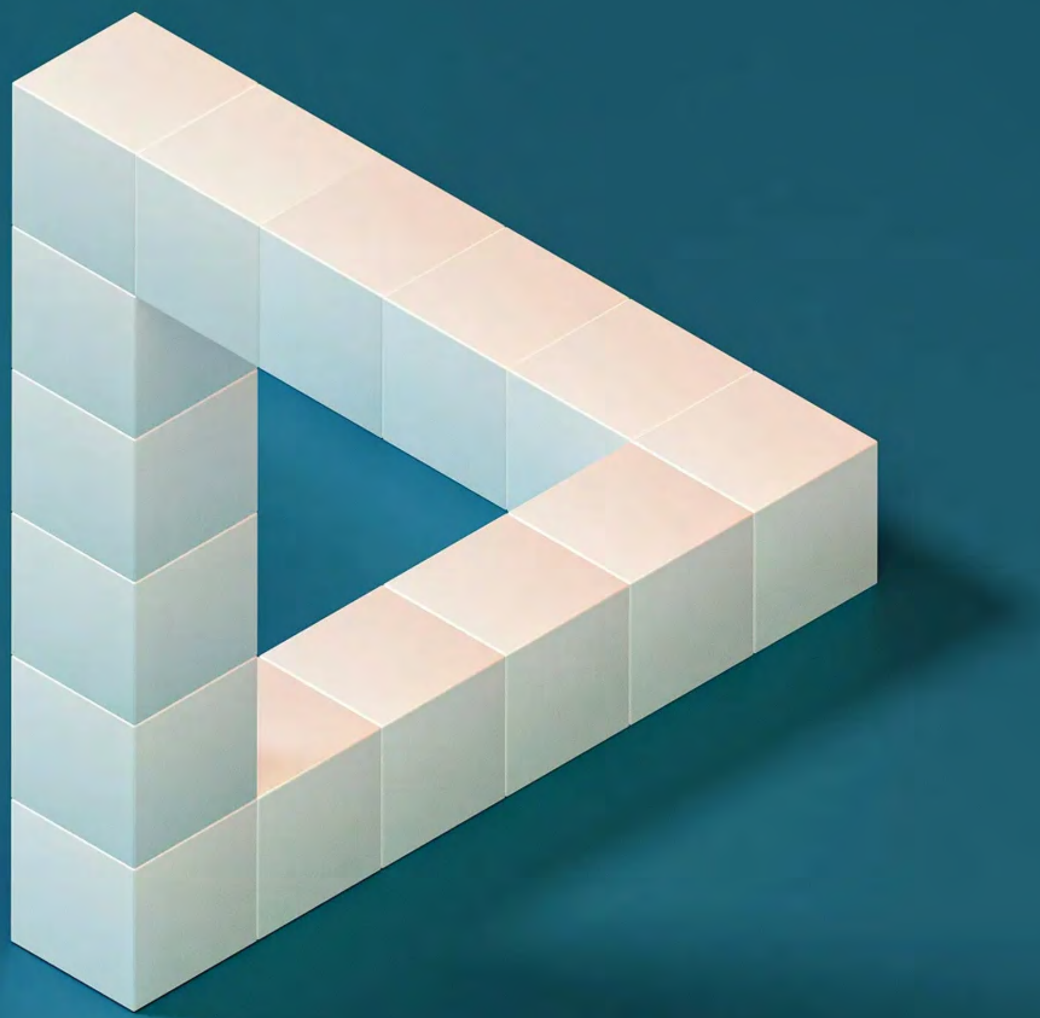
Mathematics

Year 1

SECTION

5

ANGLES & THE PYTHAGOREAN THEOREM



GEOMETRY AROUND US

Spatial Sense

INTRODUCTION

This section focuses on hands-on activities for learning all about angles and the Pythagorean theorem. You will engage in practical measurements, collaborate in teams and work in mixed-groupings to enhance your comprehension. For example, you will identify and measure angles in their surroundings to understand referents for angles. You will use protractors and paper folding to replicate and bisect angles, categorising them based on measurements. Real-life angle problems will be tackled collaboratively to develop critical thinking. You will also explore parallel lines, perpendicular lines, and transversals by creating and measuring angles in geometric shapes and engage in hands-on tasks to grasp complementary and supplementary angles and analyse angles formed by parallel lines and a transversal. Furthermore, you will solve real-world problems involving these concepts. The section also covers the exterior angle theorem of a triangle and the sum of interior angles of polygons. You will learn all about the Pythagorean theorem through practical activities, showcasing its applications in architecture, engineering, and navigation. You will apply this theorem to identify right triangles, reinforcing your understanding through practical applications

At the end of this section, you will be able to:

- Draw and describe angles with various measures, including acute, right, straight, obtuse and reflex angles.
- Solve problems that involve parallel lines, perpendicular lines and transversal and pairs of angles formed between them.
- State and use the exterior angle theorem of a triangle and identify various properties of special triangles.
- State and use the properties of quadrilaterals and calculate the sums of interior angles and exterior angles of a polygon.
- Solve problems on Pythagorean Theorem by identifying situations that involve right triangles, verify the formula and apply it.

Key ideas

- A triangle with one angle measuring **90°**. The Pythagorean Theorem applies exclusively to right-angled triangles.
- **The Pythagorean Theorem**: This is a theorem stating that in a right-angled triangle: $a^2 + b^2 = c^2$ where a and b are the lengths of the two shorter sides (legs), and c is the length of the hypotenuse (the side opposite the right angle).
- **Hypotenuse**: This is the longest side of a right-angled triangle, opposite the right angle. It is the side used in the Pythagorean Theorem as cc.
- **Legs of a Triangle**: This is the two shorter sides of a right-angled triangle that form the right angle. It is represented as a and b in the Pythagorean Theorem.
- **Special Right Triangles**: Right-angled triangles with specific angle measures and fixed side ratios, such as 45°-45°-90° and 30°-60°-90° triangles. It helps simplify calculations using known ratios, like $1:1:\sqrt{2}$ for 45°-45°-90° triangles.

REFERENTS FOR ANGLES

A referent is an object/item that can be used to help understand/represent a concept. Some referents of angles are corners of rooms and doors, the human palm, tree branches, adjustable chairs, etc.

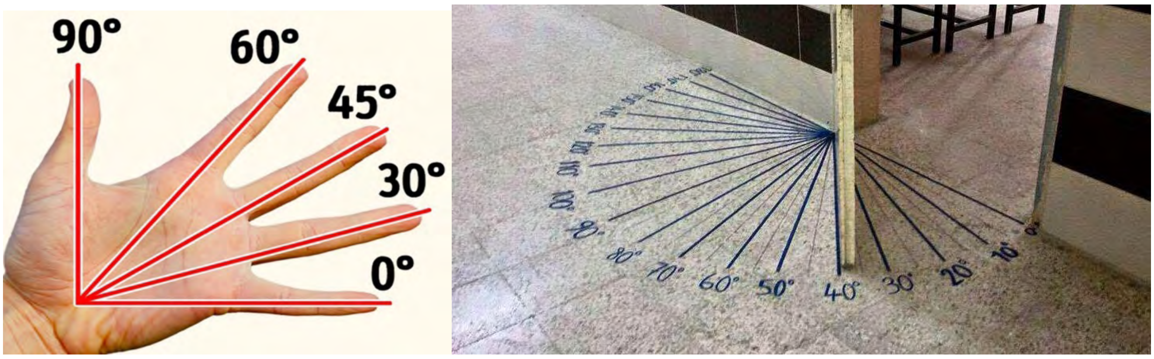


Figure 1: Angles in real Life

Sketching Common Angles Around Us

We can make free hand sketch of angles such as acute, right, straight, obtuse and reflex angles and verify the closeness of your sketch with the actual angle.

Measuring Angles such as 30°, 45°, 60°, 75°, 90° and 180°

We can measure angle sizes using a protractor.

Examples

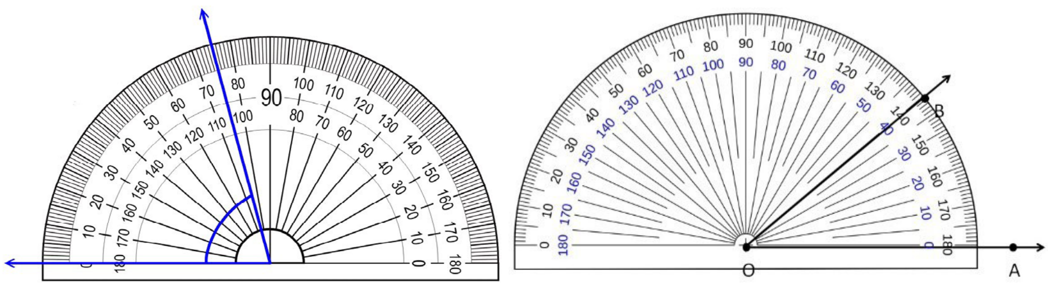


Figure 2: Measuring angles with protractor

The first angle is read clockwise from the 0 mark on the left hand side, so it is 75°. The second angle is read anticlockwise from the 0 mark on the right hand side, so it is 40°.

Replicating Angles

Angles can be replicated in a variety of ways; examples include, protractor, compass and straightedge, carpenter's square, geometry software (e.g., Geiger).

Constructing Angles

We can construct various angles using the protractor as well as a pair of compass just as you learnt in JHS. Let us try an example of each process. Now, let us undertake activities to help us remember how to construct angles using a protractor.

Activity 1

Method 1: Constructing Angles Using a Protractor

For constructing angles of any given measure, be it an acute, an obtuse or a right-angle, the simplest method is by using a protractor.

Let us say, you are asked to **construct an angle of 120 degrees**. The required steps are:

Step 1: Draw a line segment BC, which is one of the arms of the angle that is to be constructed.

Figure 3 Figure 1: Line segment BC

Step 2: Place the protractor with its point O on point B of the line segment BC.

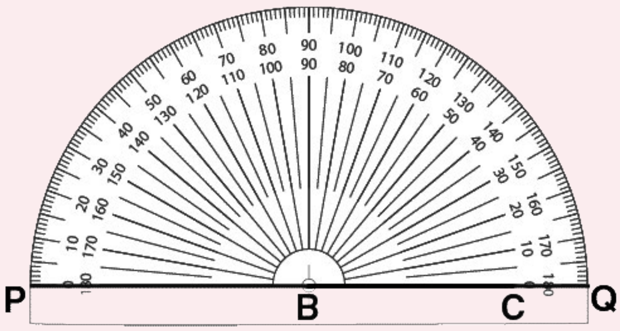


Figure 4: Protractor on the line PQ

Step 3: Align OQ along the edge BC.

