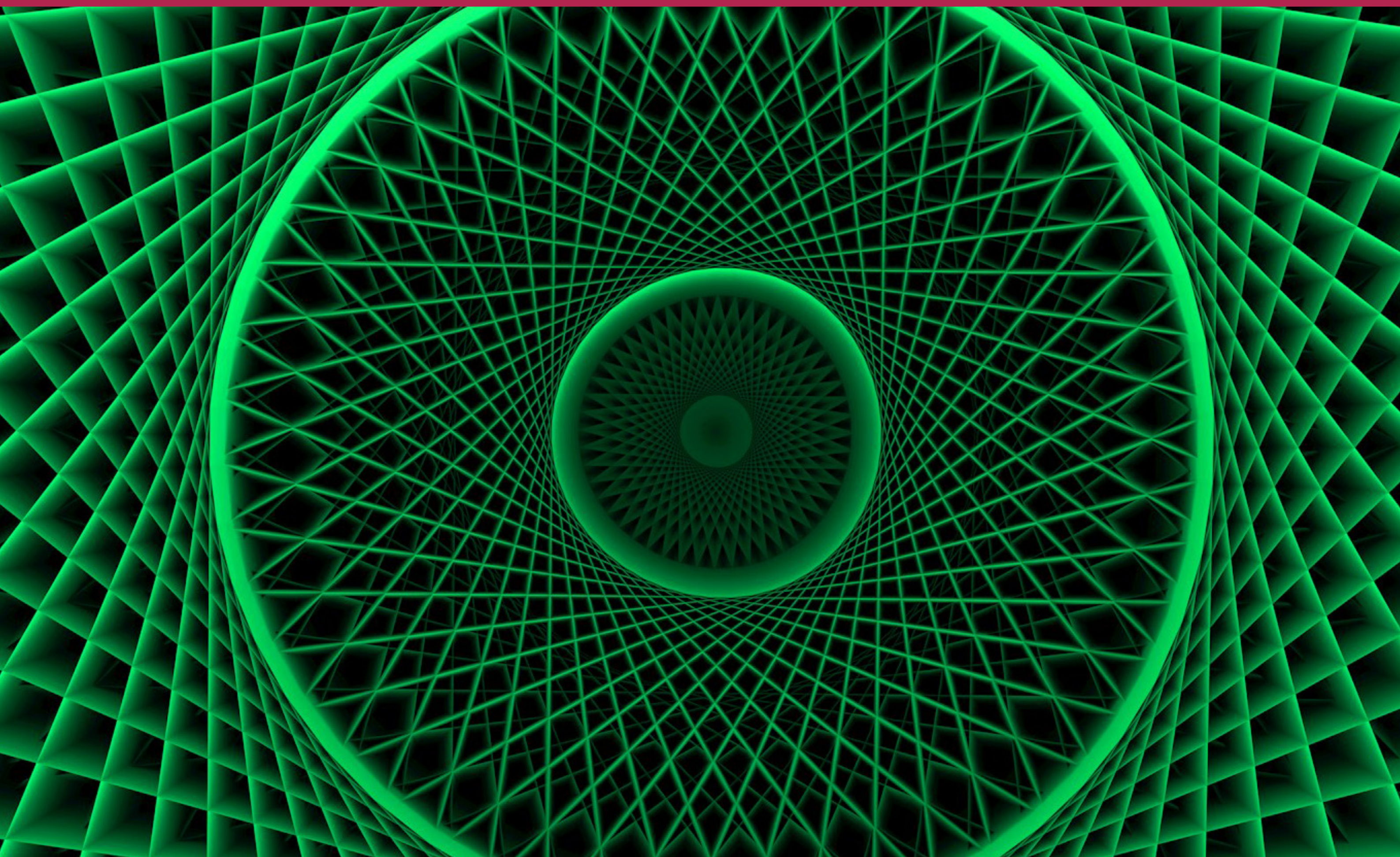
MINISTRY OF EDUCATION
MATHEMATICS ASSOCIATION OF GHANA

Additional Mathematics
for Senior High Schools

Year 2



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• Isaac Buabeng • Stella Awinipure • Joseph Fancis Kittah
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MAG

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CONTENTS

FOREWORD

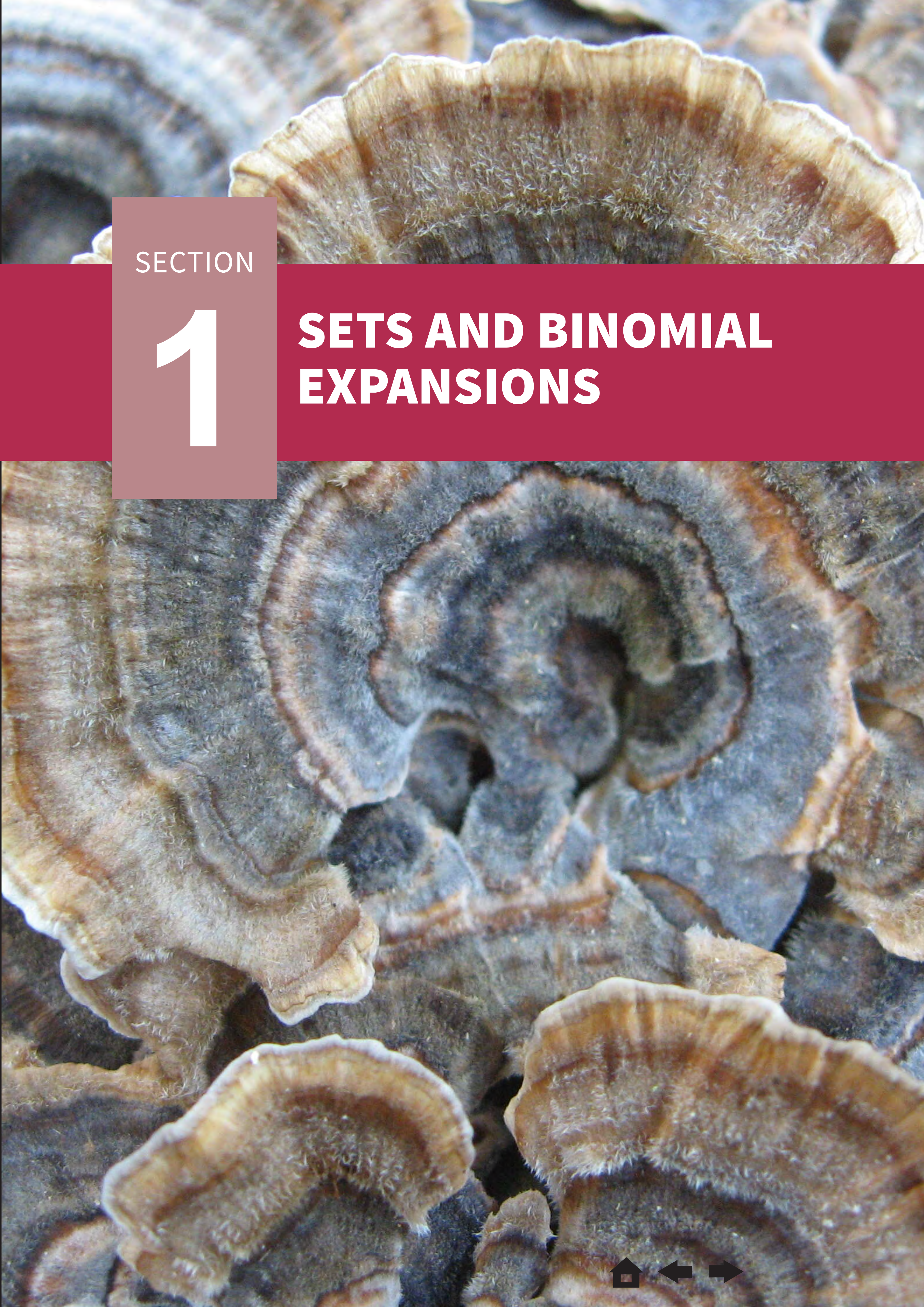
Ghana's new Senior High School Curriculum aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, knowledge, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, progress to further studies and enter the world of work. This is the first time that Ghana has developed a Senior High School Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

The Ministry of Education is proud to have overseen the production of these Learner Materials which can be used in class and for self-study and revision. These materials have been developed through a partnership between the Ghana Education Service, teacher unions (Ghana National Association of Teachers-GNAT, National Association of Graduate Teacher -NAGRAT and the Coalition of Concerned Teachers- CCT) and National Subject Associations. These materials are informative and of high quality because they have been written by teachers for teachers with the expert backing of each subject association.

I believe that, if used appropriately, these materials will go a long way to transforming our Senior High Schools and developing Ghana so that we become a proud, prosperous and values-driven nation where our people are our greatest national asset.

Haruna Iddrisu MP

Minister for Education



SECTION

1 SETS AND BINOMIAL EXPANSIONS

MODELLING WITH ALGEBRA

Application of Algebra

INTRODUCTION

This section is a continuation of what was discussed in year 1, sets and binomial expansion. De Morgan's Laws is an important theorem in sets as its application transcends beyond union, intersection and complements. These laws are central to Boolean Algebra where it is used to simplify expressions. In Digital Circuit Design, it is used to transform logic gates with the intention of not only optimising digital circuitry but also reducing cost and saving space. In Computer programming, it is used to simplify logical expressions in code, which can make code cleaner, simpler, and more readable. Database systems use these laws to optimise queries in SQL (Structured Query Language), which enhances data retrieval more efficiently.

The Binomial Theorem is a strong tool that makes the process of expanding binomial expressions much simpler. In developing this theorem, you further your algebra skills and prepare yourselves for higher classes of mathematics. The Binomial Theorem is applied in finance, investment, probability, statistics, computer science, genetics, engineering, game theory, marketing, and physics. It helps in calculating compound interest, risk assessment, modeling success/failure outcomes, understanding data structure complexities, predicting inheritance probabilities and enhancing decision-making in real-world scenarios.

KEY IDEAS

- Binomial Coefficients: The coefficients ^nC_r represent the number of ways to choose r elements from $\boldsymbol{n}$ elements and can be found using Pascal's Triangle. $^nC_r = \frac{!}{(n-r)!r!}$
- **Terms in the Expansion**: Each term in the expansion is of the form: ^nC_r $\boldsymbol{a^{n-k}b^k}$
- The ***Binomial Theorem*** provides a formula for expanding expressions of the form $(a + b)^n$ where n is a non-negative integer.
- The complement of an empty set (ϕ) is equal to the universal set (μ).

- The intersection of a set A and its complement (A') is equal to the null set ($\varnothing$)
- The union of a set A and its complement (A') is equal to the universal set (μ)

ESTABLISHING DE MORGAN'S LAWS OF SET THEORY

De Morgan's Laws are rules which are used to relate intersections and unions through complements. Generally, according to De Morgan's Laws:

1. The complement of the **union** of sets is equal to the **intersection** of the individual complements.
   This means that if A, B and C are sets,

$(A \cup B)'$	=	$A' \cap B'$
$(A \cup B \cup C)'$	=	$A' \cap B' \cap C'$
Complements of the unions	*Equals*	*The intersection of the individual complements*

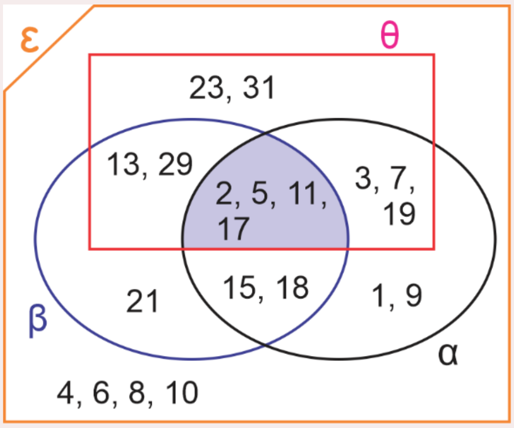
2. The complement of the **intersection** of sets is equal to the **union** of the individual complements.
   This means that if A, B and C are sets,

$(A \cap B)'$	=	$A' \cup B'$
$(A \cap B \cap C)'$	=	$A' \cup B' \cup C'$
Complements of the intersection	*Equals*	*The union of the individual complements*

Let us use the activity below to validate these rules.

Activity 1.1: De Morgan's Laws

1. Make a copy of the diagram below:



2. List the elements in α, β and θ

3. List the elements in α′, β′ and θ′

4. Find (a) $(\alpha \cup \beta \cup \theta)'$ (b) $\alpha' \cap \beta' \cap \theta'$ (c) $(\alpha \cap \beta \cap \theta)'$ (d) $\alpha' \cup \beta' \cup \theta'$

5. Compare your result in 4(a) to 4(b). What conclusion can you draw?

6. Compare your result in 4(c) to 4(d). What conclusion can you draw?

Compare your answer to the **suggested solution** below.

2. $\alpha = \{1, 2, 3, 5,7,9,11, 15,17,18,19\}$,

$\beta = \{2, 5, 11, 13, 15, 17, 18, 21, 29\}$,

$\theta = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$

3. $\alpha' = \{4, 6, 8, 10, 13, 21, 23, 29, 31\}$,

$\beta' = \{1, 3, 4, 6, 7,$ w8, $9, 10, 19, 23, 31\}$,

$\theta' = \{1, 4, 6, 8, 9, 10,15, 18, 21\}$

4. (a) $(\alpha \cup \beta \cup \theta)' = \{4, 6, 8, 10\}$

(b) $\alpha' \cap \beta' \cap \theta' = \{4, 6, 8, 10\}$

(c) $(\alpha \cap \beta \cap \theta)' = \{1, 3, 4, 6, 8, 9, 10, 13, 19, 21, 23, 29, 31\}$

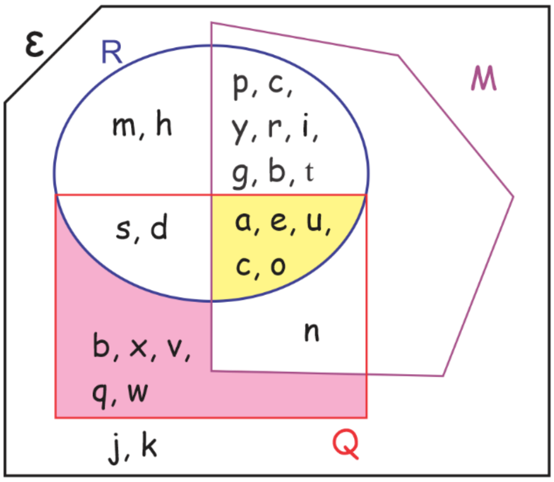
(d) $\alpha' \cup \beta' \cup \theta' = \{1, 3, 4, 6, 8, 9, 10, 13, 19, 21, 23, 29, 31\}$

5. Results in 4a and 4b are equal. This confirms De Morgan's Law

6. Results in 4c and 4d are equal. This confirms De Morgan's Law.

Example 1.1

Study the diagram carefully and use it to answer the questions.



1. List the elements in R, M, Q and ε
2. Show that $R' \cap M' = (R \cup M)'$
3. Show that $(R \cap M \cap Q)' = R' \cup M' \cup Q'$

Solution

1. R = {s, u, b, d, e, r, m, a, t, o, g, l, y, p, h, i, c},
   M = {u, n, c, o, p, y, r, i, g, h, t, a, b, l, e} and
   Q = {a, e, u, c, o, n, x, q, v, w} are subsets of
   ε = {s, u, b, d, e, r, m, a, t, o, g, l, y, p, h, i, c, n, x, q, v, w, j, k}
2. $R' \cap M' = (R \cup M)'$
   Let us first solve for $R' \cap M'$
   {b, x, v, n, q, w, j, k}∩ {m, h, s, d, b, x, v, q, w, j, k}
   $R' \cap M'$= {b, x, v, q, w, j, k}……………………..(1)
   Next, let us solve for $(R \cup M)'$
   $(R \cup M)'$ = (s, u, b, d, e, r, m, a, t, o, g, l, y, p, h, i, c, n)′
   $(R \cup M)'$ = {b, x, v, q, w, j, k}……………………..(2)
   Since equation (1) is equal to equation (2), it shows that $R' \cap M' = (R \cup M)'$
3. Let's first solve for $(R \cap M \cap Q)'$
   $(R \cap M \cap Q)' = (a, e, u, c, o)'$
   ={s, b, d, r, m, t, g, l, y, p, h, i, n, x, q, v, w, j, k} ……………..(1)

Next, let's find $R' \cup M' \cup Q'$

$R' \cup M' \cup R'$

$= \{b, x, v, n, q, w, j, k\} \cup \{m, h, s, d, b, x, v, q, w, j, k\} \cup \{m, h, p, c, y, r, i, g, b, t, j, k\}$

$= \{s, b, d, r, m, t, g, l, y, p, h, i, n, x, q, v, w, j, k\}$ ……………..(2)

Since equation (1) is equal to equation (2), it shows $(R \cap M \cap Q)' = R' \cup M' \cup Q'$

APPLYING DE MORGAN'S LAWS OF SET THEORY

We have learned about De Morgan's Laws of Set Theory. Now, we will look at its application. Before we consider this application, let's familiarise ourselves with a few properties of complements as it will be central to our understanding.

Properties of Complement

1. Union of a set and its complement.

   The union of a set A and its complement (A') is equal to the universal set (μ)

   i.e $A \cup A' = \mu$
2. Intersection of a set and its complement

   The intersection of a set A and its complement (A') is equal to the null set $(\varnothing)$

   i.e. $A \cap A' = \{\ \} = \phi$
3. Double complement.

   The complement of the complement of set A will give the same set, A. i.e. $(A')' = A$
4. Complement of an empty set.

   The complement of an empty set (ϕ) is equal to the universal set (μ). i.e. $\phi' = \mu$
5. Complement of a universal set.

   The complement of the universal set is equal to the null set. i.e. $\mu' = \phi$

Example 1.2

Rewrite: a. $(X' \cup Y)'$ b. $(X \cap Y')'$

Solution

a. $(X' \cup Y)'$

Recall that from De Morgan's laws, we have $(A \cup B)' = A' \cap B'$

So, for $(X' \cup Y)'$, we will replace A with X' and B with Y

This implies $(X' \cup Y)' = (X')' \cap Y'$

From the double complement property, $(X')' = X$

Therefore, $(X' \cup Y)' = X \cap Y'$

b. $(X \cap Y')'$

Recall that from De Morgan's laws, we have $(A \cap B)' = A' \cup B'$

So, for $(X \cap Y')'$, we will replace A with X and B with Y'

This implies $(X \cap Y')' = X' \cup Y''$

From the double complement property, $(Y')' = Y$

Therefore, $(X \cap Y')' = X' \cup Y$

Example 1.3

Show that:

a. $(A \cup B' \cup C')' = A' \cap B \cap C$

b. $(A' \cap B \cap C')' = A \cup B' \cup C$

Solution

a. Recall that $(A \cup B \cup C)' = A' \cap B' \cap C'$

This means, $(A \cup B' \cup C')' = A' \cap B'' \cap C''$

$= A' \cap B \cap C$

b. Recall that $(A \cap B \cap C)' = A' \cup B' \cup C'$

So, $(A' \cap B \cap C')' = A'' \cup B' \cup C''$

$= A \cup B' \cup C$

Example 1.4

Rewrite $[(P \cup Q')' \cap P]'$

Solution

Method 1

Let us start with the outer bracket.

Using $[A \cap B]' = A' \cup B'$

$[(P \cup Q')' \cap P]' = (P \cup Q')'' \cup P'$

$[(P \cup Q')' \cap P]' = (P \cup Q') \cup P'$

Since the union is commutative, $P \cup Q' = Q' \cup P$

$[(P \cup Q')' \cap P]' = (Q' \cup P) \cup P'$

Since the Union of sets is associative, we have $(Q' \cup P) \cup P' = Q' \cup (P \cup \;')$

$[(P \cup Q')' \cap P]' = Q' \cup (P \cup P')$

Recall that $(P \cup P') = \mu$ (universal set)

$[(P \cup Q')' \cap P]' = Q' \cup \mu$

$= \mu$

Method 2

We will start with the inner bracket and work our way out.

$[(P \cup Q')' \cap P]'$

$(P \cup Q')' = P' \cap Q''$

$= P' \cap Q$

$[(P \cup Q')' \cap P]' = [(P' \cap Q) \cap P]'$

Since the intersection of sets is commutative $P' \cap Q = Q \cap P'$

$[(P \cup Q')' \cap P]' = [(Q \cap P') \cap P]'$

Since intersection is associative, $(Q \cap P') \cap P = Q \cap (P' \cap P)$

$[(P \cup Q')' \cap P]' = [Q \cap (P' \cap P)]'$

Recall that $P' \cap P = \varnothing$

$[(P \cup Q')' \cap P]' = [Q \cap \varnothing]'$

$[(P \cup Q')' \cap P]' = [\varnothing]'$

$[(P \cup Q')' \cap P]' = \mu$

Example 1.5

Using the Venn diagram, shade the region represented by:

$(X \cup Y)' \cap Z$

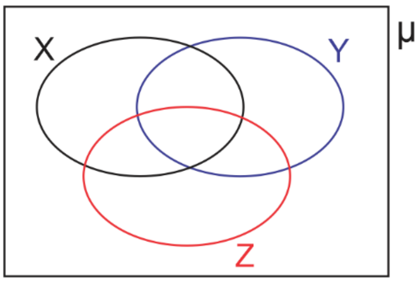
$X' \cap (Y \cup Z)$

Solution

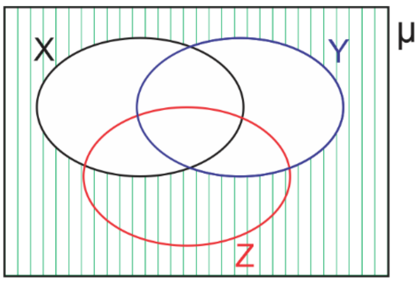
$(X \cup Y)' \cap Z$

Since the expression involves three sets, X, Y and Z, we will construct a three-set Venn diagram. We will represent the universal set with μ

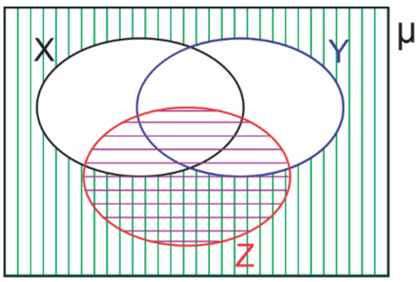
An example is shown below:



$(X \cup Y)'$ represent elements which are not in $X \cup Y$. We will use vertical lines to shade this region.

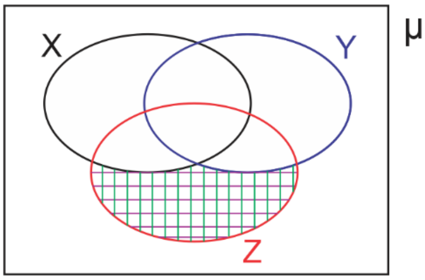


On the same diagram let us shade set Z. We will use horizontal lines.



The intersection of $(X \cup Y)'$ and Z is the double-shaded area.

Therefore, $(X \cup Y)' \cap Z =$



In year 1, we learned about set theory laws. Examples of these laws include commutative, associative and distributive. We also learned to identify the regions in a three-set Venn diagram. In this lesson, we will apply these concepts to answer real-life problems.

How to complete a three-set Venn diagram

1. Start with the intersection of the three sets and proceed to the regions involving two sets.
2. Then work on the regions involving one set and finally the regions in the universal set that do not intersect with any of the three sets.

To summarise, start from the **centre** (the region with the most overlaps) and work your way out.

Example 1.6

Set A = {odd numbers less than 15}

Set B = $\{4 < W \leq 11\}$

Set D = {Prime numbers less than 14}

All of which are subsets of:

Set R = {0, 1, 2, 3, 4, … , 16}.

a. List the members of the following sets:

b. **i.** A

ii. B

iii. D

iv. $A \cup B$

v. $(A \cup B) \cap D'$

vi. $(A' \cap B) \cup D$

c. Represent A, B, D and R using a Venn diagram

Solution

a. **i.** A={1, 3, 5, 7, 9, 11, 13}

ii. B={5, 6, 7, 8, 9, 10, 11}

iii. D={2, 3, 5, 7, 11, 13}

iv. $A \cup B$= {1, 3, 5, 7, 9, 11, 13} ∪ {5, 6, 7, 8, 9, 10, 11}

v. = {1, 3, 5, 6, 7, 8, 9, 10, 11, 13}

vi. (A ∪ B) ∩ D′ = {1, 3, 5, 6, 7, 8, 9, 10, 11, 13} ∩ {1, 4, 6, 8, 9, 10, 12, 14, 15, 16}

vii. = {1, 6, 8, 9, 10}

viii. $(A' \cap B)$ = {2, 4, 6, 8, 10, 12, 14, 15, 16} ∩ {5, 6, 7, 8, 9, 10, 11}

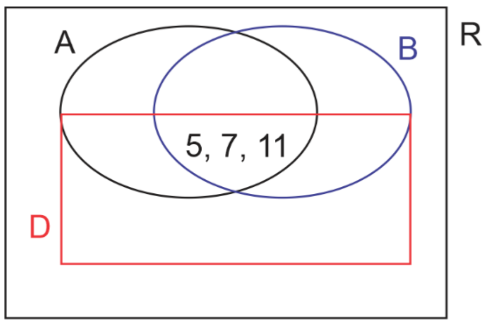
ix. = {6, 8, 10}

$(A' \cap B) \cup D$ = {6, 8, 10} ∪ {2, 3, 5, 7, 11, 13}

= {2, 3, 5, 6, 7, 8, 10, 11, 13}

b. We first find the three intersections. i.e. A ∩ B ∩ D = {5, 7, 11}

c.



Then proceed to the two intersections. There are three of these.

They are $A \cap B$ only $= A \cap B \cap D'$,

$A \cap D$ only $= A \cap B' \cap D$ and

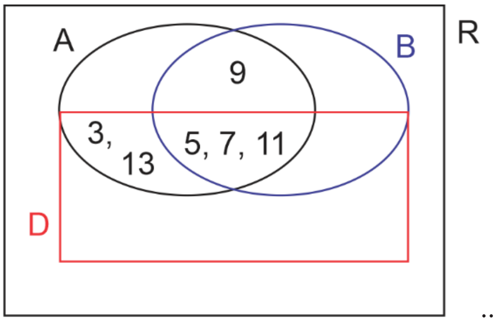
$B \cap D$ only $= A' \cap B \cap D$

$A \cap B$ only $= A \cap B \cap D' = \{9\}$

$A \cap D$ only $= A \cap B' \cap D = \{3, 13\}$

$B \cap D$ only $= A' \cap B \cap D = \{\ \}$

Let's update our table with these values



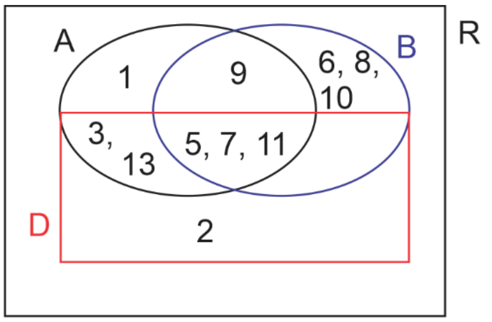
...

Next, find the single intersections. There are three of them.

A only = { 1 }

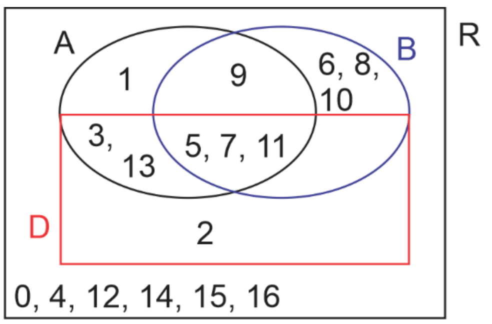
B only = { 6, 8, 10}

D only = { 2 }



Finally, we find $(A \cup B \cup D)' = \{0, 4, 12, 14, 15, 16\}$

Find below the completed diagram



Example 1.7

A survey of 70 investors reveals the following.

30 invest in bonds, 40 in stocks and 21 in Annuities.

10 had investments in bonds and annuities, 15 in Bonds and Stocks, 12 in stocks and annuities, and 7 had investments in all three asserts. The remaining investors had investments in other assets.

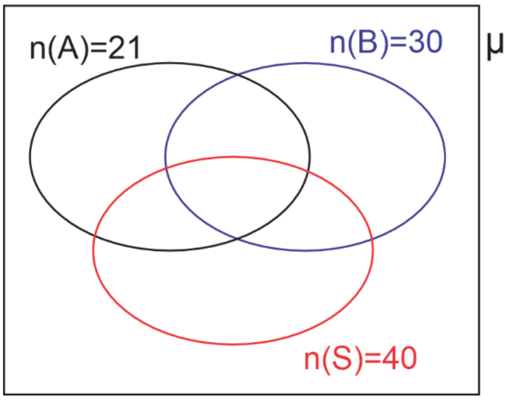
Create a Venn diagram for this information.

Solution

First, draw a Venn diagram with three intersecting sets representing the three main sets: Bonds, Stocks, and Annuities.

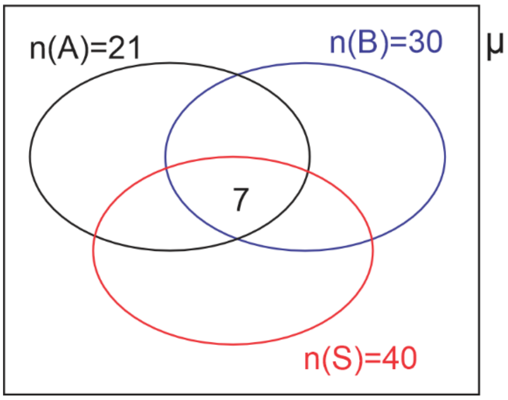
We will let B represent Bonds, A represent Annuities and S represent stocks.

This means $n(B) = 30$, $n(A) = 21$ and $n(S) = 40$



Next, we fill in the region where all the three sets intersect.

From the question, this figure is 7.



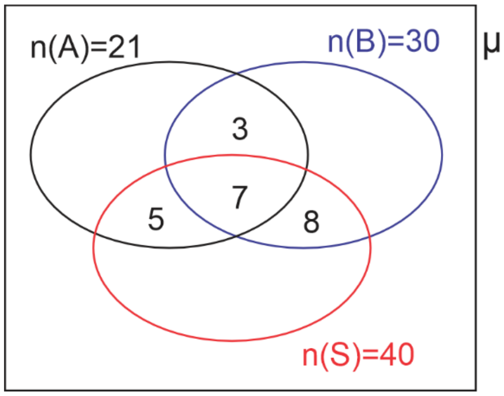
Next, we fill in the region where the two sets intersect. These are:

a. Bonds and Annuities. A total of 10 had investments in bonds and annuities but 7 of them has investments in all three asserts. This leaves $10 - 7 = 3$ investing in **only** Bonds and Annuities.

b. Bonds and Stocks. A total of 15 had investments in bonds and Stocks but 7 of them had investments in all three assets. This leaves $15 - 7 = 8$ investing in **only** Bonds and Stocks.

c. Stocks and Annuities. A total of 12 had investments in stocks and annuities but 7 of them has investments in all three asserts. This leaves $12 - 7 = 5$ investing in **only** stocks and Annuities.

Let us add these values to the Venn diagram.



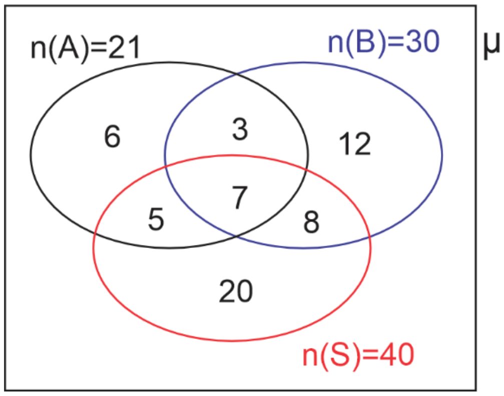
You will realise there is only one region remaining in each set. These are the regions that represent only one set.

a. Only Bonds. A total of 30 had investments in bonds but we have already accounted for 3 + 7 + 8 = 18 of them, leaving 30 − 18 = 12 having investments in only Bonds.

b. Only Annuities. A total of 21 had investments in annuities but we have already accounted for 3 + 7 + 5 = 15 of them, leaving 21 − 15 = 6 having investments in only Annuities.

c. Only Stocks. A total of 40 had investments in stocks but we have already accounted for 5 + 7 + 8 = 20 of them, leaving 40 − 20 = 20 investing in only stocks.

We will add these values to the Venn diagram.



Finally, we calculate the number of investors who have no investments in the three assets. To do this, find the sum of all values we have already calculated for the intersecting sets and deduct the result from the total number of investors surveyed.

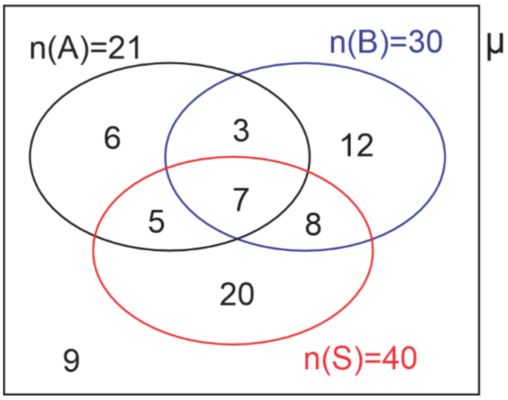
The sum of all values already calculated in the intersecting sets:

$= 12 + 3 + 6 + 8 + 5 + 7 + 20$

$= 61$

This means $70 - 61 = 9$ investors did not invest in any of the three asserts.

Record this value outside the circles but inside the universal set. This completes the Venn diagram.



Example 1.8

A survey of 97 countries that participated in the Olympics reveals that 80 won bronze, 57 won silver and 47 won gold. It also reveals that 30 won gold and silver, 50 won silver and bronze, 38 won gold and bronze and 3 countries did not win any medal.

a. Illustrate the information on a Venn diagram.

b. Calculate the number of countries who won all three medals.

c. Calculate the number of countries who won exactly two medals.

Solution

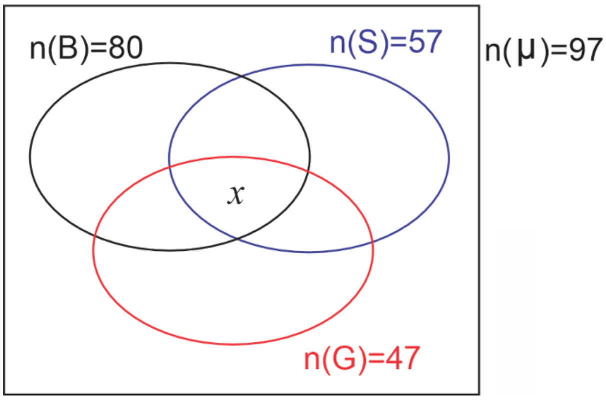
a. Let B=Bronze medal, G=Gold medal, and S=Silver medal.

b. This implies the number of countries that won bronze medals is $n(B) = 80$,

c. The number of countries that won silver medals is $n(S) = 57$ and

d. The number of countries that won Gold medals is $n(G) = 47$

We will first draw a Venn diagram and fill the region where all three sets intersect. Since we were not provided with this information, Let x be the number of countries who won all three medals.

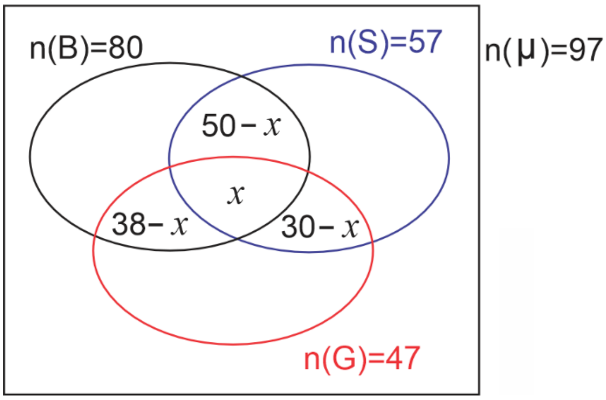


Next, we fill in the region where the two sets intersect (countries that won exactly two of the three medals). These are

A total of 30 countries won **Gold and Silver**. Out of these, x won all three medals. This shows that $(30 - x)$ countries won only Gold and Silver medals.

A total of 50 countries won **Silver and Bronze**. Out of these, x won all three medals. This shows that $(50 - x)$ countries won only Silver and Bronze medals.

A total of 38 countries won Gold and Bronze. Out of these, x won all three medals. This shows that $(38 - x)$ countries won only Gold and Bronze.



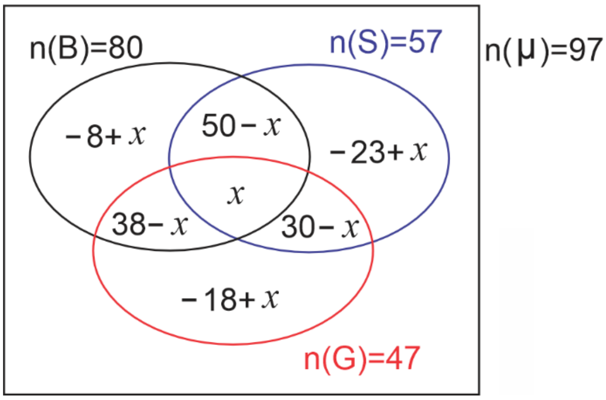
At this point, we have only one region remaining in each set. These are the regions that represent only one set.

Only Bronze. A total of 80 countries won bronze but we have already accounted for $x + 50 - x + 38 - x = (88 - x)$ of them, leaving $80 - (88 - x) = (-8 + x)$ winning only bronze.

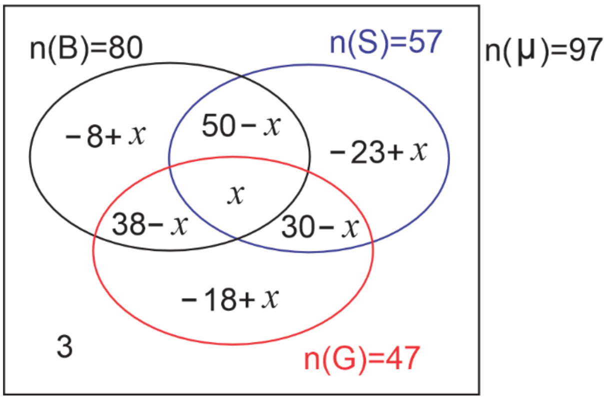
Only Silver. A total of 57 countries won silver but we have already accounted for $x + 50 - x + 30 - x = (80 - x)$ of them, leaving $57 - (80 - x) = (-23 + x)$ winning only silver.

Only Gold. A total of 47 countries won bronze but we have already accounted for $x + 30 - x + 38 - x = (68 - x)$ of them, leaving $47 - (68 - x) = (-21 + x)$ winning only bronze.

Update the diagram with these values.



According to the question, 3 countries did not win any medals. This shows that the set $(G \cup S \cup B)' = 3$



The next step is to solve for x

To do this, add all entries and equate the result to the total number of countries under consideration (97).

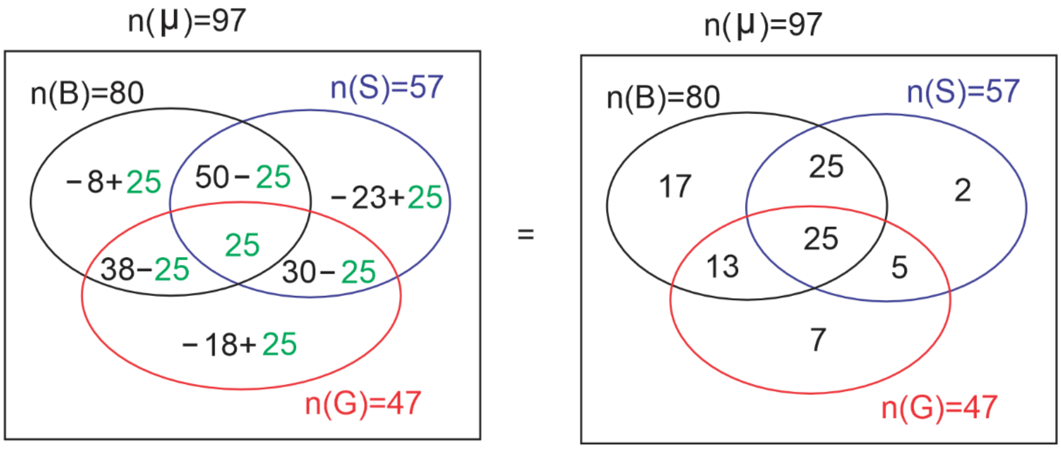
$x + (30 - x) + (50 - x) + (38 - x) + (-8 + x) + (-23 + x) + (-21 + x) + 3 = 97$

$x + 69 = 97$

$x = 97 - 69$

$x = 28$

e. Therefore, 28 countries won all three medals.
We can choose to update the diagram with this value.



f. Number of countries that won exactly two medals = 2 + 22 + 10 = 34

Expanding binomial expressions

Hopefully you recall the work we did in year one on the binomial expansion. We used the Pascal triangle and the Combination method to expand and simplify certain expressions. Let's remind you of this in the activity below.

Activity 1. 2: Revision

1. In groups or pairs, write down or generate the Pascal triangle up to $(x + y)^5$
2. Use your results in step 1 to simplify the expressions,
   a. $(2x + y)^3$,
   b. $(a - 3b)^4$
   c. iii $(2 + \sqrt{x})^5$.
3. Apply the combination method to also simplify the above task (i – ii)
4. Compare your results in (2) and (3) and write down any observations.
5. Show your solutions, or write up, to classmates in other groups or to your mathematics teacher.

The Binomial Expansion helps us to expand and simplify complex expressions, which would have been difficult using the idea of algebraic expressions. Under algebraic expressions we could only expand with a positive power, but the binomial theorem will help us to expand any expression with a rational number as the power.

Activity 1.3 – Derivation of Binomial Theorem

1. Look for the ${}^{n}C_{r}$ button on your calculator.
2. Task: use the ${}^{n}C_{r}$ on your calculator to evaluate
   - **i)** ${}^{3}C_{0}$
   - **ii)** ${}^{3}C_{1}$
   - **iii)** ${}^{3}C_{2}$
   - **iv)** ${}^{3}C_{3}$
3. Hopefully the answers you have are: 1 3 3 1.

This was used in year one to expand certain expressions.

The combination of n items taking r at a time is:

$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{(n-r)!r!}$ where n! is read as n factorial and is explained as:

$n! = n(n-1)(n-2)(n-3)\ldots 2 \times 1$

For example, $5! = 5(5-1)(5-2)(5-3)(5-4) = 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{120}$

You could also find 5!, using a calculator.

Look for ! on your calculator to find 5!, to confirm the answer obtained above.

Example 1.9

In groups or in pairs, solve the following without using calculator

a. ${}^{6}C_{3}$

b. ${}^{4}C_{2}$

Confirm your answers with a calculator.

Solution

a. ${}^{6}C_{3} = \frac{6!}{(6-3)!3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = \frac{120}{6} = 20$

b. ${}^{4}C_{2} = \frac{4!}{(4-2)!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = \frac{4 \times 3}{2 \times 1} = \frac{12}{2} = 6$

Activity 1.4: General Form of Binomial Theorem

1. Using the definition $\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$, work in pairs or groups and discuss how the following evaluations were done:

   a. ${}^nC_1 = \frac{n!}{(n-1)!\times 1!} = \frac{n(n-1)!}{(n-1)!\times 1} = n$

   b. ${}^nC_2 = \frac{n!}{(n-2)!\times 2!} = \frac{n\times(n-1)(n-2)!}{(n-2)!\times 2\times 1} = \frac{n(n-1)}{2}$

   c. Iii ${}^nC_3 = \frac{n!}{(n-3)!\times 3!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\times 3\times 2\times 1} = \frac{n(n-1)(n-2)}{6}$

2. Write down your challenges or observations for a class discussion with your classmates and teacher.

   It must be noted that $\boldsymbol{n}_{C_0} = 1$ ***and*** $0! = 1$

   In general, the **Binomial Theorem** is written as;

   $$(a+x)^n = n_{C_0 a^n x^0} + n_{C_1 a^{n-1} x^1} + n_{C_2 a^{n-2} x^2} + n_{C_3 a^{n-3} x^3} + \ldots + n_{C_r a^{n-r} x^r} + n_{C_n a^0 x^n}$$

   $$= \boldsymbol{a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n}$$

   There is a **special case when** $\boldsymbol{a} = 1$

   Substituting $a = 1$ into:

   $$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n$$

   $$= 1^n + n.1^{n-1}x + \frac{n(n-1)}{2!}1^{n-2}.x^2 + \frac{n(n-1)(n-2)}{3!}1^{n-3}.x^3 + \ldots + x^n$$

   Therefore, we have:

   $$\boldsymbol{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n}$$

   Let us go through the following worked examples to assist us in the expansion of the form $(1-x)^n$ or $(1+x)^n$

Example 1.10

By using the Binomial Theorem, fully expand the expression $(1+x)^5$.

Solution

Using the binomial theorem,

$$(1+x)^n = \boldsymbol{1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n}$$

In this case $n = 5$ so we substitute this in:

$$(1 + x)^5 = 1 + 5x + \frac{5 \times 4}{2!} \times x^2 + \frac{5 \times 4 \times 3}{3!} 1 \times x^3 + \frac{5 \times 4 \times 3 \times 2}{4!} x^4 + x^5$$

Simplify the coefficients to obtain:

$$= 1 + 5x + \frac{5 \times 4}{2 \times 1} \times x^2 + \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times x^3 + \frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1} \times x^4 + x^5$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Example 1.11

By using the Binomial Theorem, fully expand the expression$(1 + 2x)^4$.

Solution

Using the binomial theorem:

$$(1 + x)^n = \mathbf{1 + nx} + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots + x^n$$

In this case $n = 4$ and 'x' $= 2x$, so we substitute these in:

$$(1 + 2x)^4 = 1 + 4(2x) + \frac{4 \times 3}{2!} \times (2x)^2 + \frac{4 \times 3 \times 2}{3!} 1 \times (2x)^3 + (2x)^4$$

Simplify the coefficients to obtain:

$$= 1 + 4(2x) + \frac{4 \times 3}{2 \times 1} \times 4x^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} \times 8x^3 + 16x^4$$

$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

Example 1.12

Find the first four terms of the binomial $\sqrt{1 + 2x}$ using the binomial theorem.

Solution

$$\sqrt{1 + 2x} = (1 + 2x)^{\frac{1}{2}}$$

In this case $n = \frac{1}{2}$ and 'x' $= 2x$, so we substitute these in:

$$(1 + 2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(2x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}(2x)^3$$

$$= 1 + \frac{1}{2}(2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2 \times 1}\left(4x^2\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \times 2 \times 1}\left(8x^3\right)$$

$$= 1+x - \frac{1}{2}x^2 + \frac{1}{2}x^3$$

Example 1.13

Find the first four terms of the binomial $\sqrt[3]{1-3x}$ using the binomial theorem.

Solution

$\sqrt[3]{1-3x} = (1-3x)^{\frac{1}{3}}$

In this case $n = \frac{1}{3}$ and 'x' $= -3x$, so we substitute these in:

$$(1-3x)^{\frac{1}{3}} = 1+\frac{1}{3}(-3x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-3x)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-3x)^3$$

$$= 1+\frac{1}{3}(-3x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2\times 1}(-3x)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3\times 2\times 1}(-3x)^3$$

$$= 1+\frac{1}{3}(-3x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2\times 1}9x^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3\times 2\times 1}(-27x^3)$$

$$= 1+\frac{1}{3}(-3x) + \frac{-\frac{2}{9}}{2}9x^2 + \frac{\frac{10}{27}}{6}(-27x^3)$$

$$= 1 - x - x^2 - \frac{5}{3}x^3$$

Example 1.14

Expand up to the fifth term the expression $\frac{1}{(1-x)^2}$

Solution

Rewriting, $\frac{1}{(1-x)^2}$, using the rules of indices $1_{\overline{(1-x)^2}} = (1-x)^{-2}$

$$(1-x)^{-2} = 1 - 2(-x) + -\frac{2(-2-1)}{2!}(-x)^2 + -\frac{2(-2-1)(-2-2)}{3!}(-x)^3 + -\frac{2(-2-1)(-2-2)(-2-3)}{4!}(-x)^4$$

$$(1-x)^{-2} = 1 - 2(-x) + -\frac{2(-3)}{2\times 1}(-x)^2 + -\frac{2(-3)(-4)}{3\times 2\times 1}(-x)^3 + -\frac{2(-3)(-4)(-5)}{4\times 3\times 2\times 1}(-x)^4$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4$$

Example 1.15

Find the first four terms of $(1-16x)^{\frac{1}{4}}$

Solution

$$(1-16x)^{\frac{1}{4}} = 1+\frac{1}{4}(-16x) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}(-16x)^2 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}(-16x)^3$$

$$= 1+\frac{1}{4}(-16x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2\times 1}(-16x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3\times 2\times 1}(-16x)^3$$

$$= 1+\frac{1}{4}(-16x) + \frac{-\frac{3}{16}}{2}\left(256x^2\right) + \frac{\frac{21}{64}}{6}\left(-4096x^3\right)$$

$$= 1 - 4x - 24x^2 - 224x^3$$

Activity 1.5: Deciding whether binomial theorem can be used in all cases

Discuss in groups whether the binomial theorem can be used in all cases.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

1. Consider $(2+4x)^4$, can the binomial theorem be used?
2. How will you write the terms in the bracket to have 1 as one of the terms?
3. Let us go through the following tasks to help us write $(2+4x)^4$ in the form $(1+x)^n$ and use the binomial theorem to expand and simplify.

Task 1: Take the given binomial expression $(2+4x)$

Task 2: Factorise the constant out from the binomial to obtain $2(1+2x)$

Task 3: Rewrite your expression obtained with the power as given i.e. $2^4(1+2x)^4$

Task 4: Expand the binomial $(1+2x)^4$ after which you multiply your result by 2^4

Now follow the task to obtain all the terms for the given expression.

Expected Answer: $16 + 128x + 348x^2 + 512x^3 + 256x^4$

Let's solve the following example using the binomial theorem

Example 1.16

1. Use binomial theorem to expand fully $(3 + 12x)^5$
2. Find the first three terms of $\left(\frac{4}{9} + 6x\right)^{\frac{1}{2}}$ using the binomial theorem.
3. Use the binomial theorem to expand fully and simplify $(10 + 4x)^3$
4. Use the binomial theorem to find the first four terms of the expression $\sqrt[4]{81 + 162x}$

Solution

1. $(3 + 9x)^5 = 3^5(1 + 3x)^5$

$$3^5(1 + 3x)^5 = 3^5\left[1 + 5(3x) + \frac{5(5-1)}{2!}(3x)^2 + \frac{5(5-1)(5-2)}{3!}(3x)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(3x)^4 + \frac{5(5-1)(5-2)(5-3)(5-4)}{5!}(3x)^5\right.$$

$$= 3^5\left[1 + 53x + \frac{5(4)}{2 \times 1}(3x)^2 + \frac{5(4)(3)}{3 \times 2 \times 1}(3x)^3 + \frac{5(4)(3)(2)}{4 \times 3 \times 2 \times 1}(3x)^4 + \frac{5(4)(3)(2)(1)}{5 \times 4 \times 3 \times 2 \times 1}(3x)^5\right]$$

$$= 3^5\left[1 + 53x + \frac{5(4)}{2 \times 1}(9x^2) + \frac{5(4)(3)}{3 \times 2 \times 1}(27x^3) + \frac{5(4)(3)(2)}{4 \times 3 \times 2 \times 1}(81x^4) + \frac{5(4)(3)(2)(1)}{5 \times 4 \times 3 \times 2 \times 1}(243x^5)\right]$$

$$= 243[1 + 5(3x) + 10(9x^2) + 10(27x^3) + 5(81x^4) + (243x^5)]$$

$$= (3 + 9x)^5 = 243(1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5).$$

Compute the final coefficients

- $243 \cdot 1 = 243$
- $243 \cdot 15 = 3645$
- $243 \cdot 90 = 21870$
- $243 \cdot 270 = 65610$
- $243 \cdot 405 = 98415$
- $243 \cdot 243 = 59049$

Therefore the correct expansion is

$(3+9x)^5 = 243 + 3645x + 21870x^2 + 65610x^3 + 98415x^4 + 59049x^5$.

2. $\left(\frac{4}{9}+6x\right)^{\frac{1}{2}}=\left(\frac{4}{9}\right)^{\frac{1}{2}}\left(1+\frac{27}{12}x\right)^{\frac{1}{2}}$

$=\left(\frac{4}{9}\right)^{\frac{1}{2}}\left[1+\frac{1}{2}\left(\frac{27}{2}x\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{27}{2}x\right)^{2}\right]$

$=\left(\frac{4}{9}\right)^{\frac{1}{2}}\left[1+\frac{1}{2}\left(\frac{27}{2}x\right)+\frac{-\frac{1}{4}}{2}\left(\frac{27}{2}x\right)^{2}\right]$

$=\left(\frac{4}{9}\right)^{\frac{1}{2}}\left[1+\frac{1}{2}\left(\frac{27}{2}x\right)+\frac{-\frac{1}{4}}{2}\left(\frac{729}{4}x^{2}\right)\right]$

$=\frac{2}{3}\left[1+\left(\frac{27}{4}x\right)+\frac{-\frac{1}{4}}{2}\left(\frac{729}{4}x^{2}\right)\right]$

$=\frac{2}{3}\left[1+\frac{27}{4}x-\frac{729}{32}x^{2}\right]$

$=\frac{2}{3}+\frac{9}{2}x-\frac{243}{16}x^{2}$

3. $(10+4x)^{3}$

$=10^{3}(1+\frac{2}{5}x)^{3}$

$=10^{3}\left[1+3(\frac{2}{5}x)+\frac{3(3-1)}{2!}(\frac{2}{5}x)^{2}+\frac{3(3-1)(3-2)}{3!}(\frac{2}{5}x)^{3}\right]$

$=10^{3}\left[1+3(\frac{2}{5}x)+\frac{3(2)}{2\times 1}(\frac{2}{5}x)^{2}+\frac{3(2)(1)}{3\times 2\times 1}(\frac{2}{5}x)^{3}\right]$

$=10^{3}\left[1+3(\frac{2}{5}x)+3(\frac{2}{5}x)^{2}+(\frac{2}{5}x)^{3}\right]$

$=10^{3}\left[1+3\left(\frac{2}{5}x\right)+3\left(\frac{4x^{2}}{25}\right)+\left(\frac{8}{125}x^{3}\right)\right]$

$=10^{3}\left[1+\frac{6}{5}x+\frac{12x^{2}}{25}+\frac{8x^{3}}{125}\right]$

$=1000\left[1+\frac{6}{5}x+\frac{12x^{2}}{25}+\frac{8x^{3}}{125}\right]$

$=1000+1200x+480x^{2}+64x^{3}$

4. $\sqrt[4]{81+x}$

$\sqrt[4]{81+162x}=(81+162x)^{\frac{1}{4}}$

$(81+162x)^{\frac{1}{4}}=81^{\frac{1}{4}}(1+2x)^{\frac{1}{4}}$

$=81^{\frac{1}{4}}\left[1+\frac{1}{4}(2x)+\frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}(2x)^{2}+\frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{3!}\right]$

$=81^{\frac{1}{4}}\ 1+\frac{1}{4}(2x)+\frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}(2x)^{2}+\frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{3!}$

$$= 81^{\frac{1}{4}}\left[1 + \frac{1}{4}(2x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2} \times 1\,(2x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3 \times 2 \times 1}(2x)^3\right]$$

$$= 81^{\frac{1}{4}}\left[1 + \frac{1}{4}(2x) + \frac{-\frac{3}{16}}{2 \times 1}(4x^2) + \frac{\frac{21}{64}}{3 \times 2 \times 1}(8x^3)\right]$$

$$= 81^{\frac{1}{4}}\left[1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{7}{16}x^3\right]$$

$$= 3\left[1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{7}{16}x^3\right]$$

$$= 3 + \frac{3}{2}x - \frac{9}{8}x^2 - \frac{21}{16}x^3$$

Example 1.17

The first three terms in the expansion of $\left(1 + \frac{x}{p}\right)^n$ in ascending powers of x are $1 + x + \frac{9}{20}x^2$.

Find the values of n and p.

Solution

Expand $\left(1 + \frac{x}{p}\right)^n$ using the binomial theorem;

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$= \left(1 + \frac{x}{p}\right)^n = 1 + n\frac{x}{p} + \frac{n(n-1)}{2!}\left(\frac{x}{p}\right)^2$$

$$= 1 + nx + \frac{n(n-1)}{2}\frac{x^2}{p^2}$$

Equating it to $1 + x + \frac{9}{20}x^2$ and comparing the coefficient of x and x^2

$\frac{n}{p} = 1$, $\frac{n(n-1)}{2p^2} = \frac{9}{20}$

$n = p$……..(1)

$20n(n-1) = 18p^2$……..(2)

Substituting equation $[n = p]$ from into equation (2)

$20n(n-1) = 18p^2$

$20p(p-1) = 18p^2$

$20p^2 - 20p = 18p^2$

$20p^2 - 18p^2 = 20p$

$2p^2 = 20p$

$\frac{2p^2}{2p} = \frac{20p}{2p}$

$\boldsymbol{p} = 10$

$\boldsymbol{n} = 10$

Example 1.18

Find the value of t and n in the equation $(1 + tx)^n = 1 - 6x + \frac{33}{2}x^2$

Solution

Expand $(1 + tx)^n$ using the binomial theorem;

$\boldsymbol{(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n}$

$(1 + tx)^n = 1 + ntx + \frac{n(n-1)}{2!}(tx)^2$

$1 + ntx + \frac{n(n-1)}{2}t^2x^2$

Equating it to $1 - 6x + \frac{33}{2}x^2$ and comparing the coefficient of x and x^2

$nt = -6, \frac{n(n-1)}{2}t^2 = \frac{33}{2}$

$nt = -6$……..(1)

$2n(n-1)t^2 = 66$……..(2)

$(2n^2 - 2n)t^2 = 66$……..(2)

$2n^2t^2 - 2nt^2 = 66$

$2(nt)^2 - 2(nt)t = 66$

But $nt = -6$

$2(-6)^2 - 2(-6)t = 66$

$2(36) + 12t = 66$

$72 + 12t = 66$

$12t = 66 - 72$

$12t = -6$

$\frac{12t}{12} = -\frac{6}{12}$

$\boldsymbol{t} = -\frac{1}{2}$

$nt = -6$

$n\left(-\frac{1}{2}\right) = -6$

$-n = -12$

$\frac{-n}{-1} = -\frac{12}{-1}$

$\boldsymbol{n} = 12$

Example 1.19

Expand $(1 + 2x - x^2)^6$as far as the x^2 term.

Solution

How will you use binomial theorem to solve a question of this nature?
Discuss your views with a classmate or in your groups.
You need to write the trinomial $1 + 2x - x^2$ to a binomial in the form $(1 + x)^n$
How we do that?
We represent $2x - x^2$ by a variable say y, i.e. $y = 2x - x^2$

$(1 + 2x - x^2)^6 = (1 + y)^6$

Expanding $(1 + y)^6$, using the binomial theorem,

$(1 + \boldsymbol{x})^n = \mathbf{1} + \mathbf{nx} + \frac{\boldsymbol{n}(\boldsymbol{n} - 1)}{2!}\boldsymbol{x}^2 + \frac{\boldsymbol{n}(\boldsymbol{n} - 1)(\boldsymbol{n} - 2)}{3!}\boldsymbol{x}^3 + \ldots + \boldsymbol{x}^n$

$(1 + y)^6 = 1 + 6\boldsymbol{y} + 6\frac{(6 - 1)}{2!}\boldsymbol{y}^2 + \ldots$

$= 1 + 6\boldsymbol{y} + 6\frac{(5)}{2}\boldsymbol{y}^2$

$= 1 + 6y + 15y^2$

But $y = 2x - x^2$, so we substitute this in:

$= 1 + 6(2x - x^2) + 15(2x - x^2)^2 + \ldots$

$= 1 + 6(2x - x^2) + 15(4x^2 - 4x^3 + x^4) + \ldots$

$= 1 + 12x - 6x^2 + 60x^2 - 60x^3 + 15x^4 + \ldots$

$=1 + 12x + 54x^2 + \ldots$

Example 1.20

Expand $(1 - 3x + x^2)^5$ as far as the x^3 term.

Solution

We represent $-3x + x^2$ by a variable say y, i.e. $y = -3x + x^2$

$$(1 - 3x + x^2)^5 = (1 + y)^5$$

Expanding $(1 + y)^5$ using the binomial theorem,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$(1 + y)^5 = 1 + 5y + \frac{(5-1)}{2!}y^2 + \frac{5(5-1)(5-2)}{3!}y^3$$

$$= 1 + 5y + \frac{5(4)}{2}y^2 + \frac{5(4)(3)}{6}y^3$$

$$= 1 + 5y + 10y^2 + 10y^3$$

But $y = -3x + x^2$

$$= 1 + 5(-3x + x^2) + 10(-3x + x^2)^2 + 10(-3x + x^2)^3$$

$$= 1 + 5(-3x + x^2) + 10(9x^2 - 6x^3 + x^4) + 10(-27x^3 + 27x^4 - 9x^5 + x^6)$$

$$= 1 - 15x + 5x^2 + 90x^2 - 60x^3 + 10x^4 - 270x^3 + 270x^4 - 90x^5 + 10x^6$$

$$= 1 - 15x + 5x^2 + 90x^2 - 60x^3 - 270x^3$$

$$= 1 - 15x + 95x^2 - 330x^3 + \ldots$$

Did you know we can use binomial expansion to approximate exponential numbers? This powerful technique allows us to simplify complex expressions and make calculations easier. We will explore how to apply binomial expansion to approximate exponential functions.

APPLYING BINOMIAL EXPANSION TO APPROXIMATE EXPONENTIAL NUMBERS

Activity 1.6: Approximating Exponential Numbers Using Binomial Expansion $(a + b)^n$.

Ensure you have your notebook, calculator and pen ready for the activity.

1. Choose a number to work with.
2. Examples include:1.5, 3.5, 10 etc.
3. Express your chosen number in the form of $(a + b)$. For example: If you chose 1.5, you could write it as $(1 + 0.5)$.
4. Select any integer to represent n. Examples include: 2, 4, −4,
5. Write your decimal and n in the form $(a + b)^n$. For example: Using 1.5 and $n = 4$, you would write $(1 + 0.5)^4$.
6. Apply the binomial theorem to expand and evaluate your expression from step 4
7. Choose different sets of numbers and repeat steps 1 to 5. Experiment with various decimal values and n to see how the approximations change.
8. Use your calculator to find the actual value of your chosen decimal raised to the power of n. For example, calculate 1.5^4
9. Look at the results from your binomial expansion and the calculator.
10. Discuss:
    a. How close were your approximations to the actual values?
    b. What did you observe about the accuracy of your approximations with different n values?

Great work! You have effectively used binomial expansion to approximate exponential numbers. Now, let us solve more examples together to deepen your understanding!

Example 1.21

Expand $(1 + 2y)^5$ using binomial theorem.

Hence use your results to evaluate $(1.04)^5$, correct to 4 decimal places

Solution

The binomial theorem is given by

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$(1 + 2y)^5 = 1 + 5(2y) + \frac{5(5-1)}{2!}(2y)^2 + \frac{5(5-1)(5-2)}{3!}(2y)^3 + \frac{5(5-1)(5-2)(5-3)}{4!}(2y)^4 + \frac{5(5-1)(5-2)(5-3)(5-4)}{5!}(2y)^5$$

$$= 1 + 5(2y) + \frac{5(4)}{2\times 1}\times 4y^2 + \frac{5(4)(3)}{3\times 2\times 1}\times 8y^3 + \frac{5(4)(3)(2)}{4\times 3\times 2\times 1\times 1}\times 16y^4 + \frac{5(4)(3)(2)(1)}{5\times 4\times 3\times 2\times 1}\times 32y^5$$

$$= 1 + 5(2y) + 10\times 4y^2 + 5\times 2\times 8y^3 + 5\times 16y^4 + 1\times 32y^5$$

$$= 1 + 10y + 40y^2 + 80y^3 + 80y^4 + 32y^5$$

The (1.04) can be written as a binomial, i.e. (1 + 0.04)

The expression $(1.04)^5 = (1 + 0.04)^5$

Comparing $(1 + 0.04)^5$ to $(1 + 2y)^5$

$2y = 0.04$

$\frac{2y}{2} = \frac{0.04}{2}$

$y = 0.02$

Now substitute $y = 0.02$ into your results; $1 + 10y + 40y^2 + 80y^3 + 80y^4 + 32y^5$

$1 + 10(0.02) + 40(0.02)^2 + 80(0.02)^3 + 80(0.02)^4 + 32(0.02)^5$

$1 + 10(0.02) + 40(0.0004) + 80(0.000008) + 80(0.00000016) + 32(0.0000000032)$

$= 1 + 0.2 + 0.016 + 0.00064 + 0.0000128 + 0.0000001024 = 1.2166529024$

$= 1.2167$ (to 4dp)

Thus $(1.04)^5 \approx 1.2167$

Example 1.22

Find the first five terms of the expression $\sqrt{1 + 3x}$.

Hence find an estimate of the value of $\sqrt{1.12}$ to 5 decimal places

Solution

The binomial theorem is given by

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$\sqrt{1+3x} = (1+3x)^{\frac{1}{2}}$$

$$(1+3x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(3x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(3x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(3x)^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}(3x)^4$$

$$= 1 + \tfrac{1}{2}(3x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(3x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)-\left(\frac{3}{2}\right)}{6}(3x)^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24}(3x)^4$$

$$= 1 + \frac{3x}{2} + \frac{-\frac{1}{4}}{2}(3x)^2 + \frac{\frac{3}{8}}{6}(3x)^3 + \frac{\frac{-15}{16}}{24}(3x)^4$$

$$= 1 + \frac{3x}{2} + \frac{-\frac{1}{4}}{2}(9x^2) + \frac{\frac{3}{8}}{6}(27x^3) + \frac{\frac{-15}{16}}{24}(81x^4)$$

$$= 1 + \frac{3x}{2} - \frac{1}{8}(9x^2) + \frac{1}{48}(27x^3) - \frac{5}{128}(81x^4)$$

$$= 1 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{27x^3}{16} - \frac{405x^4}{128}$$

The (1.12) can be written as binomial, i.e. (1 + 0.12)

The expression $(1.12)^{\frac{1}{2}} = (1 + 0.12)^{\frac{1}{2}}$

Comparing $(1 + 0.12)^{\frac{1}{2}}$ to $(1 + 3x)^{\frac{1}{2}}$

$3x = 0.12$

$\frac{3x}{3} = \frac{0.12}{3}$

$y = 0.04$

Now substitute $y = 0.04$ into your results; $1 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{27x^3}{16} - \frac{405x^4}{128}$

$$= 1 + \frac{3(0.04)}{2} - \frac{9(0.04)^2}{8} + \frac{27(0.04)^3}{16} - \frac{405(0.04)^4}{128}$$

$$= 1 + \frac{0.12}{2} - \frac{9(0.0016)}{8} + \frac{27(0.000064)}{16} - \frac{405(0.00000256)}{128}$$

$$= 1 + \frac{0.12}{2} - \frac{0.0144}{8} + \frac{0.001728}{16} - \frac{0.00020736}{128}$$

$= 1 + 0.06 - 0.0018 + 0.000108 - 0.0000081 = 1.0582999$

$= 1.05830$ (to 5dp)

Thus $\sqrt{1.12} \approx 1.05830$

Example 1.23

Using the binomial theorem, find an estimate for the value of $\sqrt{101}$

Solution

The binomial theorem is given by

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$

$\sqrt{101} = \sqrt{100 + 1} = (100 + 1)^{\frac{1}{2}}$

$(100 + 1)^{\frac{1}{2}} = 100^{\frac{1}{2}}\left(1 + \frac{1}{100}\right)^{\frac{1}{2}} = 100^{\frac{1}{2}}\left[\left(1 + \frac{1}{100}\right)^{\frac{1}{2}}\right]$

$= 10\left[1 + \frac{1}{2}(0.01) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(0.01)^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!}(0.01)^3 + \ldots\right]$

$= 10\left[1 + \frac{1}{2}(0.01) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(0.01)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(0.01)^3 + \ldots\right]$

$= 10\left[1 + \frac{1}{2}(0.01) + \frac{-\frac{1}{4}}{2}(0.01)^2 + \frac{\frac{3}{8}}{6}(0.01)^3\right]$

$= 10\left[1 + 1/2(0.01) - \frac{1}{8}(0.0001) + \frac{1}{16}(0.000001)\right]$

$= 10[1 + 0.005 - 0.0000125 + 0.0000000625\ ]$

$= 10[1.004987563\ ] = 10.04987563$

Thus $\sqrt{101} \approx 10.0499$

Example 1.24

Use binomial theorem to find approximations for:

a. $(3.97)^{\frac{1}{2}}$

b. $(0.994)^{\frac{1}{4}}$

c. $\sqrt{25.05}$

Solution

a. $(3.97)^{\frac{1}{2}} = (4 - 0.03)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}(1 - 0.03)^{\frac{1}{2}} = 4^{\frac{1}{2}}[(1 - 0.0075)^{\frac{1}{2}}]$

$= 2[(1 - 0.0075)^{\frac{1}{2}}] = 2\left[1 + \frac{1}{2}(-0.0075) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(-0.0075)^2 + \ldots\right]$

$= 2\left[1 + \frac{1}{2}(-0.0075) + \frac{-\frac{1}{4}}{2}(-0.0075)^2 + \ldots\right]$

$= 2\left[1 + \frac{1}{2}(-0.0075) - \frac{1}{8}(-0.0075)^2 + \ldots\right]$

$= 2\left[1 - 0.00375 - \frac{1}{8}(0.0056625) + \ldots\right]$

$= 2[1 - 0.00375 - 0.00003125 + \ldots]$

$= 2[0.99621875] = 1.9924375$

$\therefore (3.97)^{\frac{1}{2}} \approx 1.992 \ (to\ 4sf)$

b. $(0.994)^{\frac{1}{4}} = (1 - 0.006)^{\frac{1}{4}}$

$= \left[1 + \frac{1}{4}(-0.006) + \frac{\frac{1}{4}\left(\frac{1}{4} - 1\right)}{2!}(-0.006)^2 + \ldots\right]$

$= \left[1 + \frac{1}{4}(-0.006) + \frac{-\frac{1}{16}}{2}(-0.006)^2 + \ldots\right]$

$= \left[1 + \frac{1}{4}(-0.006) - \frac{1}{32}(-0.006)^2 + \ldots\right]$

$= \left[1 - 0.0015 - \frac{1}{32}(0.000036) + \ldots\right]$

$= [1 - 0.0015 - 0.000001125 + \ldots]$

$= [0.998498875]$

$\therefore (0.994)^{\frac{1}{4}} \approx 0.9985 \ (to\ 4sf)$

c. $\sqrt{25.05} = (25 + 0.05)^{\frac{1}{2}} = 25^{\frac{1}{2}}(1 + 0.002)^{\frac{1}{2}}$

$= 5[(1 + 0.02)^{\frac{1}{2}}] = 5\left[1 + \frac{1}{2}(0.002) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(0.002)^2 + \ldots\right]$

$= 5\left[1 + \frac{1}{2}(0.002) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(0.002)^2 + \ldots\right]$

$= 5\left[1 + \frac{1}{2}(0.002) + \frac{-\frac{1}{4}}{2}(0.002)^2 + \ldots\right]$

$= 5\left[1 + \frac{1}{2}(0.002) - \frac{1}{8}(0.002)^2 + \ldots\right]$

$= 5\left[1 + 0.001 - \frac{1}{8}(0.000004) + \ldots\right]$

$= 5\,[1 + 0.001 - 0.0000005] = 5[1.0009995]$

$= 5.0049975$

$\therefore \sqrt{25.05} \approx 5.005\ (to\ 4sf)$

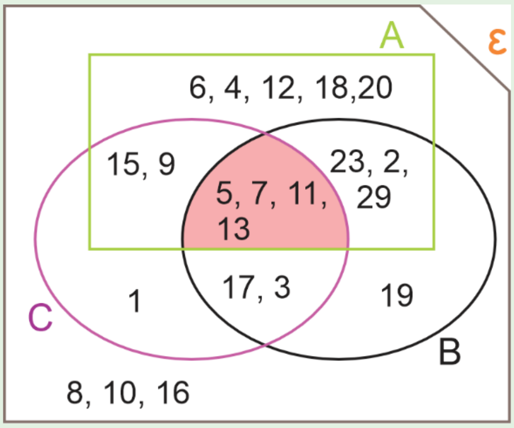
EXTENDED READING

- Cambridge Additional Mathematics by Michael Haese, Sandra Haese, Mark Humphries, Chris Sangwin. Haese Mathematics (2014).
- Goldie, S. (2012). *Pure Mathematics*. Hodder Education.
- Talbert, J. F., & Heng, H. H. (1995). *Additional Mathematics: Pure & Applied*. Pearson Education South Asia.

REVIEW QUESTIONS

Part A

1. Use the Venn diagram to answer the questions.



a. List the elements in A, B and C

b. List the elements in A′, B′ and C′

c. Find (i) (A ∪ B ∪ C)′ (ii) A′ ∩ *B*′ ∩ C′ (iii) (A ∩ B ∩ C)′ (iv) A′ ∪ B′ ∪ C′

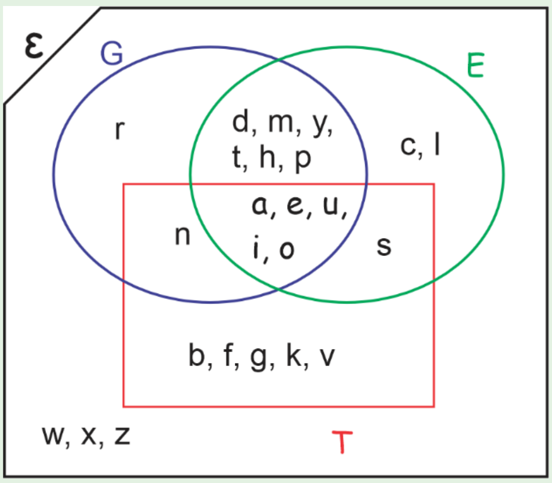
d. Compare your result in 4(a) to 4(b). What conclusion can you draw?

e. Compare your result in 4(c) to 4(d). What conclusion can you draw?

f. Describe set B

g. Describe set C

2. Study the diagram carefully and use it to answer the questions.



a. List the elements in G, E, T and *ε*

b. Show that $G' \cap E' = (G \cup E)'$

c. Show that $(G \cap E \cap T)' = G' \cup E' \cup T'$

d. Show that $(G \cup E \cup T)' = G' \cap E' \cap T'$

For questions 3 – 7, sets X, Y and Z are subsets of the universal set μ.
Use the information to choose the correct answer to the questions.

3. Find $X \cup X'$

A. $\varnothing$ **B.** μ

C. X **D.** Y

4. Find $Z \cap Z'$

A. $\varnothing$ **B.** μ

C. Z **D.** X

5. Find $(Y')'$

A. $\varnothing$ **B.** μ

C. X D. Y

6. Find $(X \cup X')'$

A. $\varnothing$ **B.** μ

C. X **D.** $X' \cup X$

7. $(Z \cap Z')'$

A. $\varnothing$ **B.** μ

C. $X \cup Y$ **D.** $Z' \cap Z$

8. Rewrite

a. $(R \cup S')'$ **b.** $(M' \cap N)'$

c. $[(A' \cup B)' \cap C]'$ **d.** $[(P \cap Q')' \cup P]'$

e. $[A \cap (A' \cup B)']'$

9. Show that

(a) $[(R \cup M)' \cap P]' = R \cup M \cup P'$

(b) $[(R \cap M) \cup M']' = R' \cap M$

10. On a Venn diagram, shade the region represented td by:

a. $(A \cap B)' \cap C$

b. $A' \cup (Y \cap Z')$

11. Sets X = {prime numbers}, Y = $\{0 < W \leq 11\}$ and

Z = {1, 2, 3, 4, 5, 6, 7, 13, 14, 15, 16, 18} are subsets of A = {0, 1, 2, 3, 4, … , 21}.

a. List the members of the following sets

b. **i.** X **ii.** Y

iii. $X \cap Y \cap Z$ **iv.** $X \cup Y'$ (v) $(X \cup Y) \cap Z'$

vi. $(X' \cap Y) \cup Z$

c. Use a Venn diagram to represent X, Y, Z and A

12. A survey of 90 teachers reveals the following.

45 teach Management, 57 Accounting and 50 Costing.

25 teach Management and Accounting, 35 teach Accounting and Costing and 22 teach Management and Costing. 5 teachers cannot teach any of the three subjects.

a. Represent the information on a Venn diagram

b. calculate the number of teachers who can teach only one of the three subjects.

13. A survey of 41 employees shows that 25 are skilled in Carpentry, 15 in Brick laying and 24 in Tiling. 12 are skilled in carpentry and brick laying, 9 in brick laying and tilling, 17 in carpentry and tilling, and 8 are not skilled in any of the three occupations.

Find the number of employees who are skilled in

a. all three occupations

b. exactly two of the occupations

14. A class of 60 students were each required to have textbooks in Accounting, Management and Costing. However, when the class teacher checked their books, she found out that:

16 had Accounting and Management books

14 had Accounting and Costing books

18 had only two of the three books and 20 had accounting plus at least one other book

Every student had at least one of the three books.

Find the number of students who had

a. All three books.

b. Only Costing and Accounting books

c. Exactly one of the three books

d. If 16 students had neither Accounting nor Costing books, find the probability that a student selected at random from the class had Management books.

15. The analysis of results of a sample of students who sat for the WASSCE in Biology, Physics and Mathematics revealed that:

80% passed in Biology

65% passed in Mathematics

70% passed in Physics

52% passed in Biology and Mathematics

58% passed in Biology and Physics

55% passed in Physics and Mathematics.

45% passed in all three subjects.

What percentage of the students:

a. passed only Mathematics

b. passed only physics

c. passed only Biology

d. passed Biology and Physics but not mathematics

e. did not pass any of the subjects.

16. A shop owner kept records of purchases made by customers over the weekend. The shop sells food, clothing and hardware. She found that, out of the customers who visited her shop, 210 purchased food products, 100 hardware and 147 clothing. 30 purchased food and hardware. 128 purchased two of the three products. She also found that of those who purchased two of the three products, 14 did not buy clothing and 20 did not buy food. 13 of the customers who visited the store did not buy any of the products.

a. How many customers visited the store over the weekend?

b. How many of the customers who visited the store:

i. purchased less than two products.

ii. all three products.

iii. Food and other products.

17. A group of 210 tourists were asked to indicate which of the languages, German, French and English they can speak. It was found that 55 can speak German, 76 can speak French and 200 can speak English. 103 can speak two of the three languages and 6 tourists cannot speak any of the three languages.

Find the number of tourists who can speak:

a. all three languages

b. only one of the three languages.

18. 87 schools in Tamale have subscriptions to at least one of the following newspapers: Graphic, Mirror and Times. 69 subscribe to Graphic, 62 to Mirror and 55 to Times. 32 have subscriptions to all three newspapers.

Find the number of schools who have subscriptions to:

a. exactly two of the newspapers

b. only one of the newspapers.

19. A publishing company surveyed its employees about their proficiency in three software: Photoshop, CorelDraw and InDesign. The results show that all the employees were proficient in at least one of the three software and 10 were proficient in all three. The number of employees proficient in CorelDraw and other software were twice those proficient in only CorelDraw. Of the employees who were proficient in two of the software, 14 were not proficient in InDesign, 8 were not proficient in CorelDraw and 6 were not proficient in Photoshop. The same number of employees proficient in only InDesign are proficient in only Photoshop. There were more employees proficient in only Photoshop and InDesign than employees proficient in only Photoshop and there were fewer employees proficient in only InDesign and CorelDraw than employees proficient in only InDesign.

a. Use a Venn diagram to represent the above data.

b. How many employees were involved in the survey?

c. How many employees were proficient in only one of the software?

Part B

1. Write down the first five terms of the binomial expansion of:

a. $(1 + 4x)^{12}$

b. $(1 - 3x)^{9}$

c. $(3 + 2x)^{7}$

d. $(1 - x \_ 2)^{8}$

e. $\left(2 - \frac{5}{2}x\right)^{6}$

2. Write down the term indicated in the binomial expansion of the following functions:

a. $(1 + 3x)^{6}$: 3rd term

b. $(1 + x \_ 3)^{20}$: 7th term

c. $(a + 4b)^{12}$ term containing a^{4}

3. Expand the following binomial expansions up to and including the terms in x^{2}:

a. $(1 - 2x)(1 + x)^{8}$

b. $(2 + 3x)(1 + 3x)^{10}$

c. $(5 + 2x)\left(1 - \frac{2}{3}x\right)^{12}$

d. $(1 + 4x)(1 - 3x)^{20}$

4. Expand $(1 - 2x)^{7}$ and hence find the value of $(0.98)^{7}$ correct to 4 decimal places.

5. Find the first four terms of the following using the binomial theorem.

a. $\left(1 - \frac{1}{2}x\right)^{\frac{1}{2}}$

b. $(1 - 2x)^{-3}$

c. $(3 - x)^{-2}$

6. Use appropriate binomial expansions to find the approximate values of:

a. $\sqrt[3]{8.7}$ correct to 5 significant figures.

b. $(81.162)^{\frac{1}{4}}$ to 4 decimal places

c. $\sqrt{4.02}$ to 3 decimal places

d. $\frac{1}{\sqrt{10}}$ to 4 decimal places

e. $\sqrt{101}$ to 4 decimal places

f. $(0.995)^5$ to 4 decimal places

7. Show that in ascending powers of x of $\left(\frac{1-x}{1+x}\right)^{\frac{1}{3}}$ is $1 - \frac{2}{3}x + \frac{2}{9}x^2 - \frac{4}{21}x^3$ ….

8. Expand the following as far as the x^3 term

a. $(1 + x + 2x^2)^3$

b. $(1 + 3x + x^2)^4$

9. a. find the 1st 3 terms in ascending powers of x of the binomial expansion$(1 + py)^9$, where p is a non zero constant.

b. If the first 3 terms are 1, 36x and qx^2 where q is a constant, find the value of p and q.

10. a. Find the first 3 terms in ascending powers of x of the binomial expansion$(2 + kx)^7$, where k is a constant.

b. Given that the coefficient of x^2 is 6 times the coefficient of x, find the value of k.

SECTION

2 SEQUENCES AND INEQUALITIES

MODELLING WITH ALGEBRA

Application of Algebra

INTRODUCTION

Studying sequences and inequalities is important because arithmetic and geometric sequences provide statistical understanding into central tendencies, growth rates, etc. Also, quadratic inequalities help in decision-making processes where certain conditions or constraints must be met. For example, in business and economics, they are used to model profit maximisation or loss minimisation under specific constraints. In physics and engineering, quadratic inequalities arise when determining safe operating conditions, or analysing forces. In architecture and design, they help define areas, volumes or structural integrity within specific boundaries. This section introduces you to the concept of the sum of sequences, convergent and divergent series, recursive sequences, arithmetic and geometric sequences. You will also learn how to solve quadratic inequalities, systems of quadratic inequalities and real-life problems involving quadratic inequalities.

KEY IDEAS

- A **sequence** is a set of numbers in a given order.
- Each of the numbers in a sequence is called a term.
- Maximum values are the largest possible values of variables within a system that satisfy a set of constraints, each represented by an inequality.
- Minimum values are the least possible values of variables within a system that satisfy a set of constraints, each represented by an inequality.
- When finding the solution to a quadratic inequality, it is advisable to start by sketching the associated quadratic graph.

SUM OF SEQUENCES

In year one we identified number patterns and classified them into arithmetic and geometric sequences. While sequences are a list of numbers with defined patterns, a series is the sum of the terms of a sequence. This year we will be finding the sum of sequences and analysing the convergence or divergence of series.

If the general term of a series with n term is known, then the complete series can be written down in short notation as indicated by the following:

1. $a+(a+d)+(a+2d)+\ldots+(a+[n-1]d)=\sum_{r=1}^{n}(a+[r-1]d)$

2. $a+ar+ar^2+\ldots ar^{n-1}=\sum_{k=1}^{n}ar^{k-1}$

3. $1^2+2^2+3^2+\ldots n^2=\sum_{r=1}^{n}r^2$

4. $-1^3+2^3-3^3+4^3+\ldots(-1)^n n^3=\sum_{r=1}^{n}(-1)^r r^3$

Notes:

a. It is sometimes more convenient to count the term of a series from zero rather than 1.

For example:

$a+(a+d)+(a+2d)+\ldots+(a+[n-1]d)=\sum_{r=0}^{n-1}(a+rd)$ and

$$a+ar+ar^2+\ldots ar^{n-1}=\sum_{k=0}^{n-1}ar^k$$

In general, for a series with n terms starting at u_0 :

$$u_0+u_1+u_2+u_3+\ldots u_{n-1}=\sum_{r=0}^{n-1}u_r$$

b. We may also use the sigma notation for "infinite series" such as those already encountered in the sum to infinity of a geometric series.

For example: $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots=\sum_{r=1}^{\infty}\frac{1}{3^{r-1}}$ or $\sum_{r=0}^{\infty}\frac{1}{3^r}$

Activity 2.1 – Sums of Sequences

Perform this activity in pairs or groups.

In year one we learnt about Arithmetic and Geometric Progression. Let us revise and build on that knowledge by creating the sequence terms and find the sum of the terms of the expressions below.

1. Evaluate $\sum_{n=0}^{6}2(n-3)$
2. Find the sum $\sum_{k=1}^{50}(3k+2)$

3. Evaluate $\sum_{r=1}^{3}\left(\frac{1}{2}\right)^r$

Solution

1. $\sum_{n=0}^{6} 2(n-3) = 2(0-3) + 2(1-3) + 2(2-3) + 2(3-3) + 2(4-3) + 2(5-3) + 2(6-3)$

$= 2(-3) + 2(-2)+2(-1) + 2(0) + 2(1) + 2(2) + 2(3)$

$= (-6) +(-4) + (-2) + 0 + 2 + 4 + 6$

$= 0$

2. $\sum_{k=1}^{50}(3k+2)$

$u_1 = 3(1) + 2 = 5$

$u_{50} = 3(50) + 2 = 152$

$s_k = \frac{k(u_1 + u_{50})}{2}$

$s_k = \frac{50(5 + 152)}{2} = 3925$

3. $\sum_{r=1}^{3}\left(\frac{1}{2}\right)^r$

$when\ r = 1, \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$when\ r = 2, \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$when\ r = 3, \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$\Longrightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

Method of undetermined coefficients

The method of undetermined coefficients is a powerful technique for finding the general solution of linear recurrence relations with constant coefficients

If the *nth* term of a linear sequence is denoted by x_n, then the recurrence relation is of the form: $x_{n+1} = f(x_n)$ for some of the function *f*.

For example, in $x_{n+1} = 4 - \frac{x_n}{2}$, when $x_2 = 2$, then x_3 is expressed as $x_{2+1} = 4 - \frac{x_2}{2}$

$x_3 = 4 - \frac{2}{2}$

$x_3 = 4 - 1$

$x_3 = 3$

Example 2.1

The *nth* term of a sequence is U_{n}. If $U_1 = 3$ and $U_{n+1} = U_n + 5$,find U_3.

Solution

$\boldsymbol{U_{n+1}} = U_n + 5$

$\mathrm{U}_1 = 3$

$U_2 = U_{1+1.} = U_1 + 5$, but $U_1 = 3$

$U_2 = 3 + 5 = 8$

$U_3 = U_{2+1.} = U_2 + 5$

$U_3 = 8 + 5 = 13$

Sum of an Arithmetic Progression (AP)

Activity 2.2

1. Working in pairs or small groups, work through the following steps to discover how to find the general sum of an arithmetic sequence, ie a sequence with a common difference.
2. Let ***Sn*** denote the sum of the first ***n*** terms of a linear sequence, where "*a*" is the first term and "*d*" the common difference.
3. $Sn = a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d]$ ………………….(1)
4. Writing the terms in the reverse order, we get :
5. $Sn = [a + (n - 1)d] + [a + 2d(n - 2)] + \ldots + (a + 2d) + (a + d) + a$ ……………….(2)

   Adding (1) and (2), we get

   $2Sn = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)] + \ldots = n[2a + (n - 1)d]$

   (Since we are adding *n* terms we have *n* lots of this expression.)

$\therefore \boldsymbol{Sn} = \frac{\boldsymbol{n}}{2}\left[2\boldsymbol{a} + \left(\boldsymbol{n} - 1\right)\boldsymbol{d}\right]$

6. Let $U_1 = a$,

7. Then, $Sn = \frac{n}{2}\left[2\,U_1 + \left(n - 1\right)d\right]$

8. Similarly, $Sn = \frac{n}{2}\left[U_1 + \left\{U_1 + (n - 1)d\right\}\right]$

9. But $U_n = U_1 + \left(n - 1\right)d$

10. By substitution, we get

11. $\boldsymbol{S_n} = \frac{\boldsymbol{n}}{2}\left[\boldsymbol{U_1} + \boldsymbol{U_n}\right]$ (Remember that U_1 = the first term and U_n = the last term)

Example 2.2

a. Find the sum of the first 35 terms of the linear sequence 2, 5, 8…

Solution

$\boldsymbol{a} = 2, \boldsymbol{d} = 3$ and $\boldsymbol{n} = 35$

$\boldsymbol{Sn} = \frac{\boldsymbol{n}}{2}\left[2\boldsymbol{a} + \left(\boldsymbol{n} - 1\right)\boldsymbol{d}\right]$

$S_{35} = \frac{35}{2}\left[2(2) + (35 - 1)(3) = 1855\right.$

b. A sequence is defined by $U_{n+1} = 3 + U_n$ and $U_1 = 1$

c. Find the sum of the first ten term of the sequence.

Solution

$U_{n+1} = 3 + U_n$

$U_1 = 1\ (= a)$

$U_2 = 4$

Common difference $d = u_2 - u_1 = 4 - 1 = 3$

$\boldsymbol{S_n} = \frac{\boldsymbol{n}}{2}[2\boldsymbol{a} + (\boldsymbol{n} - 1)\boldsymbol{d}]$**,**

where $\boldsymbol{a}$ = first term, $\boldsymbol{d}$ the common difference and $\boldsymbol{n}$ the number of terms

$S_{10} = \frac{10}{2}[2(1) + (10 - 1)3]$

$S_{10} = 5[2 + 9(3)]$

$S_{10} = 5[29]$

$S_{10} = 145$

d. Find the the sum of the first 10 positive even numbers:

Solution

$S_{10} = 2 + 4 + 6 + 8 + 10 \cdots + 18 + 20$

$\boldsymbol{S_n = \frac{n}{2}[U_1 + U_n] = \frac{n}{2}[\textit{first term} + \textit{last term}]}$

$S_{10} = \frac{10}{2}[2 + 20] = 5 \times 22 = 110$

Example 2.3

A conference hall has 189 seats, arranged in 9 rows. The organiser wants to arrange the seats such that the difference in the number of seats between any two consecutive rows is 5 seats. If there are 9 rows in total, how many seats should be in the third row?

Solution

Number of rows $n = 9$

Capacity of the room $S_9 = 189$

Difference $d = 5$

Let the number of seats in the first row $= a$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_9 = \frac{9}{2}[2a + (9 - 1)5]$

$189 = \frac{9}{2}[2a + (8)5]$

$378 = 9[2a + (8)5]$

$378 = 18a + 360$

$18a = 378 - 360$

$18a = 18$

$a = 1$

Number of seats in the third row $a + 2d \Rightarrow 1 + 2(5) = 11$

Sum of a Geometric Progression (GP)

Remember that a geometric progression has the first term, a and a common ratio, r.

The position of terms represented by n can be written as:

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$ ....(1)

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^{n-1} + ar^n$............(2)

$eqn\ (2) - eqn\ (1)$

$rSn - Sn = -a + ar^n$

$S_n(r - 1) = ar^n - a$

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{-a(1 - r^n)}{-(1 - r)}$$

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1}$$

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{(1 - r)}, r \neq 1}$$

Example 2.4

Find the sum of the first seven terms of the sequence 2, 8, 32…

Solution

$$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$a = 2, r = 4\ and\ n = 7$

$$S_7 = \frac{2(4^7 - 1)}{4 - 1}$$

$$S_7 = \frac{2(16384 - 1)}{3}$$

$$S_7 = \frac{2 \times 16\,382}{3}$$

$S_7 = 10922$

Example 2.5

Find the sum of the first 10 terms of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Solution

$S_n = \frac{a(1 - r^n)}{(1 - r)}, r \neq 1$

$a = \frac{1}{2}, r = \frac{1}{2}$ and $n = 10$

$S_{10} = \frac{\frac{1}{2}(1 - (\frac{1}{2})^{10})}{(1 - \frac{1}{2})}$

$S_{10} = \frac{1}{2}\left(1 - \frac{1}{1024}\right)$

$S_{10} = \frac{1024 - 1}{1024}$

$S_{10} = \frac{1023}{1024}$

CONVERGENCE AND DIVERGENCE OF SERIES

A series is a sum of a sequence of numbers. For example, $1 + 2 + 3 + 4 + \ldots$

Convergence of series

A series is **convergent** if its sum approaches a finite value. Think of it like a target.

1. The series get closer and closer to the target (the finite value).
2. The sum of the series stabilises.
3. Example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
4. As n increases the nth term is converging to 0 and the sum of the series is converging to 2.
5. If the limit of the *nth* term is 0, the series might converge.
6. If the ratio between the terms is constant, then the series converges when $|r| < 1$.

Divergence

A series is **divergent** if its sum grows without bounds or approaches infinity. Think of it like a runaway train:

1. The series gets farther and farther away from a finite value.
2. The sum of the series increases indefinitely.
3. Example; $1 + 2 + 4 + 8 + \ldots$ this series diverges.

If the limit of the *nth* term is not 0, the series diverges.

If the ratio between term is greater than 1, the series diverges, ie $|r| > 1$

Sum to infinity of a convergent Geometric Progression

For the general exponential sequence (GP):

$S_n = \frac{a(1 - rn)}{(1 - r)}$, *if* $-1 < r < 1$, then r^n becomes smaller and smaller as n becomes larger and larger, so as $n \longrightarrow \infty$ then $r^n \to 0$ and $(1 - r^n) \to 1$.

