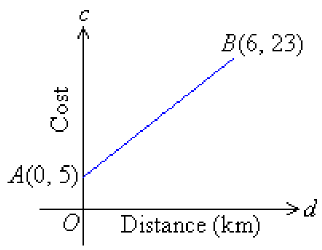


Figure 12: Gradient of a line

Solution

Let $(d_1, c_1) = (0, 5)$ and $(d_2, c_2) = (6, 23)$

$$\text{Now, } m = \frac{d_2 - c_2}{d_1 - c_1}$$

$$= \frac{23 - 5}{6 - 0}$$

$$= \frac{18}{6}$$

$$= 3$$

So, the gradient of the line is 3. This means that the cost of transporting documents is GH¢ 3 per km plus a fixed charge of GH¢ 5, i.e. it costs GH¢5 for the courier to arrive and GH¢ 3 for every kilometre travelled to deliver the documents.

Hello learners,

Now that we have explored the applications of gradients, let's take the next step and learn how to find the gradient of a straight line when given its equation.

Finding the Gradient of a Straight Line Given the Equation

We can determine the gradient from a given equation of a straight line when the equations are given in the form or can be rearranged to be in the form $y = mx + c$, where $\boldsymbol{m}$ is the gradient.

Example 20

Given the equation $2y - 6x = 12$, first rewrite the equation in the general form $y = mx + c$, where m is the gradient.

Therefore, $2y - 6x = 12$ can be written as $2y = 6x + 12$

Now, making one y the subject we have $y = 3x + 6$.

Since our new equation is in the general form, we compare and identify the gradient. Hence, the gradient of the equation $2y - 6x = 12$ is 3.

Now that you have grasped the example above, please proceed with this activity

Activity 12

Find the gradient of the equation $3y + 2x = 7$.

Let's solve it with the steps below:

Step 1: Write the equation in slope-intercept form ($y = \mathrm{m}x + \mathrm{c}$)

To find the gradient (m), we need to rewrite the equation in slope-intercept form.

Step 2: Subtract $2x$ from both sides

$3y + 2x - 2x = 7 - 2x$

Step 3: Simplify

$3y = 7 - 2x$

Step 4: Divide both sides by 3

$\frac{3y}{3} = \frac{7 - 2x}{3}$

Step 5: Simplify

$\mathrm{y} = \frac{7}{3} - \frac{2}{3}x$ or $y = -\frac{2}{3}x + \frac{7}{3}$

Step 6: Identify the gradient (m)

The gradient (m) is the coefficient of x, which is $-\frac{2}{3}$.

Therefore, the gradient of the equation $3y + 2x = 7$ is $-\frac{2}{3}$.

Now that we've learned how to determine the gradient of a given equation, let's move on to finding the equation of a straight line.

Finding the Equation of a Straight Line

Example 21

Find the equation of the line with gradient -2 that passes through the point $(3, -4)$.

Solution

Put m $= -2$, $x_1 = 3$ and $y_1 = -4$ into the formula $\boldsymbol{y - y_1 = m(x - x_1)}$

$y - y_1 = m(x - x_1)$

$y - -4 = -2(x - 3)$

Expand the brackets and simplify,

$y + 4 = -2x + 6$

$\mathbf{y = -2x + 2}$

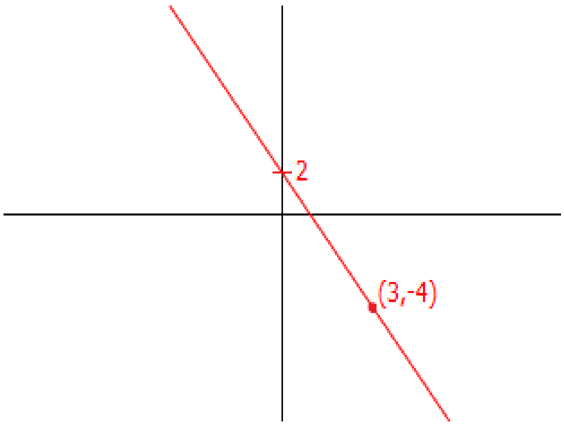


Figure 13: Gradient of a line

MAGNITUDE OF A LINE SEGMENT (INCLUDING PARALLEL, PERPENDICULAR LINES AND MIDPOINT)

Perpendicular Lines

Definition: Perpendicular lines are two lines that intersect at a 90° angle, forming right angles.