Step 4: The protractor has two-way markings. We consider the scale which has 0 degrees near point C for construction. Follow this around to mark point A next to the 120 degrees' mark on the scale.

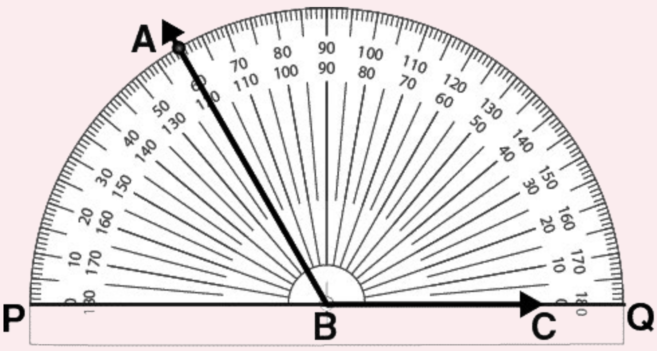


Figure 5: Protractor showing angle 120°

Step 5: Join points A and B. ∠ABC = 120 degrees is the required angle.

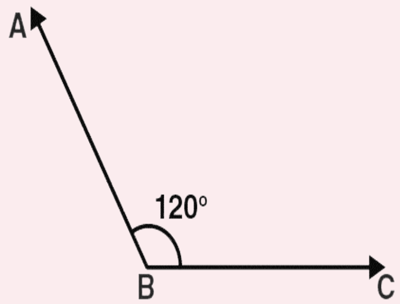


Figure 6: Angle 120°

After we complete constructing a 120° angle using a protractor, we'll move on to another activity to construct a 45° angle. This will further enhance your understanding of angle construction

Activity 2

To **construct a 45° angle** using a protractor, follow these steps:

Step 1: Start by drawing a straight-line segment (label it AB).

Figure 7: Line segment AB

Step 2: Place the centre of the protractor at point A, aligning the baseline of the protractor with line AB.

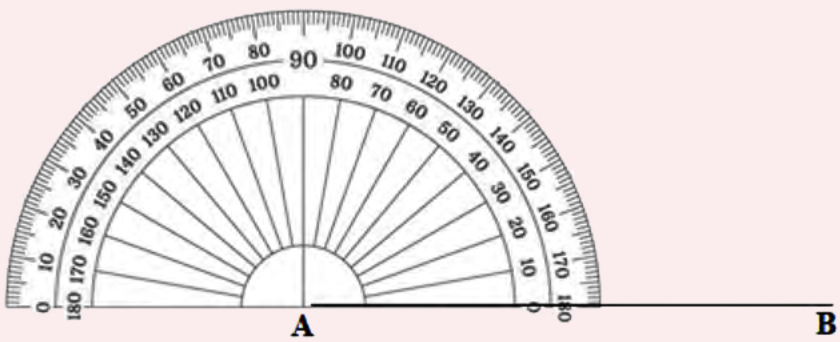


Figure 8: Protractor on the line AB

Step 3: Look for the 45°mark on the protractor (moving Anti-clockwise) and make a small mark at this point (label it C).

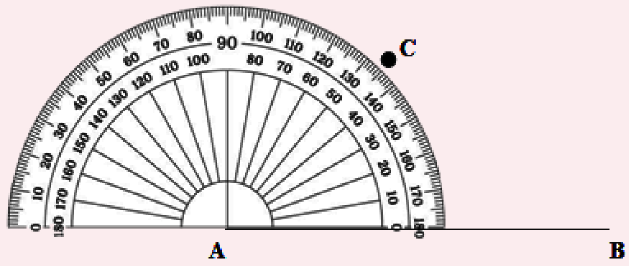


Figure 9: Protractor showing angle 45°

Step 4: Draw the Angle; Use a ruler to draw a line from point A through point C. The angle CAB is 45°

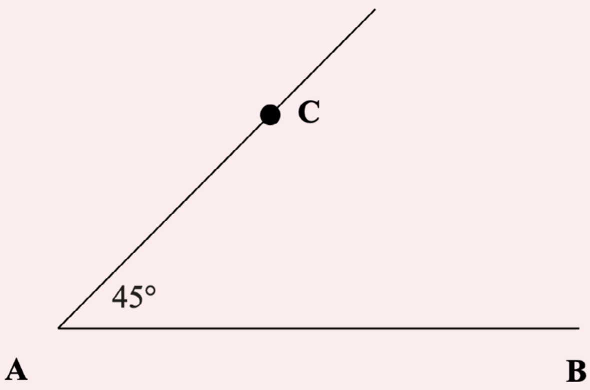


Figure 10: Angle 45°

Click on the links below to watch How to Use a Protractor to Measure and Draw Angles Explained from the Right and Left Side or scan the QR code

(i) https://youtu.be/CHJ7_q4cCuE

(ii) https://youtu.be/nrFOrFPpn2o

(iii) https://youtu.be/c3ILHIJXj4o

(i) (ii) (iii)

Practice

Use a Protractor to construct the following angles and compare your work with your classmates

(a) 75°

(b) 60°

(c) 135°

Now you should have angles sorted with protractors, but there's more to explore. In many geometric situations, a compass is the perfect tool for constructing precise angles. Now it's time to transition to this new technique and learn how to create accurate angles with a compass. Get ready to enhance your geometric skills!

Method 2: Constructing Angles Using a Compass

Constructing angles of unknown measure is basically copying a given angle whose measure is not known. We accomplish this task using compasses. Let us say that you are given ∠BAC that you want to copy.

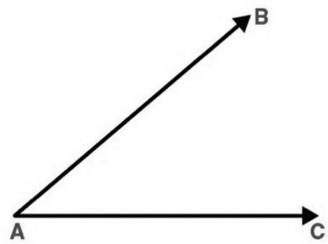


Figure 11: Angle BAC

The steps to construct angles using compass are given below:

Step 1: Draw a line PQ. Point P is the vertex of the copied angle.

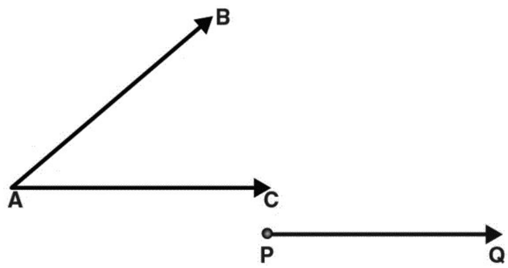


Figure 12: Line PQ and Point P showing the vertex of the copied angle

Step 2: Place the compass pointer at point A and make an arc that cuts arms AC and AB at points K and J respectively.

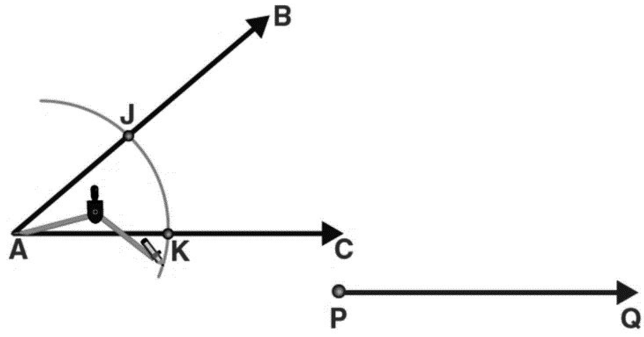


Figure 13: An arc cutting AC and AB at points K and J

Step 3: Without changing the radius of the compass, cut an arc on PQ at point M.

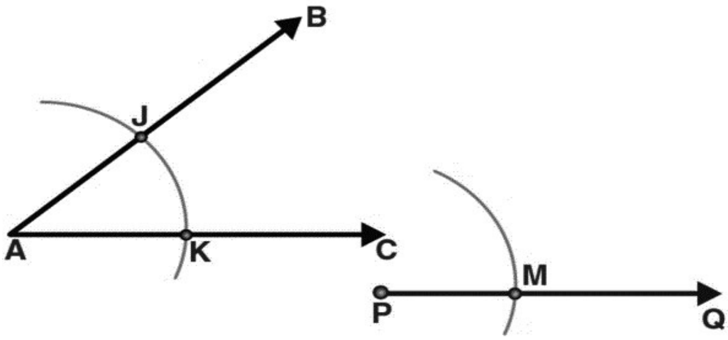


Figure 14: An arc on PQ at point M

Step 4: Adjust the compass such that the pointer is placed at K and the pencil head at J

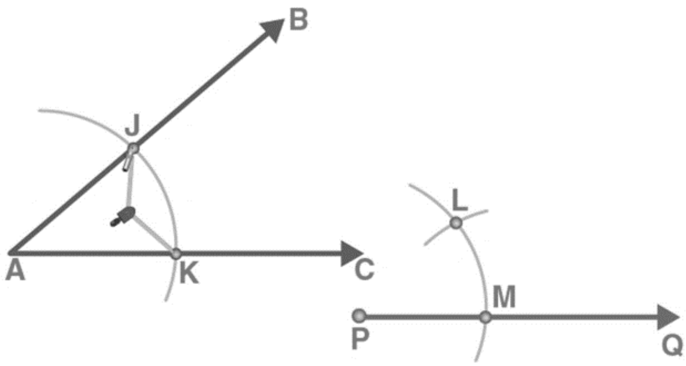


Figure 15: An arc labelled L

Step 5: With the same radius, draw an arc on the first arc with the compass pointer at M. Mark the intersecting point as L.

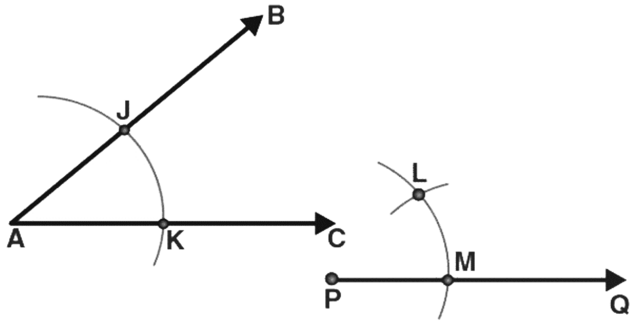


Figure 16: An arc labelled L

Step 6: Join the points P and L using a ruler. Extend the line.