We say the limiting value is denoted S_∞ is $\frac{a}{1 - r}$.

Hence, the sum to infinity of $a, ar\, ar^2, \ldots$ is given by $\boldsymbol{S_\infty = \frac{a}{1-r}}$, provided $-1 < r < 1$.

Example 2.6

Determine the sum to infinity of the geometric series $5, -1 + \frac{1}{5}, \ldots$

Solution

$5, -1 + \frac{1}{5}, \ldots$

$a = 5,\ r = -\frac{1}{5}$

The sum to infinity is

$$S\infty = \frac{a}{1 - r}$$

$$= \frac{5}{1-\left(-\frac{1}{5}\right)} = \frac{5}{\left(\frac{6}{5}\right)} = \frac{25}{6} = 5\frac{1}{6}$$

Note: $S_1, S_2, S_3, \ldots$ are the partial sums of the series. If there is a number L such that: $\boldsymbol{S_n = \sum_{r=1}^{\infty} Sn = L}$. The number L is called the sum of the infinity series. If there is no such number, the series is said to diverge.

RECURSIVE SEQUENCES

A recursive sequence is a sequence where each term is defined using previous terms.

For example, for an Arithmetic Sequence:

First term $a_1 = 2$

Recursive formula: $a_n = a_{n-1} + 3$

1. Start with $a_1 = 2$

2. To find a_2, add 3 to a_1 : $a_2 = 2 + 3 = 5$
3. To find a_3, add 3 to a_2 : $a_3 = 5 + 3 = 8$
4. To find a_4, add 3 to a_3 : $a_4 = 8 + 3 = 11$

   Sequence: 2, 5, 8, 11, 14, 17…

For example, for a Geometric Sequence:

First Term: $a_1 = 2$

Recursive formula: $a_n = 2 \times a_{n-1}$

1. Start with $a_1 = 2$
2. To find a_2, multiply a_1 *by* 2 $\rightarrow a_2 = 2 \times 2 = 4$
3. To find a_3, multiply a_2 *by* 2 $\rightarrow a_3 = 2 \times 4 = 8$
4. To find a_4, multiply a_3 *by* 2 $\rightarrow a_4 = 2 \times 8 = 16$

   Sequence: 2, 4, 8, 16, 32, 64, …

In both examples;

- The first term (a_1) is given
- The recursive formula defines how to find the next term (a_n)using the previous term (a_{n-1})

Recurrent decimals

Recurrent decimals are repeated decimals. It can be seen that every recurrent decimal represents a rational number. Every rational number can be represented by a terminal decimal or a recurrent decimal. If the denominator of a rational number (written in its simplest form) has prime factors, other than 2 and 5, then the rational number is recognised as a recurrent decimal.

For example; $\frac{5}{9} = 0.5555\ldots$ *or* $0.\dot{5}$, $\frac{2}{11} = 0.181818\ldots$*or* $0.\dot{1}\dot{8}$, $\frac{8}{15} = 0.53333\ldots$ *or* $0.5\dot{3}$

Example 2.7

Write the recurring decimals as a series. Identify the common ratio, the first term and an explicit formula for the *nth* term:

a. 0.222222…
b. 0.323232…
c. 5.414141…

Solution

a. $0.2222... = 0.2 + 0.02 + 0.002 + 0.0002 + \cdots$

$= 0.2 + 0.2(0.1) + 0.2(0.01) + 0.2(0.001) + \cdots$

$= 0.2 + 0.2(0.1) + 0.2(0.1)^2 + 0.2(0.1)^3 + \cdots$

The first term is 0.2, and the common ratio between each successive term is 0.1. The formula for the *nth* term will be $U_n = 0.2\left(\frac{1}{10}\right)^{n-1}$

b. $0.323232\ldots = 0.32 + 0.0032 + 0.000032 + 0.00000032 + \cdots$

$= 0.32 + 0.32(0.01) + 0.32(0.0001) + 0.32(0.000001) + \cdots$

$= 0.32 + 0.32(0.01) + 0.32(0.01)^2 + 0.32(0.01)^3 + \cdots$

The first term is 0.32, and the common ratio between each successive term is 0.01. The formula for the *nth* term will be $U_n = 0.32\left(\frac{1}{100}\right)^{n-1}$

c. $5.414141\ldots = 5 + 0.41 + 0.0041 + 0.000041 + 0.00000041 + \cdots$

$= 5 + 0.41 + 0.41(0.01) + 0.41(0.1)^2 + 0.41(0.1)^3 + \cdots$

From the second term (0.41), a geometric sequence with first term 0.41, and the common ratio 0.01 can be observed. The formula for the *nth* term will be

For $0.414141\ldots$, $U_n = 0.41\left(\frac{1}{100}\right)^{n-1}$

Thus, $5.414141\ldots = 5 + 0.41\left(\frac{1}{100}\right)^{n-1}$ for $n = 1, 2, 3, \ldots$

Example 2.8

Write 0.9999… as an exponential series and show that in the limit, the sum of series equals one.

Solution

$0.9999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$

This is a G.P. with first term, $a = 0.9$ and common ratio, $r = \frac{0.09}{0.9} = 0.1$

The series is an infinite series with sum; $S_\infty = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.1} = 1$

Example 2.9

Express $0.1\dot{6}$ as an infinite geometric series and find the sum of the geometric series.

Solution

$0.1\dot{6} = 0.166666\ldots$

$= 0.1 + 0.06 + 0.006 + 0.0006 + \ldots$

$= \frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \ldots$

From the series, the term after $\frac{1}{10}$ from geometric series with $= \frac{6}{100}$, $r = \frac{6}{100} \div \frac{6}{1000}$ $= \frac{1}{10}$

Sum of the geometric series:

$$0.0\dot{6} = S_\infty = \frac{\frac{6}{100}}{1 - \frac{1}{10}}$$

$$= \frac{\frac{6}{100}}{\frac{9}{10}}$$

$$= \frac{1}{15}$$

$\therefore 0.1\dot{6} = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$

Example 2.10

If a sequence $U_1, U_2, U_3\ldots$ is define by the relation $U_{n+1} = 3 + U_n$ for $n \geq 1$ and $U_1 = 1$, find;

a. U_2, U_3 and U_4

b. a formula for U_n in terms of n.

c. the sum of the $1st$ n terms

Solution

a. Using the relation

$U_{n+1} = 3 + U_n$

If $n = 1$

$U_2 = 3 + U_1 = 3 + 1 = 4$

$U_3 = 3 + U_2 = 3 + 4 = 7$

$U_4 = 3 + U_3 = 3 + 7 = 10$

The sequence is 1, 4, 7, 10, …, so it is a linear sequence

b. $U_n = a + (n - 1)d$ and $a = 1, d = 3$

$U_n = 1 + (n - 1)3$

$U_n = 1 + 3n - 3$

$U_n = 3n - 2$

c. The sum of n terms of 1, 4, 7, 10, …

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_n = \frac{n}{2}[2(1) + (n - 1)3]$

$S_n = \frac{n}{2}[2 + 3n - 3]$

$S_n = \frac{n}{2}[3n - 1]$

ARITHMETIC AND GEOMETRIC MEANS OF SEQUENCES

The mean of any two numbers, for example, a and b, is obtained by finding a half of the sum of the numbers thus: $Mean, (\mu) = \frac{1}{2}(a + b)$. The difference between a and μ is equal to the difference, or distance, between μ and b i.e., $\mu - a = b - \mu$.

Now, a, μ, b form an arithmetic sequence since there is a common difference and that for any arithmetic sequence with a and b as terms, some means namely, μ_1, μ_2, μ_3, … μ_k can be found between a and b such that a, μ_1, μ_2, μ_3, … μ_k, b, remains an arithmetic sequence as $\mu_1 - a = \mu_2 - \mu_1 = .. \mu_k - b$ and the means are equally spaced.

Example 2.11

1. Find the arithmetic mean sequence of students' scores in mathematics test: 80, 75,90,85,95.
2. A baker wants to increase the amount of sugar in a recipe from 200g to 500g in four equal steps. What arithmetic sequence represents the amount of sugar (in grammes) added and what are the intermediate values?

Solution

1. Arithmetic mean $= \frac{80 + 75 + 90 + 85 + 95}{5} = \frac{425}{5} = 85$
2. The baker wants to increase the sugar from 200g to 500g

   $500 - 200 = 300g$

$d = \frac{300}{4 - 1}$

$= \frac{300}{3}$

$= 100g$

Arithmetic mean

$200 + 100 = 300g$

$300 + 100 = 400g$

$400 + 100 = 500g$

The arithmetic sequence: 200, 300, 400, 500.

The intermediate sequence 300 *and* 400.

Geometric mean

The geometric mean of a sequence is the measure of the central tendency of the sequence, similar to the arithmetic mean. However, it is calculated differently.

For a sequence of n numbers $(a_1 \times a_2 \times a_3 \ldots a_n)^{\frac{1}{n}}$

Example 2.12

1. Find the geometric mean of the sequence 2, 4, 8, 16
2. Find the geometric mean of the sequence 10, 20, 40.

Solution

1. 2, 4, 8, 16

   $(a_1 \times a_2 \times a_3 \ldots a_n)^{\frac{1}{n}}$

   $(2 \times 4 \times 8 \times 16)^{\frac{1}{4}} = \sqrt[4]{1024} = 5.65685\ldots \approx 5.7$

2. $(10 \times 20 \times 40)^{\frac{1}{3}} = \sqrt[3]{8000} = 20$

DETERMINING MAXIMUM/MINIMUM VALUES

In year one, we learnt how to solve linear equations and inequalities using graphing and algebraic reasoning. With this knowledge, we will be able to graph polygons when given specific linear equations or inequalities which will help us determine the maximum/minimum value for a function.

Working in pairs or groups, go through this activity to review graphing of linear inequalities.

Activity 2.3 – Graphing Linear Inequalities

1. Pick a graph book/sheet
2. Represent the following inequalities graphically
   a. $y \leq 0.67x + 2$
   b. $y \leq -x + 2$
   c. $y \geq -2$
3. Write out the coordinates for the points at which the inequalities intersect.
4. What polygon did the intersecting lines create?
5. Does your graph look like the one in Figure. 2.1: Graphing Linear Inequalities?

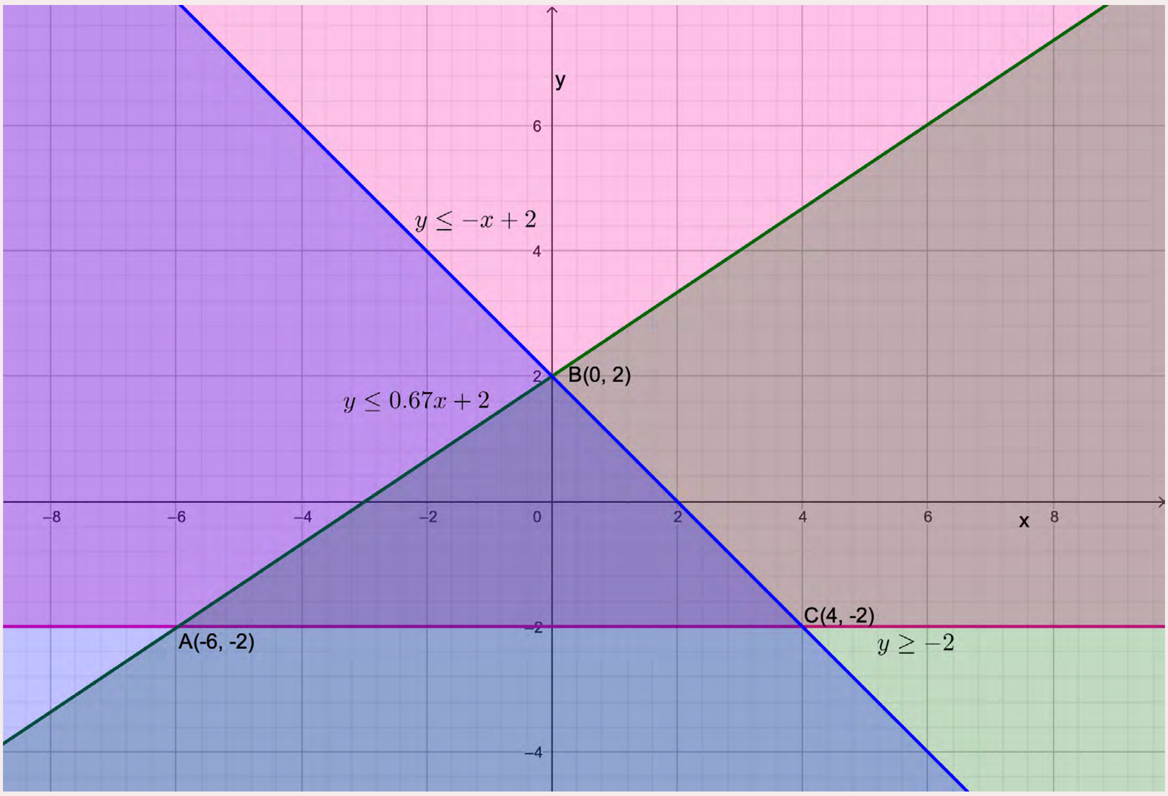


Figure. 2.1: Graphing Linear Inequalities

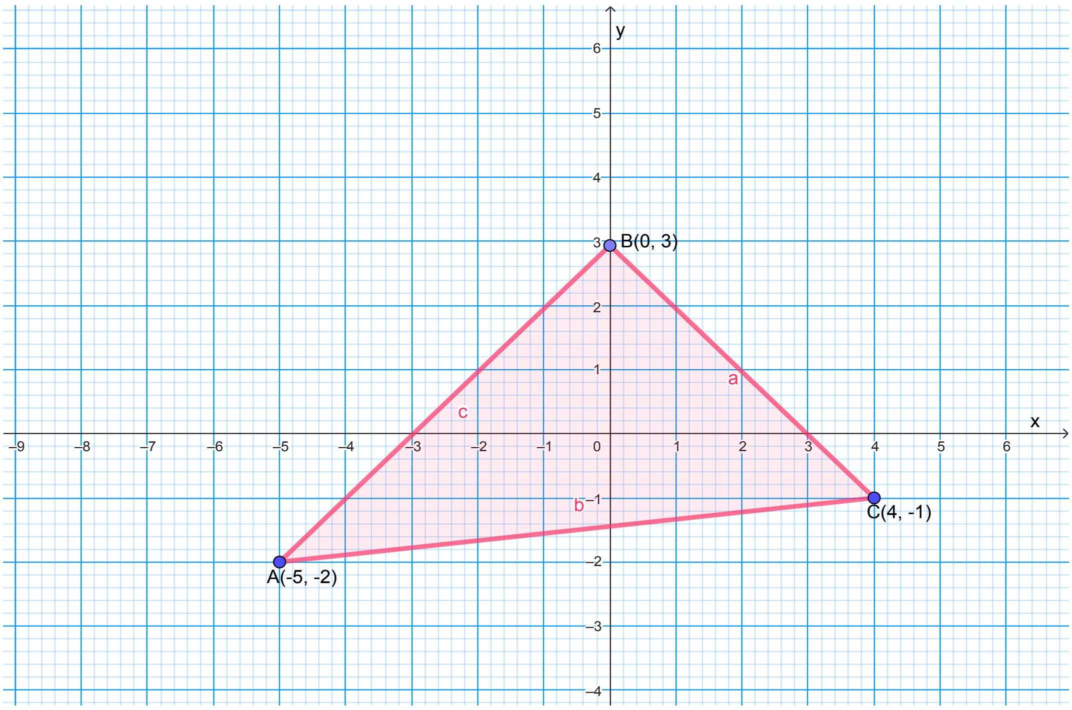
Congratulations! You have adequately remembered how to graph linear inequalities.

Now, we will learn how to determine the maximum/minimum value(s) of a given function.

We can determine the maximum or minimum value of a given function based on a specific objective. Maximum values are the highest points of a function while minimum values are the lowest points of a function. In finding maximum or minimum values of a function algebraically, you have to substitute all the coordinates of the vertices of the region of interest that forms the solution to the function. For example, if you are interested in getting the maximum/minimum area of a given rectangle, you would evaluate the function at each corner or edge by substituting the coordinates. If the polygon of interest (triangle, square, etc.) is represented graphically, you need to identify the pair of coordinates and substitute in the given function. On the other hand, if equations of lines are given, you solve these equations simultaneously and determine the values for the pair of coordinates.

Example 2.13

Evaluate the expression $V = 2x - 3y$ for the given feasible region to determine the point at which 'V' has a maximum value and the point at which 'V' has a minimum value.



Solution

Step 1: Identify all the vertices in Figure. 2.2

A (−5, −2), B(0, 3) and C (4, −1)

Step 2: Substitute the coordinates of each of the vertices in the function $V = 2x - 3y$ and compute.

For (−5, −2), $x = -5$, $y = -2$.

$V = 2(-5) - 3(-2)$
$= -4$

For (0, 3), $x = 0$, $y = 3$.

$V = 2(0) - 3(3)$
$= -9$

For (4, −1), $x = 4$, $y = -1$.

$V = 2(4) - 3(-1)$
$= 11$

Step 3: Choose the highest result as the maximum value and the least as the minimum value

Maximum value is 11 at vertex (4, −1).

Minimum value is −9 at vertex (0, 3).

Example 2.14

On the same graph, illustrate the inequalities, $y \geq \frac{7}{3}x - 5$, $y \leq x - 1$ and $x \geq 0$.

Using the polygon created, determine the minimum and maximum values of the function $m = 7x - y$.

Solution

Step 1: Draw the three linear inequalities on a graph sheet.

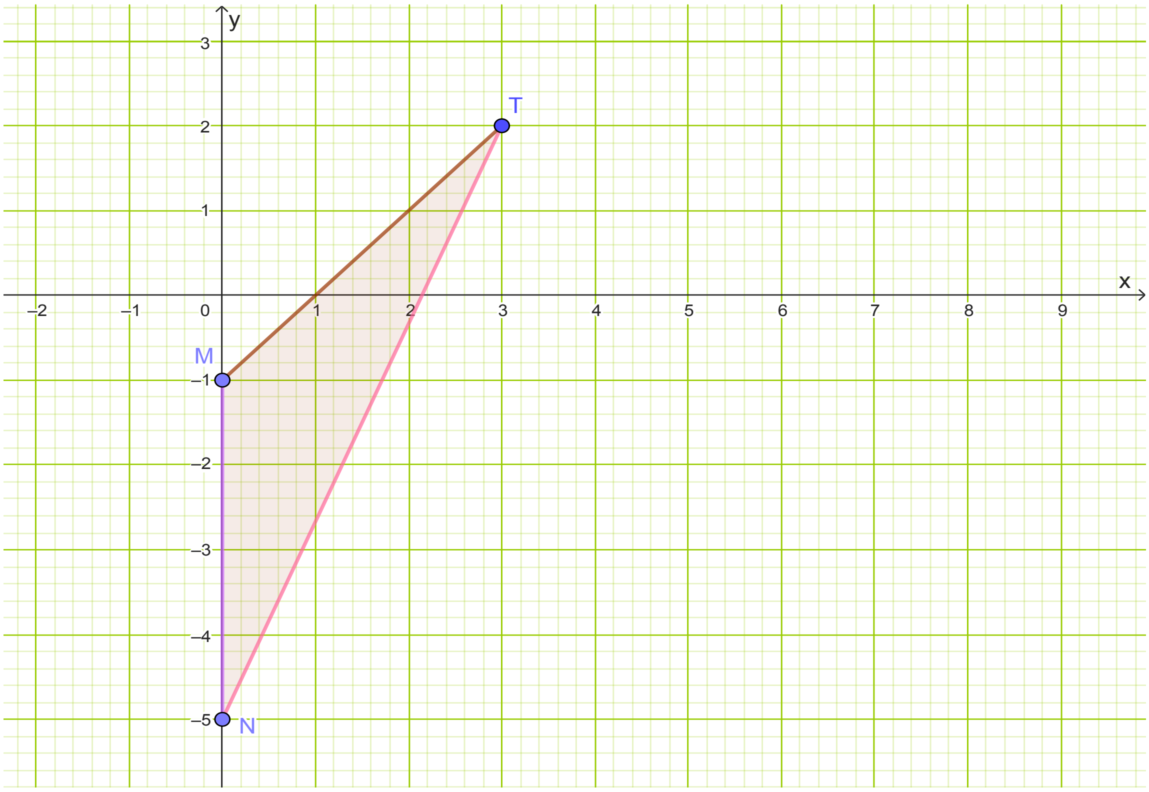


Figure 2.3 : Graphical representation of inequalities in Example 2.14

Step 2: Identify the coordinates of the vertices polygon created as a result of the intersection of the lines.

M(0, –1), T(3, 2) and N(0, –5)

Step 3: Substitute the values of the coordinates in the function $m = 7x - y$.

For (0, −1), $x = 0$, $y = -1$.

$m = 7(0) - (-1)$

$= 1$

For (3, 2), $x = 3$, $y = 2$.

$m = 7(3) - 2$

$= 19$

For (0, −5), $x = 0$, $y = -5$.

$m = 7(0) - (-5)$

$= 5$

Step 4: Select the highest result as the maximum value and the least as the minimum value

Maximum value is 19 at vertex (3, 2).

Minimum value is 1 at vertex (0, −1).

Example 2.15

On the same graph, illustrate the inequalities, $x + y \leq 8$, $y \geq 2$, $x - y \leq 3$, $-x + 2y \leq 10$ and $x \geq 1$. Using the boundaries of the polygon created, determine the minimum and maximum values of the function $w = x^2 + y^2$.

Solution

Step 1: Draw the five linear inequalities on a graph sheet.

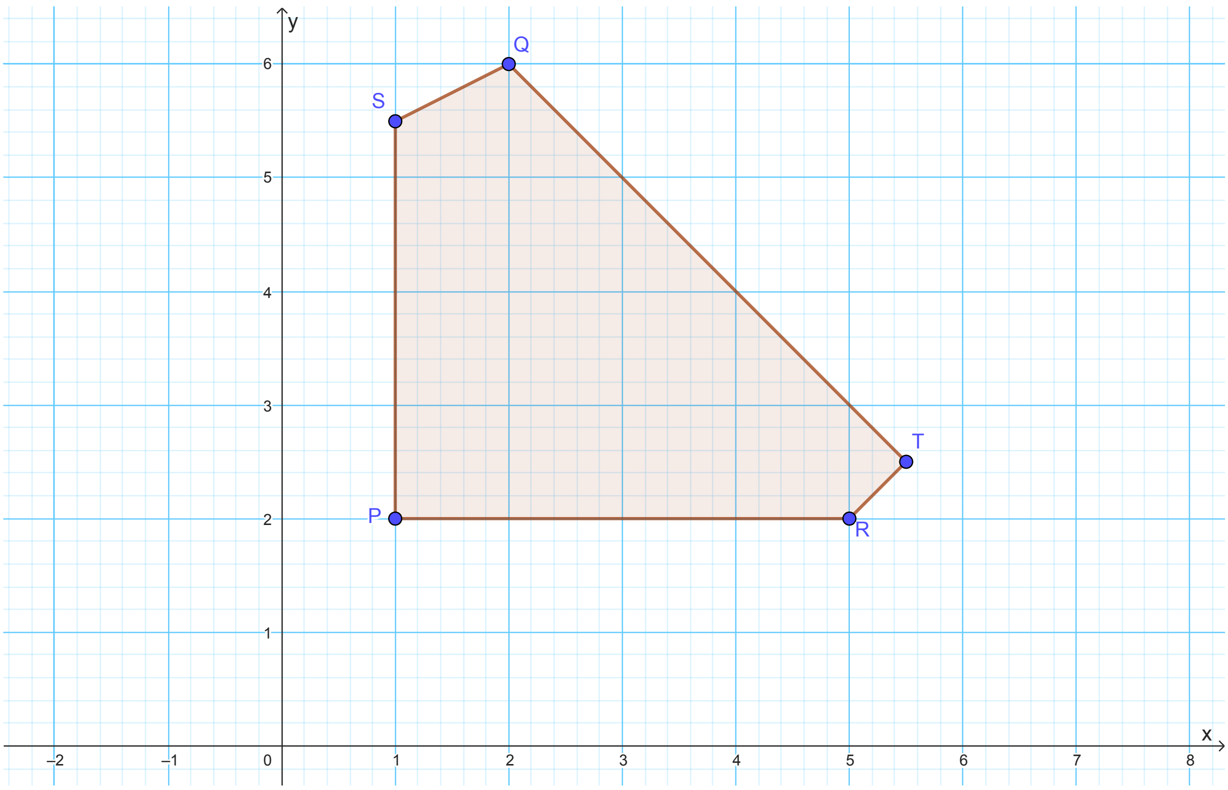


Figure2.4: Graphical representation of inequalities in Example 2.15

Step 2: Identify the coordinates of the vertices polygon created as a result of the intersection of the lines.

P(1, 2), S(1, 5.5), Q(2, 6), T(5.5, 2.5) and R(5, 2)

Step 3: Substitute the values of the coordinates in the function $w = x^2 + y^2$.

For (1, 2), $x = 1$, $y = 2$.

$w = (1)^2 + (2)^2$

$\quad = 5$

For (1, 5.5), $x = 1$, $y = 5.5$.

$w = (1)^2 + (5.5)^2$

$\quad = 31.25$

For (2, 6), $x = 2$, $y = 6$.

$m = (2)^2 + (6)^2$

$\quad = 40$

For (5.5, 2,5), $x = 5.5$, $y = 2.5$.

$m = (5.5)^2 + (2.5)^2$

$\quad = 36.5$

For (5, 2), $x = 5$, $y = 2$.

$m = (5)^2 + (2)^2$

$\quad = 29$

Step 4: Select the highest result as the maximum value and the least as the minimum value

Maximum value is 40 at vertex (2, 6).

Minimum value is 5 at vertex (1, 2).

Now, let's see how our knowledge on solving linear inequalities can be used to solve real life problems.

SOLVING REAL LIFE PROBLEMS INVOLVING SYSTEMS OF LINEAR INEQUALITIES

Knowledge for knowledge's sake is commendable. However, utility of the knowledge is even more satisfying. Let us go through this activity to explore the usefulness of knowledge on solving systems of linear equations.

Activity 2.4 – Optimising the Production of Furniture

Working in pairs, or small groups, work on this activity.

Scenario

Kpogas company manufactures two types of products (chairs and tables).

Each chair requires 2 hours of carpentry and 1 hour of painting, while each table requires 3 hours of carpentry and 2 hours of painting. The company has a maximum of 60 hours of carpentry and 40 hours of painting available per week.

They make a profit of GH¢50.00 per chair and GH¢80.00 per table.

How many chairs (c) and tables (t) should be produced each week to maximise profit, while staying within the carpentry and painting constraints?

Solution

1. Write out algebraic expressions for the hours of carpentry and painting.

Hours for carpentry: $2c + 3t \leq 60$

Hours for painting: $c + 2t \leq 40$

2. Write out algebraic expressions for the number of chairs and tables.

Number of chairs cannot be negative: $c \geq 0$

Number of tables cannot be negative: $t \geq 0$

3. Write out an equation for the expected profit.

Let P represent profit

$P = 50c + 80t$

4. Express the system of linear inequalities as the constraints of production.

$$\begin{cases} c \geq 0 \\ t \geq 0 \\ 2c + 3t \leq 60 \\ c + 2t \leq 40 \end{cases}$$

5. Note that the equation of profit is the objective function.

6. Graph the system of linear inequalities

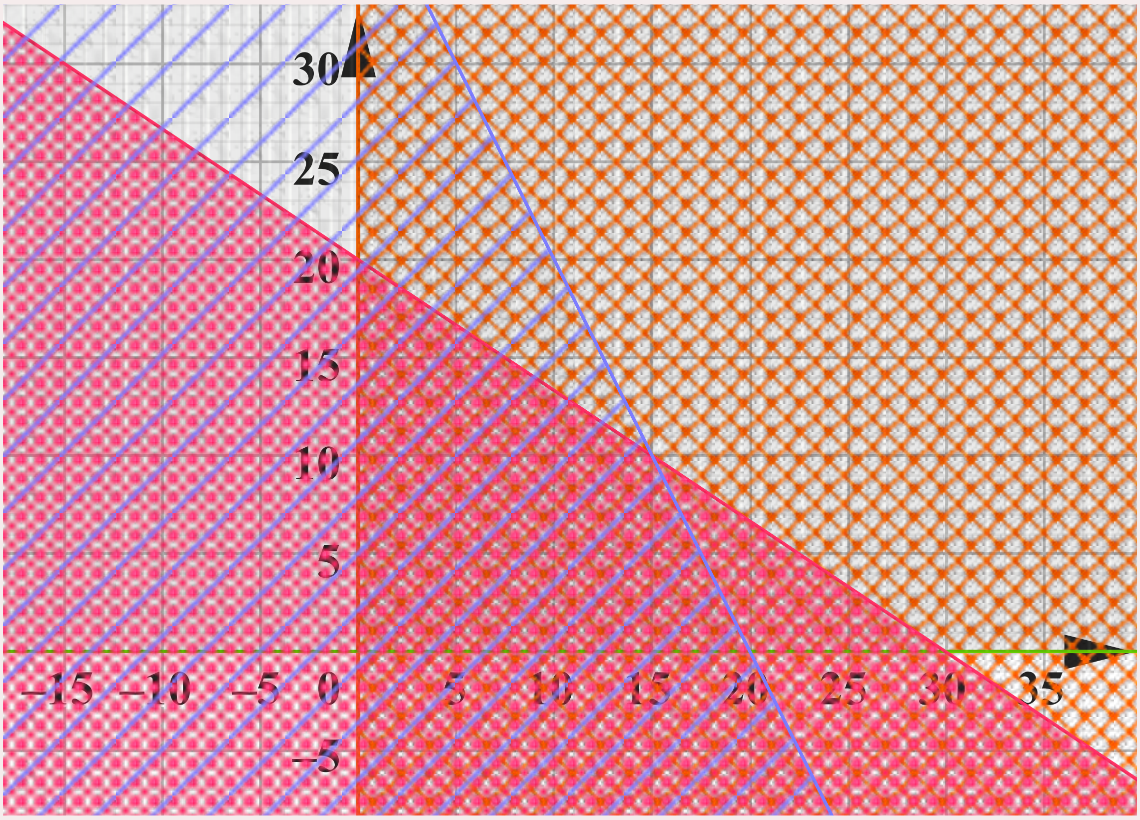


Figure 2.5 : Graphical representation of Kpogas company production constraints

7. Identify the coordinates of the vertices to the solution region.
(20, 0), (0, 0), (15, 10) and (0, 20)

8. Substitute the coordinates of the vertices into the profit equation.

At (20, 0), P = 50(20) + 80(0)
= 1000

At (0, 0), P = 50(0) + 80(0)
= 0

At (15, 10), P = 50(15) + 80(10)
= 1550

At (0, 20), P = 50(0) + 80(20)
= 1600

9. Write out your conclusion
The maximum profit is GH¢1600.00, and it occurs when Kpogas produces no chairs and 20 tables.

Example 2.16

A bakery produces cakes (x) and cookies (y) daily. A cake takes 2 hours to make while a batch of cookies takes 1 hour to make. The bakery can spend at most 12 hours per day baking. Each cake requires 3 units of flour and a batch of cookies requires 1 unit of flour. The bakery has at most 15 units of flour available each day. The bakery earns GH¢5.00 profit per cake and GH¢3.00 profit per batch of cookies. Without an increase in the units of flour and hours of baking, how many of each product should be baked daily to make the most profit?

Solution

Step 1: Identify the constraints communicated by the bakery.

$$\begin{cases} 2x + y \leq 12 \\ 3x + y \leq 15 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Step 2: Write out the equation of the profit (objective function).

$P = 5x + 3y$

Step 3: Graph the inequalities.

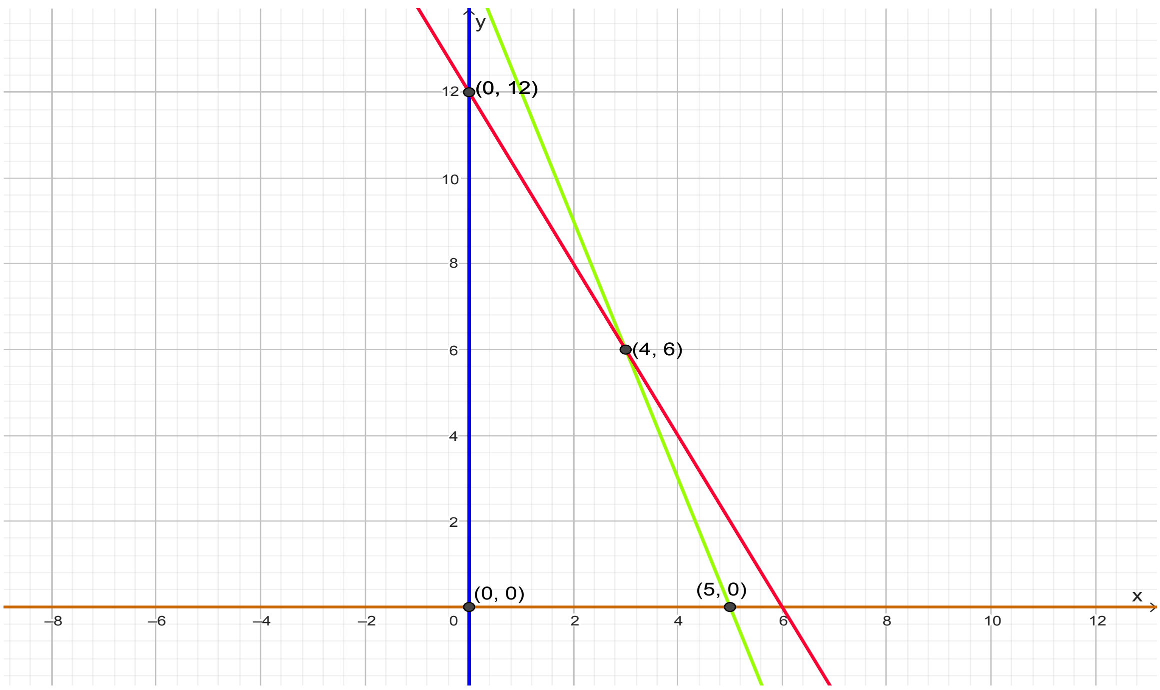


Figure 2.6: Graphical representation of constraints for baking

Step 4: Identify the coordinates of the vertices to the solution region.

(0, 12), (5, 0), (4, 6) and (0,0)

Step 5: Substitute the coordinates of the vertices into the profit equation.

At (0, 12), $P = 5(0) + 3(12)$

$= 36$

At (5, 0), $P = 5(5) + 3(0)$

$= 25$

At (4, 6), $P = 5(4) + 3(6)$

$= 38$

At (0, 0), $P = 5(0) + 3(0)$

$= 0$

Step 6: Write out your conclusion.

The bakery should bake 4 cakes and 6 batches of cookies daily to make the most profit.

Example 2.17

A farmer has 300 metres of fencing material to enclose a rectangular field for planting crops. The farmer wants to maximise the area of the field to get the best crop yield, but there is a restriction. The length of the field must be 10 metres longer than the width to accommodate the farm machinery.

What should be the dimensions of the field that will maximise the enclosed area, keeping in mind the restriction on the total fencing available?

Solution

Step 1: Represent the statement algebraically.

Let w be the width of the field

Length of the field $= w + 10$

Step 2: Write the equation for perimeter of the field.

Since Perimeter(P)= 2(width) + 2(length),

$P = 2w + 2(w + 10)$

$P = 2w + 2w + 20$

$P = 4w + 20.$

Step 3: Set up the linear inequality

Since the material available for fencing is 300m, the total distance around the field can be less or equal to 300.

$4w + 20 \leq 300$

Step 4: Simplify the inequality.

$4w \leq 300 - 20$

$4w \leq 280$

$w \leq \frac{280}{4}$

$w \leq 70$

So, the width of the field must be less than or equal to 70 metres.

Step 5: Test values for width.

Since the highest value for the width will be 70, the length will be $70 + 10 = 80$

Testing for perimeter, $2(70) + 2(80) = 300$, satisfying perimeter requirement.

Testing for maximum area of the rectangular field: length $\times$ width

Area $= 70 \times 80 = 5600m^2$

Step 6: State your conclusion

The area can be maximised when the rectangle has dimensions of 70 by 80, satisfying all constraints.

Activity 2.5 – Real Life Project Involving Linear Inequalities

1. Identify an industry or company that produces/manufactures goods in your locality.
2. Obtain the restrictions/constraints guiding the production of their goods.
3. Identify the profit made on the items of production.
4. Write a two-page report comprising:
   - **a.** where the company is located.
   - **b.** goods produced and constraints communicated by the company officials.
   - **c.** expected profit on each of the goods.
   - **d.** an algebraic representation of constraints and objective function.
   - **e.** a graphical representation of constraints (manually or by software).
   - **f.** a suggestion on the decision the company should make to minimise profit.

g. a suggestion on the decision the company should make to maximise profit.

h. conclusion based on your observations and findings.

SOLVING QUADRATIC INEQUALITIES

In year one we learnt about quadratic functions, how to solve them algebraically and how to illustrate them graphically. Remember that quadratic functions are of the form

$M(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$.

Combining this knowledge and what we know about linear inequalities, we can represent general quadratic inequalities in the forms;

1. $ax^2 + bx + c < w$
2. $ax^2 + bx + c > w$
3. $ax^2 + bx + c \leq w$
4. $ax^2 + bx + c \geq w$, where w is a constant.

To find the solution to a quadratic inequality of any of the forms above:

1. First determine the value of w
2. Simplify the inequality so that one side is zero
3. Factorise to determine the range of values for x
4. Represent it graphically to know the solution region

Activity 2.6 – Individual/Group Exploration of Quadratic Inequalities

1. Brainstorm how you would solve the inequalities:
2. $x^2 - 3x - 10 > 0$ and $-2x^2 + 5x \leq 3$.
3. Discuss multiple approaches (factorising, completing the square or using the quadratic formula).
4. Consider when the expression is equal to zero and how this can help determine the solution.
5. Draw separately the graph of each of the quadratic inequalities to understand the regions where the inequality is true or false.
6. Share your observations from the process with a peer.

Example 2.18

Find the solution to the inequality $x^2 - 2x - 15 < 0$.

Solution

Step 1: Factorise the left-hand side of the inequality

$x^2 - 2x - 15 = (x + 3)(x - 5)$

Hence $(x + 3)(x - 5) < 0$

Step 2: Find the critical values

$x = 5$ and $x = -3$

Step 3:

Method 1: Represent the extreme values of the expected solution on a number line and test

them.

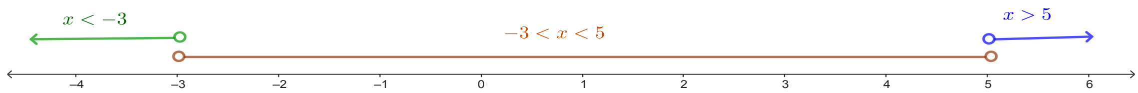


Figure2.7: Representing possible solutions to quadratic inequalities on number lines

Possible Solution Regions	Test Values	$x + 3$	$x - 5$	$(x + 3)(x - 5)$	Sign
$x < -3$	-4	-1	-9	9	Positive
$-3 < x < 5$	3	6	-2	-12	Negative
$x > 5$	6	9	1	9	Positive

Step 4: Select the solution region that corresponds to the sign satisfying the inequality.

Since $x^2 - 2x - 15 < 0$ holds for only values of x for which the product of the factors is negative (less than zero), $-3 < x < 5$ is the solution to the inequality.

OR

Step 3:

Method 2: Graph the quadratic inequality

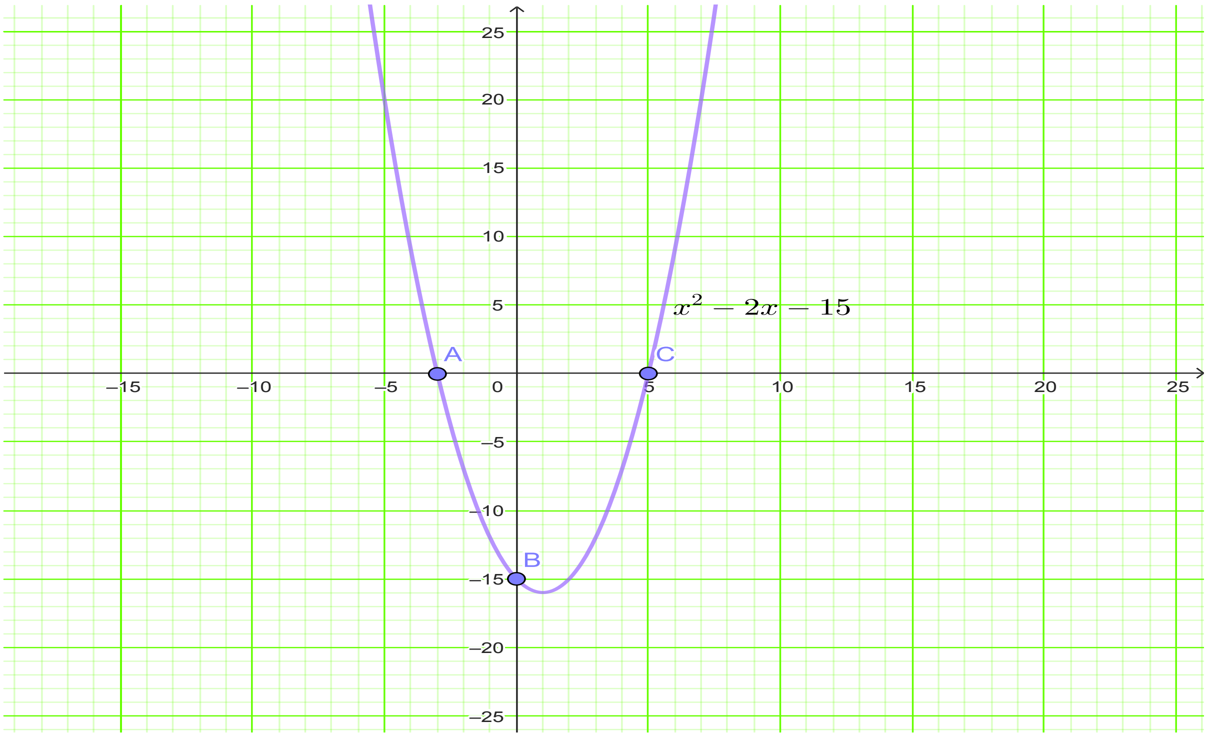


Figure2. 8: Quadratic graph of $x^2 - 2x - 15$

Step 4: Identify the portion of the graph that satisfies the inequality. In this case, the portion below the x-axis.

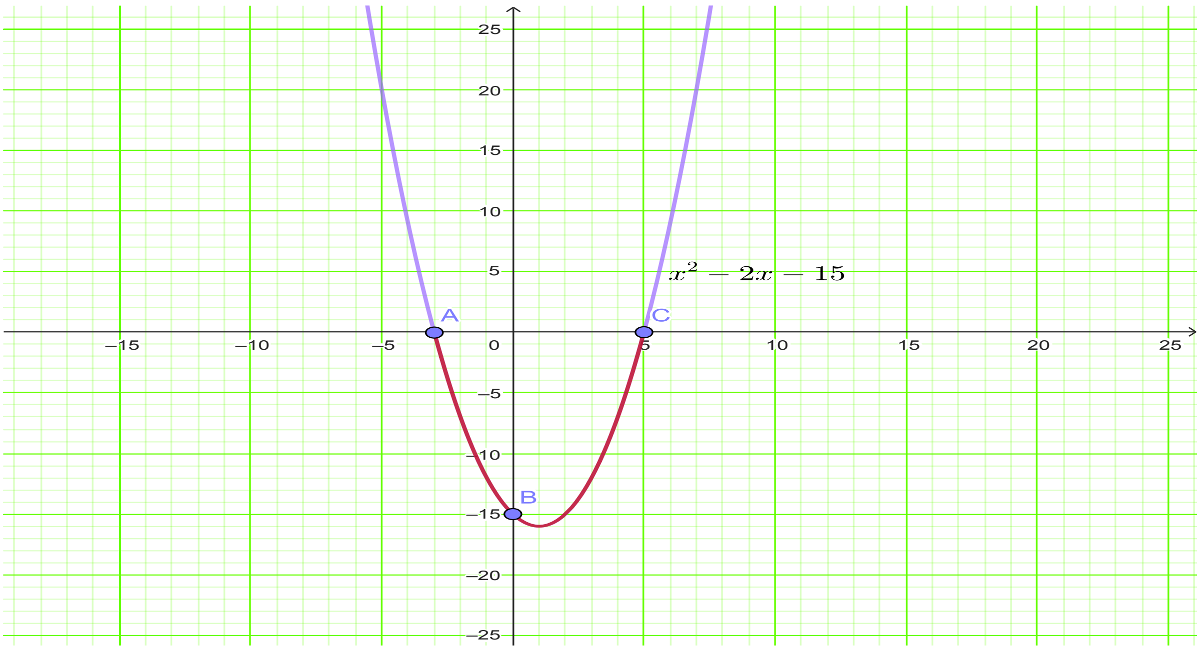


Figure2.9: Quadratic graph of $x^2 - 2x - 15$ with solution

Thus $-3 < x < 5$

Example 2.19

Find the solution to the inequality $-5x^2 + 12x - 7 > 0$.

Solution

Step 1: Factorise the left-hand side of the inequality

$-5x^2 + 12x - 7 = (-5x + 7)(x - 1)$

Hence $(-5x + 7)(x - 1) > 0$

Step 2: Find the critical values

$x = \frac{7}{5}$ and $x = 1$.

Step 3: Represent the extreme values of the expected solution on a number line and test them.

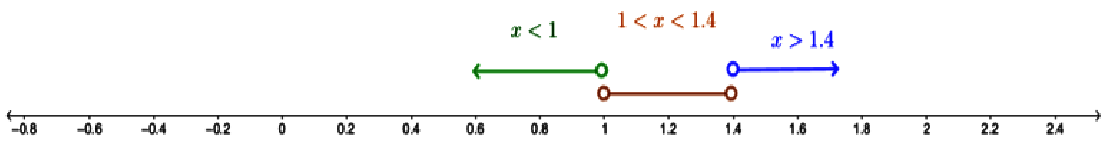


Figure 2.10: Representing quadratic solution on number line 2

Possible Solution Regions	Test Values	$-5x + 7$	$x - 1$	$(-5x + 7)(x - 1)$	Sign
$x < 1$	0	7	-1	-7	Negative
$1 < x < 1.4$	1.2	1	0.2	0.2	Positive
$x > 1.4$	3	-8	2	-16	Negative

Step 4: Select the solution region that corresponds to the sign satisfying the inequality.

Since $-5x^2 + 12x - 7 > 0$ holds for only values of x for which the product of the factors is positive (greater than zero), $1 < x < 1.4$ is the solution to the inequality.

Activity 2.7- Graphical Solution to $-5x^2 + 12x - 7 > 0$

Represent the solution to $-5x^2 + 12x - 7 > 0$ graphically and share the results with a classmate.

Activity 2.8 – Research on Quadratic Inequalities

1. Conduct research on the behaviour of solutions to maximum and minimum curves of quadratic functions.

Instructions:

a. Explore the nature of the solution when a quadratic function with a minimum curve is:

- **i.** greater than zero
- **ii.** less than zero

b. Explore the nature of the solution when a quadratic function with a maximum curve is:

- **i.** less than zero
- **ii.** greater than zero

c. Include graphical representations to support your argument.

d. Consult open education resources, YouTube, textbooks, etc.

e. Provide references to the sources you consult and use.

SOLVING SYSTEMS OF QUADRATIC INEQUALITIES

We have explored how to solve single quadratic inequalities algebraically and graphically. Now, we will learn how to solve systems of quadratic inequalities (two or more quadratic inequalities) through algebra and graphing.

In finding solutions to quadratic inequalities algebraically, you solve them simultaneously.

Example 2.20

Graph the system of quadratic inequalities $y \geq 4x^2 + 5x$, $y < -3x^2 - 7x$.

Solution

Solving the systems of inequalities simultaneously

Step 1: Equate the two expressions

$4x^2 + 5x = -3x^2 - 7x$

Step 2: Simplify and make x the subject.

$4x^2 + 3x^2 + 7x + 5x = 0$

$7x^2 + 12x = 0$

$x(7x + 12) = 0$

$x = 0, x = -\frac{12}{7}$.

Step 3: Identify whether the point of intersection is inclusive in the solution to the systems.

Since the $y < -3x^2 - 7x$ has the strictly $<$ symbol, the points of intersection will not satisfy this inequality.

For $y \geq 4x^2 + 5x$,

At $x = 0$, $y \geq 4(0)^2 + 5(0)$,

$y \geq 0$.

For $y < -3x^2 - 7x$,

At $x = 0$, $y < -3(0)^2 - 7(0)$,

$y < 0$

At $x = 0$, y must be 0 in both inequalities, but since the second inequality requires $y < 0$, there is no solution at this point.

At $x = \frac{-12}{7}$, $y \geq 4\left(\frac{-12}{7}\right)^2 + 5\left(\frac{-12}{7}\right)$,

$y \geq 3\frac{3}{49}$.

At $x = 0$, $y < -3\left(\frac{-12}{7}\right)^2 - 7\left(\frac{-12}{7}\right)$,

$y < \frac{156}{49}$.

$y < 3\frac{9}{49}$.

At $x = \frac{-12}{7}$, y must be $3\frac{9}{49}$ in both inequalities, but since the second inequality requires $y < 3\frac{9}{49}$, there is no solution at this point.

Step 4: State the solution

Since the points of intersection will not satisfy one of the inequalities, the solution becomes $-1\frac{5}{7} < x < 0$. That is, the region between the points of intersection.

Solving the systems of inequalities graphically

Step 1: Draw the graphs of the two inequalities.

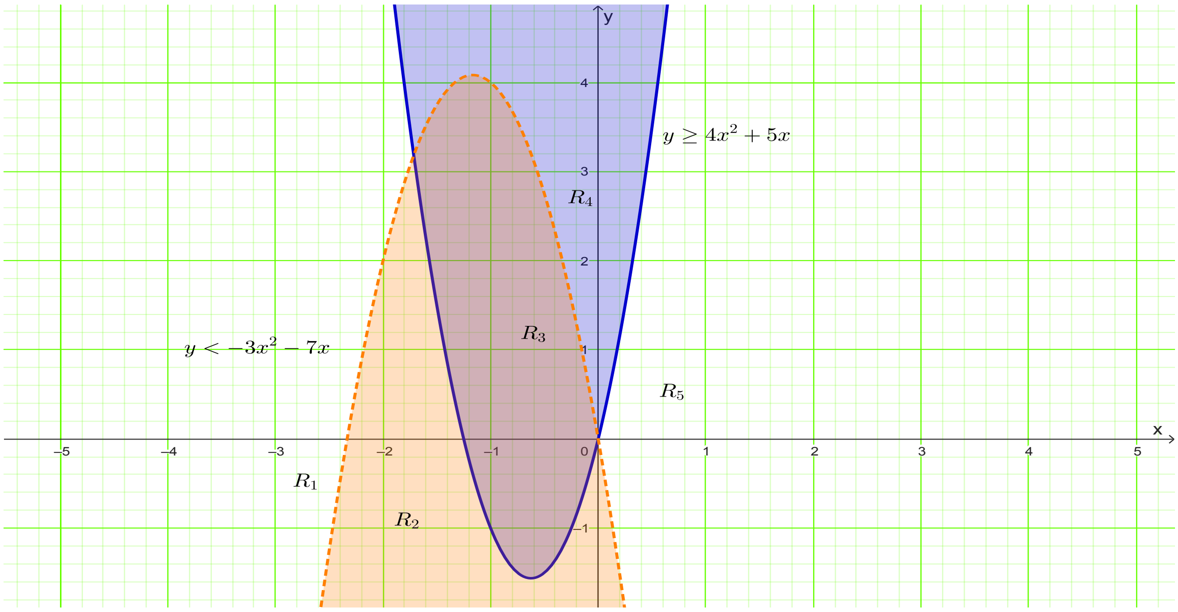


Figure 2.11: Graphical Solution to $y \geq 4x^2 + 5x$, $y < -3x^2 - 7x$

Step 2: Identify the region that is common to both inequalities.

R_3

Step 3: State the solution to the system of inequalities.

$-1\frac{5}{7} < x < 0$

Step 3: State the conclusion.

All points in the region R_3 including but not exclusive to $(-1, 0)$, $(-1, 3)$ $(-0.5, 2)$, $(-1, 1)$ and $(-0.5, -1)$ are in the solution set for the system.

Self-assessment task

Find the solution to $y < -7x^2 - 8x + 3$ and $y > 3x^2 - 7x$.

You should find $-0.6 < x < 0.5$

SOLVING REAL-LIFE PROBLEMS INVOLVING QUADRATIC INEQUALITIES

Quadratic inequalities are a powerful tool used to model and solve various real-world problems that involve limits, boundaries, or optimisation of quantities. These inequalities describe relationships where one side of the equation is a quadratic expression and are used in situations where the possible outcomes form a range or region rather than a single point.

By learning how to solve quadratic inequalities, we can:

1. Predict outcomes that must fall within thresholds like safety limits, financial losses, etc.
2. Identify ranges of values that satisfy conditions in optimisation problems such as finding the most efficient design.
3. Understand the critical points where changes in conditions occur, allowing for better decision-making and problem-solving.

Let's go through these examples to see how the concept of quadratic inequalities is useful in real-life.

Example 2.21

An artist is planning an art exhibition and needs to create rectangular canvases with a specific aesthetic balance.

The total area available for displaying the artworks is limited to 80 square feet. The artist wants the length of each canvas to be 2 feet less than its width.

To ensure that the artworks are visually appealing and fit within the assigned space, determine the maximum width for each canvas such that the total area used does not exceed 80 square feet.

Solution

Step 1: Represent the statement algebraically

Let w be the width of each canvas

Length of each canvas = w– 2

Step 2: Write the equation for area of a rectangle

Since Area(A) = width × length,

$A = w(w - 2)$

$A = w^2 - 2w$

Step 3: Set up the linear inequality

Since the space available for display is maximum 80 sq. feet, the total space covered by the art can be less or equal to 80 square feet.

width $\times$ length ≤ 80

Step 4: Set up the quadratic inequality and simplify.

$w^2 - 2w \leq 80$

$w^2 - 2w - 80 \leq 0$

$(w - 10)(w + 8) \leq 0$ (Sketch the inequality to confirm the required values)

$\therefore -8 \leq w \leq 10$

Since width can only be positive, the width of each canvas must be less than or equal to 10 feet.

Step 6: State your conclusion

The artist can create canvases with a width up to 10 feet, ensuring the canvases fit within the available display area at the exhibition.

Example 2.22

Millicent would like to design a necklace for her mother and decides it should resemble an eye. She makes a drawing as shown in Figure. 2.12. Identify the systems of quadratic inequalities that helped her draw the necklace.

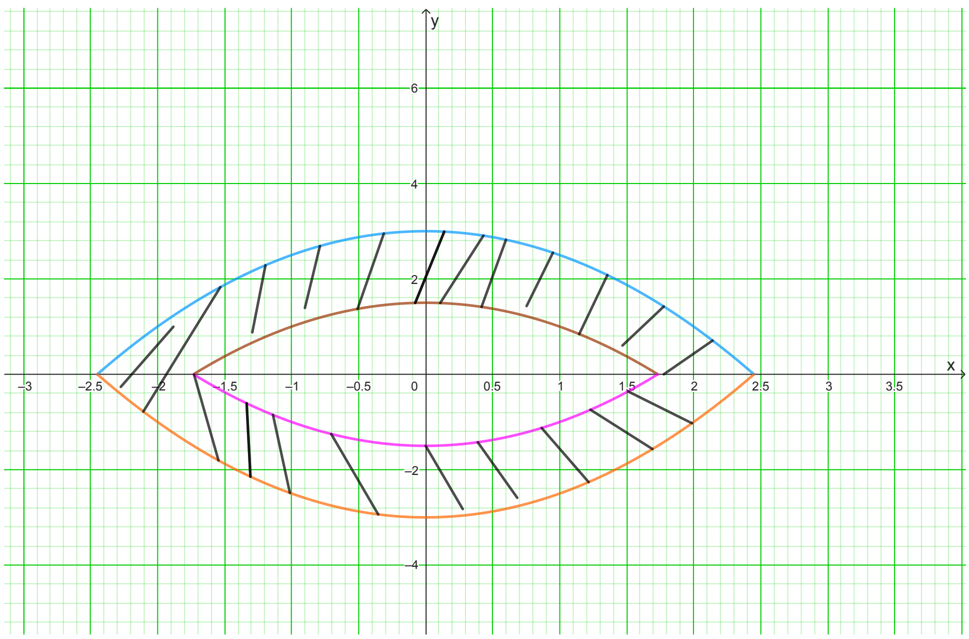


Figure 2.12: Necklace design

Solution

The blue quadratic inequality curve was drawn with $y \leq -0.5x^2 + 3$

The brown quadratic inequality curve was drawn with $y \geq -0.5x^2 + 1.5$

The magenta quadratic inequality curve was drawn with $y \leq 0.5x^2 - 1.5$

The orange quadratic inequality curve was drawn with $y \geq 0.5x^2 - 3$

Activity 2.9 - Research a Real-Life Problem

1. Identify a real-world scenario where quadratic inequalities can be applied. Use any of the following themes to base your research on:
   a. **Agriculture:** Maximising the area of a crop field, given constraints such as fencing materials.
   b. **Sports:** Determining the best angle and force to kick a ball in football for a successful shot.
   c. **Architecture:** Designing the maximum height of an arch or doorway while ensuring structural safety.
   d. **Business:** Managing production costs to ensure profit margins while minimising loss, such as determining the optimal number of products to produce.
   e. **Physics:** Understanding projectile motion when an object is thrown into the air.
2. Research how quadratic functions and inequalities play a role in this real-world context. Use reliable online sources, textbooks, or interview experts if possible.
3. Once you've chosen your problem, model it using a quadratic inequality.

   Identify the variables in your scenario (e.g., height, distance, time, cost) and set up the quadratic inequality that best represents the situation.
4. Solve the quadratic inequality you have written. Use algebraic methods or graphical tools to determine the solution set. You can use graphing software, calculators, or sketch graphs by hand to visualise the solution.
5. Discuss the real-life implications of your solution.
   a. What do the solutions of the quadratic inequality mean in terms of the real-world problem?
   b. What decisions can be made based on your results?

6. Create a visual presentation of your findings to share with your classmates or teacher. Your presentation should include:

   a. The real-world problem you chose.
   b. The quadratic inequality you modelled from the problem.
   c. The steps you took to solve the inequality.
   d. The real-world interpretation of the solution.
   e. Any challenges or insights you encountered while working on this problem.

You can create a poster, a digital slide presentation or even a short video to present your work.

EXTENDED READING

- Baffour Ba series for schools and Colleges. (Pages 444 – 472).
- Effective Elective Mathematics for Senior High School; Christian Akrong Hesse (Pages 202 – 232).
- Effective Elective Mathematics for Senior High School, Mathematics Association of Ghana. (Pages 247 – 267).
- Talbert, J. F., & Heng, H. H. (1995). *Additional Mathematics: Pure & Applied*. Pearson Education South Asia. Page(s) 90 – 98.

REVIEW QUESTIONS

1. The sum of the $2nd$ and the $4th$ term of a geometric sequence is 10 and the sum of the $3nd$ and $5th$ term is 20. Find the first term and the common ratio.
2. a. Given that $a_1 = 25$ and $a_n = a_{n-1} + 4$ for $n > 1$

   Find:

   (i) a_2, a_3, and a_4

   (ii) a formula for a_n in terms of n

   b. Use your formula for a_n to deduce a formula for S_n, the sum of the first n terms of the sequence $a_1, a_2, a_3, \ldots a_n$ and hence find S_{10}
3. The sum of the three consecutive term of an Arithmetic Progression is -3 and the product is 24. Find the terms.
4. The *nth* term of an exponential sequence (G.P) is 2^{1-n}.

   Calculate:

   a. the sum S_n, of the first four terms of the sequence.

   b. the sum of the sequence to infinity.
5. Calculate the difference between the sum to infinity of the series $1 + \frac{1}{3} + \frac{1}{9} + \ldots$, and the sum of the first 5 terms.
6. A sequence defined by $U_1 = 2$, $U_{n+1} = 2U_n - 1$, $n = 2, 3, 4, \ldots$

   a. Find the first five terms of the sequence

   b. Show that $U_{n+1} = 3U_n - 2U_{n-1}$
7. The sum of the $5th$ and $9th$ terms of an A.P, is -40 and and the $11th$ term is -32. Find the first term and the common difference.
8. A man's salary is increased by GH¢24 000 each year. His total salary at the end of 14 years is GH¢65 184 000.

   Find:

   a. his initial salary

   b. his salary in the $14th$ year
9. The sum of the first n terms of a series is given by $(n + 1)^2 - 1$. Find the first three terms and an expression in terms of n for U_n, $(n > 1)$.

10. A tennis ball is released from a height h cm, it rebounds one-third of the distance it has fallen after each fall. Find the height of;

a. the third bounce **b.** the *nth* bounce

11. Find the sum of the first n terms of a linear sequence whose common difference is 2 is 120 and the sum of the $2n$ term is 440. Calculate the first term.

12. Evaluate $\sum_{r=1}^{6}(r^2 - 2)$

13. Renie's outlet manufactures two types of clothing (shirts and trousers). The profit is GH¢50.00 for each shirt and GH¢80.00 for each pair of trousers.

14. Due to limited resources, the company can produce a combined total of no more than 120 pieces. Additionally, to meet demand, they must produce at least 20 shirts and 10 pairs of trousers. How many of each of the clothing should be produced to make the most profit?

15. Find the solution(s) to:

a. $3x^2 + 5x \geq 2.$ **b.** $x^2 - 2x - 3 \geq 0.$

16. A company determines that the cost in cedis C, of producing m bags of maize is $C = 170m + 150.$

17. The revenue R, in cedis, from selling all of the bags of maize is $R = m^2 + 530.$

18. How many bags of maize should the company produce and sell if the company wishes to earn a profit of at least GH¢4 000.00?

19. Mr. Buabeng and Ms. Osei run business assembling Hyundai Toyota cars.

The cost of parts and labour needed for each type of car is shown in the table below.

Car type	Cost of Parts (GH¢ thousand)	Labour (man-hours)
Hyundai	18	15
Toyota	36	30

The business has GH¢150 (thousand) available to buy parts each week.

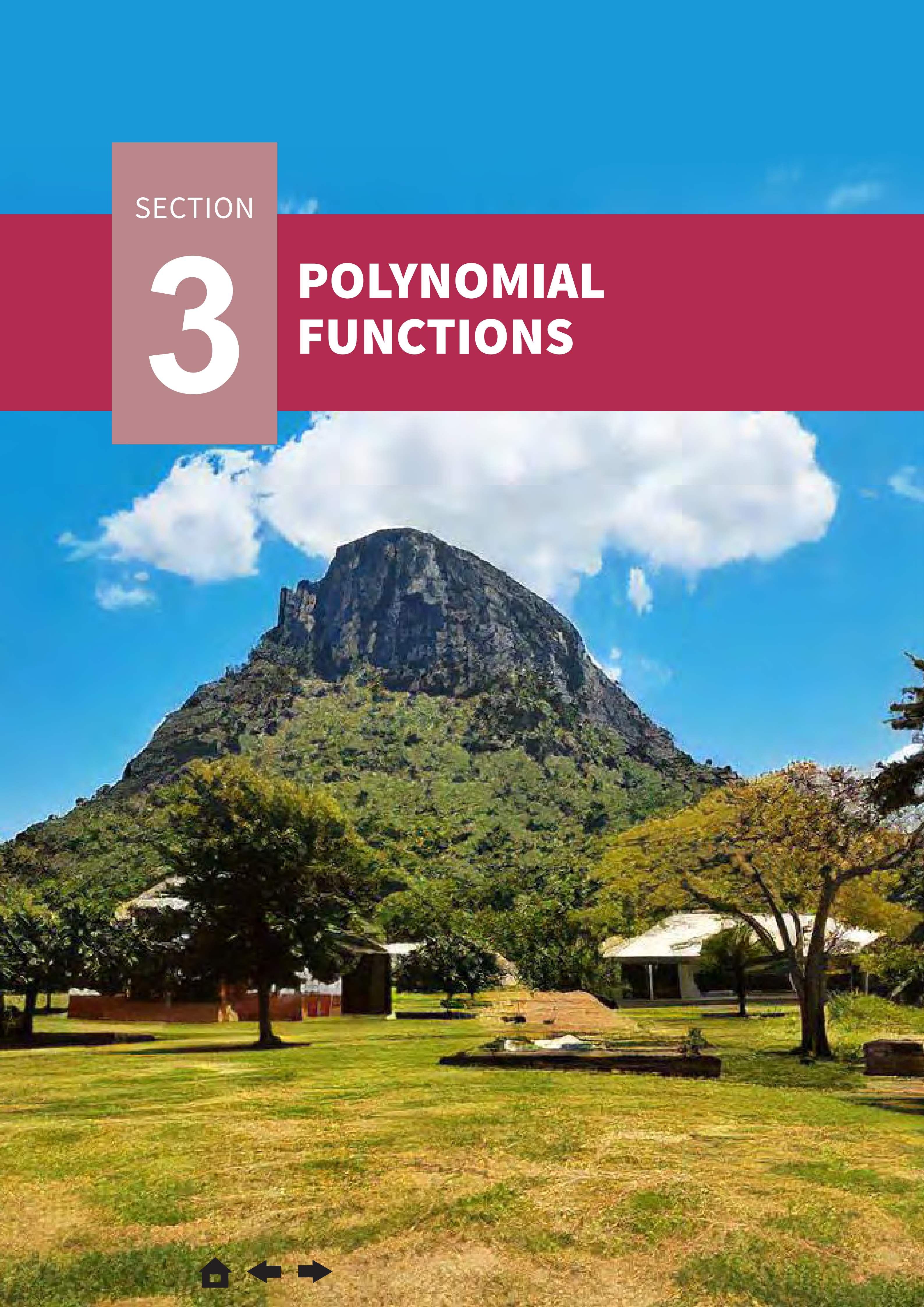
However, the total labour available each week is 85 man-hours. If they make h of Hyundai and t of Toyota each week:

a. Write down all the inequalities on h and t.

b. Determine the solution to the set of inequalities.

SECTION

3 POLYNOMIAL FUNCTIONS



MODELLING WITH ALGEBRA

Application of Algebra

INTRODUCTION

This lesson is continuation of what we learnt in year one functions. We will explore the properties, graphing techniques and basic theorems of polynomial functions. Polynomial functions are crucial in mathematics and have applications in science, engineering and economics. It covers factorising, finding zeros, graphing, Descartes' Rule of Signs, Fundamental Theorem of Algebra and Linear and Quadratic Factor Theorems. By the end of the section, you will have a deeper understanding of polynomial functions and be prepared to solve a variety of mathematical problems.

KEY IDEAS

- **Factorising polynomials** involves linear and irreducible quadratic factors.
- **Factors** and **zeros** are variables that make a **polynomial function** equal to **zero**.
- Graphing polynomials with higher degrees reveal complex shapes, key features, and roots.
- The **degree** of a polynomial function determines the maximum number of solutions a function could have and the number of times a function will cross the x-axis when graphed.
- The **degree** of a polynomial function is the total number of **factors**.

FINDING FACTORS AND ZEROS OF POLYNOMIAL FUNCTIONS

Activity 3.1: Revision of finding quadratic factors

In pairs or small groups or individually, perform these activities.

1. Factorise the following quadratic expressions:
   - **a.** $y^2 + 3y + 2$
   - **b.** $3y^2 + 11y + 10$
   - **c.** $6x^2 + 5x + 1$
   - **d.** $x^2 - 9$
2. Write down the factorising process clearly, showing each step taken to arrive at the factors.
3. Exchange your work with other groups or peers to compare answers.
4. Discuss any differences in methods used and validate the correctness of each group's results.

An expression **$(x + y)$** is a factor if you can multiply it by another expression **$(e + f)$** to get the original expression **$(ex + fx + ey + fy)$**. In simple terms, if you can find two or more expressions that multiply together to make a certain expression, then those are its factors.

For example, $(x + 2)(x - 3)$ are factors of $x^2 - x - 6$ because expanding $(x + 2)(x - 3)$:

$x(x - 3) + 2(x - 3)$

$x^2 - 3x + 2x - 6$

$x^2 - x - 6$

Therefore, we see that $(x + 2)(x - 3)$ is indeed equal to $x^2 - x - 6$ confirming that they are factors of the expression. If you found activity 3.1 challenging, use the task below to help you.

To **factorise quadratics** of the form $ax^2 + bx + c$:

Example

Task	$2x^2 + 7x + 6$
Task 1 Find two numbers that multiply to give ac (the product of a and c) and add to give b.	a = 2, b = 7 and c = 6 ac = 2 × 6 = 12 4 + 3 = 7 and 4 × 3 = 12
Task 2 Rewrite the middle term using these two numbers as coefficients of x.	$2x^2 + 4x + 3x + 6$
Task 3 If the polynomial has four terms, group the terms in pairs.	$(2x^2 + 4x) + (3x + 6)$
Task 4 Factorise out the common factor from each group.	$2x(x + 2) + 3(x + 2)$
Task 5 Look for a common binomial factor and factorise it out and you have the factorised solution.	$(2x + 3)(x + 2)$

Factor Theorem and Remainder Theorem

In addition to quadratic functions, there are several other types of polynomial functions that can be factorised using the Factor Theorem and the Remainder Theorem. Remember that if $(x \pm h)$ is a factor of a polynomial, then the remainder will be zero. Conversely, if the remainder is zero, then $(x \pm h)$ is a factor.

Here are some common types of polynomials:

Types of Polynomials

1. **Cubic Polynomials:** Polynomials of ***degree*** three, in the form: $f(x) = ax^3 + bx^2 + cx + d$
2. **Quartic Polynomials:** Polynomials of degree four, in the form:
   $f(x) = ax^4 + bx^3 + cx^2 + dx + e$
3. **Quintic Polynomials:** Polynomials of degree five, in the form:
   $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

Activity 3.2: Revision on factorisation

In groups or individually, use the idea of the factor and remainder theorems to find the factors or the remainders of the following polynomials.

1. Find the remainder when:

a. $f(x) = 3x^3 - 4x^2 + 2x + 3$ is divided by $(x + 1)$

b. $f(x) = 8x^3 - 3x^2 + 6x + 4$ is divided by $(x - 2)$

c. $f(x) = 7x^3 + 3x^2 - x + 6$ is divided by $(x + 3)$

d. $f(x) = x^3 - x^2 + x + 1$ is divided by $(x - 1)$

2.

a. show that $x + 2$ is a factor of $f(x) = x^3 - 2x^2 - 5x + 6$

b. show that $x - 3$ is a factor of $f(x) = x^3 - 3x^2 - 4x + 12$

3. Compare your answers with other groups.

Zero-Product Property

This property of real numbers says that if you multiply two numbers together and the result is zero, then at least one of those numbers must be zero. In simpler terms, if you have two numbers, **a** and b, and when you multiply them (a×b), and you get zero, it means that either a is zero, or b is zero or both are zero. This property helps us to find the **zeros** or **roots** of functions of quadratic functions.

The **zeros** or **roots** of a function are the values of the input (usually represented as x) that make the function equal to zero. In other words, if you have a function f(x), the zeros are the values of x for which f(x) = 0. For example, given the function $f(x) = 2x^2 - x - 6$, You:

1. equate f(x) = 0

$2x^2 - x - 6 = 0$

2. factorise the function

$2x^2 - 4x + 3x - 6 = 0$

$2x(x - 2) + 3(x - 2) = 0$

$(2x + 3)(x - 2) = 0$

Either $(2x + 3) = 0$ or $(x - 2) = 0$

3. find the solution for each factor

$(2x + 3) = 0$

$2x = -3$

$x = -\frac{3}{2}$

$(x - 2) = 0$

$x = 0 - 2$

$x = 2$

Therefore, the zeros, or roots, of the function $f(x) = 2x^2 - x - 6$ are $x = -\frac{3}{2}$ and $x = 2$ because both values make the function equal to zero.

We will apply the Zero-Product Property to solve polynomial equations in the next task.

Solving a Polynomial Equation

There are a number of ways that can be used to find the zeros, or the roots, of polynomial functions. We can use a calculator, plot a graph, use GeoGebra (software) and we can also calculate by using the zero property.

To find the zeros, or roots, by the calculation method:

Step 1: use the factor theorem approach to factorise the expression completely.

Step 2: equate each factor to 0 and solve.

In pairs or in groups, let's work through the following examples using the steps above.

Example 3.1

Find the zeros of the cubic function $f(x) = x^3 - 2x^2 - 5x + 6$

Solution

$x^3 - 2x^2 - 5x + 6 = 0$

Possible factors of 6 = ±1, ±2, ±3, ±6

Testing with these factors,

When x = 1,

$f(x) = (1)^3 - 2(1)^2 - 5(1) + 6$

$f(x) = 1 - 2 - 5 + 6$

$f(x) = 0$

$\therefore (x - 1)$ is a factor. If x=1 was not a factor, try another option, eg x=-1.

Using the long division method:

$$
\begin{array}{r}
x^2 - x - 6 \\
x - 1 \overline{)\, x^3 - 2x^2 - 5x + 6} \\
-\underline{(x^3 - x^2 + 0x + 0)} \\
-x^2 - 5x + 6 \\
-\underline{(-x^2 + x - 0)} \\
-6x + 6 \\
-\underline{(-6x + 6)} \\
0
\end{array}
$$

Factorising, $x^2 - x - 6$

$x^2 - 3x + 2x - 6$

$(x^2 - 3x) + (2x - 6)$

$x(x - 3) + 2(x - 3)$

$(x + 2)(x - 3)$

$\therefore \text{f(x)} = (x - 1)(x + 2)(x - 3)$

$f(x) = 0$

$(x - 1)(x + 2)(x - 3) = 0$

For $(x - 1) = 0$

$x = 1$

For $(x + 2) = 0$

$x = -2$

For $(x - 3) = 0$

$x = 3$

Hence the zeros are when x = −2, 1, 3

Example 3.2

Find the roots, or zeros, of the cubic function $f(x) = x^3 + x^2 - 10x + 8$

Solution

$x^3 + x^2 - 10x + 8 = 0$

Possible factors of 8 = ±1, ±2, ±4, ±8

Testing with these factors,

When x = 1,

$f(x) = (1)^3 + (1)^2 - 10(1) + 8$

$f(x) = 1 + 1 - 10 + 8$

$f(x) = 0$

$\therefore (x - 1)$ is a factor.

Using the long division method:

$$\begin{array}{r} x^2 + 2x - 8 \\ x - 1 \overline{\smash{\big)} x^3 + x^2 - 10x + 8} \\ -\underline{(x^3 - x^2 + 0x + 0)} \\ 2x^2 - 10x + 8 \\ -\underline{(2x^2 - 2x - 0)} \\ -8x + 8 \\ -\underline{(-8x + 8)} \\ 0 \end{array}$$

Factorising, $x^2 + 2x - 8$

$x^2 + 4x - 2x - 8$

$(x^2 + 4x) + (2x - 8)$

$x(x + 4) - 2(x + 4)$

$(x + 4)(x - 2)$

$\therefore$ f(x) $= (x - 1)(x - 2)(x + 4)$

$f(x) = 0$

$(x - 1)(x - 2)(x + 4) = 0$

For $(x - 1) = 0$

$x = 1$

For $(x - 2) = 0$

$x = 2$

For $(x + 4) = 0$

$x = -4$

Hence the roots, or zeros, are when x $= -4, 2, 1$

Example 3.3

Find the zeros of the cubic function. $f(x) = x^3 - 3x^2 - 4x + 12$

Solution

$x^3 - 3x^2 - 4x + 12 = 0$

Possible factors of 12 = ±1, ±2, ±3, ±4, ±6, ±12

Testing with these factors,

When x = 1,

$f(x) = (1)^3 - 3(1)^2 - 4(1) + 12$

$f(x) = 1 - 3 - 4 + 12$

$f(x) = 6$

∴ $(x - 1)$ is not a factor and we need to try another potential factor.

When $x = 2$,

$f(x) = (2)^3 - 3(2)^2 - 4(2) + 12$

$f(x) = 8 - 12 - 8 + 12$

$f(x) = 0$

∴ $(x - 2)$ is a factor.

Using the long division method:

$$
\begin{array}{r}
x^2 - x - 6 \\
x - 2\,\overline{)\,x^3 - 3x^2 - 4x + 12} \\
-\underline{(x^3 - 2x^2 + 0x + 0)} \\
-x^2 - 4x + 12 \\
-\underline{(x^2 + 2x - 0)} \\
-6x + 12 \\
-\underline{(6x + 12)} \\
0
\end{array}
$$

Factorising, $x^2 - x - 6$

$x^2 - 3x + 2x - 6$

$(x^2 - 3x) + (2x - 6)$

$x(x - 3) + 2(x - 3)$

$(x + 2)(x - 3)$

∴ $f(x) = (x - 2)(x + 2)(x - 3)$

$f(x) = 0$

$(x - 2)(x + 2)(x - 3) = 0$

For $(x - 2) = 0$

$x = 2$

For $(x + 2) = 0$

$x = -2$

For $(x - 3) = 0$

$x = 3$

Hence the zeros are when x = −2, 2, 3

Following the remainder theorem is the Rational Zero Theorem which helps to identify exactly which of the factor will give us zero.

The Rational Zero Theorem

The Rational Zero Theorem helps us identify potential rational zeros of a polynomial by looking at the factors of two specific parts of the polynomial:

1. Factors of the Constant Term: This is the term without a variable (the last number in the polynomial when arranged in descending powers).
2. Factors of the Leading Coefficient: This is the coefficient (the number in front of the term) of the term with the highest power of x.

Steps to Find Possible Rational Zeros

1. Identify the Constant Term and Leading Coefficient.

   For example, in the polynomial $4x^3 + 7x^2 + 2x + 5$:

   The constant term is 5.

   The leading coefficient is 4.

2. List the Factors.

   Factors of the Constant Term (5): These are: ± 1 , ± 5

   Factors of the Leading Coefficient (4): These are ± 1 , ± 2, ± 4.

3. Create Possible Rational Zeros.

   To find the possible rational zeros, take each factor of the constant term and divide it by each factor of the leading coefficient.

   Possible rational zeros for our example would be:

   From ± 1: $\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{-1}{4}, = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4},$

   From ± 5: $\frac{5}{1} - \frac{5}{1}, \frac{5}{2}, -\frac{5}{2}, \frac{5}{4}, \frac{-5}{4}, = \pm 5, \pm\frac{5}{2}, \pm\frac{5}{4}$

4. List of Possible Rational Zeros:

   Combining all these, the possible rational zeros for our polynomial are:

   ±1 , ±5 , $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, , $\pm\frac{5}{2}$, $\pm\frac{5}{4}$

Now that we have a list of possible rational zeros, we can use the Factor Theorem to test each one.

a. For each rational number c from the list, substitute it into the polynomial f(x)

b. If $f(c) = 0$, then c is a zero of the polynomial.

Example 3.4

In groups or individually, use the rational zeros theorem to list all the possible rational zeros.

a. $6x^3 + 15x^2 - 8x + 2$

b. $x^3 + x^2 - 2x - 5$

c. $3x^3 + x^2 + x - 5$

d. $6x^3 - 4x^2 + 17$

Solution

a. $6x^3 + 15x^2 - 8x + 2$

   The constant term is 2.

   The leading coefficient is 6.

   Factors of the Constant Term (2): These are.±1 , ±2

   Factors of the Leading Coefficient (6): These are ±1 , ±2, ±3 , ±6.

   Possible Rational Zeros:

   From ±1: $\frac{1}{1}$, $-\frac{1}{1}$, $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{3}$, $\frac{-1}{3}$, $\frac{1}{6}$, $-\frac{1}{6} = \pm1$, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$,

   From ±2: $\frac{2}{2}$, $-\frac{2}{2}$, $\frac{2}{3}$, $\frac{-2}{3}$, $\frac{2}{6}$, $-\frac{2}{6} = \pm1$, $\pm\frac{2}{3}$, $\pm\frac{1}{6}$,

b. $x^3 + x^2 - 2x - 8$

   The constant term is -8.

   The leading coefficient is 1.

   Factors of the Constant Term (-8): These are.±1 , ±2 , ±4, ±8

   Factors of the Leading Coefficient (1): These are ±1 .

   Possible Rational Zeros:

   From ±1: $\frac{1}{1}$, $-\frac{1}{1} = \pm1$

From ±2: , $\frac{2}{1}$, $-\frac{2}{1}$ = ±2

From ±4: $\frac{4}{1}$, $-\frac{4}{1}$,= ±4

From ±8: , $\frac{8}{1}$, $-\frac{8}{1}$ = ±8,,

Possible Rational Zeros (combined): ±1, ±2, ±4, ±8

c. $9x^3 + x^2 + x - 10$

The constant term is −10.

The leading coefficient is 9.

Factors of the Constant Term (-10): These are.±1 , ±2 , ±5, ±10

Factors of the Leading Coefficient (9): These are ±1, ±3, ±9 .

Possible Rational Zeros:

From ±1: $\frac{1}{1}, -\frac{1}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{9} = \pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}$

From ±2: $\frac{2}{1}, -\frac{2}{1}, \frac{2}{3}, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{9} = \pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}$

From ±4: $\frac{5}{1}, -\frac{5}{1}, \frac{5}{3}, -\frac{5}{3}, \frac{5}{9}, -\frac{5}{9} = \pm 5, \pm\frac{5}{3}, \pm\frac{5}{9}$

From ±10: , $\frac{10}{1}, -\frac{10}{1}, \frac{10}{3}, -\frac{10}{3}, \frac{10}{9}, -\frac{10}{9} = \pm 10, \pm\frac{10}{3}, \pm\frac{10}{9}$

Possible Rational Zeros (combined): ±1, ±2, ±5, ±10 $\pm\frac{1}{3}, \pm\frac{1}{9}, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm\frac{5}{3}$, $\pm\frac{5}{9}, \pm\frac{10}{3}, \pm\frac{10}{9}$

d. $6x^3 - 4x^2 + 17$

The constant term is 17.

The leading coefficient is 6.

Factors of the Constant Term (17): These are.±1 , ±17

Factors of the Leading Coefficient (6): These are ±1, ±2, ±3, ±6 .

Possible Rational Zeros:

From ±1: $\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$

From ±17: $\frac{17}{1}, -\frac{17}{1}, \frac{17}{2}, -\frac{17}{2}, \frac{17}{3}, -\frac{17}{3}, \frac{17}{6}, -\frac{17}{6} = = \pm 17, \pm\frac{17}{2}, \pm\frac{17}{3}, \pm\frac{17}{6}$

Possible Rational Zeros (combined): $\pm 1, \pm 17, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{17}{2}, \pm\frac{17}{3}, \pm\frac{17}{6}$

Using the rational zero theorem to find the zeros of a polynomial function

Steps to Follow:

1. Identify Possible Rational Zeros

   Use the Rational Zero theorem to list all possible rational zeros. These are of the form p and q, where p is a factor of the constant term (the last term of the polynomial) and q is a factor of the leading coefficient (the coefficient of the term with the highest degree).

2. Substitute and Evaluate

   a. Substitute each possible rational zero into the polynomial function.

   b. Calculate the value of the polynomial for each substitution.

3. Determine Actual Zeros

   a. Identify which substitutions result in a value of zero.

   b. The values for which the polynomial equals zero are the rational zeros of the function.

Example 3.5

Given the polynomial $3x^3 + 8x^2 + 7x + 2$

1. List the factors of the constant term and leading coefficient.
2. Determine possible rational zeros
3. Substitute each value into the polynomial to find which ones yield zero.

Solution

The constant term is 2.

The leading coefficient is 3.

Factors of the Constant Term (2): These are.± 1 , ± 2

Factors of the Leading Coefficient (3): These are ± 1, ± 3 .

Possible Rational Zeros:

From ± 1: $\frac{1}{1}, -\frac{1}{1}, \frac{1}{3}, -\frac{1}{3} = \pm 1, \pm \frac{1}{3}$

From ± 2: $\frac{2}{1}, -\frac{2}{1}, \frac{2}{3}, -\frac{2}{3}, = = \pm 2, \pm \frac{2}{3}$

Possible Rational Zeros (combined): $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

Substitute and evaluate:

$3x^3 + 8x^2 + 7x + 2$

For $x = \pm 1$:

$x = 1$

$3(1)^3 + 8(1)^2 + 7(1) + 2$

$3 + 8 + 7 + 2 = 20$

$\boldsymbol{x} = -1$

$3(-1)^3 + 8(-1)^2 + 7(-1) + 2$

$-3 + 8 - 7 + 2 = 0$

For $x = \pm 2$

$x = 2$

$3(2)^3 + 8(2)^2 + 7(2) + 2$

$24 + 32 + 14 + 2 = 71$

$x = -2$

$3(-2)^3 + 8(-2)^2 + 7(-2) + 2$

$-24 + 32 - 14 + 2 = -4$

For $\pm \frac{1}{3}$

$x = \frac{1}{3}$

$3\left(\frac{1}{3}\right)^3 + 8\left(\frac{1}{3}\right)^2 + 7\left(\frac{1}{3}\right) + 2$

$\frac{1}{9} + \frac{8}{9} + \frac{7}{3} + 2 = \frac{16}{3}$

$x = -\frac{1}{3}$

$3\left(-\frac{1}{3}\right)^3 + 8\left(-\frac{1}{3}\right)^2 + 7\left(-\frac{1}{3}\right) + 2$

$-\frac{1}{9} + \frac{8}{9} - \frac{7}{3} + 2 = \frac{4}{9}$

For $\pm \frac{2}{3}$

$x = \frac{2}{3}$

$3\left(\frac{2}{3}\right)^3 + 8\left(\frac{2}{3}\right)^2 + 7\left(\frac{2}{3}\right) + 2$

$\frac{8}{9} + \frac{32}{9} + \frac{14}{3} + 2 = \frac{100}{9}$

$x = -\frac{2}{3}$

$3\left(-\frac{2}{3}\right)^3 + 8\left(-\frac{2}{3}\right)^2 + 7\left(-\frac{2}{3}\right) + 2$

$-\frac{8}{9} + \frac{32}{9} - \frac{14}{3} + 2 = 0$

The values 1 , 2, $-2, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}$ did not yield zero and are not zeros of the function.

However, -1 and $-\frac{2}{3}$, yielded zero and therefore represent the zeros of the function

SKETCHING POLYNOMIAL FUNCTIONS WITH DEGREES HIGHER THAN 2

1. In mathematics, the terms "sketch" and "draw" have distinct meanings.
2. Sketch is a rough representation of a graph or function without precise measurements, used to illustrate its behaviour. It often lacks scales and coordinates.
3. Draw, on the other hand, creates a more accurate representation with specific values, scales, and tools, providing a detailed visualization of a mathematical function or data set.

We will be sketching polynomial functions with a degree higher than 2.

These will be cubics, quartics etc.

To sketch a polynomial of degree greater than 2, for example, $ax^3 + bx^2 + cx + d$, we:

Step 1: Determine the sign (+ or -) of **a**; if a > 0 or a < 0

If a < 0, the curve will have the shape:

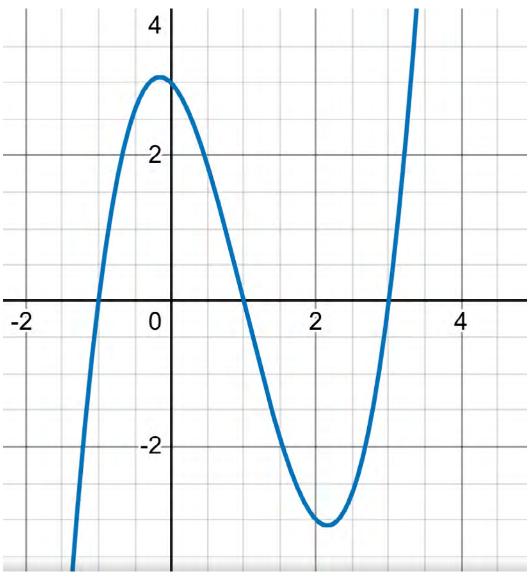


Figure 3.1: Curve of a polynomial

If a < 0, the curve will have the shape:

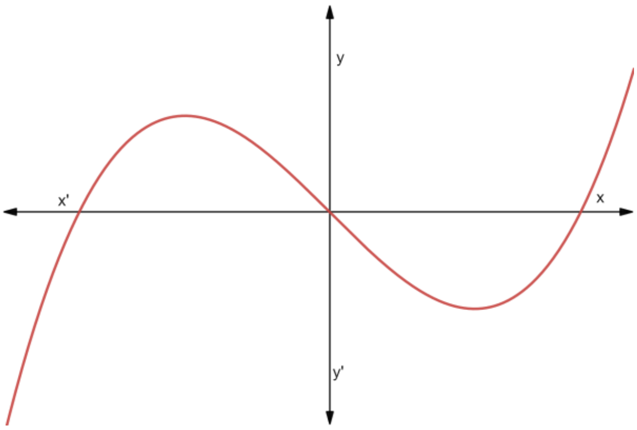


Figure 3.2: Curve of a polynomial

Step 2: Use the factor theorem or any known method to find the zeros of the polynomial

Step 3: Find the y- intercept of the function by substituting x = 0 into the given polynomial function. (Note that this equates to the constant.)

Step 4: Find the intervals of the curve based on the zeros obtained

Step 5: Sketch the curve.

Individually, or in groups, let's use the steps for sketching polynomials to solve the examples below.

Example 3.6

Sketch the curve $2x^3 + 3x^2 - 5x - 6$

Solution

Step 1: a is 2, which is greater than 0. Therefore, we know the shape that the curve will take. The extremes will go from bottom left to top right.

Step 2: Using the factor theorem, the factors of the constant, - 6 are ±1, ±2, ±3, ±6

Testing when x = - 1, we have:

$2(-1)^3 + 3(-1)^2 - 5(-1) - 6 = -2 + 3 + 5 - 6 = 0$

Which means $(x + 1)$ is a factor.

We can use long division to find the other factors.

$$
\begin{array}{r}
2x^2 + x - 6 \\
x + 1 \overline{\big) 2x^3 + 3x^2 - 5x - 6} \\
-\underline{(2x^3 + 2x^2 + 0 + 0)} \\
x^2 - 5x - 6 \\
-\underline{(x^2 + x + 0)} \\
(-6x - 6) \\
\underline{-6x - 6}
\end{array}
$$

Factorising:

$2x^2 + x - 6$

$2x^2 + 4x - 3x - 6$

$2x(x + 2) - 3(x + 2)$

$(2x - 3)(x + 2)$

Factors are:

$(x + 1)(2x - 3)(x + 2)$

The zeros:

$(x + 1)(2x - 3)(x + 2) = 0$

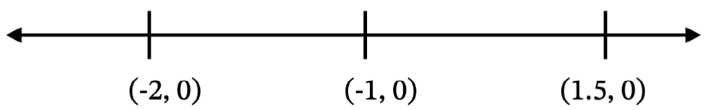
$x = -1, x = -2, x = \frac{3}{2}$

Step 3: y intercept

When x = 0

$2(0)^3 + 3(0)^2 - 5(0) - 6 = -6$

Step 4: finding the intervals using the zeros



The intervals are:

1. $-2 < x$
2. $-2 < x < -1$
3. $-1 < x < 1.5$
4. $x > 1.5$

Next is to determine the signs or test for the signs in each of the intervals obtained

Interval	$-2 > x$	$-2 < x < -1$	$-1 < x < 1.5$	$x > 1.5$
Chosen values (you can choose any number within the interval)	**-3**	**-1.5**	**1**	**2**
y value (substitute your chosen number into the given function)	$2(-3)^3 + 3(-3)^2 - 5(-3) - 6$ **= -18**	$2(-1.5)^3 + 3(-1.5)^2 - 5(-1.5) - 6$ **= 1.5**	$2(1)^3 + 3(1)^2 - 5(1) - 6$ **= -6**	$2(2)^3 + 3(2)^2 - 5(2) - 6$ **= 12**
Sign	Negative	Positive	Negative	Positive
Behaviour of curve	Below x - axis	Above x - axis	Below x - axis	Above x - axis

Step 5: sketch the curve

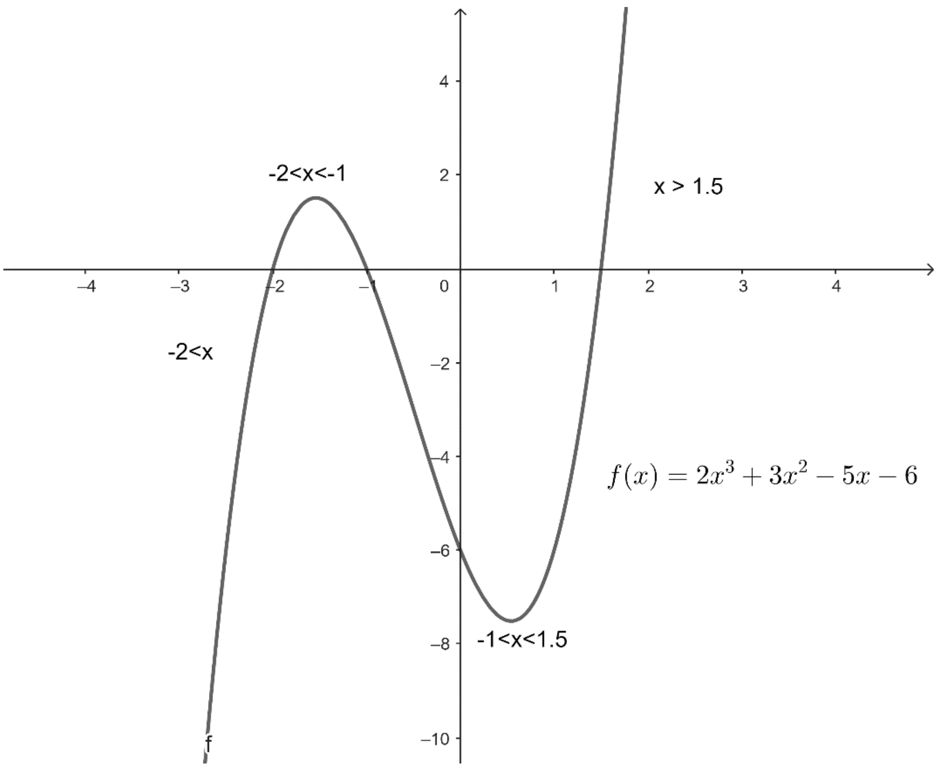


Figure 3.3: Curve of $2x^3 + 3x^2 - 5x - 6$

Example 3.7

Sketch the curve $x^3 - 6x^2 + 5x + 12$

Solution

Step 1: a is 1 which is greater than 0. This means we know the shape the curve will take.