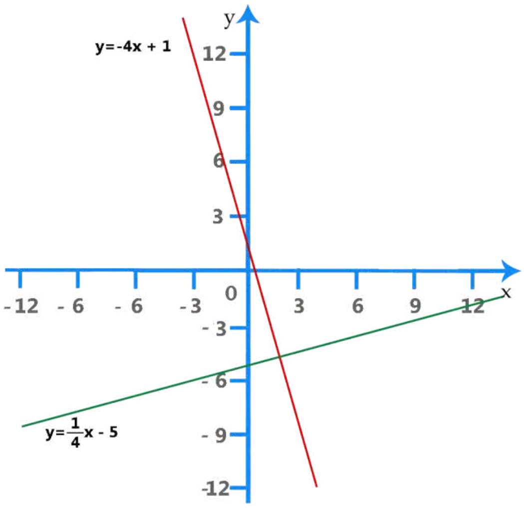


Figure 14: Perpendicular lines

Real-Life Example: Consider a door frame and the floor. The door frame (vertical line) and the floor (horizontal line) intersect to form a right angle, making them perpendicular.

Figure 15: Perpendicular lines

Parallel Lines

Definition: Parallel lines are two or more lines that never intersect. They have the same slope and are equidistant from each other.

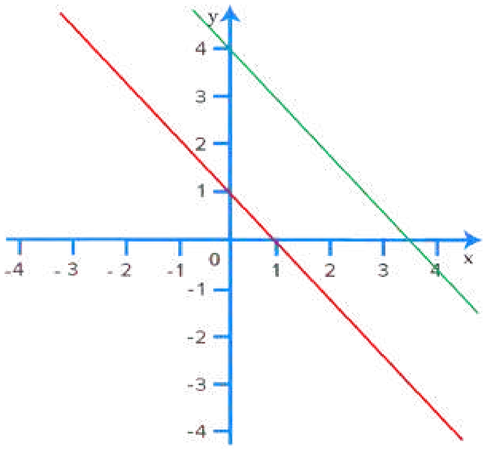


Figure 16: Parallel lines

Real-Life Example: Look at railroad tracks. The two tracks run alongside each other, never converging or intersecting. This demonstrates parallel lines in real life.

Figure 17: Railway lines showing Parallel lines

Example 22

Identify which of the lines are parallel and which perpendicular.

i. $y = 3x + 1$

ii. $y = 3x + 12$

iii. $y = \frac{1}{4}x - 5$

iv. $y = -4x + 1$

Solution

Parallel lines have the same slope. Since the functions $y = 3x + 1$ and $y = 3x + 12$ each have the same gradient ($= 3$), they represent parallel lines.

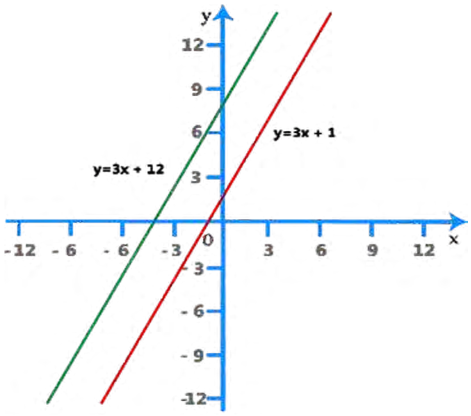


Figure 18: Parallel lines

Perpendicular lines have negative reciprocal slopes. Since -4 and $\frac{1}{4}$ are negative reciprocals the equations

$y = \frac{1}{4}x - 5$ and $y = -4x + 1$, they represent perpendicular lines.

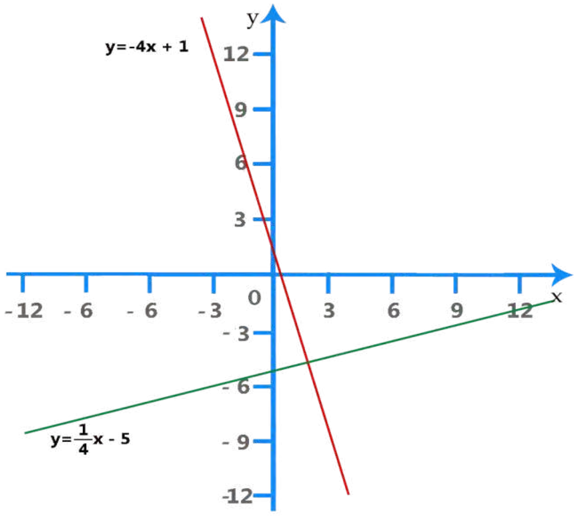


Figure 19: Perpendicular lines

Activity 13

Example 23

Find the equation of a line that is perpendicular to the line $y = 2x - 2$ and goes through the point (1,3).

Solution

We know that the general equation of a straight line is $y = mx + c$.

Firstly, we need to find the gradient of the line $y = 2x - 2$.