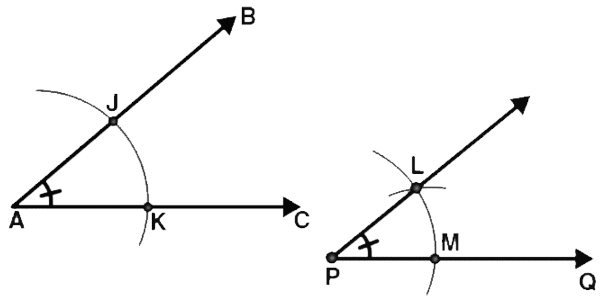


Figure 17: Copied angle

Step 7: ∠RPQ is the required angle.

Activity 3

Constructing a 45°angle using a compass

Materials Needed

- A compass
- A straightedge (ruler)
- A pencil
- A piece of paper

Step 1: Draw a line segment (let's call it AB with A being towards one end of the line, not at the end).

Step 2: Open your compass to a comfortable width and place the point end on point A.

Step 3: Draw arcs either side of point A on the line AB.

Step 4: Placing your compass point on each newly drawn arc in turn (without changing the compass width between arcs), draw an arc above line segment AB, making sure the new arcs intersect. If you draw a line up from A to the point of intersection of your arcs you will see that you have drawn a right angle.

Step 5: Now place your compass point on the new arc intersection and draw an arc somewhere above the line AB.

Step 6: Without changing your compass width, place your compass point on the arc you drew originally on the line between A and B. Draw another arc above line segment AB, making sure it intersects the first arc from step 5. Label this intersection C.

Step 7: Draw a line segment from point A to the intersection point C.

Step 8: Measure the angle formed by line segments AC and AB. It should be 45 degrees. If you carefully follow the instructions provided, you will achieve a construction similar to the images shown:

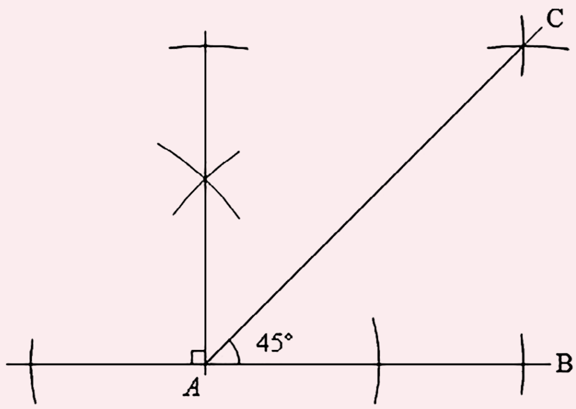


Figure 18: Constructed Angle 45°

Bisecting Angles

Angles can be bisected to generate other angles that are required to carry out a particular activity. In real life situations, we bisect angles in order to be able to make models such as building construction, carpentry works, etc. (If you look at the angle we constructed above, of 45^0, you will see that what we actually did was bisect the right angle we created.)

Let's carry out this activity to bisect the given angle AOB.

Activity 4

Step 1: Given Angle AOB

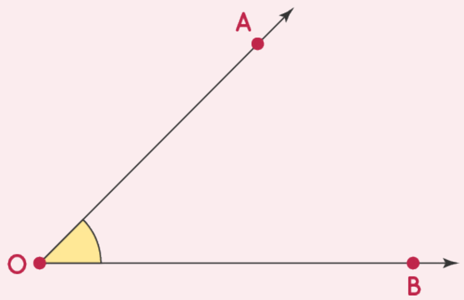


Figure 19: Angle AOB

Step 2: Span any width of radius in a compass and with O as the centre, draw two arcs such that it cut the rays OA and OB at points C and D respectively.

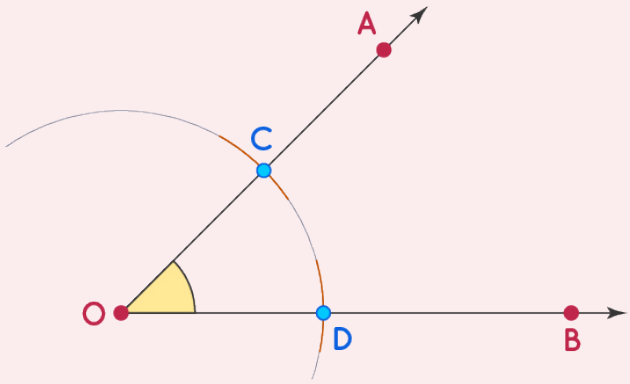


Figure 20: An arc cutting the rays OA and OB

Step 3: Without changing the distance between the legs of the compass, draw two arcs with C and D as centres, such that these two arcs intersect at a point named E.

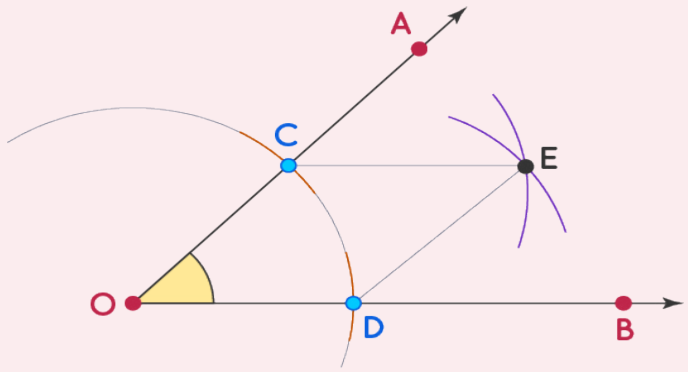


Figure 21: Arcs intersecting at C

Step 4: Join the ray OE. This is the required angle bisector of angle AOB.

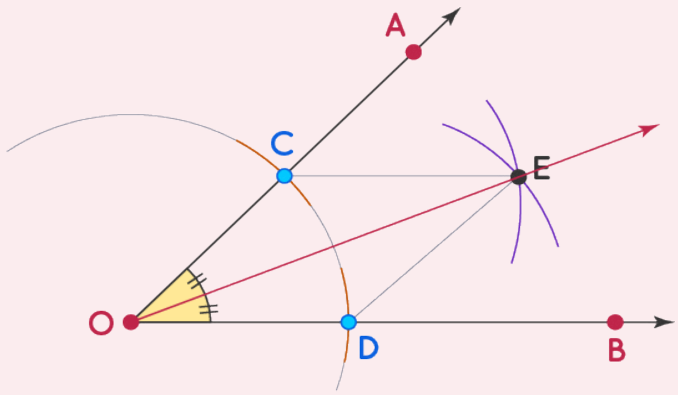


Figure 22: Angle bisector of AOB

Task

Complete this task and share your results with your teacher so they can check if you're on the right track.

Practice

Use Compass to construct the following angles and compare your work with your classmates. Discuss how you might manage this with only compasses and a straight edge.

(a) 75°

(b) 60°

(c) 120^0

(d) 135°

Types of Angles

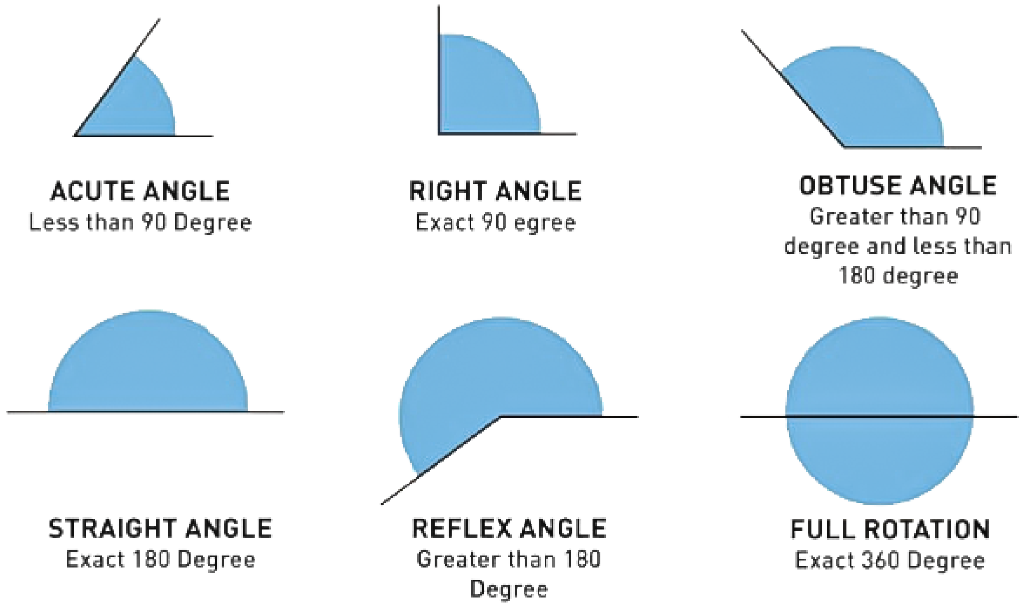


Figure 23: Types of angles

Pair of Angles

When two angles are paired, then there exist different angles, such as;

- Complementary angles
- Supplementary angles
- Linear Pair
- Adjacent angles
- Vertically Opposite angles

Parallel lines, Perpendicular lines and Transversal

Perpendicular Lines: Perpendicular lines are two lines that intersect at a 90-degree angle, forming right angles.

Real-Life Example*:* Consider a door frame and the floor. The door frame (vertical line) and the floor (horizontal line) intersect to form a right angle, making them perpendicular.

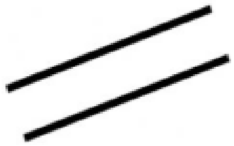
Parallel Lines: Parallel lines are two or more lines that never intersect. They have the same gradient and are equidistant from each other.

Real-Life Example: Look at railroad tracks. The two tracks run alongside each other, never diverging or intersecting. This demonstrates parallel lines in real life.

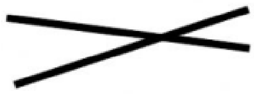
Transversal: A transversal is a line that intersects with two or more other lines at distinct points.

Real-Life Example: Imagine a set of power lines suspended on poles at either side of a road. The road acts as the transversal line that intersects the power lines at multiple points.

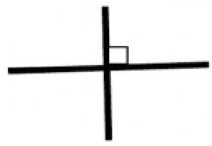
Determine whether these lines as perpendicular, parallel or neither, and justify it.



Parallel lines



Neither



Perpendicular lines

Complementary and Supplementary Angles

Complementary Angles: Complementary angles are two angles that add up to 90 degrees.