Step 2: Using the factor theorem, the factors of the constant, 12 are ±1, ±2, ±3, $\pm4 \pm 6$, ±12

Testing when x = - 1, we have:

$(-1)^3 - 6(-1)^2 + 5(-1) + 12$

$-1 - 6 - 5 + 12 = 0$

Which means $(x + 1)$ is a factor

We can use long division to find the other factors.

$$
\begin{array}{r}
x^2 - 7x + 12 \\
x + 1 \overline{\smash{)}\, x^3 - 6x^2 + 5x + 12} \\
-\underline{(x^3 + x^2 + 0 + 0)} \\
-7x^2 + 5x + 12 \\
(-7x^2 - 7x + 12) \\
12x + 12 \\
\underline{-(12x + 12)} \\
0
\end{array}
$$

Factorising:

$x^2 - 7x + 12$

$x^2 - 4x - 3x + 12$

$x(x - 4) - 3(x - 4)$

$(x - 3)(x - 4)$

Factors are

$(x + 1)(x - 3)(x - 4)$

The zeros

$(x + 1)(x - 3)(x - 4) = 0$

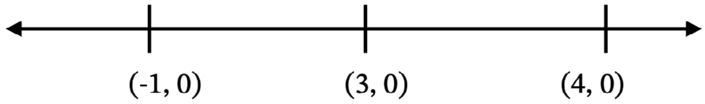
$x = -1, x = 3, x = 4$

Step 3: *y* intercept

At x = 0

$(0)^3 - 6(0)^2 + 5(0) + 12 = 12$

Step 4: finding the intervals using the zeros



The interval are

1. $-1 < x$
2. $-1 < x < 3$
3. $3 < x < 4$
4. $x > 4$

Next is to determine the signs or test for the signs in each of the intervals obtained

Table 3.1: Test for signs in each of the intervals obtained

Interval	$-2 > x$	$-2 < x < -1$	$-1 < x < 1.5$	$x > 1.5$
Chosen values (you can choose any number within the interval)	**-2**	**1**	**3.5**	**5**
y value (substitute your chosen number into the given function)	$(-2)^3 - 6(-2)^2 + 5(-2) + 12$ **= -30**	$(1)^3 - 6(1)^2 + 5(1) + 12$ **= 12**	$(3.5)^3 - 6(3.5)^2 + 5(3.5) + 12$ **= -1.125**	$2(5)^3 + 3(5)^2 - 5(5) - 6$ **= 294**
Sign	Negative	Positive	Negative	Positive
Behaviour of curve	Below x - axis	Above x - axis	Below x - axis	Above x - axis

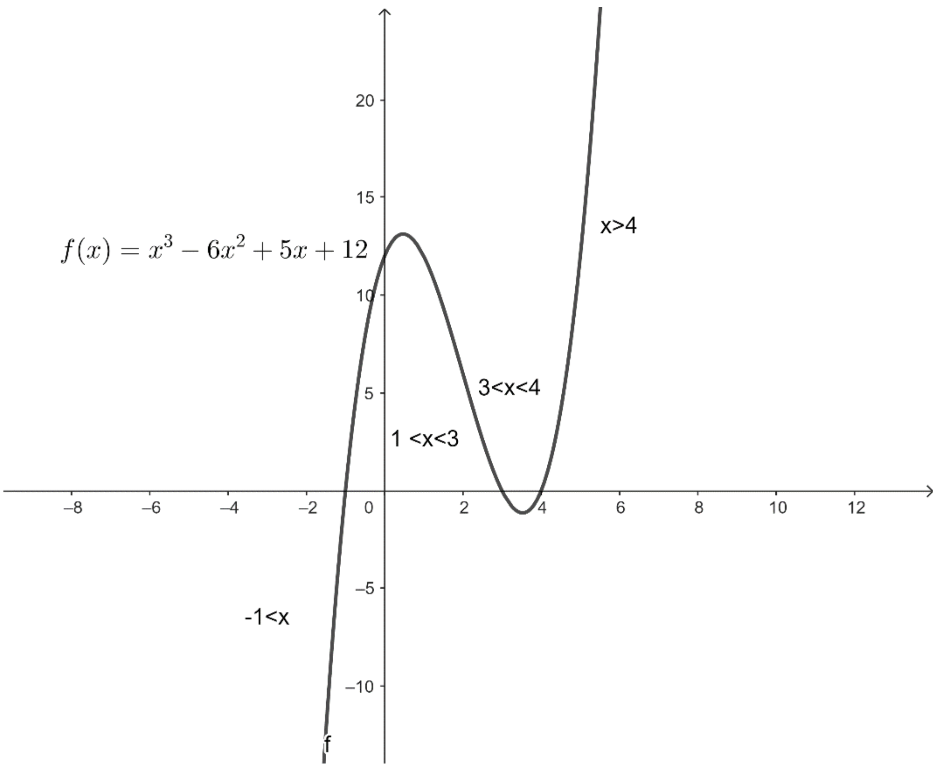


Figure 3.4: Curve of $x^3 - 6x^2 + 5x + 12$

Example 3.8

Sketch the curve $-x^3 + 4x^2 - x - 6$

Solution

Step 1: a is -1 which is less than 0. This means that the extremes of the graph go from top left to bottom right.

Step 2: Using the factor theorem, the factors of the constant, -6 are ± 1, ± 2, $\pm 3, \pm 6$

Testing when x = 2, we have:

$-(2)^3 + 4(2)^2 - 2 - 6 = -8 + 16 - 2 - 6 = 0$

Which means $(x - 2)$ is a factor.

We can use long division to find the other factors.

$$
\begin{array}{r}
-x^2 + 2x + 3 \\
x-2\overline{\big)\,-x^3 + 4x^2 - x - 6} \\
\underline{-(-x^3 + 2x^2 + 0 + 0)} \\
2x^2 - x - 6 \\
-(2x^2 - 4x - 6) \\
3x - 6 \\
\underline{-(3x - 6)} \\
0
\end{array}
$$

Factorising:

$-x^2 + 2x + 3$

$-x^2 + 3x - x + 3$

$-x(x-3) - 1(x-3)$

$(-x-1)(x-3)$

Factors are

$(x-2)(x-3)(-x-1)$

The zeros

$(x-2)(x-3)(-x-1) = 0$

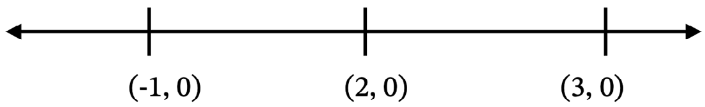
$x = -1, x = 3, x = 2$

Step 3: y intercept:

When x = 0

$-(0)^3 + 4(0)^2 - (0) - 6 = -6$

Step 4: finding the intervals using the zeros



The intervals are

1. $-1 < x$
2. $-1 < x < 2$
3. $2 < x < 3$
4. $x > 3$

Next is to determine the signs or test for the signs in each of the intervals obtained

Table 3.2: Test for signs in each of the intervals obtained

Interval	$-1 > x$	$-1 < x < 2$	$2 < x < 3$	$x > 3$
Chosen values (you can choose any number within the interval)	**-3**	**1**	**2.5**	**4**
y value (substitute your chosen number into the given function)	$-(-3)^3 + 4(-3)^2 - (-3) - 6$ **= 60**	$-(1)^3 + 4(1)^2 - (1) - 6$ **= -4**	$-(2.5)^3 + 4(2.5)^2 - (2.5) - 6$ **= 0.875**	$-(4)^3 + 4(4)^2 - (4) - 6$ **= -10**
Sign	Positive	Negative	Positive	Negative
Behaviour of curve	Above x - axis	Below x - axis	Above x - axis	Below x - axis

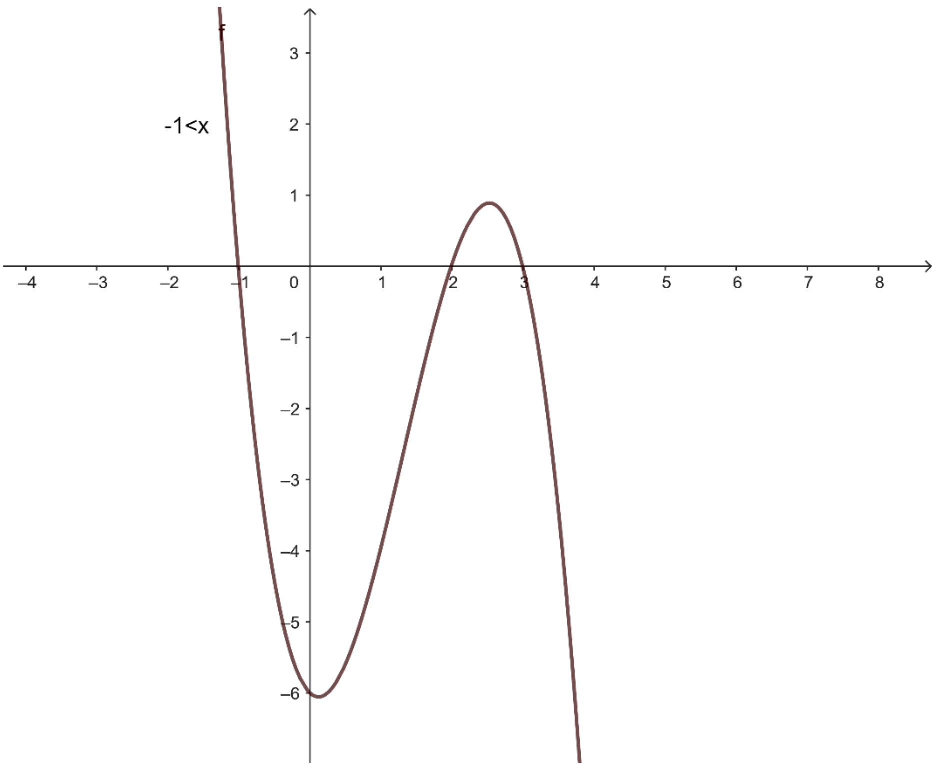


Figure 3.5: Curve of $-x^3 + 4x^2 - x - 6$

What polynomial function is represented in the graph below?

a. How many factors does it have?

b. Share your answers with the class.

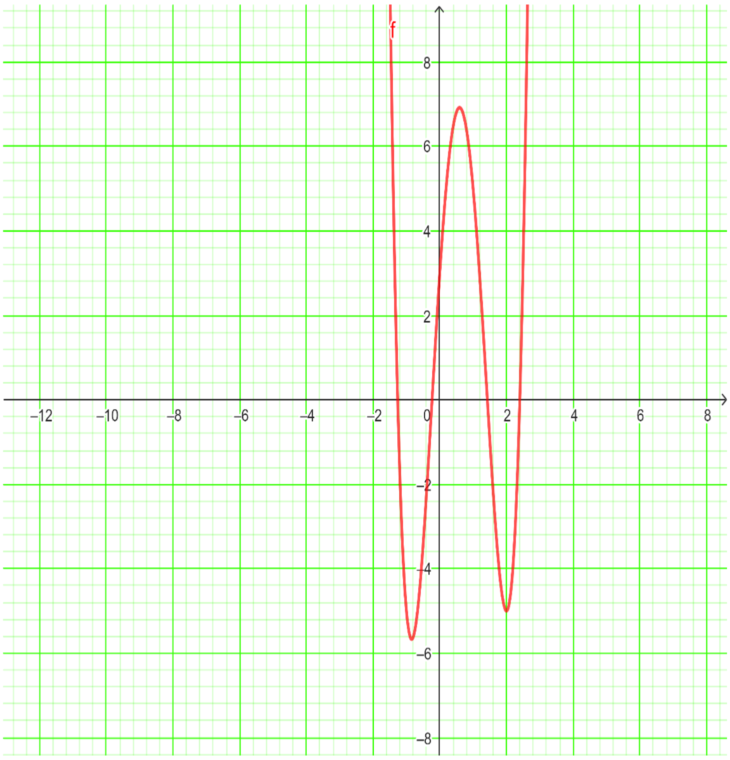


Figure 3.6: Graph of a polynomial function

Remember that it is an important skill to be able to sketch graphs by hand. However, when appropriate technology is there to assist. For example, you can use apps or web-based programs such as GeoGebra, Demos, PhET Simulations and Geometer's Sketch Pad.

APPLYING DESCARTES' RULE OF SIGNS THEOREM

There is another way that can be used to determine how many positive and negative real zeros a polynomial function might have. To do this, first write the polynomial in descending order (from the highest degree to the lowest) if it is not already. We use **Descartes' Rule of Signs**. This rule helps you see how the number of times the signs change in the polynomial is related to the number of positive real zeros.

For example, if you look at the polynomial function below, you will notice it has one sign change, which gives you information about its positive zeros.

Let us go through an example using Descartes' Rule of Signs to determine the possible number of positive and negative real zeros.

Example 3.9

Consider the polynomial:

$f(x) = 3x^3 - 2x^2 + 4x - 5$

Solution

Step 1: *Identify Sign Changes for Positive Real Zeros*

1. Write the polynomial in descending order:
2. $f(x) = 3x^3 - 2x^2 + 4x + 5$

   Look at the coefficients:

   3 (positive)

   –2 (negative)

   4 (positive)

   −5 (negative)
3. Determine the sign changes:

   From 3 to -2 (1 change)

   From −2 to 4 (2 changes)

   From 4 to −5 (3 changes)

   Total sign changes for positive zeros = 3. This means there could be 3, (3 – 2 = 1) positive real zeros. Note, the number of possible roots is found by subtracting 2 each time until we would get a negative number.

Step 2: *Identify Sign Changes for Negative Real Zeros*

Now, let's find the negative real zeros by evaluating f(−x):

$f(-x) = 3(-x)^3 - 2(-x)^2 + 4(-x) - 5$

$f(-x) = -3x^3 - 2x^2 - 4x - 5$

Step 3: *Look at the Signs of f(−x)*

The coefficients are:

−3 (negative)

– 2 (negative)

– 4 (negative)

– 5 (negative)

Step 4: *Determine Sign Changes for Negative Real Zeros*

There are **no sign changes** in f(−x)

Total sign changes for negative zeros = 0. This means there are no negative real roots or zeros.

This example illustrates how to use Descartes' Rule of Signs to analyse a polynomial and determine the possible numbers of positive and negative real zeros.

Let us solve more examples.

Example 3.10

Use Descartes' Rule of Signs to determine how many possible positive and negative real zeros of the following:

a. $g(x) = 2x^4 - 11x^3 + 4x^2 - 8x - 40$

b. $g(x) = 9x^3 - 10x^2 + 5x + 3$

c. $f(x) = -x^5 + 3x^4 - 5x^3 - 12x^2 + 6x + 4$

Solution

a. $g(x) = 2x^4 - 11x^3 + 4x^2 - 8x - 40$

Step 1: *Identify Sign Changes for Positive Real Zeros*

Write the polynomial in descending order:

$g(x) = 2x^4 - 11x^3 + 4x^2 - 8x - 40$

Look at the coefficients:

2 (positive)

–11 (negative)

4 (positive)

−8 (negative)

−40 (negative)

Look at the coefficients:

Determine the sign changes:

From 2 to -11 (1 change)

From −11 to 4 (2 changes)

From 4 to −8 (3 changes)

From -8 to −40 (no change)

Total sign changes for positive zeros = 3. This means there could be 3 or 1 positive real zeros.

Step 2: *Identify Sign Changes for Negative Real Zeros*

Now, let's find the negative real zeros by evaluating f(−x):

$g(x) = 2(-x)^4 - 11(-x)^3 + 4(-x)^2 - 8(-x) - 40$

$g(x) = 2x^4 + 11x^3 + 4x^2 + 8x - 40$

Step 3: *Look at the Signs of f(−x)*

The coefficients are:

2 (positive)

11 (positive)

4 (positive)

8 (positive)

−40 (negative)

Step 4: *Determine Sign Changes for Negative Real Zeros*

Determine the sign changes:

From 2 to 11 (no change)

From 11 to 4 (no changes)

From 4 to 8 (no changes)

From 8 to −40 (1 change)

Total sign changes for negative zeros = 1. This means there is 1 negative real zero.

b. $g(x) = 9x^3 - 10x^2 + 5x + 3$

Step 1: *Identify Sign Changes for Positive Real Zeros*

Write the polynomial in descending order: $g(x) = 9x^3 - 10x^2 + 5x + 3$.

Look at the coefficients:

9 (positive)

−10 (negative)

5 (positive)

3 (negative)

Determine the sign changes:

From 9 to -10 (1 change)

From −10 to 5 (2 changes)

From 5 to 3 (no change)

Total sign changes for positive zeros = 2. This means there could be 2 or 0.

Step 2: *Identify Sign Changes for Negative Real Zeros*

Now, let's find the negative real zeros by evaluating g(−x):

$g(x) = 9(-x)^3 - 10(-x)^2 + 5(-x) + 3$

$g(x) = -9x^3 - 10x^2 - 5x + 3$

Step 3: *Look at the Signs of f(−x)*

The coefficients are:

−9 (negative)

−10 (negative)

−5 (negative)

3 (positive)

Step 4: *Determine Sign Changes for Negative Real Zeros*

Determine the sign changes:

From -9 to -10 (no change)

From -10 to -5 (no changes)

From -5 to 3 (1 change)

Total sign changes for negative zeros = 1. This means there is 1 negative real zero.

c. $f(x) = -x^5 + 3x^4 - 5x^3 - 12x^2 + 6x + 4$

Step 1: *Identify Sign Changes for Positive Real Zeros*

Write the polynomial in descending order:

$f(x) = -x^5 + 3x^4 - 5x^3 - 12x^2 + 6x + 4$

Look at the coefficients:

−1 (negative)

3 (positive)

−5 (negative)

−12 (negative)

6 (positive)

4 (positive)

Determine the sign changes:

From −1 to 3 (1 change)

From 3 to −5 (2 changes)

From −5 to −12 (no changes)

From −12 to 6 (3 changes)

From 6 to 4 (no change)

Total sign changes for positive zeros = 3. This means there could be 3 or 1 positive real zeros.

Step 2: *Identify Sign Changes for Negative Real Zeros*

Now, let's find the negative real zeros by evaluating f(−x):

$f(x) = -(-x)^5 + 3(-x)^4 - 5(-x)^3 - 12(-x)^2 + 6(-x) + 4$

$f(x) = x^5 + 3x^4 + 5x^3 - 12x^2 - 6x + 4$

Step 3: Look at the Signs of f(−x)

The coefficients are:

1 (positive)

3 (positive)

5 (positive)

–12 (negative)

–6 (negative)

4 (positive)

Step 4: *Determine Sign Changes for Negative Real Zeros*

Determine the sign changes:

From 1 to 3 (no change)

From 3 to 5 (no changes)

From 5 to −12 (2 changes)

From −2 to −6 no changes)

From −6 to 4 (3 change)

Total sign changes for negative zeros = 2. This means there is 2 or 0 negative real zeros.

APPLYING THE FUNDAMENTAL THEOREM OF ALGEBRA

The **Fundamental Theorem of Algebra** tells us something very important about polynomial functions. It says: 'Every polynomial function of degree n (where $n > 0$) has at least one complex zero.'

This might sound complicated at first, but here is what it means:

1. The degree of a polynomial is the highest power of x in the expression.
2. For example, $x^3 - 2x^2 + 4x - 8$ has a degree of 3.
3. Complex numbers include real numbers and imaginary numbers (numbers with i, where $i = \sqrt{-1}$)
4. In factorising the polynomial, each root c_i corresponds to a linear factor of the form $(x - c_i)$. This means we can write the polynomial as $f(x) = a\,(x - c_1)\,(x - c_2) \cdots (x - c_n)$.
   The, a is a non-zero number that affects how the polynomial behaves, and $c_1, c_2, \ldots, c_n$, are the roots, or zeros.

Let us go through these examples to show how it works.

Example 3.11

Factorise the polynomial $f(x) = 3x^5 - 48x$ completely and find all its zeros.

State the multiplicity of each zero.

Solution

$f(x) = 3x^5 - 48x$

Factorising:

$3x(x^4 - 16)$

$3x[(x^2)^2 - 4^2]$

$3x[(x^2 + 4)(x^2 - 4)]$

$3x[x^2 - (-4)(x^2 - 4)]$

$3x[(x + 2i)(x - 2i)(x + 2)(x - 2)]$

To find the zeros, equate the factors to zero:

$3x[(x + 2i)(x - 2i)(x + 2)(x - 2)] = 0$

$3x = 0,\ x = 0$

$x + 2i = 0,\ x = -2i$

$x - 2i = 0,\ x = 2i$

$x + 2 = 0,\ x = -2$

$x - 2 = 0,\ x = 2$

Therefore, the zeros of $f(x)$ are 0, 2, – 2, $2i$ and – $2i$. Since each factor occurs only once, all the zeros are of multiplicity 1 and the total number of zeros is five.

Example 3.12

Factorise the polynomial $f(x) = x^4 - 625$ completely and find all its zeros.

State the multiplicity of each zero.

Solution

$f(x) = x^4 - 625$

Factorising:

$(x^4 - 625)$

$[(x^2)^2 - 25^2]$

$[(x^2 + 25)(x^2 - 25)]$

$3x[x^2 - (-25)(x^2 - 25)]$

$[(x + 5i)(x - 5i)(x + 5)(x - 5)]$

To find the zeros, equate the factors to zero:

$[(x + 5i)(x - 5i)(x + 5)(x - 5)] = 0$

$x + 5i = 0, x = -5i$

$x - 5i = 0, x = 5i$

$x + 5 = 0, x = -5$

$x - 5 = 0, x = 5$

Therefore, the zeros of $f(x)$ are 5, -5, $5i$ and $-5i$. Since each factor occurs only once, all the zeros are of multiplicity 1 and the total number of zeros is four.

Example 3.13

Factorise the polynomial $f(x) = 15x^4 + 8x^2 + 1$ completely and find all its zeros.

State the multiplicity of each zero.

Solution

$f(x) = 15x^4 + 8x^2 + 1$

in this scenario, we can represent x^2 by any other letter, for example, u

$f(x) = 15(x^2)^2 + 8x^2 + 1$

$f(x) = 15(u)^2 + 8u + 1$

$15u^2 + 8u + 1$

Factorising:

$15u^2 + 5u + 3u + 1$

$5u(3u + 1) + (3u + 1)$

$(5u + 1)(3u + 1)$

Substituting x^2 back,

$(5x^2 + 1)(3x^2 + 1)$

To find the zeros, equate the factors to zero:

$(5x^2 + 1)(3x^2 + 1) = 0$

$5x^2 + 1 = 0$ or $3x^2 + 1 = 0$

For $5x^2 + 1 = 0$

$5x^2 = -1$

$x^2 = -\frac{1}{5}$

$x = \pm\sqrt{-\frac{1}{5}}$

$x = -\sqrt{-\frac{1}{5}}$ or $x = \sqrt{-\frac{1}{5}}$

For $x = -\sqrt{-\frac{1}{5}}$

$x = -\sqrt{\frac{1}{5}} \times \sqrt{-1}$

$x = -i\sqrt{\frac{1}{5}}$

$x = -\frac{\sqrt{5}}{5}i$

For $x = \sqrt{-\frac{1}{5}}$

$x = \sqrt{\frac{1}{5}} \times \sqrt{-1}$

$x = i\sqrt{\frac{1}{5}}$

$x = \frac{\sqrt{5}}{5}i$

For $3x^2 + 1 = 0$

$3x^2 = -1$

$x^2 = -\frac{1}{3}$

$x = \pm\sqrt{-\frac{1}{3}}$

$x = -\sqrt{-\frac{1}{3}}$ or $x = \sqrt{-\frac{1}{3}}$

For $x = -\sqrt{-\frac{1}{3}}$

$x = -\sqrt{\frac{1}{3}} \times \sqrt{-1}$

$x = -i\sqrt{\frac{1}{3}}$

$x = -\frac{\sqrt{3}}{3}i$

For $x = \sqrt{-\frac{1}{3}}$

$x = \sqrt{\frac{1}{3}} \times \sqrt{-1}$

$x = i\sqrt{\frac{1}{3}}$

$x = \frac{\sqrt{3}}{3}i$

Therefore, the zeros of $f(x)$ $\frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i, -\frac{\sqrt{5}}{5}i, \frac{\sqrt{5}}{5}i$. Since each factor occurs only once, all the zeros are of multiplicity 1 and the total number of zeros is four.

Example 3.14

Factorise the polynomial $f(x) = x^4 - 4x^2 - 21$ completely and find all its zeros.

State the multiplicity of each zero.

Solution

$f(x) = x^4 - 4x^2 - 21$

in this scenario, we can represent x^2 by u

$f(x) = (x^2)^2 - 4x^2 - 21$

$f(x) = (u)^2 - 4u - 21$

$u^2 - 4u - 21$

Factorising:

$u^2 - 7u + 3u - 21$

$u(u - 7) + 3(u - 7)$

$(u + 3)(u - 7)$

Substituting x^2 back,

$(x^2 + 3)(x^2 - 7)$

To find the zeros, equate the factors to zero:

$(x^2 + 3)(x^2 - 7) = 0$

$x^2 + 3 = 0$ or $x^2 - 7 = 0$

For $x^2 + 3 = 0$

$x^2 = -3$

$x^2 = -3$

$x = \pm\sqrt{-3}$

$x = -\sqrt{-3}$ or $x = \sqrt{-3}$

For $x = -\sqrt{-3}$

$\sqrt{-3} = -\sqrt{3} \times \sqrt{-1}$

$x = -\sqrt{3} \times \sqrt{-1}$

$x = -i\sqrt{3}$

For $x = \sqrt{-3}$

$x = \sqrt{3} \times \sqrt{-1}$

$x = i\sqrt{3}$

For $x^2 - 7$

$x^2 = 7$

$x^2 = 7$

$x = \pm\sqrt{7}$

$x = -\sqrt{7}$ or $x = \sqrt{7}$

For $x = -\sqrt{7}$

$x = -\sqrt{7}$

For $x = \sqrt{7}$

$x = \sqrt{7}$

$x = \sqrt{7}$

Therefore, the zeros of $f(x)$ are $x = i\sqrt{3}$, $x = -i\sqrt{3}$, $x = -\sqrt{7}$, $x = \sqrt{7}$. Since each factor occurs only once, all the zeros are of multiplicity 1 and the total number of zeros is four.

Note: Multiplicity refers to the number of times a particular zero (or root) appears in the factorisation of a polynomial. It indicates how many times a specific value $x = r$ is a solution to the polynomial equation $P(x) = 0$.

Key Points about Multiplicity:

1. Simple Roots: If a root appears once, it has a multiplicity of 1. For example, in the polynomial $(x - 2)$, the root $x = 2$ has multiplicity 1.
2. Repeated Roots: If a root appears more than once, it has a higher multiplicity. For example, in the polynomial $(x - 2)^2$, the root $x = 2$ has a multiplicity of 2.
3. Multiplicity and Behaviour: This can be even or odd.

Odd Multiplicity: If a root has an odd multiplicity, the graph of the polynomial crosses the x - axis at that root.

Even Multiplicity: If a root has an even multiplicity, the graph of the polynomial touches the x-axis at that root but does not cross it.

For instance, for the polynomial $P(x) = (x - 1)(x - 1)(x + 2)$:

The root $x = 1$ has a multiplicity of 2 (it appears twice), so the graph just touches the x-axis at x = 1 and does not cross it.

The root $x = -2$ has a multiplicity of 1 (it appears once), so the graph crosses the x-axis at x = -2.

COMPLEX CONJUGATES THEOREM

Complex Numbers: A complex number is expressed in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit defined as $i^2 = -1$, or $i = \sqrt{-1}$

Roots of Polynomials: When dealing with polynomial equations, the Fundamental Theorem of Algebra states that every non-constant polynomial has at least one complex root. This means that if a polynomial has real coefficients, any non-real roots must occur in conjugate pairs.

If $a + bi$ (where b $\neq 0$) is a root of a polynomial, then its conjugate $a - bi$ is also a root.

Example: For a polynomial like $x^2 + 4$:

The roots can be found by rearranging to $x^2 = -4$.

Taking the square root gives $x = \pm 2i$

Here, $2i$ and $-2i$ are complex roots that come in conjugate pairs.

Note that:

Complex roots always come in **conjugate pairs**.

If $a + bi$ is a root of a polynomial, then $a - bi$ is also a root.

This property helps us find all the roots of polynomials, ensuring we account for both real and complex solutions.

The **Complex Conjugate Theorem** states that if a polynomial has a complex root, its conjugate must also be a root. This is an important concept in algebra that helps us understand the nature of polynomial equations and their solutions.

Let us go through the following examples to explore how to use the **Complex conjugate theorem.**

For example, if we are given one complex factor of a polynomial, we know another factor must be the complex conjugate. This gives us two factors. We can then form a quadratic equation and use algebraic division to find another factor. Or we may be given two real factors, for example, $x = 2$ and $x = -3$. This means that $x - 2 = 0$ or $x + 3 = 0$

$\therefore (x - 2)(x + 3) = 0$

$x^2 + 3x - 2x - 6 = 0$

$x^2 + x - 6 = 0$

Example 3.15

Find all the roots of $f(x) = x^3 + 8x^2 + 21x + 20$ if one root is $-2 - i$.

Solution

$f(x) = x^3 + 8x^2 + 21x + 20$

Since the factor given $(-2 - i)$ is complex in nature, we need the conjugate as well: $-2 + i$

$[x-(-2-i)][(x-(-2+i)]$

$[x+2+i][x+2-i]$

$(x+2)^2-(i)^2$

$x^2+4x+4-i^2$

NB: $-i^2=1$

$x^2+4x+4-(-1)$

x^2+4x+5

We can use long division to find the other factors:

$$\begin{array}{r}
x+4 \\
x^2+4x+5\overline{\big)\,x^3+8x^2+21x+20} \\
-\underline{(x^3+4x^2+5x+20)} \\
4x^2+16x+20 \\
-\underline{(4x^2+16x+20)} \\
0
\end{array}$$

$x+4=0$

$x=-4$

Therefore, the roots of (x) are -4, $-2+i$ and $-2-i$

Example 3.16

Find all the roots of $f(x)=2x^3-13x^2+26x-10$ if one root is $3+i$.

Solution

$f(x)=2x^3-13x^2+26x-10$

Since the factor given $3+i$. is complex in nature, we need the conjugate as well: $3-i$

$[x-(3+i)][(x-(3-i)]$

$[x-3-i][x-3+i]$

$(x-3)^2-(i)^2$

$x^2-6x+9-i^2$

NB: $-i^2=-1$

$x^2-6x+9-(-1)$

$x^2-6x+10$

We can use long division to find the other factor.

$$\begin{array}{r} 2x - 1 \\ x^2 - 6x + 10 \overline{\smash{\big)}\, 2x^3 - 13x^2 + 26x - 10} \\ -\underline{(2x^3 - 12x^2 + 20x + 0)} \\ -x^2 + 6x - 10 \\ -\underline{(-x^2 + 6x - 10)} \\ 0 \end{array}$$

$2x - 1 = 0$

$x = \frac{1}{2}$

Therefore, the roots of (x) are $= \frac{1}{2}$, $3 + i$ and $3 - i$

LINEAR AND QUADRATIC FACTOR THEOREMS

The linear quadratic factor theorem is another method which can be used to solve polynomial functions. This is when we break polynomial functions into linear and quadratic factors to make solving it simpler.

For example, to use the linear and quadratic factor approach to find the factors of $x^4 - 9x^2 + 14$, we:

Step 1: Let $x^2 = u$

Step 2: Using this substitution it means $x^4 - 9x^2 + 14 = (u)^2 - 9(u) + 14$

Step 3: Factorise $u^2 - 9u + 14$:

$u^2 - 7u - 2u + 14$

$u(u - 7) - 2(u - 7)$

$(u - 2)(u - 7)$

Step 4: Substitute x^2 back in for u:

$(x^2 - 2)(x^2 - 7)$

$x^2 - 2 = 0$ and $x^2 - 7 = 0$

For $x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$x = -\sqrt{2}$ or $x = \sqrt{2}$

$x + \sqrt{2}$ and $x - \sqrt{2}$ are factors

For $x^2 - 7 = 0$

$x^2 = 7$

$x = \pm\sqrt{7}$

$x = -\sqrt{7}$ or $x = \sqrt{7}$

$x + \sqrt{7}$ and $x - \sqrt{7}$ are factors

Hence the factors are $+\sqrt{2}$, $x - \sqrt{2}$, $x + \sqrt{7}$ and $x - \sqrt{7}$

Example 3.17

Given $f(x) = x^4 - 1$, find all the factors using the linear quadratic factor approach.

Solution

Let $u = x^2$

$f(x) = (x^2)^2 - 1$

$(u)^2 - 1$

$u^2 - 1$

Factorising:

$(u - 1)(u + 1)$

Substituting x^2 back in:

$(x^2 - 1)(x^2 + 1)$

$(x^2 - 1)(x^2 + 1) = 0$

$x^2 - 1 = 0$ or $x^2 + 1 = 0$

For $x^2 - 1 = 0$

$x^2 = \pm 1$

$x = 1$ or $x = -1$

For $(x^2 + 1)$

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm\sqrt{-1}$

NB: $\sqrt{-1} = \boldsymbol{i}$

$x = \pm i$

$x = -i$ or $x = i$

Hence the factors are $(x + 1)$, $(x - 1)$, $(x - i)(x + i)$

Example 3.18

Given $f(x) = 2x^4 + x^2 - 10$, find all the factors using the linear quadratic factor approach

Solution

Let $u = x^2$

$f(x) = 2(x^2)^2 + u - 10$

$2(u)^2 + u - 10$

$2u^2 + u - 10$

Factorising:

$2u^2 - 4u + 5u - 10$

$2u(u - 2) + 5(u - 2)$

$(2u + 5)(u - 2)$

Substituting:

$u = x^2$

$(2x^2 + 5)(x^2 - 2)$

$(2x^2 + 5)(x^2 - 2) = 0$

$2x^2 + 5 = 0$ or $x^2 - 2 = 0$

For $2x^2 + 5 = 0$

$x^2 = -\frac{5}{2}$

$x = \pm\sqrt{-\frac{5}{2}}$

$x = \frac{5}{2}i$ or $x = -\frac{5}{2}i$

For $(x^2 - 2)$

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$x = -\sqrt{2}$ or $x = \sqrt{2}$

Hence the factors are $(x + \frac{5}{2}i)\ (x - \frac{5}{2}i)\ \left(x - \sqrt{2}\right)\ \left(x + \sqrt{2}\right)$

We can use what we have learned to help in determining the maximum or minimum profit or productivity in the real world.

In small groups, or individually, discuss how the following problem can be solved.

Example 3.19

A water production company made sales of x bags of water and the profit was given by $2x^2 + 7x - 15$ where $0 \leq x \leq 10$.

Help the company determine their maximum profit and loss.

Solution

Since the range was given ($0 \leq x \leq 10$), we do our calculation within this range.

Evaluating the function in the given interval, we obtain:

x	0	1	2	3	4	5	6	7	8	9	10
$2x^2 + 7x - 15$	-15	-6	7	24	45	70	99	132	169	210	255

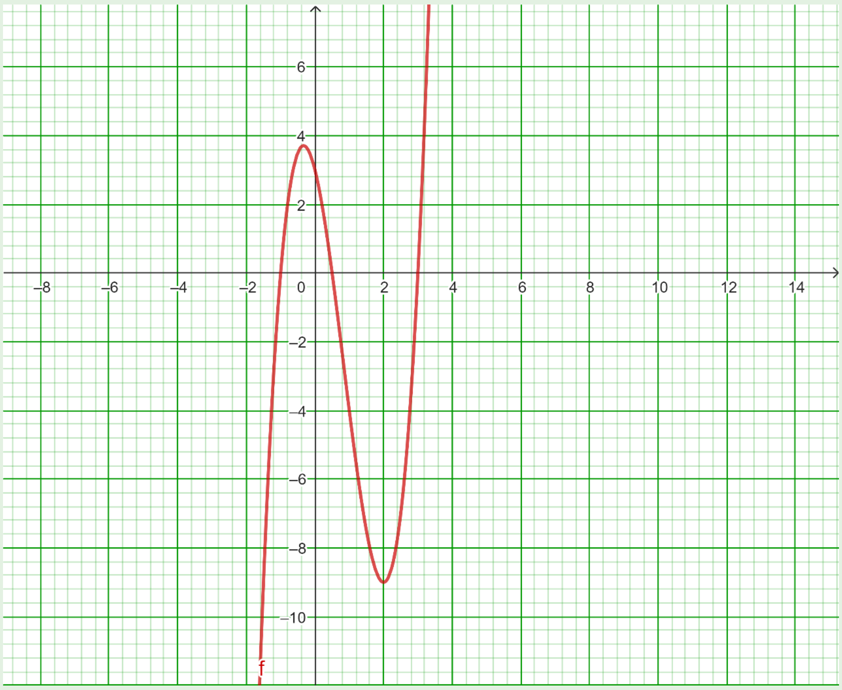
The maximum profit occurs when x=10 with a profit of 255 while the loss will occur when there is no production, i.e., $x = 0$ at a loss of 15 (-15).

EXTENDED READING

- Lial, M. L., Hornsby, E. J., & McGinnis, T. (2012). Algebra for college students. (7th Ed. Pearson Education, Inc). Pages 346 – 375.

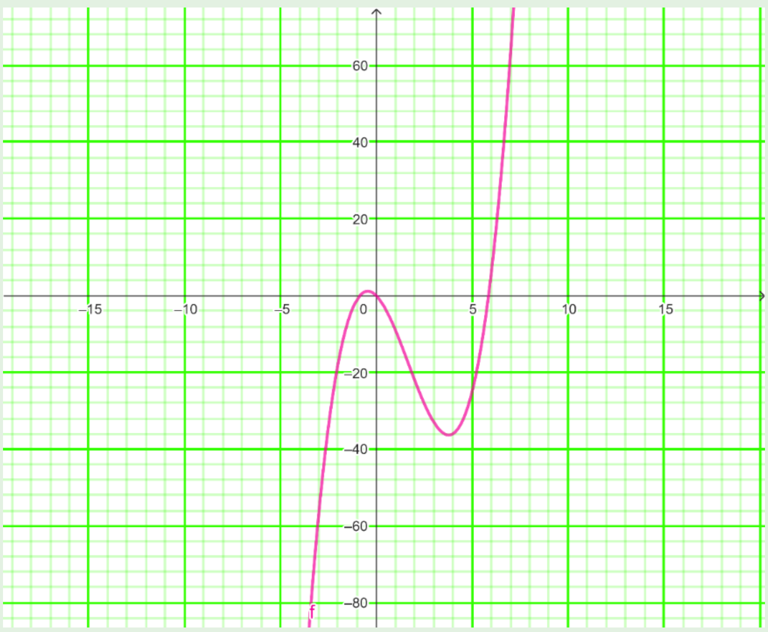
REVIEW QUESTIONS

1. Use the Rational Zero Theorem to find the rational zeros of:
   - a. $6x^3 - 13x^2 - 14x - 3$
   - b. $5x^3 + 2x^2 - 5x - 2$



2. Carefully, study the graph above which represents a polynomial function f(x).
   - a. State the zeros
   - b. Find the factors
   - c. State the interval on the curve.
   - d. Find the polynomial function f(x)
3. Sketch the following curves.
   - a. $f(x) = x^3 + 2x^2 - 9x - 18$
   - b. $f(x) = x^3 - 2x^2 - x + 2$
   - c. $f(x) = x^3 + 3x^2 - 4x - 12$
   - d. $f(x) = x^3 - 5x^2 - x + 5$
   - e. $f(x) = \frac{1}{5}(x-2)(x-4)(x+5)$

4. Factorise:
   a. $x^3 + 5x^2 + 2x - 8$
   b. $2x^3 + 3x^2 - 17x - 30$
   c. $4x^3 - 4x^2 - 11x + 6$
   d. $6x^3 + 13x^2 - 4$
   e. $3x^3 + 16x^2 - 13x - 6$
5. Find the zeros of the following:
   a. $x^3 + 4x^2 + x - 6$
   b. $2x^3 + 7x^2 - 14x + 5$
   c. $2x^3 + x^2 - 5x + 2$
   d. $4x^3 - 8x^2 - 9x + 18$
   e. $4x^3 - 4x^2 - x + 1$
   f. $4x^3 - 12x^2 + 11x - 3$
6. Find the roots of:
   a. $x^3 + 6x$
   b. $2x^4 + 17x^2 + 35$
   c. $x^4 - 14x^2 + 36$
   d. $x^4 - 1$
   e. $x^3 + 27x$



7. Find the polynomial function f(x) as represented by the curve above.

8. $Q(x) = 10x^3 + ax^2 - 10x + b$, where a and b are integers, is divisible by $(2x + 1)$. When $Q(x)$ is divided by $x + 1$, the remainder is -2.

    a. Find the values of a and of b

    b. find the expression for $Q(x)$ as a product of the three linear factors

9. Given that $x_1 = 1 + \sqrt{2}\,i$ and $x_2 = 2 + 3i$ are roots of a polynomial function $P(x)$, determine the $P(x) = 0$

10. Applying Descartes' Rule of Signs, identify the potential number of positive and negative roots.

    a. $f(x) = 10x^6 - 5x^5 + 3x^4 - 6x^3 + 24x^2 - 38x$

    b. $f(x) = 19x^5 + 6x^4 + 17x^3 + 18x^2 - 38x - 70$

    c. $f(x) = x^6 - 729$

    d. $f(x) = 12x^6 + 13x^3 + 1$

    e. $f(x) = 8x^8 - 13x^4 + 100$

11. Write down the equation having only the following roots.

    a. -5 , -1, 3

    b. $3, -\frac{1}{5}, -\frac{1}{3}$

    c. $2, -2, 2 + \sqrt{3}, 2 - \sqrt{3}$

    d. 0, 1 + 4i, 1- 4i

12. Find the four roots of $y^4 + 3x^2 + 2$

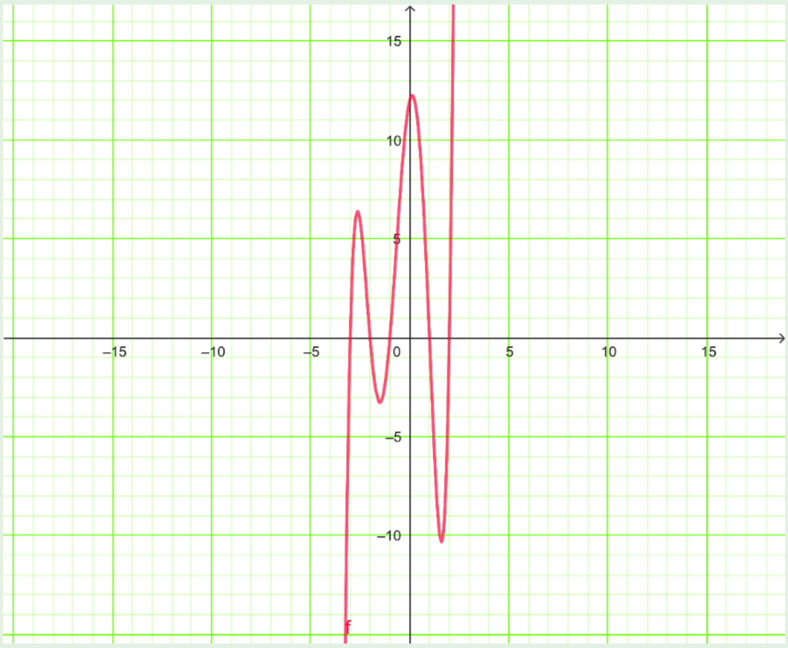
13. Study the graph below and use it to answer the following questions:

    a. state the factors

    b. find the polynomial function

    c. how many turning points

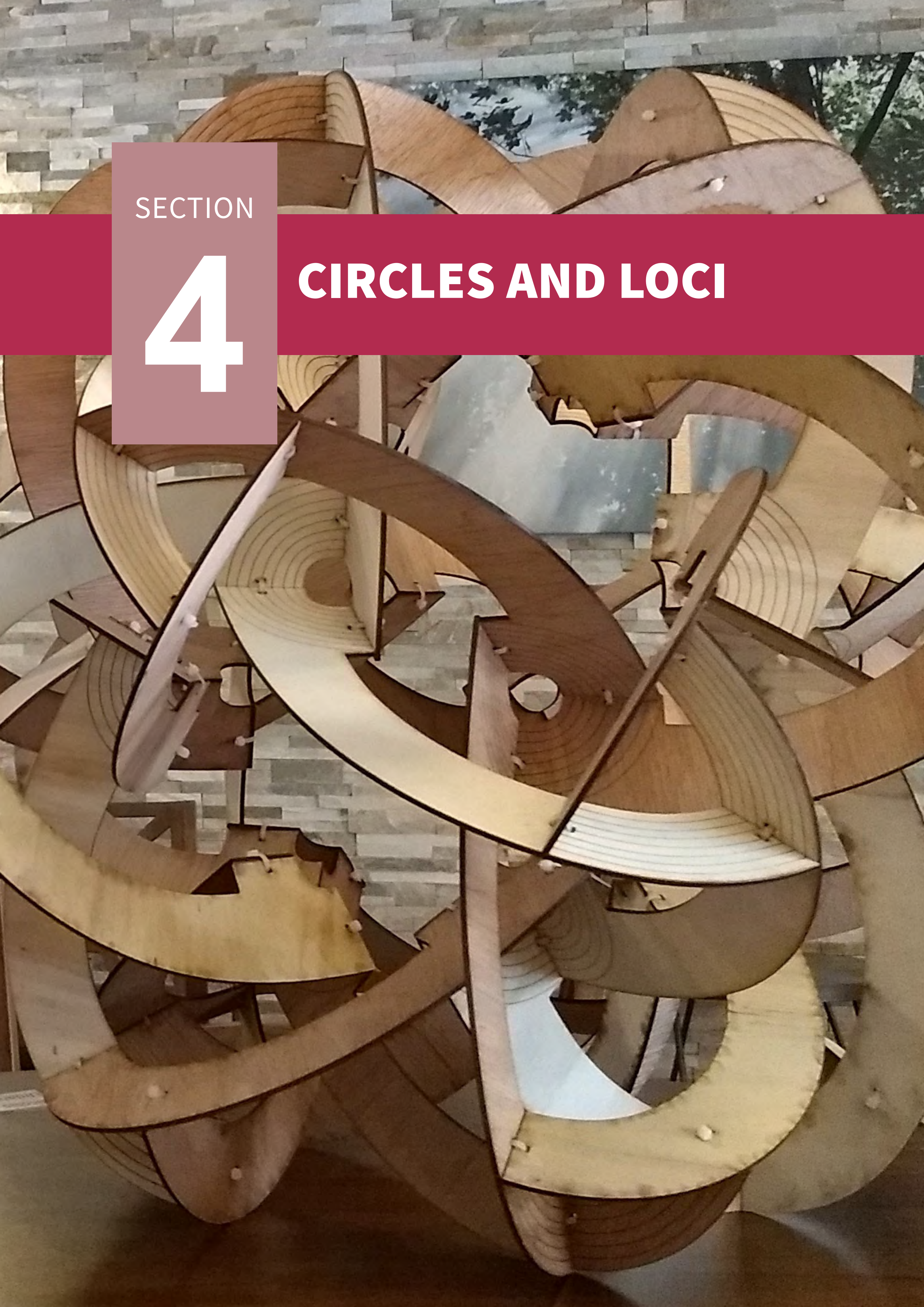
    d. what is the relationship between the degree of the function and the number of turning points



14. A new bakery in Accra specialises in creating various types of bread. The bakery wants the volume of a small rectangular loaf of bread to be 4 800 cm^3. The loaf is to be shaped like a rectangular solid. The length of the loaf is to be 4cm longer than the width. The height of the loaf is one-half of the width. Determine the dimensions of the loaf.

SECTION

4 CIRCLES AND LOCI



GEOMETRIC REASONING AND MEASUREMENT

Spatial Sense

INTRODUCTION

In this section, you will learn how to derive the circle equation. You will also learn how to manipulate this equation under given conditions and apply it to solve real-life problems. The content of this section is imperative as it forms the foundation on which other advanced topics in mathematics are built. Its real-life applications are vast so why not delve further through your own research.

KEY IDEAS

- A circle is a path traced by all points in a plane which are equidistant from a fixed point in the plane. The fixed point is called the centre.
- The standard equation of a circle with centre (h, k) and radius r is given as:
- $(x - h)^2 + (y - k)^2 = r^2$
- The general equation of a circle with centre $(-g, -f)$ and radius r is given as:
- $x^2 + y^2 + 2gx + 2fy + c = 0$

EXPLORING THE PROPERTIES OF A CIRCLE AND ITS PARTS

A Circle

The circle is the path traced by all points in a plane which are equidistant from a fixed point in the plane. The fixed point is called the centre.

Let us use the activity below to revise the parts of the circle. You will need a graph sheet and a set of mathematical instruments to do this activity.

Activity 4.1- Parts of a circle

Step 1: Make a copy of the graph below.

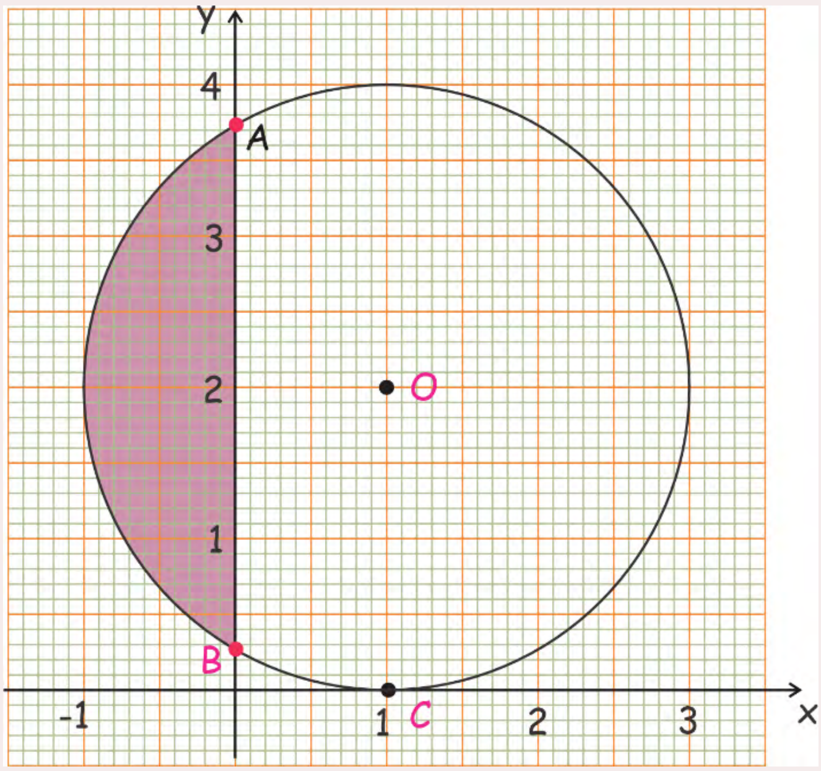


Figure 4.1: A circle

Step 2: Calculate |OA|, |OB| and |OC|

Step 3: Comment on any observation from your result in Step 2

Step 4: Aside from points A, B and C, list the coordinates of any two points on the circle's circumference. Find the distance between these points and point O. Comment on your observations.

Step 5: Join points O and C with a straight line.

Measure the angle between OC and the x–axis.

Let us use our diagram to revise the parts of the circle.

Lines OA, OB and OC are the radii of the circle. From the activity, |OA|=|OB|=|OC|.

Line AB is an example of a chord. Any straight line that connects any two points on the circumference of the circle is a chord.

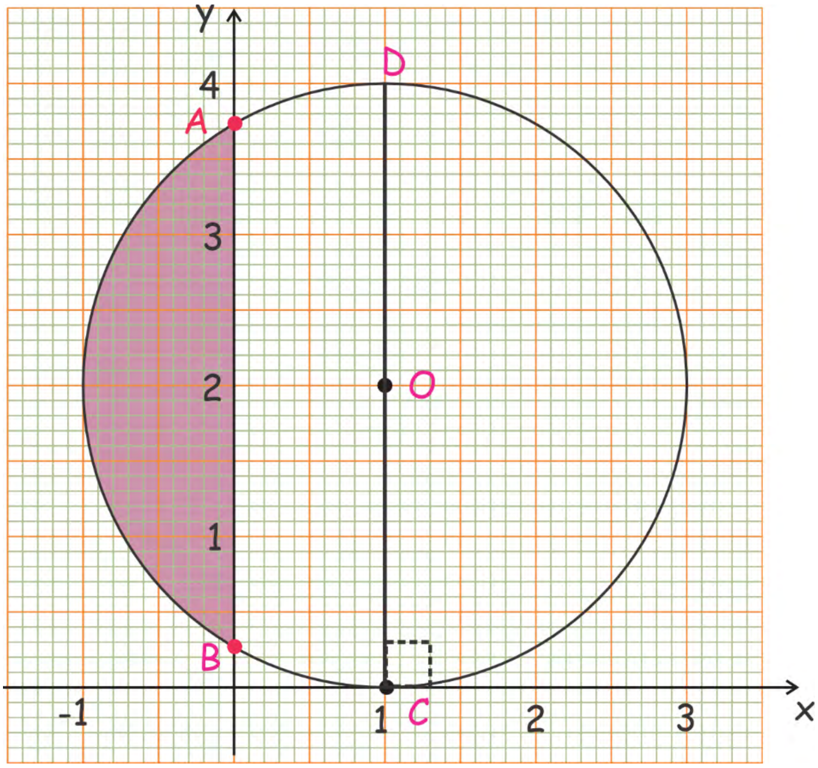


Figure 4.2: A circle

The diameter is an example of a chord. It is the longest chord. The diameter is a straight line from a point on the circumference through the centre to another point on the circumference of the circle. In the above diagram, line COD is a diameter.

The x−axis is tangent to the circle at point C(1, 0). The tangent touches the circle at one point. The angle between the tangent and the radius or the diameter at the point of tangency is 90°. In the diagram OCX = 90°

The area, shaded red, between the arc AB and the line AB is called a segment.

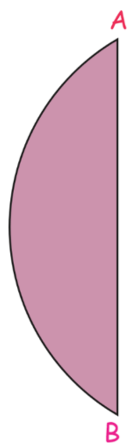


Figure 4.3: A segment

The area bounded by the arc AB and the lines OA and OB is called a sector.

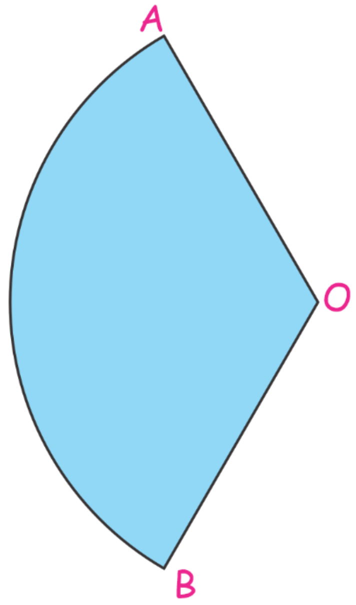


Figure 4.4: A sector

Example 4.1

The diagram below shows the graph of a circle. Use it to answer the questions.

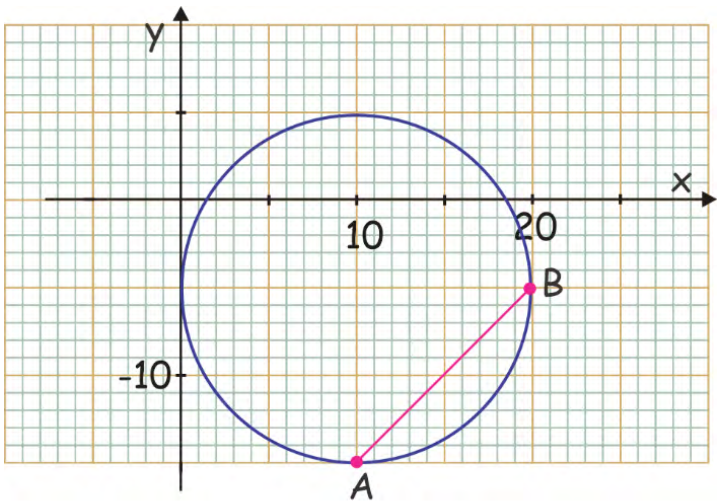


Figure 4.5: Graph of a circle

a. State the coordinates of the centre.

b. Find the radius of the circle

c. Find the length of chord AB

Solution

a. Coordinate of the centre is (10, -5)

b. The radius of the circle is 10 units

c. The coordinate of point A is (10, -15) and that of B is (20, -5)

d. $|AB| = \sqrt{(20-10)^2 + (-5—15)^2}$

e. $|AB| = \sqrt{10^2 + 10^2}$

f. $= \sqrt{100 + 100}$

g. $= \sqrt{200}$

h. $= \sqrt{100 \times 2}$

i. $= \sqrt{100} \times \sqrt{2}$

j. $= 10\sqrt{2}$ units.

DERIVING THE EQUATION OF A CIRCLE

The equation of a circle gives information about the centre and radius of the circle.

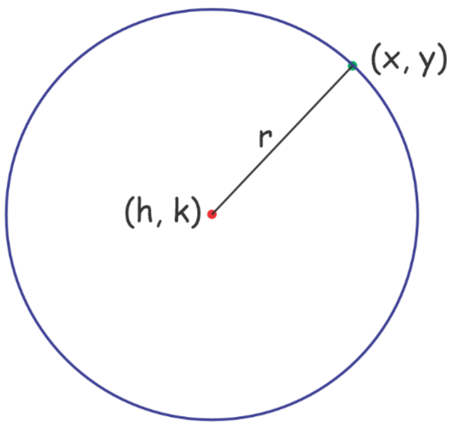


Figure 4.6: A circle

The equation of a circle with centre (h, k) and radius r is given as:
$(x-h)^2 + (y-k)^2 = r^2$, where (x, y) is any point on the circumference of the circle.

How did we come by this formula? The following activity will demonstrate.

Activity 4.2: Standard Equation of a Circle

Step 1: Make a copy of the graph below. (a, b) is the centre of the circle and (x, y) is any point on the circumference of the circle.

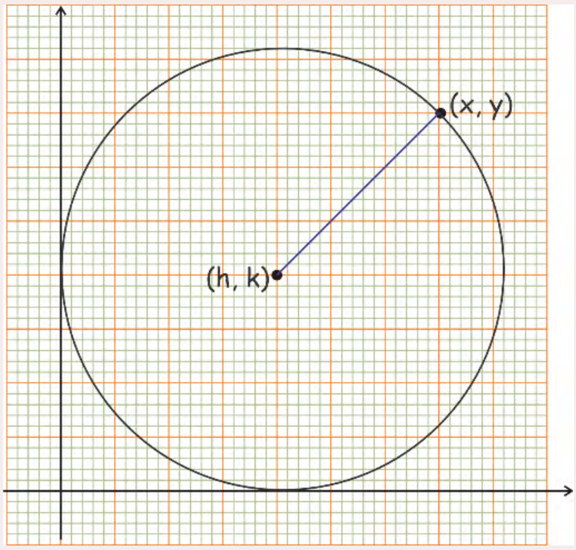


Figure 4.7: Graph of a circle

Step 2: Using the radius as the hypotenuse, draw a right-angle triangle. Find its horizontal and vertical distance.

Step 3: Using the Pythagoras theorem, write an equation to connect the lengths of sides of the triangle.

Your answer should be similar to the one below:

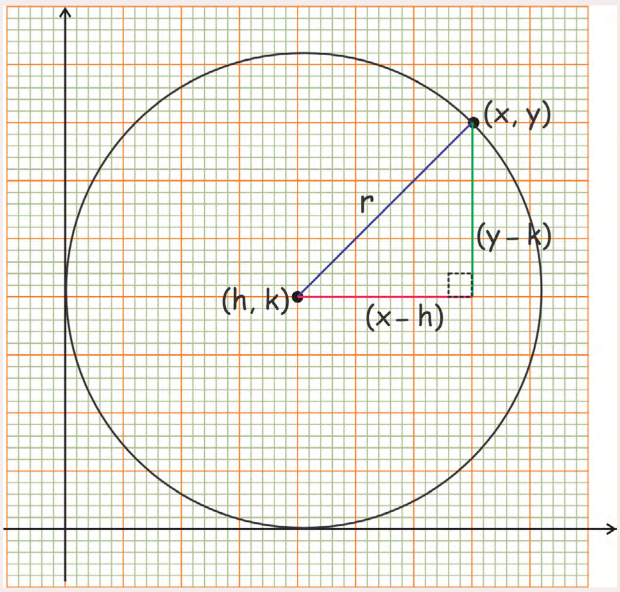


Figure 4.8: Graph of a circle

From the figure, we have a right-angle triangle with hypotenuse, r, and the lengths of the other sides as $(x - h)$ and $(y - k)$

Applying Pythagoras theorem, this gives:

$$(\boldsymbol{x} - \boldsymbol{h})^2 + (\boldsymbol{y} - \boldsymbol{k})^2 = \boldsymbol{r}^2$$

Alternatively, you can find the distance between the centre (h, k) and any point on the circumference (x, y) and you will achieve the same result

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

Square both sides of the equation:

$$\left(r\right)^2 = \left[\sqrt{(x - h)^2 + (y - k)^2}\right]^2 \quad r^2 = (x - h)^2 + (y - k)^2$$

Rearranging it, we have:

$$(\boldsymbol{x} - \boldsymbol{h})^2 + (\boldsymbol{y} - \boldsymbol{k})^2 = \boldsymbol{r}^2$$

This gives the standard equation of a circle.

Let us modify this equation for a few special scenarios.

Scenario 1: When the centre of the circle is the origin (0, 0).

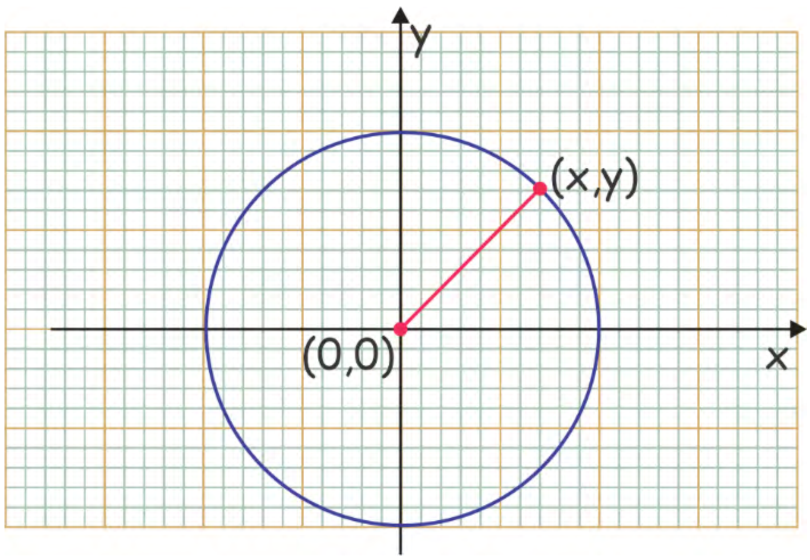


Figure 4.9: Graph of a circle

When $(h, k) = (0, 0)$, the equation simplifies to:

$$(x - 0)^2 + (y - 0)^2 = r^2$$

$$x^2 + y^2 = r^2$$

Scenario 2: When the centre is any point on the x-axis other than (0, 0). Remember that on the x-axis, y = 0.

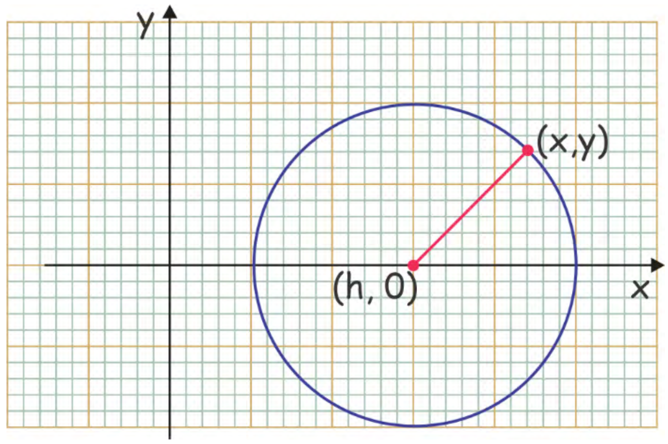


Figure 4.10: Graph of a circle

So, we will use $(h, k) = (h, 0)$

$(x - h)^2 + (y - 0)^2 = r^2$

$(x - h)^2 + y^2 = r^2$

Scenario 3: When the centre is any point on the y-axis other than (o, 0). Remember that on the y-axis, x=0.

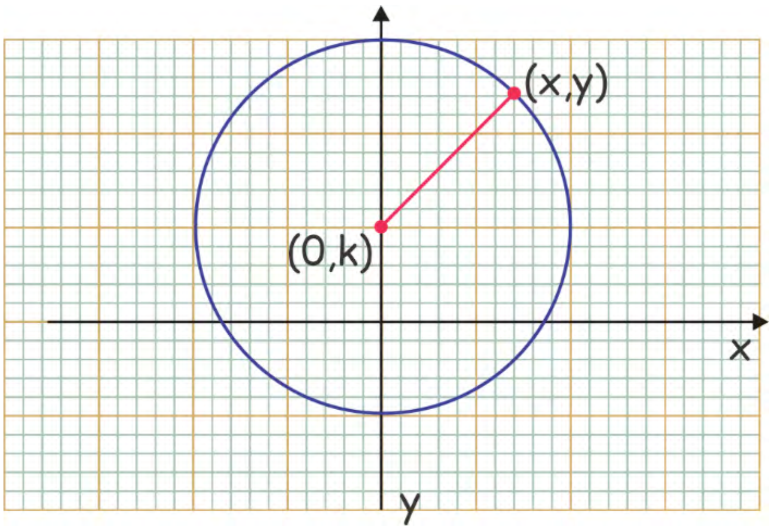


Figure 4.11: Graph of a circle

So, we will use $(h, k) = (0, k)$

$(x - 0)^2 + (y - k)^2 = r^2$

$x^2 + (y - k)^2 = r^2$

Example 4. 2

Write the standard form of the circle equation with the following properties:

a. Centre (1, 7) and radius 5

b. Centre (3, 0) and radius $\frac{2}{3}$

c. Centre (0, -5) and radius 2

d. Centre (-6, -7) and radius 1

Solution

The standard equation of a circle is given by $(x-h)^2+(y-k)^2=r^2$

a. Centre (1, 7) and radius 5

Using the standard equation, replace h with 1, k with 7 and r with 5

$(x-1)^2+(y-7)^2=5^2$

$(x-1)^2+(y-7)^2=25$

b. Centre (3, 0) and radius $\frac{2}{3}$

c. using the standard equation, replace h with 3, k with 0 and r with $\frac{2}{3}$

d. $(x-3)^2+(y-0)^2=\left(\frac{2}{3}\right)^2$

e. $(x-3)^2+y^2=\frac{4}{9}$

f. Centre (0, -5) and radius 2

g. $(x-0)^2+(y-(-5))^2=2^2$

h. $x^2+(y+5)^2=4$

i. Centre (-6, -7) and radius 1

$(x-(-6))^2+(y-(-7))^2=1^2$

$(x+6)^2+(y+7)^2=1$

Example 4.3

1. The diagram shows the graph of circles A, B and C. Use them to answer the following questions.

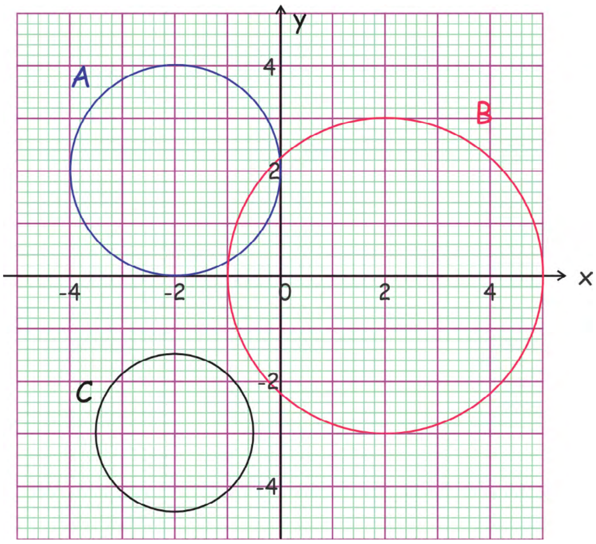


Figure 4.12: Graph of circles A, B and C

a. Find the diameter of circle:

i. A

ii. B

iii. C

b. Write in standard form the equation of:

i. A

ii. B

iii. C

2. A phone's WI-FI has a range of 10m in all directions. If the phone is 2m West and 5m North, write an equation to represent the area within which the phone's Wi-Fi can operate.

Solution

1. a.

i. Diameter of A= 4 units

ii. Diameter of B= 6 units

iii. Diameter of C= 3 units

b.

i. The coordinates of the centre of circle A is $(-2, 2)$.

Since the diameter is 4 units, it means the radius $= \frac{1}{2} \times 4 = 2$

So, the equation of circle A is: $(x-(-2))^2 + (y-2)^2 = 2^2$

$(x+2)^2 + (y-2)^2 = 4$

ii. The coordinates of the centre of circle B is $(2, 0)$.

Since the diameter is 6 units, it means the radius $= \frac{1}{2} \times 6 = 3$

So, the equation of circle B is: $(x-2)^2 + (y-0)^2 = 3^2$

$(x-2)^2 + y^2 = 9$

iii. The coordinates of the centre of circle C is $(-2, -3)$.

Since the diameter is 3 units, it means the radius $= \frac{1}{2} \times 3 = \frac{3}{2}$

So, the equation of circle C is: $(x-(-2))^2 + (y-(-3))^2 = \left(\frac{3}{2}\right)^2$

$(x+2)^2 + (y+3)^2 = \frac{9}{4}$

2.

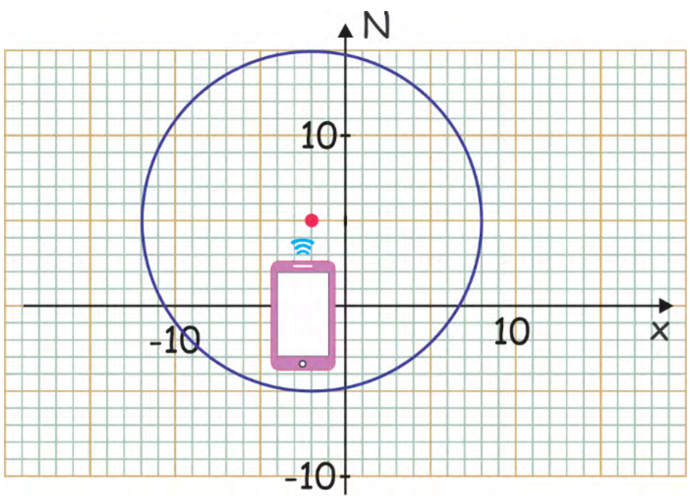


Figure 4.13: Graph of the range of the wi-fi signal

The range of the wi-fi signal is in the form of a circle with a radius of 10.

2 units west and 5 units north are the same as the coordinates $(-2, 5)$. This point gives the centre of the wi-fi signal.

The equation of the circle representing the range of the phone's wi-fi signals is:

$(x-(-2))^2 + (y-5)^2 = 10^2$

$(x+2)^2 + (y-5)^2 = 100$

Example 4.4

A Trotro driver operates within 40km at all angles from a lorry station. The lorry station is located 15km south and 8km east of the driver's house.

a. Write an equation to represent the driver's travel boundary from the lorry station using his house as the origin.

b. Find the furthest distance between the driver's house and his travel boundary.

c. Find the shortest distance between the driver's house and his travel boundary.

Solution

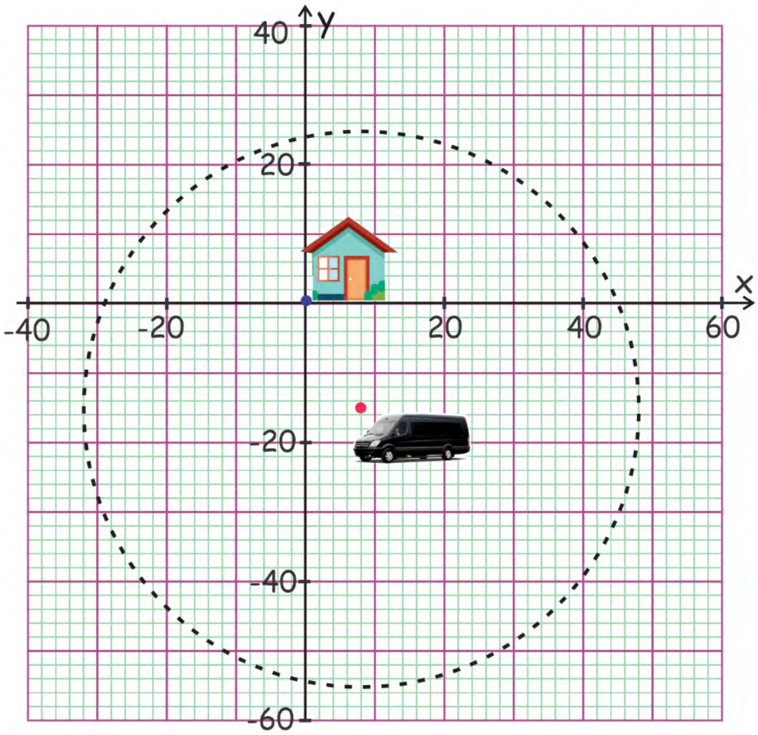


Figure 4.14: Graph of the lorry station and the driver's house

a. The driver's travel boundary is in the form of a circle with a radius of 40km. The centre is the lorry station. From his house, the lorry station is 15km south which represents $y = -15$ and 8km east which represents $x = 8$

b. Thus, the centre = $(8, -15)$

The equation of the driver's travel boundary is:

$(x - 8)^2 + (y - (-15))^2 = 40^2$

$(x - 8)^2 + (y + 15)^2 = 40^2$

c. We first find the distance between the driver's house and the lorry station.

Distance $= \sqrt{8^2 + (-15)^2}$

$= \sqrt{64 + 225}$

$= \sqrt{289} = 17\text{km}$

Since his travel distance is 40km, it means the furthest the driver travels from his house is 17km +40km = **57km**

The shortest distance between the driver's house and his travel boundary is

40 − 17 = 23 **km**

The general equation of a circle

The general equation of a circle is: $x^2 + y^2 + 2gx + 2fy + c = 0$

To derive this formula, we will modify the standard equation.

We are going to replace the centre (h, k) with $(-g, -f)$.

Recall that the standard equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$ where (h, k) is the centre and r is the radius.

Replacing (h, k) with $(-g, -f)$ we have:

$(x - (-g))^2 + (y - (-f))^2 = r^2$

$(x + g)^2 + (y + f)^2 = r^2$

Expanding the resulting equation, we get:

$(x + g)(x + g) + (y + f)(y + f) = r^2$

$x^2 + gx + gx + g^2 + y^2 + fy + fy + f^2 = r^2$

$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = r^2$

$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 - r^2 = 0$

Since $g^2 + f^2 - r^2$ will simplify into a constant, replace it with C

This gives,

$\boldsymbol{x^2 + y^2 + 2gx + 2fy + c} = 0$

Note that when the equation is written in the general form,

the centre $= (-g, -f) = \left(\frac{2g}{-2}, \frac{2f}{-2}\right)$

$= \left(\frac{\text{Coefficient of } x}{-2}, \frac{\text{Coefficient of } y}{-2}\right)$

$= -\frac{1}{2}(\text{Coefficient of } x, \text{Coefficient of } y)$

Also, $C = g^2 + f^2 - r^2$. Solving for r, we have

$r^2 = g^2 + f^2 - c$

$\sqrt{r^2} = \sqrt{g^2 + f^2 - c}$

$\boldsymbol{r = \sqrt{g^2 + f^2 - c}}$

Example 4.5

Write the general equation of a circle with:

a. Centre (1, 3) and radius 2.

b. Centre (− 2, 4) and radius $\frac{1}{2}$

Solution

Start from the standard equation and work through it to get the general equation.

a. Centre (1, 3) and radius 2.

$(x - 1)^2 + (y - 3)^2 = 2^2$

$(x - 1)(x - 1) + (y - 3)(y - 3) = 4$

$x^2 - 2x + 1 + y^2 - 6y + 9 = 4$

$x^2 + y^2 - 2x - 6y + 9 + 1 - 4 = 0$

$x^2 + y^2 - 2x - 6y + 6 = 0$

Alternatively, we can solve for the variables in the general equation and then substitute the answer into the general equation.

$x^2 + y^2 + 2gx + 2fy + c = 0$

The centre is $= (-g, -f) = (1, 3)$

$\Longrightarrow g = -1$ and $f = -3$

$C = (-1)^2 + (-3)^2 - 2^2$

$C = 1 + 9 - 4$

$= 6$

Substituting these values into the general equation gives:

$x^2 + y^2 + 2(-1)x + 2(-3)y + 6 = 0$

$x^2 + y^2 - 2x - 6y + 6 = 0$

b. Centre (−2, 4) and radius $\frac{1}{2}$

$(x - (-2))^2 + (y - 4)^2 = \left(\frac{1}{2}\right)^2$

$(x + 2)(x + 2) + (y - 4)(y - 4) = \frac{1}{4}$

$x^2 + 4x + 4 + y^2 - 8y + 16 = \frac{1}{4}$

$x^2 + y^2 + 4x - 8y + 20 - \frac{1}{4} = 0$

$x^2 + y^2 + 4x - 8y + \frac{79}{4} = 0$

Multiply through by 4, so we only have integer values:

$4x^2 + 4y^2 + 16x - 32y + 79 = 0$

Alternatively, we can solve for the variables in the general equation and then substitute the answer into the general equation.

$x^2 + y^2 + 2gx + 2fy + c = 0$

The centre is $(-g, -f) = (-2, 4)$

$\Longrightarrow g = 2$ and $f = -4$

$C = (2)^2 + (-4)^2 - \left(\frac{1}{2}\right)^2$

$C = 4 + 16 - \frac{1}{4}$

$C = 20 - \frac{1}{4} = \frac{79}{4}$

Substituting these values into the general equation gives:

$x^2 + y^2 + 2(2)x + 2(-4)y + \frac{79}{4} = 0$

$x^2 + y^2 + 4x - 8y + \frac{79}{4} = 0$

Multiply through by 4

$4\boldsymbol{x}^2 + 4\boldsymbol{y}^2 + 16\boldsymbol{x} - 32\boldsymbol{y} + 79 = 0$

Example 4.6

Find the centre and radius of the following circle equations

a. $x^2 + y^2 - 4x - 2y - 4 = 0$

b. $x^2 + y^2 - 8x + 6y = 0$

c. $x^2 + y^2 + 6x - 40 = 0$

d. $4x^2 + 4y^2 - 8x + 3 = 0$

Solution

a. $x^2 + y^2 - 4x - 2y - 4 = 0$

Method 1: Using completing the square method

$x^2 - 4x + y^2 - 2y = 4$ group corresponding terms

$(x^2 - 2x - 2x + 4) + (y^2 - y - y + 1) = 4 + 4 + 1$

Expand and complete the square

$(x - 2)^2 + (y - 1)^2 = 9$

$(x - 2)^2 + (y - 1)^2 = 3^2$

This shows that the centre is (2, 1) and the radius is 3

Method 2: Alternatively, compare the given equation to the general equation of the circle and solve for the centre and the radius.

The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

The given equation is $x^2 + y^2 - 4x - 2y - 4 = 0$

$x^2 + y^2 + 2(-2)x + 2(-1)y + (-4) = 0$

By comparing coefficients, we have $g = -2 \Longrightarrow -g = 2$

Also, $f = -1 \Longrightarrow -f = 1$ and $c = -4$

Recall that when the circle equation is written in the general form, the centre is $(-g, -f)$

Therefore, the centre is (2, 1).

You will achieve the same result if you divide the coefficient of x and y by -2

Centre $= (-4 \_ -2, -2 \_ -2) = (2, 1)$

Radius $= r = \sqrt{g^2 + f^2 - c}$

$r = \sqrt{2^2 + 1^2 - (-4)} = \sqrt{9} = 3$

b. $x^2 + y^2 - 8x + 6y = 0$

Method 1: Using the completing the square method, we have:

$x^2 - 8x + 16 + y^2 + 6y + 9 = 16 + 9$

$(x - 4)^2 + (y + 3)^2 = 25$

$(x - 4)^2 + (y + 3)^2 = 5^2$

Therefore, the centre is (4, -3) and the radius is 5

Method 2: Alternatively, we can re-write the equation as

$x^2 + y^2 + 2(-4)x + 2(3)y + 0 = 0$

Comparing this equation to $x^2 + y^2 + 2gx + 2fy + c = 0$, we have:

$g = -4 \Longrightarrow -g = 4$, $f = 3 \Longrightarrow -f = -3$ and C=0

Therefore, the centre is (4, -3)

Radius $= r = \sqrt{(-4)^2 + 3^2 - 0} = \sqrt{25} = 5$

c. $x^2 + y^2 + 6x - 40 = 0$

Method 1: Completing the square, we have:

$x^2 + 6x + 9 + y^2 + 2(0)x + 0 = 40 + 9 + 0$

$(x + 3)^2 + (y + 0)^2 = 49$

$(x + 3)^2 + (y + 0)^2 = 7^2$

Centre is (-3, 0) and radius is 7

Method 2: Alternatively, we can re-write the equation as

$x^2 + y^2 + 2(3)x + 2(0)y - 40 = 0$

Comparing this equation to $x^2 + y^2 + 2gx + 2fy + c = 0$, we have:

$g = 3 \Longrightarrow -g = -3$, $f = 0 \Longrightarrow -f = 0$ and C= -40

Therefore, the centre is (-3, 0)

Radius $= r = \sqrt{(3)^2 + 0^2 - (-40)} = \sqrt{49} = 7$

d. $4x^2 + 4y^2 - 8x + 3 = 0$

Method 1: Completing the square, we will first divide the equation by 4

$x^2 + y^2 - 2x + \frac{3}{4} = 0$ $x^2 - 2x + 1 + y^2 + 0 = -\frac{3}{4} + 1 + 0$

$(x - 1)^2 + (y + 0)^2 = 1 \_ 4$

$\left(x - 1\right)^2 + \left(y + 0\right)^2 = \left(\frac{1}{2}\right)^2$

Centre is (1, 0) and radius is $\frac{1}{2}$

Method 2: Alternatively, we can re-write the equation as:

$x^2 + y^2 + 2(-1)x + 2(0)y + \frac{3}{4} = 0$

Comparing this equation to $x^2 + y^2 + 2gx + 2fy + c = 0$, we have:

$g = -1 \Longrightarrow -g = 1$, $f = 0 \Longrightarrow -f = 0$ and C= $\frac{3}{4}$

Therefore, the centre is (1, 0)

Radius $= r = \sqrt{(-1)^2 + 0^2 - \frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

DERIVING THE EQUATIONS OF A CIRCLE FROM GIVEN POINTS

How to Find the Equation of the Circle Given the Endpoints of the Diameter

Method 1:

Step 1: Find the centre of the circle by finding the mid-point of the diameter.

Step 2: Calculate the length of the radius. This is half the diameter and is the same as the distance between the centre and any of the endpoints of the diameter.

Step 3: Substitute the centre and the radius into the standard equation of the circle.

Step 4: Simplify the resulting equation to achieve the desired result.

Method 2:

An alternative method involves using the relationship between gradients of two perpendicular lines.

Recall that if m_1 and m_2 are gradients of any two lines which intersect at right angles, then $m_1 \times m_2 = -1$

Also, if one side of a triangle inscribed in a circle is a diameter, then the triangle is right-angled. The angle opposite the diameter is the right angle. This property of circles is illustrated in the diagram below.

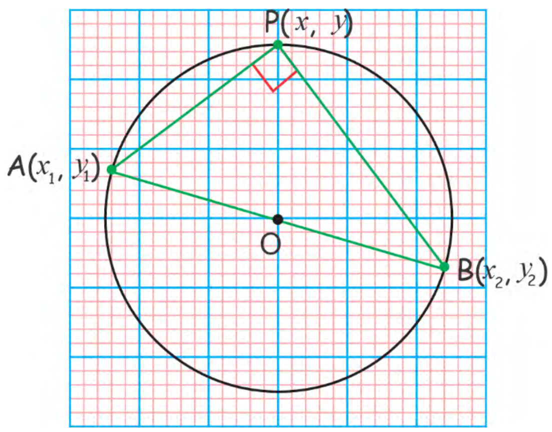


Figure 4.15: Graph of circle with centre O

From the figure, $A(x_1, y_1)$ and $B(x_2, y_2)$ are the endpoints of the diameter and $P(x, y)$ is any point on the circumference of the circle. The angle opposite the diameter is AOB= 90°. This means that the product to the gradients of lines AP and BP is equal to -1. Using this relation will generate the equation of the circle.

Gradient (m_1) of |AP| $= \frac{y - y_1}{x - x_1}$

Gradient (m_2) of |BP| $= \frac{y - y_2}{x - x_2}$

Using $m_1 \times m_2 = -1$, we have

$\frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$

Example 4.7

The endpoints of the diameter of a circle are (-2, 2) and (4, -6).

Find the;

a. radius and centre of the circle

b. general equation of the circle.

Solution

The graph below shows a sketch of the circle.

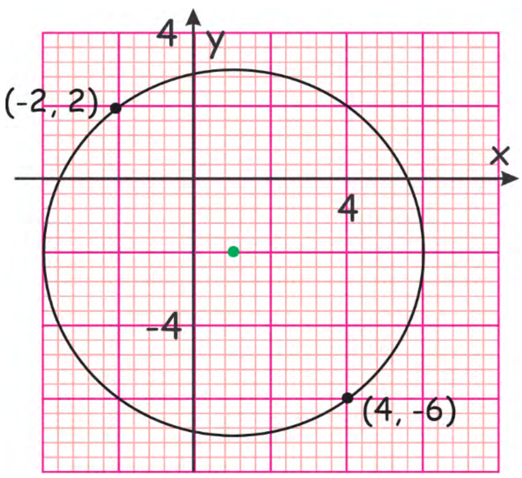


Figure 4.16: Graph of circle with diameter (-2, 2) and (4, -6)

Method 1:

1. The centre is equal to the mid-point of the endpoints of the diameter

2. Midpoint $= \frac{1}{2}(-2 + 4, 2 + (-6))$

3. $= \frac{1}{2}(2, -4)$

4. $= \left(\frac{2}{2}, -\frac{4}{2}\right) = (1, -2)$

Radius is the distance between the centre and any of the endpoint/point on the circumference.

$r = \sqrt{(-6-(-2))^2+(4-1)^2}$

$r = \sqrt{(-4)^2+3^2}$

$r = \sqrt{16+9}$

$= \sqrt{25} = 5$

Using $(x-h)^2+(y-k)^2=r^2$, we have:

$(x-1)^2+(y-(-2))^2=5^2$

$(x-1)^2+(y+2)^2=25$

$x^2-2x+1+y^2+4y+4=25$

$x^2+y^2-2x+4y+5=25$

$x^2+y^2-2x+4y+5-25=0$

$\mathbf{x^2+y^2-2x+4y-20=0}$

Method 2:

Alternatively, let any point on the circumference of the circle be P(x, y) as shown in the figure below.

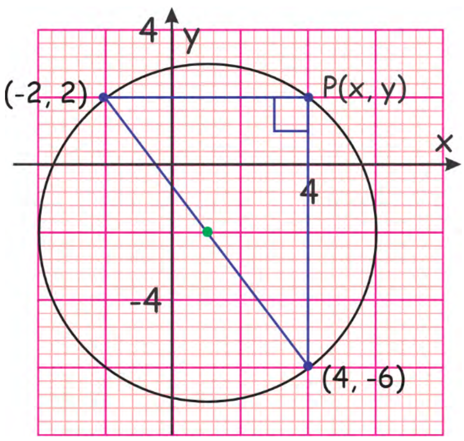


Figure 4.17: Graph of circle with diameter (-2, 2) and (4, -6)

The gradient of the line connecting (-2, 2) and $(x, y) = \frac{-2}{x-(-2)} = \frac{y-2}{x+2}$

The gradient of the line connecting (4, -6) and $(x, y) = \frac{y-(-6)}{x-4} = \frac{y+6}{x-4}$

Since the two lines intersect at 90°,

$\left(\frac{y-2}{x+2}\right) \times \left(\frac{y+6}{x-4}\right) = -1$

$$\frac{(y-2)(y+6)}{(x+2)(x-4)} = -1$$

$$\frac{y^2+6y-2y-12}{x^2-4x+2x-8} = -1$$

$$\frac{y^2+4y-12}{x^2-2x-8} = -\frac{1}{1}$$

Cross multiply:

$y^2 + 4y - 12 = -1(x^2 - 2x - 8)$

$y^2 + 4y - 12 = -x^2 + 2x + 8$

Regroup the terms:

$y^2 + x^2 - 2x + 4y - 12 - 8 = 0$

$\mathbf{y^2 + x^2 - 2x + 4y - 20 = 0}$

Equation of a Circle Given Three Points on the Circumference of the Circle

Method 1:

Substitute each of the points into the general equation of the circle $(x^2 + y^2 + 2gx + 2fy + c = 0)$.

This results in 3 simultaneous equations with 3 unknowns to be solved for g, f and c.

Method 2:

Alternatively, follow the following steps.

Step 1: Find the equation of the perpendicular bisectors of any two of the chords.

In the Figure 4.18 A, B and C are points on the circumference of the circle.

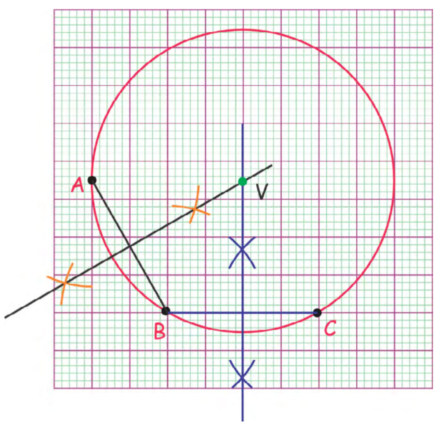


Figure 4.18: Graph of circle with equation $(x^2 + y^2 + 2gx + 2fy + c = 0)$

Step 2: Solve the equations obtained in step 1 simultaneously. This will give the coordinates of the centre of the circle.

Step 3: Using the centre and any point on the circumference, calculate the radius.

Step 4: Use the radius and the centre to calculate the equation of the circle.

Example 4.8

Find the equation of the circle which satisfies the points (2, -4), (-6, 4) and (2, 12).

Solution

Method 1:

In the diagram below, A, B and C are points on the circumference of the circle. The perpendicular bisectors of chords AB and BC intersect at point V. Point V is the centre of the circle.

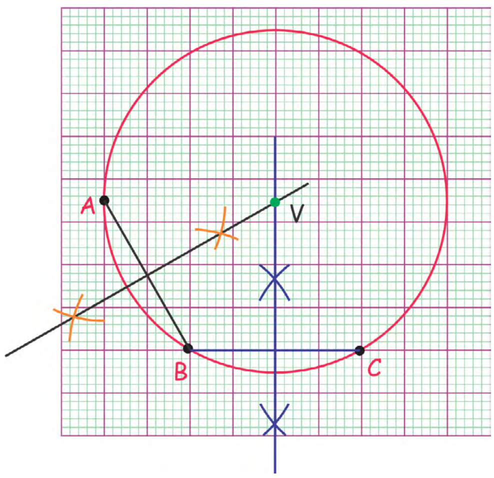


Figure 4.19: Graph of circle with points (2, -4), (-6, 4) and (2, 12)

Substitute the points into the general equation of the circle.

Substituting (-6, 4) into $x^2 + y^2 + 2gx + 2fy + c = 0$ gives:

$(-6)^2 + 4^2 + 2g(-6) + 2f(4) + c = 0$

$36 + 16 - 12g + 8f + c = 0$

$-12g + 8f + c = -52$ ……………………..(1)

Substituting (2, 12) into the same equation:

$(2)^2 + (12)^2 + 2g(2) + 2f(12) + c = 0$

$4 + 144 + 4g + 24f + c = 0$

$4g + 24f + c = -148$ ....(2)

Substituting (2, -4) into the same equation:

$(2)^2 + (-4)^2 + 2g(2) + 2f(-4) + c = 0$

$4 + 16 + 4g - 8f + c = 0$

$4g - 8f + c = -20$ .... (3)

Subtract (2) from (3). That is (3) – (2)

$(4g - 8f + c) - (4g + 24f + c) = -20 - (-148)$

$4g - 8f + c - 4g - 24f - c = -20 + 148$

$-32f = 128$

$f = -\frac{128}{32} = -4$

Subtract (1) from (2):

$(4g + 24f + c) - (-12g + 8f + c) = -148 - (-52)$

$4g + 24f + c + 12g - 8f - c = -148 + 52$

$16g + 16f = -96$

$16g + 16(-4) = -96$

$16g - 64 = -96$

$16g = -96 + 64$

$\frac{16g}{16} = -\frac{32}{16}$

$g = -2$

Substitute $g = -2$ and $f = -4$ into (3):

$4(-2) - 8(-4) + c = -20$

$-8 + 32 + c = -20$

$c = -20 + 8 - 32$

$c = -44$

Finally, substitute g, f and c into the general equation.

$x^2 + y^2 + 2(-2)x + 2(-4)y - 44 = 0$

$\boldsymbol{x^2 + y^2 - 4x - 8y - 44 = 0}$ gives the equation of the circle which satisfies (2, −4), (− 6, 4) and (2, 12).

Method 2:

First plot the points on the x-y plane and connect the points with chords.