If we label the gradient of our line as m_1 and the gradient of the line that is perpendicular with our line m_2 then we know that the product of those two gradients should be -1.

The gradient of our line is $m_1 = 2$

Meanwhile the gradient of the line perpendicular to our line is:

$$m_2 = -\frac{1}{m_1}$$

$$m_2 = -\frac{1}{2}$$

After finding the gradient the equation of the line we want to find takes the form $y = -\frac{1}{2}x + c$

To find the value of c, we substitute the point (1,3) on our equation, since the graph of this line passes through this point.

$$y = -\frac{1}{2}x + c$$

$$3 = -\frac{1}{2} \times 1 + c$$

$$3 = -\frac{1}{2} + c$$

$$c = 3 + \frac{1}{2}$$

$$c = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$$

The final form of our line that is perpendicular with the given line is: $y = -\frac{1}{2}x + \frac{7}{2}$

Example 24

Find the equation of a line that is parallel to the line $x + y - 1 = 0$ and goes through the point (-1, 1).

Solution

Firstly, we rewrite our line $x + y - 1 = 0$ in the correct form $y = -x + 1$

Then we find the gradient of our line that is $m_1 = -1$

We know that parallel lines have the same gradient so the gradient of the line we are going to find is $m_1 = m_2 = -1$

The line takes the form $y = -x + c$

To find the value of c, we substitute the point (-1,1) into our equation, since the graph passes through this point.

$y = -x + c$

$1 = -(-1) + c$

$1 = 1 + c$

$c = 1 - 1 = 0$

The final form of our parallel line is $y = -x$.

Finding the Magnitude of Line Segment

The magnitude of a line also known as its "length", "distance" or "modulus" describes the length of a line linking two points. Relating length of objects discussed from activities, if P and Q have coordinates (x_1, y_1) and (x_2, y_2). From the figure below:

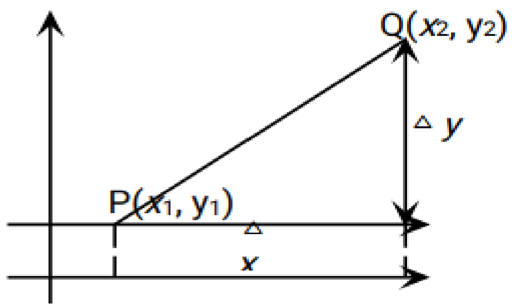


Figure 20: Magnitude of Line Segment

$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

Where Δ means a change

By Pythagoras theorem,

$|PQ|^2 = \Delta x^2 + \Delta y^2$

$|PQ| = \sqrt{\Delta x^2 + \Delta y^2}$

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

So, if P (x_1, y_1) and Q (x_2, y_2) are two points in the x-y plane, then the distance between P and Q is

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note: The symbol (D) called delta as used here implies change in x_2, x_1 and y_2, y_1 or the differences in their values.

Example 25

Determine the distance between the points

(a) P(2, 1) and Q(5, 5)

(b) A(7, -3) and B(-1, 5)

(c) D(4, 1) and E(-3, -5)

Solution

The distance between two points given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(a) P(2, 1) and Q(5, 5)

$|PQ| = \sqrt{(5-2)^2 + (5-1)^2} = \sqrt{3^2 + 4^2}$

$|PQ| = \sqrt{25} = 5 \text{ } units$

(b) A (7, -3) and B(-1, 5)

$\Rightarrow |AB| = \sqrt{(-1-7)^2 + (5+3)^2} = \sqrt{(-8)^2 + 8^2}$

$|AB| = \sqrt{128} \text{ } units$

(c) D(4, 1) and E(-3, -5)

$|DE| = \sqrt{(-3-4)^2 + (-5-1)^2} = \sqrt{(-7)^2 + (-6)^2}$

$|DE| = \sqrt{49 + 36} = \sqrt{85} \text{ } units$

Determine the Midpoint of a Line

What is a midpoint?

In geometry, the midpoint is the middle point of a line segment. It is equidistant from both endpoints, and it is the midpoint of both the segment and the endpoints. It bisects the segment.

For example: Consider the following line segment and the points A (3, 4) and B (5, 10).

We can determine the mid-point.

HOW?

If you add both x co-ordinates and then divide by two you get

$\frac{(3+5)}{2} = \frac{8}{2} = 4$

If you add both y co-ordinates and then divide by two you get

$\frac{(4 + 10)}{2} = \frac{14}{2} = 7$

This gives a new point with co-ordinates (4, 7). This point is exactly halfway between A and B.

Activity 14

Tom and Alex are planning a road trip from City A to City B. City A is located at (30, 40) on a map, and City B is located at (60, 80). If they stop for lunch at the midpoint of their journey, what are the coordinates of the lunch spot?