Real-Life Examples: Consider the door opening. The two angles made by the door will always add up to 90, as the door swings open.

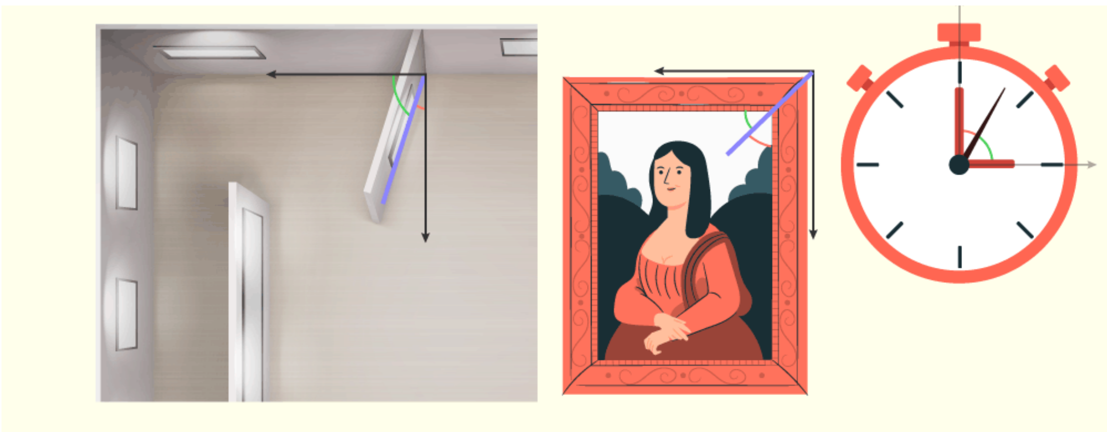
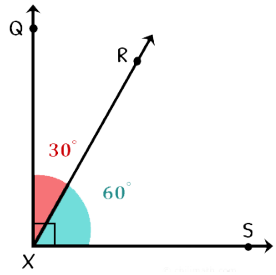
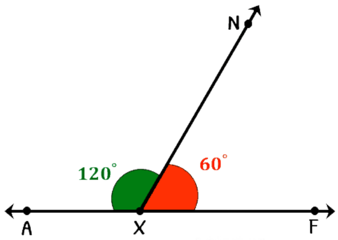


Figure 24: Real-life examples of complementary angles

Supplementary Angles: Supplementary angles are two angles that add up to 180 degrees.

Real-Life Example: Think about a straight road with a street sign or a lamppost. If you stand at the base of the sign and measure the angle to the left and the angle to the right of the road, those two angles will be supplementary because they add up to a straight angle of 180 degrees.

Let's take a look at these examples.

Type of Angles	Description	Example
Complementary Angles	Angles that add up to 90°	
Supplementary Angles	Angles that add up to 180°	

Real-life Problems Involving Angles

Example 1

Kwamena cut the straight edge of a piece of wood to make a 45^0 angle. At what angle was the other piece cut?

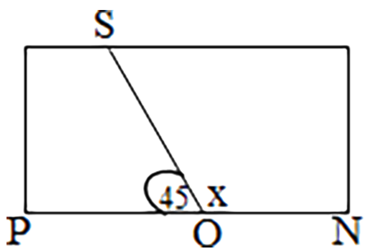


Figure 25: Angle 45°

Think:

We need to find ∠SON

Since PON is a straight line, its angle is 180º

Therefore, ∠PON = ∠POS + ∠ SON

∠PON – ∠POS = ∠SON

180º – 45º = 135º

Therefore, the other piece cut is at an angle of 135º

Example 2

Two angles are complementary. If one of the angles is double the other angle, find the two angles.

Solution

Let x be one of the angles

Then the other angle is $2x$.

Because x and $2x$ are complementary angles, we have;

$x + 2x = 90$

$3x = 90$

Divide each side by 3.

$x = 30$

$2x = 2(30) = 60$

So, the two angles are 30° and 60°

Pairs of angles formed by parallel lines and a transversal

Study the diagram carefully,

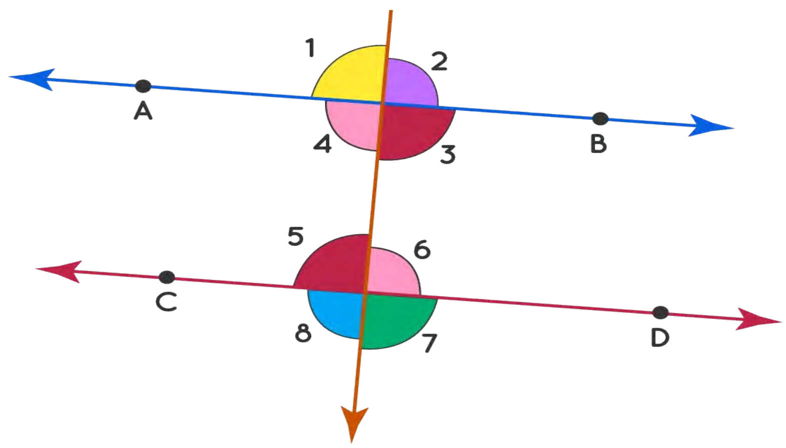


Figure 26: Parallel lines cut by transversal

A) In the figure given above, the *corresponding angles* formed by the intersection of the transversal are:

- $\angle 1$ and $\angle 5$
- $\angle 2$ and $\angle 6$

- $\angle 3$ and $\angle 7$
- $\angle 4$ and $\angle 8$

It should be noted that the **pair of corresponding angles are equal** in measure, that is: $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, and $\angle 4 = \angle 8$

B) In the figure given above, there are two pairs of *alternate interior angles.*

- $\angle 3$ and $\angle 5$
- $\angle 4$ and $\angle 6$

It should be noted that the **pair of alternate interior angles are equal** in measure, that is: $\angle 3 = \angle 5$, and $\angle 4 = \angle 6$

C) In the figure given above, there are two pairs of *alternate exterior angles*.

- $\angle 1$ and $\angle 7$
- $\angle 2$ and $\angle 8$

It should be noted that the **pair of alternate exterior angles are equal** in measure, that is: $\angle 1 = \angle 7$, and $\angle 2 = \angle 8$

D) In the given figure, there are two pairs of *consecutive interior angles*.

- $\angle 4$ and $\angle 5$
- $\angle 3$ and $\angle 6$

It should be noted that unlike the other pairs given above, the **pair of consecutive interior angles are supplementary,** that is: $\angle 4 + \angle 5 = 180°$, and $\angle 3 + \angle 6 = 180°$.

E) In the figure given above, there are four pairs of *vertically opposite angles.*

- $\angle 1$ and $\angle 3$
- $\angle 2$ and $\angle 4$
- $\angle 5$ and $\angle 7$
- $\angle 6$ and $\angle 8$

It should be noted that the **pair of vertically opposite angles are equal** in measure, that is: $\angle 1 = \angle 3$, $\angle 2 = \angle 4$, $\angle 5 = \angle 7$ and $\angle 6 = \angle 8$.

Click on the links below to watch Angles formed by Parallel Lines Cut by a Transversal

(i) https://youtu.be/5PcMbN46NMA

(ii) https://youtu.be/IyCOaXMFGLE

(i) **(ii)**

Solve problems on parallel lines, perpendicular lines and transversal

Example 1

A teacher asked Kofi to draw two parallel lines. With the help of his set squares and ruler, he drew a straight-line segment AB and then placed the set square on this line and drew two line segments XY and PQ, by changing the position of the set squares as shown.

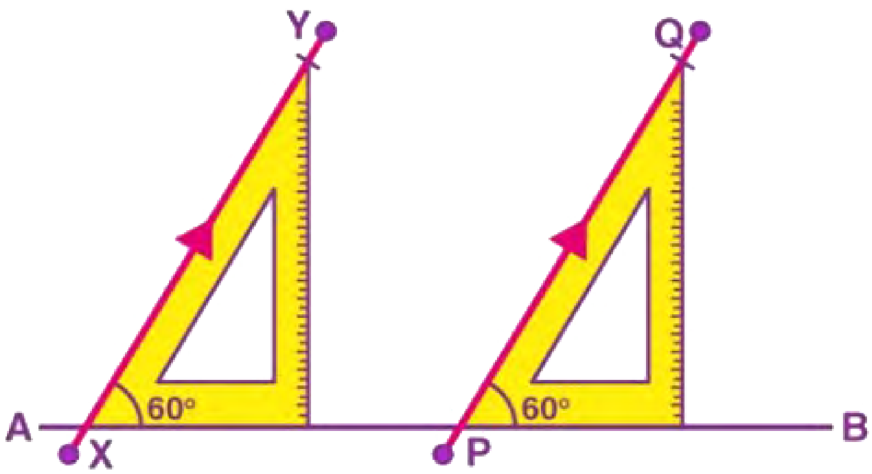


Figure 27: Set square showing parallel lines

Kofi claimed that XY and PQ are parallel. Can you tell how?

Let's now engage in activities that will help you measure angles involving parallel lines and a transversal, using their relationships.

Activity 5

Determine the measures of angles involving parallel lines and a transversal, using angle relationships.

Find the value of x in the given parallel lines ‘a’ and ‘b’, cut by a transversal ‘t’.

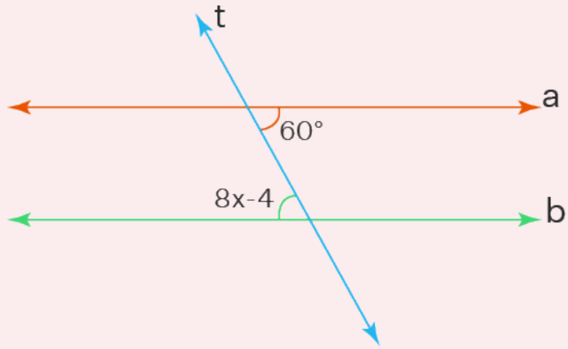


Figure 28: Angles involving parallel lines and a transversal

Solution:

The given parallel lines are cut by a transversal, therefore, the marked angles in the figure are the *alternate interior angles* which are equal in measure. This means:

$8x - 4° = 60°$

$8x = 64°$

$\therefore x = 8°.$

Therefore, the value of $x = 8°$.

Activity 6

Observe the figure given below: Find the values of x and y.