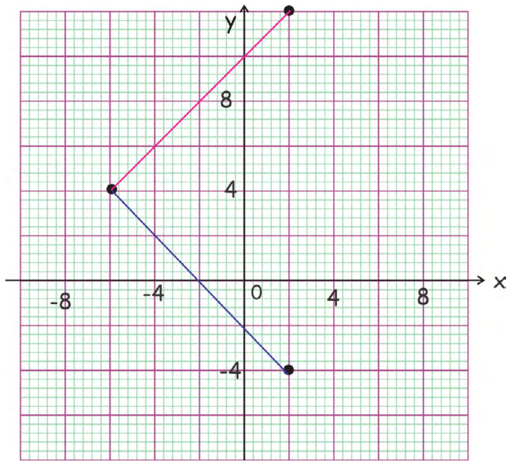


Figure 4.20: Graph of the *x-y* plane

Mid-point of $(-6, 4)$ and $(2, 12)$ is $\left(\frac{-6+2}{2}, \frac{4+12}{2}\right) = \left(-2, 8\right)$

The gradient of the line connecting the points $(-6, 4)$ and $(2, 12)$ is

$\frac{12-4}{2-(-6)} = \frac{8}{8} = 1$

The gradient of the perpendicular bisector is -1

The equation of the perpendicular bisector is $y - 8 = -1(x - (-2))$

$y = -1(x + 2) + 8$

$y = -x - 2 + 8$

$y = -x + 6$ ....(1)

Mid-point of $(-6, 4)$ and $(2, -4)$ is $\left(\frac{-6+2}{2}, \frac{4-4}{2}\right) = \left(-2, 0\right)$

The gradient of the line connecting the points $(-6, 4)$ and $(2, -4)$ is

$\frac{-4-4}{2-(-6)} = \frac{-8}{8} = -1$

The gradient of the perpendicular bisector is 1

The equation of the perpendicular bisector is $y - 0 = 1(x - (-2))$

$y = (x + 2)$

$y = x + 2$

$y = x + 2$ ....(2)

Add equation 1 to equation 2:

$y + y = -x + x + 6 + 2$

$2y = 8$

$y = 4$

Substitute $y = 4$ into equation 2 and solve for x:

$4 = x + 2$

$x = 4 - 2$

$x = 2$

This means the coordinates of the centre of the circle is (2, 4).

The distance between the centre and any point on the circle's circumference gives the radius.

Distance between (2, 4) and (2, 12)= $r = \sqrt{(2-2)^2 + (12-4)^2}$

$= \sqrt{0^2 + 8^2}$

$= \sqrt{64}$

$= 8$

Finally, substitute the centre, (2, 4), and the radius, 8, into the standard equation of the circle.

$(x - 2)^2 + (y - 4)^2 = 8^2$

$x^2 - 4x + 4 + y^2 - 8y + 16 = 64$

$x^2 + y^2 - 4x - 8y + 20 - 64 = 0$

$\boldsymbol{x^2 + y^2 - 4x - 8y - 44 = 0}$

A construction of the circle is shown below

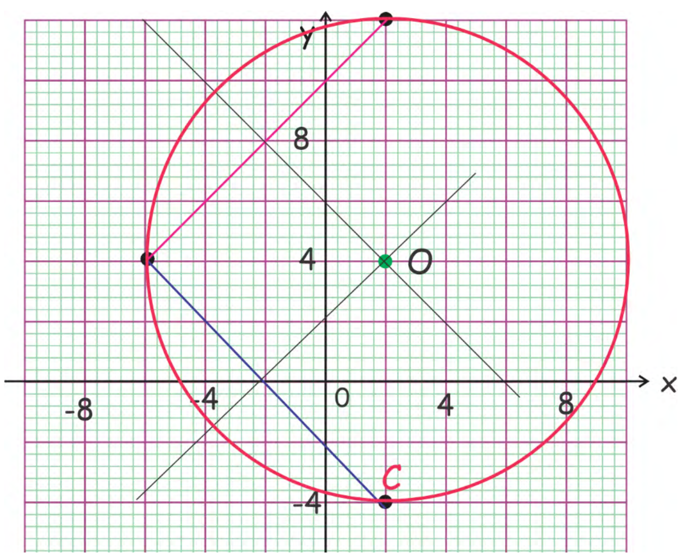


Figure 4.21: Graph of circle $x^2 + y^2 - 4x - 8y - 44 = 0$

TANGENTS AND NORMALS

Study Figure 4.22 carefully:

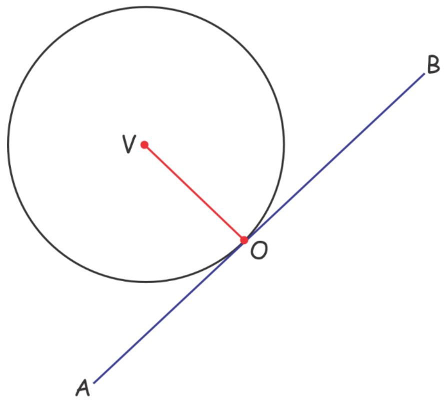


Figure 4.22: Tangent of a circle

In Figure 4.22, V is the circle's centre, |VO| is a radius and AOB is a tangent to the circle at O. The tangent is a line that touches the circle at only one point.

The equation of the radius, VO, or its extension is the normal to tangent, AOB. The normal and the tangent intersect at right angles at the point of tangency.

Since |VO| and |AOB| intersect at right angles,

We have the (gradient of |OV|)× (gradient of $|AOB|$) $= -1$

Let us use this relation to answer a few questions.

How to find the equation of a tangent and normal given the circle equation and a point, (a, b), on the circumference of the circle

Step 1: Using the given circle equation, find the centre, (h, k)

Step 2: Find the gradient of the normal. This is the gradient (m) of the line connecting points (h, k) and (a, b)

Step 3: Find the gradient of the tangent. The gradient of the tangent is $-\frac{1}{m}$, where m is the gradient of the normal.

Step 4: Find the equation of the tangent. This is given as:

$y - b = -\frac{1}{m}(x - a)$

The equation of the normal is $y - k = m(x - h)$

Where (a, b) is a point on the circumference of the circle and (h, k) is the centre of the circle.

Example 4.9

Find the equations of the line tangent and normal to the circle $x^2 + y^2 = 169$ at point $(5, -12)$

Solution

First find the centre of the circle.

To do this, we will write the equation in the standard form:

$(x - 0)^2 + (y - 0)^2 = 13^2$

This is a circle with a centre (0, 0) and a radius of 13 units.

Next, we will check if the given point satisfies the circle equation.

$5^2 + (-12)^2 = 25 + 144 = 169$

This shows that $(5, -12)$ is on the circumference of the circle.

Next, find the gradient of the normal. This is the gradient of the line connecting the points (0, 0) and (5, -12)

$m = -\frac{12 - 0}{5 - 0} = -1\frac{2}{5}$

Equation of the normal is: $y - 0 = -\frac{12}{5}(x - 0)$

Multiply through by 5:

$5y = -12x$

$5y + 12x = 0$ is the equation of the normal.

The gradient of the tangent is $= -\frac{1}{m} = -\frac{1}{\frac{12}{5}} = 1 \times \frac{5}{12} = \frac{5}{12}$

The equation of the tangent is: $y - (-12) = \frac{5}{12}(x - 5)$

$(y + 12) = \frac{5}{12}(x - 5)$

Multiply through by 12:

$12y + 144 = 5x - 25$

$12y - 5x + 144 + 25 = 0$

$12\boldsymbol{y} - 5\boldsymbol{x} + 169 = 0$ is the equation of the tangent.

Example 4.10

The point (5, 3) satisfies the equation $x^2 + y^2 - 6x - 4y + 8 = 0$.

a. Find the radius of the circle.

b. Fine the equation of the normal to the circle at point (5, 3)

c. Find the equation of the tangent to the circle at (5, 3)

Solution

a. To find the radius, write the equation in standard form

$x^2 - 6x + 9 + y^2 - 4y + 4 = -8 + 9 + 4$

$(x - 3)^2 + (y - 2)^2 = 5$

$\left(x - 3\right)^2 + \left(y - 2\right)^2 = (\sqrt{5})^2$

This means the **radius is** $\sqrt{5}$ **and the centre is** (3, 2).

Gradient of the normal.

This is the gradient of the line connecting the points (3, 2) and (5, 3)

b. The gradient of the normal $= \frac{3-2}{5-3} = \frac{1}{2}$

The equation of the normal is: $y - 2 = \frac{1}{2}(x - 3)$

$2y - 4 = x - 3$

$2y - x - 4 + 3 = 0$

$\mathbf{2y - x - 1 = 0}$

Gradient of the tangent $= -1 \div \frac{1}{2} = -2$

c. The equation of the tangent is:

$y - 3 = -2(x - 5)$

$y - 3 = -2x + 10$

$\mathbf{y + 2x - 13 = 0}$

Example 4.11

Find the equation of the normal to the circle $y^2 + x^2 + 8x + 7 = 0$ that passes through the point (1, 3)

Solution

First, verify if the point lies on the circumference of the circle:

$(3)^2 + (1)^2 + 8(1) + 7 \neq 0$

Therefore, the point (1, 3) is not on the circle's circumference.

Next, find the coordinates of the centre of the circle.

The centre is given as $= \left(\frac{\textit{coefficient of } x}{-2}, \frac{\textit{coefficient of } y}{-2}\right)$

$= \left(\frac{8}{-2}, \frac{0}{-2}\right)$

$= (-4, 0)$

The gradient of the normal $= \frac{-3}{-4-1} = \frac{3}{5}$

Equation of the normal: $y - 3 = \frac{3}{5}(x - 1)$

$5y - 15 = 3x - 3$

$5y - 3x - 15 + 3 = 0$

$5\boldsymbol{y} - 3\boldsymbol{x} - 12 = 0$

The diagram below shows the construction of the circle and the normal

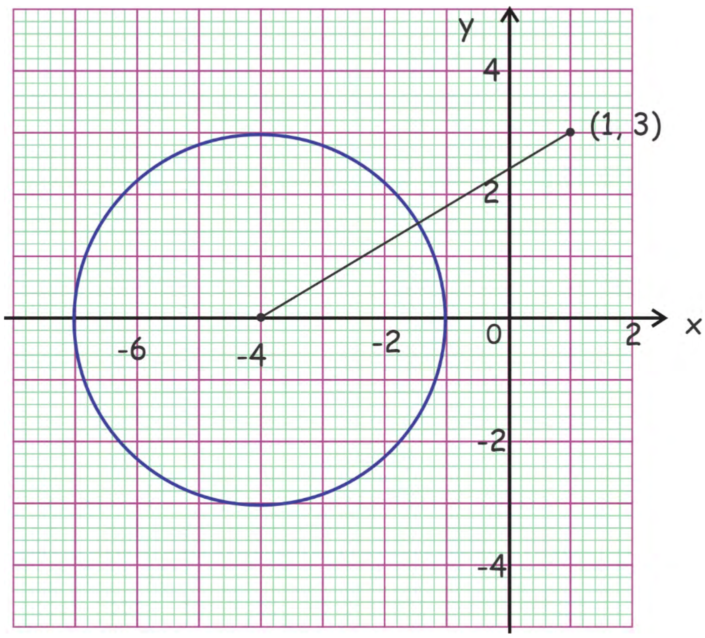


Figure 4.23: Construction of a circle and normal

Length of a tangent (L)

Given the equation of a circle, it is possible to find the length of a tangent to the circle from a point (x, y) outside the circle.

Figure 4.24 illustrates this:

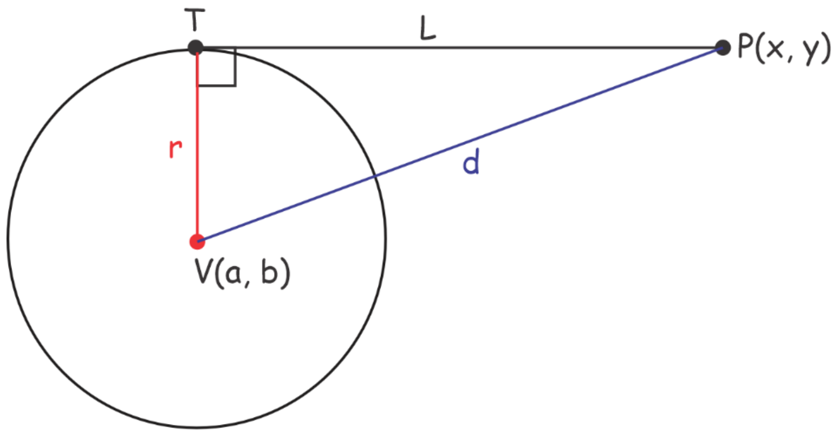


Figure 4.24: Length of a tangent

In Figure 4.24, V(a, b) is the centre of the circle. T is a point of tangency and P(x, y) is any point outside the circle. The length of the tangent is |PT| = L.

From Pythagoras' theorem, we have $L^2 + r^2 = d^2$, since we are interested in finding L, make it the subject.

$L^2 = d^2 - r^2$

$\sqrt{L^2} = \sqrt{d^2 - r^2}$

$L = \sqrt{d^2 - r^2}$

Example 4.12

Find the length of the tangent from the point (0, 4) to the circle $x^2 + y^2 - 2x + 12y - 3 = 0$

Solution

First find the centre and radius of the circle:

$x^2 - 6x + 9 + y^2 + 4y + 4 - 3 = 9 + 4$

$(x - 3)^2 + (y + 2)^2 = 16$

$(x - 3)^2 + (y + 2)^2 = 4^2$

The centre of the circle is (3, -2) and the radius is 4

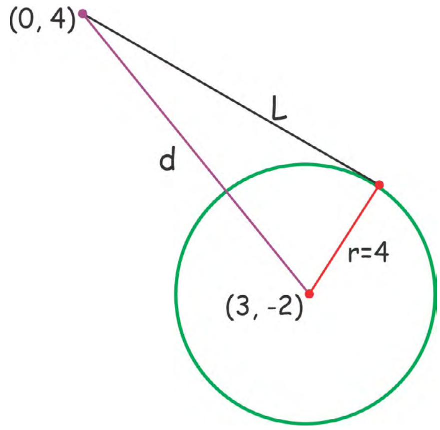


Figure 4.25: Length of the tangent to the circle $x^2 + y^2 - 2x + 12y - 3 = 0$

Distance between the (0, 4) and the centre (3, -2), $d = \sqrt{(3-0)^2 + (-2-4)^2}$

$d = \sqrt{(3)^2 + (-6)^2}$

$d = \sqrt{9 + 36}$

$d = \sqrt{45}$

Length of the tangent $= \sqrt{d^2 - r^2}$

Length of the tangent $= \sqrt{(\sqrt{45})^2 - (4)^2}$

$= \sqrt{45 - 16}$

$= \sqrt{29}$

Example 4.13

Figure 4.26 shows a graph of a circle. Line OBA is a tangent to the circle at B.

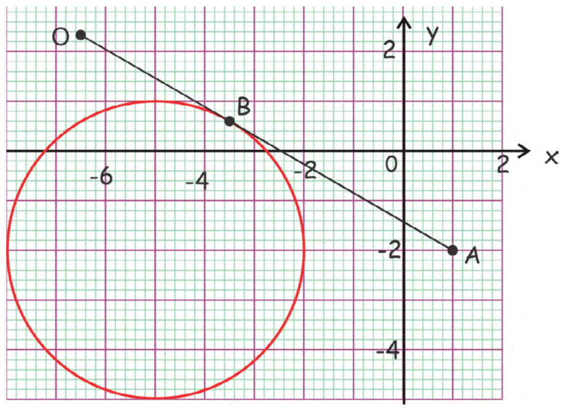


Figure 4.26: Graph of a circle with Line OBA as tangent

a. Find the equation of the circle.

b. Find |AB|, leave your answer in surds

Solution

a. The centre of the circle is (-5, -2) and the radius is 3.

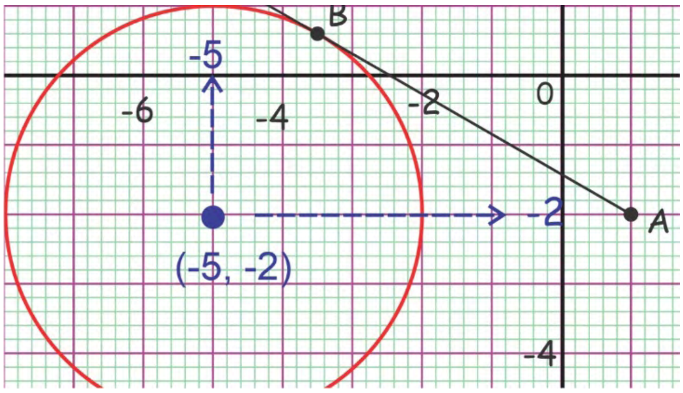


Figure 4.27: Graph of a circle with Line OBA as tangent

The equation of the circle is: $(x-(-5)^2+(y-(-2))^2=3^2$

$(x+5)^2+(y+2)^2=9$

$x^2+10x+25+y^2+4y+4-9=0$

$\boldsymbol{x^2+y^2+10x+4y-5=0}$

b. Coordinates of point A is (1, -2)

Distance between point A and the centre (-5, -2), $d=$ $\sqrt{(-5-1)^2+(-2-(-2))^2}$

$d=\sqrt{(-6)^2+(0)^2}$

$d=\sqrt{36}=6$

$d=6$

Length of the tangent $=\sqrt{6^2-3^2}=\sqrt{36-6}=\sqrt{30}$

DEDUCING THE RELATION OF VARIOUS LOCI UNDER GIVEN CONDITION

Locus

A locus (plural = loci) is a set of points which satisfies a given condition or criterion. The result of a locus will either be a complete circle, a part of a circle (arc), a line or a single point. Here, we will learn about some conditions or criteria a set of points satisfies.

Condition 1:

The Locus of points are **equidistant** from one point.

The result of this locus is a circle which will be r units away from the given point. The given point is the centre of the circle. The criterion or the condition is that: "it is equidistant from the centre"

Figure 4.28 shows the locus of all points r units from point (h, k)

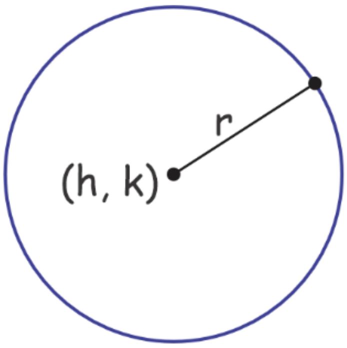


Figure 4.28: Locus of all points r units from point (h, k)

Example 4.14

Find the equation of all points 4 units away from (− 1, 2)

Solution

This is the same as finding the equation of a circle with centre (−1, 2) and radius = 4.

Using the general equation of the circle, we have:

$(x - (-1))^2 + (y - 2)^2 = 4^2$

$(x + 1)^2 + y^2 - 4y + 4 = 16$

$x^2 + 2x + 1 + y^2 - 4y + 4 = 16$

$x^2 + y^2 + 2x - 4y + 1 + 4 - 16 = 0$

$\boldsymbol{x^2 + y^2 + 2x - 4y} - 11 = 0$ is the locus of all points 4 units away from the point $(-1, 2)$

Condition 2:

The locus of points equidistant from two fixed points.

The result of this locus is a line which will perpendicularly bisect the line joining the two points.

Figure 4.29 below shows the locus of points, L, equidistant from points P and Q. Line L bisects PQ perpendicularly at m, the midpoint of line PQ.

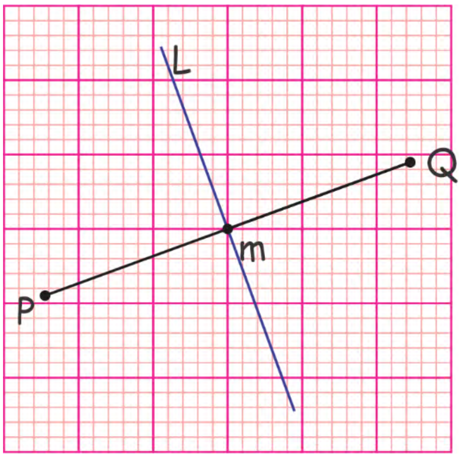


Figure 4.29: Locus of points, L, equidistant from points P and Q

Example 4.15

Calculate the equation of all points equidistant from A(-5, 0) and B(3, -6)

Solution

Midpoint of line AB $= \left(\frac{-5+3}{2}, \frac{0+-6}{2}\right)$

$= \left(-\frac{2}{2}, -\frac{6}{2}\right)$

$= (-1, -3)$

Gradient of line AB $= \frac{-6-0}{3 - 5}$

$= -\frac{6}{8}$

$= -\frac{3}{4}$

$$\text{Gradient of the locus} = -1 \div \left(-\frac{3}{4}\right)$$
$$= -1 \times -\frac{4}{3}$$
$$= \frac{4}{3}$$

Equation of the locos is:

$y - -3 = \frac{4}{3}(x - (-1))$

$y + 3 = \frac{4}{3}(x + 1)$

Multiply through by 3:

$3 \times (y + 3) = 3 \times \frac{3}{4}(x + 1)$

$3y + 9 = 4(x + 1)$

$3y + 9 = 4x + 4$

$3y - 4x + 9 - 4 = 0$

$3y - 4x + 5 = 0$

Therefore, $3\boldsymbol{y} - 4\boldsymbol{x} + 5 = 0$ is the locus of all points equidistant from (-5, 0) and (3, -6)

Condition 3:

The locus of points equidistant from two intersecting lines.

The result of this locus is a pair of angle bisectors. For any point (x, y) on the angle bisector, the perpendicular distance is equidistant from the two lines.

In Figure 4.30, l and m are the loci equidistant from line AB and line CD.

Line L bisects angle BOD and M bisects angle AOD.

Also, for any point P(x, y) on the locus, L, the perpendicular distance between point P and line AB is equal to the perpendicular distance between point P and line CD.

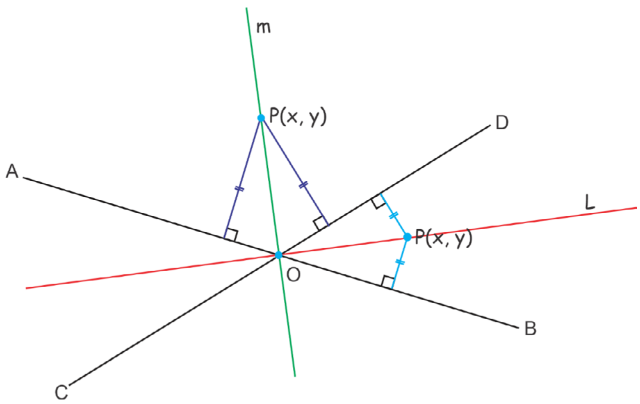


Figure 4.30: Locus of points equidistant from two intersecting lines

Example 4.16

Find the equation of the locus of points which moves so that its distance from the lines

$y = 2x - 3$ and $x + 2y = 5$ is the same.

Let P(x, y) be any point on the loci. This means that the perpendicular distance between P(x, y) and the line $y = 2x - 3$ is equal to the perpendicular distance between point P(x,y) and the line $x + 2y = 5$

Recall that the perpendicular distance between a point (m,n) and a line $ax + by + c = 0$ is

$$\left|\frac{a(m) + b(n) + c}{\sqrt{a^2 + b^2}}\right|$$

The equation of the loci is given by:

$$\left|\frac{y - 2x + 3}{\sqrt{1^2 + (-2)^2}}\right| = \left|\frac{2y + x - 5}{\sqrt{2^2 + 1^2}}\right|$$

$$\left|\frac{y - 2x + 3}{\sqrt{5}}\right| = \left|\frac{2y + x - 5}{\sqrt{5}}\right|$$

multiply both sides by $\sqrt{5}$,which effectively removes the denominator:

$$|y - 2x + 3| = |2y + x - 5|$$

Remove the absolute value sign by multiplying **any side** of the equation by $\pm$

In this example, we chose to remove the absolute value sign by multiplying the RHS of the equation by $\pm$

$$y - 2x + 3 = \pm(2y + x - 5)$$

So, either

$y - 2x + 3 = 2y + x - 5$	or	$y - 2x + 3 = -2y - x + 5$
$0 = 2y - y + x + 2x - 5 - 3$		$y + 2y - 2x + x + 3 - 5 = 0$
$y + 3x - 8 = 0$		$3y - x - 2 = 0$

Therefore, the equation of the locus is either $\boldsymbol{y + 3x - 8 = 0}$ **or** $\boldsymbol{3y - x - 2 = 0}$

Condition 4:

The locus equidistant from a line.

The result of this locus is a pair of lines equidistant from the given line.

For instance, the locus of points ***d*** units away from line AB is either line M or line L as shown in Figure 4.31.

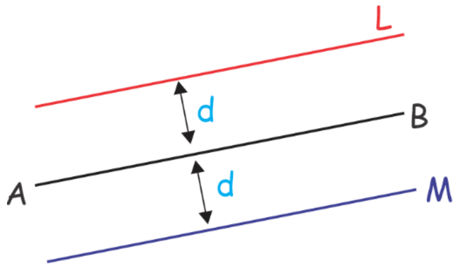


Figure 4.31: Locus equidistant from a line

So, if the equation of AB is $ax + by + c = 0$, then $d = \left|\frac{ax + by + c}{\sqrt{a^2 + b^2}}\right|$

Example 4.17

Find the locus of the point R(x, y) such that it is always 3 units from the line $y + 1 = 0$

Solution

$3 = \left|\frac{0x + y + 1}{\sqrt{0^2 + 1^2}}\right|$

$3 = \left|\frac{y + 1}{\sqrt{1}}\right|$

$3 = \pm|y + 1|$

$3 = y + 1$ or $3 = -y - 1$

$y - 2 = 0$ or $y + 4 = 0$

Therefore, the locus R(x, y) of points which is 3 units away from $y + 1 = 0$ is either $\boldsymbol{y - 2 = 0}$ **or** $\boldsymbol{y + 4 = 0}$

Example 4.18

If P(1, 4) and Q(5, -1) are two fixed points and A(x,y) moves such that |AP|:|AQ| = 1:2. Find the equation of the locus.

Solution

From the condition given, |AP|:|AQ| = 1:2

$\frac{|AP|}{|AQ|} = \frac{1}{2}$

$2|AP| = 1 \times |AQ|$

$2\sqrt{(y-4)^2+(x-1)^2} = \sqrt{(y-(-1))^2+(x-5)^2}$

$2\sqrt{(y-4)^2+(x-1)^2} = \sqrt{(y+1)^2+(x-5)^2}$

Square both sides:

$4[y^2-8y+16+x^2-2x+1] = y^2+2y+1+x^2-10x+25$

$4y^2-32y+64+4x^2-8x+4 = y^2+2y+1+x^2-10x+25$

$4y^2-y^2-32y-2y+4x^2-x^2-8x+10x+68-26=0$

$3\boldsymbol{x}^2+3\boldsymbol{y}^2+2\boldsymbol{x}-34\boldsymbol{y}+42=0$ is the equation of the locus.

Example 4.19

A point A(x, y) moves such that AC and AD are perpendicular.
If C(-3, 0) and D(3, 5), find the equation of the locus A.

Solution

Since AC and AD are perpendicular, (gradient of AC)× (gradient of AD)= -1

$\left(\frac{y-0}{x-(-3)}\right) \times \left(\frac{y-5}{x-3}\right) = -1$

$\left(\frac{y}{x+3}\right) \times \left(\frac{y-5}{x-3}\right) = -1$

$\frac{y^2-5y}{x^2-9} = -\frac{1}{1}$

Cross multiply:

$y^2-5y = -1(x^2-9)$

$y^2-5y = -x^2+9$

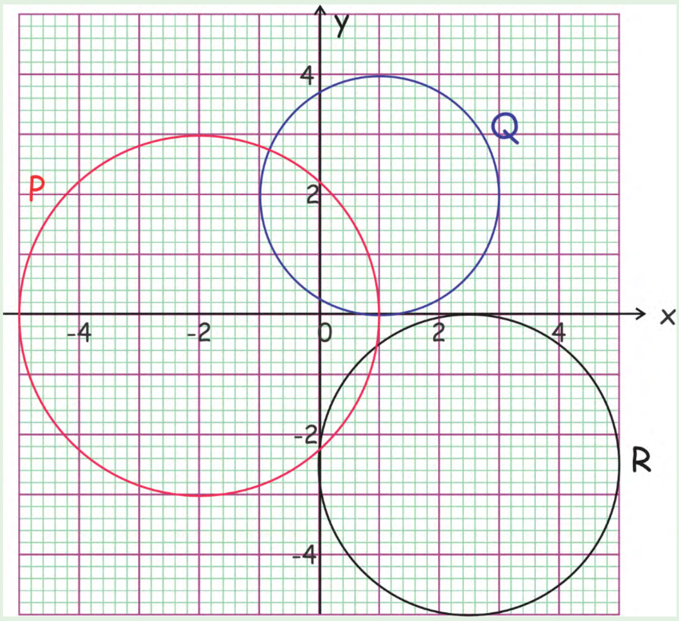
$\boldsymbol{y}^2+\boldsymbol{x}^2-5\boldsymbol{y}-9=0$ is the locus A

EXTENDED READING

- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. The diagram below shows the graph of three circles, P, Q and R.
2. Use it to answer the questions.

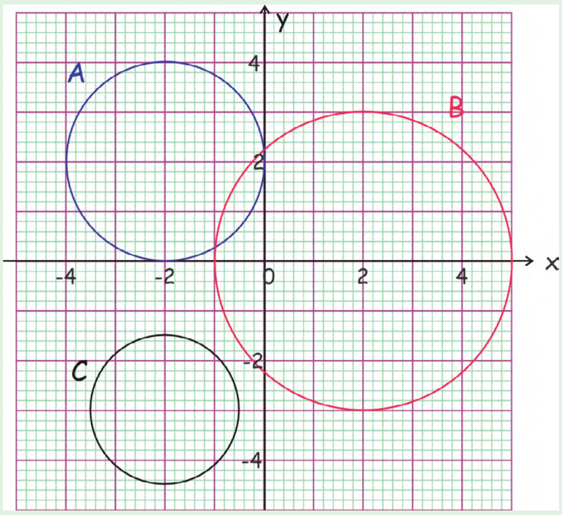


   a. State the coordinates of the centre of P, Q and R.
   b. Find the radii of circles P, Q and R
   c. Write the equation of circle R in general form
   d. Write the equation of circle P in the standard form
   e. Write the equation of circle Q in the standard form
   f. Find the perimeter of the triangle formed by the centres of the three circles.
   g. Give your answer correct to three significant figures.

3. Write the general form of the circle equation with the following properties.
   a. Centre (1, -1) and radius 7
   b. Centre (-2, 1) and radius 5
   c. Centre (0, -3) and radius 2
   d. Centre (-3, -4) and radius 8

4. The diagram shows the graph of circles A, B and C.

   Use them to answer the following questions.



   a. Write the equation of each circle in the general form

   b. Find the perimeter of the triangle formed by the centres of the circle. Leave your answer in surd form.

5. A TV remote has a range of 8m in all directions. If the TV is 5m East and 3m South, write an equation to represent the area within which the TV receives signals from the remote. Can the TV receive signals from the remote if the remote is located 2m West? Justify your response.

6. An Uber driver operates within 30km of a post office at all angles. His house is 5km West and 12km south of a post office.

7. a. Write an equation to represent the driver's travel boundary from the post office using the driver's house as the origin. Leave your equation in standard form.

   b. Find the furthest distance between the driver's house and his travel boundary.

   c. Find the shortest distance between the driver's house and the driver's travel boundary.

8. Write the general equation of a circle with:

   a. Centre $(4, -1)$ and radius $\sqrt{11}$.

   b. Centre $(-2, -3)$ and radius $2\frac{1}{2}$

   c. Centre $(0, -3)$ and radius $\frac{1}{3}$

   d. Centre $\left(\frac{1}{2}, -1\right)$ and radius $1\frac{1}{2}$

9. Write each of the following circles in the standard form. Also, state the centre and radius of each circle.

   a. $x^2 + y^2 - 2x + 2y - 47 = 0$

   b. $x^2 + y^2 + 6y = 0$

   c. $x^2 + y^2 - 12x - 64 = 0$

   d. $x^2 + y^2 + 4x - 8y + 4 = 0$

   e. $2x^2 + 2y^2 - 2x - 2y - 7 = 0$

   f. $3x^2 + 3y^2 + 12x - 2y + 4 = 0$

10. The endpoints of the diameter of a circle are (0, 5) and (0, -7).

11. Find the

   a. radius and centre of the circle

   b. general equation of the circle.

12. The endpoints of the diameter of a circle are (6, 0) and (-2, 0).

13. Find the

   a. radius and centre of the circle

   b. general equation of the circle.

14. The endpoints of the diameter of a circle are (-4, 3) and (4, -3).

15. Find the

   a. radius and centre of the circle

   b. general equation of the circle.

16. In each of the following questions, coordinates of three points are provided. Write the general equation of the circle that satisfies all three points. State the centre and radius of the resulting circle.

   a. $(-5, 0)$, $(-1, -4)$ and $(-1, 4)$

   b. $(6, -1)$, $(3, -4)$ and $(0, -1)$

17. Find the equations of the line tangent and normal to the circle.

   a. $x^2 + y^2 = 100$ at point $(-8, -6)$

   b. $x^2 + y^2 = 49$ at point $(7, 0)$

   c. $x^2 + y^2 - 2x - 24 = 0$ at $(1, -5)$

18. Find the length of the tangent from the point:

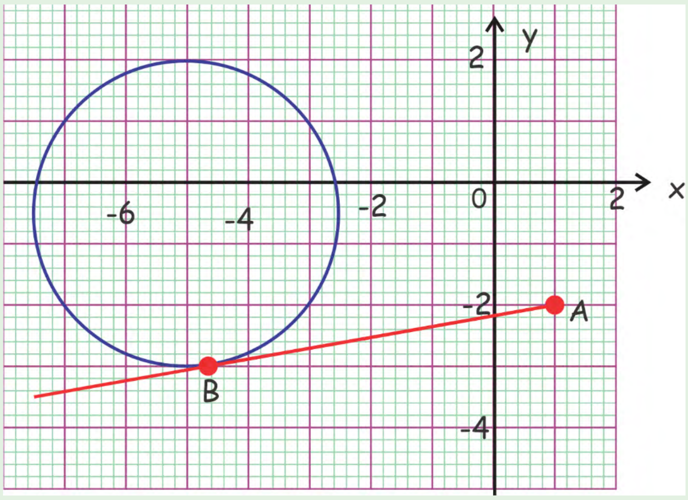
a. (-8, 6) to the circle $x^2 + y^2 - 25 = 0$, leave your answer in surd form

b. (3, 6) to the circle $x^2 + y^2 - 4y - 5 = 0$

c. $(7, -3)$ to the circle $x^2 + y^2 + 10x - 4y + 13 = 0$, leave your answer in surd form

d. $\left(5, -\frac{1}{2}\right)$ to the circle $4x^2 + 4y^2 + 12x + 4y - 15 = 0$

19. The diagram below shows a graph of a circle. Line BA is a tangent to the circle at B.



a. Find the equation of the circle.

b. Find |AB|, leave your answer in surds

20. Calculate the equation of all points equidistant from A(-4, 0) and B(2, 8))

21. Find the equation of the locus of points which moves so that its distance from the lines $3y - x = 0$ and $x - 3y = 0$ is the same.

22. Find the locus of point R(x, y) that is always 2 units from the line $x + 5 = 0$

23. If P(-2, 1) and Q(4, -1) are two fixed points and A(x, y) moves such that |AP|:|AQ| = 2:1. Find the equation of the locus.

24. A point A(x, y) moves so that AC and AD are perpendicular. C(-2, 0) and D(6, 4).

25. Find the equation of the locus A.

SECTION 5
VECTORS

GEOMETRIC REASONING AND MEASUREMENT

Spatial Sense

INTRODUCTION

Learning about vectors and their projections is important because vectors allow us to understand and describe quantities that have both magnitude and direction such as forces, velocities and electric fields. In this section you will learn about transposing vectors, dividing a vector internally based on a ratio and finding the dot (scalar) product of vectors. You will also learn about finding angles between two vectors, the application of the sine and cosine laws and the projection of one vector onto another.

KEY IDEAS

- In transposing a vector, change the orientation from a column to a row and vice versa.
- The angle between two vectors, **a** and **b**, is given by θ in $\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||b|}$
    - o where $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2$ (in two dimensions) $= a_1b_1 + a_2b_2 + a_3b_3$ (in three dimensions).
- The vector $\overrightarrow{AB}$ is given by $\boldsymbol{b} - \mathbf{a}$ where **a** and **b** are the position vectors of A and B.

TRANSPOSING VECTORS

In year one we learnt about straight lines and their properties (section 5), vectors (section 6) and trigonometry (section 7). In this section we will make use of the stated concepts learned in year one to help us understand and solve problems in relation to vectors. Let us work through this activity to refresh our minds on lines and vectors.

Activity 5.1: Treasure Hunt on the Coordinate Plane (in small groups, pairs or individually)

1. Imagine you are on a treasure hunt! Start by marking the point (0, 0) on your graph paper as your starting point. Each vector will guide you closer to the treasure!
2. Follow the Directions:
   a. Move from (0, 0) to the first position with $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
   b. Add the vector $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ to get to the next position
   c. Add the vector $\overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$
   d. Subtract the vector $\overrightarrow{OD} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ to go to the next position
   e. Add the vector $\overrightarrow{OE} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for your final move
   f. After each move, mark your new spot and draw a line from your last point to your current one.
3. Congratulations! You've found the treasure.
4. Write out the coordinates of the treasure position. Check that your finishing position is the same as your classmates.
5. Look for Patterns
   a. Check out your line segments:
      i. Notice the gradient for each line segment and see if any gradients (slopes) are the same or opposite.
      ii. Are any segments parallel or perpendicular?
      iii. Observe the distance between each point you marked.
   b. Do the segments form any special shapes, like triangles or parallelograms?
6. How did adding and subtracting vectors help you reach the treasure?
7. What straight-line properties did you notice along the way?

*Now that we've gone through this activity, let us talk about **transposing vectors**!*

Transposition simply means to change something from one position to another. In transposing a vector, you change the orientation from a column vector to a row vector or change a row vector to a column vector.

For example, the vector, W = $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_n \end{pmatrix}$ when transposed becomes

$W^T = (w_1\ w_2\ w_3\ w_4 \ldots\ w_n)$.

Example 5.1

Transpose the vector $M = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}$.

Solution

Step 1: Identify whether the given vector is a column or row vector.

In this case it is a *column* vector

Step 2: Identify the vector orientation it should change to.

In this case, a *row* vector

Step 3: Rearrange the elements of the vector to suit the new orientation.

$M^T = (2\ 5\ -7)$

Example 5.2

Transpose the vector $H = (-5\ 6\ 8\ 3)$.

Solution

Step 1: Identify whether the given vector is a column or row vector.

Row vector

Step 2: Identify the vector orientation it should change to.

Column vector

Step 3: Rearrange the elements of the vector to suit the new orientation.

$H^T = \begin{pmatrix} -5 \\ 6 \\ 8 \\ 3 \end{pmatrix}$

DIVIDING A LINE OR VECTOR IN A GIVEN RATIO

In year one, we learnt how to divide straight lines internally based on a ratio and how to extend a line segment based on a ratio. We will apply this knowledge to divide a vector in a given ratio.

Let us use the concept of unit vectors to assist in dividing vectors through these examples.

Example 5.3

R and T are points on the position vectors $8i + 3j$ and $-3i + 5j$, respectively on a vector $\overrightarrow{RT}$, S is a point on $\overrightarrow{RT}$such that $|RS| : |ST| = 3:1$. Find the position vector of the point S.

Solution

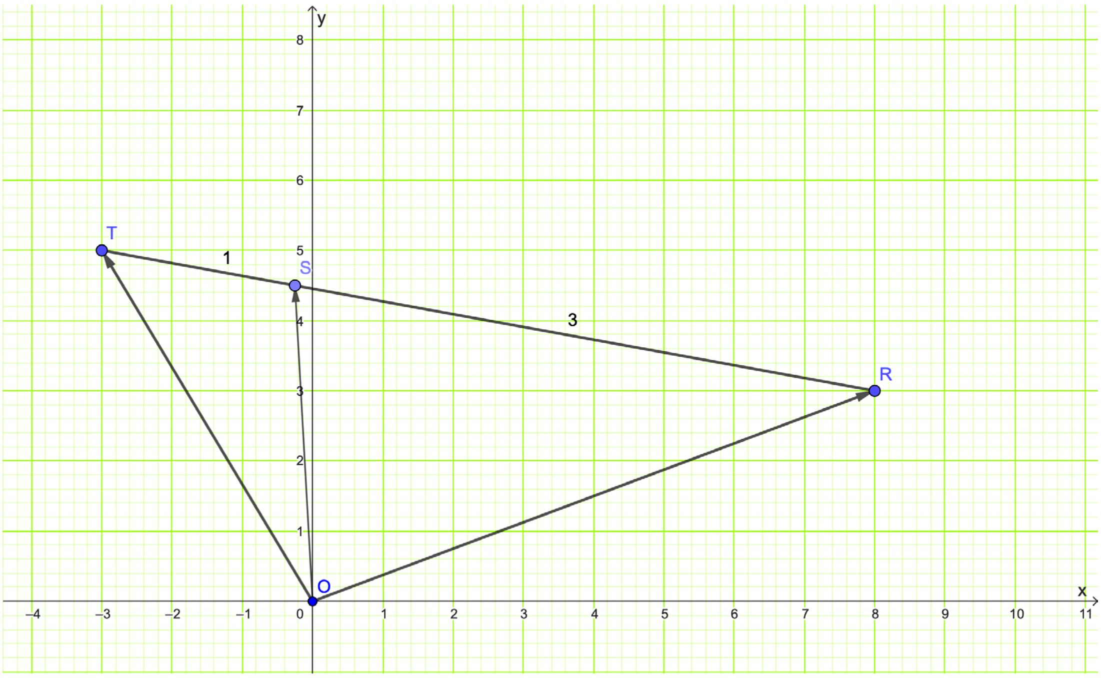


Figure 5.1: Dividing ($\overrightarrow{RT}$) by ratio 3:1

$$\overrightarrow{OS} = \frac{1\left(\overrightarrow{OR}\right) + 3\left(\overrightarrow{OT}\right)}{3+1}$$

$$= \frac{1(8i+3j) + 3(-3i+5j)}{4}$$

$$= \frac{8i+3j-9i+15j}{4}$$

$$= -\frac{i + 18j}{4}$$

$$= -0.25i + 4.5j$$

Example 5.4

A and E are points on the position vectors $-6i + 5j$ and $2i - 9j$, respectively on a vector $\overrightarrow{AE}$, B is a point on $\overrightarrow{AE}$ such that $|AB| : |BE| = 7{:}3$. Find the position vector of the point B.

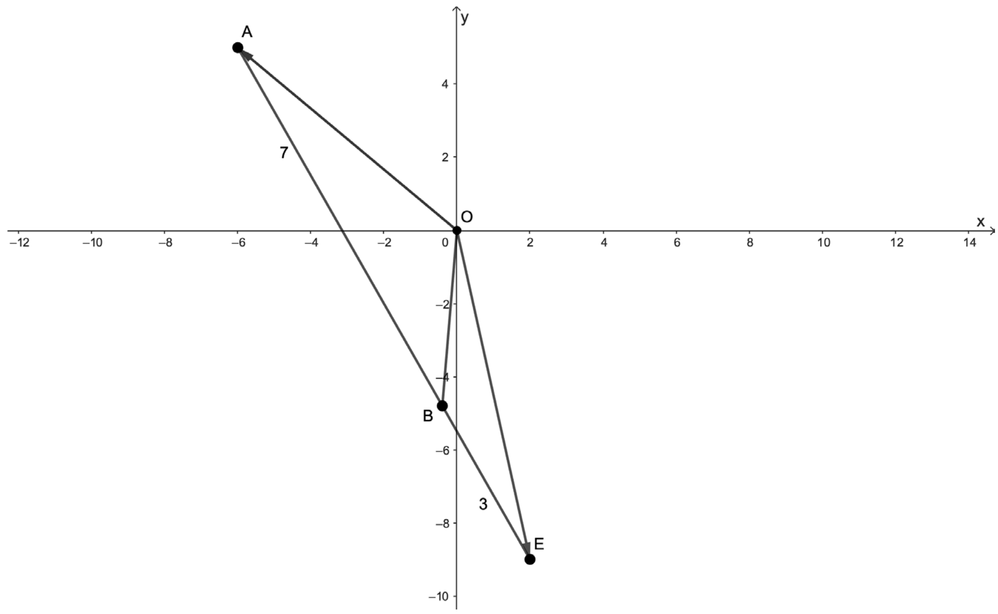


Figure 5.2: Division of $\overrightarrow{AE}$ in the ratio 7:3

Solution

$$\overrightarrow{OB} = \frac{3\left(\overrightarrow{OA}\right) + 7\left(\overrightarrow{OE}\right)}{3 + 7}$$

$$= \frac{3(-6i + 5j) + 7(2i - 9j)}{10}$$

$$= -\frac{18i + 15j + 14i - 63j}{10}$$

$$= -\frac{4i - 48j}{10}$$

$$= -0.4i - 4.8j$$

FINDING AND APPLYING THE DOT PRODUCT OF VECTORS

As the name implies anything about products is to do with multiplication. In this case, the dot product of vectors focuses on multiplying the corresponding coordinates of given vectors and summing the results.

For example, given the vectors $\boldsymbol{m} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ the dot product will be

$\boldsymbol{m} \cdot \boldsymbol{n} = (a_1 \times a_2) + (b_1 \times b_2)$ which will result in

$= a_1\, a_2 + b_1\, b_2$

Note that $\boldsymbol{m} \cdot \boldsymbol{n}$ is read as "$\boldsymbol{m}$ dot $\boldsymbol{n}$"

Let us go through this activity to explore what the dot product of a vector and its transposed vector will be.

Activity 5.2: Investigating Dot Products (working individually or in pairs)

1. Choose a vector $\left(\text{e.g. } \boldsymbol{m} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\right)$
2. Transpose the vector (e.g. $\boldsymbol{m}\ \boldsymbol{T}$)
3. Find the dot product of the original vector and its transposed vector
4. Discuss your results with a classmate.

Hopefully you found, in this case, that

$\boldsymbol{m}.\boldsymbol{m}^T = (u_1 \times u_1) + (v_1 \times v_1) = (u_1)^2 + (v_1)^2 = \boldsymbol{m}.\,\boldsymbol{m}$

Let's go ahead to explore how an angle between two vectors relates to the dot product of the vectors.

Activity 5.3: Investigating Angles Between Vectors (working in small groups)

1. Create two vectors of your choice.
2. Represent these vectors on a graph (manually or electronically).
3. Join the vectors from a third point, the origin.

4. Measure the angle formed between the two vectors at the origin.
5. Find the cosine of the angle formed and document it.
6. Now find the magnitude of each of the two vectors.
7. Find the product of the two magnitudes and the documented result in step 5
8. At this point, compute the dot product of the two vectors.
9. Compare the results in steps 7 and 8.
10. What conclusion can be drawn from the results?

Hopefully you concluded that the dot product is the same as the product of the two magnitudes and cosine of the angle between the vectors.

Generalisations

1. $\boldsymbol{m}\,.\boldsymbol{n} = |\boldsymbol{m}||\boldsymbol{n}|\cos\theta$
2. $\cos\theta = .\frac{\boldsymbol{n}}{|\boldsymbol{m}||\boldsymbol{n}|}$
3. $\boldsymbol{m}\,.\boldsymbol{n} = \boldsymbol{n}\,.\boldsymbol{m}$ (Dot product is commutative)

Activity 5.4: Algebraic Proof of the Dot Product (work in groups)

Carry out research on the algebraic proof for $\boldsymbol{a}\,.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$.

Example 5.5

Given the vectors $\boldsymbol{t} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$, find the dot product of $\boldsymbol{t}$ and $\boldsymbol{k}$.

Solution

Step 1: Identify corresponding coordinates

7 and 5, −3 and 8

Step 2: Multiply the identified pairs and find their sum:

$$\boldsymbol{t}\,.\boldsymbol{k} = (7 \times 5) + (-3 \times 8)$$
$$= 35 + (-24)$$

Step 3: Simplify the results:

$\boldsymbol{t}.\boldsymbol{k} = 35 - 24 = 11$

Therefore, $\boldsymbol{t}.\boldsymbol{k} = 11$.

Example 5.6

Given the vectors $\boldsymbol{p} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Find the:

a. dot product

b. angle formed between $\boldsymbol{p}$ and $\boldsymbol{q}$

Solution

a. $\boldsymbol{p}.\boldsymbol{q} = (1 \times (-2)) + (4 \times 3) = 10$

b. $\cos\theta = \frac{\cdot \mathbf{q}}{|\boldsymbol{p}||\boldsymbol{q}|}$,

Step 1: Make θ the subject.

$\theta = cos^{-1}\left(\frac{\boldsymbol{p}\cdot\mathbf{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\right)$

Step 2: Find the magnitude of $\boldsymbol{p}$ and $\boldsymbol{q}$.

$|\boldsymbol{p}| = \sqrt{(1)^2 + (4)^2}$

$= \sqrt{17}$

$|\boldsymbol{q}| = \sqrt{(-2)^2 + (3)^2}$

$= \sqrt{13}$

Step 3: Substitute the values into $\theta = cos^{-1}\left(\frac{\boldsymbol{p}\cdot\mathbf{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\right)$.

$\theta = cos^{-1}\left(\frac{10}{(\sqrt{17})(\sqrt{13})}\right)$

$\theta = 47.7263^0$

You can confirm the size of the angle by plotting the vectors and measuring the angle using GeoGebra Software.

Properties of the scalar (dot) product

1. The dot product of any two vectors is a scalar and not a vector, hence it can be known as the scalar product, rather than dot product.
2. The dot product of a vector with itself is the square of its magnitude.

   $\boldsymbol{a}\ .\ \boldsymbol{a} = |\boldsymbol{a}|^2$
3. *Parallelism Property*

In year one we learnt about parallel vectors. If you remember, parallel vectors are scalar multiples of each other and do not intersect at any point. This helps to establish that no angle can be formed between two parallel vectors. Hence,

$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos(0)$ but $\cos(0) = 1$ leading to;

$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|$ for parallel vectors.

So, the dot product becomes same as the product of the magnitudes for parallel vectors.

Curious about the validity of this property? Choose parallel vectors and explore using GeoGebra software or manually.

4. *Perpendicular Property*

   You know that $\cos 90° = 0$ and perpendicular vectors have an angle of 90° formed between them.

   Therefore, it can be said that if vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. Also, if the scalar product of two non-zero vectors is zero, they must be perpendicular.

5. Multiplication by a constant

   $m\boldsymbol{a} \bullet n\boldsymbol{b} = mn|\boldsymbol{a}||\boldsymbol{b}|$ where m and n are constants and $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors

6. The scalar(dot) product is distributive over addition.

   Hence $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{b} \bullet \boldsymbol{c}$ where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors.

ESTABLISHING AND APPLYING THE SINE AND THE COSINE RULE

The sine and cosine rules are very useful. They provide ways of finding angles and lengths when the triangle is not necessarily right-angled.

Let's go through this activity to investigate these laws.

Activity 5.5: Investigating the Cosine and Sine Rules (work in small groups)

1. Draw three different triangles (1 acute, 1 obtuse, 1 right angled triangle)
2. Label the vertices of each triangle as Q, R, and T.
3. Label the side opposite each angle as q, r, and t respectively (ie, q is opposite Q, r is opposite R, and t is opposite T).

4. Use a ruler to measure each side q, r and t in centimetres (cm).
5. Use a protractor to measure each angle Q, R, and T in degrees (°).
6. Record your measurements in a table like this:

Triangle type	q(cm)	r(cm)	t(cm)	Q(°)	R(°)	T(°)
Acute						
Obtuse						
Right-angled						

7. For each triangle, check if $t^2 = q^2 + r^2 - 2qr\cos(T)$ holds.
8. For each triangle, calculate and record the following ratios:

   $\frac{q}{\sin Q}, \frac{r}{\sin R}, \frac{t}{\sin T}$
9. What do you notice about the relationships?
10. Does the equation with cos hold true for all the triangles?
11. Does the equation with sin hold true for all triangles?
12. Discuss with a classmate or write a short paragraph explaining what this relationship reveals about any triangle. Consider why this must be useful in maths.
13. These are the Cosine and Sine Rules.

Generalisations:

1. The **Cosine Rule** states that, in any triangle, the length of one side can be calculated if we know the lengths of the other two sides and the angle between them:

   $\boldsymbol{t^2 = q^2 + r^2 - 2qr.\cos(T)}$
2. The **Sine Rule** helps us find missing sides or angles in any triangle if we have some known values:

   $\boldsymbol{\frac{q}{\sin Q} = \frac{r}{\sin R} = \frac{t}{\sin T}}$.

Example 5.7

Two boats named Triumph and Mayflower leave a harbour at the same time. Triumph heads east and travels 5km while Mayflower heads north-east and travels 8km. Find the distance between the two boats after the journey.

Solution

Step 1: Make a rough sketch of the positions and movement of the boats.

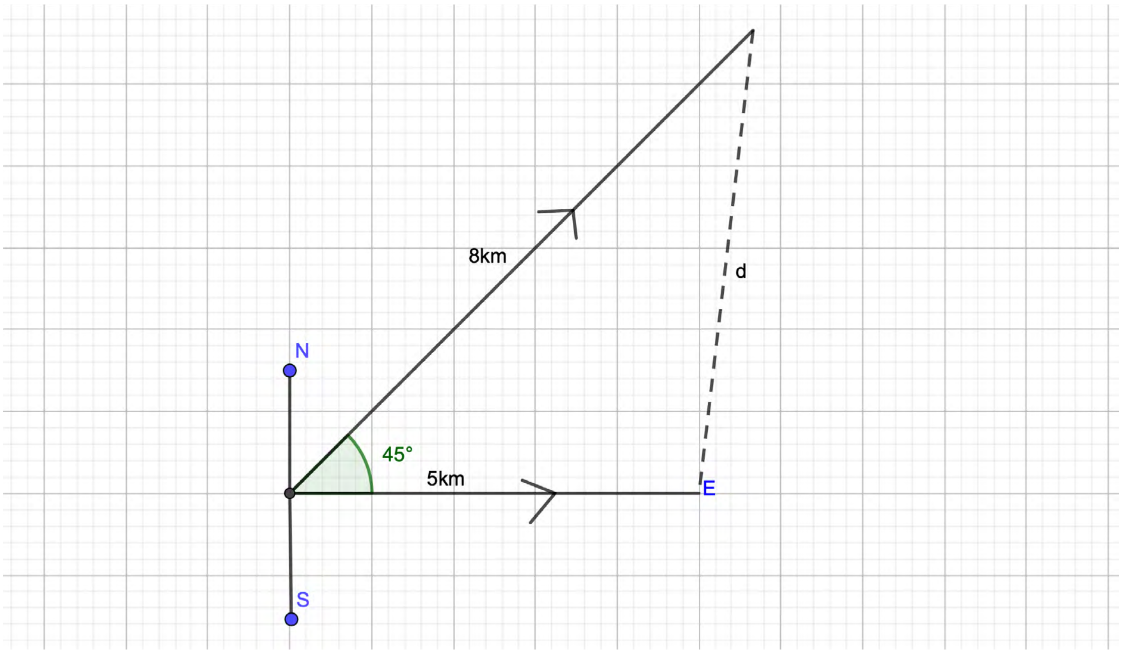


Figure 5.3: Representation of boat movements

Step 2: Choose a variable to represent the distance between the two boats.
Let d be the distance between the two boats.

Step 3: Identify the angle formed between the boats

$\alpha = 45^0$

Step 4: Apply the cosine rule:

$d^2 = 5^2 + 8^2 - 2(5)(8).\cos(45)$

$d^2 = 89 - 40\sqrt{2}$

$= \sqrt{32.43}$

$= 5.69km$

Example 5.8

Two observation towers are located 500 metres apart along a coastline at points T_1 and T_2. From T_1 a ship is sighted at a bearing such that the angle between the line T_1T_2 and T_1S is 40^0. From T_2 the ship is sighted at a bearing such that the angle between T_2T_1 and T_2S is 75^0. Find the distance between the tower T_1 and the ship using the sine law.

Solution

Step 1: Make a rough sketch of the positions and points of the Tower.

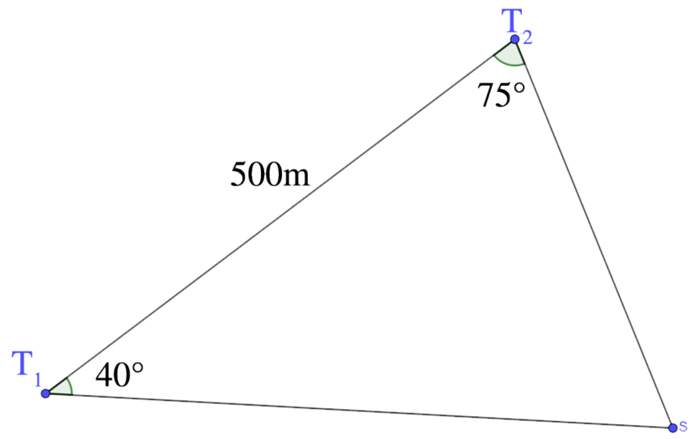


Figure 5.4: Representation of Towers and ships

Step 2: Since we have two angles, determine the third angle in the triangle.

$< T_1ST_2 = 180^0 - 40^0 - 75^0 = 65^0$

Step 3: Apply the sine law

$$\frac{ST_1}{\sin 75^0} = \frac{T_1T_2}{\sin 65°}$$

Step 4: Substitute the value for T_1T_2 and simplify the equation

$$\frac{ST_1}{\sin 75^0} = \frac{500}{\sin 65^0}$$

$$ST_1 = \frac{\sin 75^0\,(500)}{\sin 65^0}$$

$$ST_1 = 532.89\ metres$$

Therefore, the distance between tower T_1 and the ship is 532.89 metres.

Area of a triangle using the Sine ratio

You already know that area of triangle is $\frac{1}{2}$ base × perpendicular height? We can use this information to find the area of triangles using the sine rule.

Let us work through this activity to find out how.

Activity 5.6

1. Draw an isosceles or scalene triangle.
2. Label the points of the triangle.
3. Indicate the perpendicular height of the triangle.

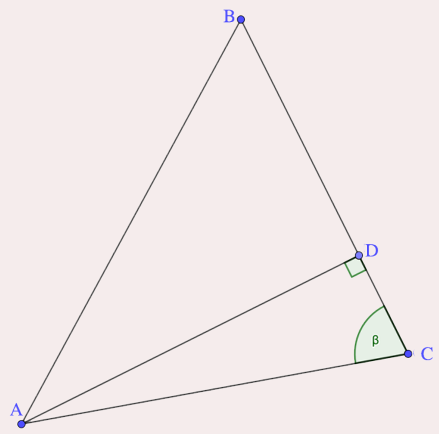


Figure 5.5: Exploring area of a triangle

4. Express the angle β in terms of the height using the sine trigonometry identity.

   For instance, $\sin\beta = \frac{|AD|}{|AC|}$.
5. Now make the perpendicular height the subject (in this case, $|AD| = |AC|\sin\beta$).
6. From our triangle, the base is $|BC|$, now substitute the base and height into the known formula for area of a triangle.
7. So, we have: $\frac{1}{2} \times |\boldsymbol{BC}| \times |\boldsymbol{AC}|\sin\boldsymbol{\beta}$
8. This is area of a triangle in terms of sine and it does not matter if we do not have the perpendicular height making it very useful.

Generalisation

The area of a triangle equals a half of the product of the magnitudes of the lengths of two sides (or two vectors which bound the triangle) and the sine of the angle between the two known sides (or vectors).

Let's go through these examples to consolidate knowledge.

Example 5.9

If Q(2, -3), R(6, 2) and T(-4, 1), are the vertices of ΔQRT, calculate:

a. <QRT to two significant figures.

b. the area of triangle QRT to two significant figures.

Solution

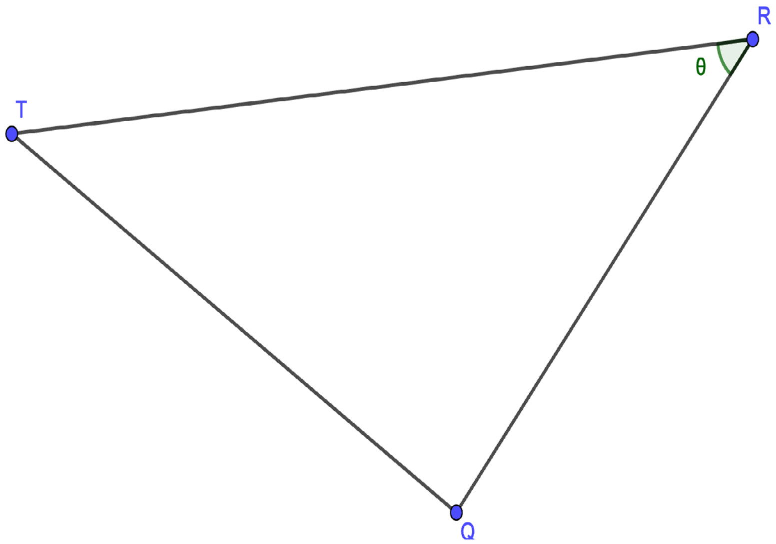


Figure 5.6: Graphical Representation of example 5.9

Step 1: Represent <QRT by θ

Step 2: Find the dot product of $\overrightarrow{RQ}$ and $\overrightarrow{RT}$ and make the angle the subject.

$\overrightarrow{RQ} = \begin{pmatrix} 2-6 \\ -3-2 \end{pmatrix}, \overrightarrow{RT} = \begin{pmatrix} -4-6 \\ 1-2 \end{pmatrix}$

$\overrightarrow{RQ} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \overrightarrow{RT} = \begin{pmatrix} -10 \\ -1 \end{pmatrix}$

$\overrightarrow{RQ} \cdot \overrightarrow{RT} = \left|\overrightarrow{RQ}\right| \left|\overrightarrow{RT}\right| \cos\theta$

$40 + 5 = \sqrt{(-4)^2 + (-5)^2} \times \sqrt{(-10)^2 + (-1)^2} \cos\theta$

$45 = 64.3506 \cos\theta$

$\theta = \cos^{-1}\left(\frac{45}{64.3506}\right)$

$\theta = 45.63^0$

$\theta \approx 46^0$ to two significant figures.

Step 3: Use the sine rule to find the area

$$\text{Area} = \frac{1}{2} \times \left|RQ\right| \times |RT|\sin\theta$$

$$= \frac{1}{2} \times \sqrt{41} \times \sqrt{101} \times \sin 46^{0}$$

$$= 23.1450$$

The area of triangle QRT is 23 square units to two significant figures.

PROJECTION OF ONE VECTOR ON A GIVEN VECTOR

Do you know how a projector works? When we project a presentation or movie onto another screen, we allow for what is on the original screen (e.g., computer) to be seen on the projected screen (e.g. whiteboard). An image of whatever is on the original screen is projected onto a new screen.

According to the Cambridge dictionary, a projection is a calculation or guess about the future based on information you have. In studying vectors, we can also project one vector onto another vector allowing us to determine the shortest distance between that point and the line.

When you project one vector onto another vector, you can either compute the scalar projection or the vector component.

Let us go through this activity to see how the projection works.

Activity 5.7 – Exploring Vector Projections

1. Draw a vector **A** from **O** going up and to the right, at a 45° angle, about 6 cm long.
2. Draw a second vector **B** from **O** going horizontally to the right (at a 0° angle), about 8 cm long.
3. Label both vectors and mark their directions with arrows.
4. Use your protractor to measure the angle between vector OA and vector OB.
5. Label this angle as θ.
6. Imagine shining a light straight down on vector A. The shadow of A will fall along B.

7. Imagine OA as a shadow, how much of OA falls along OB
8. The goal is to draw how long this shadow would be if A was stretched along the direction of B.
9. Place your ruler along vector OB (the horizontal line).
10. From the tip of A, draw a straight line down to B, making sure this line is at a right angle (90°) to B.
11. Where this line hits B is the "shadow" tip of A on B.
12. Measure the distance from point O to the point where the shadow hits B.
13. This distance is the projection of A onto B.
14. The distance shows you how much of A falls along the same direction as B.

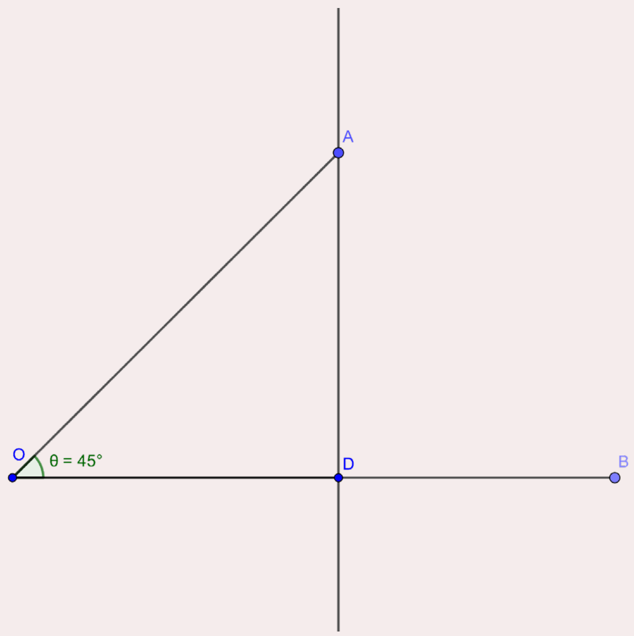


Figure 5.7: Projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$

15. Your diagram should look like *Figure 5.7: Projection of* $\overrightarrow{OA}$ *on* $\overrightarrow{OB}$.

Generalisation

1. The scalar projection of a vector **a** on vector **b** is,

   $\text{Proj}_b \boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ or $|\boldsymbol{a}|\cos\theta$

2. The vector component of a projection of a vector **a** on a vector ***b*** is

   $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} (\hat{b})$ or $(\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\, \hat{\boldsymbol{b}}$

Let's go through these examples to see how to calculate the projection of a vector on another vector.

Example 5.10

A farmer in Northern Ghana is setting up irrigation pipes. Due to wind blowing in the direction of **W** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, she needs to align the water flow direction vector **I** = $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ so that it's not significantly affected by the wind. Calculate the projection of the irrigation flow vector **I** onto the wind vector **W** and interpret.

Solution

Step 1: Identify the vectors.

$\mathbf{W} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{I} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

Step 2: Compute the dot product of the vectors.

$\mathbf{I.W} = (4 \times 6) + (3 \times 8)$

$= 24 + 24$

$= 48$

Step 3: Find the magnitude of **W**

$|\boldsymbol{w}| = \sqrt{(4)^2 + (3)^2}$

$|\boldsymbol{w}| = 5$

Step 4: Calculate the scalar projection of **I** on **W**

$\text{Proj}_w\text{I} = \frac{\boldsymbol{I}.\boldsymbol{W}}{|\boldsymbol{w}|}$

$= \frac{48}{5}$

$= 9\frac{3}{5}$

$= 9.6$

Step 5: Write out the interpretation

The irrigation flow vector has a 9.6-unit component in the direction of the wind vector. This means that the wind will push the irrigation flow slightly off its intended path in the direction of the wind by 9.6 units. Hence the farmer should adjust the angle of the irrigation pipe slightly to counteract this drift caused by the wind.

Example 5.11

A construction worker is using a ramp to transport building materials onto a platform. The ramp is inclined at an angle of 30° to the ground and the force required to move a load along the ramp is measured to be 100N. Calculate the effective force acting in the direction of the horizontal that is needed to move the load.

Solution

Step 1: Identify the angle of inclination.

$\theta = 30°$

Step 2: Identify the magnitude of the Force needed.

$|\boldsymbol{F}| = 100N$

Step 3: Calculate the scalar projection

$|\mathbf{F}|\cos\theta = 100\cos 30°$

$= 86.60\text{N}$

The force acting in the direction of the horizontal that is needed to move the load upwards is 86.60N.

EXTENDED READING

- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. A boat is positioned at point A (0, 0) on a river, and another point B(6, 0) is located 6 km downstream. A third point P on the river, representing a midway check, is supposed to divide AB internally in the ratio 2:3.

   a. Determine the coordinates of point P.

   b. If the boat travels from A to P at a speed of 2 km/h, calculate the time it takes to reach P.

2. Relative to a fixed origin O, the respective position vectors of three points A, B and

   C are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

   a. Determine, in component form, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

   b. Hence find, to the nearest degree, the angle BAC.

   c. Calculate the area of the triangle BAC.

3. Relative to the position vector O, the position vectors are given by:

4. $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix}$.

   a. Show clearly that $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BD}$

   b. State the ratio AB: BC

   c. Determine the area of triangle ABD

5. Find the angle between the vectors. Give your answers to 1 decimal place.

   a. 2i + 3j and 4i + j

   b. —i — j and —i —2j

   c. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$

6. Relative to an origin O, the position vectors of the points A and B are given by

7. $\overrightarrow{OA}$ = 2i – 8j + 4k and $\overrightarrow{OB}$ = 7i + 2j – k.

   a. Find the value of $\overrightarrow{OA} \cdot \overrightarrow{OB}$ and hence state whether angle AOB is acute, obtuse or a right angle.

b. If the point X is such that $\overrightarrow{AX} = \frac{2}{5}\overrightarrow{AB}$, find the unit vector in the direction of OX.

8. Miriam has her own small helicopter. One afternoon, she flies for 1 hour with a velocity of 120i + 160j km/h where *i* and *j* are unit vectors in the directions east and north. Then she flies due north for 1 hour at the same speed. Finally, she returns to her starting point; flying in a straight line at the same speed. Find, to the nearest degree, the direction in which she travels on the final leg of her journey and, to the nearest minute, how long it takes her.

9. An aircraft travels from City $A(0, 0)$ to City $B(250, 0)$, which are 250 km apart. A fuel check point, F is located at a point dividing AB internally in the ratio 1:4, closer to City A.

a. Find the coordinates of the fuel check point F .

b. If the airplane flies from City A at a speed of 500 km/h, how long will it take to reach F?

10. Vectors **b** and **e** are such that $|b| = 5$cm, $|e| = 12$cm and $|b + e| = \sqrt{229}$. Find

a. The angle between **b** and **e.**

b. The scalar (dot) product of **b** and **e**.

11. Transpose the vector $\boldsymbol{g} = (15\ 6\ 7)$.

SECTION

6 MATRICES

MODELLING WITH ALGEBRA

Application of Algebra

INTRODUCTION

Ever wondered how your favourite animation movie characters get to move from one place to another in the same scene? Well transformation of matrices helps the computer graphic designers with that. The matrix concept is important to study because it is applicable in fields such as genetics and machine learning, organising data for pattern recognition and supporting advancements in medicine. In year one we discussed the concept of matrices, especially 2×2 matrices, and the varied arithmetic operations that can be performed on them. This section is going to expose you to the determinant of 3×3 matrices, inverse of 2×2 matrices, solving of systems of linear equations using matrices and how to solve and model real-life problems using matrices.

Next time you watch a cartoon animation, think about matrices!

KEY IDEAS

- A cofactor is obtained by multiplying the element's place sign and minor
- A matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions arranged in rows and columns.
- In a 3×3 matrix, a minor of an element is the determinant of the 2×2 submatrix that remains after removing the row and column containing that element.
- Transposing a matrix is obtained by flipping the matrix over its diagonal, effectively switching its rows with its columns.

REVISION ON TYPES OF MATRICES AND MATRIX ALGEBRA

A matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, organised in rows and columns. Each entry in a matrix is called ***elements*** or ***entries*** and it is identified by its row and column indices.

For example, in the 3 x 3 matrix below you can see the labelling of each element according to its row and column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Everyday situations that exemplify the concept of matrices include:

a. Image compression (matrix transformation)

b. Electrical circuit analysis (matrix equation)

c. 3D transformations (matrix multiplication)

Let us look at some types of matrices.

1. A square matrix is matrix that has the same number of rows and the same number of columns. For instance, $B = \begin{bmatrix} 2 & 7 \\ 5 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 4 \\ 2 & 6 & 3 \end{bmatrix}$ are examples of square matrices. B is a 2×2 matrix and E is a 3 x 3 matrix.
2. A rectangular matrix is a matrix in which the number of rows is not equal to the number of columns. For example, $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix}$ and $D = \begin{pmatrix} 4 & 2 \\ 3 & 5 \\ 2 & 6 \end{pmatrix}$ are rectangular matrices. B is a 2×3 matrix and D is a 3×2 matrix. The number of rows are put first, followed by the number of columns.
3. A zero matrix is a matrix whose entries are all zeros or are equivalent to zero. Examples are, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $(0 \quad 0)$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i+i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x-x \end{pmatrix}$.
4. A diagonal matrix is a square matrix in which all elements outside the diagonal (top left corner to bottom right corner) are zero. It has non-zero elements on its main diagonal. For example, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

5. An identity/unit matrix is a unique square matrix with the element one (1) as entries on the main diagonal and zero (0) everywhere else.
6. For a 2 × 2 matrix, the identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For a 3 × 3 matrix, the identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is denoted by I.
7. A triangular matrix has the entries above and/or below the diagonal as zeros. When only the entries **above the diagonal are zero**, we refer to the matrix as a **lower triangular matrix**. On the other hand, if the entries **below the diagonal are zero**, we refer to them as **upper triangular matrices.** For example, $\begin{bmatrix} 4 & 3 & 6 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix while $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 3 & 4 & 3 \end{bmatrix}$ is a lower triangular matrix.

Now, work through these examples to consolidate your knowledge. You can work individually, or in pairs.

Example 6.1

Which of the following matrices are zero matrices?

a. $A = \begin{pmatrix} x - x & 0 & 0 \\ 0 & -b + b & 0 \end{pmatrix}$

b. $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

c. $C = \begin{pmatrix} m - m & 0 \\ 0 & 0 \end{pmatrix}$

Solution

a. Simplifying the entries in matrix A gives $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, thus A is a zero matrix

b. Matrix B has two (2) non-zero entries, thus B is not a zero matrix

c. Matrix C can be simplified as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ making it a zero matrix

Hence options A and C are zero matrices.

Example 6.2

If $P = \begin{pmatrix} \frac{3}{2} & x \\ y - x & -3 \end{pmatrix}$ and $Q = \begin{pmatrix} x & 1.5 \\ 3.5 & -3 \end{pmatrix}$ are two equal matrices, find the values of x and y.

Solution

Since $P = Q$, the entries for each column and row need to be the same.

$\frac{3}{2} = x$ and $y - x = 3.5$

$y - \frac{3}{2} = 3.5$

$y = 5$

Example 6.3

Find the product of the two matrices, $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution

$$\boldsymbol{AI} = \begin{vmatrix} 2 \times 1 + 3 \times 0 & 2 \times 0 + 3 \times 1 \\ 4 \times 1 + 5 \times 0 & 4 \times 0 + 5 \times 1 \end{vmatrix} = \boldsymbol{AI} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \boldsymbol{A}$$

Example 6.4

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$, find;

a. $A + B$

b. $B + A$

c. $A + (B + C)$

d. $(A + B) + C$

Solution

a. $A + B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} = \begin{pmatrix} -5 + 4 & 3 - 3 \\ 2 + 7 & -1 - 5 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$

b. $B + A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 - 5 & -3 + 3 \\ 7 + 2 & -5 - 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$

c. $B + C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4-1 & -3+3 \\ 7+5 & -5+4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix}$

$A + (B + C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix} = \begin{pmatrix} -5+3 & 3+0 \\ 2+12 & -1-1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$

d. $(A + B) + C = \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -1-1 & 0+3 \\ 9+5 & -6+4 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$

Example 6.5

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$, evaluate the following;

a. $A - B$

b. $B - A$

c. $A - (B - C)$

d. $(A - B) - C$

Solution

a. $A - B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} = \begin{pmatrix} -5-4 & 3-(-3) \\ 2-7 & -1-(-5) \end{pmatrix} = \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix}$

b. $B - A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} - \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4-(-5) & -3-3 \\ 7-2 & -5-(-1) \end{pmatrix} = \begin{pmatrix} 9 & -6 \\ 5 & -4 \end{pmatrix}$

c. $B - C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4-(-1) & -3-3 \\ 7-5 & -5-4 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix}$

$A - (B - C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix} = \begin{pmatrix} -5-5 & 3-(-6) \\ 2-2 & -1-(-9) \end{pmatrix}$

$= \begin{pmatrix} -10 & 9 \\ 0 & 8 \end{pmatrix}$

d. $(A - B) - C = \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -9-(-1) & 6-3 \\ -5-5 & 4-4 \end{pmatrix} = \begin{pmatrix} -8 & 3 \\ 0 & 0 \end{pmatrix}$

Example 6.6

Given that $A = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and the matrix $P = -2A$, write out the matrix P.

Solution

$$P = -2A = -2\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ -4 \end{pmatrix}$$

Example 6.7

If $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$, find $2\mathbf{A} - 3\mathbf{B} + 4\mathbf{C}$.

Solution

$$2\mathbf{A} = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix},$$

$$3\mathbf{B} = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix}$$

$$4\mathbf{C} = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$\therefore 2\mathbf{A} - 3\mathbf{B} + 4\mathbf{C} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix} = \begin{pmatrix} -8 & 3 \\ 27 & -36 \end{pmatrix}$$

Example 6.8

Given that $A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}$

Evaluate the following:

a. AB

b. BA

Solution

a. $$AB = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix} = \begin{pmatrix} 4(-2) + (-2)(-2) & 4(3) + (-2)(-7) \\ 1(-2) + 3(-2) & 1(3) + 3(-7) \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 26 \\ -8 & -18 \end{pmatrix}$$

b. $$BA = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} (-2)(4) + (3)(1) & (-2)(-2) + (3)(3) \\ (-2)(4) + (-7)(1) & (-2)(-2) + (-7)(3) \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 13 \\ -15 & -17 \end{pmatrix}$$

Example 6.9

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$, find p and q if $AB = \begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + 3\begin{pmatrix} 4 & 2 \\ -3 & q \end{pmatrix}$

Solution

$$\text{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix}$$

$$\text{And AB} = \begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + 3\begin{pmatrix} 4 & 2 \\ -3 & q \end{pmatrix} = \begin{pmatrix} p & -2 \\ -6 & 4 \end{pmatrix} + \begin{pmatrix} 12 & 6 \\ -9 & 3q \end{pmatrix}$$

$$= \begin{pmatrix} p + 12 & 4 \\ -15 & 4 + 3q \end{pmatrix}$$

Equating corresponding entries, it follows that:

$$\begin{pmatrix} -7 & 4 \\ -15 & 10 \end{pmatrix} = \begin{pmatrix} p + 12 & 4 \\ -15 & 4 + 3q \end{pmatrix}$$

$-7 = p + 12 \Longrightarrow \boldsymbol{p} = -19$

$10 = 4 + 3q \Longrightarrow 3q = 6 \therefore \boldsymbol{q} = 2$

Example 6.10

If $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, find the determinant.

Solution

Using the formula, $\text{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determinant of A, $\det(A) = ad - bc$

$\therefore \det(A) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

Example 6.11

Evaluate the determinants of the following matrices:

a. $\begin{vmatrix} 21 & 4 \\ 17 & 9 \end{vmatrix}$

b. $\begin{vmatrix} 2 & -8 \\ -3 & 6 \end{vmatrix}$

c. $\begin{vmatrix} a+3 & 7-a \\ a & 7 \end{vmatrix}$

Solution

a. $det\begin{vmatrix} 21 & 4 \\ 17 & 9 \end{vmatrix} = (21 \times 9) - (4 \times 17) = 121$

b. $\det\begin{vmatrix} 2 & -8 \\ -3 & 6 \end{vmatrix} = ((2 \times 6) - (-8 \times -3)) = -12$

c. $\det\begin{vmatrix} a+3 & 7-a \\ a & 7 \end{vmatrix} = (7(a+3) - a(7-a)) = 7a + 21 - 7a + a^2 = a^2 + 21$

DETERMINANT OF 3 × 3 MATRICES

In year one we learnt and practised finding the determinant of a 2×2 matrix:

Determinant $= ad - bc$ given the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, we will apply that concept to help find the determinant of a 3×3 matrix. In order to work out the determinant of a 3×3 matrix we need to define minors and cofactors of a square matrix.

Minor of a Square Matrix

In a 3×3 matrix, a minor of an element is the determinant of the 2×2 submatrix that remains after removing the row and column containing that element. If we let $B = (b_{ij})$ be a matrix of order $n \times n$. Then, the minor of element (b_{ij}) (denoted by (B_{ij})) is the determinant of the $(n-1) \times (n-1)$ matrix obtained after removing row i and column j from B.

Taking a 3×3 square matrix, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$,

the minor of B can be found by crossing out the row and column of b_{11}, b_{12}, b_{13}

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Illustration 1 Illustration 2 Illustration 3

From the illustrations, the minor of b_{11} is $\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix}$, b_{12} *is* $\begin{pmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{pmatrix}$ and b_{13} *is* $\begin{pmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$.
Note that each element in the square matrix has its own minor.

Example 6.12

Given $M = \begin{pmatrix} 4 & 8 & -3 \\ 10 & 5 & 11 \\ -6 & 7 & 4 \end{pmatrix}$, determine the minor of M_{12}.

Solution

The minor of M_{12} is the determinant of the matrix formed by crossing out row one and column two from M.

The minor of $M_{12} = \begin{pmatrix} 10 & 11 \\ -6 & 4 \end{pmatrix} = (10 \times 4) - (11 \times -6) = 106$.

Cofactors of a square matrix

Every entry in a square matrix has its own cofactor. The cofactor is obtained by multiplying the element's place sign and minor. The place sign of N_{ij} is given by $(-1)^{i+j}$. Thus, the cofactor of an element, N_{ij} in a square matrix is given $(-1)^{i+j}$. N_{ij}. Each entry in the square matrix has its own cofactor.

Let us go through these examples to practise how to determine a cofactor. You can work individually or in pairs.

Example 6.13

Given $B = \begin{pmatrix} 4 & 8 & -3 \\ 10 & 5 & 11 \\ -6 & 7 & 4 \end{pmatrix}$, determine the cofactor of 7 (B_{32}) and 10 (B_{21}).

Solution

The minor of 7 = $\begin{pmatrix} 4 & -3 \\ 10 & 11 \end{pmatrix} = (11 \times 4) - (10 \times -3) = 74$

The place sign of 7 is $(-1)^{3+2} = -1$

$\therefore$ the cofactor of 7 is $(-1)(74) = -74$

The cofactor of 10 is $(-1)^{2+1} \times \det of \begin{pmatrix} 8 & -3 \\ 7 & 4 \end{pmatrix}$

$(-1)[(8 \times 4) - (-3 \times 7)] = -53$

$\therefore$ the cofactor of $10 = -53$

Having defined the minor and cofactor of square matrices, let's talk about using the theorem of expanding by cofactors to find the determinant of a matrix.

The theorem states that the value of a determinant can be found by expanding by cofactors of any row or column.

Note: No matter which row or column is chosen, the value for the determinant will remain the same. It is advised to use the row or column which has the most zeros simply because it makes the calculations easier.

Generally, to evaluate the determinant of a matrix $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, by expanding the cofactors.

Choose, for example, the second row then

$|B| = b_{21}\,(-1)^{2+1}(M_{21}) + b_{22}\,(-1)^{2+2}(M_{22}) + b_{23}\,(-1)^{2+3}(M_{23})$,

where M represents the minor.

Example 6.14

Evaluate the determinant of $C = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 3 \\ 0 & -1 & -3 \end{pmatrix}$ by expanding the cofactors.

Solution

Finding the determinant of C from row one, $C = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 3 \\ 0 & -1 & -3 \end{pmatrix}$

$|C| = 2\,(-1)^{1+1}(C_{11}) + 5\,(-1)^{1+2}(C_{12}) + 1\,(-1)^{1+3}(C_{13})$

$|C| = 2\begin{vmatrix} 8 & 3 \\ -1 & -3 \end{vmatrix} - 5\begin{vmatrix} 3 & 3 \\ -0 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & 8 \\ 0 & -1 \end{vmatrix}$

$|C| = 2[(8 \times -3) - (-1 \times 3)] - 5[(3 \times -3) - (0 \times 3)] + 1[(3 \times -1) - (0 \times 8)]$

$|C| = 2 \times -21 - 5 \times -9 - 3$

$|C| = 0$

Activity 6.1 – Verifying the determinant using different rows/columns

1. Working in small groups, find the determinant of C by choosing different rows/columns.
2. Did you get the same determinant or not? Check each other's workings as each route to finding the determinant should give the same answer.
3. Discuss the results with your group.

Using the Sarrus rule to find the determinant

Another way to find the determinant of matrices is to use the Sarrus rule.

To find the determinant of $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, extend the matrix by writing the first and second columns to the right-hand side of B.

i.e., $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{22} \end{matrix}$

Add the product of the diagonals

$b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}$

Subtract the product of the opposite diagonals

$-b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} - b_{13}b_{22}b_{31}$

Put the two formulae together to find the determinant.

$\det(B) = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} - b_{13}b_{22}b_{31}$

Example 6.15

Evaluate the determinant of $C = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 3 \\ 0 & -1 & -3 \end{pmatrix}$.

Solution

Using the Sarrus rule:

$$\begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 3 \\ 0 & -1 & -3 \end{pmatrix}\begin{matrix} 2 & 5 \\ 3 & 8 \\ 0 & -1 \end{matrix}$$

$$\begin{aligned} \det(A) &= (2)(8)(-3) + (5)(3)(0) + (1)(3)(-1) - (5)(3)(-3) - (2)(3)(-1) - (1)(8)(0) \\ &= -48 + 0 - 3 - (-45) - (-6) - 0 \\ &= 0 \end{aligned}$$

Example 6.16

Evaluate the determinant of the matrix $D = \begin{pmatrix} 2 & 4 & 3 \\ 6 & 2 & 3 \\ 2 & 5 & 4 \end{pmatrix}$

Solution

$$D = \begin{pmatrix} 2 & 4 & 3 \\ 6 & 2 & 3 \\ 2 & 5 & 4 \end{pmatrix} = 2\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} - 4\begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix}$$

$$\begin{aligned} &= 2(8 - 15) - 4(24 - 6) + 3(30 - 4) \\ &= -14 - 72 + 78 \\ &= -8 \end{aligned}$$

Singular and Non-Singular Matrices

A square matrix is classified as a singular matrix if it has a determinant of zero and it has no inverse. On the other hand, a non-singular matrix is a square matrix with a non-zero determinant and it has an inverse.

Tabular representation on the differences between singular and non-singular matrix

	Non-singular	Singular
B is	invertible	Not invertible
Rows	independent	dependent
Columns	independent	dependent
det(B)	$\neq 0$	$= 0$
$Bx = 0$	one solution $x = 0$	infinitely, many solutions
$Bx = c$	one solution	No solution

To find out if a square matrix is singular or non-singular find the determinant of the given matrix. If the determinant is zero, it is a singular matrix and if the determinant is not zero, the matrix is non-singular.

Let us work through some examples.

Example 6.17

Find the value of x, if $A = \begin{pmatrix} 4 & 12 \\ x & 6 \end{pmatrix}$ is a singular matrix.

Solution

If A is a singular matrix then $ad - bc = 0$

$(4)(6) - (x)(12) = 0$

$24 - 12x = 0$

$24 = 12x$

$x = 2$

Example 6.18

Find the value of x, if $B = \begin{vmatrix} 5x + 2 & 6x - 3 \\ 4 & 3 \end{vmatrix}$ is a singular matrix.

Solution

$B = \begin{vmatrix} 5x+2 & 6x-3 \\ 4 & 3 \end{vmatrix}$

$(5x+2)(3) - (6x-3)(4) = 0$

$3(5x+2) - 4(6x-3) = 0$

$15x + 6 - 24x + 12 = 0$

$18 - 9x = 0$

$x = 2$

Transpose of a matrix

In section 5 we looked at transposing vectors. We can also transponse matrices in a similar manner. The transpose of a matrix is determined by interchanging its rows into columns or columns into rows. For example, given $G = \begin{pmatrix} 3 & 2 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, then $G^T = \begin{pmatrix} 3 & 5 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}$.

Let us work through these examples.

Example 6.19

Given that $P = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$, find P^T.

Solution

$P^T = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$

Example 6.20

What is the transpose of a 4×3 matrix?

Solution

The transpose of a 4×3 is a 3×4 matrix.

Activity 6.2

1. Given that $Q = \begin{pmatrix} -2 & -1 & 4 \\ 4 & 3 & -9 \end{pmatrix}$, find Q^T.
2. Now find the transpose of Q^T.
3. What is your observation?
4. Discuss with your peers.

Hopefully you found that $(Q^T)^T = Q$

Addition Property of Transpose

If $R = \begin{pmatrix} 3 & 2 \\ -5 & 8 \end{pmatrix}$ and $S = \begin{pmatrix} 7 & 1 \\ -6 & 3 \end{pmatrix}$, find $(R + S)^T$ and $R^T + S^T$.

What is the relationship between $(R + S)^T$ and $R^T + S^T$?

Solution

$$(R + S) = \begin{pmatrix} 3 & 2 \\ -5 & 8 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ -11 & 11 \end{pmatrix}$$

$$(R + S)^T = \begin{pmatrix} 10 & -11 \\ 3 & 11 \end{pmatrix}$$

$$R^T = \begin{pmatrix} 3 & -5 \\ 2 & 8 \end{pmatrix} \text{ and } S^T = \begin{pmatrix} 7 & -6 \\ 1 & 3 \end{pmatrix}$$

$$\therefore R^T + S^T = \begin{pmatrix} 3 & -5 \\ 2 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -11 \\ 3 & 11 \end{pmatrix}$$

The sum of $(R + S)^T$ and $R^T + S^T$ are the same.

This communicates that adding the transpose of two individual matrices say M *and* N i.e., $M^T + N^T$ is equal to the transpose of the sum of the two matrices $(M + N)^T$.

INVERSES OF 2 × 2 MATRICES

The real numbers 6 and $\frac{1}{6}$ are referred to as multiplicative inverses because their product is a multiplicative identity 1 (i.e., $6 \times \frac{1}{6} = 1$). If we have the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

) multiplying by $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$, results in $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is the 2 x 2 identity matrix (**I**).

Hence, we call $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$ multiplicative inverses of each other.

Let us go through this activity to know how to find the inverse of a 2×2 matrix.

Activity 6.3 – Proof of inverse of a matrix formula

Work through these steps in small groups where possible.

1. Choose a 2×2 matrix such as $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
2. Assume the inverse of M is $M^{-1} = \begin{pmatrix} m & p \\ n & v \end{pmatrix}$.
3. Multiply M by its inverse
4. Equate the result to a 2×2 Identity matrix.
5. Using your idea in equality of matrices, equate each entry in the matrix on the LHS to the corresponding entry on the RHS.
6. Solve the equations containing *m* and *n* simultaneously.
7. Now, solve the equations containing *p* and *v* simultaneously.
8. Create a new matrix using the results for *m, n, p* and *v*.
9. Factorise $\left(\frac{1}{\text{ad} - \text{bc}}\right)$ from the new matrix.
10. Hopefully you have $\frac{1}{\text{d} - \text{bc}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$?
11. Congratulations, you have derived the formula for finding the inverse of a 2 x 2 matrix.

Note: $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is what is referred to as the **adjoint** of a matrix while ad – bc is the determinant for the matrix M.

Let us work through these examples to practise finding adjoints and inverses of a matrix.

Example 6.21

Find the inverse of matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{etA}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{(1)(2)-(3)(3)}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$$

$$\frac{-1}{7}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & -\frac{2}{7} \end{pmatrix}$$

Example 6.22

Given the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find the inverse matrix B^{-1}.

Solution

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$B^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Example 6.23

Find the inverse of the matrices of the following:

$$C = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$$

Hence, find:

a. CC^{-1}

b. $D^{-1}D$

Solution

$\det C = 2 - 9 = -7$

$$\therefore C^{-1} = -\tfrac{1}{7}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & -\frac{2}{7} \end{pmatrix}$$

Hence:

a. $$CC^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \times -\tfrac{1}{7}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix} = -\tfrac{1}{7}\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$$
$$= -\tfrac{1}{7}\begin{pmatrix} 2-9 & -6+6 \\ 3-3 & -9+2 \end{pmatrix}$$
$$= -\tfrac{1}{7}\begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-7}{-7} & \frac{0}{7} \\ \frac{0}{-7} & \frac{-7}{-7} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\therefore CC^{-1} = \boldsymbol{I}$$

b. $D^{-1}D$

$$D^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$D^{-1}D = \begin{pmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4-4 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\therefore D^{-1}D = \boldsymbol{I}.$$

USING MATRICES TO SOLVE SYSTEMS OF LINEAR EQUATIONS

In matrices, a system of linear equations can be written in the form MX = C where M is the matrix of coefficients, X is the unknown column matrix and C is a column matrix of constants. The matrix method for solving systems of linear equations is more helpful when the number of equations is many. We will explore how the matrix method and Cramer's rule can be used to solve systems of linear equations.

Let us start with examples on equations involving 2×2 matrices.

Example 6.24

Find the value of x and y in $\begin{pmatrix} 2x+2 & 3 \\ 4 & 6x-3 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 4 & 9 \end{pmatrix}$.

Solution

$$\begin{pmatrix} 2x+2 & 3 \\ 4 & 6y-3 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ -3 & 9 \end{pmatrix}$$

$2x + 2 = 10$

$\boldsymbol{x} = 4$

Also, $6y - 3 = 9$

$6y = 12$

$y = 2$

Example 6.27

$$\begin{pmatrix} 3x-2 & 2 \\ 1 & 2y+1 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ 1 & -6 \end{pmatrix}$$

$3x - 2 = -7$

$x = -5-$

Also, $2y + 1 = -6$

$y = -\frac{7}{2}$

Note: In Examples 6.26 and 6.27, the entries of the matrices on the left hand and the right hand were compared to each other. Since the matrices are equal, the entries with the variables were equated and a solution was found.

Example 6.28

Given that $\begin{pmatrix} 4 & -1 \\ 3 & x \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 19 \\ 8 \end{pmatrix}$, find x.

Solution

$$\begin{pmatrix} (4\times 4)+ & (-1\times -3) \\ (3\times 4)+ & (-3\times x) \end{pmatrix} = \begin{pmatrix} 19 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 19 \\ 12-3x \end{pmatrix} = \begin{pmatrix} 19 \\ 8 \end{pmatrix}$$

$$12 - 3x = 8$$

$$3x = 8 + 12$$

$$3x = 20$$

$$x = \frac{20}{3}$$

Example 6.29

Given that $\begin{pmatrix} -3 & 4 \\ y & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$, find y

Solution

$$\begin{pmatrix} -3 & 4 \\ y & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 2y+6 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$2y + 6 = 12$$

$$2y = 12 - 6$$

$$2y = 6$$

$$y = 3$$

Note: In Example 6.28 and 6.29, the concept of multiplication of matrices were applied before equating the entry with the variable to its corresponding entry.