Step 1: Find the midpoint formula

The midpoint formula is:, $\left(\frac{x_1 + x_2}{2}\right), \left(\frac{y_1 + y_2}{2}\right)$

Step 2: Identify the coordinates

City A: $(x_1, y_1) = (30, 40)$

City B $(x_2, y_2) = (60, 80)$

Step 3: Plug in the values

Midpoint $= (\frac{30 + 60}{2}, \frac{40 + 80}{2})$

Step 4: Calculate the coordinates

Midpoint $= (\frac{90}{2}, \frac{120}{2})$

Midpoint = (45, 60)

The coordinates of the lunch spot are (45, 60).

REVIEW QUESTIONS

Review Questions 4.1

1. Solve $\frac{1}{2}x - \frac{1}{3}(x + 4) > 4x + \frac{2}{3}$
2. If Kofi's age is 30 years now, what is his age in 5 years' time?
3. If Ama is 40 years now, what was her age 4 years ago?
4. Solve the following equations:
   i. $6 + y - 2 = 12$
   ii. $14y - 5 = 2y$
   iii. $10(y + 2) = 14$
5. Find the solution set of the following inequalities and illustrate your answer on the number line;
   i. $3x - 2 \geq 12 - x$
   ii. $x - \frac{2}{3}(x + 1) \leq \frac{1}{2}(4 - x) - 5$
6. Answer the following questions;

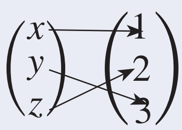
i. The sum of four consecutive even numbers is 36. Find the numbers.

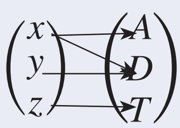
ii. The perimeter of a football field in a rectangular form of a certain school is 296m. If the breadth is $\frac{2}{3}$ of the length, find the length.

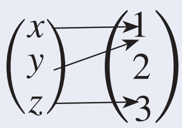
iii. Find the number N, such that when $\frac{1}{3}$ of it is added to 8, the result is the same as 18 from $\frac{1}{2}$ of it.

Review Questions 4.2

1. Identify whether each of the mappings below are functions. Give a reason for each answer.

i. 

ii. 

iii. 

2. Which of the following relations defined on the set of real numbers are functions? Give a reason for your answers.

$A = (x, y) : y = 3x + 1$

$B = (x, y) : y = 2x^2$

$C = (x, y) : y^2 = x$

3. Draw mapping diagrams to represent the following

i. $y = 2x^2 - 1$, where $\{x : x = 0, 1, 2, 3, 4\}$

ii. $y = 3x + 2$, where $\{x : x = 0, 1, 2, 3, 4, 5\}$

4. Find the rule of the following mapping

$$\begin{array}{ccccc} \mathbf{P} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{q} & \mathbf{17} & \mathbf{21} & \mathbf{25} & \mathbf{29} \end{array}$$

5. Draw the following linear graphs and interpret them

i. $y = 2x + 1$

ii. $y = -3x + 1$

iii. $y = -5x + 1$

6. A Bakery sells a total of 250 loaves of bread per day. The number of whole wheat loaves (x) and white loaves (y) sold are related by the equation $y = 150 - x$. How many whole wheat loaves are sold if 75 white loaves are sold?

7. A person's height in inches (h) is related to their shoe size (s) by the equation, $h = 5s + 20$.

What is the person's height if their shoe size is 8 ?

8. Find the rule for the mapping below.

$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 4 & 8 & 16 \end{array}$$

9. Find the rule for the mapping below.

$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \end{array}$$

Review Questions 4.3

1. Suppose A is the point (3, 4) and B is the point (8, 14). What is the gradient/ straight line joining these points?
2. Calculate the gradient of the straight line given the coordinates A (2,6) and B (8,24)
3. Find the midpoint of the line joining the points or with end points of (1,3) and (5,6).
4. Show that the line segment joining the points (1, 4) and (3, 10) is parallel to the line segment joining the points (−5, −10) and (−2, −1).
5. Are the lines L1 through (2, 3) and (4, 6) and L2 through (-4, 2) and (0, 8) parallel, or do they intersect? Give reason for your answer.
6. Determine the equation of the line perpendicular to $2y + 3x = 6$ which goes through the point (5, 2)

EXTENDED READING

- Akrong Series: Core mathematics for Senior High Schools New International Edition (Pages 164 – 170)
- Aki – Ola series : Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 197 – 203)
- Baffour – Ba Series: Core Maths for Schools and Colleges, (Pages 140 - 149)

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- Akrong Series: Core mathematics for Senior High Schools New International Edition

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