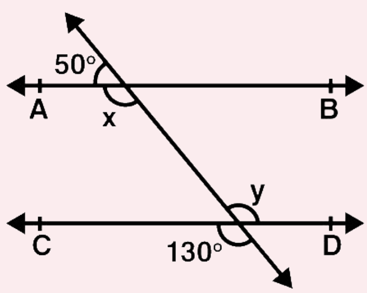


Figure 29: Angles involving parallel lines and a transversal

Solution:

The given parallel lines are cut by a transversal, therefore,

$y = 130°$ (Vertically opposite angles)

$x = 130°$ (Corresponding angles)

Did you notice any alternative methods to find the values of x and y besides the methods we used above? There are many different ways to arrive at the correct answer.

Now we shall identify some special triangles and their respective properties, explore the exterior angle theorem and, finally, determine the sum of interior angles of polygons. Let's start with a quick recall.

In Junior High School, you were introduced to some types of triangles and their properties. Revising this will help you to better under this new topic. Now, let us identify some special triangles and their respective properties.

What are special triangles?

Special triangles are triangles with unique properties and characteristics, making them particularly useful in areas such as; geometry, trigonometry and other mathematical applications. Here are some common ones:

1. **Equilateral Triangle:**

   All sides are equal.

   All angles are equal and measure 60°.

   Symmetrical about all altitudes (a perpendicular line from a vertex to the opposite side of a triangle), medians, and angle bisectors.

2. **Isosceles Triangle**:

   Two sides are equal.

   Two angles are equal (at the base of the equal sides).

   Symmetrical about the altitude and median to the base.

3. **Right Triangle**:

   One angle is 90° (a right angle) and the remaining two angles are acute.

   It obeys the Pythagorean Theorem: $a^2 + b^2 = c^2$ (where c is the hypotenuse).

   It has Trigonometric ratios: sine, cosine, and tangent.

4. **Scalene Triangle**:
   All sides are unequal.
   All angles are unequal.
5. **Obtuse Triangle**:
   One angle is greater than 90°, the remaining two angles are acute.
   Sum of the other two angles is less than 90°.
6. **Acute Triangle**:
   All angles are less than 90°.
   Sum of any two angles is greater than the third angle.
7. **Similar Triangles**:
   Same shape, but not necessarily the same size.
   Corresponding angles are equal.
   Corresponding sides are proportional.
8. **Congruent Triangles:**
   Same shape and size.
   All corresponding angles and sides are equal.

I hope this has helped you recall all you have learnt about triangles. What about how to calculate unknown angles in a triangle? For example, if an interior angle is an angle located within a shape, what then is an exterior angle?

Let us identify exterior angles and how to calculate them using the "Exterior Angle Theorem"

Exterior Angles of a Triangle

Exterior angles are angles formed outside a triangle by extending one of its sides. They are also known as external angles.

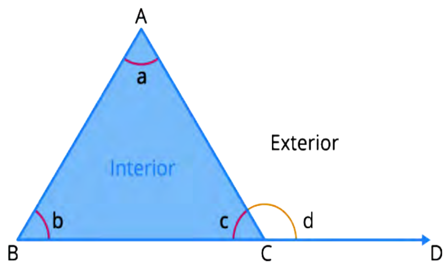


Figure 30: Exterior angle of a triangle

EXTERIOR ANGLE THEOREM OF A TRIANGLE

According to the exterior angle theorem, the exterior angle that results from stretching a triangle's side is equal to the sum of the dimensions of the triangle's two opposed interior angles. The theorem can be used to find the measure of an unknown angle in a triangle.

Example 2

Three internal angles in a triangle always add up to 180°. This theorem is applied to each of the outer angles, which total six. As they constitute a linear pair of angles, take note that an exterior angle is supplementary to the neighbouring interior angle. Exterior angles are those that are created between a polygon's side and its extended neighbouring side.

Proof of the exterior angle theorem

We can verify the exterior angle theorem with the known properties of a triangle. Carefully go through this proof.

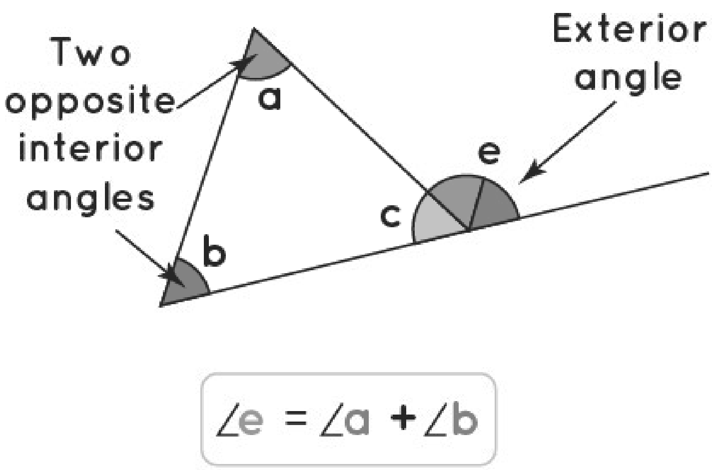


Figure 31: Exterior angle and two opposite interior angles

Consider a Δ ABC. The three angles have a sum of 180^0

$a + b + c = 180°$ *Equation 1*

$c = 180° - (a + b)$ *Equation 2* (rewriting equation 1)

$e = 180° - c$ *Equation 3* (linear pair of angles)

Substituting the value of c in *equation 3*, we get

$e = 180° - [180° - (a + b)]$

$e = 180° - 180° + (a + b)$

$e = a + b$

Hence proven.

Let us perform the following activities on the exterior angle theorem.

Activity 7

Materials to use are; Paper or whiteboard, Ruler, mathematical set.

1. Consider a triangle ABC

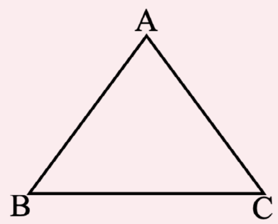


Figure 31: Triangle ABC

2. Extend side BC to point D to form an exterior angle ACD

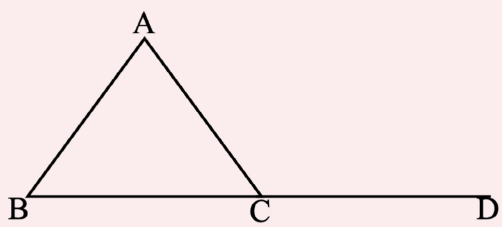


Figure 32: Triangle ABC with extended side

3. Draw a line through point C, parallel to line AB.

Applying the angle properties, write down your observations, discuss them with your classmates and compare them to the following;

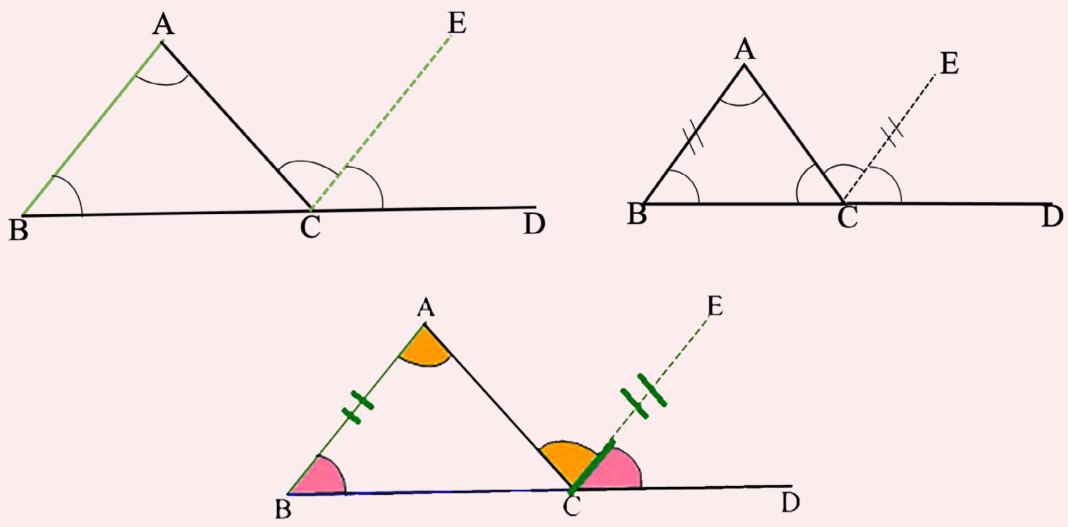


Figure 33: Triangles ∠ACD = ∠ABC + ∠BAC

- ∠ACE is an alternate angle to ∠BAC. Meaning ∠ACE=∠BAC
- ∠ECD is a corresponding angle to ∠ABC. This means ∠ECD=∠ABC
- ∠ACD = ∠ACE + ∠ECD
- This implies ∠ACD = ∠ABC + ∠BAC

Therefore, the measure of the exterior angle ACD is equal to the sum of the measures of the two opposite interior angles, ∠ABC and ∠BCA.

Activity 8

Materials to use are;

Paper or whiteboard

Ruler, protractor

pencil

1. Consider a triangle ABC

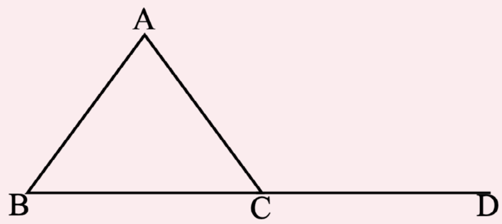


Figure 34: Triangle ABC with extended side

2. Extend side BC to point D to form an exterior angle ACD
3. With the help of a protractor, measure the three interior angles of the triangle; ∠ABC, ∠BCA and ∠BAC. Record the measurement for each.
4. Measure the exterior angle ∠ACD using a protractor. Record the measurement.
5. Add the measurements of the two opposite interior angles (∠A and ∠B) to see if they are equal to the measurement of the exterior angle ∠ACD.
6. Repeat steps 4-5 for the other two exterior angles (by extending lines AB and AC respectively).
7. Compare your results with your classmates and draw your conclusion on whether the Exterior Angle Theorem holds true for this triangle.

With these two activities, we can therefore conclude that: ***an exterior angle of a given triangle is equal to the sum of the two opposite interior angles of that triangle***

I hope these activities were interesting and helpful. Let us work out some examples.

Example 3

Use the properties of exterior angles of a triangle to determine the measure of the angles represented by x and y.