Using Cramer's rule for involving 2 × 2 matrix

In using the Cramer's rule, you need to know the determinant of a matrix to find the solution of the system MX = C.

Given two linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, rewrite the equations in the form MX = C.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Keeping in mind that M is the coefficient matrix, X is the variable and matrix C is the constant matrix, you have to find the determinants so that

$\det(M) = a_1 b_2 - b_1 a_2$.

Thus, $\det(M_x) = \begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix} = c_1 b_2 - b_1 c_2$ and

$\det(M_y) = \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} = a_1 c_2 - c_1 a_2$

Therefore $x = \dfrac{\det(M_x)}{\det(M)}$, and $y = \dfrac{\det(M_y)}{\det(M)}$.

Example 6.30

Solve for x and y in the equations

a. $3x - 4y = 2$ *and* $x + 2y = 4$

b. $3x - y = 7$ *and* $5x - 4y = 0$

Solution

a. $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Find the determinant $\det(M) = 6 + 4 = 10$

To find the determinant of M_{x}, use the constant $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and the last column in the matrix $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$M_x = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix} = 4 + 16 = 20$

Repeat the same process to find M_y

To find $M_y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = 12 - 2 = 10$

$x = \dfrac{det\,(M_x)}{\det(M)} = \dfrac{20}{10} = 2$

$y = \dfrac{det\,(M_y)}{\det(M)} = \dfrac{10}{10} = 1$

$\therefore$ The solution to the simultaneous equation is $x = 2$ *and* $y = 1$.

b. $\begin{pmatrix} 3 & -1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

Find the determinant $\det(M) = -12 + 5 = -7$

To find the determinant of M_y use the constant $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ and the last column in the matrix $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$

$M_x = \begin{pmatrix} 7 & -1 \\ 0 & -4 \end{pmatrix} = -28 - 0 = -28$

$M_y = \begin{pmatrix} 3 & 7 \\ 5 & 0 \end{pmatrix} = 0 - 35 = -35$

Now, $x = \frac{det\,(M_x)}{\det(M)} = \frac{-28}{-7} = 4$

$y = \frac{det\left(M_y\right)}{\det(M)} = \frac{-35}{-7} = 5$

$\therefore$ The solution to the simultaneous equation is $x = 4 and\ y = 5$.

Now, let us talk about solving equations involving **3** $\times$ **3** matrices.

Using Cramer's rule for equations involving a 3 × 3 matrix

In using the Cramer's rule, you need to know the determinant of a matrix to find the solution of the system MX = C.

Given three linear equations, say $a_1x + b_1y + c_1z = e_1$, $a_2x + b_2y + c_2z = e_2$, and $a_3x + b_3y + c_3z = e_3$, rewrite the equations in the form MX = C.

Where, $M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$.

Now you have to find det(M).

Using $\begin{pmatrix} e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \\ e_3 & b_3 & c_3 \end{pmatrix}$ find $\det\left(M_x\right)$, $\begin{pmatrix} a_1 & e_1 & c_1 \\ a_2 & e_2 & c_2 \\ a_3 & e_3 & c_3 \end{pmatrix}$ to find $\det\left(M_y\right)$ and

$\begin{pmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{pmatrix}$ to find $\det\left(M_z\right)$. You will notice that you are swapping the constant column with the corresponding, x, y or z column for M_x, M_y, M_z.

From Cramer's rule, $x = \frac{\det(M_x)}{\det(M)}$, $y = \frac{\det(M_y)}{\det(M)}$ and $z = \frac{\det(M_z)}{\det(M)}$.

Let us go through these examples to practise solving systems of linear equations.

Example 6.31

Solve for x, y and z in the equations:

a.
$$\begin{aligned} 5x + 3y + z &= 2 \\ 4x + 3y + 2z &= 1 \\ 6x + 4y + z &= 1 \end{aligned}$$

b.
$$\begin{aligned} 2x + y - 4z &= 4 \\ x + y + z &= 6 \\ x - 3y + 2z &= 8 \end{aligned}$$

Solution

a. $\begin{pmatrix} 5 & 3 & 1 \\ 4 & 3 & 2 \\ 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 3 & 2 \\ 6 & 4 & 1 \end{pmatrix}$, and find its determinant.

$$det(A) = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 3 & 2 \\ 6 & 4 & 1 \end{pmatrix} = 5(3-8) - 3(4-12) + 1(16-18)$$

$$= -25 + 24 - 2$$

$$= -3$$

To find the $det(A_x)$ replace the first column matrix $\begin{pmatrix} 5 \\ 4 \\ 6 \end{pmatrix}$ by the constant $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$det(A_x) = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 1 \end{pmatrix} = 2(3-8) - 3(1-2) + 1(4-3)$$

$$= -10 + 3 + 1$$

$$= -6$$

To find the $det(A_y)$ replace the second column matrix $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ by the constant $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$det(A_y) = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 1 & 2 \\ 6 & 1 & 1 \end{pmatrix} = 5(1-2) - 2(4-12) + 1(4-6)$$
$$= -5 + 16 - 2$$
$$= 9$$

To find the $det(A_z)$ replace the third column matrix $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ by the constant $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$det(A_z) = \begin{pmatrix} 5 & 3 & 2 \\ 4 & 3 & 1 \\ 6 & 4 & 1 \end{pmatrix} = 5(3-4) - 3(4-6) + 2(16-18)$$
$$= -5 + 6 - 4$$
$$= -3$$

Hence; $x = \frac{det(A_x)}{det(A)} = \frac{-6}{-3} = 2$, $y = \frac{det(A_y)}{det(A)} = \frac{9}{-3} = -3$, $z = \frac{det(A_z)}{det(A)} = \frac{-3}{-3} = 1$

$\therefore$ The solution to the simultaneous equation is $x = 2$, $y = -3$ *and* $z = 1$.

b. $\begin{pmatrix} 2 & 1 & -4 \\ 1 & 1 & 1 \\ 4 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$

Let A= $\begin{pmatrix} 2 & 1 & -4 \\ 1 & 1 & 1 \\ 4 & -3 & 2 \end{pmatrix}$

det A= $2(2+3) - 1(2-4) - 4(-3-4) = 10 + 2 + 28 = 40$

$$det(A_x) = \begin{pmatrix} 4 & 1 & -4 \\ 6 & 1 & 1 \\ 8 & -3 & 2 \end{pmatrix} = 4(2+3) - 1(12-8) - 4(-18-8)$$
$$= 20 - 4 + 104$$
$$= 120$$

$$det(A_y) = \begin{pmatrix} 2 & 4 & -4 \\ 1 & 6 & 1 \\ 4 & 8 & 2 \end{pmatrix} = 2(12-8) - 4(2-4) - 4(8-24)$$
$$= 8 + 8 + 64$$
$$= 80$$

$$det(A_z) = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 6 \\ 4 & -3 & 8 \end{pmatrix} = 2(8+18) - 1(8-24) + 4(-3-4)$$
$$= 52 + 16 - 28$$
$$= 40$$

Hence, $x = \frac{det(A_x)}{det(A)} = \frac{120}{40} = 3,\ y = \frac{det(A_y)}{det(A)} = \frac{80}{40} = 2,\ z = \frac{det(A_z)}{det(A)} = \frac{40}{40} = 1$

$\therefore$ The solution to the simultaneous equation is $x = 3,\ y = 2$ *and* $z = 1$.

USING MATRICES TO MODEL AND SOLVE REAL LIFE PROBLEMS

Just as we learnt in the section on vectors, knowing how the mathematical concepts we learn are applicable in solving real life problems is very important. Matrices are useful in industries that deal with input-output analysis. Considering this, let us go through examples showing how matrices are applied in real life.

Activity 6.4 – Group Investigation

1. In small groups, discuss and come up with circumstances in real life where matrices can be used in solving problems.
2. Prepare a presentation (use PowerPoint if accessible) and share with your classmates and your teacher.

Example 6.32

A dietitian wants to create a daily plan for a patient using three food supplements, x, y and z. The patient requires exactly 22 units of calcium, 18 units of iron and 40 units of vitamin A.

The nutrient content per ounce of supplements is represented on the table below.

	X	Y	Z
Calcium (units)	2	3	1
Iron (units)	1	2	3
Vitamin A (units)	4	5	2

How much of each supplement (x, y, z) should the patient consume daily to meet the nutrient requirements exactly.

Solution

Let x, y and z represent the amount of supplement to be taken daily. The following can be derived from the table.

$$\begin{pmatrix} 2x+ & 3y+ & z=22 \\ x+ & 2y+ & 3z=18 \\ 4x+ & 5y+ & 2z=40 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 22 \\ 18 \\ 40 \end{pmatrix}$$

By finding the determinant

Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 2 \end{pmatrix}$

$\det(A) = 2(4-15) - 3(2-12) + 1(5-8) = -22 + 30 - 3 = 5$

Replace x-column with constants:

Let $A_x = \begin{pmatrix} 22 & 3 & 1 \\ 18 & 2 & 3 \\ 40 & 5 & 2 \end{pmatrix}$

$\det(A_x) = 22(4-15) - 3(36-120) + 1(90-80) = -242 + 252 + 10 = 20$

Replace y-column with constants:

Let $A_y = \begin{pmatrix} 2 & 22 & 1 \\ 1 & 18 & 3 \\ 4 & 40 & 2 \end{pmatrix}$

$\det(A_y) = 2(36-120) - 22(2-12) + 1(40-72) = -168 + 220 - 32 = 20$

Replace z-column with constants:

Let $A_z = \begin{pmatrix} 2 & 3 & 22 \\ 1 & 2 & 18 \\ 4 & 5 & 40 \end{pmatrix}$

$\det(A_z) = 2(80-90) - 3(40-72) + 22(5-8) = -20 + 96 - 66 = 10$

$$\frac{det(A_x)}{\det(A)} = \frac{20}{5} = 4, \frac{det(A_y)}{\det(A)} = \frac{20}{5} = 4, \frac{det(A_z)}{\det(A)} = \frac{10}{5} = 2$$

∴ The dietitian's solution is to have 4 lots of x, 4 lots of y and 2 lots of z.

Example 6.33

A Company produces two products, A and B. The ideal combination is when: $2A + 3B = 85$ and $A + 2B = 50$.

Find the values of A and B which makes both of these true.

Solution

Coefficient matrix

$$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}85\\50\end{pmatrix}$$

Let M= $\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$

Determinant M = (2)(2) – (3)(1) = 1

Let $M_A = \begin{pmatrix}85 & 3\\50 & 2\end{pmatrix}$

$det(M_A) = (85)(2) - (3)(50) = 170 - 150 = 20$

Let $M_B = \begin{pmatrix}2 & 85\\1 & 50\end{pmatrix}$

$det(M_B) = (2)(50) - (85)(1) = 100 - 85 = 15$

Hence, $\frac{\det(M_A)}{\det(M)} = \frac{20}{1} = 20$ and $\frac{\det(M_B)}{\det(M)} = \frac{15}{1} = 15$

∴ A should be 20 and B should be 15.

Example 6.34

Forces F_1 and F_2 act on an object with the following force equations:

$F_1 + F_2 = 8$ and $2F_1 - F_2 = 7$.

Find the values of F_1 and F_2.

Solution

Coefficient matrix

$$\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$$

Let A = $\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$

Determinant A = (1)(–1) – (1)(2) = –3

Determinant $(A_x) = \begin{pmatrix}8 & 1\\7 & -1\end{pmatrix} = -8 - 7 = -15$

Determinant of $(A_x) = \begin{pmatrix}1 & 8\\2 & 7\end{pmatrix} = 7 - 16 = -9$

Hence, $F_1 \frac{\det(A_x)}{\det(A)} = \frac{-15}{-3} = 5$

$F_2 \frac{\det(A_y)}{\det(A)} = \frac{-9}{-3} = 3$

$\therefore F_1 = 5N \; and \; F_2 = 3N$

EXTENDED READING

- Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952. (Pages 779 – 809)
- Effective Elective Mathematics for Senior High School; Christian Akrong Hesse Reloaded Edition, Third Revision in 2003 (Pages 367-375).
- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740. (Pages 477 – 489)

REVIEW QUESTIONS

1. If $T = \begin{pmatrix} 4 & -2 \\ 3x & 5 \end{pmatrix}$, find the value of x, if the determinant of $T = 32$.
2. If $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, find A^{-1}.
3. Evaluate the determinant of $P = \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$
4. Determine whether $Q = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 4 & 3 \\ 3 & 2 & -1 \end{pmatrix}$, is singular or not.
5. $B = \begin{pmatrix} 2 & x+5 \\ 2x-2 & -9 \end{pmatrix}$ Given that B is a singular matrix, find the value of x.
6. Find the determinant of the matrix $\begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 3 \\ 0 & -1 & -3 \end{pmatrix}$
7. **a.** If $A = \begin{pmatrix} 3 & 1 \\ x & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 2 \\ 4 & y \end{pmatrix}$, find the values of x and y given that $AB = BA$

   b. Show that the inverse of the matrix AB is $B^{-1}A^{-1}$.
8. If $A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}$, find A^{-1}.
9. Hence find the matrix B such that $BA = C$, where $C = \begin{pmatrix} 12 & 0 \\ 1 & 5 \end{pmatrix}$.
9. Use Cramer's rule to find the value x and y in the equations:

   $3x + 2y = 10$ and $7x + 5y = 23$.
10. Use Cramer's rule to find the value x, y and z in the equations:

    $2x - 3y + 4z = 10$, $x + y - z = 1$ and $x - 6y + 3z = -1$.
11. Use Cramer's rule to find the value x, y and z in the equations:

    $2x + 3z = -1$, $y + 2z = 5$ and $x + y = 1$.
12. A financial advisor wants to invest in three stocks A, B and C. The advisor requires a portfolio with a total value of exactly GH¢ 1920, annual returns

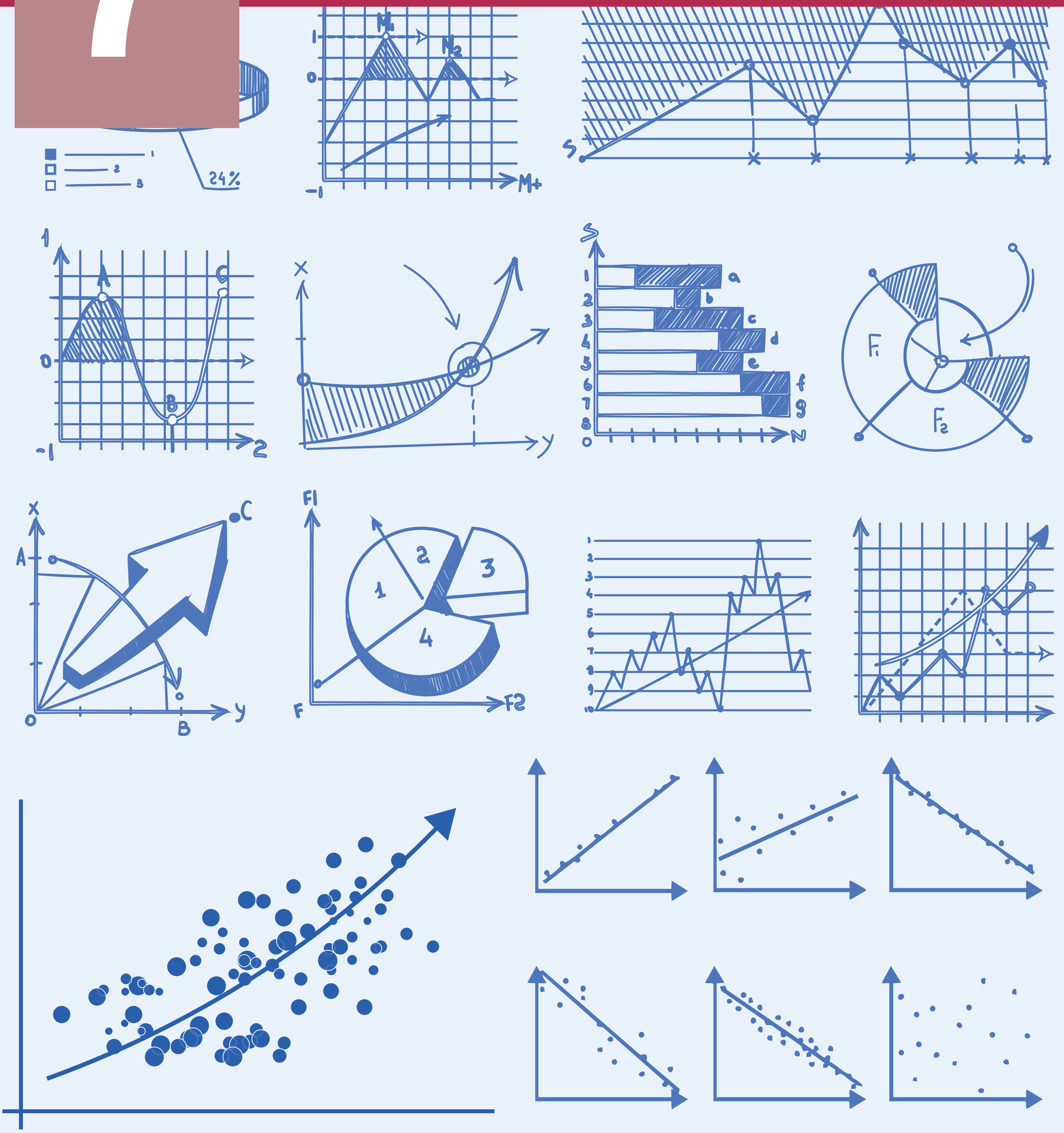
of exactly GH¢ 66 and Beta value (market risk) of exactly 4.2. The characteristics of each stock value are:

	A	B	C
Stock value	100	60	80
Annual returns	2	3	4
Beta value	0.1	0.3	0.2

How many shares of each stock A, B and C should the advisor buy to meet the portfolio requirement exactly.

SECTION

7 CORRELATION



HANDLING DATA

Organising, Representing and Interpreting Data

INTRODUCTION

The ability to organise, analyse and present data is an important skill as this is essential in real life. In this section we will develop these skills whilst learning about the concept of correlation and statistics, along with details on univariate and bivariate data, scatter plots, analysing scatter plots and spearman rank's correlation.

KEY IDEAS

- A measure of the correlation is called the **correlation coefficient**.
- A measure of the nature and strength of the relationship between two or more variables is called **correlation**.
- A **scatter plot** is used to visually show the relationship between variables of bivariate data.
- A **variable** is a characteristic or measurement that can be determined for a population.
- An estimate of a linear function for bivariate data is called the **line of best fit**.
- **Univariate data** involves only one variable while **bivariate data** involves exactly two variables.

DISTINGUISHING BETWEEN UNIVARIATE AND BIVARIATE DATA AND THE CONCEPT OF CORRELATION

Univariate Data

In statistics, a variable is a characteristic or measurement that can be determined for a population. Variables may describe values like weight in kg, height in metres or favourite food.

The word; "Uni" means "one" and "variate" is another word for "variable". So, "univariate data" means data with only one variable or a single characteristic. Examples are the age of learners, the height of trees, the weight of babies at birth or the wages of construction workers.

You can describe patterns in univariate data using central tendencies (mean, mode and median) and dispersion (range, interquartile range, standard deviation and variance). Frequency distribution tables, bar charts, histograms, pie charts and frequency polygons can be used to present univariate data.

Bivariate Data

"Bi" means "two". So, "bivariate data" means data that involves exactly two variables. Examples:

1. Amount of money spent on advertisement and total revenue.
2. Hours of work and wages
3. Hours of study and grades
4. Years of schooling and annual income
5. Total rainfall and amount of cocoa harvested.
6. Income and Expenditure

The table below shows an example of bivariate data. It gives the hours of study and marks of ten students in Mathematics.

Name	Abi	Ato	Fafa	Bushra	Akos	Yaa	Oko	Mbo	Kasi	John
Hours	7.5	4.5	6	9	4	6	5	7	8	7
Marks	65	50	75	95	45	52	70	83	90	50

This data contains two variables, hours and marks.

To present bivariate data, we can use a scatter plot. It allows us to visualise the relationship between the two variables.

Difference between Univariate and Bivariate data

Univariate data	*Bivariate data*
Involves a single variable	Involves two variables
Does not deal with cause and relationship	Deals with cause and relationship
The purpose is to describe	The purpose is to explain
Does not have any dependent variable	Contains only one dependent variable

Concept of Correlation

Correlation is a measure of the nature and strength of the relationship between two or more variables. It describes how well two variables go together. The relationship between two variables may be positive, negative or non-existent (no correlation). The strength of the relationship ranges from -1 to 1 inclusive.

A positive correlation between two variables means that an increase in the value of one variable is likely to increase the value of the other. Likewise, a decrease in one of the variables will cause a reduction in the other. It is a relationship that moves in tandem (in the same direction). It shows a direct relation between two variables. An example of a positive correlation is the relationship between Temperature and human water consumption. Generally, as temperature increases, people consume more water. Another example is deforestation and erosion. As we cut down more trees, the likelihood of erosion increases. Other examples are hours of study and test scores, demand and price, cleanliness and life expectancy, prices of fuel and prices of transportation, investment and interest, hours of work and pay. Figure 7.1 below shows a scatter plot with positive correlation.

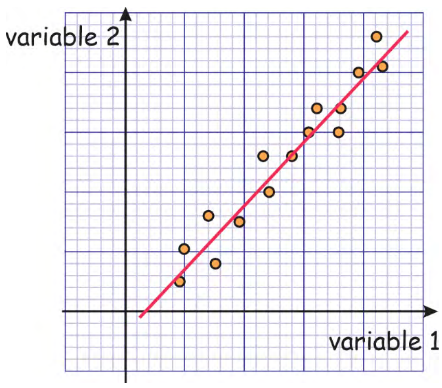


Figure 7.1: Positive Correlation

A positive perfect correlation means that 100% of the time, the variables in question move together by the same percentage and direction and all points on the scatter plot lie on a perfect straight line, as shown in Figure 7.2.

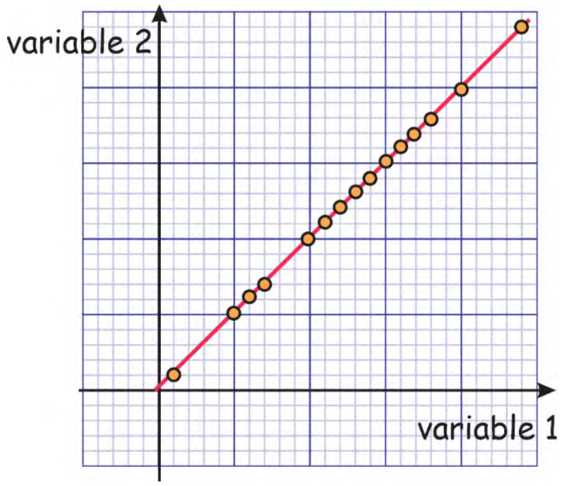


Figure 7.2: Perfect Positive Correlation

Negative Correlation

A negative correlation describes an inverse relationship. As one variable increases, the other decreases. It describes the relationship between two variables that change in opposite directions. An example is the relationship between age and agility. As your age increases, your agility decreases. Another example is speed and travel time. The higher the speed, the shorter the travel time. Other examples include preparation and mistakes, supply and price, exercise and body weight, inflation and purchasing power. Figure 7.3 shows negative correlation on a scatter plot, whilst Figure 7.4 shows a perfect negative correlation with all points on a perfect straight line.

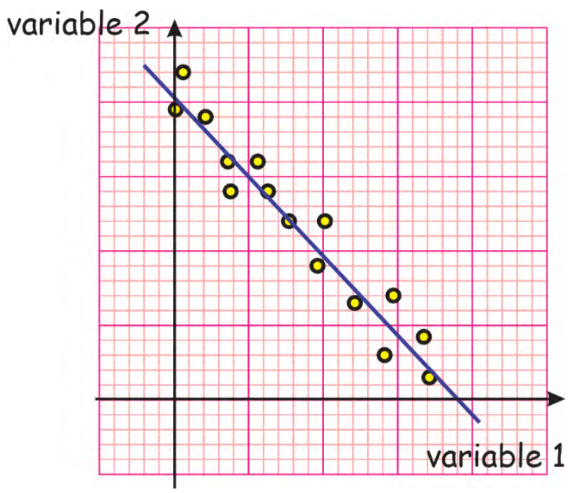


Figure 7.3: Negative correlation

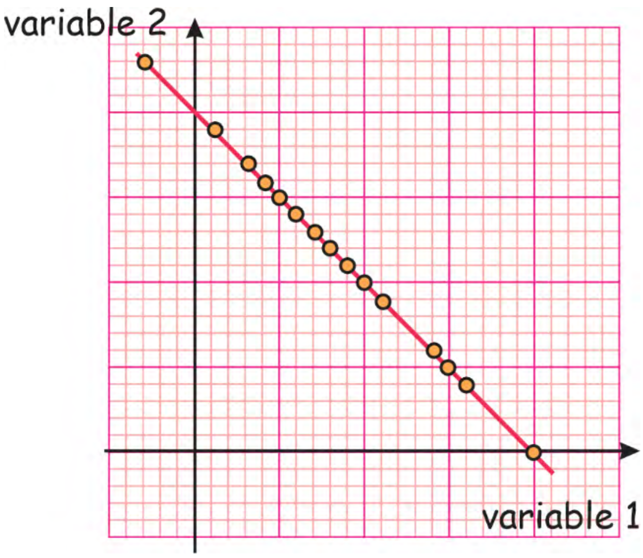


Figure 7.4: Perfect Negative Correlation

Zero or no correlation

This occurs when there is no relationship between two variables. It means changes in one variable do not predict changes in the other. An example is weight and intelligence. Your weight has nothing to do with your intelligence and an increase in your weight has nothing to do with your intelligence. Other examples are blood type and prosperity, shoe size and favourite colour, weight and income. Figure 7.5 shows a random array of dots on a scatter plot which shows no correlation.

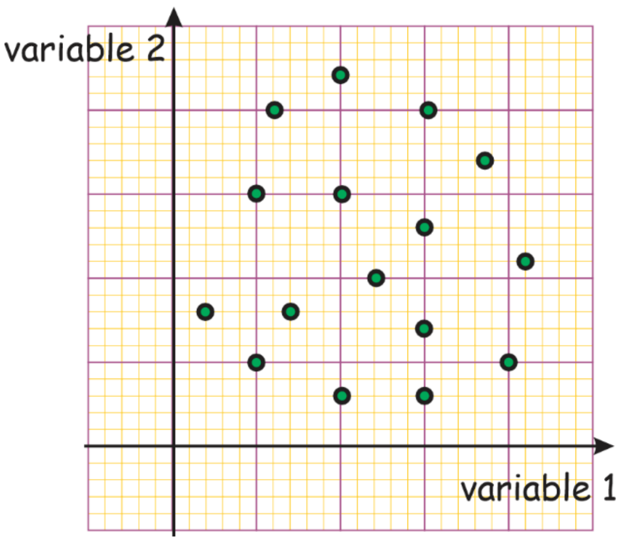


Figure 7.5: No Correlation

CONSTRUCTING A SCATTER PLOT USING GIVEN DATASET

Scatter Plot

The graph below is an example of a scatter plot

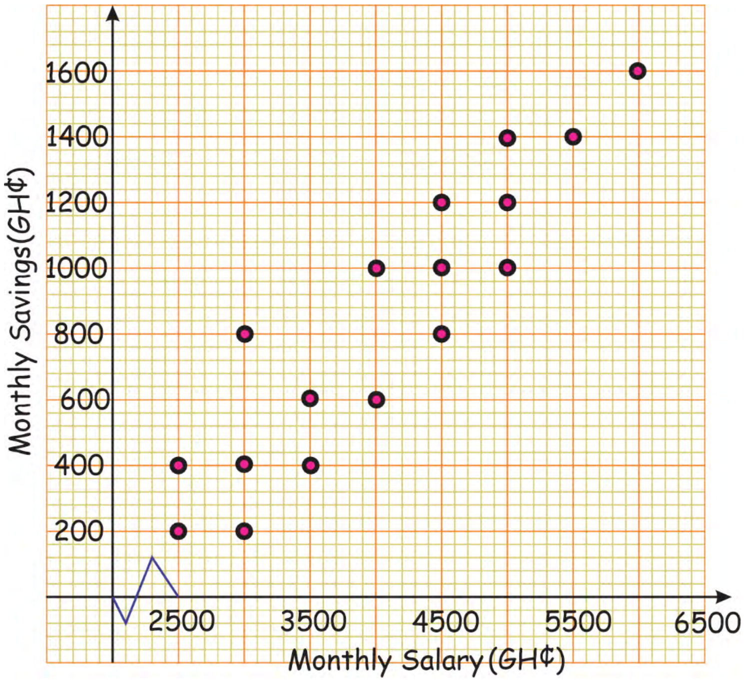


Figure 7.6: Scatter plot

Figure 7.6 shows bivariate data of monthly salary and savings of a sample of 17 teachers. Such a graph helps us to see the relationship between the two variables under consideration. In this example, the variables being compared are monthly salary and savings.

One important thing to note is that correlation does *not* imply that one variable causes the other. For example, there is a strong correlation between accidents and speeding. This does not imply that accidents are caused by speeding. The correlation simply tells us that when speeding increases, we are likely to have more accidents. Likewise, data may give a strong negative correlation between inflation and purchasing power. Does this mean we can conclude that inflation causes a decrease in purchasing power? No! Having a strong correlation does not give us the yardstick to make that conclusion.

Activity 7.1: How to construct a scatter plot manually (Work in pairs)

Use the steps below

Step 1: Identify the variables.

Since we are dealing with bivariate data, it will involve two variables, an independent variable and a dependent variable.

The independent variable is the variable that is being manipulated in the data. It is also called the input variable. This variable is plotted on the x-axis, the horizontal axis.

The dependent variable is the variable being measured. It is the output or the outcome. It is plotted on the y-axis, the vertical axis.

In the example above, the independent variable is Monthly salary and the dependent variable is Monthly savings. This is because savings depend on the monthly salary or savings in this instant are obtained from the monthly salary.

Step 2: Label the axes and scale them.

In the example above, we used an interval of 2cm on the graph to represent GH¢ 1000 on the x axis (monthly salary) and 1cm on the graph to represent GH¢ 200.00 on the y axis.

Step 3: Convert each data point into (x, y) coordinates and plot them on the graph. If two or more points fall on the same point, place them side-by-side.

Example 7.1

Determine the independent and dependent variables in the following scenarios.

a. The effect of exercise on agility

b. The effect of motivation on output.

c. Grades and hours of study

d. Life expectancy and amount of sleep

e. Grades of students in English and Mathematics.

f. Relationship between the rainfall and yield of crops.

g. Relationship between the working time and salary.

Solution

a. The effect of exercise on agility:

b. independent variable = Exercise

c. dependent variable = agility

d. The effect of motivation on output:

e. independent variable = Motivation

f. dependent variable = Output

g. Grades and hours of study:

h. independent variable = hours of study

i. dependent variable = Grades

j. Life expectancy and amount of sleep:

k. independent variable = Amount of sleep

l. dependent variable = Life expectancy

m. Grades of students in English and Mathematics:

n. Either of the subjects can be taken as the independent variable and the other will be the dependent variable.

o. Relationship between the rainfall and yield of crops:

independent variable = Rainfall

dependent variable = Yield of crops

p. Relationship between the working time and salary:

independent variable = Working time

dependent variable = Salary

Example 7.2

State the type of correlation in the following real-life scenarios:

a. Motivation and output of workers

b. Age and Intelligence test scores among adults

c. Being in the choir and academic performance

d. Education and life expectancy

e. Distance and time

f. Speed and distance

g. Distance and time

Solution

a. Motivation and output of workers.

b. Generally, an increase in motivation results in an increase in the output of workers

c. This is a positive correlation

d. Age and Intelligence test scores among adults

e. Generally, as age increases from about 20 years there is a gradual and continuous decline in intelligence test scores

f. This represents a negative correlation

g. Being in the choir and academic performance

h. There is no correlation between being in the choir and academic performance

i. Education and life expectancy

j. Generally the higher the level of education the higher the life expectancy

Positive correlation

k. Income and number of children

l. Generally, people who earn more have fewer children

m. Negative correlation

n. Speed and distance

o. The higher the speed, the greater the distance covered

p. Positive correlation

q. Distance and time

r. Generally, you need more time to cover longer distance, depending on the same mode of transport being used

s. Positive correlation

Example 7.3

The test scores of ten students in English and Mathematics are as shown:

English (x)	20	25	15	28	10	35	10	24	23	15
Mathematics (y)	25	30	20	35	15	43	20	35	30	15

Draw a scatter diagram for the data.

Solution

Let us use the steps to answer the question

1. From the data provided, the independent variable is being taken as English Marks and dependent variable as Mathematics marks.
2. We will use 2cm to represent 10marks on both axes.
3. Finally plot the ordered pairs (20, 25), (25, 30), (15, 20) (28, 35), (10, 15), (35, 43), (10, 20), (24, 35), (23, 30) and (15, 15)

The graph is as shown:

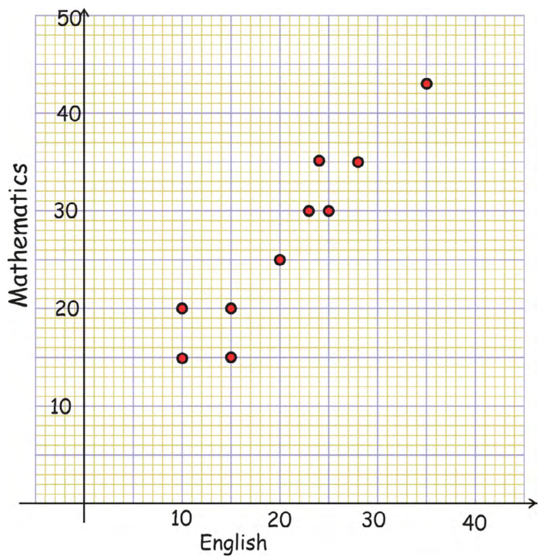


Figure 7.7: A scatter graph showing the correlation between student's English and Maths scores

Example 7.4

The data below shows the temperature of the day and the number of people wearing jackets.

Temperature (x)	35	6	17	15	21	20	10	7
Number wearing jackets (y)	11	45	22	30	20	25	32	42

Draw a scatter plot for the data.

Solution

Steps involved in drawing a scatter diagram:

1. Draw two perpendicular lines to from the x-y plane on a sheet of graph paper.

2. Choose an appropriate scale for the axes (x-axis – independent variable - representing temperature and y-axis – dependent variable - representing number wearing jackets)
3. The ordered pairs (35, 11), (6, 45), (17, 22), (15, 30), (21, 20), (20, 25), (10, 32) and (7, 42) are plotted in the x-y plane.

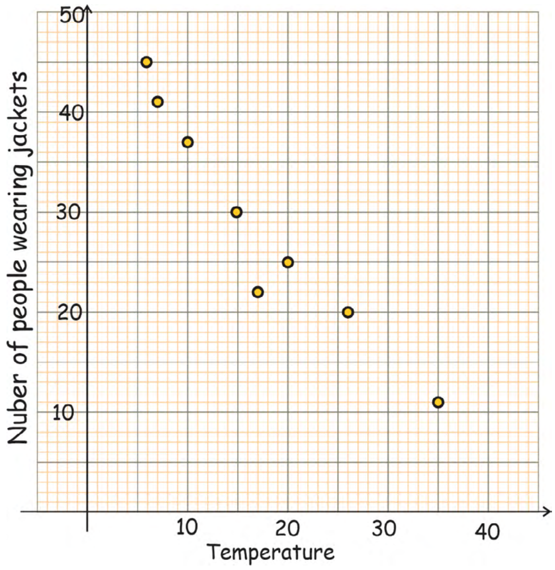


Figure 7.8: A scatter graph showing the temperature and the number of people wearing jackets

Activity 7.2: How to construct a scatter diagram using Excel

In small groups, work through the following steps to produce a scatter diagram in Excel of the data above.

Step 1: Open a new document in the Excel spreadsheet.

A section of the interface should look like the picture below

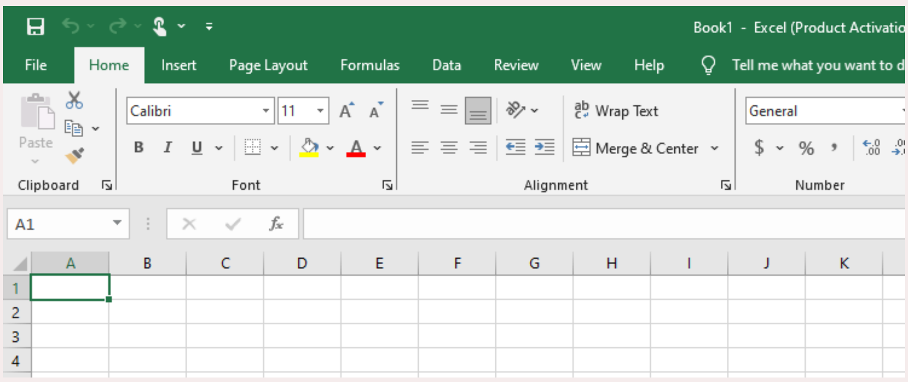


Figure 7.9: Excel spreadsheet

Step 2: Input your data in the first and second columns of the spreadsheet.

We will use the data below

Temperature (x)	35	6	17	15	21	20	10	7
Number wearing jackets (y)	11	45	22	30	20	25	32	42

Your spreadsheet should be similar to the one below

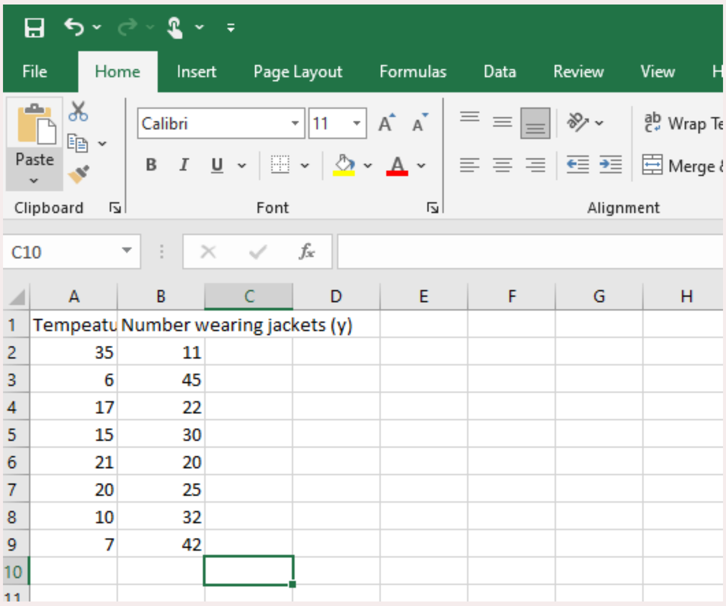


Figure 7.10

Step 3: Select the data, go to **Insert** and choose **Scatter** from the drop-down menu.

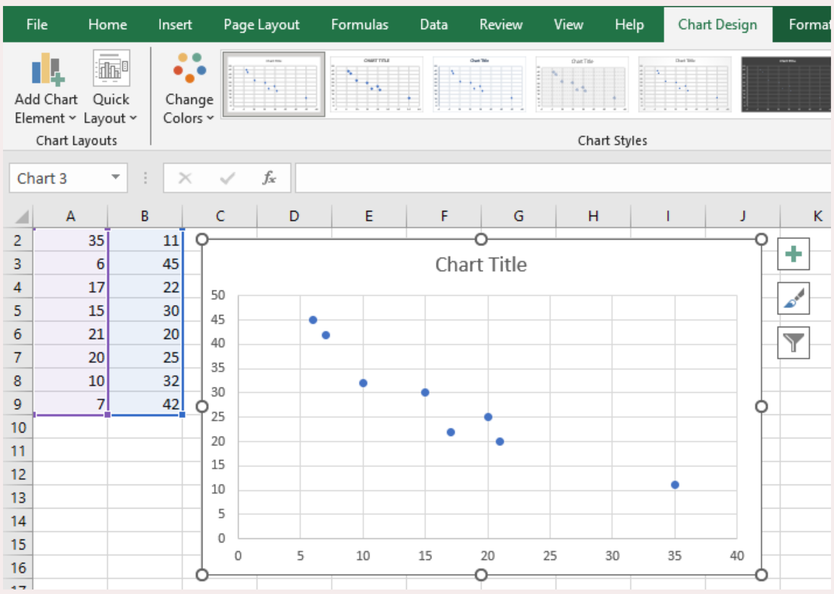


Figure 7.11

Step 4: Insert the title and the description of the x and y axes.

To do this, click on the plus sign beside the scatter graph. This will open a dropdown menu titled Chart Elements. Choose "Axis Titles" from the dropdown menu.

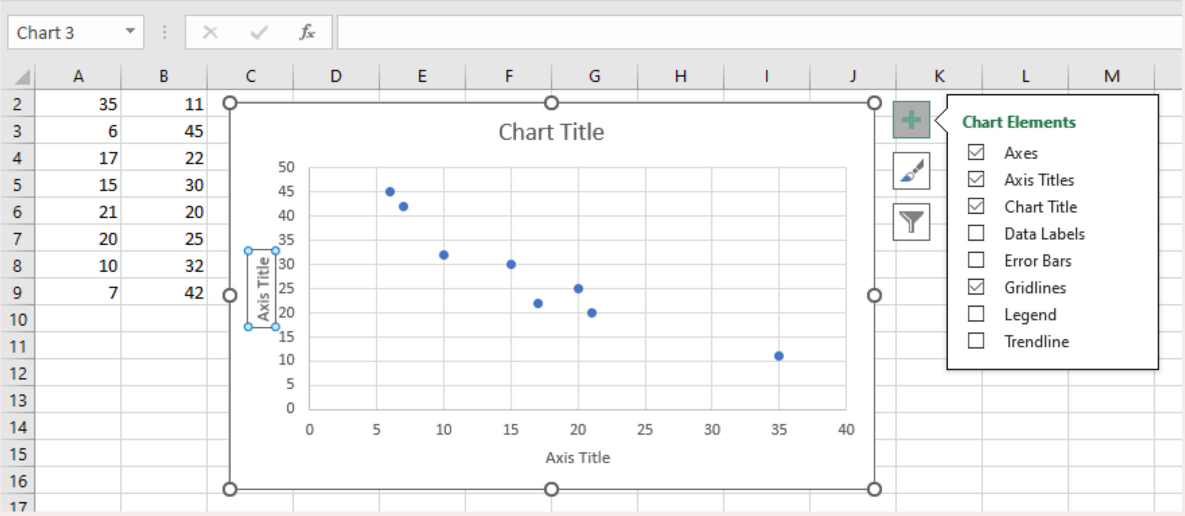


Figure 7.12

Step 5: Your final plot should be similar to the chart below.

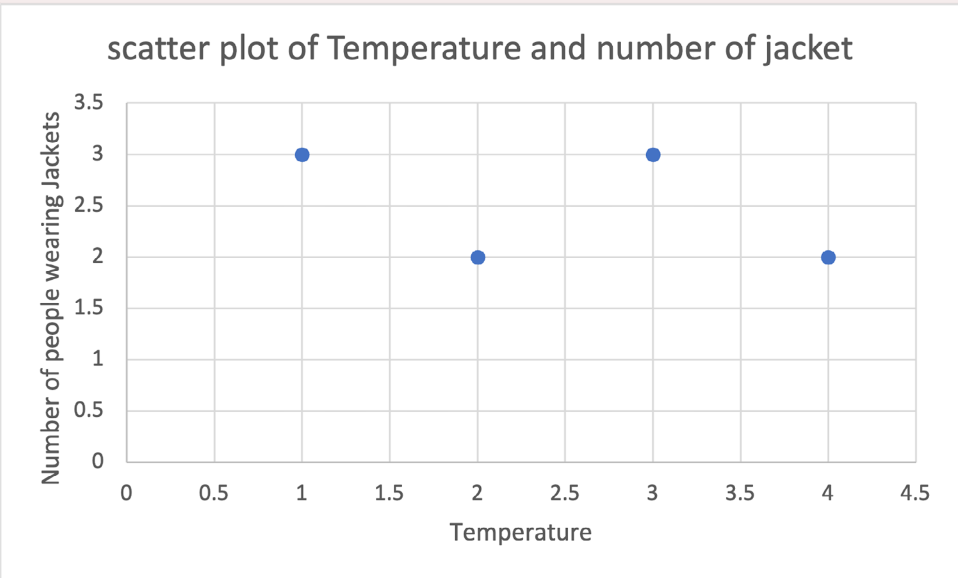


Figure 7.13: A scatter graph showing the temperature and the number of people wearing jackets

DESCRIBING THE RELATIONSHIP BETWEEN TWO VARIABLES

The scatter plot helps us to visualise the relationship between two variables. An alternative way to describe the relationship is to use a linear function (a straight line). Note that this is used to approximate the relationship between the variables being compared as the data points do not form a straight line in most cases. Hence the approximated line is called a **line of best fit**. The line of best fit should either pass through most of the points or closest to most of the points. Because of this, the line of best fit is usually used to predict data values within the given data. This is known as **interpolation**. When used to predict data values outside a given data, it is called **extrapolation**. Pairs of data values (points) relatively far from the line of best fit are called outliers. We will obtain the line of best fit through **observation**.

Example 7.5

The scatter graph shows a sample of students' mid-term and end-of-term physics scores.

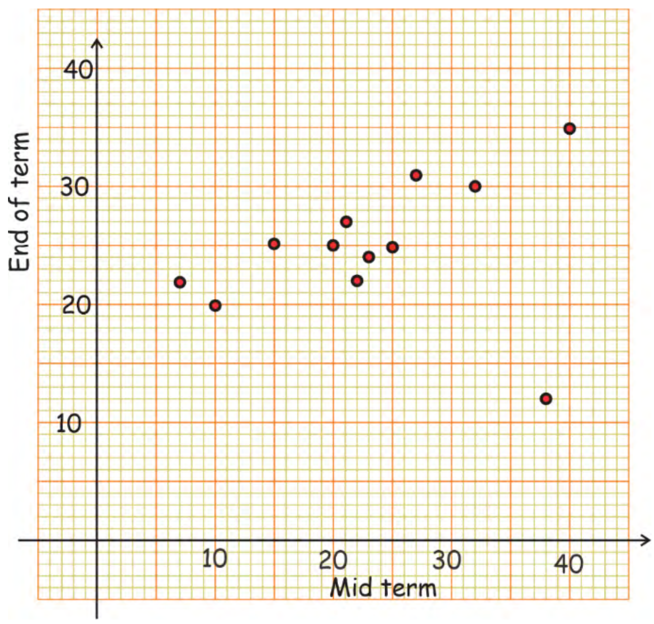


Figure 7.14: A scatter diagram showing student's mid-term scores and their end of term scores in physics

a. Make a copy of the graph and draw a line that best fits the data.

b. Use your graph to find the equation of the line of best fit.

c. Do you identify any outliers in the data?

d. Use your line of best fit to estimate the End of term score of a student who scored 30 in the midterm.

Solution

a.

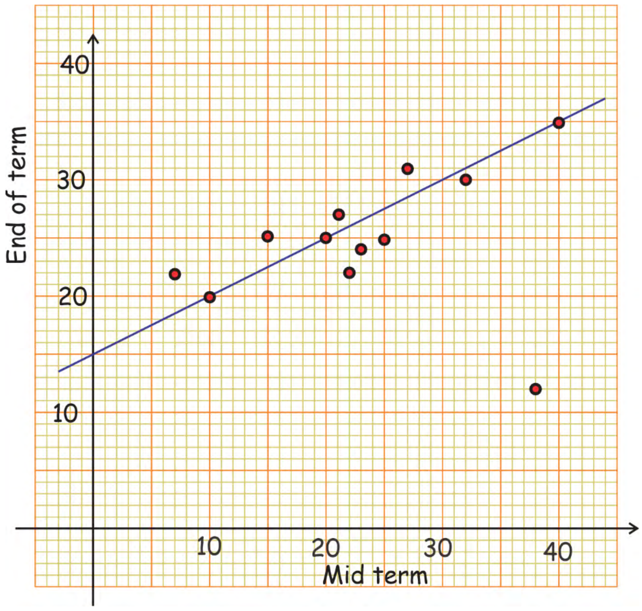


Figure 7.15: A scatter diagram showing student's mid-term scores and their end of term scores in physics with a line of best fit

The line of best fit in this example passes through points (40,35) and (10, 20). It looks close to most of the points except for (38, 12).

b. The equation of the line of best fit is of the form $y = mx + c$

c. Where x =mid-term scores and y =end-of-term scores, m is the slope or gradient of the line and c is the y-coordinate of the y-intercept.

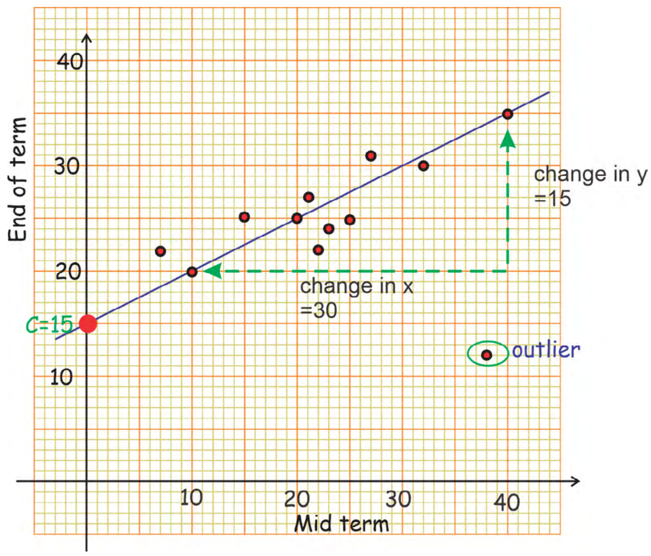


Figure 7.16: Calculating the equation of the line of best fit

From the graph $c = 15$ since the graph intersects the y-axis at (0, 15)

The gradient, $m = \frac{change\ in\ y}{change\ in\ x} = \frac{15}{30} = \frac{1}{2}$

Therefore, the equation of the line of best fit is: $y = \frac{1}{2}x + 15$

d. (38, 12) is an outlier since it is far away from the line of best fit and it shows a student who performed very well in the mid terms, but poorly in the end of term exams. All the other students show positive correlation.

e. To do this, trace a vertical line from the mid-term axis at 30 to meet the line and then trace a horizontal line to the end-of-term axis and record the value.

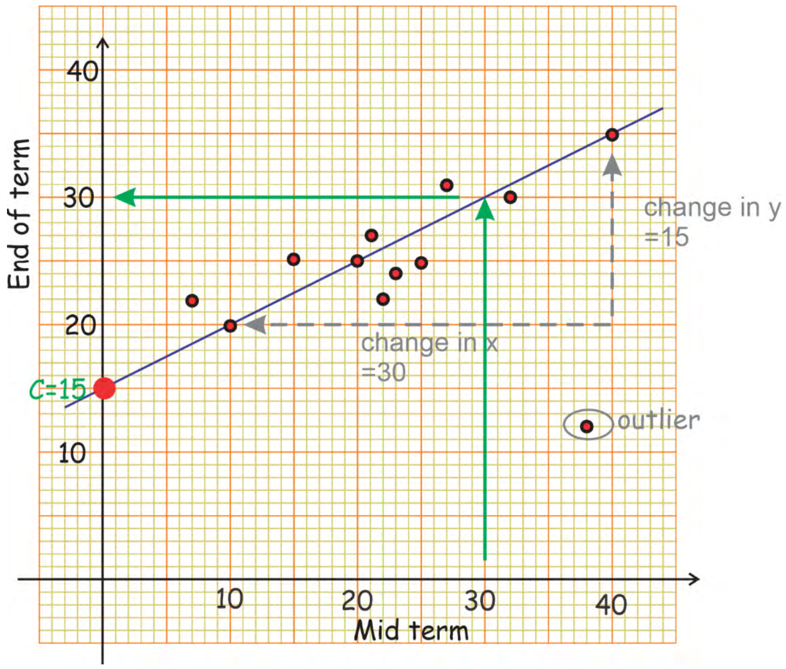


Figure 7.17: Calculating an estimate for end of term score for a student who scored 30 in their mid term

From the graph, a student who scored 30 in the midterm is expected to have a score of 30 at the end of the term. Note that this answer can vary depending on where you have drawn your line of best fit by eye, but it should be around this score.

ANALYSING AND DESCRIBING VISUAL DATA IN A SCATTER PLOT BY INTERPRETING THE RELATIONSHIP BETWEEN GIVEN BIVARIATE DATASETS

We have learnt that the relationship between variables of bivariate data can be positive linear, negative linear, non-linear and no association. We will now apply this knowledge to a few examples.

Example 7.6

Study the graphs below and for each of them determine the type of correlation between the variables they represent.

1. Data showing hours 11 sportswomen hours of exercise and their weight.

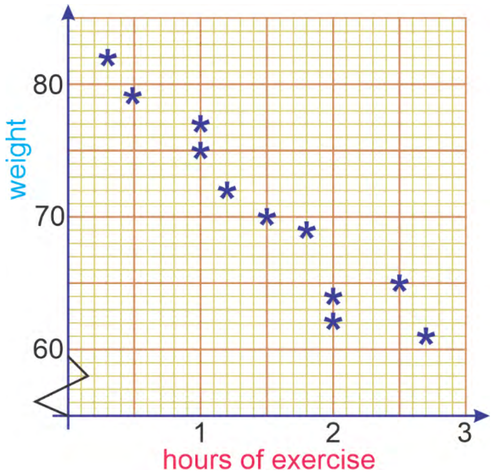


Figure 7.18

2. Data showing the velocity-time plot of a vehicle.

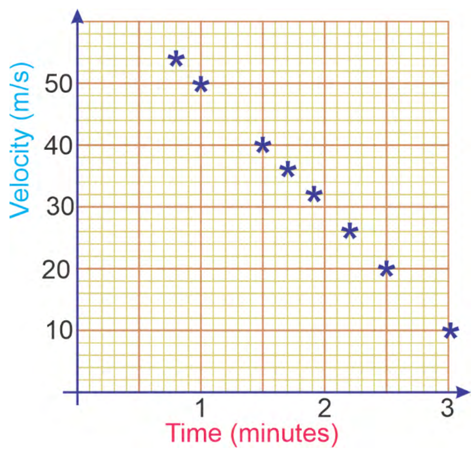


Figure 7.19

3. Marks of 10 students in Mathematics and Physics

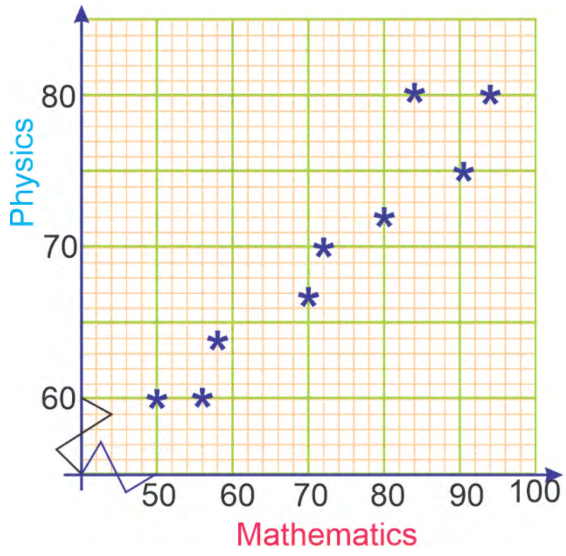


Figure 7.20

4. Age of 11 companies and their annual profit.

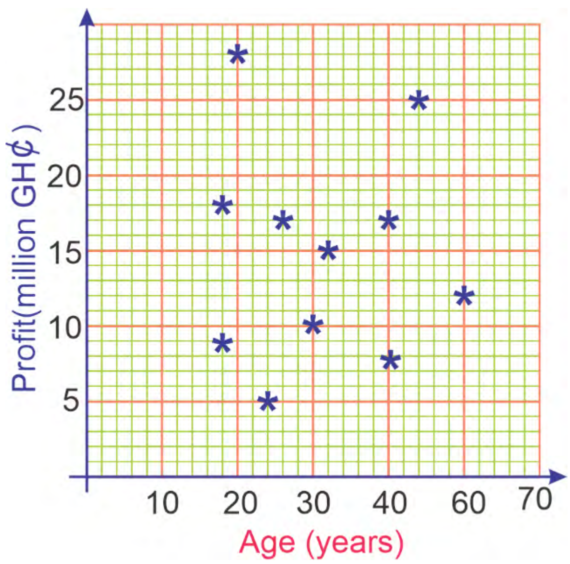
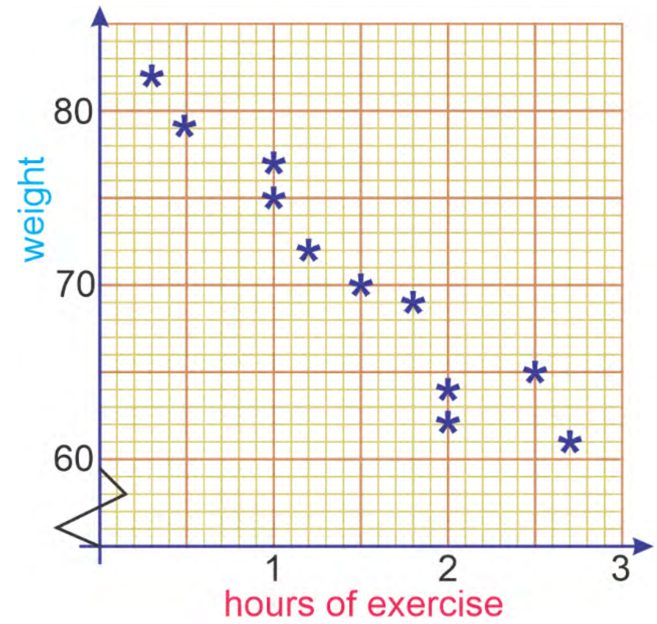


Figure 7.21

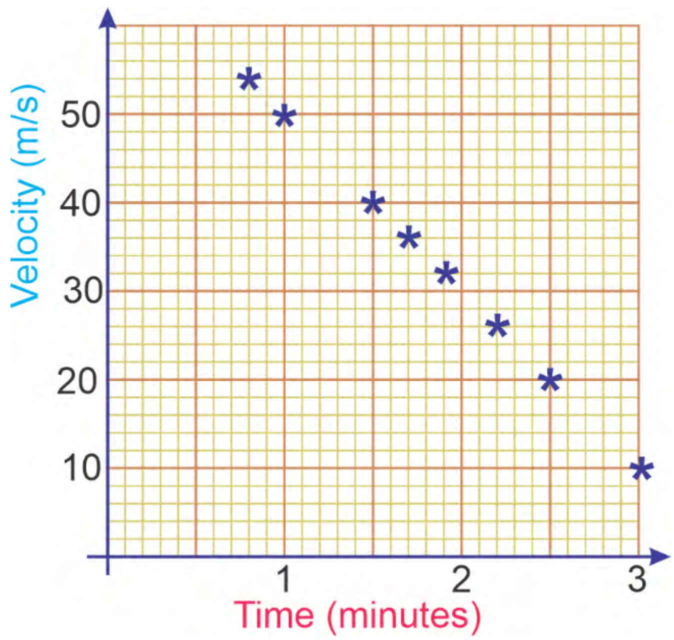
Solution

1.



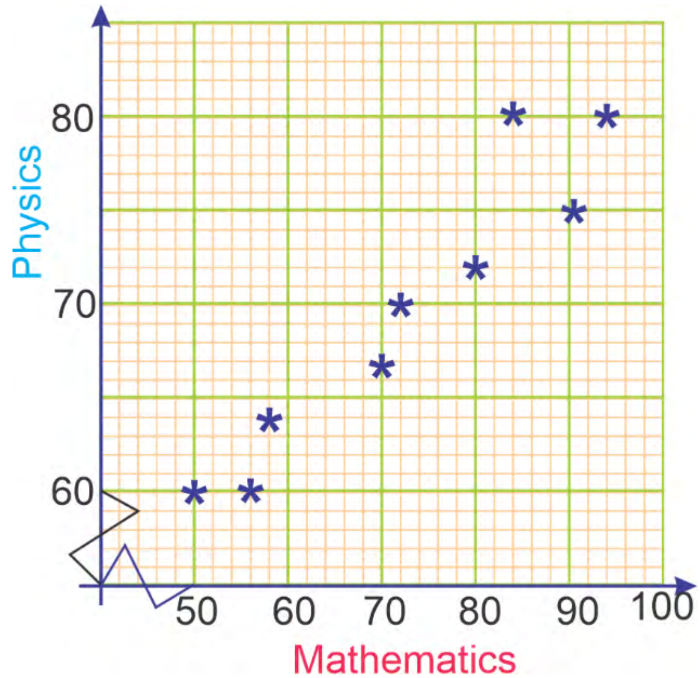
This shows a negative correlation between hours of exercise and weight. This suggests that an increase in hours of exercise is associated with a lower weight.

2.



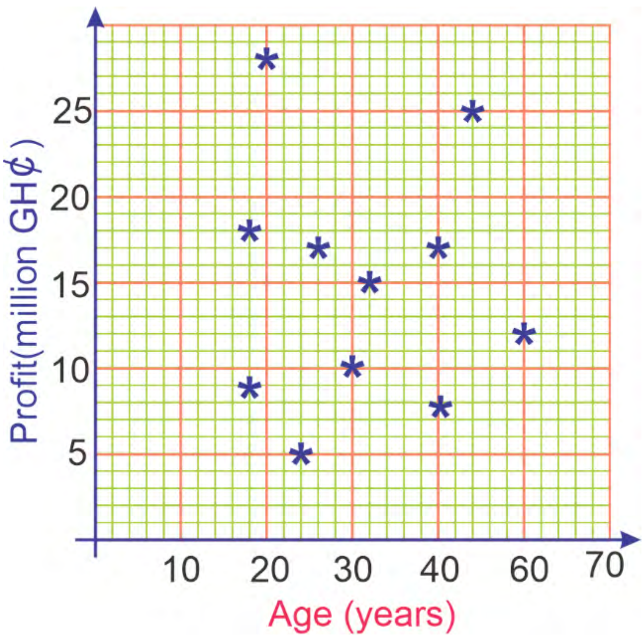
This shows a perfect negative correlation between time and velocity. Every 20m/s decrease in velocity is associated with 1 minute increase in time.

3.



The scatter plot shows a positive correlation between mathematics scores and Physics scores. Generally, a higher mathematics score is associated with a higher Physics score. Likewise, lower Mathematics scores are associated with lower Physics scores.

4.



This shows no correlation between the age of a company and its yearly profit.

Example 7.7

The table gives information about the distance from a school and the monthly rent of 11 houses.

Distance from the school (km)	Monthly rent (GH¢)
12.5	250
20.0	100
18.0	200
10.0	400
7.5	500
5.5	470
5.0	550
14.5	240
10.0	280
8.0	420
12.5	360

a. Draw a scatter graph for the data.

b. Draw a line of best fit for the data.

c. Find the gradient of the equation of the line of best fit and interpret your answer.

d. Find the equation of the line of best fit and use it to find the monthly rent of a building which is 15km from the school.

e. Determine the type of correlation between distance and monthly rent.

Solution

a.

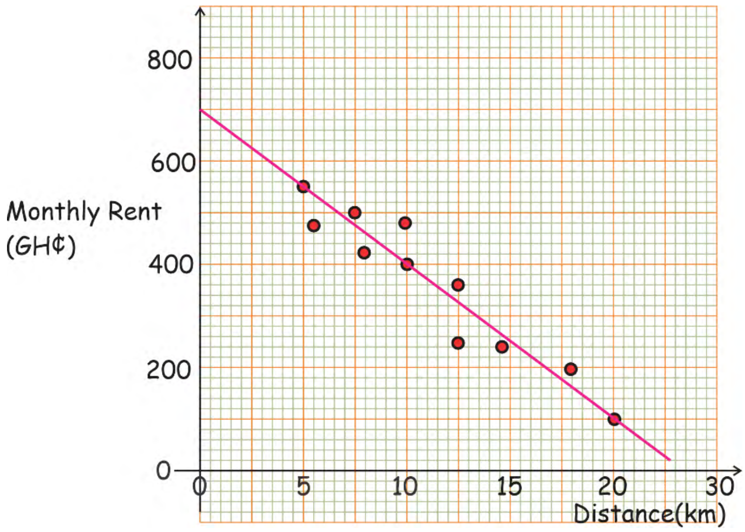


Figure 7.22: A Scatter graph showing the distance of a house from a school and the monthly rent

b. The equation is given as $R = mD + C$ where R=monthly rent in Ghana Cedis, D = distance from the school (kilometres) and C is the value of R when D = 0km

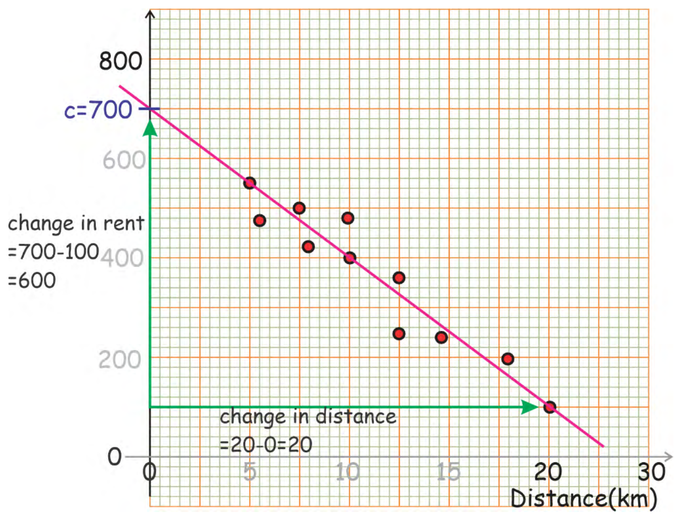


Figure 7.23: Calculating the equation of the line of best fit

From the graph C =700

c. Gradient $= -\frac{\textit{change in rent}}{\textit{change in distance}} = -\frac{600}{20} = -30$ GH¢ per km

d. Interpretation of the gradient:

e. For each km further from the school, the cost of rent is reduced by GH¢ 30.00

f. Therefore, the equation of the line of best fit is: $R = -30D + 700$

g. When the distance is 15km, Rent $= -30 \times 15 + 700 = -450 + 700 =$ GH¢ 250.00 Or you could read this from the graph directly. Remember that this answer can vary slightly depending on where you have drawn your line of best fit.

h. The graph shows a negative correlation between distance from the school and monthly rent. The further away from the school a house is located, the cheaper the cost of rent.

Example 7.8

The scatter graph below shows data on daily sales and profits of a sample of traders.

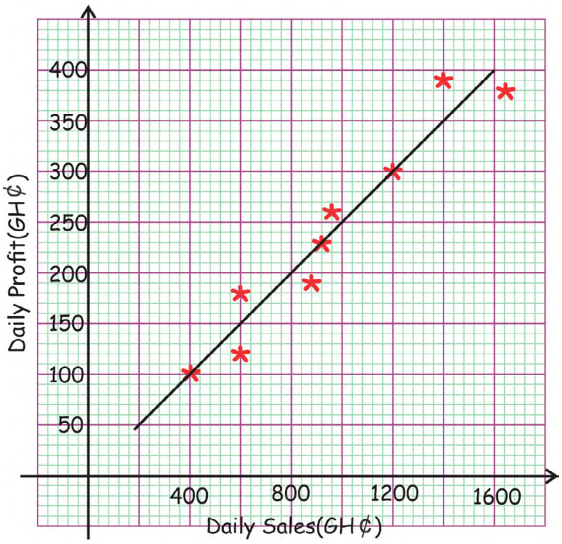


Figure 7.24: A scatter graph showing the daily sales against the daily profits of some traders

a. Use the graph to complete the data below

Daily sales (GH¢)	400	600	600	880	960	880	1200		1640
Daily profit (GH¢)		120	180		260	230		390	380

b. Find the equation of the line of best fit and interpret the gradient of the equation.

c. Determine the type of correlation between daily sales and daily profit.

Solution

a.

Daily sales (GH¢)	400	600	600	880	960	880	1200	1400	1640
Daily profit (GH¢)	100	120	180	190	260	230	300	390	380

b. The equation passes through points (400, 100) and (1200, 300). We will use these points to calculate the gradient of the line of best fit.

c. The best-fit equation is: $P - = mS + C$ where P=daily profit and S=daily sales

Gradient $= \frac{300 - 100}{1200 - 400} = \frac{200}{800} = \frac{1}{4}$

This means that for every GH¢ 4.00 of sales, the trader makes GH¢ 1.00 profit.

Or a quarter of sales is profit.

Or profit is equal to 25% of sales.

The equation of the line of best fit is $P = \frac{1}{4}S + C$

C is the y-intercept and we can see that the line will meet at the origin. Therefore, the equation of the line of best fit is $P = \frac{1}{4}S$

d. The correlation between sales and profit is positive. Generally, higher sales are associated with higher profit.

DESCRIBING THE SPEARMAN'S RANK CORRELATION COEFFICIENT AND INTERPRETING THE RESULTS WITHIN A GIVEN SITUATION

Spearman's rank correlation coefficient

To understand this concept, you must know what a monotonic function is.

A monotonic function is a function which is either:

1. always increasing as the independent variable increases OR
2. always decreasing as the independent variable increases

In a nutshell, a strictly increasing function or a strictly decreasing function is monotonic.

Let us use the following graphs to illustrate the monotonic function.

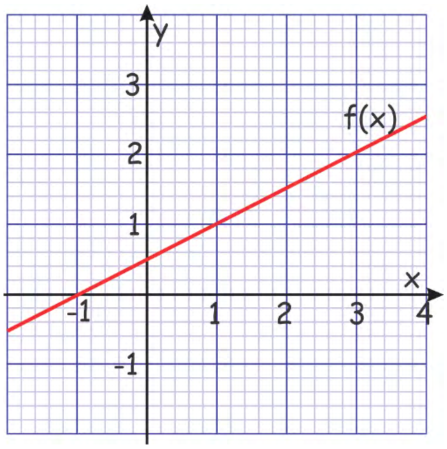


Figure 7.25

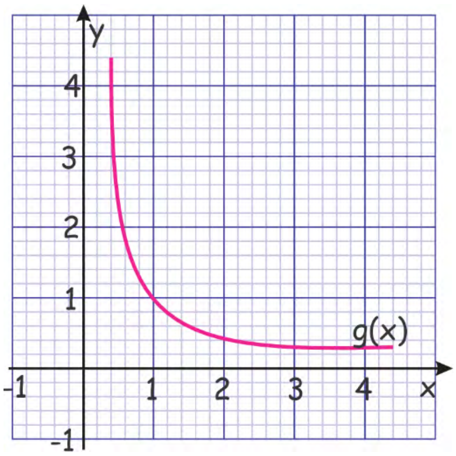


Figure 7.26

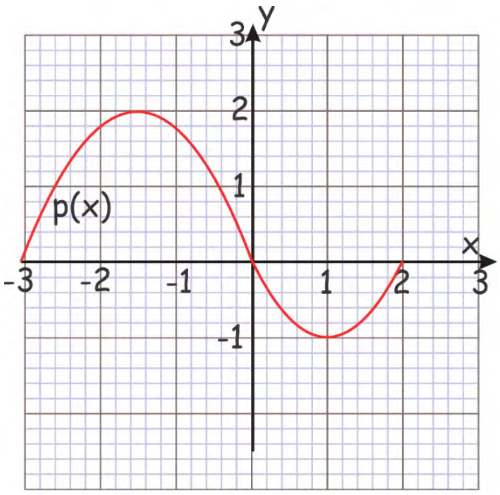


Figure 7.27

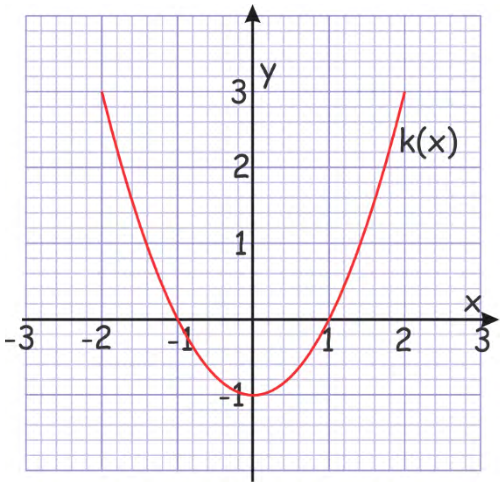


Figure 7.28

f(x) is monotonic increasing. As x increases, f(x) increases.

g(x) is monotonic decreasing. As x increases, g(x) decreases.

p(x) and k(x) are not monotonic functions. As the x variable increases, the y variable sometimes increases and sometimes decreases.

A Spearman's rank correlation coefficient is a statistical measure of a **monotonic relationship** between **ranked** variables.

Steps to Calculate Spearman's Rank Correlation Coefficient:

1. Rank the independent variables (R_x)
2. Rank the dependent variables (R_y)
3. For each pair ranks (R_x, R_y), calculate the deviation, $d = R_x - R_y$

4. Calculate the squared deviation, d^2
5. Calculate Spearman's Rank Correlation Coefficient using the formula:

   $$r_s = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

   Where n = total number of paired ranks

The coefficient of the spearman's rank correlation coefficient (r_s) is between the interval [– 1, 1], that is to say, $-1 \leq r_s \leq 1$. The coefficient (r_s) is interpreted as follows:

1. The absolute value of r_s indicates the strength of the relationship between the variables being compared. The larger the magnitude of the value, the stronger the relationship.
2. The sign of r_s indicates the direction of the relationship. A positive r_s means that an increase in the value of one variable is likely to increase the value of the other. Also, a decrease in one of the variables will cause a reduction in the other.
3. A negative r_s means that an increase in the value of one variable will likely result in a decrease in the other variable and a decrease will increase the other.
4. When $r_s = 0$, it means there is no relationship between the variables.

Table 7.1 shows the size of a correlation coefficient (r) and their respective interpretations. Values close to -1 or 1 represent a very strong correlation and values close to 0 indicate negligible correlation.

Table 7.1: Size of a correlation coefficient (r) and their interpretations

	Size of Correlation	Interpretation
1	$0.9 \leq$ correlation coefficient < 1.0	Very high positive correlation
2	$-1.0 <$ correlation coefficient ≤ -0.9	Very high negative correlation
3	$0.7 \leq$ correlation coefficient ≤ 0.9	High positive correlation
4	$-0.9 \leq$ correlation coefficient ≤ -0.7	High negative correlation
5	$0.5 \leq$ correlation coefficient ≤ 0.7	Moderate positive correlation
6	$-0.7 \leq$ correlation coefficient ≤ -0.5	Moderate negative correlation
7	$0.3 \leq$ correlation coefficient ≤ 0.5	Low positive correlation
8	$-0.5 \leq$ correlation coefficient ≤ -0.3	Low negative correlation
9	$0.3 \leq$ correlation coefficient < 0.0	Negligible positive correlation

	Size of Correlation	Interpretation
10	$0.0 <$ correlation coefficient ≤ -0.3	Negligible negative correlation
11	Correlation Coefficient $= 0$	No correlation
12	Correlation Coefficient$= 1$	Perfect positive correlation
13	Correlation Coefficient$= -1$	Perfect negative correlation

The examples below will enhance our understanding.

Example 7.9

The data shows the test scores of 7 students in Maths and Costing

Maths (x)	30	34	40	44	57	68	78
Costing (y)	42	54	57	68	83	82	90

a. Illustrate the relationship between the scores with a scatter plot,

b. Calculate the Spearman's rank correlation

c. Describe the correlation between the scores

Solution

a. *Scatter plot of Maths and Costing scores of a sample of 7 students.*

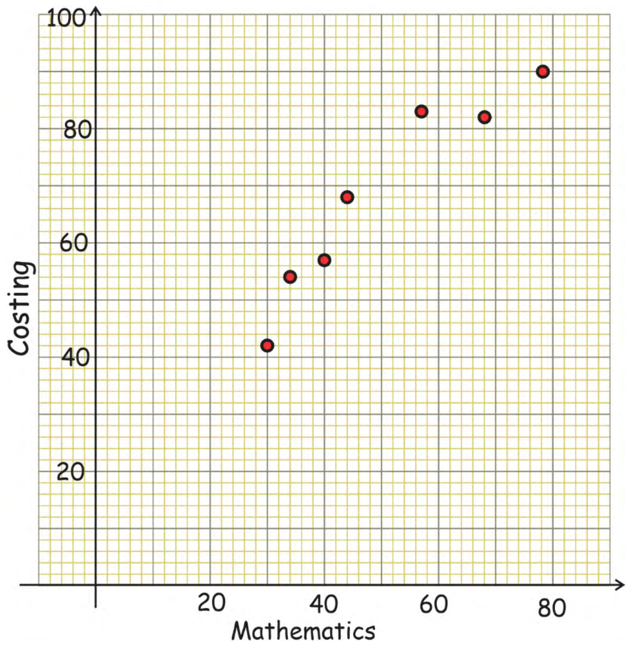


Figure 7.29: Scatter plot

b. Table 7.2: Generated values for Spearman's rank correlation

Mathematics (x)	Costing (y)	R_x	R_y	$R_x - R_y = d$	d^2
30	42	7	7	0	0
34	54	6	6	0	0
40	57	5	5	0	0
44	68	4	4	0	0
57	83	3	2	1	1
68	82	2	3	−1	1
78	90	1	1	0	0
Total					2

Now using the formula:

$$r_s = 1 - \frac{6\sum d^2}{n(n^2 - 1)}, \quad n = 7, \text{ and } \quad \sum d^2 = 2$$

$$r_s = 1 - \frac{6(2)}{7(7^2 - 1)} = 1 - \frac{1}{28} = 0.964$$

c. This shows a **strong positive** correlation between mathematics and costing scores. This shows that a student scoring high in mathematics will likely score high in Costing. Likewise, a student scoring low in mathematics will score low in Costing.

Example 7.10

Mr. Senyo and Mrs Awudi ranked the paintings of 12 students. Table 7.3 gives information about their ranks.

Table 7.3: Ranks of students

Student	Mr. Senyo	Mrs Awudi
Addo	5	5
Nii	6	6
Aba	2	1
Wan	8	7
Afi	3	3
Fofo	1	4
Ali	4	2

Student	Mr. Senyo	Mrs Awudi
Ata	9	9
Ago	10	8
Dua	7	12
Yaw	11	10
Ayi	12	11

a. Calculate the Spearman's rank correlation

b. Describe the correlation between the scores

Solution

a. Table 7.4: Values for calculating Spearman's rank correlation

Student	Mr. Senyo (R_s)	Mrs Awudi (R_A)	$d = R_s - R_A$	d^2
Addo	5	5	0	0
Nii	6	6	0	0
Aba	2	1	1	1
Wan	8	7	1	1
Afi	3	3	0	0
Fofo	1	4	-3	9
Ali	4	2	2	4
Ata	9	9	0	0
Ago	10	8	2	4
Dua	7	12	-5	25
Yaw	11	10	1	1
Ayi	12	11	1	1
				$\sum d^2 = 46$

Since there are 12 pairs of ranks, $n = 12$

$$r_x = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

$$r_x = 1 - \frac{6 \times 46}{12(12^2 - 1)}$$

$$= 1 - \frac{6 \times 46}{12 \times 143}$$

$$= 1 - 0.160839 = 0.83916 = 0.84$$

b. A correlation coefficient of 0.84 shows a strong positive correlation between Mr. Senyo and Mrs Awudi's scores of the students. This means that the paintings which Mr Senyo scored highly, Mrs Awudi was likely to score highly. Conversely, the paintings which Mr Senyo scored low were also likely to be scored low by Mrs Awudi.

Data with tied values

This is when two or more observations have the same value. There are many ways of dealing with tied values. One method is to assign the mean rank of the tied values for each tied observation.

The next example will focus on tied values.

Example 7.11

The data below gives information about the price and mileage of 10 cars.

Mileage (in 1 000km)	Price (in GH¢ 1 000)
80	20
70	40
40	50
30	60
20	70
90	20
50	46
20	80
40	60
20	100

a. Draw a scatter graph for the data

b. Calculate the Spearman's rank correlation coefficient for the data.

c. Comment on the strength of the correlation

d. Interpret the correlation between price and mileage of the used cars.

Solution

Scatter plot of Mileage and Price of Used Cars

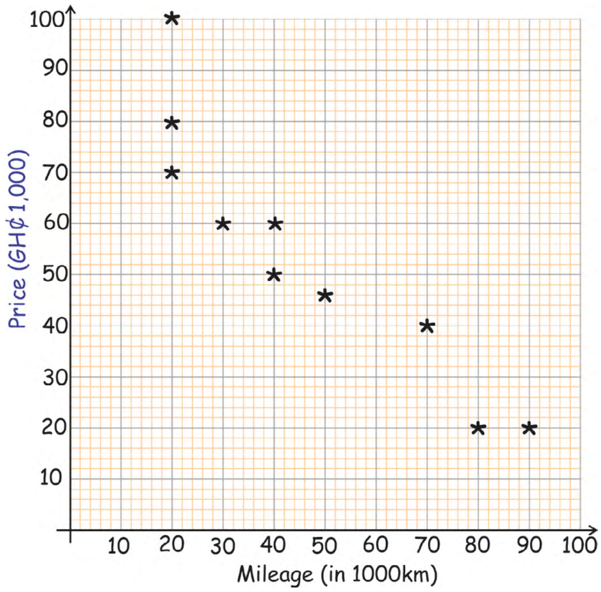


Figure 7.30

a. We will first rank the mileage. To do this, we will arrange the values in ascending order:

90	80	70	50	40	40	30	20	20	20
1^{st}	2^{nd}	3^{rd}	4^{th}			7^{th}			

40 is tied at the 5^{th} and the 6^{th} positions. So, we will assign the average of the 5^{th} and the 6^{th} positions $= \frac{5+6}{2}th = \frac{11}{2} = 5.5th$ position to 40

Also, 20 is tied at the 8^{th}, 9^{th} and 10^{th} positions. We will assign $\frac{8+9+10}{3} = \frac{27}{3}$ $= 9^{th}$ position to 20.

90	80	70	50	40	40	30	20	20	20
1^{st}	2^{nd}	3^{rd}	4^{th}	5.5^{th}	5.5^{th}	7^{th}	9^{th}	9^{th}	9^{th}

Next, let's use the same method to rank the Price.

First, arrange the data values in descending order:

100	80	70	60	60	50	46	40	20	20
1^{st}	2^{nd}	3^{rd}			6^{th}	7^{th}	8^{th}		

60 is tied at the 4^{th} and the 5^{th} positions. So, we will assign the average of the 4^{th} and the 5^{th} positions $= \frac{4+5}{2}th = \frac{9}{2} = 4.5th$ position to 60

Likewise, 20 is tied at the 9th and 10th positions. We will assign $\frac{9+10}{2} =$ 9.5*th* position to 20.

100	80	70	60	60	50	46	40	20	20
1st	2nd	3rd	4.5th	4.5th	6th	7th	8th	9.5th	9.5th

Let us update our table with the ranks. Take care when doing this that you are giving the correct ranks to the data. Then we will use it to calculate the deviations (d) and the squared deviations (d^2)

Mileage (in 1,000km)	Price (in GH¢ 1,000)	R_m	R_p	$d = R_m - R_p$	d^2
80	20	2	9.5	$2 - 9.5 = -7.5$	$(-7.5)^2 = 56.25$
70	40	3	8	$3 - 8 = -5$	$(-5)^2 = 25$
40	50	5.5	6	$5.5 - 6 = -0.5$	$(-0.5)^2 = 0.25$
30	60	7	4.5	$7 - 4.5 = 2.5$	$(2.5)^2 = 6.25$
20	70	9	3	$9 - 3 = 6$	$6^2 = 36$
90	20	1	9.5	$1 - 9.5 = -8.5$	$(-8.5)^2 = 72.25$
50	46	4	7	$4 - 7 = -3$	$(-3)^2 = 9$
20	80	9	2	$9 - 2 = 7$	$7^2 = 49$
40	60	5.5	4.5	$5.5 - 4.5 = 1$	$1^2 = 1 \times 1 = 1$
20	100	9	1	$9 - 1 = 8$	$8^2 = 64$
					$\sum d^2 = 319$

b. Finally, use the formula:

c. $r_x = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$

$r_x = 1 - \frac{6 \times 319}{10(10^2 - 1)}$

$r_x = 1 - \frac{1914}{10(99)}$

$r_x = 1 - \frac{1914}{990}$

$r_x = 1 - 1.933$

$r_x = -0.93$

d. A correlation coefficient of -0.93 shows a strong negative correlation between the mileage or distance a car has covered and the price.

e. This means that as the mileage of the car increases, the price decreases. Thus, a high mileage car will have a lower price, compared with a low mileage car which will have a higher price.

EXTENDED READING

- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. Determine the independent and dependent variables in the following scenarios.
   a) The effect of absenteeism on academic performance
   b) The effect of advertisement on sales.
   c) Grades and hours of study
   d) Expenditure and income
   e) Midterm and end-of-term History scores.
   f) Height and weight of students
   g) Amount of rainfall and bags of cocoa harvested
2. State the type of correlation in the following real-life scenarios.
   a) Relationship between husband's age and wife's age
   b) Height and occupation
   c) Age and life expectancy
   d) Gender and intelligence
   e) Alcohol consumption and driving ability
   f) Water consumption and temperature
   g) Hours of study and grades
   h) Wages earned and number of days worked
3. The data below shows the mid-term and end-of-term accounting scores of a sample of students.:

Mid-term (x)	60	75	45	80	30	85	40	58	75	60
End-of-Term(y)	75	90	60	90	45	92	50	70	84	60

   (a) Construct the data on a scatter plot manually.
   (b) Use Excel to construct a scatter plot for the data.
4. Aba experimented to determine whether different drug dosages affect the duration of relief from malaria. She used a random sample of 12 patients and recorded the following observations.

Dosage	3	3	4	5	6	6	7	8	8	9	6	5
Duration of relief (hours)	7	5	10	7	12	14	20	16	22	20	13	8

(a) Manually draw a scatter plot for the data.

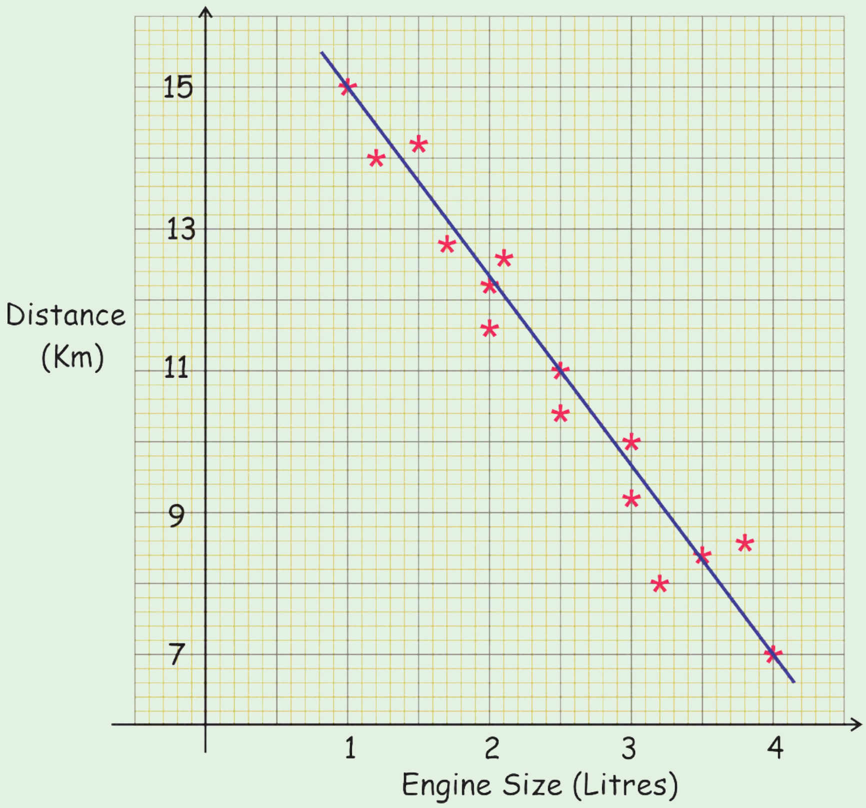
(b) Calculate the Spearman's rank correlation coefficient for the data and interpret the value.

5. The data below shows the income and expenditure of 10 Ghanaian workers.

Income (in GH¢ 100)	60	35	40	63	75	35	20	70	60	65	48
Expenditure (in GH¢ 100)	55	50	55	40	68	40	35	70	70	50	40

Draw a scatter plot for the data.

6. The scatter diagram below provides information about 15 cars. It shows their engine size (litres) and the distance (km) they can travel on one litre of fuel. [Take the blue line as a line of best fit]

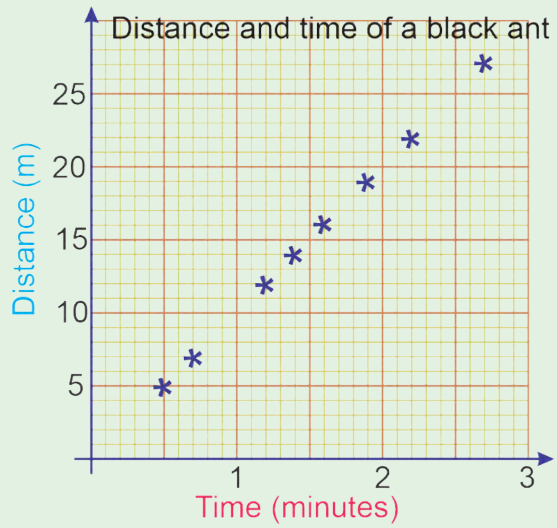


(a) Use your graph to find the equation of the line of best fit.

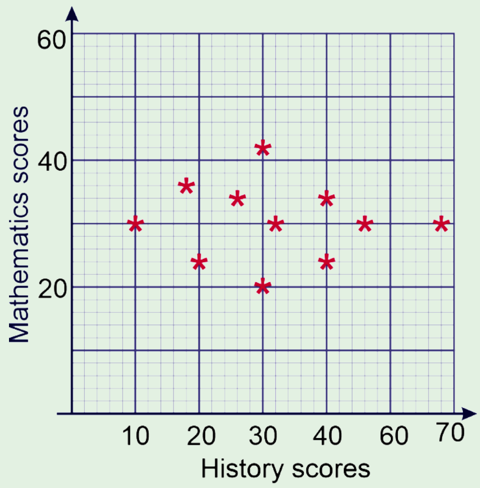
(b) Use your line of best fit to find the distance a car with an engine size of 1.5 litres travels on a litre of fuel.

7. Study the graphs below carefully and for each of them determine the type of correlation between the variables they represent.

a Distance and time travelled by a black ant.

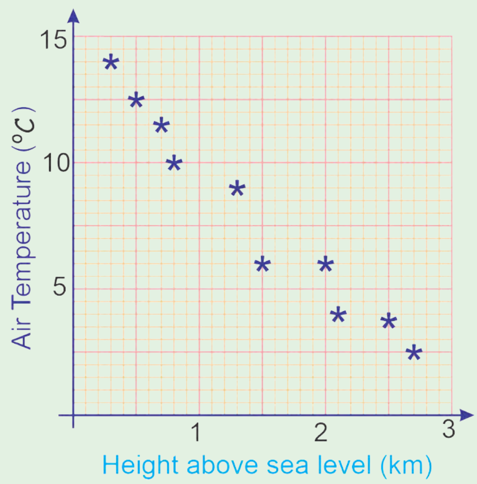


b Mathematics and history scores of students



c Price and demand of laptops

d Height above sea level and air temperature.



8. The table gives information about the distance from a city centre and the monthly rent of 11 houses.

Distance from the city centre (km)	Monthly rent (GH¢)
14.5	400
22.0	300
20.0	400

Distance from the city centre (km)	Monthly rent (GH¢)
12.0	600
19.5	700
7.5	670
7.0	750
16.5	440
12.0	480
10.0	520
14.5	560

a. Draw a scatter graph for the data.

b. Draw a line of best fit for the data.

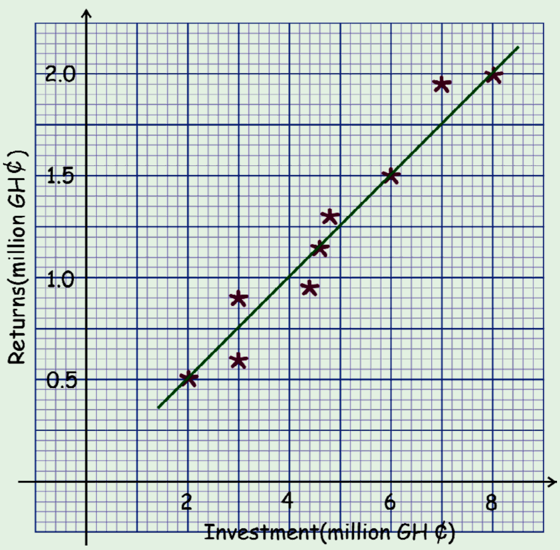
c. Find the gradient of the equation of the line of best fit and interpret your answer.

d. Find the equation of the line of best fit and use it to find the monthly rent of a building which is 17km from the city centre.

e. Determine the type of correlation between distance from the city centre and monthly rent.

f. Calculate the Spearman's rank correlation coefficient between distance from the city centre and monthly rent.

9. The scatter graph shows data on the amount of investment and returns earned by a sample of businesswomen. [Take the green line as a line of best fit]



a. Use the graph to complete the data below

Investment (million GH¢)									
Returns (million GH¢)									

b. Find the equation of the line of best fit. Interpret the gradient of the equation.

c. Madam Malik made 1 million Ghana Cedis returns on his investment, estimate the amount she invested.

d. Determine the type of correlation between Investment and Returns.

e. Calculate the Spearman's rank correlation coefficient between investment and returns.

f. Interpret your correlation coefficient.

10. Mrs Paintsil and Mrs Blay ranked the performance of 12 Adowa dancers.

The table below gives information about their ranks.