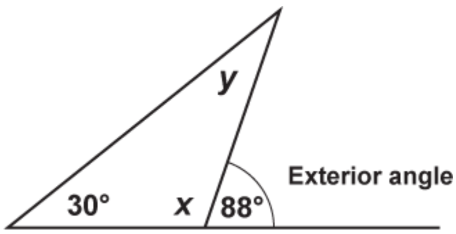


Figure 35: Triangle

Solution

$x + 88^o = 180^o$ (angles on a straight line)

$x + 88^o - \mathbf{88^o} = 180^o - \mathbf{88^o}$

$x = 92^o$

$y + 30^o = 88^o$ (exterior angle theorem)

$y + 30^o - 30^o = 88^o - 30^o$

$y = 58^o$

Alternatively:

$x + 88^o = 180^o$ (angles on a straight line)

$x + 88^o - \mathbf{88^o} = 180^o - \mathbf{88^o}$

$x = 92^o$

$x + y + 30^o = 180^o$ (sum of interior angles of a triangle)

$92^o + y + 30^o = 180^o$

$y + 122^o = 180^o$

$y = 180^o - \mathbf{122^o}$

$y = 58^o$

Example 4

Find the value of the angle marked x in the figure below

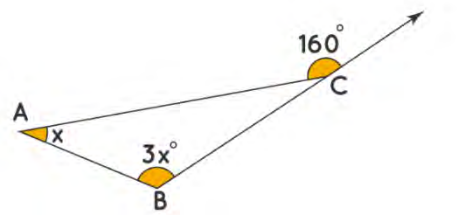


Figure 36: Triangle ABC

Solution

By the exterior angle theorem,

$x + 3x = 160^\circ$

$4x = 160^\circ$

$\frac{4x}{4} = \frac{160^\circ}{4}$

$x = 40^\circ$

Alternatively:

$\angle ACB + 160^\circ = 180^\circ$(angles on a straight line)

$\angle ACB = 180^\circ - 160^\circ$

$\angle ACB = 20^\circ$

$\angle ABC + \angle ACB + \angle BAC = 180^\circ$ (sum of interior angles of a triangle)

$x + 20^\circ + 3x = 180^\circ$

$4x + 20^\circ = 180^\circ$

$4x = 180^\circ - 20^\circ$

$4x = 160^\circ$

$\frac{4x}{4} = \frac{160^\circ}{4}$

$x = 40^\circ$

Look for more examples, solve them and show them to your classmates or teacher.

I hope you can also recall some properties of quadrilaterals, four sided shapes, for example, square, rectangle, rhombus, kite, trapezium and many others.

Let's revise some of the general properties of quadrilaterals.

- Number of sides: A quadrilateral has 4 sides.
- Number of angles: A quadrilateral has 4 angles.
- Sum of angles: The sum of the interior angles of a quadrilateral is always 360°.

Now, we shall explore some other types of polygons, their interior and exterior angles and also delve into how to find the sum of their interior angles.

Sum of the Interior Angles of Polygons

Interior angles of a polygon are the angles formed inside a closed plane figure with straight sides, known as a polygon. The interior angles are formed at each vertex or corner of the polygon.

Key points about the interior angles of a polygon

Sum of Interior Angles: The sum of the interior angles of a polygon with n sides can be calculated using the formula $(n - 2)\,180^\circ$. This formula holds true for all polygons, regardless of their size or shape.

Regular Polygons: In a regular polygon, all interior angles have the same measure. For example, in a regular hexagon, each interior angle measures 120 degrees because $\frac{(6-2)\times 180^\circ}{6} = 120^\circ$

This implies, An interior angle of a regular polygon:

$$= \frac{(n-2)180^\circ}{n} = \frac{\textit{Sum of interior ofangles of a polygon}}{\textit{number of sides } (n)}$$

Irregular Polygons: In an irregular polygon the interior angles can have different measures. The sizes of the interior angles depend on the specific lengths of the polygon's sides and the arrangement of its vertices.

Relationship between Number of Sides and Interior Angles

As the number of sides in a polygon increases, the sum of the interior angles also increases. However, the measure of individual interior angles increases at a slower rate in larger polygons. To determine the sum of interior angles of a polygon, let's go through the following activities.

Note that every polygon can be divided into triangles.

The sum of interior angles of a pentagon

Perform the activity below on how to determine the sum of interior angles of a pentagon.

Activity 9

Materials to use are; Paper or whiteboard, Ruler, Pentagon printout or drawing, pencil, etc.

Procedure:

- Draw a pentagon of sides ABCDE, with a ruler
- Use your ruler to divide the pentagon into triangles by drawing diagonals from one vertex say E
- Count the number of triangles obtained
- Number each of the triangles and their respective interior angles
- calculate the sum of interior angles for each triangle. That is, each triangle is 180°
- multiply the number of triangles by 180° and that gives us $3 \times 180° = 540°$

Similarly, to determine the sum of the interior angles of a pentagon ABCDE, we have;

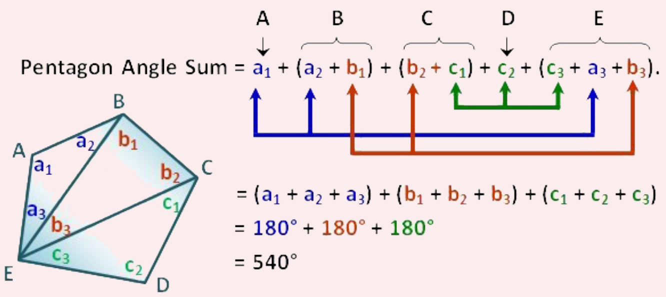


Figure 37: Pentagon ABCDE

Now try that on a hexagon, heptagon etc. Share what you have done and your observations with your classmate(s).

Explanation:

You realise that the angle measured in the first line of our equation is just a rearrangement of the measures of the interior angles of the three triangles. Hence, the sum of the interior angles of the pentagon is equal to the angle sum of the three triangles. Therefore, we can conclude that the sum of the interior angles of a polygon is equal to the angle sum of the number of triangles that can be formed by dividing it using the method described above.

Using this conclusion, we will now relate the number of sides of a polygon, the number of triangles that can be formed by drawing diagonals and the polygon's angle sum.

Picture	Name	Sides	Triangles	Degrees inside
	Quadrilateral	4	2: 	$2(180^\circ) = 360^\circ$
	Pentagon	5	3: 	$3(180^\circ) = 540^\circ$
	Hexagon	6	4: 	$4(180^\circ) = 720^\circ$
	Heptagon	7	5: 	$5(180^\circ) = 900^\circ$
	Octagon	8	6: 	$6(180^\circ) = 1{,}080^\circ$
	...	...	...	...
	n-gon	n	$n - 2$	$(n - 2)180^\circ$

Figure 38: Triangles made from given pentagons

Polygon	Number of Vertices (n)	Number of triangles	Sum of Angles (m°)
Triangle	3	1	1(180) = 180
Quadrilateral	4	2	2(180) = 360
Pentagon	5	3	3(180) = 540
Hexagon	6	4	4(180) = 720
Heptagon	7	5	5(180) = 900
…	…	…	…
Decagon	10	8	8(180) = 1440
100-gon	100	?	?
n-gon	n	n-2	(n-2)180

From the table, we observe that the number of triangles formed is 2 less than the number of sides of the polygon. This is true, because $n - 2$ triangles can be formed by drawing diagonals from one of the vertices to $n - 3$ non-adjacent vertices. Therefore, the angle sum m of a polygon with n sides is given by the formula $m = 180°(n - 2)$.

Watch this video on "sum of interior angles of a polygon" by clicking on the link https://youtu.be/BG1HpadfiKw

Let us work out some examples.

Example 5

The diagram below is a polygon. Find the value of the angle marked x.

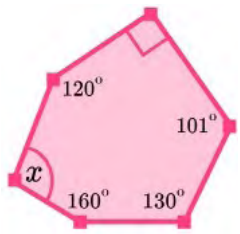


Figure 39: Hexagon

Solution

The above figure is a polygon with 6 sides (hexagon)

The sum of interior angles of a polygon = $(n - 2)180°$

Number of sides (n) is 6 or n = 6 sides

Therefore, the sum of interior angles of a hexagon $= (6 - 2)180° = 4 \times 180° = 720°$

This implies that the sum of all the angles in the hexagon above is 720°

$\implies 120° + 90° + 101° + 130° + 160° + x = 720°$

$601° + x = 720°$

$x = 720° - 601°$

$x = 119°$

Therefore, the value of x is 119°

Example 6

A graphic designer creates a school's logo in the form of a pentagon. What is the sum of the interior angles of the logo?

Solution

Number of sides/ interior angles of a pentagon is 5. Since the sides are equal, the interior angles are also equal.

Sum of interior angles of a pentagon = (5 – 2) x 180 = 3 x 180 = 540^0

Therefore, the sum of interior angles of the logo is 540^o

Look for more examples, solve them and share what you have done with your classmates.

Now let us look at the exterior angles of a polygon with n sides?

Exterior Angles of a Polygon

Exterior angles of a polygon are the angles formed outside the polygon by extending its sides.

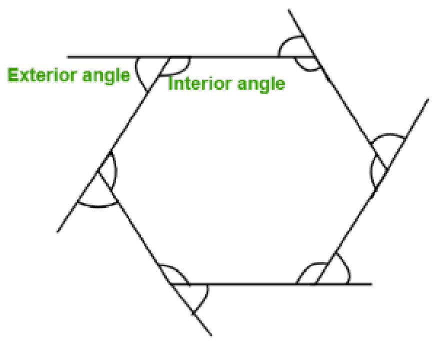


Figure 40: Hexagon

Observe the figure above and perform the following activity below.

Activity 10

1. Draw any polygon of your choice
2. Count the number of sides of the polygon and record it
3. Count the number of angles and record it
4. Extend the sides of the polygon to create exterior angles as shown in the figure above.
5. Count the number of exterior angles created and record it
6. Compare the numbers obtained in steps 2, 3 and 5

I am sure you had the same number for all. Thumbs up!

Try the same activity for other polygons and discuss your findings with your classmate(s).

Here are some key points to note about Exterior angles;

- A polygon has the same number of exterior angles as the number of sides. For example, a hexagon has 6 sides so its exterior angles are also 6 in number as shown in the figure above.
- The sum of the exterior angles of any polygon is always 360°.
- Exterior angles are supplementary to their corresponding interior angles (they add up to 180°).
- An exterior angle of a regular polygon is obtained by dividing the sum of the exterior angles of polygon (360°) by the number of sides (n) of the given polygon.

That is, an exterior angle of a regular polygon $= \frac{360°}{n}$, where n is the number of sides of the polygon.

Watch this video on exterior angles of a polygon https://youtu.be/CAoaZxvmG48

Let us consider some examples.