Student	Mrs Paintsil	Mrs Blay
Adwoa	2	4
Nhyiraba	4	1
Antoa	8	8
Asantewaa	3	2
Yaa	11	12
Dufie	1	3
Abena	7	7
Ataa	12	9
Akos	10	11
Ohenewaa	9	10
Naana	6	5
Morowa	5	6

(a) Calculate the Spearman's rank correlation

(b) Describe the correlation between Mrs Paintsil and Mrs Blay's ranks.

11. The data below gives information about the price and age of 10 used cars.

Age (years)	Price (in GH¢ 1 000)
10	20
9	40
6	50
5	60
4	70
11	20
7	46
4	80
6	60
4	100

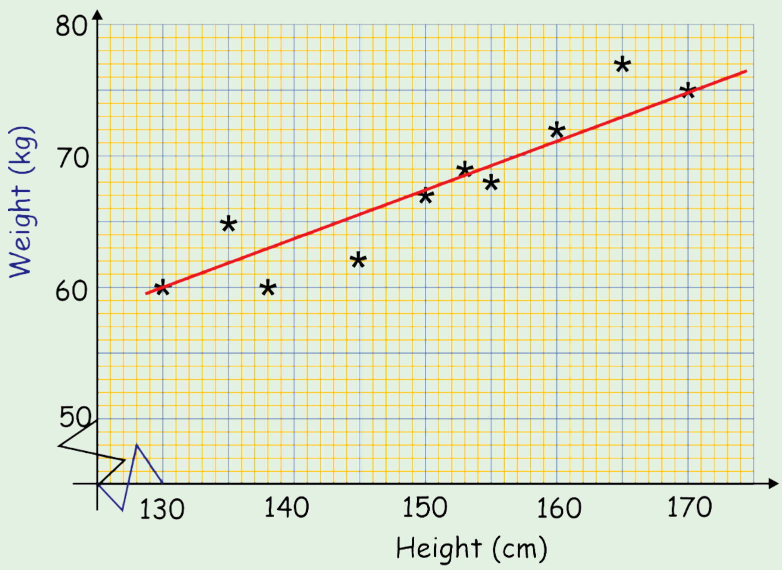
a. Calculate the Spearman's rank correlation coefficient for the data.

b. Comment on the strength of the correlation.

c. Interpret the correlation between price and mileage of the used cars.

12. The scatter plot below shows the height and weight of 10 students.

13. Take the red line as the line of best fit.



a. Use your graph to complete the table below

Height (cm)	170	150	130				153	145		155
Weight(kg)	75	67		72	65	60			67	

b. Find the gradient of the line of best fit and interpret your answer.

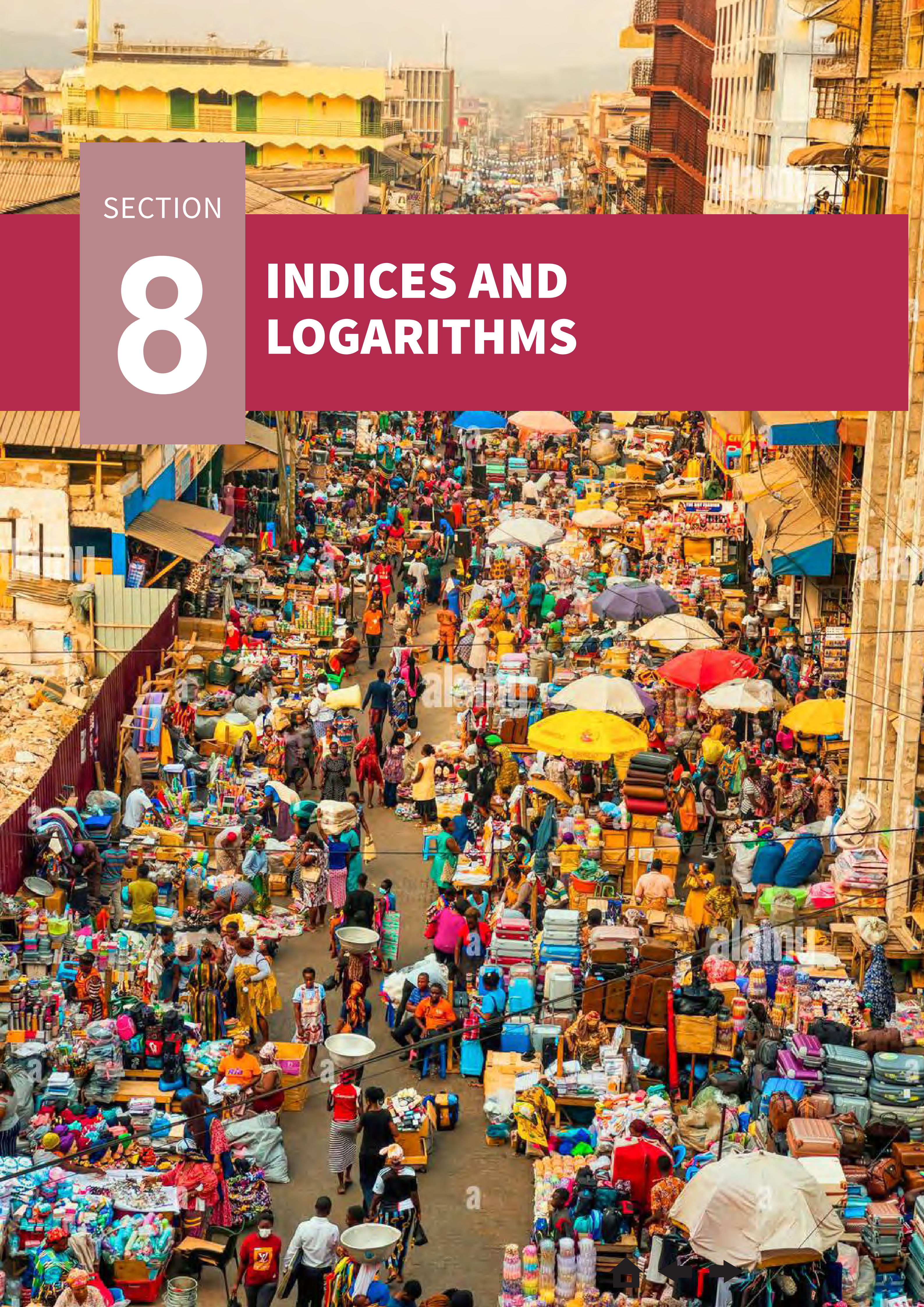
c. Find the equation of the line of best fit and use it to estimate the height of a student who is 70kg.

d. Find the Spearman rank correlation between height and weight.

e. Describe the correlation between height and weight.

SECTION

8 INDICES AND LOGARITHMS



MODELLING ALGEBRA

Application of Algebra

INTRODUCTION

Some phenomena in everyday life, like population growth and decay, asset depreciation and the loudness of sound, can be modelled with exponential or logarithmic functions. In year one, we covered logarithms and their inverse functions (exponential functions). Now we will reinforce your understanding and introduce you to how they are applied in real life. Before we look at the applications of Indices and logarithms, let us first review the laws associated with these concepts.

Laws of indices

They provide rules for simplifying calculations or expressions involving powers of the same base.

1. Multiplying indices: $a^m \times a^n = a^{m+n}$
2. Dividing indices: $a^m \div a^n = a^{m-n}$
3. Multiplying exponents: $(a^m)^n = a^{m \times n}$

Laws of logarithms

They provide rules for combining, splitting and evaluating logarithms.

1. Product Law: $\log a + \log b = \log ab$
2. Quotient Law: $\log a - \log b = \log \frac{a}{b}$
3. Power Law: $\log a^n = n \log a$

KEY IDEAS

- **Exponential Decay** can be modelled by the equation $A = P(1 - r)^t$.
- Natural Phenomena that follow this relation include depreciation, population decay and radioactive decay.
- **Exponential Growth** can be modelled by the equation $A = P(1 + r)^t$.
- Natural phenomena that follow this relationship include inflation, compound interest and population growth.

- ***A*** in the equation represents future value,
- ***r*** represents the rate of growth,
- **P** is the starting value and
- **t** is the time duration over which the natural phenomenon is being considered.
- To draw a linear graph for $Y = ab^X$, plot $LogY$ against X
- To reduce an exponential function of the form $Y = aX^n$ to linear, plot LogY against LogX. The gradient of the resulting line = n.

APPLICATIONS OF THE LAWS OF INDICES AND LOGARITHMS

Solving Equations Involving Logarithms

Express terms on both sides of the equation as a single logarithm to the same base. Equate the numbers and solve for the unknown variable or variables.

In general, if $\log_b x = \log_b y \Longrightarrow x = y$

Also, if $\log_x a = \log_y a \Longrightarrow x = y$

Remember the laws of logarithms:

$\log_b \boldsymbol{x} + \log_b \boldsymbol{y} = \log_b \boldsymbol{xy}$

$\log_b \boldsymbol{x} - \log_b \boldsymbol{y} = \log_b \frac{\boldsymbol{x}}{\boldsymbol{y}}$

$\log_b \boldsymbol{x}^{\boldsymbol{n}} = \boldsymbol{n}\log_b \boldsymbol{x}$

$\log_b \boldsymbol{a} = \frac{\log_x \boldsymbol{a}}{\log_x \boldsymbol{b}}$

Example 8.1

Solve $\log_{10}(2x + 1) - \log_{10}(1 - x) = \log_{10}2$

Solution

$\log_{10}(2x + 1) - \log_{10}(1 - x) = \log_{10}2$

$\log_{10}\left(\frac{2x + 1}{1 - x}\right) = \log_{10}2$

As we now have single logarithmic terms on both side to the same base, we can equate the terms.

$\frac{2x + 1}{1 - x} = 2$

$2x + 1 = 2(1 - x)$

$2x + 1 = 2 - 2x$

$2x + 2x = 2 - 1$

$4x = 1$

$x = \frac{1}{4}$

Example 8.2

Solve simultaneously $\log_2 y = 4 + \log_2 x$ and $y\log_4 8 = 2x + 3$

Solution

Simplify equation 1:

$\log_2 y = 4 + \log_2 x$

$\log_2 y = 4\log_2 2 + \log_2 x$

$\log_2 y = \log_2 2^4 + \log_2 x$

$\log_2 y = \log_2 16x$

$y = 16x$ ……………………*equation (1)*

Simplify equation 2:

$y\log_4 8 = 2x + 3$

$y\,\frac{\log_2 8}{\log_2 4} = 2x + 3$

$\frac{3}{2}y = 2x + 3$……………*equation (2)*

Substitute (1) into (2) to solve the simultaneous equations:

$\frac{3}{2} \times 16x = 2x + 3$

$24x = 2x + 3$

$22x = 3$

$\boldsymbol{x} = \frac{3}{22}$

Substitute $x = \frac{3}{22}$ into (1)

$y = 16 \times \frac{3}{22}$

$\boldsymbol{y = \frac{24}{11}}$

Confirm you have done this correctly by substituting both values into (2).

Example 8.3

Find the truth set of the equation $\log_{\frac{1}{2}} x + \log_2(\frac{1}{x}) + 6 = 0$

Solution

$\log_{\frac{1}{2}} x + \log_2\left(\frac{1}{x}\right) + 6 = 0$

The base of the logs are not the same, so we change it to a uniform base. This will help us apply the laws to simplify the expression. We will change it to base 2, although we could have chosen to change it to base ½

$$\frac{\log_2 x}{\log_2\left(\frac{1}{2}\right)} + \log_2\left(\frac{1}{x}\right) + 6 = 0$$

$$\frac{\log_2 x}{\log_2(2^{-1})} + \log_2\left(\frac{1}{x}\right) + 6 = 0$$

$$\frac{\log_2 x}{-\log_2 2} + \log_2\left(\frac{1}{x}\right) = -6$$

$$\frac{\log_2 x}{-1} + \log_2\left(\frac{1}{x}\right) = -6$$

$$-\log_2 x + \log_2\left(\frac{1}{x}\right) = -6$$

$$\log_2(x)^{-1} + \log_2\left(\frac{1}{x}\right) = -6$$

$$\log_2 \frac{1}{x} \times \frac{1}{x} = -6$$

$$\log_2\left(\frac{1}{x^2}\right) = -6$$

$$\frac{1}{x^2} = 2^{-6}$$

$$\frac{1}{x^2} = \frac{1}{64}$$

$$x^2 = 64$$

$$x = \pm\sqrt{64} = 8 \; or -8$$

Truth set $= \{x : x = -8, 8\}$

Compound Interest

The amount of money a lender or a financial institution receives for lending out money is called *interest*. Interest is usually calculated as a percentage of the amount borrowed. The person giving the money is the lender. The person receiving the *loan* is called the borrower.

Simple interest (SI) $= P \times T \times r$

Where:

$P =$ The principal or the amount borrowed or the amount invested

T = Time the borrower will keep the money until he/she pays.

$r =$ rate of interest in percentage (per annum or yearly)

The *amount* of money the borrower will pay the lender = *Principal* + *Interest*

Compound Interest is the interest calculated on the principal and the interest accumulated over the period. The compound interest formula is given below:

Compound interest = *Amount* − *Principal*

Amount (A)= $P\left(1 + \frac{r}{n}\right)^{nt}$

Where A = Amount, P = Principal, $r =$ interest rate, $n =$ number of times interest is compounded per year, $t =$ time (in years)

How did we arrive at this formula? The following activity will explain.

Activity 8.1: Compound Interest

In small groups, work through the following steps to discover the formula for compound interest.

Step 1: Calculate the amount at the end of the first year (A_1) using the simple interest formula.

Step 2: Calculate the amount at the end of the second year (A_2) using the amount obtained in step 1 as the principal.

Step 3: Calculate the amount at the end of the third year (A_3) using the amount obtained in step 2 as the principal.

Step 4: Repeat the process for 2 or 3 more terms. This will generate A_4, A_5, …….

Step 5: Find the t^{th} term of the sequence $A_1, A_2, A_3, A_4, A_5, \ldots\ldots\ldots, A_t$

Compare your answer with the one below:

Step 1: Interest at the end of the first year $= P \times 1 \times r = Pr$

Amount = Principal + Interest

Amount at the end of the first year $\left(A_1\right) = P + Pr = P(1 + r)$

Step 2: Interest at the end of the second year$= P(1 + r) \times 1 \times r = P(1 + r)r$

Amount at the end of the second year (A_2)

= Amount at the beginning of the first year + interest at the end of the second year

$= P(1 + r) + P(1 + r)r$

$= P(1 + r)(1 + r)$

$A_2 = P\left(1 + r\right)^2$

Step 3: $A_3 = P\left(1 + r\right)^2 + P\left(1 + r\right)^2 \times r$

$A_3 = P\left(1 + r\right)^2\left(1 + r\right)$

$\quad = P(1 + r)^3$

Step 4: This follows that:

$A_4 = P\left(1 + r\right)^4$

$A_5 = P\left(1 + r\right)^5$

Step 5: This will generate the sequence: $P(1 + r), P(1 + r)^2, P(1 + r)^3, P(1 + r$ $)^4, \ldots\ldots.$

This is a geometric sequence with $a = P(1 + r)$ and common ratio $(R) = \dfrac{P(1 + r)^2}{P(1 + r)}$ $= (1 + r)$

The tth term, $U_t = A_t = aR^{t-1}$

$A_t = [P(1 + r)]\left(1 + r\right)^{t-1}$

$A_n = P\left(1 + r\right)^{t-1+1}$

$A_t = P\left(1 + r\right)^t$ gives the amount at the end of the tth year

If the rate of interest is compounded n times per annum, then we the number of compounding periods $= n \times t = nt$ and the interest rate $= r$–

$$\boldsymbol{A_t = P\left(1 + \frac{r}{n}\right)^{nt}}$$

Other than the first year, interest compounded annually is greater than that of simple interest.

Most transactions in the banking and financial sector use compound interest. Other applications include population growth, bacteria growth, inflation and depreciation.

What part do logarithms play in these formulas, $A = P\left(1 + \frac{r}{n}\right)^{nt}$ and $A = P(1 + r)^{t}$?

They prove essential when we are solving for t or r.

Example 8.4

Ama borrowed GH¢ 10 000 from Bobo Bank for 3 years at an interest of 12% compounded annually. Find the compound interest and amount he has to pay at the end of 3 years.

Solution

Amount borrowed = *Principal* = 10 000

Number of years = 3

Interest rate $= \frac{12}{100} = 0.12$

Using the formula, Amount $= P(1 + r)^n$,

Amount owed at the end of the third year $= 10\,000 \times (1 + 0.12)^3$

$= 10\,000 \times 1.12^3$

$= 10\,000 \times 1.404928$

$= 14049.28$

Therefore, the amount at the end of the third year= *GH¢* 14 049.28

Compound interest owed = 14 049.28 – 10 000

= *GH¢* 4 049.28

Example 8.5

A bank decides to offer a business owner a GH¢ 750 000 loan at a compound interest of 8% per year.

Find the total amount, the bank will receive when the loan is repaid after 4 years.

How much interest will the business owner pay?

Solution

Principal = 750 000

Interest rate = $\frac{8}{100} = 0.08$

Time = 4 years.

Amount at the end of 4 years = $750\ 000 \times (1 + 0.08)^4$

$= 750{,}000 \times 1.08^4$

$= 750{,}000 \times 1.36048896$

$= GH¢\ 1\ 020\ 366.72$

Interest paid by the business owner = $1{,}020{,}366.72 - 750{,}000$

$= GH¢\ 270\ 366.72$

Example 8.6

Sweety invested $2 400 into a fund that pays 5 percent interest each year, compounded semi-annually. Find the value of the investment after 4 years.

Solution

Principal = $2400

Interest rate = $\frac{5}{100} = 0.05$

interest is compounded semi-annually

Time = 4

Since interest is compounded more than once in a year, we will use the formula

Amount = $\boldsymbol{P}\left(1 + \frac{\boldsymbol{r}}{\boldsymbol{n}}\right)^{nt}$

$n = 2$ since interest is compounded semi-annually (twice a year)

Amount = $2\ 400 \times \left(1 + \frac{0.05}{2}\right)^{2\times4}$

$= 2\ 400 \times (1 + 0.025)^8$

$= 2\ 400 \times 1.025^8$

$= 2\ 924.1669$

Therefore, the value of the investment after 4 years= $GH¢$ 2 924.17

Example 8.7

Baba invested €50 000 into a fund that pays 10% interest each year, compounded once every two years.

a. Find the total amount of his investment after 6 years.

b. After how many years will his investment double?

Solution

a. Principal = €50 000

Interest rate = $\frac{10}{100} = 0.1$

Time = 6years

Since interest is compounded less than once a year, we will use the formula

Amount = $\boldsymbol{P}\left(1+\frac{\boldsymbol{r}}{\boldsymbol{n}}\right)^{nt}$

$n = \frac{1}{2} = 0.5$ since interest is compounded once every two years (1 in every $2 = 1{:}2 = \frac{1}{2}$)

Intuitively, once every two years translates to three times in six years.

$$\text{Amount} = 50\,000 \times \left(1 + \frac{0.1}{0.5}\right)^{0.5 \times 6}$$
$$= 50\,000 \times (1 + 0.2)^3$$
$$= 50\,000 \times 1.2^3$$
$$= 50\,000 \times 1.728$$
$$= 86\,400$$

Therefore, the value of the investment after 6 years= $GH¢$ 86 400.

b. Doubling his investment means that his investment will amount to:

$50\,000 + 50\,000 = 100\,000$

Let the time it will take to double his investment be T.

This means that we seek to find T such that:

$$100\,000 = 50\,000 \times \left(1 + \frac{0.1}{0.5}\right)^{0.5 \times \boldsymbol{T}}$$

First divide bot sides by 50 000

$$\frac{100\,000}{50\,000} = \frac{50\,000}{50\,000} \times (1 + 0.2)^{0.5 \times \boldsymbol{T}}$$

$2 = 1.2^{0.5T}$

Take log to base 10 on both sides of the equation:

$\log_{10}2 = \log_{10}1.2^{0.5T}$

$\log_{10}2 = 0.5T\log_{10}1.2$

$\frac{\log_{10}2}{\log_{10}1.2} = 0.5T$

$3.80178 = 0.5T$

$\frac{3.80178}{0.5} = \frac{0.5T}{0.5}$

$T = 7.604$

$T = 7.604$ years $= 7$ years, $0.604 \times 12 = 7.24$ months

This means that it will take Baba about 7.6 years or 7 years and 7.24 months to double his investment.

Let us look back on our answer to check if it is correct.

$$\text{Amount} = 50\,000 \times \left(1 + \frac{0.1}{0.5}\right)^{0.5 \times 7.604}$$

$$= 50\,000 \times 1.2^{3.802}$$

$$= 50\,000 \times 2$$

$$= 100\,000$$

This confirms that our answer is correct.

Example 8.8

Calculate the annual rate of compound interest that will allow a loan of \$30 000 to amount to \$43 923 in four years.

Solution

We have:

Principal = \$30 000.00

Amount = \$43 923.00

Time = 4 years.

We have to find r = rate of interest

Substitute these values into $A = P(1 + r)^t$

$43\,923 = 30\,000(1 + r)^4$

Divide through by 30 000

$\frac{43\,923}{30\,000} = 30\,000\ \_\ 30\,000\ (1 + r)4$

$1.4641 = (1 + r)^4$

Find the 4^{th} root on both sides of the equation or raise each side of the equation to the exponent $\frac{1}{4}$

$\sqrt[4]{1.4641} = \sqrt[4]{(1+r)^4}$ *or* $(1.4641)^{\frac{1}{4}} = ((1+r)^4)^{\frac{1}{4}}$

$1.1 = 1 + r$

$1.1 - 1 = r$

$0.1 = r$

$r = 0.1 = \frac{10}{100} = 10\%$

Therefore, the annual rate of compound interest would need to be 10% to make the amount to be repaid raise to \$43 923.

Population Growth and Decay

The growth and decay of certain populations follow an exponential equation. The following examples will show us how the exponential equation is applied when dealing with a population problem.

Example 8.9

Currently, the population of cats in a country is 12 million. This figure is expected to increase at a constant rate of 4% each year. Estimate the population of cats in the country in 5 years.

Solution

We can use the compound interest formula to answer this question. Since the principal (here population) compounds once every year, we will use:

$A = P(1 + r)^t$, where:

P = current population of cats

A = Future population of cats

T = Time = 5 years

Population in 5 years $= 12\,000\,000 \times \left(1 + \frac{4}{100}\right)^5$

$= 12\,000\,000 \times 1.04^5 = 14\,599\,834.8288$

This means that, in 5 years, there will be approximately 14.6 million cats in the country.

Example 8.10

The population of elephants in a country is currently 640 000 and is predicted to decrease at a rate of 6% per year.

a. What will the population be in 4 years? Give your answer to 3 significant figures.

b. How long will it take for the population to decrease to 390 120?

c. At this rate of decrease, Hannah thinks there will be no more than 300 000 elephants in a decade. Is she correct? Justify your response.

Solution

We can use the compound interest formula to answer this question. Since the principal (population of elephants) *decreases* every year, we will use:

$A = P(1 - r)^t$, where:

A = population of elephants in 4 years

P = current population of elephants

$r = 6\% = 0.06$ = the rate of decrease

$t = 4$ years

a. Population of elephants in 4 years= 640 000 × (1 – 6 _ 100)^4

b. = 640 000 × 0.94^4 = 499 679.3344 = 500 000 (3 sf)

The population of elephants in 4 years is about 500 000

c. We seek to find t such that A = 390,120. This will give us the equation

$$390120 = 640000 \times \left(1 - \frac{6}{100}\right)^t$$

$$\frac{390120}{640000} = \frac{640000}{640000} \times 0.94^t$$

$$0.6095625 = 0.94^t$$

$$\log 0.6095625 = \log 0.94^t$$

$$\log 0.6095625 = t\log 0.94$$

$$\frac{\log 0.6095625}{\log 0.94} = t$$

$$\therefore t = 8$$

So, it will take about 8 years for the population to decrease to 390120.

In 10 years (a decade), the population of elephants

$= 640\,000 \times \left(1 - \frac{6}{100}\right)^{10}$

$= 640\,000 \times 0.94^{10}$

$= 344\,713.673$

This figure is greater than 300 000. So, Hannah is not correct.

Example 8.11

A body eliminates the concentration of drugs in the blood at a constant rate per hour. The initial concentration of the drug in the blood is 8.46mg/ml. After 5 hours, the concentration is 5mg/ml.

a. Find the rate at which the body eliminates the drug.

b. The concentration is considered negligible if it is 0.125 mg/ml. When will the concentration be negligible?

Solution

a. Using $A = P(1 - r)^t$, where:

A = Concentration of the drug after 5 hours

P = initial concentration of the drug = 8.46mg/ml

r is the rate at which the body eliminates the drug from the blood

$t = 5$ hours

We seek to find r, such that:

$5 = 8.46 \times (1 - r)^5$

$\frac{5}{8.46} = \frac{8.46}{8.46} \times (1 - r)^5$

$\frac{5}{8.46} = (1 - r)^5$

Take the fifth root on both sides of the equation

$\sqrt[5]{\frac{5}{8.46}} = \sqrt[5]{(1 - r)^5}$ *or* $\left(\frac{5}{8.46}\right)^{\frac{1}{5}} = ((1 - r)^5)^{\frac{1}{5}}$

$(0.591)^{\frac{1}{5}} = 1 - r$

$0.9 = 1 - r$

$r = 1 - 0.9$

$r = 0.1 = \frac{10}{100} = 10\%$

Therefore, the rate at which the body eliminates the drug is 10%

b. We have to find t such that $0.125 = 8.46 \times (1 - 0.1)^t$

$$\frac{0.125}{8.46} = \frac{8.46}{8.46} \times 0.9^t$$

$$0.014775 = 0.9^t$$

$$\log 0.014775 = t \times \log 0.9$$

$$\frac{\log 0.014775}{\log 0.9} = t \times \frac{\log 0.9}{\log 0.9}$$

$$t = 40$$

It will be 40 hours before the concentration will be negligible.

Depreciation

Most things we use do not last forever. This is because they are being used up little by little. The item will not continue to look new or perform the same way as time passes. This is even true with some natural things. The "using up" or "the reduction in value" or "reduction in performance" is called *depreciation*. If the item is expected to last for 10 years, this 10 years is called the item's "useful life". The value of the item at the end of its useful life is called *scrap value*.

There are many methods of calculating depreciation. Here we will apply percentages to calculate depreciation.

Depreciation = Rate of depreciation × Value at the beginning of the year.

The item's value at the beginning of the year is often called the original value.

If the rate of depreciation is r, then depreciation $= r \times$ original value

The rate of depreciation for each year may vary.

If the rate of depreciation is same for every year, then

Current Value or Scrap Value $= P(1 - r)^t$ where:

P = Purchase price

r = rate of depreciation

t = time in years.

The current value of the item = Original value – Depreciation.

Depreciation = Purchase price – Current Value

Example 8.12

The value of a generator which originally cost GH¢7 500.00 depreciates by 10% of its original value each year. Find the value of the computer at the end of the third year.

Solution

Current Value $= P(1 - r)^t$

$P =$ GH¢ 7500.00

$r = 10\,\% = 0.1$

$t = 3.$

$$\begin{aligned}\text{Current Value} &= 7\,500(1 - 0.1)^3 \\ &= 7\,500 \times 0.9^3 \\ &= 7\,500 \times 0.729 \\ &= 5\,467.5\end{aligned}$$

The value of the generator after 3 years is GH¢ 5 467.50

Example 8.13

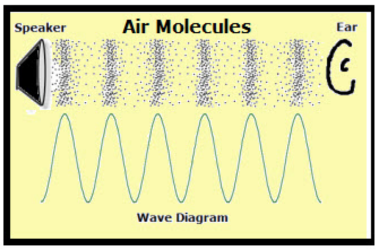
Find the scrap value of a machine costing *GH¢* 50 000.00, having a useful life of fifteen years and a constant annual rate of depreciation of 12%

Solution

$$\begin{aligned}\text{Scrap Value} &= 50\,000 \times (1 - 0.12)^{15} \\ &= 50\,000 \times -0.88^{15} \\ &= 50\,000 \times 0.1469738539 \\ &= GH₵\ 348.69\end{aligned}$$

Logarithmic scales are used when calculating the loudness of sound.

Loudness of Sound



The loudness of sound is the intensity or the amount of energy in a sound wave. It shows how loud or soft a listener perceives a sound. It is measured in decibels (dB)

The loudness of a sound, L, perceived by the human ear depends on the ratio of the intensity (I) of the sound, to the threshold (l_o) of hearing for the average human ear. The intensity of sound (I) is the average power of sound wave per unit area. The formula for calculating loudness of sound is given by:

$$L = 10\text{Log}\left(\frac{I}{l_o}\right)$$

Example 8.14

Find the loudness of a sound that has an intensity 100 000 times the threshold of hearing for the average human ear.

Solution

Let the threshold of hearing be l_o $Intensity,\ I = 100\ 000 l_o$

$$Loudness,\ L = 10\text{Log}\left(\frac{I}{l_o}\right)$$

$$L = 10\text{Log}\left(\frac{100\ 000\, l_o}{l_o}\right)$$

$$= 10 \log (100\ 000)$$

$$= 10 \times 5 = 50dB$$

MODELLING WITH LOGARITHMIC FUNCTIONS

1. **Equation of the form Y = aX^n**
2. Any relation of a non-linear form Y = aX^n, where x and y are variables and a, n are constants can be reduced to a linear equation of the form $y = mx + c$. How do we achieve this? The following activity will explain:

Activity 8.2: Reducing the Exponential function to a linear function

In small groups, work through the following steps to see how this is done.

Part A

Step 1: Take the logarithm on both sides of Y = aX^n

Step 2: Apply the addition rule to expand $Log(aX^n)$ into two terms

Step 3: Apply the power rule to rewrite $LogX^n$ as $nLogX$

Step 4: Compare the resulting Log equation to the general equation of a straight line.

Step 5: Based on your comparison, find the values of the unknown constants.

Compare your answer to the one below:

Step 1: Take the logarithm on both sides of Y = aX^n

$LogY = Log(aX^n)$

Step 2: Apply the addition rule to expand $Log(aX^n)$ into two terms

Log Y= $Loga + LogX^n$

Step 3: Apply the power rule to rewrite $LogX^n$ as $nLogX$

$LogY = Loga + nLogX$

Step 4: Compare the resulting Log equation to the general equation of a straight line.

$LogY = nLogX + Loga$

$y = m\,x + c$

Step 5: Based on your comparison, find the values of the unknown constants.

Comparing terms, we have $y = LogY$, $m = n$ and $c = Loga$.

This implies that $a = 10^c$

This shows that to reduce an exponential function of the form $Y = aX^n$ to linear, plot LogY against LogX. The gradient of the resulting line = n.

Part B

Graphs of the form $Y = ab^X$, where a and b are constants, can also be reduced to linear form. We will use the same procedure to evaluate this one too:

Step 1: Take the logarithm on both sides of Y = aX^n

$\log_{10}Y = \log_{10}ab^X$

Step 2: Apply the addition rule to expand $\text{Log}_{10} ab^X$ into two terms

$\text{Log}_{10}Y = \text{Log}_{10}a + \text{Log}_{10}b^X$

Step 3: Apply the power rule to rewrite $\text{Log}_{10}b^X$ as $X\text{Log}_{10}b$

$\text{Log}_{10}Y = \text{Log}_{10}a + X\text{Log}_{10}b$

Step 4: Compare the resulting Log equation to the general equation of a straight line.

$\text{Log}_{10}Y = X\text{Log}_{10}b + \text{Log}_{10}a$

$y = x\, m + c$

Step 5: Based on your comparison, find the values of the unknown constants.

From the comparisons

We have $\log_{10}Y = y$, y coordinates of the resulting line

$X = x$, x coordinates of the resulting line

$\text{Log}_{10}b = m$, gradient of the resulting line

This implies that $b = 10^m$

$\text{Log}_{10}a = C, y - cordinate$ of the y-intercept

This implies that $a = 10^C$

To draw a linear graph for $Y = ab^X$, plot $LogY$ against X

Example 8.15

Convert the following to linear functions.

a. $y = 3x^{1.5}$

b. $y = \frac{2}{x}$

c. $V = 5P^2$

Solution

a. $y = 3x^{1.5}$

Take logs on both sides of the equation

$Log\ y = Log(3x^{1.5})$

Log y = log 3 + log $x^{1.5}$

Log y = Log 3 + 1.5 Log x

Log y = 1.5 Log x + log 3

b. $y = \frac{2}{x}$

Log y = Log $\left(\frac{2}{x}\right)$

Log y = Log 2 – Log x

Log y = -Log x + Log 2

c. $V = 5P^2$

Log V = log $(5P^2)$

Log V = log 5 + Log P^2

Log V = 2logP + Log 5

Example 8.16

Convert the following to linear functions

a. $M = 5 \times 3^r$

b. $\frac{100}{5^x}$

Solution

a. $M = 5 \times 3^r$

Take logs on both sides of the equation

Log $M = Log\ (5 \times 3^r)$

Log M = Log 5 + Log 3^r

Log M = Log 5 + rLog 3

Log M = rLog 3 + Log5

b. $y = \frac{100}{5^x}$

$Log\ y = Log\left(\frac{100}{5^x}\right)$

$Log\ y = Log\ 100 - Log\ 5^x$

$Log\ y = 2 - xLog5$

$Logy = -Log5 + 2$

Example 8.17

Convert $P = 10t^3$ to a linear function and graph it.

Solution

$P = 10t^3$

$\text{Log P} = \log(10t^3)$

$\text{Log P} = \log 10 + \text{Log } t^3$

$\text{Log P} = 3\text{Log } t + \text{Log}10$

$\text{Log P} = 3\text{Log } t + 1$

$y=3x+1$

This is the same as the line $y = 3x + 1$, where y=Log P and x=Log t

Figure 8.1: Graph of linear function $y = 3x+1$

Example 8.18

The table below gives some values of two related variables, x and y.

x	1	1.2	1.4	1.6	1.8	2
y	2	3.46	5.49	8.19	11.66	16

The relationship between y and x is of the form $y = Ax^b$ where A and b are constants.

a. Draw a suitable linear graph for $y = Ax^b$

b. Use your graph to find A and b, correct to the nearest whole number.

c. Use your graph to find, correct to the nearest whole number, the value of y when $x = 1.5$

Solution

a. To reduce $y = Ax^b$ to a linear graph, take logs of both sides:

$\log_{10}y = \log_{10}Ax^b$

$\log_{10}y = \log_{10}A + b\log_{10}x$

$\log_{10}y = b\log_{10}x + \log_{10}A$ ………………… (1)

Comparing (1) to the general equation of a line $Y = mX + C$, we have:

$\log_{10}y = Y$, y coordinates of the resulting line

$\log_{10}x = X$, x coordinates of the resulting line

$b = m$, gradient of the resulting line

$\log_{10}A = C$, $y -$ *cordinate* of the $y -$ intercept.

Using the values given, we construct a table for $\log_{10}y$ and $\log_{10}x$

$\log_{10}x$	0	0.079	0.146	0.204	0.255	0.301
$\log_{10}y$	0.30	0.54	0.74	0.91	1.07	1.20

The graph of $\log_{10}y = b\log_{10}x + \log_{10}A$ is presented below

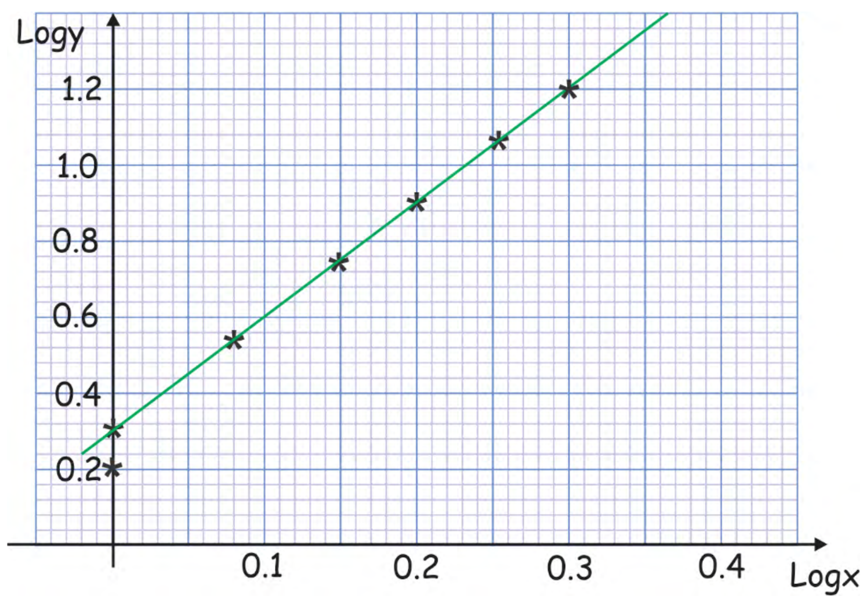


Figure 8.2: Graph of $\log_{10}y = b\log_{10}x + \log_{10}A$

b.

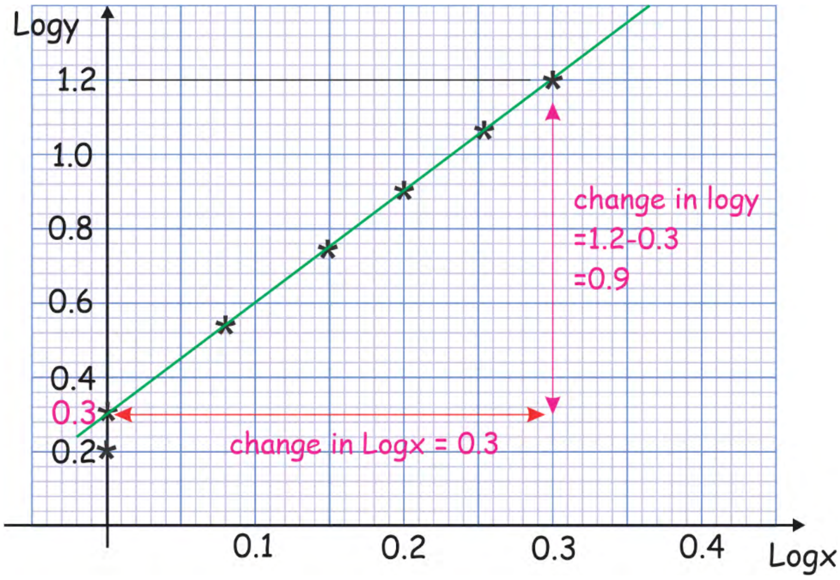


Figure 8.3: Graph of $\log_{10}y = b\log_{10}x + \log_{10}A$

From the graph, the y-coordinates intersect the line at 0.3, $\Longrightarrow \log_{10}A = 0.3$

$A = 10^{0.3} = 1.995 \approx 2$

Also, the gradient of the line is equal to b, $\Longrightarrow b = \dfrac{\textit{change in } \log_{10}y}{\textit{change } \log_{10}x} = \dfrac{0.9}{0.3} = 3$

Hence, $A = 2$ and $b = 3 \Longrightarrow y = 2x^3$

c. Note that the graph has $\log_{10}x$ values not x values. Hence, to read $x = 1.5$ on the graph, we have to convert it to a logarithm of the base 10.

$\text{Log}_{10}1.5 = 0.18$

Locate 0.18 on the $\log_{10}x$ axis. Draw a vertical line to intersect the green line, from this point of intersection, draw a horizontal line to $\log_{10}y$ axis. Find the corresponding $\log_{10}y$ value of 0.18

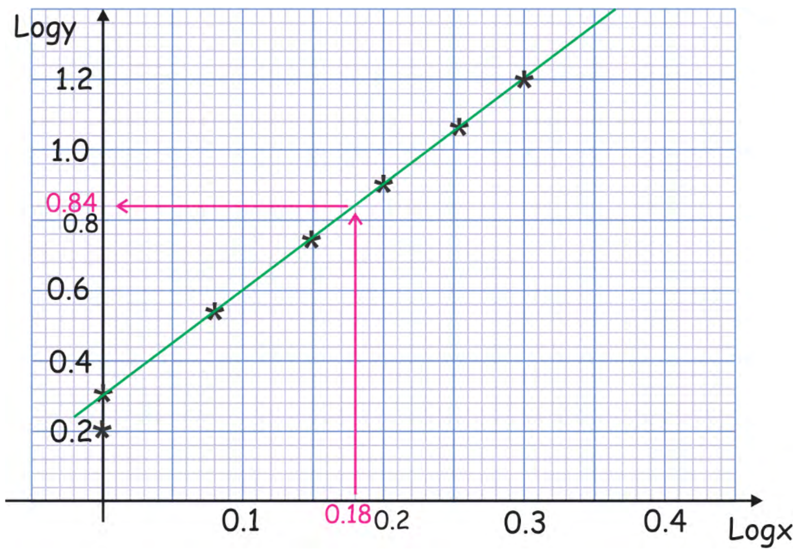


Figure 8.4: graph of $\log_{10}y = b\log_{10}x + \log_{10}A$

From the graph, when $\log_{10}x = 1.18$,

$\text{Log}_{10}y = 0.84$

$\Longrightarrow y = 10^{0.84} = 6.9183 \approx 7$

Therefore, when $x = 1.5$, $y = 7$

Example 8.19

The population of trees in a forest has been decreasing since 1970. A tree population census was conducted every 5 years to assess the decline. It is believed that the decline follows the relation $P = a\,b^t$. Where P = population of trees after 1970 and t = time (years) after 1970. The census data is shown below.

t	5	10	15	20	25	30	35	40	45
P	108470	98049	88628	80113	72416	65458	59169	53484	48345

a. Use the table to draw a suitable **linear graph** for the relation.

b. Use your graph to find a and b, correct your answers to two significant figures.

c. Use your graph to estimate the population of trees in 1997.

d. From your graph, in which year did the population of trees decrease to about 85 000

Solution

a. To draw a line graph for the relation $P = a\,b^t$, take logs on both sides of the equation.

b.

$LogP = Log(a\,b^t)$

$LogP = Loga + tLogb$

$LogP = tLogb + Loga$

The equation is similar to $y = mx + c$, where

$y = LogP$, $t = x$, $m = Logb$ and $c = Loga$

To obtain a linear equation, we have to plot $LogP$ against t.

Let us first create a table of values for Log P and t.

t	5	10	15	20	25	30	35	40	45
Log$P$	5.04	4.99	4.95	4.90	4.86	4.82	4.77	4.73	4.68

Find below the graph of *LogP* against *t*

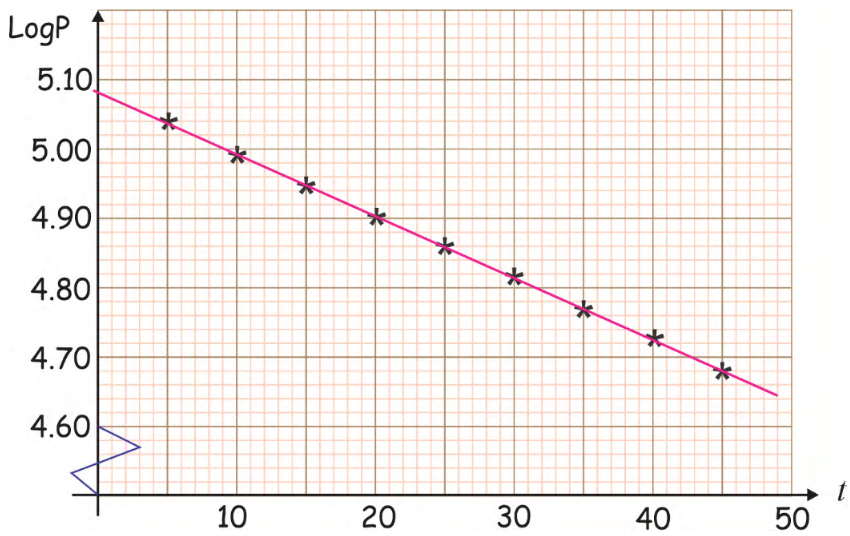


Figure 8.5: Graph of the relation $P = a\,b^t$,

b. From our relation, $m = Logb$ and $c = Loga$

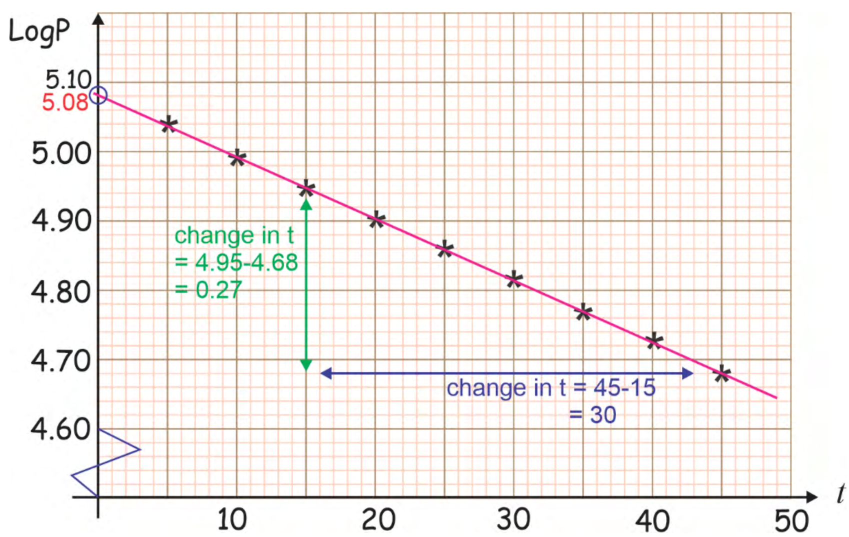


Figure 8.6: Graph of the relation $P = a\,b^t$,

Since the line graph is a decreasing function,

$$m = -\frac{change\ in\ LogP}{change\ in\ t} = -\frac{0.27}{30} = -0.009$$

From the relation, we have $m = Logb$

$$-0.009 = Logb$$

$b = 10^{-0.009} = 0.9795 = 0.98$ (to 2 significant figures)

From the relation $c = Loga$. Recall that C is the y-intercept of the graph.

From the graph $c = 5.08$

This implies that $5.08 = Log a$

$a = 10^{5.08} = 120\ 226.443$

$a = 120\ 000$ (to 2 significant figures)

The relation is $P = 120\ 000 \times 0.98^{t}$

c. From 1970 to 1997 = 27 years. This means that we will use the graph to find the value of P when t = 27.

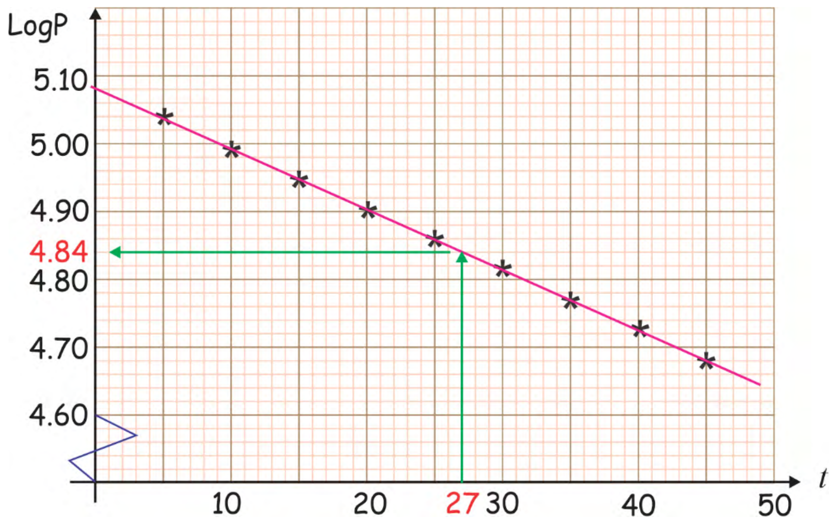


Figure 8.7: Graph of the relation $P = ab^{t}$,

From the graph, when t = 27, LogP = 4.84

This implies that $P = 10^{4.84} = 69\ 183.097$

In 1997, the population of trees was about 69 183

d. Since the graph has LogP values, we will find Log 85 000 and use the graph to calculate its t value

$\boldsymbol{Log}85\ 000 = 4.93$

$Log 85\ 000 = 4.93$

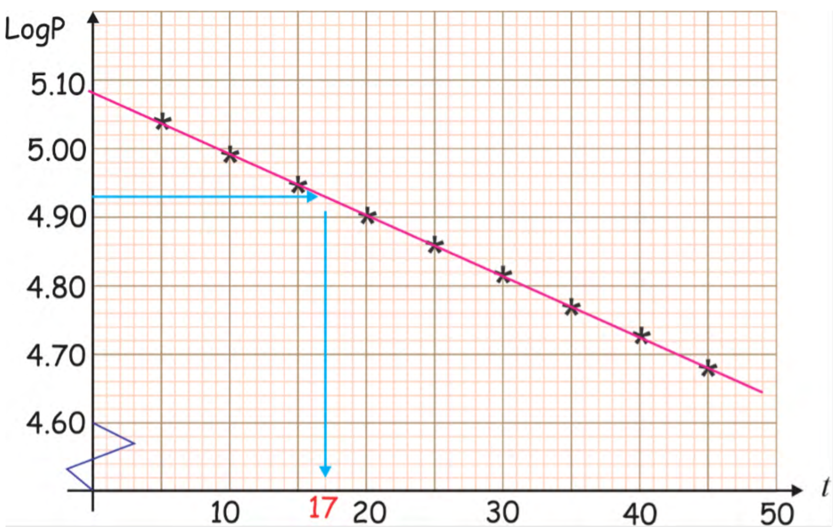


Figure 8.8: Graph of the relation $P = ab^{t}$,

This will take us to 17 years after 1970, which is the same as the year = 1970 + 17 = 1987

Therefore, the population of trees reached 85 000 in 1987

Example 8.20

The value of an investment (V) in cocoa at the end of *t* years satisfies the relation: $V = PR^t$ where P and R are constants.

The table below gives some values of V at the end of *t* years.

t	2	4	6	8	10
V($)	14 400	20 700	29 900	43 000	61 900

a. Draw a suitable linear graph for $V = PR^t$

b. Use your graph to find:

i. P, correct to the nearest whole number

ii. R, correct to two significant figures.

iii. the Value (V) of an investment at the end of 5 years, correct to two significant figures.

Solution

a. To reduce $V = PR^t$ to a linear graph, take $\log_{10}$ on both sides of the equation.

$\log_{10}V = \log_{10}PR^t$

$\log_{10}V = \log_{10}P + t\log_{10}R$

$\log_{10}V = t\log_{10}R + \log_{10}P$ .... (1)

Comparing (1) to the general equation of a line $Y = mX + C$, we have:

$\log_{10}V = Y$, y coordinates of the resulting line

$t = X$, x coordinates of the resulting line

$\log_{10}R = m$, gradient of the resulting line

$\log_{10}P = C$, y – coordinate of the y–intersept.

Using the values given, we construct a table for $\log_{10}A$ and x

x	2	4	6	8	10
$\log_{10}V$	4.16	4.32	4.48	4.63	4.79

The graph of $\log_{10}V = t\log_{10}R + \log_{10}P$ is presented below

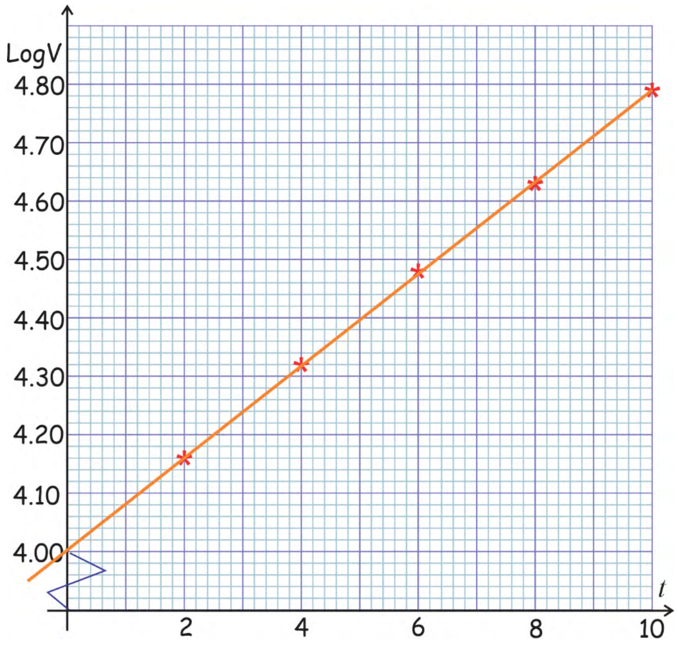


Figure 8.9: Graph of $\log_{10}V = t\log_{10}R + \log_{10}P$

b. **i.** To find P, extend the line to intersect $\log_{10}V$ axis.

From the graph, the line intersects it at 4

$P = 10^4 = 10\ 000$

ii. $\log_{10}R = m$ (gradient of the line)

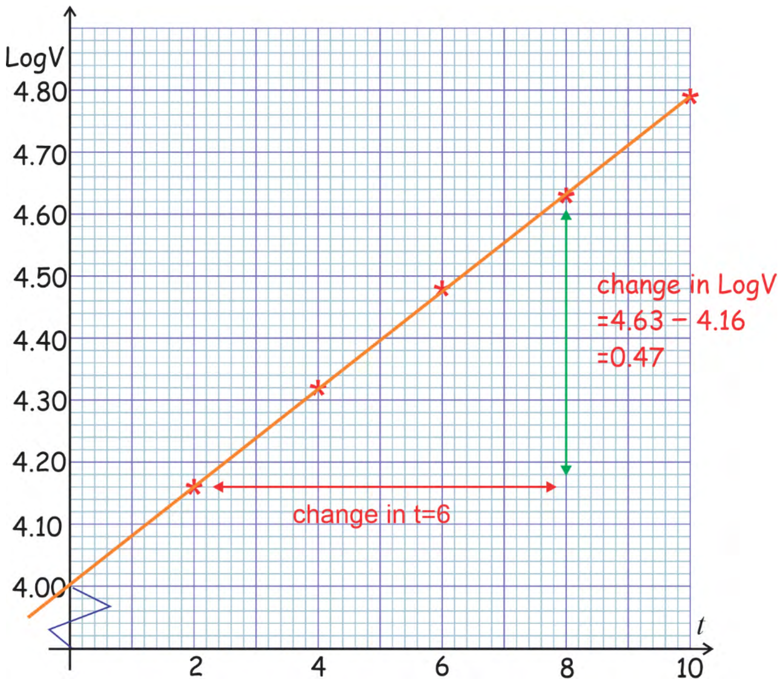


Figure 8.10: graph of $\log_{10}V = t\log_{10}R + \log_{10}P$

$m = \frac{0.47}{6} = 0.07833$

$\Longrightarrow R = 10^{0.07833} = 1.19765 = 1.2$ (correct to 2 significant figures)

Hence, $P = 10\ 000$ and $R = 1.2 \Longrightarrow V = 10\ 000(1.2)^t$

c. To find the corresponding value of $t = 5$ on the graph, locate 5 on the t - axis of your graph. Draw a vertical line to intersect your line. From this point of intersection, draw a horizontal line to $\log_{10}V$ axis and find its value.

Figure 8.11: graph of $\log_{10}V = t\log_{10}R + \log_{10}P$

From the graph when $t = 5$, $\log_{10}V = 4.4$

$\Longrightarrow V = 10^{4.4} = 25118.864315 = \$25\ 000$ (correct to 2 significant figures)

Example 8.21

Figure 8.12 below shows the graph of Log y against x

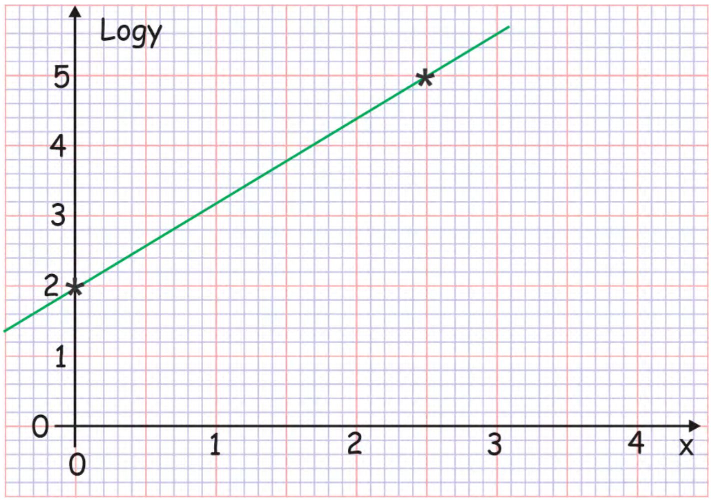


Figure. 8.12: *Graph of* $y = 10^{2+1.2x}$

a. Find the equation of Logy in terms of x.

b. Find the equation of y in terms of x.

Solution

a. the equation of the graph is of the form: $Logy = mx + c$

b. Gradient of the line $= \frac{5-2}{2.5-0} = \frac{3}{2.5} = 1.2$

The line intersects $Log\ y$ axis at 2

this implies $c = 2$

Therefore, the equation of Logy in terms of x is:

$Logy = 1.2x + 2$

c. The equation on y in terms of x is of the form $y = ab^x$, where a and b are constants.

Writing $y = ab^x$ in linear form, we have Logy = xlogb+loga.

Comparing Logy = xlogb+loga to

Logy $= 1.2x + 2$

Log y = Logy. This implies that y = y

x = x

Log b =1.2

Laking antilog on both sides, we have b $=10^{1.2}$

And Loga = 2

Taking antilog on both sides, we have $a = 10^2 = 100$

Therefore, the equation of y in terms of x is:

$y = 100 \times (10^{1.2})^x$,

$y = 10^2 \times 10^{1.2x}$

$y = 10^{2+1.2x}$

Example 8.22

Figure 8.13 shows the relationship between $LogV$ and $LogP$.

Figure 8.13: *Graph of* $V = 1000\,P^{-2}$

a. Find the equation of $LogV$ in terms of $LogP$

b. Find the equation of V in terms of P

Solution

a. The line passes through the points (0,3) and (0.5, 2)

b. The gradient of the line is $m = \dfrac{2-3}{0.5-0} = -\dfrac{1}{0.5} = -2$

The equation of $LogV$ in terms of $LogP$ is:

$LogV = -2LogP + 3$

The relation is of the form $V = a\,P^n$, where a and n are constants.

When $V = a\,P^n$ is written in linear form, it becomes $LogV = nLogP + Loga$

Comparing $LogV = nLogP + Loga$ to

$LogV = -2logP + 3$

We have:

$n = -2$ and $Log a=3 \Longrightarrow a = 10^3 = 1000$

Therefore, the relation of V in terms of P is:

$V = 1000 P^{-2}$ or $V = \frac{1000}{P^2}$

EXTENDED READING

1. Explore plotting of logarithmic graphs using GeoGebra software.
   a. Open the GeoGebra Application
   b. In the input bar, type the equation of the logarithmic function (E.g., $y = \log(x)$)
   c. Press Enter and GeoGebra will display the graph of y=log(x)
   d. Modify the logarithmic function to explore transformations:

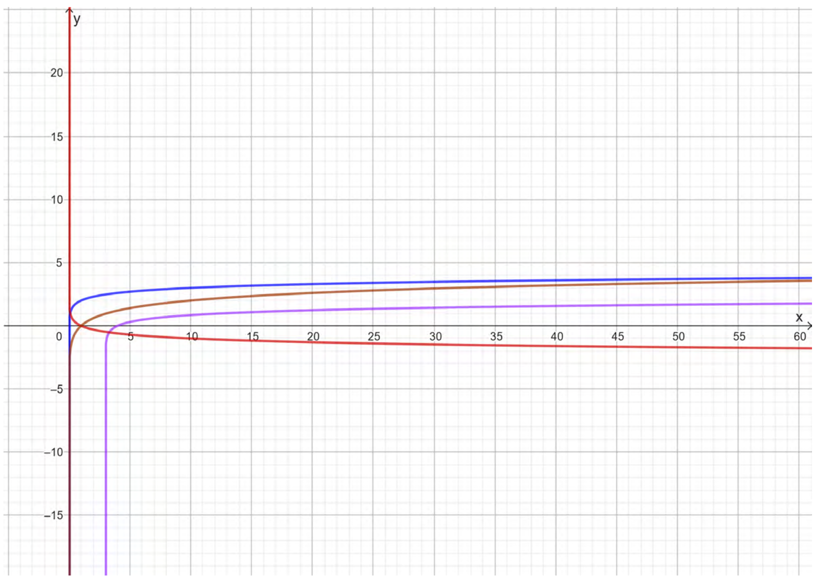
      Input the following functions:

      i. $y = \log(x)+2$.

      ii. $y = \log(x - 3)$.

      iii. $y = -\log(x)$.

      iv. $y = 2\log(x)$.
   e. Observe how the graph changes with each transformation.
   f. Discuss the changes with your classmates.
   g. Save the GeoGebra file by clicking on the Save or Export option.



2. Talbert, J.F. & Heng, H.H., (2007). Additional Mathematics (Pure and Applied). Pages 370 – 379

3. Haese M., Haese S., Humphries M., Sangwin C., (2014). Cambridge Additional Mathematics. Pages 140, 141

REVIEW QUESTIONS

1. Find the value of x that satisfies the equation:
   a. $\log_{10}(2x - 1) = \log_{10}x - \log_{10}3$
   b. $\log_{10}(3x + 1) - \log_{10}(1 - x) = 2\log_{10}2$
   c. Solve $\log_3(x + 4) - \log_2(x - 4) = 2$
   d. $\log_2\sqrt[3]{4x + 5} = \frac{1}{3}[\log_2 x - \log_2 3]$
2. Make y the subject of the relation $2 + \log_a b + 3\log_a y = 2\log_a a^2 y$
3. Find the truth set of the equation:
   a. $\log_x 4x - \log_{4x} x = 1\frac{1}{2}$
   b. $\log_4 x + \log_x 16 = 3$
4. Two equations are defined by: $\log_2 x + 2\log_4 y = 4$ and $x - y = 6$,
5. for $x > y$. Find $x + y$.
6. Ato borrowed GH¢ 12 000 from a Bank for 4 years at an interest of 15% compounded annually. Find the compound interest and amount he has to pay at the end of 4 years.
7. A bank decides to offer Fatima a GH¢ 800 000 loan at a compound interest of 10% per year. Find the total amount the bank will receive when the loan is repaid after 5 years.
8. How much interest will Fatima pay?
9. A housewife invested \$30 000 into a fund that pays 12 percent interest each year, compounded quarterly. Find the value of the investment at the end of the fourth year.
10. Ato invested €70 000 into a fund that pays 10% interest each year, compounded once every two years.
    a. Find the total amount of his investment after 10 years.
    b. After how many years will his investment double?
11. Calculate the annual rate of compound interest that will allow a loan of \$50 000 to amount to \$73 205 in four years.
12. Currently, the population of a country is 72 million. This figure is expected to increase at a constant rate of 2.5% each year. Estimate the population of the country in 5 years.

13. Find the loudness of a sound that has an intensity 10 000 times the threshold of hearing for the average human ear.

14. Find the scrap value of a generator costing *GH¢* 25 000.00, having a useful life span of fifteen years and a constant annual rate of depreciation of 10%.

15. Convert the following to linear functions:

    a. $y = \frac{1}{2}x^4$

    b. $P = \frac{10}{r}$

    c. $V = 2P^3$

16. Convert the following to linear functions

    a. $M = 10 \times 7^r$

    b. $y = \frac{1000}{2^x}$

17. Convert $P = 100t^2$ to a linear function and graph it.

18. The table below gives some values of two related variables, x and y.

x	1	1.2	1.4	1.6	1.8	2	2.2
y	10	14.4	19.6	25.6	32.4	40	48.4

The relationship between y and x is of the form $y = Ax^b$ where A and b are constants.

   a. Draw a suitable linear graph for $y = Ax^b$

   b. Use your graph to find A and b, correct to the nearest whole number.

   c. Use your graph to find, correct to the nearest whole number the value of y when $x = 1.7$

19. The value of an investment (V) at the end of t years satisfies the relation:

20. $V = PR^t$ where P and R are constants. The table below gives some values of V at the end of t years.

t	1	2	3	4	5	6	7
V (GH¢)	22400.00	25088.00	28098.56	31470.39	35246.83	39476.45	44213.63

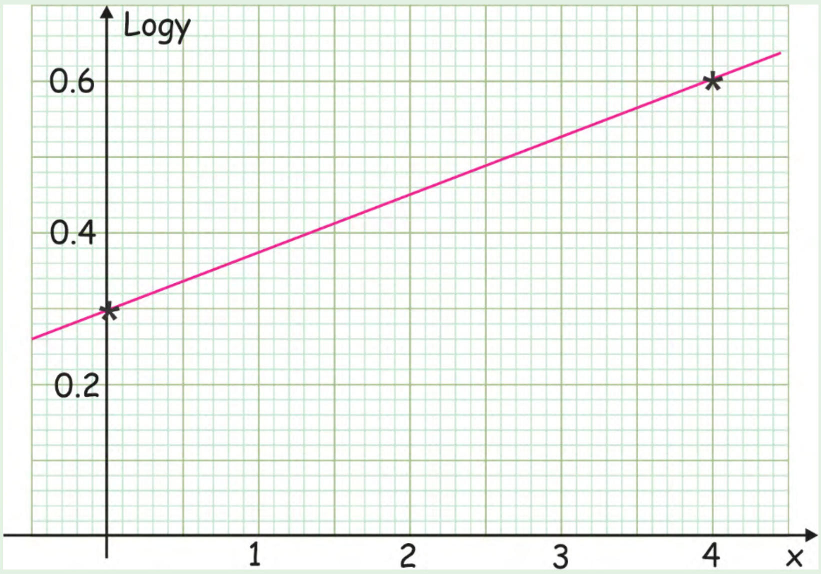
   a. Draw a suitable line graph for $V = PR^t$

   b. Use your graph to find

      i. P, correct to the nearest thousand

ii. R, correct to three significant figures.

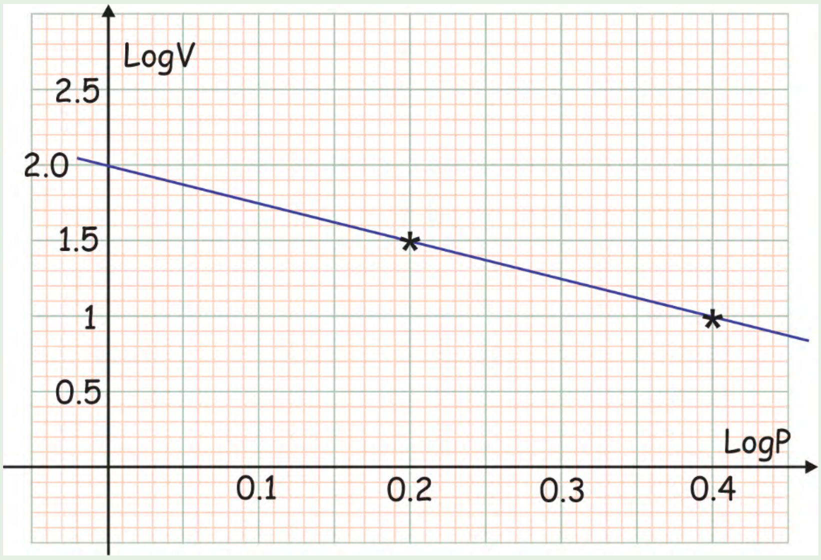
21. The figure below shows the graph of Log y against x



22. a. Find the equation of Logy in terms of x

b. Find the equation of y in terms of x

23. The figure shows the relationship between $Log V$ and $Log P$.



a. Find the equation of $Log V$ in terms of $Log P$

b. Find the equation of V in terms of P

SECTION 9

TRIGONOMETRIC IDENTITIES

GEOMETRIC REASONING AND MEASUREMENT

Measurement of Triangles

INTRODUCTION

Knowledge of trigonometric identities helps simplify complex trigonometric expressions into more manageable forms and prove mathematical results. In physics and engineering, trigonometric identities help in studying wave motion, mechanics, analysing electrical circuits and mechanical systems. In architecture and design, the concept aids in designing structures with curved surfaces. This section introduces you to the concept of trigonometric identities, deriving and applying the sine and cosine rule and solving trigonometric equations.

KEY IDEAS

- **Multiple angles** formulas express trigonometric functions of multiples of an angle (e.g., 2θ, 3θ) in terms of the trigonometric functions of the original angle.
- **Trigonometric equations** involve trigonometric functions and require solving for the angles or other variables. These equations often leverage trigonometric identities and properties for solutions.
- **Trigonometric Identities** are fundamental relationships between trigonometric functions that are true for all values of the variables involved, within their domain. These include Pythagorean identities, Reciprocal Identities and sum and difference identities.

TRIGONOMETRIC IDENTITIES

In year one, we learnt about the fundamental identities in trigonometry:

$\sin\theta = \frac{opposite}{hypotenuse}$, $\cos\theta = \frac{adjacent}{hypotenuse}$, $\tan\theta = \frac{opposite}{adjacent}$,

$\csc\theta = \dfrac{hypotenuse}{opposite}$, $\sec\theta = \dfrac{hypotenuse}{adjacent}$ and $\cot\theta = \dfrac{adjacent}{opposite}$

From this we can derive the reciprocal identities.

Activity 9.1: Reciprocal Identities

In small groups or pairs, work through the following activity.

1. Choose a number (e.g., 2)
2. Write down the reciprocal of the number $\left(\text{e.g.}, \frac{1}{2}\right)$
3. Now, write down the reciprocal of the sine identity as $\dfrac{1}{\sin\theta}$.
4. Express $\sin\theta$ as $\dfrac{opposite}{hypotenuse}$ within the reciprocal to get $\dfrac{1}{\frac{opposite}{hypotenuse}}$.
5. Note that $\dfrac{1}{\frac{opposite}{hypotenuse}}$ is the same as $\dfrac{hypotenuse}{opposite}$.
6. What relationship do you see between the result of $\dfrac{1}{\sin\theta}$ and the cosec (csc) identity?
7. Hopefully you see that $\csc\theta = \dfrac{1}{\sin\theta}$.
8. Now, repeat the steps from 3 to 6 using the identities $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$ and $\cot\theta$.
9. Hopefully, you realise that:
10. $cos\theta = \dfrac{1}{(sec\ \theta)}$; $tan\theta = \dfrac{1}{cot\theta}$, $sec\theta = \dfrac{1}{cos\theta}$, $cot\theta = \dfrac{1}{tan\theta}$, $sin\theta = \dfrac{1}{csc\theta}$.

Now that we have established the reciprocal identities through Activity 9.1, let us explore the Quotient Identities.

Activity 9.2: Quotient Identities (in small groups or individually)

In small groups or pairs, work through the following activity.

1. Write out $\dfrac{\sin\theta}{\cos\theta}$ as $\dfrac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}}$.

2. Simplify the resulting expression:

$$\frac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}} = \frac{opposite}{hypotenuse} \div \frac{adjacent}{hypotenuse} = \frac{opposite}{hypotenuse} \times \frac{hypotenuse}{adjacent} = \boldsymbol{\frac{opposite}{adjacent}}$$

3. Which trigonometric identity can be expressed as $\frac{opposite}{adjacent}$?
4. Identify the relationship between $\frac{\sin\theta}{\cos\theta}$ and $\tan\theta$.
5. Hopefully, you were able to conclude that $\frac{\sin\boldsymbol{\theta}}{\cos\boldsymbol{\theta}} = \tan\boldsymbol{\theta}$.
6. Now go through steps 1 to 3 using $\frac{\cos\theta}{\sin\theta}$ and identify which trigonometric identity is equal to $\frac{\cos\theta}{\sin\theta}$.
7. Hopefully, you came to the conclusion that $\frac{\cos\boldsymbol{\theta}}{\sin\boldsymbol{\theta}} = \cot\boldsymbol{\theta}$ and that $\cot\theta$ and $\tan\theta$are reciprocal identities as well.

Pythagorean Identities

Using knowledge of Pythagoras theorem and algebra, we can deduce three Pythagorean Identities. Let us go through this activity to deduce the three identities.

Activity 9.3: Proving Pythagorean Identities

In small groups or pairs, work through the following activity.

Identity 1

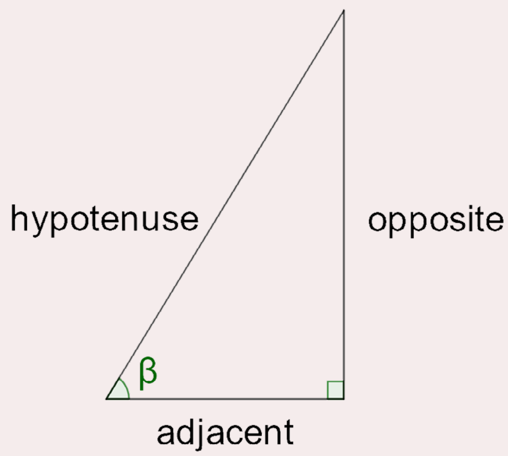


Figure 9.1: Right-angled Triangle for Activity 9.3

1. Draw a right-angle triangle.
2. Choose variables to represent the lengths of the hypotenuse, opposite and adjacent sides to the angle (β).

   r = hypotenuse, x = adjacent and w = opposite.
3. Write out the Pythagoras theorem using identified lengths.

   $r^2 = x^2 + y^2$
4. Divide both sides by r^2 and simplify.

   $$\frac{r^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

   $$1 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$
5. Rewrite $\left(\frac{x}{r}\right)^2$ and $\left(\frac{y}{r}\right)^2$ using the sides they represent.

   $$\left(\frac{adjacent}{hypotenuse}\right)^2 = cos^2\beta \text{ and } \left(\frac{opposite}{hypotenuse}\right)^2 = sin^2\beta$$
6. Substitute $cos^2\beta$ and $sin^2\beta$ into the simplified equation in Step 4

   $1 = \boldsymbol{cos^2\beta + sin^2\beta}$

Identity 2

1. From Step 3 in **Identity 1**, divide both sides by x^2 and simplify.

   $$\frac{r^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$$

   $$\left(\frac{r}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$$
2. Rewrite $\left(\frac{r}{x}\right)^2$ and $\left(\frac{y}{x}\right)^2$ using the sides they represent.

   $$\left(\frac{hypotenuse}{adjacent}\right)^2 = sec^2\beta \text{ and } \left(\frac{opposite}{adjacent}\right)^2 = tan^2\beta$$
3. Substitute $sec^2\beta$ and $tan^2\beta$ into the simplified equation in Step 1 of Identity 2.

   $\boldsymbol{sec^2\beta = 1 + tan^2\beta}$

Identity 3

1. From Step 3 in **Identity 1**, divide both sides by y^2 and simplify.

   $\frac{r^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$

   $\left(\frac{r}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$

2. Rewrite $\left(\frac{r}{y}\right)^2$ and $\left(\frac{x}{y}\right)^2$ using the sides they represent.

   $\left(\frac{hypotenuse}{opposite}\right)^2 = csc^2\beta$ and $\left(\frac{adjacent}{opposite}\right)^2 = cot^2\beta$

3. Substitute $csc^2\beta$ and $cot^2\beta$into the simplified equation in Step 1 of Identity 3.

   $\boldsymbol{csc^2\beta = cot^2\beta + 1}$

Hopefully, you were able to follow through all the steps in the activity to arrive at the Pythagorean identities, which are:

1. $\boldsymbol{cos^2\beta + sin^2\beta = 1}$
2. $\boldsymbol{sec^2\beta = 1 + tan^2\beta}$
3. $\boldsymbol{csc^2\beta = cot^2\beta + 1}$

Compound angles identities

There are other trigonometric identities that involve two variables. These identities are commonly referred to as compound identities or the sum and difference formula. The compound angle formulas are important because they help describe rotations. If an object rotates by two angles A and B, the formulas combine these rotations into a single equivalent angle. This concept is vital in designing 3D models and simulating movements in animation or robotics. Compound angle identities are used in modelling sound waves, light waves and electromagnetic waves.

The concept of compound angles is based on a unit circle. A unit circle (circle with radius of 1 unit) can be drawn such that the starting side of the angle, $(\beta - \alpha)$ lies on the positive $x-axis$ and hence intersects the circle at (1, 0) while the ending side is rotated such that instead of intersecting the circle at $(\cos\beta, \sin\beta)$, it intersects at $(\cos(\beta - \alpha)\ \sin(\beta - \alpha))$. This ensures the length of the chord between the sides of angle, $(\beta - \alpha)$ is maintained, as shown in figure 9.2.

Activity 9.4: Research on compound angle identities

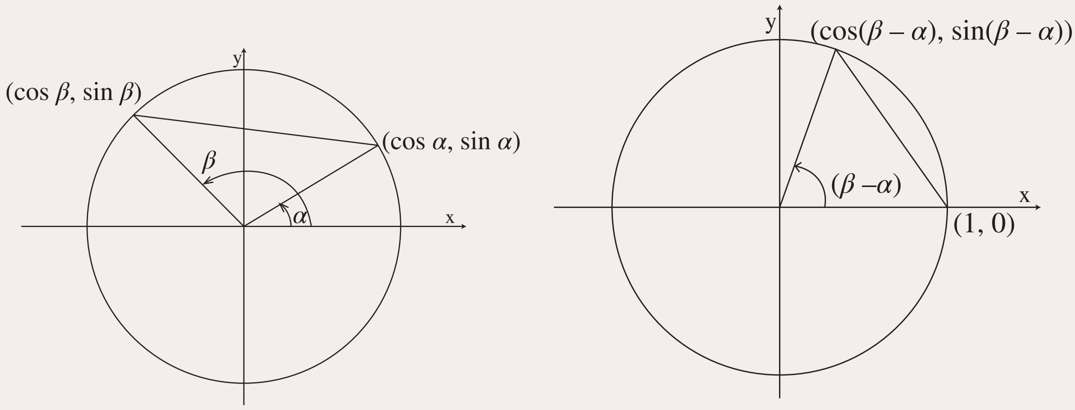


Figure 9.2: Graphical illustration of compound angles

In small groups or pairs, undertake the following research.

1. Carry out research on the compound angle identities outlined from a. – f.

a. $\cos(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \cos(\boldsymbol{\alpha})\boldsymbol{cos}(\boldsymbol{\beta}) + \sin(\boldsymbol{\alpha})\sin(\boldsymbol{\beta})$

b. $\cos(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \cos(\boldsymbol{\alpha})\boldsymbol{cos}(\boldsymbol{\beta}) - \sin(\boldsymbol{\alpha})\sin(\boldsymbol{\beta})$

c. $\sin(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \boldsymbol{sin}(\boldsymbol{\alpha})\boldsymbol{cos}(\boldsymbol{\beta}) + \boldsymbol{cos}(\boldsymbol{\alpha})\boldsymbol{sin}(\boldsymbol{\beta})$

d. $\sin(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \boldsymbol{sin}(\boldsymbol{\alpha})\boldsymbol{cos}(\boldsymbol{\beta}) - \boldsymbol{cos}(\boldsymbol{\alpha})\boldsymbol{sin}(\boldsymbol{\beta})$

e. $\tan(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \dfrac{\boldsymbol{tan}(\alpha) + \boldsymbol{tan}(\beta)}{1 - \boldsymbol{tan}(\alpha)\boldsymbol{tan}(\beta)}$

f. $\tan(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \dfrac{\boldsymbol{tan}(\alpha) - \boldsymbol{tan}(\beta)}{1 + \boldsymbol{tan}(\alpha)\boldsymbol{tan}(\beta)}$

2. Consult textbooks and online resources to work out the algebraic proofs.

3. Present your findings to your classmates.

Application of compound angle identities

In year one we learnt about special angles (30^0, 45^0, 60^0, 90^0). We will now apply this knowledge on special angles and the compound angle identities to solve problems.

Example 9.1

Determine the exact value of $\sin(120^0)$ without using calculators.

Solution

Step 1: Split the given angle into two special angles

$\sin(120^0) = \sin(60^0 + 60^0)$

Step 2: Express the split angle based on $\sin(\alpha+\beta)$ compound angle identity

$\sin(60^0 + 60^0) = \sin(60^0)\cos(60^0) + \cos(60^0)\sin(60^0)$

Step 3: Substitute the values for the trigonometric identity angles and simplify

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

Therefore, $\sin(120^0) = \frac{\sqrt{3}}{2}$.

Example 9.2

Determine the exact value of $\cos(135^0)$ without using calculators.

Solution

Step 1: Split the given angle into two special angles

$\cos(135^0) = \cos(90^0 + 45^0)$

Step 2: Express the split angle based on $\cos(\alpha+\beta)$ compound angle identity

$\cos(90^0 + 45^0) = \cos(90^0)\cos(45^0) - \sin(90^0)\sin(45^0)$

Step 3: Substitute the values for the trigonometric identity angles and simplify

$$= (0)\left(\frac{\sqrt{2}}{2}\right) - (1)\left(\frac{\sqrt{2}}{2}\right) = \frac{-\sqrt{2}}{2}$$

Therefore, $\cos(135^0) = \frac{-\sqrt{2}}{2}$.

Example 9.3

Find the value of $\tan(120^0)$ without using calculators.

Solution

Step 1: Split the given angle into two special angles

$\tan(120^0) = \tan(60^0 + 60^0)$

Step 2: Express the split angle based on $\tan(\alpha+\beta)$ compound angle identity

$$\tan(60^0 + 60^0) = \frac{an(60^0) + tan(60^0)}{1 - tan(60^0)tan(60^0)}$$

Step 3: Substitute the values for the trigonometric identity angles and simplify

$$= \frac{\sqrt{3} + \sqrt{3}}{1 - (\sqrt{3})(\sqrt{3})} = \frac{2\sqrt{3}}{-2}$$

Therefore, $\tan(120^0) = -\sqrt{3}$.

Multiple-angle identities

Multiple–angle identity corresponds to multiples of an angle, such as 2α, 3α, or *n*α, where *n* is a positive integer. These identities express trigonometric functions of a multiple of an angle in terms of the trigonometric functions of the base angle, α. Multiple angles can be double and triple angles. However, here we only look at double angles.

Double angle identities

Double angle identities focus on angles that can be expressed as a sum of two of the same angles (i.e., $\gamma = \alpha + \alpha$). In such cases, compound angle identities can be used to find the solution given that $\alpha = \beta$. Based on our knowledge of multiple angles, if we substitute α with β given that $\alpha = \beta$, we will realise that the three identities below hold:

1. $\cos(\alpha + \alpha) = \cos(\alpha)cos(\alpha) - \sin(\alpha)\sin(\alpha)$

   $\cos 2\boldsymbol{\alpha} = \boldsymbol{cos}^2\boldsymbol{\alpha} - \boldsymbol{sin}^2\boldsymbol{\alpha}$

   And using the Pythagorean identities this means that:

   $\cos 2\boldsymbol{\alpha} = 1 - 2\boldsymbol{sin}^2\boldsymbol{\alpha}$ and $\cos 2\boldsymbol{\alpha} = 2\boldsymbol{cos}^2\boldsymbol{\alpha} - 1$

2. $\sin(\alpha + \alpha) = sin(\alpha)cos(\alpha) + cos(\alpha)sin(\alpha)$

   $\sin 2\boldsymbol{\alpha} = 2\boldsymbol{sin}(\boldsymbol{\alpha})\boldsymbol{cos}(\boldsymbol{\alpha})$

3. $\tan(\alpha + \alpha) = \frac{tan(\alpha) + tan(\alpha)}{1 - tan(\alpha)tan(\alpha)}$

   $\tan 2\boldsymbol{\alpha} = \frac{2\boldsymbol{tan}(\boldsymbol{\alpha})}{1 - 2\boldsymbol{tan}(\boldsymbol{\alpha})}$

If an angle is α, half of the angle results in $\frac{\alpha}{2}$. This leads us into the concept of half-angles.

Let us go through the Activity 9.5 to prove the half-angle identities.

Activity 9.5: Research on half-angle Identities (work in groups)

In small groups, undertake the following research.

1. Conduct research to prove that:

a. $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$

b. $\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$

c. $\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$

2. Include graphical illustrations to support your algebraic proofs.

3. Present a report on your findings to your classmates and teacher.

Verifying Identities

An identity is an equation that is true for all values of the variables within its domain. It is a fundamental relationship that holds universally and does not depend on specific variable values. We can apply the trigonometric identities to verify whether an equation is an identity or not.

For example, using graphs to illustrate identities, sketch the graph of

$y = \frac{tan(x) - cot(x)}{sin(x)\ cos(x)}$ and $y = sec^2(x) - \csc^2(x)$.

Figure 9.3: Graphical illustration of $y = \frac{\tan(x)\text{-}\cot(x))}{(\sin(x)\cos(x)}$

Comparing Figures 9.3 and 9.4, it is evident that, for any value of x, the results of the two equations will be the same. Thus, $\frac{tan(x) - cot(x)}{sin(x)\, cos(x)}$ and $sec^2(x) - \csc^2(x)$ are identities.

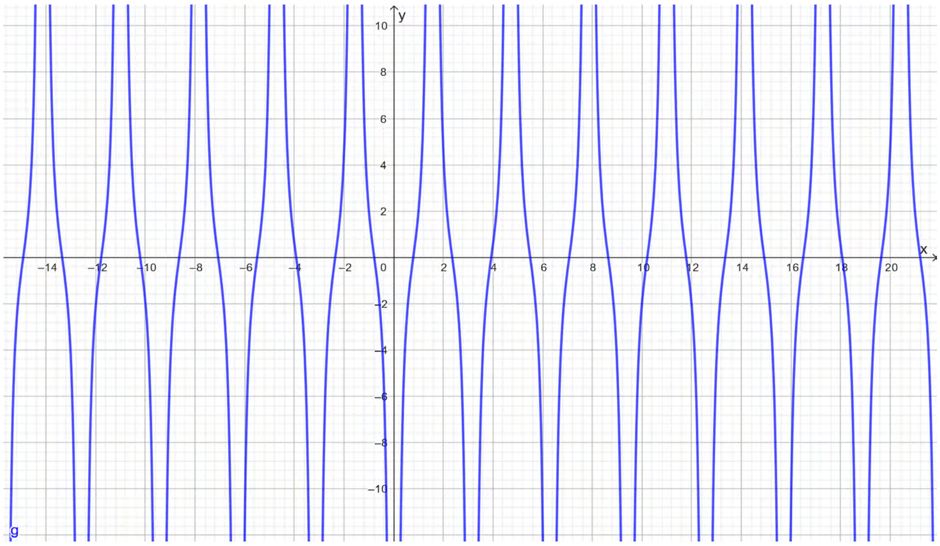


Figure 9.4: Graphical illustration of $y = sec^2(x) - csc^2(x)$

Mathematically, we can represent them as $\frac{tan(x) - cot(x)}{sin(x)\, cos(x)} \equiv sec^2(x) - \csc^2(x)$.

Example 9.4

Prove that $\frac{tan(x) - cot(x)}{sin(x)\, cos(x)} \equiv sec^2(x) - \csc^2(x)$, algebraically.

Solution

Step 1: Rewrite Left-hand side expression in terms of trigonometric identities:

$$\frac{tan(x) - cot(x)}{sin(x)\, cos(x)} = \frac{\frac{sin(x)}{cos(x)} - cos(x)\,\_\, sin(x)}{sin(x)\, cos(x)}$$

Step 2: Simplify the numerator of the fraction:

$$\frac{\frac{sin^2(x) - cos^2(x)}{cos(x) sin(x)}}{sin(x)\, cos(x)}$$

Step 3: Simplify the double fraction:

$$\frac{sin^2(x) - cos^2(x)}{cos(x) sin(x)} \times 1\frac{-}{in(x)\, cos(x)}$$

$$\frac{sin^2(x) - cos^2(x)}{cos^2(x)\, sin^2(x)}$$

Step 4: Split the fraction in two terms and simplify:

$$\frac{sin^2(x)}{cos^2(x)\,sin^2(x)} - \frac{cos^2(x)}{cos^2(x)\,sin^2(x)}$$

$$\frac{1}{cos^2(x)} - \frac{1}{sin^2(x)}$$

Step 5: Rewrite $\frac{1}{cos^2(x)}$ and $\frac{1}{sin^2(x)}$ in terms of secant (sec) and cosec (csc) trigonometric identities.

$$\frac{1}{cos^2(x)} = sec^2\left(x\right) \text{ and } \frac{1}{sin^2(x)} = csc^2\left(x\right)$$

Therefore, $\frac{1}{cos^2(x)} - \frac{1}{sin^2(x)}$ results in $sec^2(x) - csc^2(x)$.

Step 6: Compare the righthand side (RHS) of the original equation with the lefthand side (LHS) result and draw a conclusion.

Since the LHS resulted in $sec^2(x) - csc^2(x)$ which is same as the original RHS expression, $\frac{tan(x) - cot(x)}{sin(x)\,cos(x)} \equiv sec^2(x) - csc^2(x)$.

Activity 9.6: Algebraic Verification of Identities

In small groups, undertake the following research.

1. Research on the algebraic proof of $tan(x) + cot(x) \equiv \frac{csc(x)}{cos(x)}$.
2. Provide a graphical illustration (using graphing tool or manually) to support your algebraic proof.
3. Make a presentation to your classmates and teacher.

DERIVING AND APPLYING THE SINE AND COSINE RULE

When we talked about vectors in Section 5, we looked at the Sine and Cosine rule that helped in finding angles and lengths of lines of oblique (non-right-angle) triangles. While we used measurements to understand the relationships, we will use algebraic proofs to establish the Sine and Cosine rules in this section. The cases in which we can apply a sine or cosine rule are:

1. When two angles and any side length are known.

2. When two side lengths and an angle opposite one of the known side lengths are known.
3. When three side lengths are known.
4. When two side lengths and the angle included between the two sides are known.

Sine Rule

From Section 5 on vectors, we found out that $\frac{r}{\sin R} = \frac{m}{\sin M} = \frac{t}{\sin T}$ given the triangles in Figure 9.5 below.

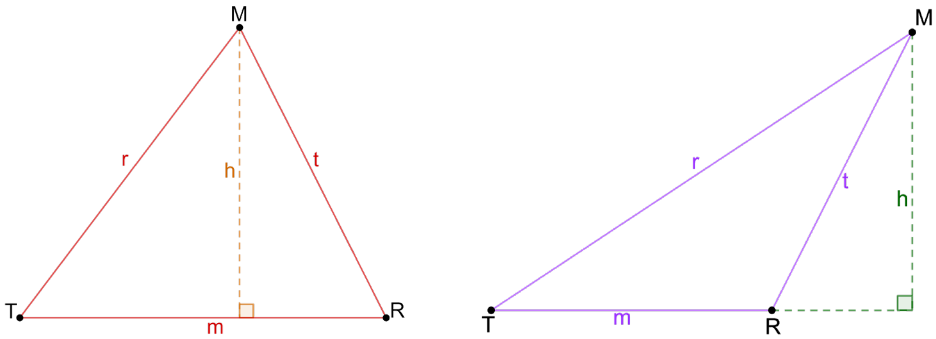


Figure 9.5: Oblique triangles supporting exploration of Sine rule

The Sine rule is applicable when two angles and any side length is known.

Let us go through the process of deriving the Sine rule using the steps outlined below.

Deriving the sine rule

This derivation is based on the triangles in Figure 9.5.

1. Using <MTR, $\sin(T) = \frac{h}{r}$ from trigonometric identities.
2. Making h the subject, we get $h = r\sin(T)$
3. Also, using <TRM, $\sin(R) = \frac{h}{t}$
4. Making h the subject, we get $h = t\sin(R)$
5. This reveals that $h = r\sin(T) = t\sin(R)$
6. Rewrite $r\sin T = t\sin(R)$ by dividing both sides by $\sin(R)\sin(T)$:

   $$\frac{r\sin(T)}{\sin(R)\sin(T)} = \frac{\sin(R)}{\sin(R)\sin(T)}$$

   Simplifying the equation results in $\frac{\boldsymbol{r}}{\sin(\boldsymbol{R})} = \frac{\boldsymbol{t}}{\sin(\boldsymbol{T})}$

Activity 9.7: Verifying the Sine Rule

In small groups, undertake the following proof.

Follow the steps used for deriving the sine rule and verify if $\frac{m}{\sin(M)} = \frac{r}{\sin(R)}$.

Application of the sine rule

Now that we are familiar with the concept of the sine rule, let us see how we can apply them using the worked examples below.

Example 9.5

A ship is moving from Point A to Point B, then to Point C. At A, the angle between the paths to B and C is 45^0. At B, the angle between the paths to A and C is 60^0. If the distance between A and B is 8km, find the distance between B and C.

Figure 9.6: Moving Ship

Solution

Step 1: Sketch the movement of the ship:

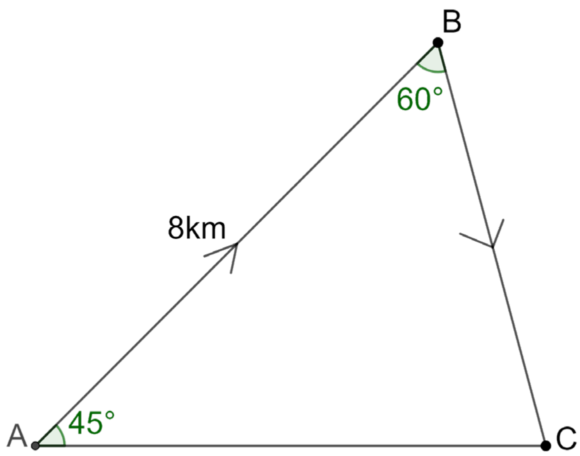


Figure 9.7: Graphical representation of the ship's movement

Step 2: Since we have two angles, we can calculate the third angle in the triangle:

$< ACB = 180^0 - 45^0 - 60^0 = 75^0$

Step 3: Applying the sine rule:

$$\frac{|AB|}{\sin 75^0} = \frac{|BC|}{\sin 45^0}$$

Step 4: Substitute the value for $|AB|$ and simplify the equation

$$\frac{8}{\sin 75^0} = \frac{|BC|}{\sin 45^0}$$

$|BC| = 8(\sin45\ 0)$ _ $\sin75\ 0$

$|BC| = 5.86\ km$ (to 3sf)

Therefore, the distance between Point B and C is 5.86 *km*.

Example 9.6

A drone needs to deliver medicine to a hospital to help save a patient's life. At the hospital (H), the angle between the drone's position (D) and the supply depot (S) is 50^0. From the drone's current position, the angle to the hospital and a nearby supply depot is 70^0. The distance between the drone and the supply depot is 12 km.

Figure 9.8: Drone delivering Medicine

a. Find the distance between the:

i. drone and the hospital.

ii. hospital and the supply depot.

b. If the drone has to pick up the medicine from the supply depot and deliver it to the hospital, how many kilometres will it travel?

Solution

a. i. Step 1: Sketch the movement of the drone

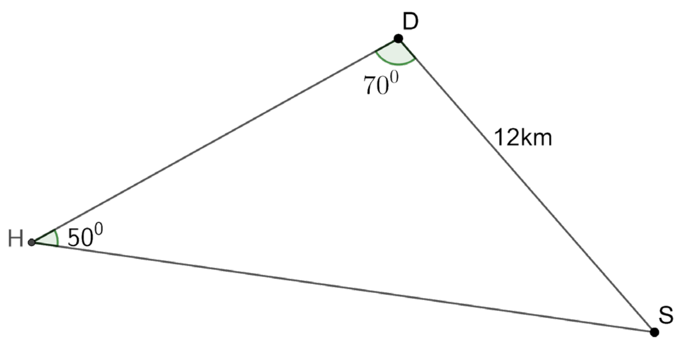


Figure 9.9: Drone movement

Step 2: Since we have two angles, we can calculate the third angle in the triangle.

$$< HSD = 180^0 - 50^0 - 70^0 = 60^0$$

Step 3: Apply the sine rule:

$$\frac{|HD|}{\sin 60^0} = \frac{|DS|}{\sin 50^0}$$

Step 4: Substitute the value for $|DS|$ and simplify the equation

$$\frac{|HD|}{\sin 60^0} = \frac{12}{\sin 50^0}$$

$$|HD| = 12(\sin 60\ 0)\ \_\ \sin 50\ 0$$

$$|HD| = 13.57\text{km}$$

Therefore, the distance between the hospital and drone's current position is 13.6km (to 3sf).

ii. **Step 1:** Apply the sine rule:

$$\frac{|HS|}{\sin 70^0} = \frac{|DS|}{\sin 50^0}$$

Step 2: Substitute the value for $|DS|$ and simplify the equation

$$\frac{|HS|}{\sin 70^0} = \frac{12}{\sin 50^0}$$

$$|HS| = 12(\sin 70\ 0)\ \_\ \sin 50\ 0$$

$$|HS| = 14.72\text{km}$$

Therefore, the distance between the hospital and supply depot is 14.7km (to 3sf).

b. Distance travelled = |DS| + |HS| = 12 + 14.7 = 26.7km (to 3sf)

The drone needs to travel 26.7km in order to get to the hospital with the medicine.

Example 9.7

Joel moves from the school entrance walking at 5 km/h on a bearing of 050^0. Edith leaves the same point (school entrance) as Joel at the same time and cycles on a bearing of 125^0 travelling at a constant speed. Find Edith's average cycling speed if she and Joel are 60km apart after 6 hours.

Solution

Step 1: Use the speed distance formula to determine the kilometres walked by Joel

$$\text{Speed} = \frac{distance}{time}$$

$$5 = \frac{distance}{6}$$

Distance $= 6 \times 5 = 30$km

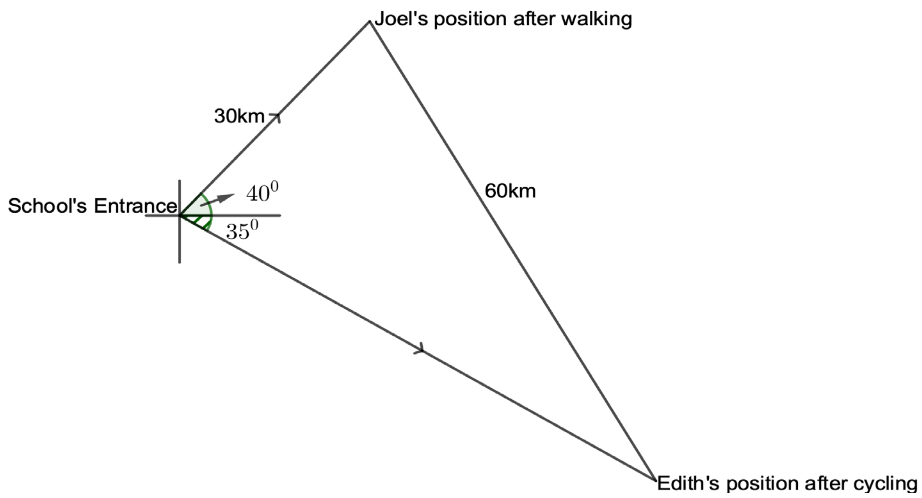


Figure 9.10: Graphical illustration of Joel and Edith's movement

Step 2: Sketch the movement of Joel and Edith

Step 3: Calculate the distance covered by Edith using the sine rule

Let θ be angle formed at Edith's final position

$$\frac{60}{\sin75^0} = \frac{30}{\sin\theta}$$

$$\sin\theta = \frac{30\sin75^0}{60}$$

$$\theta = sin^{-1}\left(\frac{30\sin75^0}{60}\right)$$

$$\theta = 28.88^0$$

Now, $180^0 - 75^0 - 28.88^0 = 76.12^0$(sum of interior angles in a triangle)

Let distance covered by Edith be represented by *d*.

$$\frac{60}{\sin75^0} = \frac{d}{\sin76.12°}$$

$$d = \frac{60\sin76.12°}{\sin75^0}$$

$d = 60.30km$

Step 4: Use the distance – speed relationship to determine Edith's cycling speed.

Edith's Speed $= \frac{60.30}{6}$

Edith's Speed $= 10.05km/h$

Edith's average cycling speed is 10.05km/h

Cosine Rule

In section 5, on vectors, we learnt about the cosine rule and proved it using measurements. We also discovered that when three sides are known or when two sides and an angle included between the two sides are known, the cosine rule can assist in finding the unknown angles and lengths. Let us see how algebra can assist us in proving the cosine rule. Suppose ΔDEF in Figure 9.11 has the coordinates of vertex F as (x, y), then $sin(E) = y$ and $cos(E) = x$. Hence, $y = dsin(E)$ and x = $dcos(E)$ as depicted in Figure 9.11

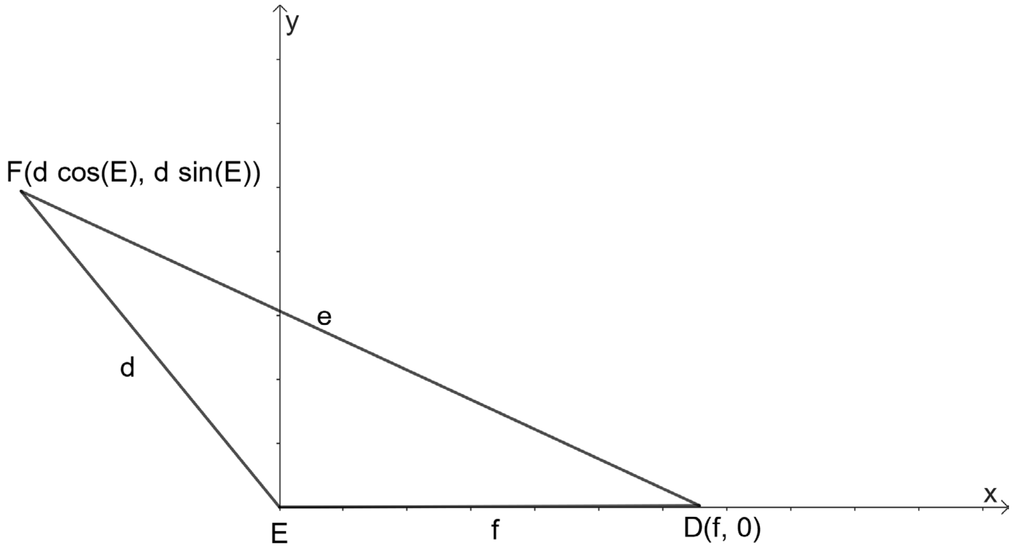


Figure 9.11: Oblique triangle supporting Cosine Rule

Activity 9.8: Algebraic proof of the cosine rule

In small groups, work through the following activity.

1. Draw a triangle such as the one in Figure 9.11
2. Define the coordinates of the vertices as E(0, 0), D(f, 0) and F($dcos(E)$, $dsin(E)$).
3. Write the distances in terms of the coordinates

$$\left(\begin{array}{c} |ED| = f, \\ |EF| = \sqrt{d\cos(E)^2 + d\sin(E)^2},\ |DF| = \sqrt{(dcos(E) - f)^2 + (dsin(E) - 0)^2} \\ . \end{array}\right)$$

4. Calculate $|DF|^2$ and simplify using
5. $|DF|^2 = (dcos(E) - f)^2 + d^2 sin^2(E)$ as a guide.
6. You should have $|DF|^2 = d^2 + f^2 - 2dfcos(E)$
7. Substitute $|DF|$ with e to get $e^2 = d^2 + f^2 - 2dfcos(E)$
8. Share your results with a classmate
9. You have successfully proved the cosine rule.
10. Investigate what happens to the cosine rule when dealing with a right-angle triangle.

Generalisation(s)

1. $\boldsymbol{e^2 = d^2 + f^2 - 2df\cos(E)}$
2. $\boldsymbol{d^2 = f^2 + e^2 - 2fe\cos(D)}$
3. $\boldsymbol{f^2 = d^2 + e^2 - 2de\cos(F)}$

Example 9.8

Given that ΔGMN has |GM| = 12m, |MN| = 17.8m and <GMN = 75^0, determine the value(s) of:

a. <MGN

b. <GNM to the nearest whole number.

Solution

Step 1: Sketch the triangle:

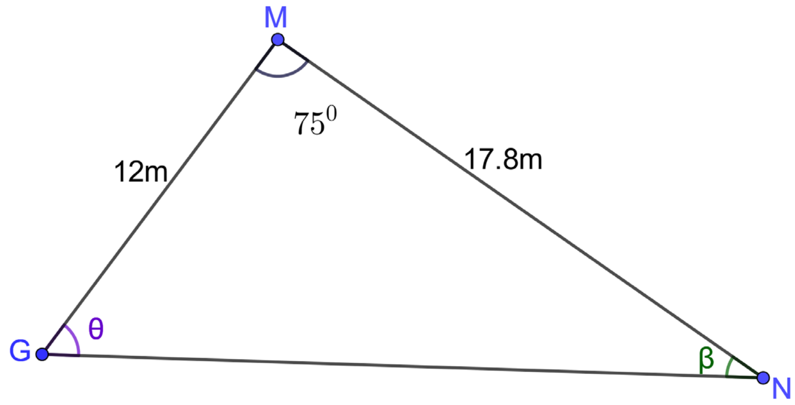


Figure 9.12: Graphical representation of ΔGMN

a. **Step 2:** Represent <MGN by θ

Step 3: Since two sides and the included angle is known, apply the cosine rule to find $|GN|$.

$|GN|^2 = |GM|^2 + |MN|^2 - 2|GM||MN|\cos(75)$

$|GN| = \sqrt{12^2 + 17.8^2 - 2(12)(17.8)\cos(75)}$

$|GN| = 18.72m$

Step 4: Substitute $|GN| = 18.72$ into the cosine rule to find θ

$|MN|^2 = |GM|^2 + |GN|^2 - 2|GM||GN|\cos(\theta)$

$17.8^2 = 12^2 + 18.72^2 - 2(12)(18.72)\cos(\theta)$

Step 5: Make $\cos(\theta)$ the subject of the equation

$449.28\cos(\theta) = 12^2 + 18.72^2 - 17.8^2$

$\cos(\theta) = \frac{177.598}{449.28}$

Step 6: Take cos^{-1} of $\frac{177.598}{449.28}$

$\theta = cos^{-1}\left(\frac{177.598}{449.28}\right)$

$\theta = 66.72^0$

Therefore <MGN = 67^0 to the nearest whole number.

b. **Step 1:** Represent <GNM by β

Step 2: Since two sides and the included angle is known, apply the cosine rule.

$|GM|^2 = |GN|^2 + |MN|^2 - 2|GN||MN|\cos(\beta)$

$12^2 = 18.72^2 + 17.8^2 - 2(18.72)(17.8)\cos(\beta)$

Step 3: Make cos(β) the subject of the equation

$666.432\cos(\beta) = 18.72^2 + 17.8^2 - 12^2$

$$\cos(\beta) = \frac{523.2784}{666.432}$$

Step 4: Take cos^{-1} of $\frac{523.2784}{666.432}$

$\beta = cos^{-1}\left(\frac{523.2784}{666.432}\right)$

$\beta = 38.26^0$

Therefore < GNM = 38^0 to the nearest whole number.

SOLVING TRIGONOMETRIC EQUATIONS

Just as we are able to solve algebraic equations, we are also able to solve trigonometric equations. In the same way as algebraic equations, trigonometric equations can come in the form of linear, quadratic, cubic form, etc. Here we will focus on linear and quadratic trigonometric equations.

Linear Equations

A linear equation is a mathematical statement that shows a relationship between variables where each term is either a constant or a product of a constant and a single variable and the equation represents a straight line when graphed on a coordinate plane.

Example 9.9

Find the value of x given that $2\cos x - 1 = 0$

Solution

Step 1: Write the equation

$2\cos x - 1 = 0$

Step 2: Group like terms

$2\cos x = 1$

Step 3: Divide both sides by 2 and simplify

$\frac{2\cos x}{2} = \frac{1}{2}$

$\cos x = \frac{1}{2}$

Step 4: Take cos^{-1} of both sides

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$x = 60^0$

Therefore, $x = 60^0$.

Example 9.10

Find the value of w given that $2\sin w - \sqrt{3} = 0$

Solution

Step 1: Write the equation

$2\sin w - \sqrt{3} = 0$

Step 2: Group like terms

$2\sin w = \sqrt{3}$

Step 3: Divide both sides by 2 and simplify

$\frac{2\sin w}{2} = \frac{\sqrt{3}}{2}$

$\sin w = \frac{\sqrt{3}}{2}$

Step 4: Take sin^{-1} of both sides

$w = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$w = 60^0$

Therefore, $w = 60^0$.

Example 9.11

Find the exact solution for $\tan 2v = \sqrt{3}$

Solution

Step 1: Write the equation

$\tan 2v = \sqrt{3}$

Step 2: Take tan^{-1} of both sides and divide by 2

$v = \frac{\tan^{-1}(\sqrt{3})}{2}$

$v = 30^0$

Therefore, when $v = 30^0$, $\tan 2v = \sqrt{3}$.

Example 9.12

Find the exact solution for $4\cos 3t + 2 = 0$

Solution

Step 1: Write the equation

$4\cos 3t + 2 = 0$

Step 2: Group like terms

$4\cos 3t = -2$

Step 3: Divide both sides by 4 and simplify

$\frac{4\cos 3t}{4} = -\frac{2}{4}$

$\cos 3t = -\frac{1}{2}$

Step 4: Take cos^{-1} of both sides and divide by 3

$3t = \cos^{-1}\left(\frac{-1}{2}\right)$

$t = \frac{120^0}{3}$

$t = 40^0$

Therefore, when $t = 30^0$, $4\cos 3t + 2 = 0$.

Quadratic Equations

A quadratic equation is a mathematical equation where the highest power of the variable is 2. It represents a parabolic curve when graphed on a coordinate plane. It is usually in the form $ax^2 + bx + c$ where a, b, c are constants.

Let us go through some examples to see how this works in trigonometric equations.

Example 9.13

Find the exact solution(s) for $2sin^2 m - \sin m = 0$.

Solution

Step 1: Write the equation

$2sin^2 m - \sin m = 0$

Step 2: Factorise sin*m*

$\sin m(2\sin m - 1) = 0$

Step 3:

$\sin m = 0$, or $(2\sin m - 1) = 0$

$\sin m = 0$, or $\sin m = 1=$

Step 4: take sin^{-1} of both sides and simplify

$\mathrm{m} = \sin^{-1}(0)$, or $\mathrm{m} = \sin^{-1}\left(\frac{1}{2}\right)$

$m = 0^0$, or $m = 30^0$

Therefore, when $m = 0^0$ and $m = 30^0$, $2sin^2 m - \sin m = 0$.

Example 9.14

Find the exact solution(s) for $sin^2 y = 2 - 2 \cos y$.

Solution

Step 1: Write the equation

$sin^2 y = 2 - 2 \cos y$

$sin^2 y + 2 \cos y = 2$

Step 2: Replace $sin^2 y$ with $1 - cos^2 y$ (trigonometric identities) and group like a quadratic equation:

$1 - cos^2 y + 2 \cos y = 2$

$cos^2 y - 2 \cos y + 1 = 0$

Step 3: Factorise:

$(\cos y - 1)(\cos y - 1) = 0$

$\cos y = 1$

Step 4: Take cos^{-1} of both sides and simplify:

$y = \cos^{-1}(1)$

$y = 0^0$

Therefore, when $y = 0^0$, $sin^2 y = 2 - 2\cos y$.

Example 9.15

Find the exact solution(s) for $2cos^2 x + \cos x - 1 = 0$

Solution

Step 1: Write the equation:

$2cos^2 x + \cos x - 1 = 0$

Step 2: Factorise:

$(\cos x + 1)(2\cos x - 1) = 0$

Step 3: Solve the equations:

$\cos x = -1$, $2\cos x - 1 = 0$

$\cos x = -1$, $\cos x = \frac{1}{2}$

Step 5: take cos^{-1} of both sides and simplify:

$x = 180^0$ or $x = 60^0$

Therefore, when $x = 180^0$ and $x = 60^0$, $2cos^2 x + \cos x - 1 = 0$.

EXTENDED READING

- Haese, M., Haese, S., Humphries & M., Sangwin, C. (2014). Cambridge Additional Mathematics IGCSE® (0606) O Level (4037) 1st Edition. Haese & Harris Publications Pages 243 – 254.
- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. Given that $\sin\rho = \frac{5}{8}$, find $\csc\rho$.
2. If $\sin\beta = \frac{12}{13}$, and $\cos\beta = \frac{5}{13}$, find the value of $\cot\beta$.
3. Swedru School of Business' rectangular football field's diagonal forms an angle θ with one of its shorter sides. If the field's length is 40 metres and its width is 30 metres:
   a. Calculate the value of $\sin\theta + \cos\theta$.
   b. Prove that $\sin^2\theta + \cos^2\theta = 1$ using the dimensions of the field.
4. Determine the exact value of $\cos(120^0)$ without using calculators.
5. Simplify $(1 - \cos^2\varphi)\sec^2\varphi$
6. Eliminate δ from the equations $v = 9\cos\delta$ and $r = 7\sin\delta$.
7. Find the value of $\tan(150^0)$ without using calculators.
8. Find the value of $\cos(45^0)$ without using calculators.
9. Find the value of $2\sin 30^0 \cos 30^0$ without using calculators.
10. Simplify $\frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0}$ without using calculators.
11. Find the angles between 0^0 and 180^0 which satisfy the equations
    a. $\sin v = 0.45$.
    b. $\cos k = -0.63$.
    c. $\tan\beta = 2.15$.
12. Prove that $\cot\sigma + \tan\sigma \equiv \operatorname{cosec}\sigma \sec\sigma$
13. Show that:
    a. $\sin^4\varphi - cos^4\varphi \equiv 1 - 2\cos^2\varphi$
    b. $\sec^4\beta - sec^2\beta \equiv \tan^2\beta + \tan^4\beta$
    c. $\tan^2\sigma - sin^2\sigma \equiv \sin^4\sigma \sec^2\sigma$
14. Find values of α in the interval $0^0 \leq \alpha \leq 180^0$ for which $\tan 2\alpha - tan\ \alpha = 2$.

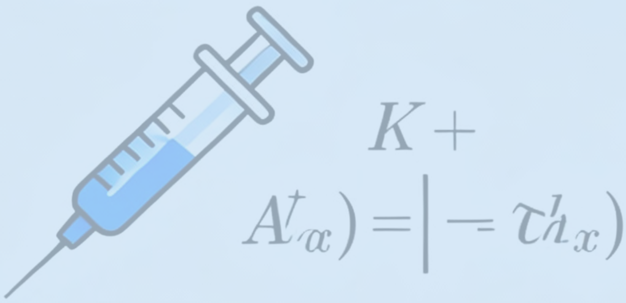
15. In a triangle MNP, it is known that $< NMP = 73^0$, $< MNP = 49^0$ and $|NP| = 12.2m$. Find the value(s) of:
    a. $< NPM$
    b. $|MN|$
    c. $|MP|$
16. Two buses (A and B) set out from a bus terminal at the same time. Bus A moves North while bus B moves on a bearing of 057^0. When bus B had travelled 10km, the two buses were 15km apart. How far was bus A from the bus terminal?
17. Find the value of y in the equation $2sin(y) - 1 = 0$.
18. Determine the value of v given that $sin(v) - 2sin(v)cos(v) = 0$.
19. A vertical tower JK of height 40m is observed from two points G and H in the same horizontal plane as K, the foot of the tower. The points K, G and H lie a straight-line KG = GH. Given the angle of elevation of J from H is 60^0, calculate:
    a. The distance of G from the foot of the tower.
    b. The angle of elevation of J from G.

SECTION

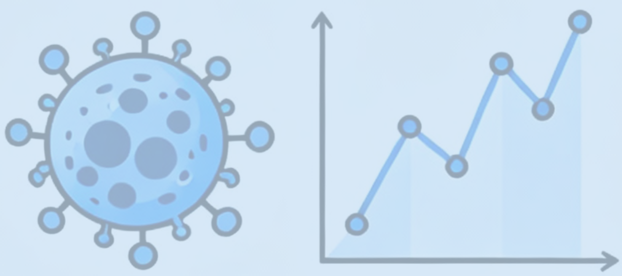
10 DIFFERENTIATION

Medicine Applications

- Drug Dosage Calculation



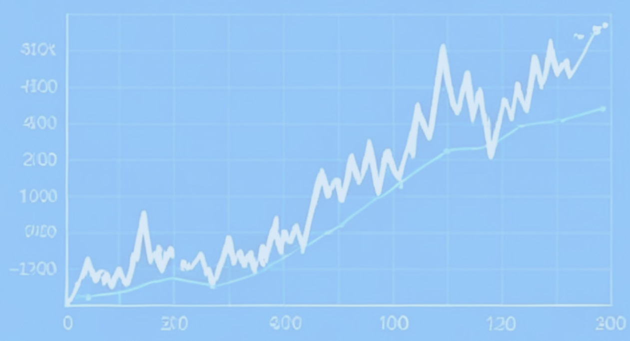
- Tumor Growth Modeling



- differential equations
- integral calculus

Finance Applications

- Options Pricing



Black-Scholes)

- Portfolio Optimization

- differential equations
- integral calculus

CALCULUS

Principles of Calculus

INTRODUCTION

In the year one, we learnt about the concept of limits and differentiating functions by first principles and by using the power rule. In this section, we will learn about the power, product, quotient and chain rules to find derivatives of functions, which are crucial for solving real-world problems involving rates of change. We will also cover Implicit differentiation which allows us to handle equations with interrelated variables and differentiating transcendental functions, such as exponential, logarithmic and trigonometric functions, which broadens your mathematical toolkit. These topics collectively build a solid foundation for advanced studies and diverse career applications.

KEY IDEAS

- **Chain Rule**: If $f(g(x))$, let $u = g(x)$, so $y = f(u)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- **Derivative of e^x** is e^x and if $g(x) = e^{f(x)}$ then $\frac{d}{dx}(g(x)) = f'(x) \times e^{f(x)}$
- **Derivative of $\cos x$** is $-\sin x$
- **Derivative of $\sin x$** is $\cos x$
- **Derivative of the natural logarithm**, $ln(x)$, is $\frac{1}{x}$ and if $f(x) = \log_a(g(x))$ then $\frac{d}{dx}fx = \frac{1}{\ln a} \times \frac{1}{gx} \times g'(x)$
- **Derivative** represents the rate of change of a function with respect to its input.
- **Differentiability** is a function which is differentiable if its derivative exists at a point.
- **Limit**: the derivative of a function $f(x)$ is defined as the limit of the difference quotient.
- **Power Rule**: If $y = ax^n$, where a is a constant, then $\frac{dy}{dx} = nax^{n-1}$
- **Product Rule**: If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

- **Quotient Rule**: if $y = \frac{u}{v}$ where u and v are function of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

IDENTIFYING DIFFERENTIATION RULES

Differentiation is a process to find the derivative or gradient function. There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use the first principles. The derivative of y with respect to x is usually written as $\frac{dy}{dx}$. This is called the *Leibniz notation*. The derivative of $f(x)$ with respect to x is usually written as $f'(x)$ or $\frac{dy}{dx}[f(x)]$.

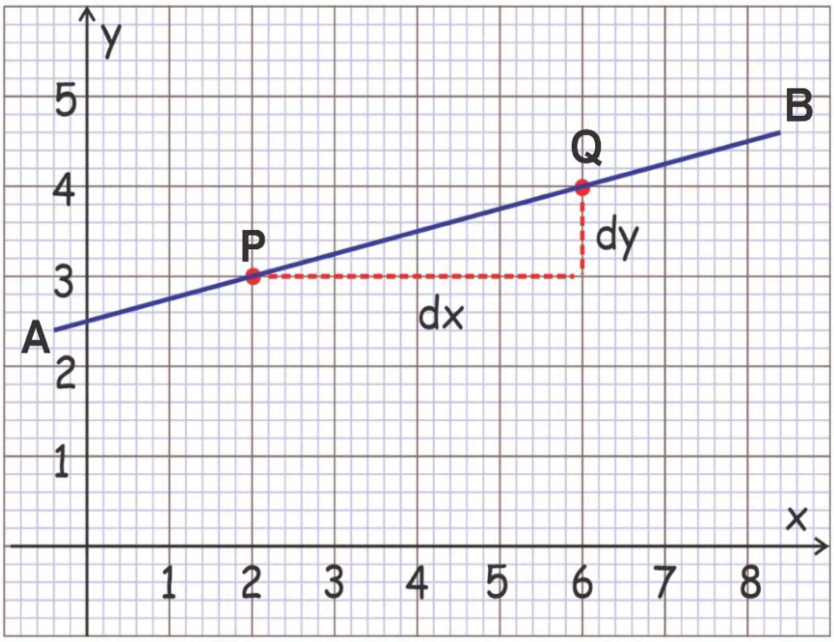


Figure 10.1: Graphical illustration of $\frac{\Delta y}{\Delta x}$

The gradient (m) of the sloping straight line shown in the diagram above is defined as:

$$m = \frac{\textit{the vertical distance the line rises or falls between two points P and Q}}{\textit{the horizontal distance between P and Q}}$$

Where P is the point to the left of point Q on the straight-line AB which slopes upwards from left to right. The changes in the x and y values of the point P and Q are denoted by dx and dy respectively. The gradient of the line is $\frac{dy}{dx}$.

We could have chosen any pair of points on the straight line for P and Q and by similar triangles; this ratio would have been worked out to the same value. *The gradient of a straight line is constant throughout its' length.* Its value is denoted by the symbol m. Therefore $m = \frac{dy}{dx}$.

For example, P is the point (2, 3) and Q is the point (6, 4), then P is to the left and below the point Q.

dy = the change in the $y-$values $= 4 - 3 = 1$

dx = the change in the $x-$values $= 6 - 2 = 4$

$$m = \frac{dy}{dx} = \frac{1}{4} = 0.25$$

Note: the sloping line rises vertically from left to right by 0.25 unit for every 1 unit horizontally.

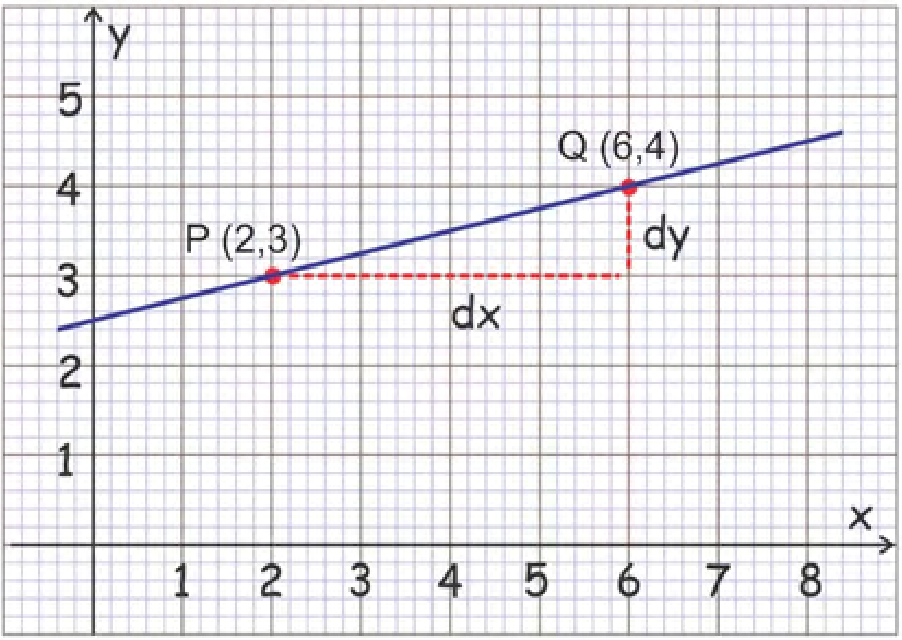


Figure 10.2: Graphical illustration of gradient of line PQ

If for another line M is the point (− 1, 3) and N is the point (0, −4), then M is to the left and above the point M.

dy = the change in the $y-$values $= 3 - -4 = 7$

dx = the change in the $x-$values $= -1 - 0 = -1$

$$m = \frac{dy}{dx} = \frac{7}{-1} = -7$$

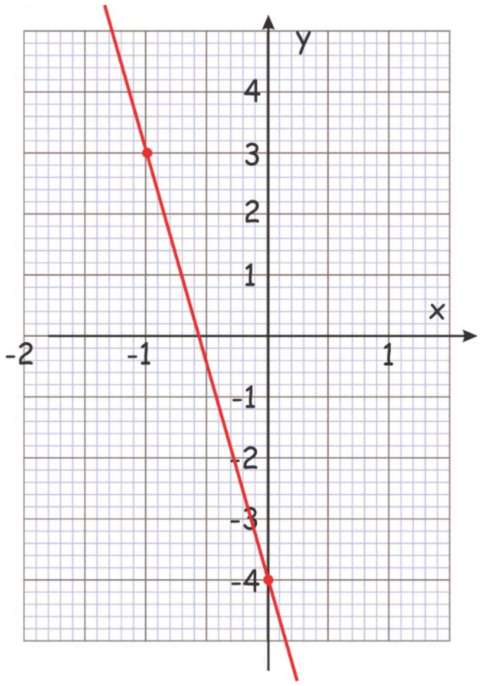


Figure 10.3: Graphical illustration of gradient of line MN

The lines going up to the right have a positive gradient, lines going down to the right have a negative gradient.

Rules of differentiation

Certain rules have been made to simplify and guide the process of differentiation. Let us work through these rules together.

Differentiation of a constant function

If $y = c$ where c is constant, then $\frac{dy}{dx} = 0$. Remember if $y = c$ then this is a horizontal line, and, like all horizontal lines, the gradient is zero.

For example, if $y = 5$, $\frac{dy}{dx} = 0$

Therefore, *the derivative of any constant is zero*.

Differentiation of *ax* where *a* is a constant

If $y = ax$ where a is a constant, then $\frac{dy}{dx} = a$.

For example, if $y = 5x$, then $\frac{dy}{dx} = 5$

Also, if $y = -2x$, then $\frac{dy}{dx} = -2$

Differentiation of x^n

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, where n is any real number. Therefore, the general case for differentiating x^n is to multiple the term by the power (exponent) and decrease the power by one.

For example, if $y = x^4$, then $\frac{dy}{dx} = 4x^3$.

Also, if $y = x^6$, then $\frac{dy}{dx} = 6x^5$.

Differentiation of ax^n

If $y = ax^n$, where a is a constant, then $\frac{dy}{dx} = nax^{n-1}$.

For example, if $y = 6x^3$, then $\frac{dy}{dx} = 18x^2$.

Also, if $y = 4x^{-3}$, then $\frac{dy}{dx} = -12x^{-4}$.

This is known as the *power rule*.

Derivative of the sum and difference

The differential coefficient of sum and difference is the sum and difference of the differential coefficients of the separate terms. Thus differentiate the sum and difference term by term.

Hence, $\frac{d}{dx}[f(x) + g(x) - h(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] - \frac{d}{dx[h(x)]}$.

Let us go through some examples to practise this rule.

Example 10.1

Find the derivative function with respect to x.

a. $5x^2 + 3x$

b. $2x^3 - 6x^2 + 4x - 2$

c. $6x^{\frac{-1}{3}} - 2x$

d. $3x^2 + 7 - \frac{4}{x}$

Solution

a. $\frac{dy}{dx} = \frac{d}{dx}(5x^2) + \frac{d}{dx}(3x)$

$= 10x + 3$

b. $\frac{dy}{dx} = \frac{d}{dx}(2x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(2)$
$= 6x^2 - 12x + 4$

c. $\frac{dy}{dx} = \frac{d}{dx}(6x^{\frac{-1}{3}}) - \frac{d}{dx}(2x)$
$= 6\left(-\frac{1}{3}\right)x^{\frac{-4}{3}} - 2$
$= -2x^{\frac{-4}{3}} - 2$

d. $\frac{dy}{dx} = \frac{d}{dx}(3x^2)) + \frac{d}{dx}(7) - \frac{d}{dx}(4x^{-1})$
$= 6x + 0 - (-4)x^{-2}$
$= 6x + \frac{4}{x^2}$

Example 10.2

Find $\frac{dy}{dx}$ if:

a. $y = \frac{4}{x^3}$

b. $y = \frac{1}{\sqrt{x}}$

c. $y = \frac{1}{x^5}$

d. $y = 3\sqrt{x}$

Solution

a. $y = \frac{4}{x^3} = 4x^{-3}$
$\frac{dy}{dx} = -12x^{-4}$

b. $y = \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$
$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$

c. $y = \frac{1}{x^5} = x^{-5}$
$\frac{dy}{dx} = -5x^{-6} = -\frac{5}{x^6}$

d. $y = 3\sqrt{x} = 3x^{\frac{1}{2}}$
$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$

Note: If the expression is such that it cannot be easily multiplied out or divided into separate terms the *product* or *quotient* rules must be used.

Derivative of linear combination of functions

The derivative of a linear combination of functions can be found using the linearity property of differentiation. This property states that the derivative of a linear combination of functions is equal to the linear combination of the derivatives of the derivatives of those functions

Definition:

If you have two differentiable functions $f(x)$ and $g(x)$ and constants a and b the linear combination of these functions is given by:

$h(x) = af(x) + bg(x)$

Derivative:

The derivative of $h(x)$ is:

$h^{I}(x) = \frac{d}{dx}[af(x) + bg(x)]$

Using the linearity property, we can separate the terms:

$h^{Ix} = a\,\frac{d}{dx}[f(x)] + b\,\frac{d}{dx}[g(x)]$

Simplify to get the result:

$a \times f'(x) + b \times g'(x)$

Example 10.3

Find the derivative of $f(x) = 4x^3 + 7x^2$

Solution

$h(x) = 4x^3 + 7x^2$

$f(x) = x^3$ and $g(x) = x^2$

Constants $a = 4$ and $b = 7$

Derivative of Individual functions:

$f^{I}(x) = \frac{d}{dx}(x^3) = 3x^2$

$g^{I}(x) = \frac{d}{dx}(x^2) = 2x$

Applying the linearity property:

$h'(x) = af'(x) + bg'(x)$

$h'(x) = 4 \times 3x^2 + 7 \times 2x$

$h'(x) = 12x^2 + 14x$

Example 10.4

Find the derivative of $f(x) = 5x^4 - 3x^3 + 2x - 1$

Solution

$f(x) = 5x^4 - 3x^3 + 2x - 1$

$u(x) = x^4$, $v(x) = x^3$, $w(x) = x$ and $c = -1$

Constants: $a = 5$, $b = -3$, $d = 2$

Derivative of Individual functions:

$u'(x) = \frac{d}{dx}(x^4) = 4x^3$

$v'(x) = \frac{d}{dx}(x^3) = 3x^2$

$w'(x) = \frac{d}{dx}(x) = 1$

The derivative of the constant is 0

Applying the linearity property:

$f'(x) = au'(x) + bv'(x) + dw'(x)$

$f'(x) = 5 \times 4x^3 - 3 \times 3x^2 + 2 \times 1$

$f'(x) = 20x^3 - 9x^2 + 2$

Example 10.5

Find the derivative of $f(x) = 7x^4 - 5x^3 + 3x - 4$

Solution

$f(x) = 7x^4 - 5x^3 + 3x - 4$

$u(x) = x^4$, $v(x) = x^3$, $w(x) = x$ and $c = -4$ (constant term)

Constants: $a = 7$, $b = -5$, $d = 3$

Derivative of Individual functions:

$u^I(x) = \frac{d}{dx}(x^4) = 4x^3$

$v^I(x) = \frac{d}{dx}(x^3) = 3x^2$

$w^I(x) = \frac{d}{dx}(x) = 1$

The derivative of the constant is 0

Applying the linearity property:

$f^I(x) = a\,u^I(x) + b\,v^I(x) + d\,w^I(x)$

$f^I(x) = 7 \times 4x^3 - 5 \times 3x^2 + 3 \times 1$

$f^I(x) = 28x^3 - 15x^2 + 3$

Chain Rule (Composite function)

A function such as $y = \frac{1}{(3x-4)^2} = (3x-4)^{-2}$ is a problem for differentiation. It cannot be expressed as separate terms in a polynomial. To differentiate this we need to use the *chain rule*. To find $\frac{dy}{dx}$, we treat $(3x-4)^{-2}$ as composite function, built up in two stages from a core function $(3x - 4)$ which we will call u and then taking $y = u^{-2}$. We have $u = 3x - 4$ and $y = u^{-2}$.

If $y = g(u)$ where $u = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

This rule is extremely important and enables us to differentiate complicated functions.

Note: *du* cannot be 'cancelled' on the right-hand side as these are not fractions but derivatives.

Example 10.6

Find $\frac{dy}{dx}$ if $y = (2x-1)^3$

Solution

$y = (2x-1)^3$

Let $u = 2x - 1$ and $y = u^3$

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = 3u^2$

But $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx} = 3u^2 \times 2$

But $u = 2x - 1$, so we substitute this in:

$\frac{dy}{dx} = 3(2x-1)^2 \times 2 = 6(2x-1)^2$

Example 10.7

Find $\frac{dy}{dx}$ if $y = (x^2+3)^{-1}$

Solution

$y = (x^2+3)^{-1}$

Let $u = x^2 + 3$ and $y = u^{-1}$

$\frac{du}{dx} = 2x$ and $\frac{dy}{du} = -u^{-2}$

But $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx} = -u^{-2} \times 2x$, but $u = x^2 + 3$

$\frac{dy}{dx} = -(x^2+3)^{-2} \times 2x = -2x(x^2+3)^{-2} = -\frac{x}{(x^2+3)^2}$

Example 10.8

Find $\frac{dy}{dx}$ if $y = \sqrt{2x+5}$

Solution

$y = \sqrt{2x+5} = (2x+5)^{\frac{1}{2}}$

Let $u = 2x + 5$ and $y = u^{\frac{1}{2}}$

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}}$

But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{\frac{-1}{2}} \times 2 = \frac{1}{2}(2x+5)^{\frac{-1}{2}} \times 2 = (2x+5)^{\frac{-1}{2}} = \frac{1}{\sqrt{2x+5}}$

Product rule

$y = (3x-1)^3(x^2+5)$ is the product of two expressions, $(3x-1)^3$ and (x^2+5). To find $\frac{dy}{dx}$, we could expand the product and then differentiate each term separately, but this would be time consuming. Therefore, let us see how $\frac{dy}{dx}$ can be found without doing this.

Let $y = uv$ where u and v are functions of x. Then uv is the product of two functions. In the above, for example, u would be $(3x - 1)$ and v would be $(x^2 + 5)$.

Take an increment δx in x which will in turn produce increments δu in u and δv in v, finally producing a change δy in y. Then:

$$y + \delta y = (u + \delta u)(v + \delta v)$$

$$= uv + v\delta u + u\delta v + (\delta u)(\delta v), \text{ so}$$

$$\delta y = v\delta u + u\delta v + (\delta u)(\delta v), \text{ and}$$

$$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\,\delta v$$

Now, let $\delta x \to 0$, then $\delta u \to 0$ as a result.

$$\therefore \frac{\delta u}{\delta x} \to \frac{du}{dx}, \frac{\delta v}{\delta x} \to \frac{dv}{dx} \text{ and } \frac{\delta y}{\delta x} \to \frac{dy}{dx}$$

So, the limiting value $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$, (the third term $\to 0$ as $\frac{\delta u}{\delta x}$ is definite, but $\delta v \to 0$).

Hence for $\boldsymbol{y = uv}$, we have the product rule: $\boldsymbol{\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$

Example 10.9

Find the derivate of $y = (3x - 2)(x^2 + 3)$

Solution

Let $u = (3x - 2)$ and $v = (x^2 + 3)$

$$\frac{du}{dx} = 3 \text{ and } \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (x^2 + 3)(3) + (3x - 2)(2x) = 3x^2 + 9 + 6x^2 - 4x = 9x^2 - 4x + 9$$

Example 10.10

Differentiate $(x^2)(x + 1)^5$ with respect to x

Solution

$y = (x^2)(x+1)^5$

Let $u = (x^2)$ and $v = (x+1)^5$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 5\left(x+1\right)^4$, differentiated using the chain rule

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = x^2[5(x+1)^4] + (x+1)^5(2x)$

$\frac{dy}{dx} = 5x^2\left(x+1\right)^4 + \left(x+1\right)^5(2x) = 5x^2\left(x+1\right)^4 + 2x\left(x+1\right)^5$

We could then factorise this and simplify:

$\frac{dy}{dx} = x(x+1)^4\left[5x + 2(x+1)\right] = x(x+1)^4\left(7x+2\right)$

Example 10.11

Differentiate $(x+1)^3(2x-5)^2$ with respect to x

Solution

$y = (x+1)^3(2x-5)^2$

Let $u = (x+1)^3$ *and* $v = (2x-5)^2$

$\frac{du}{dx} = 3\left(x+1\right)^2$ and $\frac{dv}{dx} = 2\left(2x-5\right)\left(2\right)$, differentiated using the chain rule

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (2x-5)^2(3)(x+1)^2 + (x+1)^3(4)(2x-5)$

$\frac{dy}{dx} = 3\left(2x-5\right)^2\left(x+1\right)^2 + 4(x+1)^3\left(2x-5\right)$

We could then factorise this and simplify:

$\frac{dy}{dx} = (2x-5)(x+1)^2\left[3(2x-5) + 4(x+1)\right] = (2x-5)(x+1)^2\left(10x-11\right)$

Quotient Rule

Expression like $\frac{x^2+1}{2x-5}$, $\frac{\sqrt{x}}{1-5x}$, and $\frac{x^2}{(x-4)^3}$ are called quotients because they represent the division of one function by another.

Quotient functions have the form $Q(x) = \frac{u(x)}{v(x)}$

Notice that $u(x) = Q(x)v(x)$

$\therefore u'(x) = Q'(x)v(x) + Q(x)v'(x)$, using the Product Rule

$\therefore u'(x) - Q(x)v'(x) = Q'(x)v(x)$

$$Q'(x)v(x) = u'(x) - \frac{u(x)}{v(x)}v'(x)$$

$$Q'(x)v(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)}$$

$Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when this exists

If $Q(x) = \dfrac{u(x)}{v(x)}$ then $Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$

Alternatively, if $y = \frac{u}{v}$ where u and v are function of x, then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Example 10.12

Find $\dfrac{dy}{dx}$ if $y = \dfrac{1 + 3x}{x^{2+}1}$

Solution

$y = \dfrac{1 + 3x}{x^{2+}1}$ is a quotient with $u = 1 + 3x$ and $v = x^2 + 1$

$\dfrac{du}{dx} = 3$ and $\dfrac{dv}{dx} = 2x$

Using the quotient rule: $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{3(x^2 + 1) - (1 + 3x)2x}{(x^2 + 1)^2} = \frac{3x^2 + 3 - 2x - 6x^2}{(x^2 + 1)^2} = \frac{3 - 2x - 3x^2}{(x^2 + 1)^2}$$

Example 10.13

Find $\dfrac{dy}{dx}$ if $y = \frac{2x^2}{x - 2}$

Solution

$y = \dfrac{2x^2}{x - 2}$

$u = 2x^2$ and $v = x - 2$

$u' = 4x$ and $v' = 1$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{(4x)(x-2) - 2x^2(1)}{(x-2)^2} = \frac{4x^2 - 8x - 2x^2}{(x-2)^2} = \frac{2x^2 - 8x}{(x-2)^2}$$

Example 10.14

Find $\frac{dy}{dx}$ if $y = \frac{x^2}{\sqrt{x+1}}$

Solution

$u = x^2$ and $v = \sqrt{x+1}$

$u' = 2x$ and $v = \left(x+1\right)^{\frac{1}{2}} \Rightarrow v' = \frac{1}{2}\left(x+1\right)^{\frac{-1}{2}} = \frac{1}{2\sqrt{x+1}}$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{(\sqrt{x+1})(2x) - \frac{x^2}{2\sqrt{x+1}}}{(\sqrt{x}+1)^2} = \frac{(\sqrt{x+1})(2x) - \frac{x^2}{2\sqrt{x+1}}}{(x+1)}$$

$$= \frac{2(\sqrt{x+1}) \times \sqrt{x+1} \times 2x - x^2}{2\sqrt{x+1}\;(x+1)} = \frac{4x(x+1) - x\,2}{2(x+1)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{(3x+4)}{2(x+1)^{\frac{3}{2}}}$$

Reciprocal Rule

The reciprocal rule, which is derived from the quotient rule, is a tool for finding the derivative of a function that is a reciprocal to another function. If you have a function $f(x)$ and you want to find the derivative of its reciprocal $\frac{1}{f(x)}$,we use the formula $\left(\frac{1}{f(x)}\right)^I = \frac{f'(x)}{(f(x))^2}$

Derivation:

Let $g(x) = \frac{1}{f(x)}$ to find $g^I(x)$, we can rewrite it as $g(x) = (f(x))^{-I}$

$(f(x))^{-I} : g^I(x) = -1 \times (f(x))^2 \times f^I(x)$

$= -f\,I(x)\,\_\,(f(x))\,2$

Example 10.15

Find the derivative of $h(x) = \frac{1}{x^2}$

Solution

$h(x) = \frac{1}{x^2}$

Step 1: Identify $f(x)$

Here, $f(x) = x^2$

Step 2: Find $f^I(x)$

$$f^I(x) = \frac{d}{dx}(x^2)$$
$$= 2x$$

Step 3: Apply to the reciprocal rule using the formula

$$\frac{1}{f(x)} = -\frac{f^I(x)}{(f(x))^2}$$

$$h^I(x) = -\frac{x}{(x^2)^2}$$
$$= -\frac{2x}{x^4}$$
$$= -\frac{2}{x\ 3}$$

Example 10.16

Find the derivative of $g(x) = \frac{1}{2x^3 + 1}$

Solution

$g(x) = \frac{1}{2x^3 + 1}$

Step 1: Identify the function

$f(x) = 2x^3 + 1$

Step 2: Find the derivative of $f(x)$

$$f^I(x) = \frac{d}{dx}(2x^3 + 1)) = 6x^2$$

Step 3: Apply the reciprocal rule

$$\frac{1}{f(x)} = -\frac{f\,I(x)}{(f(x))^2}$$

Step 4: Substitute $f(x)$ and $f^I(x)$ into the formula

$f(x) = 2x^3 + 1$ and $f^I(x) = 6x^2$

$$g^I(x) = -\frac{f\,I(x)}{(f(x))^2} = -\frac{6x\ 2}{(2x^3 + 1)^2}$$

DIFFERENTIATING FUNCTIONS USING DIFFERENTIATION RULES

Let us go through more examples that use the differentiation rules we have learnt.

Example 10.17

Product Rule:

Differentiate with respect to x, $y = (x^2 + 1)(x^3 + 3)$

Solution:

If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$u = x^2 + 1$ and $v = x^3 + 3$

$\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 3x^2$

$\frac{dy}{dx} = 2x(x^3 + 3) + 3x^2\left(x^2 + 1\right)$

Example 10.18

Quotient rule

$h(x) = \frac{x^2}{x + 1}$

Solution

$h^I(x) = u\,I\,v - u\,v\,I\_v\,2$

$u(x) = x^2$ and $v(x) = x + 1$

$u^I(x) = 2x$ and $v^I(x) = 1$

Applying the quotient rule:

$h^I(x) = \frac{2x(x + 1) - x^2}{(x + 1)^2}$

$h^I(x) = \frac{2x^2 + 2x - x^2}{(x + 1)^2}$

$h^I(x) = \frac{x^2 + 2x}{(x + 1)^2}$

Example 10.19

Chain rule:

Differentiate with respect to$x$, $y = (x^2 + 3)^4$

Solution

Chain rule: $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx}$

Let $u = x^2 + 3$, $\frac{du}{dx} = 2x$, $y = u^4$, $\frac{y}{du} = 4u^3$

$\frac{dy}{dx} = 4u^3 \times 2x = 8x(x^2 + 3)^3$

DIFFERENTIATING IMPLICIT FUNCTIONS

The function which can be easily written as $y = f(x)$ with the y variable on one side and the function of x on the other side, is called an explicit function. Some functions may not be given directly or explicitly. Examples include $x^2y - 3x = 2$ and $y^4 = 5x^2y + 2xy$ and such functions are called implicit functions, where the x and the y variable cannot be written in the form $y = f(x)$. An implicit function has more than one solution for the given function.

Note the following *where w.r.t x means "with respect to x".*

1. When you differentiate y $w.r.t$ x we obtain $\frac{dy}{dx}$
2. When you differentiate y^2 $w.r.t$ x we obtain $2y\frac{dy}{dx}$ *or* $\frac{dy^2}{dx} = 2y\frac{dy}{dx}$
3. When you differentiate y^3 $w.r.t$ x we obtain $\frac{dy^3}{dx} = 3y^2\frac{dy}{dx}$
4. When you differentiate y^4 $w.r.t$ x we obtain $\frac{dy^4}{dx} = 4y^3\frac{dy}{dx}$

Also:

1. When you differentiate x^2 $w.r.t$ x we obtain $\frac{dx^2}{dx} = 2x$
2. When you differentiate x^3 $w.r.t$ x we obtain $\frac{dx^3}{dx} = 3x^2$

Example 10.20

Given that $y^3 - 2y^2 + x^2 = 4$, find $\frac{dy}{dx}$.

Solution

$y^3 - 2y^2 + x^2 = 4$

$3y^2 \frac{dy}{dx} - 4y \frac{dy}{dx} + 2x = 0$

$\frac{dy}{dx}(3y^2 - 4y) = -2x$

$\frac{dy}{dx} = \frac{2x}{3y^2 - 4y}$

Example 10.21

Find $\frac{dy}{dx}$, if $6y - 3x^4 = 2y^6$

Solution

$6y - 3x^4 = 2y^6$

$6\frac{dy}{dx} - 12x^3 = 12y^5\frac{dy}{dx}$, divide through by 6

$\frac{dy}{dx} - 2x^3 = 2y^5\frac{dy}{dx}$, collect your $\frac{dy}{dx}$ terms on one side and the remaining terms on the other

$\frac{dy}{dx} - 2y^5\frac{dy}{dx} = 2x^3$

$\frac{dy}{dx}(1 - 2y^5) = 2x^3$

$\frac{dy}{dx} = \frac{x^3}{1 - 2y^5}$

Example 10.22

Find $\frac{dy}{dx}$ if $x^2 + y^2 - 2x + y = 6$

Solution

$x^2 + y^2 - 2x + y = 6$

$2x + 2y \frac{dy}{dx} - 2 + (1) \frac{dy}{dx} = 0$

$2y \frac{dy}{dx} + \frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx}(2y + 1) = 2 - 2x$

$\frac{dy}{dx} = \frac{-2x}{2y + 1}$

When differentiating the product of two functions, we apply the product rule.

- If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

We use this when we do implicit differentiation.

For example, the derivative of x^2y^3 with respect to x,

$u = x^2, \frac{du}{dx} = 2x$ and $v = y^3, \frac{dv}{dx} = 3y^2\frac{dy}{dx}$

$\therefore \frac{d(x^2y^3)}{dx} = 2xy^3 + 3x^2y^2\frac{dy}{dx}$

Example 10.23

Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$

Solution

Differentiate (w. r. t. x) term by term y^3 as a composite function and $3xy$ as a product.

Then $3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$

$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$

$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$

$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$

Example 10.24

Find $\frac{dy}{dx}$ if , $x^2y - 5x = 3$

Solution

$x^2y - 5x = 3$

We differentiate with respect to x (w.r.t x) term by term throughout.

To differentiate x^2y consider it as uv and use the product rule.

$x^2\frac{dy}{dx} + y \times 2x$

$v\frac{du}{dx} + u\frac{dv}{dx}$, so we have by differentiating $x^2y - 5x = 3$

$x^2\frac{dy}{dx} + 2xy - 5 = 0$

Now find $\frac{dy}{dx}$ by making it the subject of the formula.

$x^2\frac{dy}{dx} = 5 - 2xy$

$\frac{dy}{dx} = \frac{5 - 2xy}{x^2}$

Example 10.25

Find $\frac{dy}{dx}$ if $4y^2x - 5x^2y^2 + 4y = 0$

Solution

$4y^2x - 5x^2y^2 + 4y = 0$

Remember $4y^2x$ and $-5x^2y^2$ are products, so we must use the product rule:

$4y^2 + 4x \times 2y\frac{dy}{dx} - 10xy^2 - 10x^2y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$4y^2 + 8xy\frac{dy}{dx} - 10xy^2 - 10x^2y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

Isolate $\frac{dy}{dx}$ terms and simplify:

$(8xy - 10x^2y + 4)\frac{dy}{dx} = 10xy^2 - 4y^2$

$(4xy - 5x^2y + 2)\frac{dy}{dx} = y^2(5x - 2)$

$\frac{dy}{dx} = \frac{y^2(5x - 2)}{4xy - 5x^2y + 2}$

Everyday problems involving derivatives

Example 10.26

Two cars start from the same point. Car X travels north at 40mph and car Y travels east at 30 mph. After 2 hours, how fast is the distance between the two cars changing.

Solution

x : Distance Car Y has travelled east

y : Distance Car X has travelled north

D : Distance between the two cars

Express distance after t hours

$x = 30t$ and $y = 40t$

For the relationship between distances use Pythagoras' theorem;

$D^2 = x^2 + y^2$

Differentiate implicitly with respect to time (t)

$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ (Simplify by dividing through by 2)

$D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

After 2 hours:

$x = 30 \times 2 = 60$ miles

$y = 40 \times 2 = 80$ miles

$\frac{dx}{dt} = 30mph$ and $\frac{dy}{dt} = 40\ mph$

Solving for D

$D^2 = x^2 + y^2$

$D^2 = (60)^2 + (80)^2$

$D^2 = 3600 + 6400$

$D^2 = 10000$

$D = 100$

Substitute into the related rate equation

$$D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$100\frac{dD}{dt} = 60 \times 30 + 80 \times 40$$

$$100\frac{dD}{dt} = 1800 + 3200$$

$$100\frac{dD}{dt} = 5000$$

$$\frac{dD}{dt} = \frac{5000}{100} = 50mph.$$

Therefore, the rate of change of the distance with respect to time after 2 hours is 50 mph.

Example 10.27

A cylindrical tank has a radius of 2 metres and is filled with water at a rate of 3 cubic metres per minute. How fast is the height the water rising when the water is 4 metres deep?

Solution

1. Cylinder Dimensions: Radius(r) = 2 metres
2. Volume of Rate (water is being filled) $= \frac{dV}{dt} = 3m^3/min.$
3. Height of water (we need to find the rate of change with water height ($\frac{dh}{dt}$) when the height (h) is 4 metres.

Steps:

1. The volume of a cylinder is given by $V = \pi r^2 h$
2. Differentiate with respect to time $(t) \Rightarrow \frac{dV}{dt} = \pi r^2 \times \frac{dh}{dt}$
3. Substitute known values: $\frac{dV}{dt} = 3m^3/min$ and $r = 2m$

   $\therefore 3 = \pi \times (2)^2 \times \frac{dh}{dt}$
4. Solve for $\frac{dh}{dt} \Rightarrow 3 = 4\,\pi \times \frac{dh}{dt}$

   $\frac{dh}{dt} = \frac{3}{\pi}$

Using $\pi \approx 3.14$

$$\frac{\text{dh}}{\text{dt}} = \frac{3}{4 \times 3.14} = \frac{3}{12.56} = 0.239m/min.$$

Therefore, the rate at which the water is rising when 4m deep is 0.24m per minute (to 2sf).

Example 10.28

A conical tank has a height of 6m and a base radius of 3m. Water is pumped into the tank at a rate of 2 cubic metres per minute.

How fast is the water level rising when the water is 2m deep?

Solution

1. Conical tank dimension:

   Height (H) = $6m$ and base radius (R) = $3m$
2. Volume Rate:

   Water is being pumped in at $\frac{dV}{dt} = 2m^3/min$
3. Height of water:

   We need to find the rate of change of the water height, $\left(\frac{dh}{dt}\right)$, when the height ($h$) is 2m

Steps:

1. Volume of the water:

   The volume of cone is given by $V = \frac{1}{3}\pi r^2 h$

   Where r and h are the radius and height of the water. Since the water forms a smaller, similar cone within the tank, we have:

   $$\frac{r}{R} = \frac{h}{H} \Rightarrow r = \frac{R}{H}h = \frac{3}{6}h = \frac{h}{2}$$

   Substitute $r = \frac{h}{2}$ into the volume formula:

   $$V = \frac{1}{3} \times \pi \times (\frac{h}{2})^2 \times h$$

   $$V = \frac{1}{3} \times \pi \times \frac{h}{4^3} = \frac{\pi h^3}{12}$$
2. Differentiate with respect to time (t):

   $$\frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \times \frac{\text{dh}}{\text{dt}} = \frac{\pi h^2}{4} \times \frac{\text{dh}}{\text{dt}}$$
3. Substitute known values:

   $\frac{dV}{dt} = 2m^3/min$ and $h = 2m$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \times \frac{dh}{dt}$$

$$2 = \frac{\pi(2)^2}{4} \times \frac{dh}{dt}$$

Simplify:

$$2 = \frac{\pi \times 4}{4} \times \frac{dh}{dt}$$

$$2 = \pi \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2}{\pi}$$

Using $\pi \approx 3.14$

$$\frac{dh}{dt} = \frac{2}{3.14} = 0.636\ m/min$$

Therefore, the water level is rising at 0.64m/min (to 2sf) when h = 2m.

DIFFERENTIATING TRANSCENDENTAL FUNCTIONS

Transcendental functions are those that are not algebraic, meaning they cannot be expressed as a finite combination of the basic algebraic operations (addition, subtraction, division, multiplication, taking square roots etc). They include trigonometric, exponential and logarithmic functions.

Common Transcendental Functions:

1. Exponential Functions: $f(x) = e^x$

   These involve the constant e (approximately 2.71828). This is the base of natural logarithms.
2. Logarithmic Functions: $f(x) = ln(x)$

   The natural logarithm function is the inverse of the exponential function.
3. Trigonometric Functions:

   a. Sine: $f(x) = sin(x)$

   b. Cosine: $f(x) = \cos(x)$

   c. Tangent: $f(x) = \tan(x)$

Derivative of Exponential and Logarithmic Functions

The process of differentiating transcendental functions follows specific rules:

1. Exponential Functions:

   $\frac{d}{dx}e^x = e^x$ (The derivative of e^x is itself)

   If $g(x) = e^{f(x)}$ then $\frac{d}{dx}(g(x)) = f'(x) \times e^{f(x)}$

2. Logarithmic Functions:

   $\frac{d}{dx}ln(x) = \frac{1}{x}$ (The derivative of the natural logarithm $ln(x)$ *is* $\frac{1}{x}$.

   If $f(x) = \log_a(g(x))$ then $\frac{d}{dx}fx = \frac{1}{\ln(a)} \times \frac{1}{g(x)} \times g'(x)$

Example 10.29

Find the derivative of the following

1. $f(x) = e^{2x}$
2. $y = e^{-3x}$
3. $g(t) = e^{2t^2+t}$
4. Differentiate the function $y = xe^{-2x}$

Solution

1. $f'(x) = \frac{d(2x)}{dx} \times e^{2x} = 2e^{2x}$
2. $\frac{dy}{dx} = \frac{d(-3x)}{dx} \times e^{-3x} = -3e^{-3x}$
3. $g'(t) = \frac{d(2t^2+t)}{dt} \times e^{2t^2+t} = (4t+1)e^{2t^2+t}$
4. $y = xe^{-2x}$

   Using the product rule followed by the chain rule:

$$\frac{dy}{dx} = x\frac{d(e^{-2x})}{dx} + e^{-2x}\frac{d(x)}{dx}$$

$$= xe^{-2x}\frac{d(-2x)}{dx} + e^{-2x}$$

$$= -2xe^{-2x} + e^{-2x}$$

$$= e^{-2x}\left(1-2x\right)$$

Example 10.30

Find the derivative of $y = \sqrt[3]{lnx}$

Solution

$y = \left(\ln(x)\right)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}(\ln(x))^{-\frac{2}{3}} \times \frac{1}{x} = \frac{1}{3x} \times \frac{1}{\sqrt[3]{(\ln(x))^2}} = \frac{1}{3x\sqrt[3]{(\ln(x))^2}}$

Derivative of trigonometric functions

Trigonometric Functions:

- Sine: $\frac{d}{dx}\sin(x) = \cos\left(x\right)$
- Cosine: $\frac{d}{dx}cosx = -\sin(x)$
- Tangent: $\frac{d}{dx}tan(x) = sec^2\left(x\right)$

Differentiation of *sinx* and *cosx* from first principles

Let $f(x) = sinx$

$f(x + h) = \sin(x + h)$ (where x is in radians)

$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - sinx}{h}$

Using the trigonometry identity:

$sinA - sinB = 2cos\frac{1}{2}\left(A + B\right)sin\frac{1}{2}\left(A - B\right)$

When $A = x + h$ and $B = x$, we have

$sin(x + h) - sinx = 2cos\frac{1}{2}(x + h + x)sin\frac{1}{2}(x + h - x) = 2\cos\left(x + \frac{1}{2}h\right)\sin\left(\frac{1}{2}h\right)$

$\frac{sin(x+h) - Sinx}{h} = \frac{2cos(x + \frac{1}{2}h)Sin(\frac{1}{2}h)}{h} = \cos\left(x + \frac{1}{2}h\right) \times \frac{2\sin(\frac{1}{2}h)}{h}$

$\lim_{h\to0}\frac{\sin x + h - sinx}{h} = \lim_{h\to0}[\cos(x + \frac{1}{2}h) \times 2\sin\frac{(\frac{1}{2}x)}{2(\frac{1}{2}h)}]$

Note: $\lim_{h\to0}\left(x + \frac{1}{2}h\right) = cosx$ and $\lim_{h\to0}\sin\frac{\left(\frac{1}{2}x\right)}{\frac{1}{2}x} = 1$

This shows that if $y = sinx$, then $\frac{dy}{dx} = cosx$ or $\frac{d(sinx)}{dx} = cosx$

Let $f(x) = cosx$

$f(x + h) = \cos(x + h)$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - cosx}{h}$$

Using the trigonometry identity:

$cosA - cosB = -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$.

When $A = x + h$ and $B = x$, we have;

$$cos(x+h) - cosx = -2\sin\frac{1}{2}(x + h + x)\sin\frac{1}{2}(x + h - x) = -2\sin\left(x + \frac{1}{2}h\right)\sin\left(\frac{1}{2}h\right)$$

$$\frac{cos(x+h) - cosx}{h} = \frac{2\sin\left(x + \frac{1}{2}h\right)\sin\left(\frac{1}{2}h\right)}{h} = -\sin(x + \frac{1}{2}h) \times \frac{2\sin(\frac{1}{2}h)}{h}$$

$$\lim_{h\to 0}\frac{\cos(x+h) - cosx}{h} = \lim_{h\to 0}\left[-\sin\left(x + \frac{1}{2}h\right) \times \frac{2\sin(1\_2\ h)}{2\left(\frac{1}{2}h\right)}\right]$$

$$\lim_{h\to 0} -\sin\left(x + \frac{1}{2}h\right) \times \lim_{h\to 0}\frac{\sin\left(\frac{1}{2}h\right)}{\frac{1}{2}h}$$

Note: $\lim_{h\to 0} sin\left(x + \frac{1}{2}h\right) = sinx$ and $\lim_{h\to 0}\frac{\sin(1\_2\ x)}{\frac{1}{2}x} = 1$

$-sinx \times 1 = -sinx$

That shows that if $y = cosx$, then $\frac{dy}{dx} = -sinx$ or $\frac{d(cosx)}{dx} = -sinx$.

Example 10.31

Differentiate with respect to x, $sin3x$.

Solution

Let $y = sin3x$ and let $u = 3x$

Then $y = sinu$

$\frac{dy}{du} = cosu$ and $\frac{du}{dx} = 3$

Using the chain rule: $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = cosu \times 3 = 3cosu$, but $u = 3x$

$\frac{dy}{dx} = 3cos3x$

Example 10.32

If $y = sin(2x - 4)$, find $\frac{dy}{dx}$.

Solution

$y = sin(2x - 4)$, let $u = 2x - 4$, then $y = sinu$

$\frac{dy}{du} = cosu$ and $\frac{du}{dx} = 2$

Using the chain rule: $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = cosu \times 2 = 2cosu$, but $u = 2x - 4$

$\frac{dy}{dx} = 2\cos\left(2x - 4\right)$

Example 10.33

If $y = cos\left(\frac{\pi}{4} - 2x\right)$, find $\frac{dy}{dx}$

Solution

$y = cos\left(\frac{\pi}{4} - 2x\right)$, let $u = \frac{\pi}{4} - 2x$, then $y = cosu$

$\frac{dy}{du} = -sinu$ and $\frac{du}{dx} = -2$

Using the chain rule: $\frac{dy}{dx} = \frac{y}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -sinu \times -2 = 2sinu$, but $u = \frac{\pi}{4} - 2x$

$\frac{dy}{dx} = 2\sin\left(\frac{\pi}{4} - 2x\right)$

Example 10.34

If $y = sin\,x^0$, find $\frac{dy}{dx}$

Solution

We can only differentiate if the angle is in radians, so we must convert the x degrees to radians, $x^0 = \frac{\pi}{80}x$

$y = sin\, x^0$

$y = sin\, \frac{\pi}{80}x$

$\frac{dy}{dx} = cos\, \frac{\pi}{80}x \times \frac{\pi}{80}$

$\frac{\pi}{180} cos\, x^0$

Note that the result is NOT $Cos\, x^0$. It must be stressed that the formula for differentiating trigonometric functions is only true if the angles are expressed in radians.

Remember that angles in radian measure are written simply as $x, \theta, 3x$,etc. Angles in degree measure must have the degree symbol $^{(\circ)}$.

Example 10.35

Given that $y = \sin(2x)$, find $\frac{dy}{dx}$

Solution

$y = \sin(2x)$

Let $u = 2x \Rightarrow \frac{du}{dx} = 2$

$y = \sin u \Rightarrow d\frac{y}{dx} = \cos u$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \cos u \times 2 = 2\cos u$, but $u = 2x$

$\therefore \frac{dy}{dx} = 2cos2x$

Example 10.36

Find the derivative of the function $\frac{Sinx}{x}$.

Solution

$y = \frac{sinx}{x}$ is a quotient:

Let $u = sinx \Rightarrow \frac{du}{dx} = cosx$

$v = x \Rightarrow \frac{dv}{dx} = 1$

Using the quotient rule, $\frac{dy}{dx} = \frac{v\,du \_\, dx - u\frac{dv}{dx}}{v^2}$:

$$\frac{dy}{dx} = \frac{xcosx - sinx}{(x)^2}$$

Example 10.37

Given that $y = \frac{cosx}{\sqrt{x}}$, find $\frac{dy}{dx}$

Solution

$y = \frac{cosx}{\sqrt{x}}$ is a quotient.

Let $u = cosx \Rightarrow \frac{du}{dx} = -sinx$

$V = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow \frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\,du \_\, dx - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}(-sinx) - cosx \times (\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x})^2} = \frac{-\sqrt{x}\,sinx - \frac{cosx}{2\sqrt{x}}}{x} = \frac{\frac{-2\sqrt{x} \times \sqrt{x}\,sinx - cosx}{2\sqrt{x}}}{x}$$

$$\frac{dy}{dx} = \frac{-2\,x\,sinx - cosx}{2x\sqrt{x}}$$

Example 10.38

Given that $y = x^2 sin\left(x\right)$, find $\frac{dy}{dx}$

Solution

$y = x^2 sin(x)$ is a product.

$u = x^2, \frac{du}{dx} = 2x$

$v = sinx, \frac{dv}{dx} = cosx$

Using the product rule, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$:

$\frac{dy}{dx} = sinx \times 2x + x^2 \times cosx = 2xsinx + x^2 cosx$

EXTENDED READING

- Aki-Ola Series, Elective (Further) Mathematics for Senior High Schools in West Africa Millennium Edition 3.
- Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952.
- Godman T. A. & Ogum, G. O. Additional Mathematics for West Africa.
- Hesse, C. A. (2005). Effective Elective Mathematics for Senior High School. Akrong Publications.
- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. Find $\frac{dy}{dx}$ if $y = x^2 + 5\sqrt{x}$
2. Find $\frac{dy}{dx}$ if $y = x^4 - 2x + \frac{1}{x^2}$
3. Find $\frac{dy}{dx}$ if $y = 3\sqrt{x}$
4. Find $\frac{dy}{dx}$ if $y = \frac{3}{\sqrt{x}}$
5. Differentiate with respect to x, $(2x - 1)^2$
6. Differentiate with respect to x, $(2x + 4)(3x - 1)$
7. Find $\frac{dy}{dx}$ if $\frac{3x^4 + 2x^2 - 1}{2x^2}$
8. Differentiate with respect to x $\frac{1}{\sqrt{4x - 7}}$
9. Find $\frac{dy}{dx}$ if $y = x^2(2x - 5)^4$
10. Differentiate $y = (3x - 1)^3(x^2 + 5)$
11. Find $\frac{dy}{dx}$ if $y = \left(x^2+1\right)^{\frac{1}{2}}$
12. Find $\frac{dy}{dx}$ if $y = \frac{4}{\sqrt{1 - 2x}}$
13. If $y^4 + x^4 - 2x^2y^2 = 9$, find $\frac{dy}{dx}$
14. Differentiate with respect to x, $\tan(4x - 1)$
15. Find $\frac{dy}{dx}$,if $x\cos y + y\cos x = 2$
16. If $y \frac{5}{(1 - x^2)^3}$, show that $(1 - x^2)\frac{dy}{dx} = 6xy$
17. Differentiate with respect to x, $y = x^2\cos x$
18. Differentiate with respect to x, $y = x\sin x$ z

SECTION

11 INTEGRATION

CALCULUS

Principles of Calculus

INTRODUCTION

Welcome to this fascinating section on integration! In this section, we will uncover the basics of integration, a fundamental concept in mathematics that plays a vital role in various fields. Integration allows us to find areas, volumes and other quantities that accumulate over time, making it essential for understanding the world around us.

To fully appreciate integration, we will build on your existing knowledge of **differentiation**. Differentiation helps us understand how things change; like the speed of a car or the growth of a plant. By studying these changes, we can make predictions and decisions in our daily lives. Integration and differentiation are closely linked. While differentiation focuses on rates of change, integration helps us find the total accumulation of those changes. In physics, when we differentiate the position of an object over time, we find its velocity. Conversely, by integrating velocity, we can determine the total distance travelled. In economics, differentiation helps us understand how supply and demand change over time, while integration allows us to calculate total revenue or costs over a period. Also, Engineers use integration to design structures and analyse forces, in biology, integration helps model population growth and resource consumption and Environmental Scientists use it to calculate the total amount of resources consumed or pollutants produced over time.

KEY IDEAS

- **An antiderivative** is a fundamental concept in calculus that is closely related to integration. It is essentially the reverse process of differentiation.
- **Calculating the area of rectangles** is fundamental in understanding the approximation of areas under curves
- **Definite Integrals** represent the total accumulation of a quantity over a specific interval. It is written as $\int_a^b f(x)dx$ where f(x) is the function being integrated and **a** and **b** are the limits of integration.

- **Indefinite Integrals** represent a family of functions whose derivative is the integrand. It is written as: $\int f(x)dx = F(x) + C$ where $F(x)$ is the antiderivative of $f(x)$, and C is the constant of integration.
- **Partitioning intervals** involve dividing a given interval on the number line into smaller subintervals of equal or varying lengths. This concept is fundamental in calculus, particularly in approximation techniques like **integration**.

PARTITIONING AN AREA WITHIN AN INTERVAL

Activity 11.1: Partitioning an area within an interval

Working in small groups, carry out the following activity.

A frog is to hop into a pond, as shown below. Study the number line and answer the questions that follow in your groups.

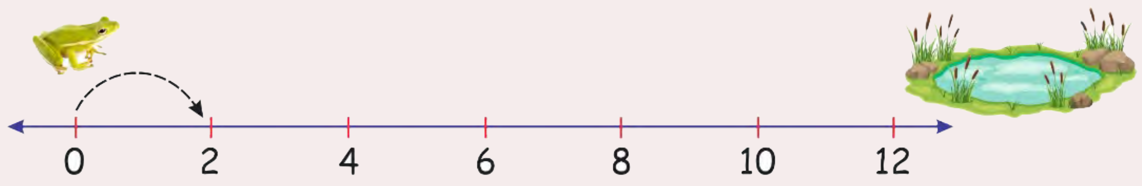


Figure 11.1: Number line

1. What is the interval from the starting point of the frog to the pond?
2. How many steps was the frog hopping from one point to the other?
3. How many steps in total did the frog hop to the pond?
4. Now perform the following task:
   a. Draw a number line with the interval [0, 18].
   b. If you move three steps each, how many sub-divisions will you have?
   c. Discuss how the sub-divisions can be found in the absence of a number line.

Understanding Intervals

An interval [a, b] represents all the numbers between **a** and **b**, including the endpoints. For example, the interval [1, 5] includes all numbers from 1 to 5, inclusive.

Introducing Partitioning

Partitioning an interval means dividing it into smaller, manageable parts called subintervals. This is particularly useful for approximating areas under curves.

Notation for Partitioning

Let $x_0, x_1, x_2, \ldots x_n$ be the points that define the partition, where:

$x_0 = a$ (the left endpoint of the interval), $x_n = b$ (the right endpoint of the interval)

The points in between represent the divisions of the interval into subintervals. For example, with n=4 for the interval [1, 5] the points would be: 1, 2, 3, 4, 5, so $x_0 = 1, x_1 = 2, x4 = 3, x_3 = 4, x_4 = 5$.

Calculating the width, Δx:

To find the width of each subinterval in the interval [a, b]:

The width of each subinterval is given by: $\Delta x = \frac{b - a}{n}$ where a is the a is the left endpoint of the interval and b is the right endpoint of the interval

For our example with a = 1 and b = 5, and n = 4: $\Delta x = \frac{5 - 1}{4}$

This means each subinterval has a width of 1.

Subintervals:

Define the subintervals as:

The first subinterval is $[x_0, x_1] = [1, 2]$

The second subinterval is $[x_1, x_2] = [2, 3]$

The third subinterval is $[x_2, x_3] = [3, 4]$

The fourth subinterval is $[x_3, x_4] = [4, 5]$

Importance of Partitioning: Partitioning an interval allows us to approximate the area under a curve by breaking it down into simpler shapes (rectangles) over these subintervals.

Example 11.1

Determine the subintervals for the interval [0, 15] for a step size of 3?

Solution

Step size of 3 means we add 3 to the left endpoint which is 0 until we get the right endpoint which is 15:

$0 + 3 = 3$

$3 + 3 = 6$

$6 + 3 = 9$

$9 + 3 = 12$

$12 + 3 = 15$

The first subinterval is [0,3]

The second subinterval is [3,6]

The third subinterval is [6,9]

The fourth subinterval is [9,12]

The fifth subinterval is [12,15]

There are 5 subintervals.

Example 11.2

Determine the subintervals for the interval [0, 6] for a step size of 0.5

Solution

The step size of 0.5 means we add 0.5 to the left endpoint which is 0 until we get to the endpoint of 6.

x	0	0.5	1	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5	5.5	6.0
x+0.5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5

We have [0, 0.5], [0.5, 1.0], [1.0, 1.5], [1.5, 2.0], [2.0, 2.5], [2.5, 3.0], [3.0, 3.5], [3.5, 4.0], [4.0, 4.5], [4.5, 5.0], [5.0, 5.5], [5.5, 6.0],

There are 12 sub intervals.

Example 11.3

Find the width for 4 subintervals for interval [1, 2], hence write the subintervals.

Solution

The width of each subinterval (step size) is given by:

$\Delta x = \frac{b - a}{n}$

where a is the a is the left endpoint of the interval and b is the right endpoint of the interval

a=1 and b=2, and n=4:

$\Delta x = \frac{2 - 1}{4} = 0.25$

Subintervals $= 1, 1 + 0.25 = 1.25, 1.25 + 0.25 = 1.5, 1.5 + 0.25$

$= 1.75, 1.75 + 0.25 = 2$

1, 1.25, 1.5, 1.75, 2

Example 11.4

Find the width for 5 subintervals for interval [0, 1], hence write the subintervals.

Solution

The width of each subinterval (step size) is given by:

$\Delta x = \frac{b - a}{n}$

where a is the left endpoint of the interval and b is the right endpoint of the interval

a = 0 and b = 1, and n = 5:

$\Delta x = \frac{1 - 0}{5} = 0.2$

Subintervals $= 0, 0 + 0.2 = 0.2, 0.2 + 0.2 = 0.4, 0.4 + 0.2 = 0.6, 0.6 + 0.2 = 0.8, 0.8 + 0.2 = 1,$

0, 0.2, 0.4, 0.6, 0.8, 1

Example 11.5

Find the width for 10 subintervals for interval [-2, 3], hence write the subintervals.

Solution

The width of each subinterval is given by:

$\Delta x = \frac{b-a}{n}$

where a is the left endpoint of the interval and b is the right endpoint of the interval

a=-2 and b=3, and n = 10:

$\Delta x = \frac{3-(-2)}{10} = 0.5$

Subintervals: -2, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0

Example 11.6

Find the width for 9 subintervals for interval [1, 4], hence write the subintervals

Solution

The width of each subinterval is given by:

$\Delta x = \frac{b-a}{n}$

where a is the left endpoint of the interval and b is the right endpoint of the interval

a= 1 and b=4, and n = 9:

$\Delta x = \frac{4-(1)}{9} = \frac{1}{3}$

Subintervals: $1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 4$

AREA UNDER A CURVE

Activity 11.2: Approximating the area under a curve

Working in pairs, or individually, carry out the following activity.

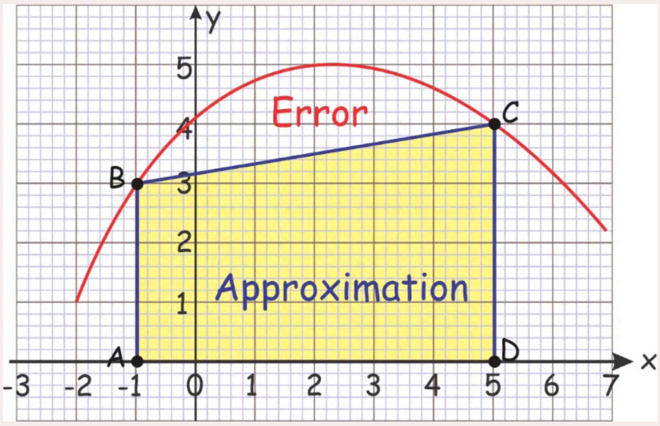


Figure 11.2: Graph of quadrilateral ABCD

1. What type of quadrilateral is ABCD
2. Find the area of the quadrilateral.
3. Is it a good approximation for the area under the curve from x = -1 to x = 5?
4. How could the approximation be improved?

Example 11.7

Find the area of the trapeziums A_1 to A_4 to estimate the area under the curve. Will this be an over or underestimation? How can our estimate be improved?

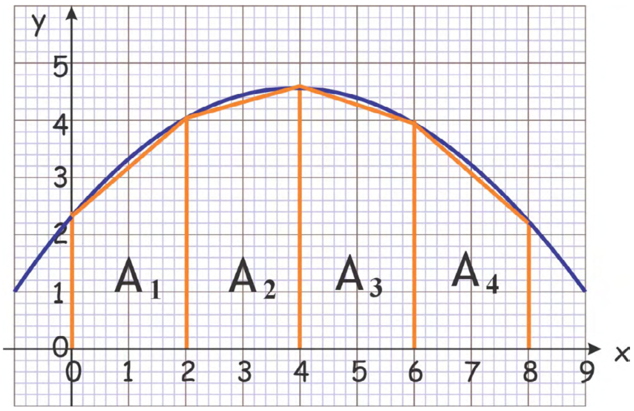


Figure 11.3: Graph of trapeziums A_1 to A_4

Solution

	a	b	h	Area $= \frac{1}{2}(a+b)h$
A_1	2.4	4	2	$\frac{1}{2}(2.4+4)(2) = 6.4$
A_2	4	4.6	2	$\frac{1}{2}(4+4.6)(2) = 8.6$
A_3	4.6	4	2	$\frac{1}{2}(4.6+4)(2) = 10.6$
A_4	4	2.2	2	$\frac{1}{2}(4+2.2)(2) = 6.2$

The approximated area is $A_1 + A_2 + A_3 + A_4 = 6.4 + 8.6 + 10.6 + 6.2 = 31.8$ *squared units*.

This will be an *underestimation* as we are not including the area immediately below the curve and the top of the trapeziums.

To improve our estimation, we could increase the number of trapeziums by increasing the number of partitions.

Example 11.8

Find the area of the function y = 2x + 2 as shown in the diagram below, by calculating the area of the individual rectangles and adding them together.

Is this an over or underestimation of the actual area?

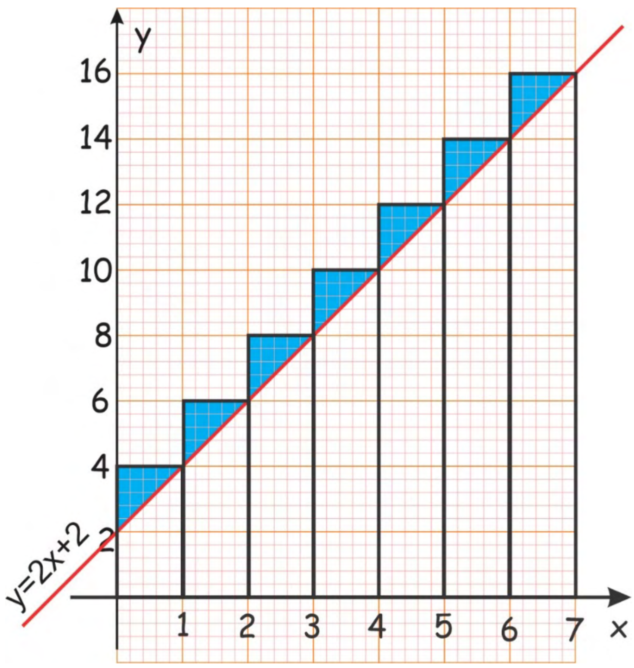


Figure 11.4: Graph of the function $y = 2x+2$

Solution

Rectangle	a	h	Area = ah
1	4	1	4 x 1 = 4
2	6	1	6 x 1 = 6
3	8	1	8 x 1 = 8
4	10	1	10 x 1 = 10
5	12	1	12 x 1 = 12
6	14	1	14 x 1 = 14
7	16	1	16 x 1 = 16

The approximated area is $A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 = 4 + 6 + 8 + 10 + 12 + 14 + 16 = 70$ *squared units*.

This is an *overestimation* of the area as we have the counted the additional blue triangles in our total.

THE CONNECTION BETWEEN LIMITS AND INTEGRALS

Introduction to Partitioning

We have learned that when we want to find the area under a curve within a specific interval, using more partitions (trapeziums or rectangles) gives us a better approximation of the actual area. As we increase the number of partitions, the approximated area gets closer to the exact area.

Step-by-Step Computation

To approximate the area under a curve f(x) from x = a to x = b using rectangles.

Number of Rectangles (= number of subintervals)

We divide the interval [a, b] into n equal parts (subintervals).

The width of each rectangle is denoted as h.

Calculating Width

The width, h, can be calculated using the formula: $h = \frac{b-a}{n}$

For example, if a = 0 and b = 10 with n = 5, then: $h = \frac{10-0}{5} = 2$

Finding the *x* Coordinates

If we choose to use the right side of the rectangles, the x coordinates of the right edges of the rectangles are:

$x_1 = a + h$

$x_2 = a + 2h$

$x_3 = a + 3h$

$x_n = a + nh$

For our example:

$x_1 = 0 + 2 = 2$

$x_2 = 0 + 2(2) = 4$

$x_3 = 0 + 3(2) = 6$

$x_4 = 0 + 4(2) = 8$

$x_5 = 0 + 5(2) = 10$

Finding the Heights:

The heights of the rectangles are given by the function values at these x coordinates:

$f(x_1) = f(a + h)$

$f(x_2) = f(a + 2h)$

$f(x_3) = f(a + 3h)$

$f(x_n) = f(b) = f(a + nh)$

Calculating the Area of Each Rectangle:

The area of each rectangle can be calculated as:

$A_1 = h \times f(x_1)$

$A_2 = h \times f(x_2)$

$A_3 = h \times f(x_3)$

$A_n = h \times f(x_n)$

Total Approximate Area

To find the total approximate area under the curve, we sum the areas of all rectangles:

$R(n) = A_1 + A_2 + A_3 + \ldots . A_n$

This can be expressed as: R(n)= $h \times \sum_{i=1}^{n} f(a + ih)$

Or as a **definite integral:**

If we want to find the area under the curve from x=a to x=b, we write $\int_a^b f(x)dx$ where a is the lower limit and b is the upper limit.

Approximation **Using Definite Integrals:**

The area can be approximated as:

Area $\int_a^b f(x) \approx [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x.......f(x_n)\Delta x]$

Where Δx is the width of each subinterval.

Example 11.10

Find the definite integral of $\int_0^4 (x^2 + 2)dx$ on the interval [0, 4], using 8 subintervals.

Solution

$h = \frac{b-a}{n}$

a = 0, b = 4, n = 8

$h = \frac{4-0}{8}$

$h = 0.5$

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
f(x) = $(x^2 + 2)$	2.25	3	4.25	6	8.25	11	14.25	18

Using, R(n)= $h \times \sum_{i=1}^{n} f(a + ih)$, where h= 0.5,

$$\sum_{i=1}^{4} f(a + ih) = [2.25 + 3 + 4.25 + 6 + 8.25 + 11 + 14.25 + 18] = 67$$

R(n)= $0.5 \times 67 = 33.5$ square units

Alternatively:

The area can be approximated as:

Area $\int_a^b f(x) \approx [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x.......f(x_n)\Delta x]$

$= [2.25(0.5) + 3(0.5) + 4.25(0.5) + 6(0.5) + 8.25(0.5) + 11(0.5) + 14.25(0.5) + 18(0.5)]$

= 33.5 square units

Thus, the approximate value of the definite integral $\int_0^4 (x^2 + 2)dx$, using 8 subintervals ≈ 33.5 square units.

Example 11.11

Find the approximate value of $\int_1^9 \left(\frac{1}{x^2}\right)dx$ using 5 subintervals

Solution

$h = \frac{b-a}{n}$

a = 1, b = 9, n = 5

$h = \frac{9-1}{5}$

$h = 1.6$

x	2.6	4.2	5.8	7.4	9
f(x) $= \frac{1}{x^2}$	0.1479	0.0566	0.0297	0.0182	0.0123

Using, R(n)= $h \times \sum_{i=1}^{n} f(a + ih)$, where h= 1.6,

$$\sum_{i=2.6}^{9} f(a + ih) = [0.1479 + 0.0566 + 0.0297 + 0.0182 + 0.0123] = 0.2467$$

R(n)= 1.6 × 0.2647

0.4235 square units

Alternatively:

The area can be approximated as:

Area $\int_a^b f(x) \approx [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x.......f(x_n)\Delta x]$

= [0.1479(1.6) + 0.0566(1.6) + 0.0297(1.6) + 0.0182(1.6) + 0.0123(1.6)]

= [0.23664 + 0.09056 + 0.04752 + 0.02912 + 0.01968] = 0.4235 square units

Example 11.12

Find the approximate value of $\int_1^3 \sqrt{1 + 2x^2}dx$ using 6 subintervals. Give your answer to 4 significant figures.

Solution

$h = \frac{b-a}{n}$

a = 1, b = 3, n = 6

$h = \frac{3-1}{6}$

$h = 1=$

x	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
$f(x) = \sqrt{1+2x^2}$	$\frac{\sqrt{41}}{3}$	$\frac{\sqrt{59}}{3}$	3	$\frac{\sqrt{107}}{3}$	$\frac{\sqrt{137}}{3}$	$\sqrt{19}$

Using, R(n)= $h \times \sum_{i=1}^{n} f(a+ih)$, where h = $\frac{1}{3}$

$$\sum_{i=2.6}^{9} f(a+ih) = \left[\frac{\sqrt{41}}{3} + \frac{\sqrt{59}}{3} + 3 + \frac{\sqrt{107}}{3} + \frac{\sqrt{137}}{3} + \sqrt{19}\right] = 19.4032$$

R(n)= $\frac{1}{3} \times 19.4032 = 6.468$ square units (to 4sf).

FINDING INDEFINITE AND DEFINITE INTEGRALS

Activity 11.3: Revision of Differentiation

In groups, pairs or individually solve the following questions and then share your results with the class.

1. Differentiate the following functions listed in the Table 11.1.

Table 11.1: Function and its derivative

	Function	Derivative
a.	$2x^3 + 6$	
b.	$2x^3 + 8$	
c.	$-3x^4 + 2.5$	
d.	$-3x^3 + 16$	
e.	$2x^3 + 5x + 3$	
f.	$2x^3 + 5x - 8$	
g.	$x^5 - 5x^2 + 4$	
h.	$x^5 - 5x^2 - 9$	

2. Once you have completed the differentiation, look at your results. Identify any pairs of functions that have the same derivative. Write them down.
3. Discuss in your groups what made the functions with the same derivatives different from each other?

Hint: *Consider aspects such as constants added or subtracted from the functions.*

Observations

- The functions $3x^2 - 8$ and $3x^2 + 7$ have the same derivative $6x$. The only difference is the constant (−8 and 7).
- Similarly, $2x^3 + 6$ and $2x^3 + 3$ share the derivative. The only difference is the constant (6 and 3).

This means that if we are given the derivatives to find the function, we must make sure that we make provision for a constant term. The process of finding the function $f(x)$ when we know the derivative $f'(x)$ is called **Integration** or **Antiderivative.**

An antiderivative of a function is the reverse process of finding a derivative. While differentiation helps us determine how a function changes, finding an antiderivative allows us to reconstruct the original function from its rate of change.

If $f'(x)$ is the derivative of a function f(x), then f(x) is called an **antiderivative** of $f'(x)$. This relationship can be expressed mathematically as:

$\int f'(x)dx = f(x) + c$ where c is the **constant of integration**. This constant is important because there are infinitely many antiderivatives for any given derivative, all differing by a constant as shown in *Activity 11.3*.

Finding the antiderivatives

Given $f(x) = x^n$, to find $f'(x)$ we:

i. Multiplied the function by the power, n
ii. Subtracted 1 from the power, n – 1.

If integration is the reverse of differentiation, then study Table 11.2 for steps in finding the antiderivative.

Table 11.2: Steps in finding the antiderivative

Differentiation	Integration (Reverse of Differentiation)
Step 1: **Multiply** the function by the power, n nx^n	Step 1: **Add** 1 to the power n, to obtain n+1 x^{n+1}
Step 2: **Subtract** 1 from the power, n nx^{n-1}	Step 2: **Divide** the function by the new power n+1 $\frac{x^{n+1}}{n+1}$

NB: Add the constant of integration, c, to all indefinite integrals.

Indefinite Integrals are when we have to find the antiderivative without limits.

Indefinite Integrals

Example 11.13

Integrate $\int(x^2 + 4x)dx$

Solution

To integrate we add one to the power and divide by the new power, so we have:

$$\int(x^2 + 4x)dx = \frac{x^{2+1}}{2+1} + \frac{4x^{1+1}}{1+1} + c = \frac{x^3}{3} + \frac{4x^2}{2} + c = \frac{x^3}{3} + 2x^2 + c$$

Example 11.14

Simplify $\int(3x^3 - 18x^2 + 5x)dx$

Solution

$$\int(3x^3 - 18x^2 + 5x)dx = \frac{3x^{3+1}}{3+1} - \frac{18x^{2+1}}{2+1} + \frac{5x^{1+1}}{2} + c = \frac{3x^4}{4} - \frac{18x^3}{3} + \frac{5x^2}{2} + c$$

$$= \frac{3x^4}{4} - 6x^3 + \frac{5x^2}{2} + c$$

Example 11.15

$\int 4dx$

Solution

4 can be written as $4x^0$

$\int 4x^0 dx = \frac{4x^{0+1}}{0+1} + c = \frac{4x^1}{1} + c = 4x + c$

This shows how a constant term in a given function is integrated.

Example 11.16

Find the integral of $\int (4x^4 + 3x^2 + \frac{1}{x^3} + 6)dx$

Solution

$\int (4x^4 + 3x^2 + \frac{1}{x^3} + 6)dx \ = \ \int (4x^4 + 3x^2 + x^{-3} + 6)dx = \frac{4x^5}{5} + x^3 - \frac{1}{2x^2} + 6x + c$

Example 11.17

Find the integral of $\int \frac{x^5 - 3x^3 + 8x - 13}{x^3} dx$

Solution

Simplify each term in the numerator with the term in the denominator

$\frac{x^5}{x^3} - \frac{3x^3}{x^3} + \frac{8x}{x^3} - \frac{13}{x^3} = x^2 - 3 + 8x^{-2} - 13x^{-3}$

$\int (x^2 - 3 + 8x^{-2} - 13x^{-3})dx = \frac{x^3}{3} - 3x - \frac{8}{x} + \frac{13}{2x^2} + c$

FUNDAMENTAL THEOREM OF CALCULUS

Suppose f(x) is a continuous function over the interval [a, b]. We can define:

$\int_a^b f(x)dx \ = \ F(b) - F(a)$ where F is any antiderivative of f(x).