Example 7

Find the value of x in the figure below.

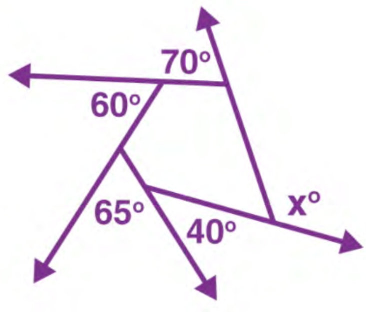


Figure 41: Pentagon

Solution

The figure above is a pentagon and has 5 sides. It is not a regular pentagon as all its sides and angles are different sizes, but it remains true that the sum of Exterior Angles of a polygon = 360°

$\Longrightarrow 70^o + 60^o + 65^o + 40^o + x = 360^o$

$235^o + x = 360^o$

$x = 360^o - 235^o$

$x = 125^o$

Example 8

A design on a table has a shape of a parallelogram with extended sides. If one extended side measures 75°, what is the measure of its corresponding interior angle?

Solution

Exterior angles are supplementary to their corresponding interior angles.

Let x represent the corresponding interior angle of the 75°

$x + 75° = 180^o$

$x = 180^o - 75^o$

$x = 105^o$

Therefore, the measure of the corresponding interior angle is 105^o

Example 9

Work out the size of the angles labelled x and y in the polygon below.

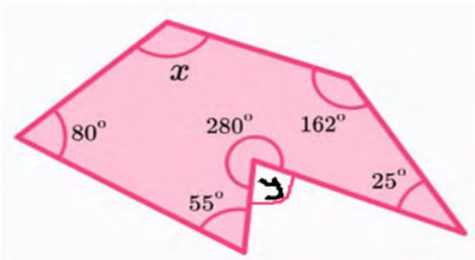


Figure 42: Hexagon

Solution

Number of sides (n) = 6 (hexagon)

Sum of interior angles of a hexagon= $(6-2)180° = 720°$

$80° + x + 162° + 25° + 280° + 55° = 720°$

$602° + x = 720°$

$x = 720° - 602°$

$x = 118°$

$280° + y = 360°$ (angles at a point)

$y = 360° - 280°$

$y = 80°$

Therefore, $x = 118°$ and $y = 80°$

Look for more examples from the extended reading materials and videos provided at the end of this section.

Well done for all your hard work. It is now time to move on to the exciting world of Pythagoras.

THE PYTHAGOREAN THEOREM

The Pythagorean Theorem states that in a right-angle triangle, the square of the length of the hypotenuse (the side opposite the right angle and the longest side in a right-angle triangle) is equal to the sum of the squares of the lengths of the other two sides.

The reason why the Pythagorean Theorem only applies to right triangles is established in the geometric properties and relationships within such triangles. In a right-angle triangle, one of the angles is always a right angle, which measures exactly 90 degrees. This characteristic allows for a specific set of relationships among the triangle's sides and angles, which the Pythagorean theorem captures.

If we consider a triangle that is not a right-angle triangle, the sum of the squares of the lengths of any two sides will not equal the square of the length of the remaining side. This is because the angle opposite the longest side, known as the hypotenuse in a right-angle triangle, will not be a right-angle, and the triangle's proportions will differ.

Verify the Pythagorean Theorem

Watch this video on Pythagoras' theorem by clicking on the link below.

https://youtu.be/YompsDlEdtc

Let's study the following examples illustrating Pythagoras' Theorem.

Example 10

Pythagoras' theorem using the following figure shows that the area of the square formed by the longest side of the right triangle (the hypotenuse) is equal to the sum of the area of the squares formed by the other two sides of the right triangle.

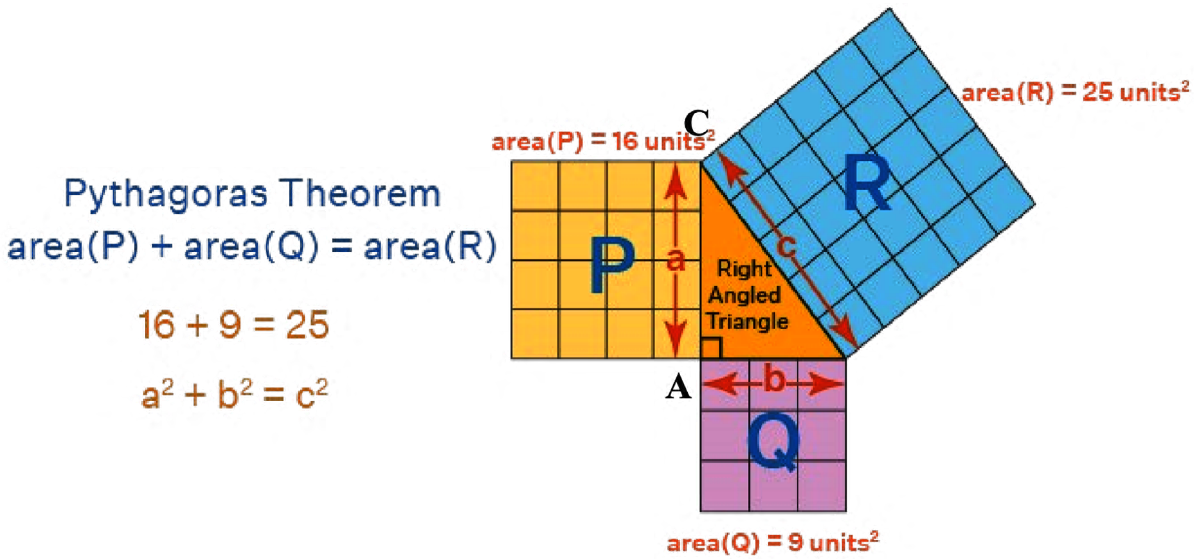


Figure 43: Squares formed by the sides of a triangle

Thus, if AB and AC are the shorter sides and BC is the hypotenuse of the triangle, then:

$BC^2 = AB^2 + AC^2$. In this case, AB is the base, AC is the altitude or the height, and BC is the hypotenuse.

Activity 11

Use the values a, b, and c as shown in the following figure and follow the steps given below:

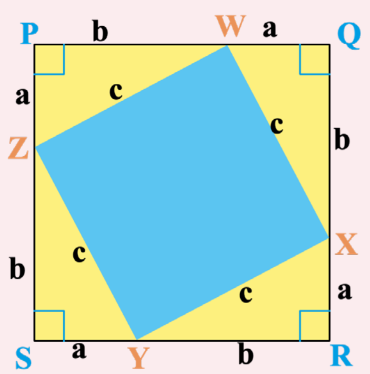


Figure 44: Right triangles in the given square PQRS

Step 1: Arrange four congruent right triangles in the given square PQRS, whose side is a + b. The four right triangles have 'b' as the base, 'a' as the height and, 'c' as the hypotenuse.

Step 2: The 4 triangles form the inner square WXYZ as shown, with 'c' as the four sides.

Step 3: The area of the square WXYZ by arranging the four triangles is c^2.

Step 4: The area of the square PQRS with side (a + b) = Area of 4 triangles + Area of the square

WXYZ with side 'c'. This means $(a + b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$

This leads to $a^2 + 2ab + b^2 = 2ab + c^2$.

$a^2 + b^2 = 2ab - 2ab + c^2$.

Therefore, $a^2 + b^2 = c^2$. **Hence proved.**

Real-life uses of Pythagorean Theorem

- The Pythagorean Theorem is useful for two-dimensional navigation. That is horizontal and vertical navigation.
- Painting on a Wall: To paint tall structures, painters make use of ladders and they frequently employ Pythagoras' theorem to carefully position the ladder's base away from the wall, so it won't topple over.
- What size of TV should you buy? The size of a television is always specified in terms of its diagonal. If a television is specified as 43 inches in size, its true size is the diagonal's or hypotenuse's measurement.

Applications of Pythagoras Theorem

- To know if the triangle is a right-angled triangle or not.
- In a right-angled triangle, we can calculate the length of any side if the other two sides are given.
- To find the diagonal length of a square or rectangle.

Using the Pythagorean Theorem to determine if a given triangle is a right triangle

Pythagoras' theorem can be used to determine whether a triangle has a right-angle. The triangle contains a right-angle if the sum of the squares of the two shorter sides equals the square of the hypotenuse.

Have a go at the following activities.

Activity 12: Does the triangle ABC contain a right angle?

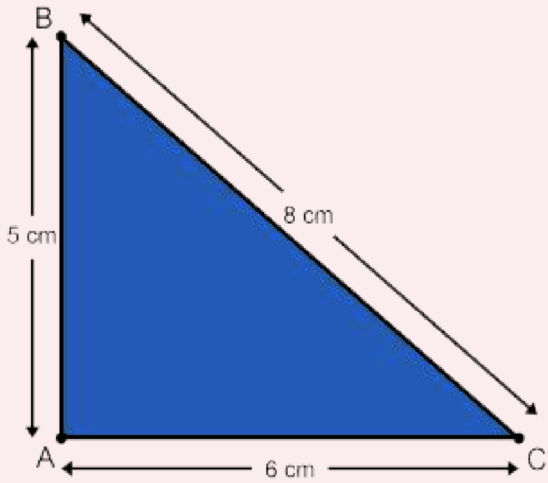


Figure 45: Triangle ABC

Step One: Find the square of line AB. That is, $/AB/^2 = 5^2 = 25$

Step Two: Find the square of line AC. That is, $/AC/^2 = 6^2 = 36$

Step Three: Find the sum of the squares of lines AB and AC.

That is, $/AB/^2 + /AC/^2 = 25 + 36 = 61$

Step Four: Find the square of line BC. That is, $/BC/^2 = 8^2 = 64$.

Step Five: Compare the answers in step three and Step four. That is, $61 \neq 64$.

Conclusion: 61 does not equal 64. Therefore, the triangle does **not** contain a right angle.

Let's look at the following examples.

Solve problems using Pythagorean theorem.

Example 11

A rectangular playing field is 20 metres long. A straight path is cut across the field along one of its diagonals. If the length of the path in metres is 25m; how wide is the playing field?

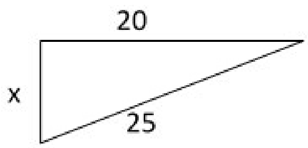


Figure 46: Triangle

Solution

This is a right-angled triangle.

Let x be the width

$20^2 + x^2 = 25^2$

$x^2 = 625 - 400$

$x^2 = 125$

$x = 15$ m

Activity 13

Provide an explanation on why a triangle with the side length ratio of 3:4:5 is a right angle triangle.

We can prove this by using the Pythagorean Theorem as follows:

Step One: Longest side = $5^2 = 25$

Step two: Sum of the squares of the other two sides = $3^2 + 4^2 = 9 + 16 = 25$

Step 3: Comparing Step One and Step two; 25 = 25

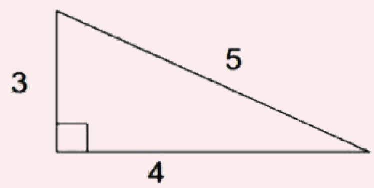


Figure 47: Triangle

Conclusion: The triangle is a right angled.

Example 12

Given the side of a square to be 4 cm. Find the length of the diagonal.

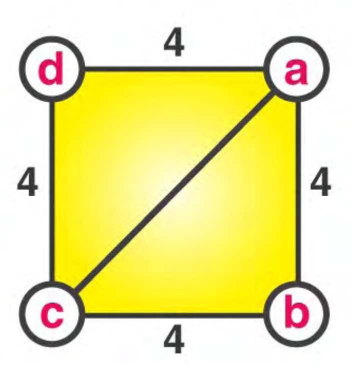


Figure 48: Square abcd

Solution

From right-angled triangle abc; $|ac|^2 = |ab|^2 + |bc|^2$

$|ac|^2 = 4^2 + 4^2$

$|ac|^2 = 16 + 16$

$|ac|^2 = 32$

$|ac| = \sqrt{32}$

$|ac| = 4\sqrt{2}$

Activity 14

Explore and prove the Pythagorean Theorem using a geometric model.

Materials Needed:

- Graph paper
- Ruler
- Compass
- Scissors
- Tape or glue
- Colored pencils

Activity Steps: Using the materials above, follow the steps below to explore and prove Pythagorean Theorem.

Step One. Draw a right-angled triangle.

Step Two: Use a ruler to draw a square on each side of the triangle. Colour each square differently to differentiate them.

Step Three: Carefully cut out each square from the graph paper.

Step Four: Calculate the area of each square (area = side length squared) and verify that the sum of the areas of the two smaller squares equals the area of the larger square.

Step Five: Rearrange the squares of the two shorter sides such that their combined area covers a region equal to the area of the square on the longer side (hypotenuse).

Step Six. Compare the total shaded area of the two smaller squares with the shaded area of the larger square.

Step Seven: Discuss with your friends how this model visually confirms the Pythagorean Theorem.

Activity 15

Perform the following activity.

Come up with different right-angled triangles with integer side lengths and verify the Pythagorean Theorem using different models. Show the results to your friend and your teacher.

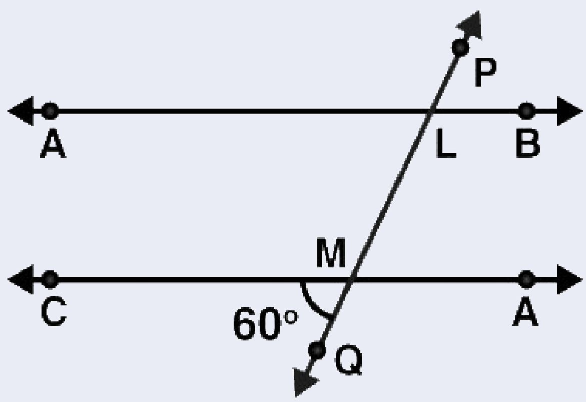
Activity 16

Picture a project where you design a small triangular park. You must use the Pythagorean Theorem to ensure the park's paths and borders form a right triangle. For example, if one path is 3 meters and another is 4 meters, you need to calculate the hypotenuse to ensure proper design. Repeat this in laying a foundation for building a room and designing a football field.

REVIEW QUESTIONS

Review Questions 5.1

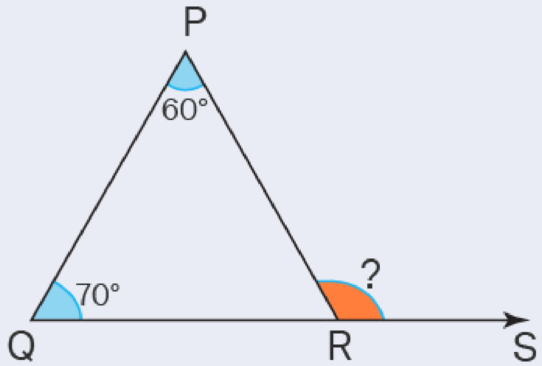
1. Give 4 examples of referents for angles.
2. Solve the following word problems on angles measurement;
   a. Find the measure of angle that is 10° more than its complement.
   b. Find the measure of an angle that is 30° less than its supplement
3. Define Complementary and Supplementary angles and give one real-life example for each.
4. Draw, analyse and name pairs of angles formed by parallel lines and a transversal by identifying corresponding angles, alternate interior angles, alternate exterior angles, and vertically opposite angles.
5. (i) In Figure, AB and CD are parallel lines intersected by a transversal PQ at L and M respectively, if ∠CMQ = 60, find all other angles in the figure.



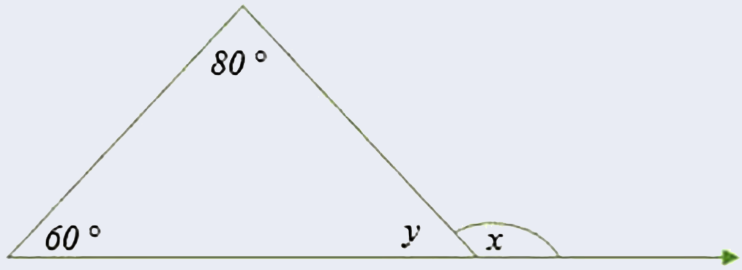
(ii) If x and y are a pair of interior angles on the same side of transversal, then what are the angles?

Review Questions 5.2

1. Find the value of the exterior angle in the triangle.

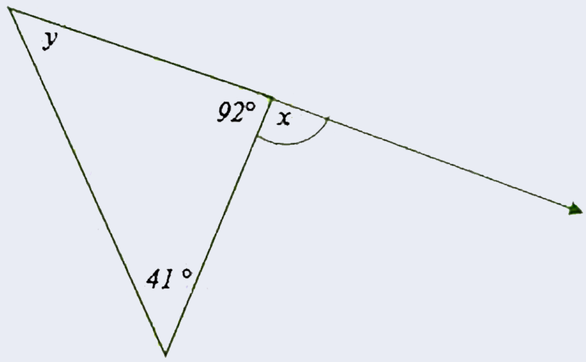


2. Calculate the values of x and y in the following triangle.



3. The exterior angle of a triangle is 120°. Find the value of x if the opposite non-adjacent interior angles are $(4x + 40)$ ° and 60°

4. Determine the value of x and y in the figure below.

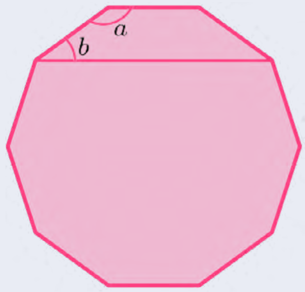


5. If each of the interior angles of a regular polygon is 140°, how many sides does the polygon have?

6. If one interior angle of a triangle is 56°, find the measure of its corresponding exterior angle.

7. A regular polygon's interior and exterior angles are in the ratio 9:1. How many sides does the polygon have?

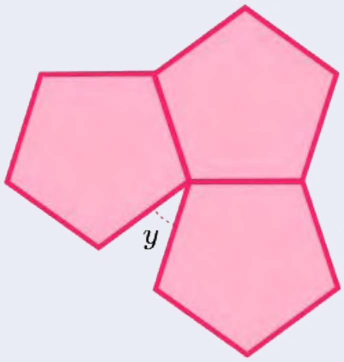
8. The diagram below shows a regular decagon.



Work out the size of angle a and b

9. If the shape of a chapel building is an octagon. Calculate the sum of its interior angles.

10. Shown below are three congruent regular pentagons. Find angle *y*.



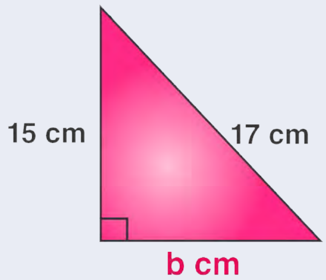
11. A park's walking path is shaped like a polygon with 18 sides. What is the sum of the interior angles?

Review Questions 5.3

1. A rectangular playing field is 20 metres long. A straight path is cut across the field along one of its diagonals. If the length of the path in metres is 25m; how wide is the playing field?
2. Given the side of a square to be 4 cm. Find the length of the diagonal.
3. In a right-angle triangle ABC, right-angled at B, the lengths of AB and BC are 7 units and 24 units, respectively. Find AC.
4. A triangle has sides of length 11 cm, 60 cm, and 61 cm. Check whether these are the sides of a right-angled triangle.
5. Suppose a triangle with sides 10cm, 24cm, and 26cm are given. Determine if this is a right-angled triangle. Give a reason for your answer.

6. An anchor line for a tower needs to be replaced. The tower is 96ft tall. The anchor line is 105ft long. How far from the tower base will the anchor line be?

7. The two sides of a right-angled triangle are given as shown in the figure. Find the third side.



8. An animal shed with a pent roof need to have some new roof beams fitted. The width of the shed is 5m and the height of the pent roof is 1.3m. Work out the length of the roof beams needed.

9. An army captain is on a hunt for a criminal. Her GPS tells her that she is 50m away from the criminal. She walks 34m due west. The GPS compass now tells her that the criminal is due south from where she is standing. How far south does she need to go to find the criminal?

EXTENDED READING

- ***Akrong Series: Core mathematics for Senior High Schools New International Edition (Pages 767 – 776)***
- Aki – Ola series : Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 326 – 343)
- Baffour Asamoah, B. A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 140 - 149 and 191 -200)
- Finding interior angles of polygons — Krista King Math | Online math help
- Watch this video on Pythagora's theorem by clicking on the link below. https://youtu.be/SKzYeyzP4rM
- Watch this video on Pythagora's theorem by clicking on the link below. https://youtu.be/YompsDlEdtc

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- Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1, 1990.
- Asiedu, P. (Millennium Edition 5). Core Mathematics for Senior High Schools in West Africa. Aki - Ola Publications.
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