The *Fundamental Theorem of Calculus* shows a fascinating relationship between differentiation and integration. In simple terms, it tells us that the process of finding an integral can be thought of as the reverse of finding a derivative.

Let us take the function $f(x) \ = \ x^3$ *as an example to illustrate this concept.*

Step 1: Finding the Derivative

To find the derivative f′(x), we use the **power rule**. The power rule states that if you have a function of the form x^n, the derivative is calculated by:

a. **Reducing the exponent by 1**.

b. **Multiplying the resulting expression by the original exponent**.

So, for our function $f(x) = x^3, f'(x) = 3x^2$

Step 2: Finding the Integral

Now, let's find the integral of f(x). To do this, we will reverse the process we used for differentiation:

a. **Increase the exponent by 1**.

b. **Divide by the new exponent**.

For the function $f(x) = x^3$:

$F(x) = \int f(x)\ dx = \int x^3 dx$.

Following the steps:

a. Increase the exponent: 3 + 1 = 4

b. Divide by the new exponent

$F(x) = \frac{x^4}{4} + c$, where c is the constant of integration.

The expression, $\int_a^b f(x)dx$ is a *definite integral* because the limits are known i.e., the upper limit is **b** and the lower limit is **a**. However, $\int f(x)dx$ is indefinite integral because the limit is not known

Let us work through the following steps to **evaluate a Definite Integral**

1. Write down the integral you want to evaluate, e.g., $\int_a^b f(x)dx$, where a and b are the limits of integration.
2. Determine the antiderivative F(x) of the function f(x). This involves using basic integration techniques
3. Compute F(b) and F(a):
   a. F(b): Substitute the upper limit **b** into the antiderivative.
   b. F(a): Substitute the lower limit **a** into the antiderivative.
4. The value of the definite integral is given by: $\int_a^b f(x)dx = F(b) - F(a)$.

Example 11.18

Evaluate $\int_1^4 (3x^2 + 2x + 6)dx$

Solution

$$\int_1^4 (3\boldsymbol{x}^2 + 2\boldsymbol{x} + 6)\boldsymbol{dx}$$

Finding the Antiderivative:

$F(x) = [x^3 + x^2 + 6x]_1^4$

Substitute the limits

$= [(4)^3 + (4)^2 + 6(4)] - [(1)^3 + (1)^2 + 6(1) = [64 + 16 + 24] - [1 + 1 + 6]$

$= [104] - [8] = 96$

Example 11.19

Evaluate $\int_{-3}^0 (t^3 + 5t^2 - 5)dt$

Solution

Finding the Antiderivative:

$F(x) = \left[\frac{t^4}{4} + \frac{5t^3}{3} - 5t\right]_{-3}^{0}$

Substitute the limits:

$= \left[\frac{(0)^4}{4} + \frac{5(0)^3}{3} - 5(0)\right] - \left[\frac{(-3)^4}{4} + \frac{5(-3)^3}{3} - 5(-3)\right]$

$= [0] - \left[\frac{81}{4} - \frac{135}{3} + 15\right] = [0] - \left[-\frac{117}{12}\right] = \frac{117}{12} = \frac{39}{4}$

Example 11.20

Evaluate $\int_1^2 (x + 3)(x - 5)dx$

Solution

$\int_1^2 (x^2 - 2x - 15)dx = \left[\frac{x^3}{3} - x^2 - 15x\right]_1^2$

Substitute the limits:

$$\left[\frac{(2)^3}{3}-(2)^2-15\left(2\right)\right]-\left[\frac{(1)^3}{3}-(1)^2-15\left(1\right)\right]=\left[\frac{8}{3}-4-30\right]-\left[\frac{1}{3}-1-15\right]$$

$$=\left[-\frac{94}{3}\right]-\left[-\frac{47}{3}\right]=-\frac{47}{3}$$

Activity 11.4: Fundamental Theorem of Calculus

Working in pairs, carry out the following activity.

1. The graph below represents the function $f(x) = 2x$ on the interval [0, 4]

a. Find the area of the figure ABC.

b. Find the integral of $f(x) = 2x$ on the interval [0, 4]

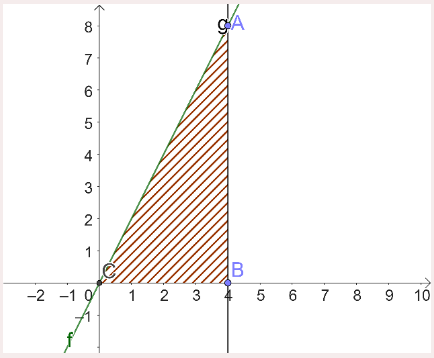


Figure 11.5: Graph of the function $f(x) = 2x$

2. Repeat the steps above for the graph $f(x) = 2x + 3$ on the interval [-1, 2]

Compare your results for (1) and (2). Look at the graph below to help with this.

Does the result of the integral match the area under the line?

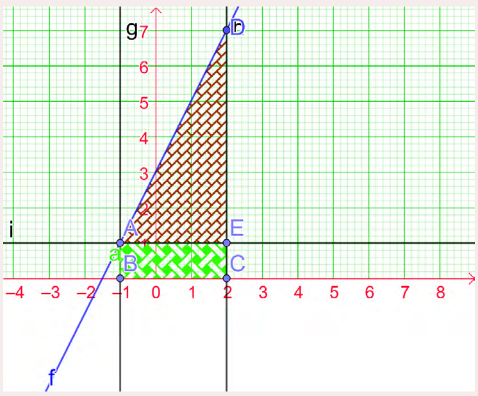


Figure 11.6: Graph of $f(x) = 2x + 3$

Therefore, we can use the idea of the *Fundamental Theorem of Calculus* to help us calculate the area under curves accurately.

Example 11.21

Evaluate the following, remembering that what you are finding is the area between the curve and the x axis between the given limits.

Note, if the answer is negative, this means the curve, and its corresponding area, is below the x-axis.

1. $\int_1^3 (2x^2 + 5x + 2)dx$
2. $\int_{-2}^0 (5x^3 + 4x^2 - 3x - 8)dx$
3. $\int_2^4 (x^4 - \frac{x^3}{2} - 4x + 9)dx$
4. $\int_{\sqrt{2}}^2 (2x - 1)dx$

Solution

1. $\int_1^3 (2x^2 + 5x + 2)dx$

$F(x) = \left[\frac{2}{3x^3} + \frac{5x^2}{2} + 2x\right]_1^3$

Substitute the limits:

$\left[\frac{2}{3}(3)^3 + \frac{5(3)^2}{2} + 2(3)\right] - \left[\frac{2}{3}(1)^3 + \frac{5(1)^2}{2} + 2(1)\right]$

$= \left[18 + \frac{45}{2} + 6\right] - \left[\frac{2}{3} + \frac{5}{2} + 2\right]$

$= \left[\frac{93}{2}\right] - \left[\frac{31}{6}\right] = \frac{248}{6} = \frac{124}{3}$

2. $\int_{-2}^0 (5x^3 + 4x^2 - 3x - 8)dx$

$= \left[\frac{5}{4x^4} + \frac{4}{3x^3} - \frac{3}{2x^2} - 8x\right]_{-2}^{0}$

$= \left[\frac{5}{4}(0)^4 + \frac{4}{3}(0)^3 - \frac{3}{2}(0)^2 - 8(0)\right] - \left[\frac{5}{4}(-2)^4 + \frac{4}{3}(-2)^3 - \frac{3}{2}(-2)^2 - 8(-2)\right]$

$= [0] - \left[\frac{5}{4}(16) + \frac{4}{3}(-8) - \frac{3}{2}(4) + 16)\right]$

$= [0] - \left[20 - \frac{32}{3} - 6 + 16\right]$

$= [0] - \left[20 - \frac{32}{3} - 6 + 16\right]$

$= 0 - \left[\frac{58}{3}\right] = -\frac{58}{3}$

3. $\int_2^4 (x^4 - \frac{x^3}{2} - 4x + 9)dx$

$= \left[\frac{x^5}{5} - \frac{x^4}{8} - 2x^2 + 9x\right]_2^4$

$= \left[\frac{(4)^5}{5} - \frac{(4)^4}{8} - 2(4)^2 + 9(4)\right] - \left[\frac{(2)^5}{5} - \frac{(2)^4}{8} - 2(2)^2 + 9(2)\right]$

$= \left[\frac{1024}{5} - \frac{256}{8} - 32 + 36\right] - \left[\frac{32}{5} - \frac{16}{8} - 8 + 18\right]$

$= \left[\frac{884}{5}\right] - \left[\frac{72}{5}\right] = \frac{812}{5}$

4. $\int_{\sqrt{2}}^{2} (2x - 1)dx$

$= [x^2 - x]_{\sqrt{2}}^{2}$

$= [(2)^2 - (2)] - [(\sqrt{2})^2 - \sqrt{2}]$

$= [4 - 2] - [2 - \sqrt{2}]$

$= [2] - [2 - \sqrt{2}] = \sqrt{2}$

EXTENDED READING

- Stewart, J. (2008). Calculus: Early transcendentals (6th ed.). Thomson Brooks/Cole Cengage Learning, page 384 – 484

REVIEW QUESTIONS

1. Given the interval [0, 20] and a step size of 4, find the various intervals and hence find the number of sub-intervals.
2. There are 8 subintervals with a step size of 1.5. If the interval is [1, t] find the value of t.
3. Find the approximated area under the curve below.

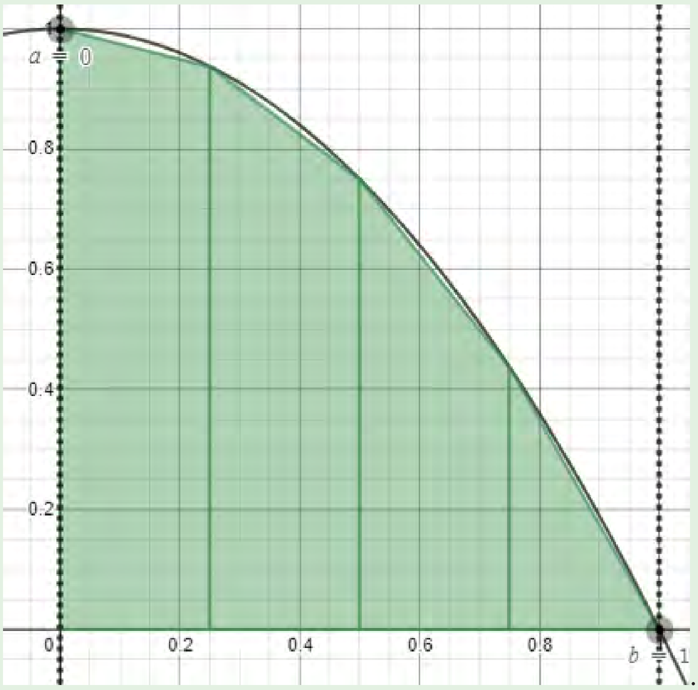


Figure 11.7

4. Find the approximate value of $\int_1^3 (x^2 + 1)dx$ using 8 subintervals.
5. Evaluate $\int_0^8 \frac{1}{x^2 + 1}dx$ using 4 subintervals.
6. Evaluate $y = \int_0^6 (x^3 + 2x^2 - 6)dx$
7. Integrate the following:
   a. $-\frac{5}{9x^4}$
   b. $8x$
   c. $\frac{6v^2 + 3\sqrt{v}}{v}$
   d. $8x^3 - 5\sqrt{x} - 6$
   e. $10 - \frac{1}{\sqrt{x}} + \frac{1}{x^4}$

f. $\frac{3x^4 + 16x}{4x^3}$

8. A businessman found that the rate of change in the cost of GH¢, of producing x thousands of calculators is given by $\frac{dy}{dx} = \frac{1500}{x^2}$ and that the overhead cost is GH¢ 10 000.00.

9. Find the cost function, y, and hence find the cost of producing 500 calculators, leaving your answer correct to the nearest Ghanaian Cedi.

10. Given that $\frac{dy}{dx} = 3x^2 + 4x + 5$ and y = 20 when x = 2, find the value of y when x = 13.

11. The rate of change of an area(A) with respect to the radius is given by $\frac{dA}{dr} = 5r + 12$.

    If A = 24 when r = 2, find r when A = 152.

12. A curve is such that $\frac{dy}{dx} = 3x^2 + 10x + m$, where m is a constant and that it passes through the points (2,6) and (4,18). Find the equation of the curve.

13. Evaluate:

    a. $\int_0^1 (2x^9 + x^4 + 1)dx$

    b. $\int_2^4 (2 + x^2)dx$

    c. $\int_{-1}^2 (3t^2 - 4)dt$

    d. $\int_1^4 \frac{1}{y^3}dy$

    e. $\int_{-3}^3 (4w^2 - 4w)dw$

    f. $\int_1^5 \frac{1}{\sqrt{t}}dt$

SECTION

12 APPLICATIONS OF DIFFERENTIATION

CALCULUS

Application of Calculus

INTRODUCTION

Imagine you are playing a football match and you kick the ball high into the air. At some point, the ball reaches its highest point before starting to fall back to the ground. How do we figure out exactly where that highest point is? This is where the first and second derivatives come into play. They help us uncover hidden details about a function, such as where it reaches its highest (maximum) or lowest (minimum) points. The second derivative helps us identify special points called saddle points, where the function flattens but does not have a peak or valley. These ideas are not just about solving equations on paper; they are the keys to understanding real-life problems in sports, economics, science and more! In this section, you will learn how to use the second derivative to classify points as maximum, minimum or saddle points and then use that knowledge to sketch the curve of a function. You will also find the maximum and minimum values of functions to solve real-world problems. Let us dive in and unlock the secrets of curves and turning points!

KEY IDEAS

- Differentiation helps to determine how a quantity changes with respect to another quantity.
- Points of inflection are where the concavity of a function changes (from concave up to concave down or vice versa).
- Saddle points occur when the graph flattens at a critical point but does not result in a local maximum or minimum in all directions.
- When the rate of change is **negative**, the graph of the function falls as you move from left to right.
- When the rate of change is **positive**, the graph of the function rises as you move from left to right.
- When the rate of change is **zero**, the graph neither rises nor falls at that specific point.

DETERMINING THE NATURE OF GRADIENTS

In year one, we learnt about the importance of finding the derivative of a function. It was discussed that the derivative of a function at a point represents the gradient of the tangent line that touches the curve of the function at that point. The gradient of a curve (other than straight lines) at different points on the curve differ, the derivatives are functions in themselves. We can therefore consider the nature of the gradient at different points on the curve.

1. Constant functions of the form $f(x) = c$, c $\in R$, representing the horizontal line, have a gradient of **zero.**
2. Functions of the form $f(x) = mx + c$,
   a. These represent a straight line that slopes upwards from left to right if the value of m is positive, $m > 0$
   b. These represent a straight line that slopes downwards from left to right if the values of m is negative,$m < 0$.

Combining all these facts which state that given a function $y = f(x)$, if y or f is a function of x, then the first derivative $\frac{dy}{dx}$ or f')x(is called the gradient function. The gradient at any point Q) x_1, y_1(is obtained by substituting the value of x_1 and y_1 into the expression for $\frac{dy}{dx}$.

Functions and derivative

1. For $f(x)$, the gradient is represented by its derivative f')x(.

   Behaviour: The derivative gives the rate of change of the function at any given point.
2. If $f'(a) > 0$, the gradient is positive.

   Behaviour: The function $f(x)$ is increasing at $x = a$
3. If $f'(a) < 0$, the gradient is negative.

   Behaviour: The function $f(x)$ is decreasing at $x = a$
4. If $f'(a) = 0$, the function has a horizontal tangent at $x = a$.

   Behaviour: This often indicate a local maximum, minimum or saddle point and is known as a stationary point.

Graphical Interpretation

1. Positive gradient means the graph of the function is rising as you move from left to right.

2. Negative gradient means the graph of the function is falling as you move from left to right.
3. Zero gradient means the graph is flat at the point, indicating no change in $f(x)$.

Activity 12.1: Graphing functions

Carry out the following activity in small groups.

Materials Needed:

1. Graph paper
2. Pen/pencil and paper
3. Ruler

Steps:

1. Review basic concepts of:
   - **a.** Derivative, the derivative of a function $f)x($ represents the gradient (slope) of the function at any point x.
   - **b.** Positive gradient indicates the function is increasing at that point.
   - **c.** Negative gradient indicates the function is decreasing at that point.
   - **d.** Zero gradient indicates a horizontal tangent, possibly at a maximum, minimum or inflection point, but a stationary point.
2. Select a variety of functions to analyse, for example:
   - **a.** $f(x) = x^2$
   - **b.** $f(x) = -x^2 + 4$
   - **c.** $f(x) = x^3 - 3x^2 + 2$
3. Calculate derivatives of the functions:
   - **a.** Find the first derivative of each function
     Example,$f(x) = x^2$, the derivative is $f'(x) = 2x$
   - **b.** Choose specific point to evaluate the gradients.
     Example, for $f(x) = x^2$, evaluate $f'(x)$ at $x = -1, x = 0$ *and* $x = 1$
   - **c.** Calculate $f'(-1) = -2, f'(0) = 0, f'(1() = 2$.
4. Interpret results:
   - **a.** Positive gradient means increasing function
   - **b.** Negative gradient means decreasing function

  c. Zero gradient means horizontal tangent and a stationary point.

5. Graph the functions:

  a. Use your graph paper and pencil and plot each function.

  b. Mark the points where you evaluated the gradient.

  c. Observe the behaviour of the function at these points.

6. Compare observations:

  a. Compare your calculated gradients with visual slopes on the graph.

  b. Confirm whether the function is increasing, decreasing or has a horizontal tangent at the chosen points.

Now let consider the following worked examples.

Example 12.1

If $f(x) = x^3 - 3x^2 + 2$, find the gradient of $f)x($ at $x = 0$, $x = 1$, $x = 3$.

Solution

$f(x) = x^3 - 3x^2 + 2$

$f')x(= 3x^2 - 6x$

At $x = 0$, $f'(0) = 0$

At $x = 1$, $f'(1) = 3(1)^2 - 6(1) = -3$ (Negative)

At $x = 3$, $f'(3) = 3(3)^2 - 6(3) = 27 - 18 = 9$ (Positive)

Interpretation:

The function has a horizontal tangent/stationary point at $x = 0$

The function is decreasing at $x = 1$ and increasing at $x = 3$.

Example 12.2

If $f(x) = 2x^2 + 3x - 5$ find the gradient of $f)x($ at $x = -1$, $x = 0$, $x = 2$.

Solution

$f(x) = 2x^2 + 3x - 5$

$f'(x) = 4x + 3$

At $x = -1, f'(-1) = 4(-1) + 3 = -1$ (Negative)

At $x = 0, f'(0) = 4(0) + 3 = 3$ (Positive)

At $x = 2, f'(2) = 4(2) + 3 = 11$ (Positive)

Interpretation:

The function is decreasing at $x = -1$ and increasing at $x = 0$ and $x = 2$

Example 12.3

Find the gradient on the curve $f(x) =)x - 2(^2 - 4$, at $x = 1$, $x = 2$ *and* $x = 3$

Solution

$f(x) =)x - 2(^2 - 4 = x^2 - 4x$

$f'(x) = 2)x - 2($

At $x = 1, f'(1) = 2(1 - 2) = 2(-1) = -2$ (Negative)

At $x = 2, f'(2) = 2(2 - 2) = 2(0) = 0$ (Zero)

At $x = 3, f'(3) = 2(3 - 2) = 2(1) = 2$ (Positive)

Interpretation:

The function is decreasing at $x = 1$, stationary at $x = 2$ and increasing at $x = 3$.

Example 12.4

Find the gradient of $x^2 + y^2 = 9$ at the point where $x = 1$

Solution

Using implicit differentiation gives:

$$2x + 2y\frac{dy}{dx} = 0$$

$2y\frac{dy}{dx} = -2x$, then divide through by $2y$

$\frac{dy}{dx} = -\,x$-, this is the gradient function

At $x = 1$, we substitute the value of $x = 1$ into the original equation to find the value of y at this point:

$(1)^2 + y^2 = 9$

$y^2 = 8$

$y = \pm 2\sqrt{2}$

Substituting into the gradient function, at $x = 1$ and $y = \pm 2\sqrt{2}$

$-\frac{1}{2\sqrt{2}} = \pm\frac{\sqrt{2}}{4}$

Worked Example 12.5

Find the coordinates of the point on the curve $y = \frac{2}{x^2}$ at which its gradient is $\frac{1}{2}$.

Solution

$y = \frac{2}{x^2} = 2x^{-2}$

$\frac{dy}{dx} = -4x^{-3} = -\frac{4}{x^3}$

When $\frac{dy}{dx} = \frac{1}{2}$

$\frac{-4}{x^3} = \frac{1}{2}$

$x^3 = -8$

$\therefore x = -2$

When $x = -2$

$y = 2/x^2 = \frac{2}{(-2)^2} = \frac{2}{4} = \frac{1}{2}$

Therefore, the gradient is $\frac{1}{2}$ at the point $) -2, \frac{1}{2}($

INVESTIGATING TURNING POINTS (STATIONARY POINTS)

A stationary point also known as a turning point is a point on the graph of a function where the gradient (slope) is zero or $\frac{dy}{dx} = 0$. At this point, the function changes direction from increasing to decreasing, or vice versa, or is saddle point. A saddle point is where the graph changes from convex to concave, or vice versa, but continues increasing or decreasing either side of the saddle. Therefore, a stationary point can be classified into three types: local maximum (increasing to decreasing), local minimum (decreasing to increasing) and saddle point (or point of inflection) where it continues increasing/decreasing either side of the saddle point.

Types of stationary point

1. Local maximum:
   a. At a local maximum, the function changes direction from increasing to decreasing.
   b. The derivative $f'(x)$ changes from positive to negative.
   c. Graphically, it looks like a peak. Such as in Figure 1.

Figure 12.1: Mountain Odwoanoma in Kwahu-Ghana

2. Local minimum:
   a. At a local minimum, the function changes direction from decreasing to increasing
   b. The derivative $f'(x)$ changes from negative to positive.
   c. Graphically, it looks like a valley.

3. Saddle point (Point of inflection)
   a. At a saddle point, the function changes concavity, but not necessarily direction.
   b. The derivative $f'(x)$ is zero, but it does not change sign
   c. Graphically, it looks like a point where the curve flattens out momentarily.

Identifying Stationary Points

1. Find the first derivative: calculate the first derivative $f'(x)$ of the function.
2. Set the first derivative to zero and solve the equation $f'(x) = 0$ to find the x -coordinates of the stationary points.
3. To classify the stationary points, use the second derivative test.
   a. Calculate the second derivative $f''(x)$.
   b. If $f''(x) > 0$ at a stationary point, it is a local minimum.
   c. If $f''(x) < 0$ at a stationary point, it is a local maximum.
   d. If $f''(x) = 0$, the test is inconclusive and the point might be a saddle point.

Example 12.6

Find the coordinates of the stationary point(s) of $f(x) = x^3 - 3x^2 + 2$.

Determine the nature of the stationary points.

Solution

$f(x) = x^3 - 3x^2 + 2$

$f'(x) = 3x^2 - 6x$

Set the first derivative to zero:

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$

When $x = 0$

$f(0) = (0)^3 - 3(0)^2 + 2 = 2$

The stationary point is (0, 2(

$f(2) = (2)^3 - 3(2)^2 + 2 = -2$

The stationary point is $(2, -2)$

Classify the nature of the stationary points:

$f''(x) = 6x - 6$

At $x = 0$

$f''(0) = 6(0) - 6 = -6$

if $f''(x) < 0 \Longrightarrow$ Maximum

Therefore, this is a local maximum.

At $x = 2$

if $f''(x) > 0 \Longrightarrow$ Minimum

Therefore, this is a local minimum.

Example 12.7

Determine the coordinates and nature of the turning point of the function: $y = -3x^2 + 18x - 20$

Solution

$y = -3x^2 + 18x - 20$

$\frac{dy}{dx} = -6x + 18$

At the stationary point $\frac{dy}{dx} = 0$

$0 = -6x + 18$

$x = 3$

When $x = 3$

$y = -3(3)^2 + 18(3) - 20 = 7$

The stationary point is $(3, 7)$

$f''(x) = -6 < 0 \Longrightarrow$ maximum

Hence $(3, 7)$ is maximum

Example 12.8

Determine the coordinates and the nature of the turning points of the function:

$y = 2x^3 - 3x^2 - 12x + 18$

Solution

$y = 2x^3 - 3x^2 - 12x + 18$

$\frac{dy}{dx} = 6x^2 - 6x - 12$

At the stationary points $\frac{dy}{dx} = 0$

$0 = 6x^2 - 6x - 12$

$0 = x^2 - x - 2$

$0 =)x + 1()x - 2($

$x = -1$ or $x = 2$

When $x = -1$

$y = 2)-1(^3 - 3)-1(^2 - 12) - 1(+ 18 = 25$

When $x = 2$

$y = 2)2(^3 - 3)2(^2 - 12)2(+ 18 = -2$

Hence the stationary points are $(-1, 25)$ and $(2, -2)$

To determine the nature of the turning points we find the second derivative:

$f^{II}(x) = \frac{d2y}{dx^2} = 12x - 6$

At $x = -1$

$\frac{d^2y}{dx^2} = 12(-1) - 6 = -18 < 0$

Hence $) - 1, 25($ is a maximum point.

At $x = 2$

$\frac{d^2y}{dx^2} = 12(2) - 6 = 18 > 0$

Hence $(2, -2)$ is a minimum point

Example 12.9

Determine the coordinates and the nature of the turning points of the function:

$y = 4x^3 - 9x^2 + 6x - 2$

Solution

$y = 4x^3 - 9x^2 + 6x - 2$

$\frac{dy}{dx} = 12x^2 - 18x + 6$

At the stationary point $\frac{dy}{dx} = 0$

$0 = 12x^2 - 18x + 6$

$0 = 2x^2 - 3x + 1$

$0 = (2x - 1)(x - 1)$

$x = \frac{1}{2}$ or $x = 1$

When $x = \frac{1}{2}$

$y = 4\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) - 2 = -\frac{3}{4}$

When $x = 1$

$y = 4(1)^3 - 9(1)^2 + 6(1) - 2 = -1$

Hence the stationary points are $\left(\frac{1}{2}, -\frac{3}{4}\right)$ and $(1, -1)$

To determine the nature of the turning points we find the second derivative:

$\frac{d^2y}{dx^2} = 24x - 18$

When $x = \frac{1}{2}$

$\frac{d^2y}{dx^2} = 24\left(\frac{1}{2}\right) - 18 = -6 < 0$

Hence $\left(\frac{1}{2}, -\frac{3}{4}\right)$ is a maximum point

When $x = 1$

$\frac{d^2y}{dx^2} = 24(1) - 18 = 6 > 0$

Hence $(1 - 1)$ is a minimum point

Example 12.10

Determine the coordinates and the nature of the turning points of the function:

$y = x^3 - 6x^2 + 12x - 5$

Solution

$y = x^3 - 6x^2 + 12x - 5$

$\frac{dy}{dx} = 3x^2 - 12x + 12$

At stationary point $\frac{dy}{dx} = 0$

$0 = 3x^2 - 12x + 12$

$0 = x^2 - 4x + 4$

$0 =)x - 2()x - 2($

Which gives $x = 2$

When $x = 2$

$y =)2(^3 - 6)2(^2 + 12(2) - 5 = 3$

Hence the stationary point is (2, 3)

To determine the nature of the turning point, we find the second derivative:

$\frac{d^2y}{dx^2} = 6x - 12$

When $x = 2$

$\frac{d^2y}{dx^2} = 6(2) - 12 = 0$

Since $\frac{d^2y}{dx^2} = 0$, it follows that the stationary point (2, 3) is a point of inflection as the function is decreasing either side.

Example 12.11

The function $f(x) = ax^2 + bx + c$ has a gradient function $4x + 2$ and is stationary when $y = 1$. Find the values of a, b and c.

Solution

$f(x) = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

But $\frac{dy}{dx} = 4x + 2$

$\therefore 2ax + b = 4x + 2$ Equating the gradient to the differentiated function

Comparing corresponding terms:

$2a = 4 \Longrightarrow a = 2$ and $b = 2$

The stationary value of $f)x($ occurs when

$\frac{dy}{dx} = 0$

$4x + 2 = 0$

$x = -\frac{1}{2}$

The stationary value

$f(x) = ax^2 + bx + c$

$fx = 2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + c$

$f(x) = -\frac{1}{2} + c$

But the stationary value is when $y = 1$.

$-\frac{1}{2} + c = 1$

$c = \frac{3}{2}$

Therefore, $a = 2, b = 2, c = \frac{3}{2}$

SKETCHING POLYNOMIAL FUNCTIONS

A polynomial function is a function that involves only positive integer exponents of a variable in an equation like the quadratic equation, cubic equation, etc. We will explore how to sketch such polynomials and the points to consider.

Follow these steps to sketch a curve.

1. Find the intercepts on the x and y axes.
   For the intercept on the x–axis, put $y = 0$ and solve for x
   For the intercept on the y–axis, put $x = 0$ and solve for y
2. Find the turning point(s). At turning point put $\frac{dy}{dx} = 0$ and solve for x. Substitute the value of x into the original equation $y = f(x)$, to find the

corresponding $y-$ coordinate values. This establishes the coordinates of the stationary points.

3. Test for maximum and minimum, use the following conditions:
   a. If $d\,2y\,\_\,dx\,2 > 0$)i.e., positive) the turning point is minimum)
   b. If $d\,2y\,\_\,dx\,2 < 0$)i.e., negative) the turning point is maximum)

Example 12.12

Sketch the curve $y = x^3 - 7x^2 + 15x - 9$ indicating clearly its points of intersection on the axes and its turning points.

Solution

The equation of the curve is $y = x^3 - 7x^2 + 15x - 9$

For the intercept on the $y-$ axis, put $x = 0$

$y = -9$

The point $(0, -9)$ is the intercept on the $y-$ axis

For the intercept on the $x-$ axis, put $y = 0$

$x^3 - 7x^2 + 15x - 9 = 0$

By factorising the polynomial

$x^3 - 7x^2 + 15x - 9 = 0$

$)x - 1()x - 3(^2 = 0$

$x = 1\ or\ 3\ (repeated)$

The intercepts on the $x-$ axis is (1, 0) and (3, 0).

Note, as (3, 0) is a repeated root this means the curve just touches the x-axis at this point, but does not cut through it.

$y = x^3 - 7x^2 + 15x - 9$

$\frac{dy}{dx} = 3x^2 - 14x + 15$

At stationary points $\frac{dy}{dx} = 0$

$3x^2 - 14x + 15 = 0$

By solving $x = \frac{5}{3}$ or 3

We investigate which of these points gives maximum or minimum points by finding the second derivative:

$\frac{dy}{dx} = 3x^2 - 14x + 15$

$\frac{d^2y}{dx^2} = 6x - 14$

At $x = 3$

$\frac{d^2y}{dx^2} = 6(3) - 14 = 4 > 0$ (Positive = local minimum)

To find the corresponding y value substitute $x = 3$ into the expression for y.

When $x = 3$, $y = 0$. Hence $(3, 0)$ is the minimum turning point.

At $x = \frac{5}{3}$

$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 14 = -4 < 0$ (Negative = local maximum)

To find the corresponding y value substitute $x = \frac{5}{3}$ into the expression for y

$y = \left(\frac{5}{3}\right)^3 - 7\left(\frac{5}{3}\right)^2 + 15\left(\frac{5}{3}\right) - 9 = \frac{32}{27}$

When $x = \frac{5}{3}$, $y = \frac{32}{27}$. Hence $\Big)\frac{5}{3}, \frac{32}{7}\Big($ is the maximum turning point.

Sketch the graph with points (1, 0), (3, 0), (1.67, 1.19) and (0, -9)

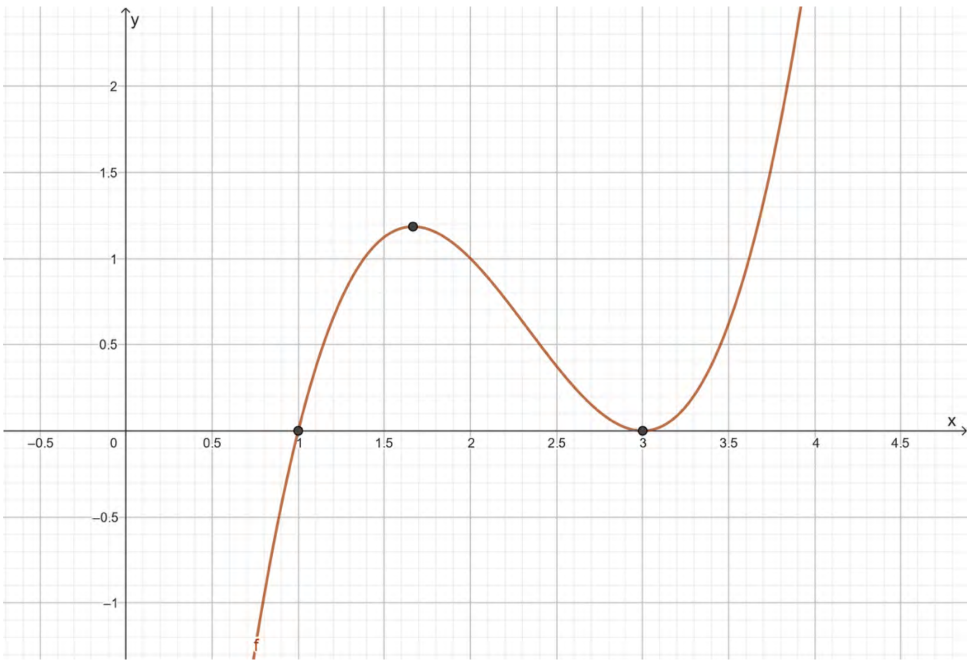


Figure 12.2: Graphical illustration of $y = x^3 - 7x^2 + 15x - 9$

Example 12.13

Sketch the curve $y = x^2 - 4x + 3$ indicating all its turning points and intercepts.

Solution

$y = x^2 - 4x + 3$

$\frac{dy}{dx} = 2x - 4$

At turning point $\frac{dy}{dx} = 0$

$0 = 2x - 4$

$x = 2$

Substituting $x = 2$ into the function to find y, we have

$y = (2)^2 - 4(2) + 3 = -1$

Hence, the turning point is $(2, -1)$

To determine the nature of the turning point we find the second derivative

$\frac{dy}{dx} = 2x - 4$

$\frac{d^2y}{dx^2} = 2 > 0$, ∴ Positive minimum point and occurs at $(2, -1)$

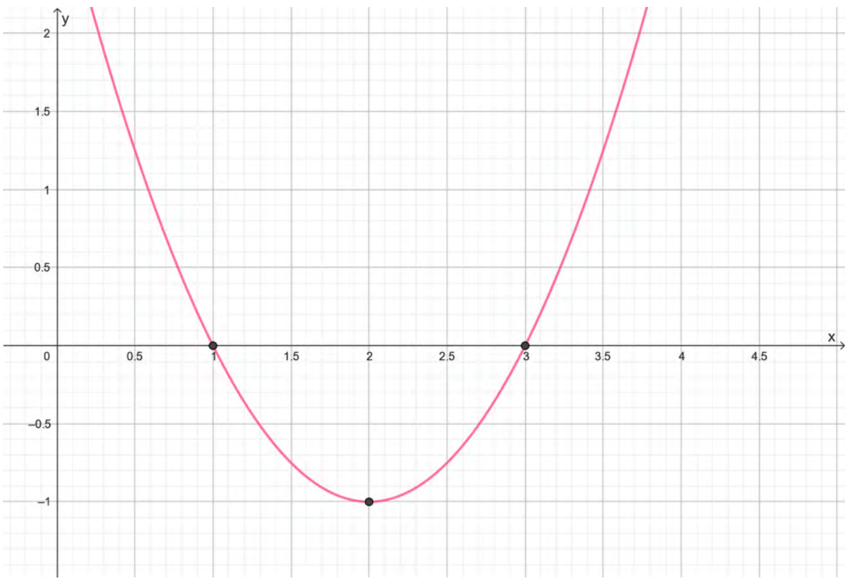


Figure. 12.3: Graphical illustration of $y = x^2 - 4x + 3$

For the y–intercept, put $x = 0$

$y = (0)^2 - 4(0) + 3 = 3$

For the $x-$intercept, put $y = 0$

$0 = x^2 - 4x + 3$

$(x - 1)(x - 3) = 0$

$x = 1$ and $x = 3$

Sketch the graph with points (1, 0), (3, 0), (0, 3) and (2, -1)

Example 12.14

Sketch the curve $y = x^3 - 9x^2 + 15x - 7$ indicating all its turning points and intercepts

Solution

$y = x^3 - 9x^2 + 15x - 7$

$\frac{dy}{dx} = 3x^2 - 18x + 15$

At turning point $\frac{dy}{dx} = 0$

$0 = 3x^2 - 18x + 15$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 5$ and $x = 1$

Substituting $x = 5$ and $x = 1$ into the function to find y, we have

$y = (5)^3 - 9(5)^2 + 15(5) - 7 = -32$

$y = (1)^3 - 9(1)^2 + 15(1) - 7 = 0$

The turning points are $(5, -32)$ and $(1,0)$

To determine the nature of the turning point we find the second derivative

$\frac{dy}{dx} = 3x^2 - 18x + 15$

$\frac{d^2y}{dx^2} = 6x - 18$

At $x = 5$

$\frac{d^2y}{dx^2} = 6(5) - 18 = 12 > 0$, Positive = minimum point and occurs at $(5, -32)$

At $x = 1$

$\frac{d^2y}{dx^2} = 6(1) - 18 = -12 < 0$, Positive = maximum point and occurs at (1,0)

For the y–intercept, put $x = 0$

$y = (0)^3 - 9(0)^2 + 15(x) - 7 = -7$

For the x–intercept, put $y = 0$

$0 = x^3 - 9x^2 + 15x - 7$

By the factor theorem:

$f)x(= x^3 - 9x^2 + 15x - 7$

$f(1) = (1)^3 - 9(1)^2 + 15(1) - 7 = 0 \Longrightarrow (x - 1)$ is a factor of $f(x)$

By the division of Polynomial

$$
\begin{array}{r l}
 & x^2 - 8x + 7 \\
x - 1 & \sqrt{x^3 - 9x^2 + 15x - 7} \\
 & \underline{x^3 - x^2} \\
 & -8x^2 + 15x \\
 & \underline{-8x^2 + 8x} \\
 & 7x - 7 \\
 & \underline{7x - 7} \\
 & 0 \quad 0
\end{array}
$$

$f(x) = (x - 1)(x^2 - 8x + 7) = 0$

Factorising the trinomial $(x^2 - 8x + 7) = 0$:

$(x - 1)(x - 1)(x - 7) = 0$

Intercepts on the x–axis are $x = 1$ and $x = 7$

Sketch the graph with points (0, -7), (1, 0), (5, -32) and (7,0).

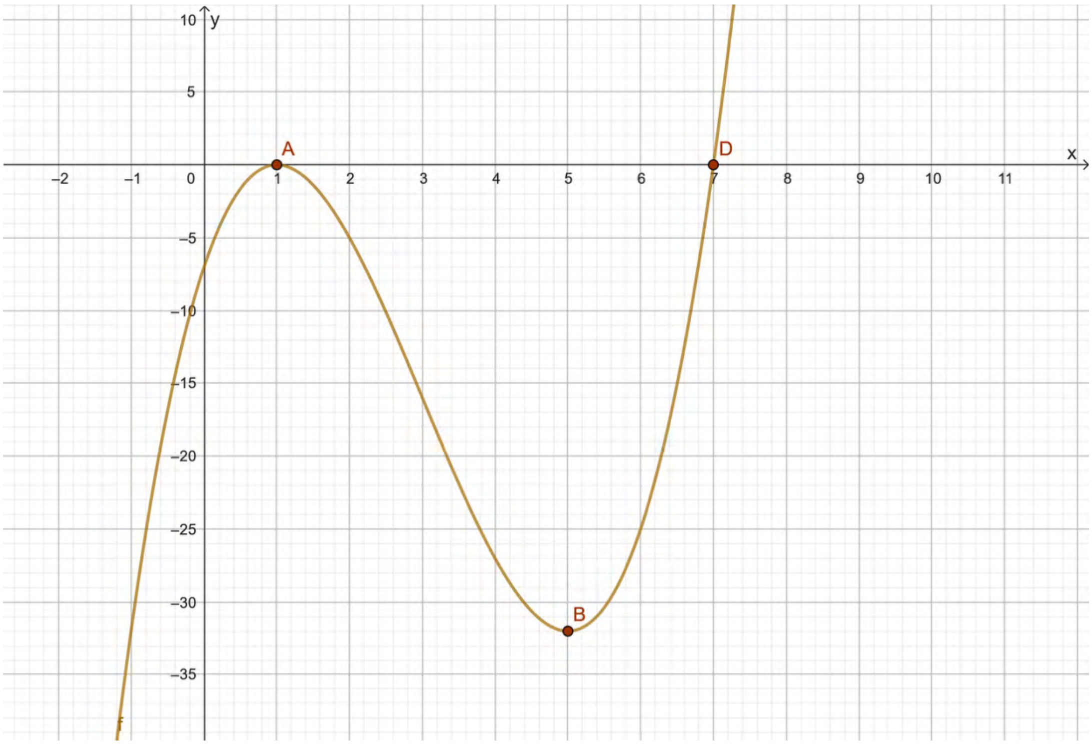


Figure 12.4: Graphical illustration of $y = x^3 - 9x^2 + 15x - 7$

APPLYING DIFFERENTIATION TO SOLVE REAL - LIFE PROBLEMS

Understanding the nature of gradients, investigating turning points and sketching polynomial functions is critical for solving real-life problems due to their ability to model and optimise real-world scenarios. Gradients help determine rates of change, essential for analysing phenomena such as speed in physics, profitability in economics and environmental changes in engineering. Turning points identify maxima or minima, which are vital in optimising resources, such as maximising agricultural yields or minimising production costs. Sketching polynomial functions provides a visual understanding of relationships and trends, aiding in decision-making, like predicting investment outcomes or modelling population growth. We will apply these concepts by solving in real-life problems.

Activity 12.2: Project on real life application of differentiation techniques

1. Working in pairs, or small groups, choose from one of the projects below:
   - **a.** Investigate how the time taken to walk from your dormitory to the dining hall varies depending on your walking speed. Use a stopwatch to record times for different walking speeds and calculate the rate of change of distance with respect to time.
   - **b.** Choose a fixed distance, use formula for speed $= \frac{distance}{time}$
   - **c.** A shopkeeper in your school wants to determine the price of sachet water that will result in the highest sales revenue. Use basic mathematics to model and determine the price that maximises revenue using hypothetical data.
   - **d.** Note: Revenue = price × quantity
   - **e.** Observe the motion of a ball thrown vertically upward on the school field. Record the time it takes to reach its highest point and use this data to calculate the maximum height using basic physics and a quadratic model.
   - **f.** Note: Use motion under gravity formula
   - **g.** i.e., height(t) = Initial velocity $-\frac{1}{2}(acceleration\ due\ to\ gravity)(time)^2$
   - **h.** Investigate how the cost of printing exercise books changes as the number of books printed increases. Collect data to model this relationship and determine the number of books that minimises the cost while balancing production efficiency.
2. After choosing your project, define the variables involved in the problem.
3. Express the relationships between these variables using equations.
4. Apply the appropriate differentiation technique to the project of choice.
5. Interpret the results obtained.
6. Create a visual using graphs (manually or by software).
7. Write a clear, concise report summarising:
   - **a.** The real-life problem you chose.
   - **b.** The mathematical formulation of the problem.
   - **c.** The differentiation techniques used to solve the problem.

d. The interpretation of the results.

e. Graphs that visualise the solution.

f. Ensure that your report includes both the mathematical steps and the practical implications of your solution.

Let us go through some examples of how differentiation techniques are applied in real life.

Example 12.15

A rectangular cake dish is made by cutting out squares from the corners of a 12.5cm by 20cm rectangle of tin-plate and then folding the metal to form the container. What size squares must be cut out to produce the cake dish of maximum volume?

Solution

Step 1: Let m be the side lengths of the squares that are cut out.

Step 2: Determine the volume

Volume = length × width × height

$= (20 - 2m)(12.5 - 2m)m$

$= (250 - 40m - 25m + 4m^2)m$

$= 250m - 65m^2 + 4m^3$

And $0 < m < 10$

Step 3: Differentiate the volume

$\frac{dV}{dm} = 12m^2 - 130m + 250$

Step 4: Equate the differentiated volume to zero and simplify

$12m^2 - 130m + 250 = 0$

$\therefore m = \frac{25}{3}, m = \frac{5}{2}$

Step 5: Find the second derivative to confirm which is the local maximum and which the local minimum. This is necessary as both values lie between 0 and 10.

$\frac{d^2V}{dm^2} = 24m - 130$

When m $= \frac{25}{3}, \frac{d2V}{dm^2} = 70 \therefore$ a local minimum

When m $= \frac{5}{2}, \frac{d^2V}{dm^2} = -70 \therefore$ a local maximum and the required value to maximise the volume of the cake tin.

Therefore, the maximum volume is obtained when 2.5cm squares are cut from the corners of the rectangle. (The minimum volume would be obtained when 8.33cm squares are cut from the corners of the rectangle.)

Example 12.16

A cocoa farmer in Ghana models the annual yield of cocoa (in tons) as a function of the amount of fertiliser applied, $y = -0.5x^2 + 10x + 20$ where x is the amount of fertiliser (in kg).

Find the amount of fertiliser needed to maximise the yield and the maximum yield this gives.

Solution

Step 1: Differentiate y with respect to x

$\frac{dy}{dx} = -x + 10$

Step 2: To determine the value of x at turning point, equate $\frac{dy}{dx}$ to zero.

$-x + 10 = 0$

$x = 10$

Step 3: Substitute $x = 10$ into the original function.

$y = -0.5(10)^2 + 10(10) + 20$

$y = 70$ tons

The amount of fertiliser needed to maximise the yield is 10kg to give a maximum yield of 70 tons.

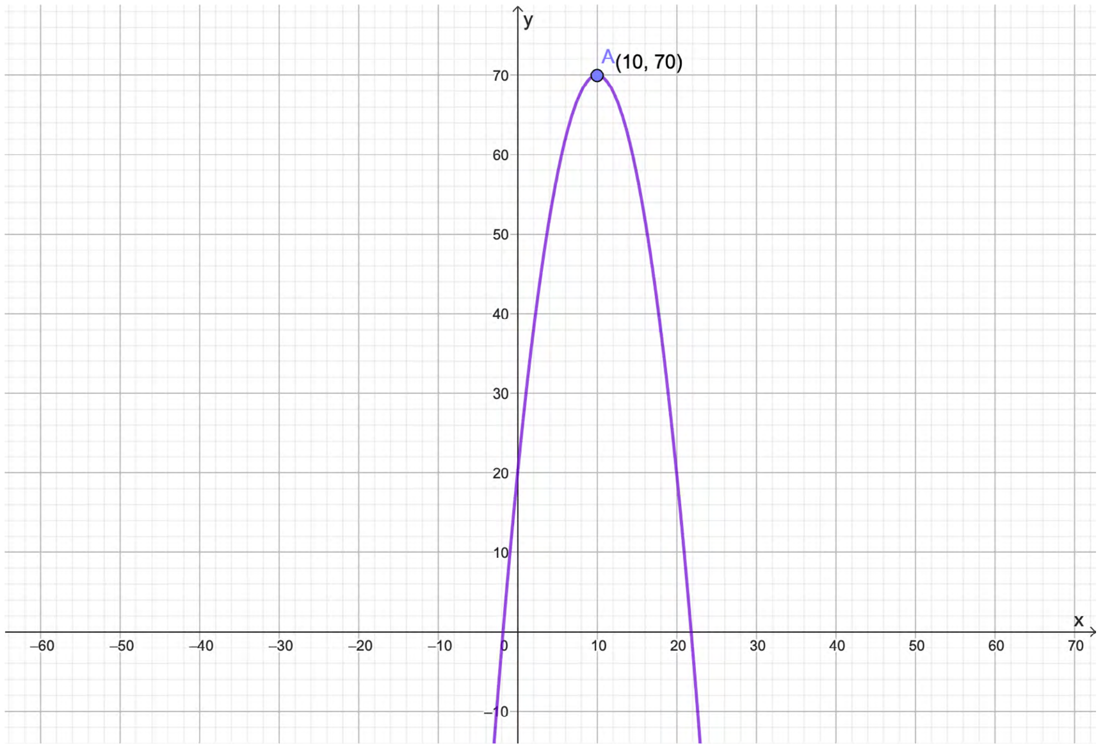


Figure 12.5: Graphical illustration of maximum cocoa yield

Example 12.17

A logistics company transports goods from Accra to Kumasi.

The cost function, $C)x(= x^2 - 8x + 1000$, represents the cost in Ghana cedis per trip, where x is the number of trips made in a month.

Determine the number of trips to minimise the cost and find the minimum cost.

Solution

Step 1: Let $C(x) = y$. Differentiate y with respect to x

$$\frac{dy}{dx} = 2x - 8$$

Step 2: To determine the value of x at turning point, equate $\frac{dy}{dx}$ to zero.

$2x - 8 = 0$

$2x = 8$

$x = 4$

Step 3: Substitute $x = 4$ into the original function.

$C(4) = 4^2 - 8(4) + 1000 = 984$

The minimum cost is **GH¢** 984.00 and four trips are required.

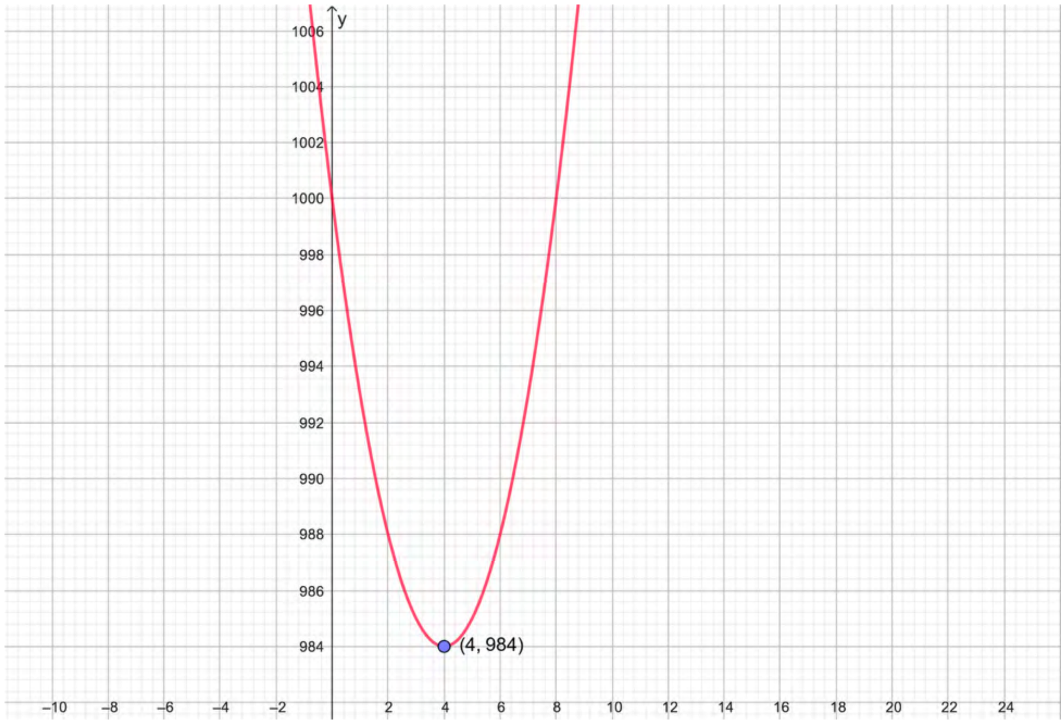


Figure 12.5: Graphical illustration of cost of transporting logistics

Example 12.18

A rice farmer in South Tongu uses water to irrigate fields.

The productivity $P)w(= -2w^2 + 40w$, where w is the water used in cubic meters per day, models the rice yield in kg. How much water should be used to maximise yield, and what is the maximum yield?

Solution

Step 1: Let $P(w) = v$. Differentiate v with respect to w

$$\frac{dv}{dw} = -4w + 40$$

Step 2: To determine the value of w at turning point, equate $\frac{dv}{dw}$ to zero.

$-4w + 40 = 0$

$-4w = -40$

$w = 10$

Step 3: Substitute $w = 10$ into the original function.

$P(10) = -2(10)^2 + 40(10) = 200$

The farmer should use $10m^3$ of water per day to irrigate the farm which will yield a maximum of 200kg of rice.

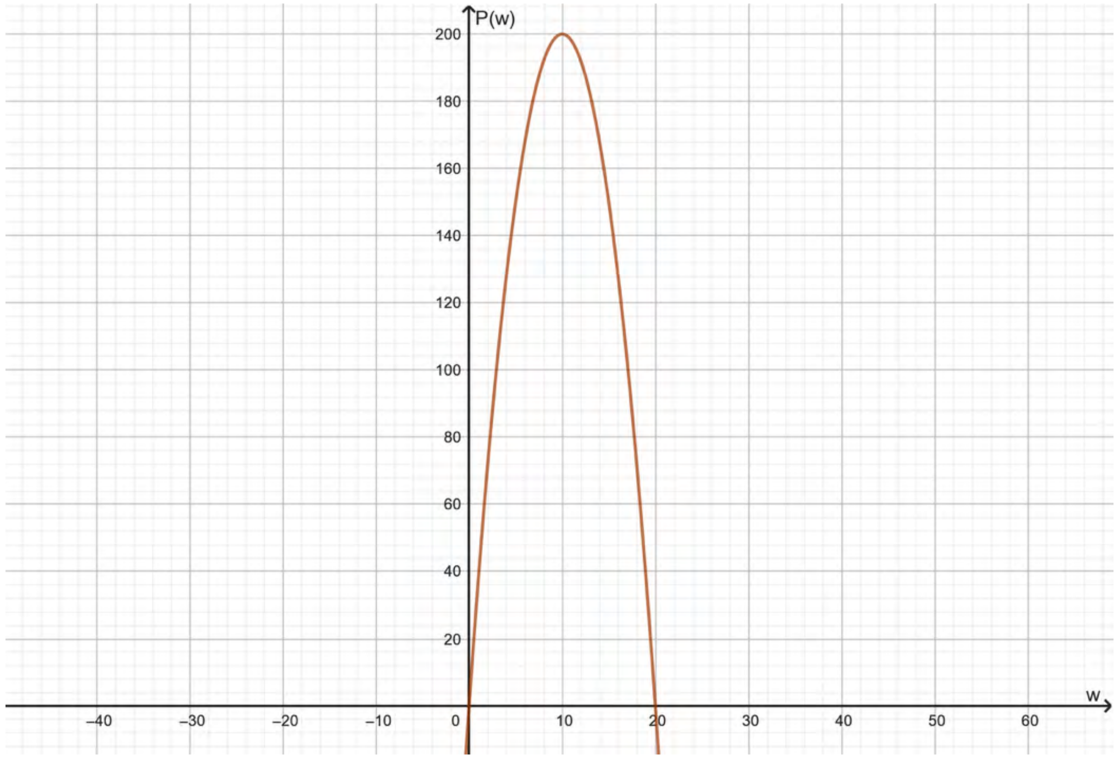


Figure 12.6: Graphical illustration of rice yield

Example 12.19

A market seller in Makola Market determines that the revenue from selling fruits is modelled using $R(p) = -3p^2 + 30p + 200$ where p is the price per kg (in cedis).

a. What price maximises the revenue?

b. What is the maximum revenue?

Solution

Step 1: Let $R(p) = w$. Differentiate w with respect to p

$$\frac{dw}{dp} = -6p + 30$$

Step 2: To determine the value of p at turning point, equate $\frac{dw}{dp}$ to zero.

$-6p + 30 = 0$

$-6p = -30$

$p = 5$

Step 3: Substitute $p = 5$ into the original function.

$R(5) = -3\,(5)^2 + 30(5) + 200 = 275$

a. A price of **GH¢** 5.00 will maximise revenue.

b. The maximum revenue will be **GH¢** 275.00.

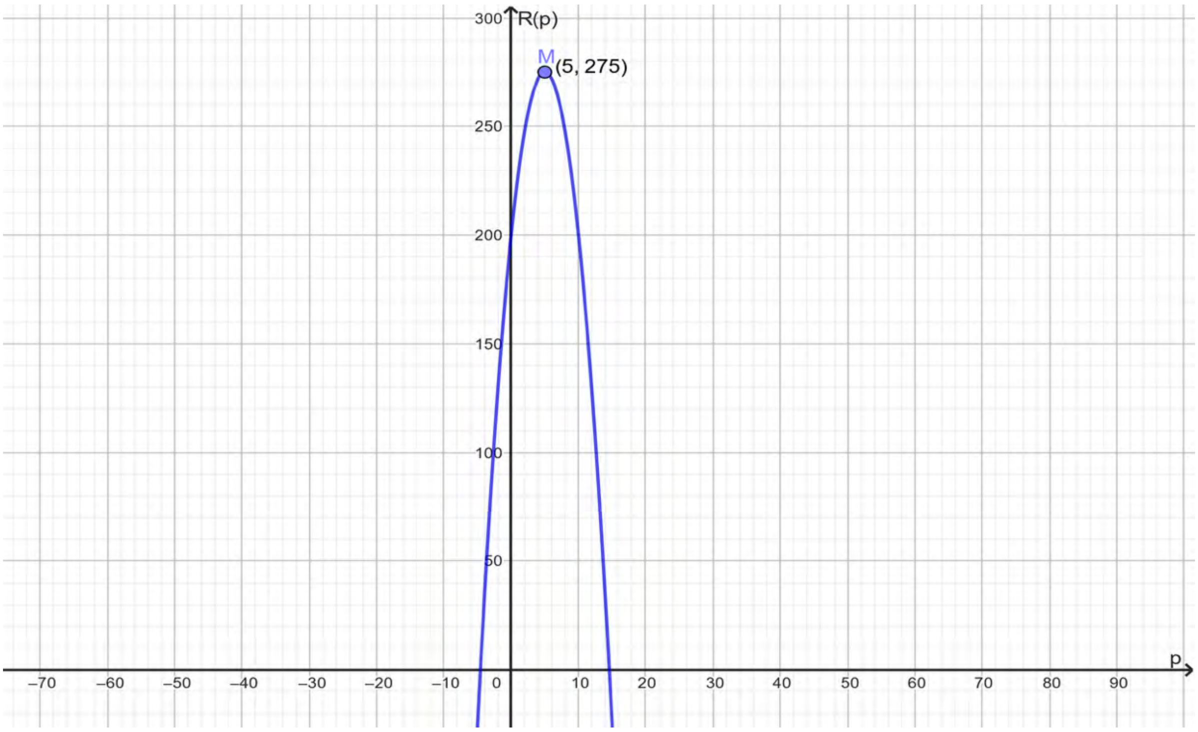


Figure 12.7: Graphical illustration of Revenue

Example 12.20

A storage company designs cylindrical containers for shipping hazardous materials. Regulations require that the total height plus the circumference of the base must not exceed 96 inches.

Determine the dimensions of the cylindrical container that has the greatest possible volume while complying with these regulations.

Solution

Let h represent the height of the cylinder and r be the radius circular base

Step 1: Determine the constraints

$h + 2\pi r \leq 96$, maximum will occur when $h + 2\pi r = 96$

$\therefore h = 96 - 2\pi r$

Step 2: State the volume formula for a cylinder

$V = \pi r^2 h$

Step 3: Substitute the expression for h and simplify

$V = \pi r^2(96 - 2\pi r) = 96\pi r^2 - 2\pi^2 r^3$

Step 4: Differentiate V with respect to r

$\frac{dV}{dr} = 192\pi r - 6\pi^2 r^2$

Step 5: Equate $\frac{dV}{dr}$ to zero and make r the subject.

$192\pi r - 6\pi^2 r^2 = 0$

$6\pi r\,(32 - \pi r) = 0$

$r = \frac{32}{\pi}$

Step 6: Substitute $r = \frac{32}{\pi}$ into V

$V = 96\,\pi\left(\frac{32}{\pi}\right)^2 - 2\pi^2\left(\frac{32}{\pi}\right)^3$

$V = \frac{32^3}{\pi}$

Step 7: Calculate the value of the height

$h = 96 - 2\pi\left(\frac{32}{\pi}\right) = 32$ inches

Step 8: Determine the value of the circumference for the circular base

$2\pi r = 2\pi\left(\frac{32}{\pi}\right) = 64$ inches

The dimensions to yield a maximum volume are when the height is 32inches, circumference of 64inches and a radius of $\frac{32}{\pi}$ inches.

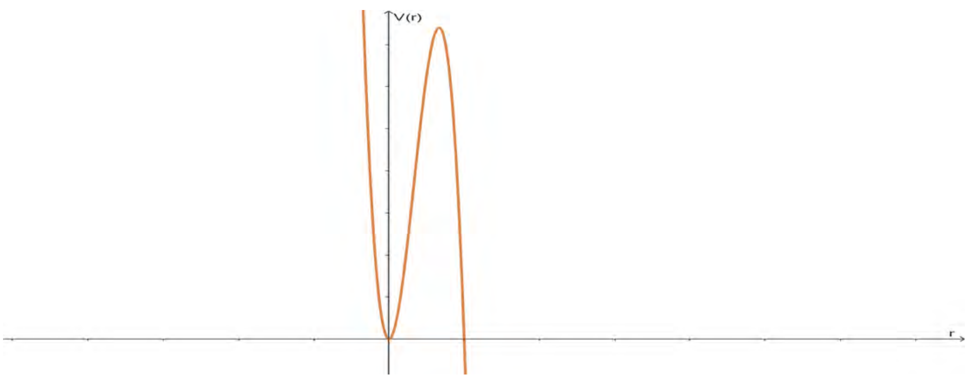


Figure 12.8: Graphical illustration of packing volume

EXTENDED READING

- Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952. (Pages 579 - 637).
- Hesse, C. A. (2005). Effective Elective Mathematics for Senior High School. Akrong Publications. (Pages 447- 475).
- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740. (Pages 339 - 352).

REVIEW QUESTIONS

1. Find the coordinates of the point(s) on the given the curve at which its gradient has the given value.
   a. $y =)x + 3()x - 5($ gradient $= 0$
   b. $y = 5 + 3x - 2x^2$ gradient $= -3$
   c. $y = x + \frac{1}{x}$ gradient $= 2$
   d. $y = \sqrt{x}$ gradient $= 2$
2. Sketch the curve $y = x^3 - 7x^2 + 15x - 9$, indicating clearly its points of intersection with the axes and its turning points.
3. The curve $y = ax^2 + bx + c$ passes through the point (1, 0), y has a minimum value of $-\frac{9}{4}$. when $x = -\frac{1}{2}$,find the values of a, b and c.
4. Find the gradient of the curve at the point (2, 1) on the curve $x^2y - 2xy^2 + y^2 = 1$.
5. Given the implicit equation $xy = 1$. Find the value of $\frac{dy}{dx}$ at the point (1, 1).
6. A curve, the gradient of which at any point is $-2x$, passes through the point (0, 1). Find the equation of the curve and sketch its graph, showing clearly the points where it cuts the x–axis.
7. The curve $y = x^2 - ax + b$ has turning point at (1, 3). Find the values of a and b .
8. Find the coordinate of the point to the curve $y = x^2 + x$ which has a gradient of -1.
9. When a manufacturer makes x items per day:

   the cost function is $C(x) = 720 + 4x + 0.02x^2$ Ghana Cedis:

   the price function is $p(x) = 15 - 0.002x$ GH¢ per item.

   Find the production level that will maximise profits.
10. A 25-foot ladder rests against a vertical wall. If the bottom of the ladder is sliding away from the base of the wall at the rate of 3 ft/s, how fast is the top of the ladder moving down the wall when the bottom of the ladder is 7 ft from the base?

11. Water is running out of a conical funnel at the rate of 1 in^3/s. If the radius of the base of the funnel is 4 *in* and the height is 8 *in*, find the rate at which the water level is dropping when it is 2 *in* from the top.

(The formula for the volume, V, of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.)

12. If a stone is thrown vertically upward with a velocity of 80ft/s, then its height after t seconds is $s = 80t - 16t^2$.

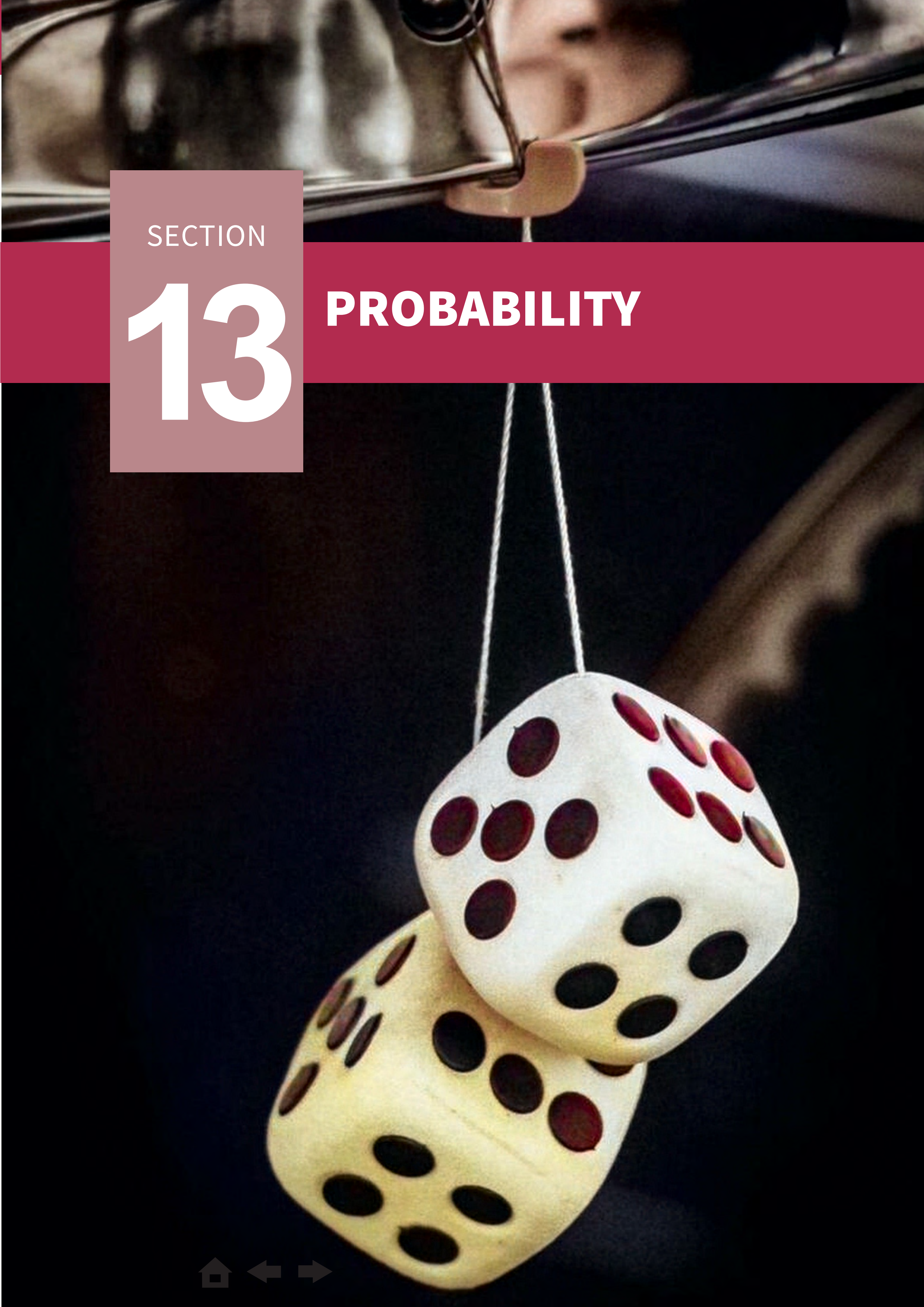
a. What is the maximum height reached by the ball?

b. What is the velocity of the ball when it is 96ft above the ground on its way up and again on its way down?

13. Sketch the graph of $y = x^3 - 3x$ indicating all the necessary points.

SECTION

13 PROBABILITY



HANDLING DATA

Making Predictions with Data

INTRODUCTION

Figure 13.1: A pile of playing cards

From understanding the odds of winning a lottery, predicting weather conditions or evaluating risks like health issues, probability provides a framework for assessing the likelihood of various outcomes. It also prevents common misconceptions, such as assuming random events are predictable. Probability empowers you to critically analyse statistics presented in the media, advertisements or health studies, ensuring you make informed choices. By grasping the basics of probability, you can approach situations with a clearer, more logical perspective. This section details the application of addition and multiplication laws and axioms of probability as well as the investigation of these axioms as a build-up of what you learnt in year one.

KEY IDEAS

- If the events are not mutually exclusive, the probability of either event A or event B occurring is the sum of their individual probabilities minus the intersection probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If two events A and B are dependent, the multiplication rule incorporates the conditional probability of event B occurring, given that event A has already occurred. That is, $P(A \cap B) = P(A) \cdot P(B|A)$
- If two events A and B are independent the probability of both events happening is the product of their individual probabilities: $P(A \cap B) = P(A) \cdot P(B)$
- If two events A and B are mutually exclusive the probability of either event A or event B occurring is the sum of their individual probabilities: $P(A \cup B)=P(A) + P(B)$

APPLYING THE ADDITION AND MULTIPLICATION LAWS AND AXIOMS OF PROBABILITY

The concept of probability assists in quantifying uncertainty, helping us evaluate the likelihood of events in a systematic way. Among its fundamental principles are the **addition law** and the **multiplication law**, which are essential for understanding and solving problems involving combinations of events. We shall explore how the addition and multiplication laws help in solving problems faster and simpler.

Addition Law of Probability

The addition law for probability is a fundamental principle in probability theory that finds the possibility of the occurrence of at least one of two, or more, mutually exclusive or not mutually exclusive events. It is also known as the "OR" rule in probability.

Remember that if events are mutually exclusive, they cannot happen at the same time.

Activity 13.1: Establishing the Addition Law of Probability

In small groups, work through the following activity to establish the addition law. Discuss each stage as you go, ensuring you understand what is happening.

1. State the De Morgan's Law, $(A \cup B)^c = A^c \cap B^c$
2. Remember that, for example, A^c, which can also be written as A', means the complement of A, or the probability that A does *not* occur. $P(A^c) = 1 - P(A)$
3. State the complement rule $P(A \cup B) = 1 - P((A \cup B)^c)$
4. Substitute the De Morgan's result into the complement

   $P(A \cup B) = 1 - P(A^c \cap B^c)$
5. Express $P(A^c \cap B^c)$ in terms of probabilities of A and B.

   From the complement rule, $P)A^c(= 1 - P)A($, $P)B^c(= 1 - P)B($
6. Connect $P(A^c \cap B^c)$ to $P(A \cap B)$.
7. Consider A^c and B^c as independent then, $P(A^c \cap B^c) = P)A^c(\bullet P(B^c($

   (As we can recall that if events are independent then $P(A \cap B) = P(A) \cdot P(B)$)

8. Substitute P)A^c(= 1 – P(A) and P(B^c(= 1 – P)B(into P)A^c(• P(B^c(.

P($A^c \cap B^c$) =)1 – P)A((• (1 – P)B()

9. Substitute the result of P($A^c \cap B^c$) into $P(A \cup B)$.

P($A \cup B$) = 1– [)1 – P)A((•)1 – P)B(( ]

10. Simplify the terms on the RHS of the equation

P(*A* ∪ *B*) = P)A(+P)B(– P)A(• *P*)B(if events A and B are independent

P(*A* ∪ *B*) = P)A(+P)B(– P)A ∩ B(

We also have **P(*A* ∪ *B*) = P)A(+P)B(** if events A and B are mutually exclusive as, in this case, P(A ∩ B) = 0, as by definition there is no overlap as the two events cannot occur simultaneously.

The addition law can also be represented in a Venn Diagram:

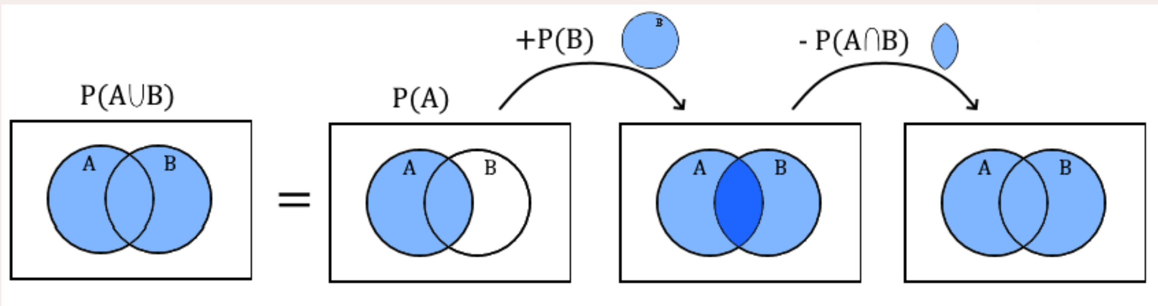


Figure 12.2: Venn diagram

Example 13.1

In a local market in Accra, a vendor sells 30 fruits: 18 are oranges and 12 are bananas. If 5 of the oranges and 3 of the bananas are ripe, what is the probability of randomly selecting either a ripe orange or a ripe banana?

Figure 12.3: A basket of fruits

Solution

Step 1: Extract the totals

Total fruits = 30

Total oranges = 18

Ripe oranges (R_O) = 5

Total Bananas = 12

Ripe bananas (R_B) = 3

Step 2: Determine the probability of randomly picking a ripe orange

$P(R_O) = \frac{5}{30}$

Step 3: Determine the probability of randomly picking a ripe banana

$P(R_B) = \frac{3}{30}$

Step 4: Analyse if a fruit can be both a ripe orange and a ripe banana at the same time

A fruit cannot be both a ripe banana and ripe orange so the intersection is zero as the events are mutually exclusive.

Step 5: Substitute the known probabilities in the addition law and simplify.

$$P(R_O \cup R_B) = P(R_O) + P(R_B) - P(R_O \cap R_B)$$

$$= \frac{5}{30} + \frac{3}{30} - 0 = \frac{8}{30} = \frac{4}{15} = 0.2\dot{6}$$

Therefore, the probability of randomly selecting either a ripe orange or a ripe banana is $0.2\dot{6}$.

Example 13.2

In East Legon residential area, the likelihood of a teenager owning a skateboard is 0.37 while the chance of owning a bicycle is 0.81.

Figure 12.4: Skateboard

If the probability that a teenager owns both a skateboard and a bicycle is 0.36, determine the likelihood that a teenager owns either a skateboard or a bicycle.

Solution

Step 1: Extract each probabilities given

P(skateboard) = 0.37,

P(bicycle) = 0.81,

P(skateboard and bicycle)= 0.36

Step 2: Substitute the known probabilities in the addition law.

P(skateboard or bicycle) = P(skateboard) + P(bicycle)− P(skateboard and bicycle)

P(skateboard or bicycle) = 0.37 + 0.81 − 0.36 = 0.82

Therefore, the probability of a teenager owning either a skateboard or a bicycle is 0.82.

Example 13.3

There are 200 first year students in a school in Tamale, 120 boys and 80 girls. Among the students, 30 boys and 20 girls are members of the science club.

If a prefect needs to be selected from the class, what is the probability of the prefect being either a boy or a student from the science club?

Solution

Step 1: Extract the needed totals

Total number of Boys = 120

Total number of boys in the science club = 30

Total number of science club members = 50

Total number of students = 200

Step 2: Determine the probability of picking a boy

$P(B) = \frac{120}{200}$

Step 3: Determine the probability of being a science club member

$P(S_M) = \frac{50}{200}$

Step 4: Determine the probability of being a boy and a science club member

$P(B \cap S_M) = \frac{30}{200}$

Step 5: Substitute the known probabilities in the addition law and simplify.

$$P(B \cup S_M) = P(B) + P(S_M) - \mathrm{P}\big)\mathrm{B} \cap S_M\big(= \frac{120}{200} + \frac{50}{200} - \frac{30}{200}$$

$$= \frac{140}{200} = 0.7$$

Therefore, the probability of the prefect being either a boy or a student from the science club is 0.7.

Example 13.4

100 tickets were sold at a national lottery point in Ghana.

20 of these tickets win a cash prize, whilst 15 tickets win a gift prize and 10 of these winning tickets win both.

What is the probability of winning any prize?

Solution

Step 1: Extract the needed totals

Number of cash prize tickets = 20

Number of gift prize tickets = 15

Number of these which are cash and gift tickets = 10

Step 2: Determine the probability of picking a cash prize ticket.

$\mathrm{P}(C) = \frac{20}{100}$

Step 3: Determine the probability of picking a gift ticket.

$\mathrm{P(G)} = \frac{15}{100}$

Step 4: Determine the probability of picking a cash and gift prize ticket.

$\mathrm{P}(C \cap G) = \frac{10}{100}$

Step 5: Substitute the known probabilities in the addition law and simplify.

$\mathrm{P})C \cup G(= \mathrm{P})C(+ \mathrm{P)G(} - \mathrm{P})C \cap G($

$= \frac{20}{100} + \frac{15}{100} - \frac{10}{100} = \frac{25}{100} = \frac{1}{4} = 0.25$

Therefore, the probability of winning any prize is 0.25.

Multiplication Laws of Probability

The multiplication rule of probability is essential for calculating the probability of two or more events occurring together. It can be applied to both independent and dependent events.

For *independent events*, where the outcome of one event has no effect on the other event happening, the probability of both events occurring is the product of their individual probabilities: $P(A \cap B) = P(A) \times P(B)$

For *dependent events*, where the outcome of one event affects the other's outcome, the formula adjusts to: $P(A \cap B) = P(A) \times P(B|A)$ where $P(B|A)$ is the probability of event B occurring given that A has occurred.

Let us go through some examples to make this clear.

Example 13.5

Suppose you are playing a game where you roll two six-sided dice.

What is the probability that both dice show a number greater than 3?

Figure 12.5: Two six-sided dice

Solution

Step 1: Find the probability of rolling a number greater than 3 on one die

The numbers greater than 3 are 4, 5, and 6.

$P(x > 3) = \frac{3}{6} = \frac{1}{2}$

Step 2: Determine whether the rolls are independent or not

They are independent as rolling a number greater than 3 on one die has no impact on whether the other die roll is greater than 3.

Step 3: Apply the independent events law of multiplication

$P(D_1 \cap D_2 > 3) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

The probability that both dice show a number greater than 3 is $\frac{1}{4}$.

Example 13.6

In Sunyani, the probability of rain on any given day is 0.6. If it rains, there is a 0.8 chance that sales at a local market will increase.

What is the probability that it rains and sales increase?

Solution

Step 1: Restate the probability that it rains(R)

$P(R) = 0.6$

Step 2: Restate the probability that sales at a local market will increase if it rains

$P(S_I|R) = 0.8$

Step 3: Determine whether rains and sales increase are independent or not

The events are dependent, as it depends on the rain to increase sales at the market.

Step 4: Apply the dependent events law of multiplication

$P(R \cap S_I(= P(R) \times P(S_I|R)$

$P(R \cap S_I(= 0.6 \times 0.8 = 0.48$

INVESTIGATING THE AXIOMS OF PROBABILITY

An axiom is a statement that is accepted as true without proof, serving as a basis for further reasoning or arguments. Axioms are crucial in various fields, particularly in mathematics and logic. They form the starting points from which theorems and other statements are derived. One common axiom is the axiom of equality in mathematics which states that "a number is always equal to itself". This is based on the idea that "things which are equal to the same thing are also equal to each other". This underpins many mathematical operations.

Let us look at the axioms of probability.

The Axioms of probability

The axioms of probability are principles that define the mathematical framework for probability theory. The axioms of focus in this section are:

1. Unit Measure Axiom:

   The probability of the entire sample space (S) is equal to one, P(S)=1.

This axiom asserts that when considering all possible outcomes of a random experiment, at least one outcome must occur, making the total probability of all outcomes equal to 1, or 100%.

2. The probability of an empty set is zero. Thus, $P(\varnothing) = 0$.
3. Non-negativity Axiom:

   For any event B, the probability of B is a non-negative real number, $P(B) \geq 0$

   This means that probabilities cannot be negative and they range from zero to one.

   Hence $0 \leq P(B) \leq 1$ where B is a proper subset of S.

4. Complement Rule:

   $P(B) = 1 - P(B')$

Example 13.7

The probabilities of Kwame and Ama solving a physics problem correctly are 0.6 and 0.9 respectively.

Find the probability that:

a. Only Ama solves it correctly.
b. Both Kwame and Ama fail to solve it correctly.
c. Only one of them solves it correctly.
d. What is the probability of at least one of them solves the problem correctly?

Solution

Let P(A) = P(Ama solving correctly)

Let P(K) = P(Kwame solving correctly)

$P(A) = 0.9, P)A'(= 0.1$

$P(K) = 0.6, P)K'(= 0.4$

a. P(Only Ama solving it correctly) $= P(A) \times P()K'($

   $= 0.9 \times 0.4 = 0.36$

b. P(both Ama and Kwame fail to solve it correctly) $= P()A'(\times P))K'($

   $= 0.1 \times 0.4 = 0.04$

c. P(only one of them solves it correctly) $= P(A) \times P)K'(+ P(K) \times P)A'($

$= 0.9(0.4) + 0.6(0.1) \; = \; 0.42$

d. P(at least one of them solves it correctly) $= P(A) \times P()K'(+ P(K) \times P)A'($
$+ P(A)P(K)$

$= 0.9(0.4) + 0.6(0.1) + 0.9$

$(0.6) \; = \; 0.96$

Alternatively:

P(at least one of them solves it correctly) = 1 – P(Neither of them solve it correctly)

$= 1 - P()A'(\times P)K'($

$= 1 - 0.1)0.4(= 0.96$

Example 13.8

A box contains 40 bulbs of which 3 are defective. Dzifa has agreed to buy the whole box of bulbs if, when she selects 3 at random, her 3 selected contains at most 1 which is defective.

Find the probability that she does not buy the box of bulbs.

Solution

Represent defective bulbs with D

Represent non-defective bulbs with D′

Calculate the probability that the selection contains 2 or more defective bulbs:

$$P(D \geq 2) = \left(\frac{3}{40} \times \frac{2}{39} \times \frac{37}{38}\right) + \left(\frac{3}{40} \times \frac{37}{39} \times \frac{2}{38}\right) + \left(\frac{37}{40} \times \frac{3}{39} \times \frac{2}{38}\right) + \left(\frac{3}{40} \times \frac{2}{39} \times \frac{1}{38}\right)$$

$$= \frac{14}{1235}$$

$$= 0.011 \text{ (to 2sf)}$$

EXTENDED READING

- Baffour, A. (2018). *Elective Mathematics for Schools and Colleges*. Baffour Ba Series. ISBN: P0002417952.
- Mathematical Association of Ghana (2009). Effective Elective Mathematics: Seddco Publishing Limited. ISBN 978 9964 72 4740.

REVIEW QUESTIONS

1. Accra has 9 coffee shops: 4 Vida e caff***é***, 2 Cuppa Cappuccino and 3 Cafe Accra.
   If a tourist selects one shop at random to buy a cup of coffee, find the probability that it is either a Cuppa Cappuccino or a Vida e caff***é***.
2. The research and development centres for three local companies have the following number of employees:
   Noguchi Memorial Institute for Medical Research 200

   CSIR Ghana 700

   Kintampo Health Research Centre (KHRC) 150

   If a research employee is selected at random, find the probability that the employee is employed by Noguchi or KHRC.
3. A single card is drawn at random from an ordinary deck of cards. Find the probability that it is either an ace or a black card.
4. The probability that Roland selects a pizza with mushrooms and/or beef is 0.55 and the probability that the he selects only mushrooms is 0.32.

   If the probability that he selects only beef is 0.17, find the probability of him selecting both items.
5. The probability that Stacy wins a 200m race is $\frac{7}{10}$ and the chances that she wins a 1500m race is $\frac{1}{2}$.
   Assuming that these events are independent, what is the probability that she wins:
   a. both races
   b. only one race?
6. Whenever, I go to the Mandela market, I bump into Kojo 3 days out of 10 and Efua 4 days out of 10.

   Assuming that these events are independent, find the probability that, on a particular market day, I shall meet:
   a. Both Kojo and Efua
   b. Neither of them
   c. At least one of them.

7. In a game of darts, the likelihood that Danful hits the target is $\frac{3}{5}$ and the chances that Alex hits the target is $\frac{2}{3}$. If they both throw the darts together, what is the probability that:
   a. neither of them hits the target
   b. at least one of them hits the target
   c. Exactly one of them hits the target
   d. both hit the target?
8. Two dice are rolled.
   Find the probability of getting:
   a. a sum greater than 9 or less than 4
   b. a sum of 7
   c. a sum of 8, 9 or 10
9. Research on 300 patients revealed that out of 120 hypertensive patients (those with high blood pressure), 67 had high cholesterol levels and of 180 non-hypertensive patients, 54 had high cholesterol levels.

   If a patient is selected at random, find the probability that the patient is;
   a. Non-hypertensive
   b. Hypertensive with high cholesterol levels
   c. Non-hypertensive with low cholesterol levels
10. One box contains 2 pink balls and 1 blue ball. A second box contains 1 pink ball and 2 blue balls.

    A fair coin is tossed and if it falls heads up, the first box is selected and a ball is randomly drawn.

    If the coin falls tails up, the second box is selected and a ball is randomly drawn.

    Find the likelihood of selecting a pink ball.
11. Akwasi is taking a mathematics test which has two papers.

    The probability that Akwasi passes both papers is 0.3.

    The probability that he passes the first paper is 0.6 and that he passes the second paper is 0.5.

    Knowing this information can you tell if passing the two papers are independent?

12. A box contains 5 red balls, 3 blue balls and 2 green balls.

Two balls are drawn randomly one after the other without replacement.

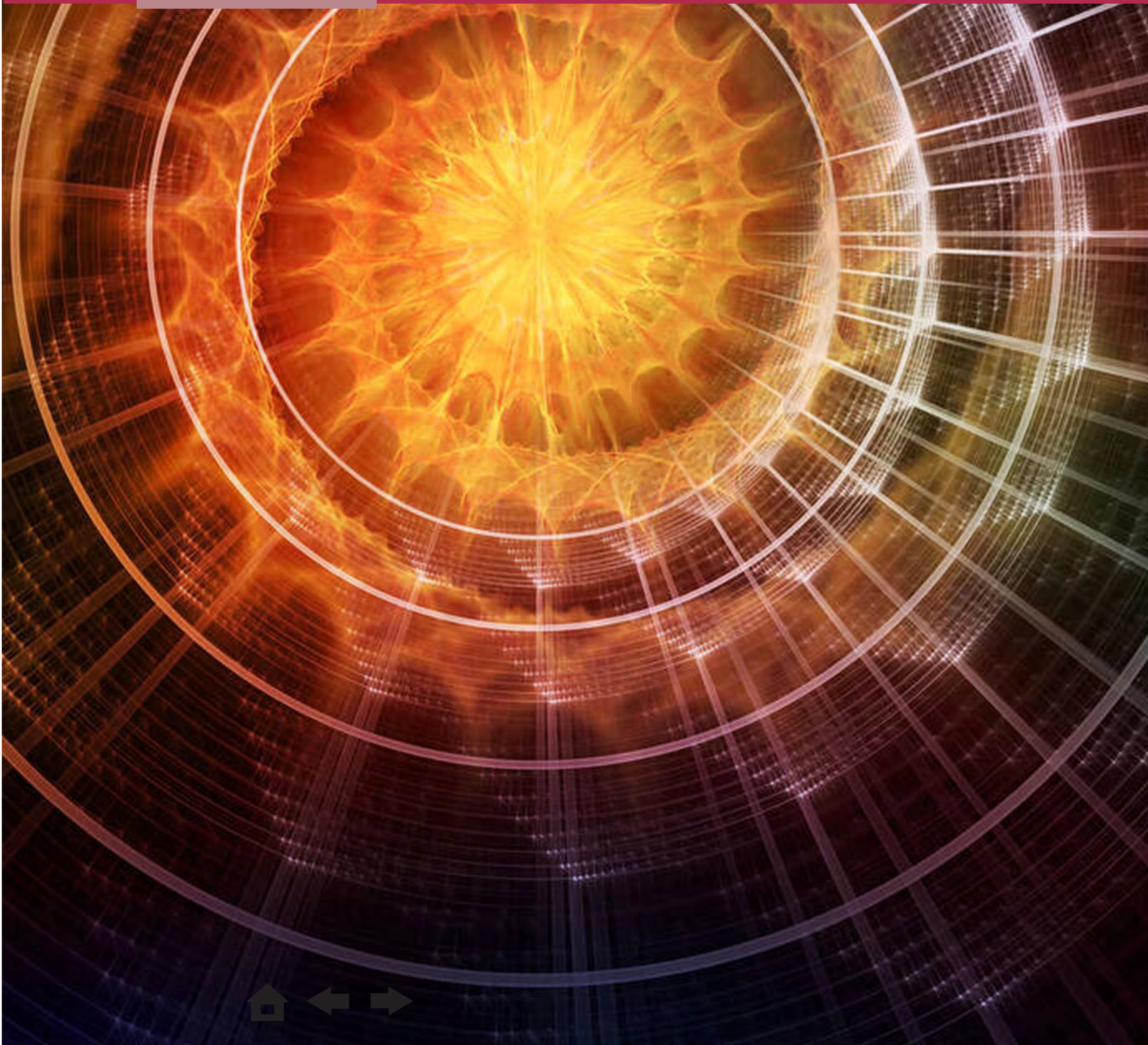
a. What is the probability that at least one of the balls drawn is red?

b. What is the probability that both balls drawn are of the same colour?

c. What is the probability that the first ball is blue, and the second ball is not green?

SECTION

14 COMBINATIONS AND PERMUTATIONS



HANDLING DATA

Making Predictions with Data

INTRODUCTION

In this section, we will explore permutations and combinations. Understanding these concepts will help you develop a strong foundation in counting principles, to enable you to solve complex problems related to arrangements and probability with confidence. Embracing these concepts will empower you to make informed decisions in event planning, sports, research and more. Let us embark on this journey together and unlock the power of counting!

KEY IDEAS

- **Combinations** involve the selection of items where the *order does not matter*. For example, choosing 2 colours from {red, blue, white} results in 3 combinations: {red, blue}, {red, white} and {blue, white}.
- **Key Differences** are that in permutations order matters, but in combinations order does not matter
- **Key Formulas:**

  Permutations: $n_{P_r} = \frac{n!}{(n-r)!}$

  Combinations: $n_{C_r} = \frac{n!}{(n-r)!r!}$
- **Permutations** are the arrangements of items where the *order matters*. For example, for the arrangement of the letters in “CAT” there are 6 permutations: CAT, CTA, ACT, ATC, TCA and TAC.

FUNDAMENTAL COUNTING RULES

Activity 14.1: Revision of Fundamental Counting Rules

Working in pairs discuss how to go about solving this problem.

A school's NSMQ main team consists of two males, A and B, and three Females, E, F and G. The school is to choose only two contestants to sit on stage. As a coordinator, show all the possible pairings and indicate how many ways you can do this.

The principle applied above is the *multiplication principle*, commonly known as the *fundamental concept of counting*, which is a straightforward method for calculating the number of possible outcomes.

This is how it works:

If you have one event that can occur in $\boldsymbol{m}$ different ways and another event that can occur in $\boldsymbol{n}$ different ways, multiply the two numbers together to determine the total number of ways both events can occur consecutively. So, the total number of ways $= \boldsymbol{m} \times \boldsymbol{n}$.

Example 14.1

You have four pairs of socks and three pairs of shoes.

How many different ways can you combine your socks and shoes?

Solution

$4 \times 3 = 12$ ways

This principle is widely used in probability, statistics and various fields of mathematics to solve counting problems efficiently.

Example 14.2

Two coins are flipped and a die is rolled.

Find the number of outcomes for the sequence of events.

Solution

Outcome when two coins are flipped: {HH, HT, TH, TT}

Outcome when a die is rolled: {1, 2, 3, 4, 5, 6}

	1	2	3	4	5	6
HH	HH1	HH2	HH3	HH4	HH5	HH6
HT	HT1	HT2	HT3	HT4	HT5	HT6
TH	TH1	TH2	TH3	TH4	TH5	TH6
TT	TT1	TT2	TT3	TT4	TT5	TT6

The total outcome is 24

Alternatively,

The outcome when two coins are rolled = 4

The outcome when a die is rolled = 6

$4 \times 6 = 24$ ways

Example 14.3

A painter wishes to paint a building with different paints. The categories include

Colour: red, green, yellow, white, blue, brown, black, violet

Type: latex, oil

Texture: flat, semi-gloss, high gloss

Use: indoors, outdoors

How many different combinations of paint are there?

Solution

There are 8 items in the colours category, 2 in the type category, 3 in the texture category and 2 in the use category.

$8 \times 2 \times 3 \times 2 = 96$ ways

Example 14.4

There are 4 blood groups, A, B, AB and O. Blood can also be Rh^+ and Rh^-. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labelled?

Solution

$4 \times 2 \times 2 = 16$ ways

SOLVING PROBLEMS INVOLVING PERMUTATIONS

If you remember back to year 1, you will recall that permutations relate to the act of arranging all the members of a set into some specific sequence or order. The order in which the items are arranged is important.

If we are asked to find the number of ways to arrange r objects from n objects, we us this formula:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Therefore, if we had n objects and we were arranging all of them, we have:

$${}^{n}P_{n} = n!$$

The number of permutations of n different objects taken r at a time with *repetitions* allowed is given by:

$$n^r$$

Let us use this knowledge in some examples.

Example 14.5

How many 3 letter words with or without meaning can be formed from the letters in the word DUST when repetitions are allowed?

Solution

Step 1: Identify the Available Letters

The letters in "DUST" are: D, U, S, T. This gives us a total of 4 distinct letters.

Step 2: Calculate the Number of Possible Arrangements

Since repetitions are allowed, for each of the 3 positions in the word, we can choose any of the 4 letters.

For the first letter: 4 choices (D, U, S, T)

For the second letter: 4 choices (D, U, S, T)

For the third letter: 4 choices (D, U, S, T)

Step 3: Multiply the Choices

The total number of 3-letter combinations can be calculated as follows: Total combinations $= 4 \times 4 \times 4 = 4^3 = 64$

Alternatively**,** we can use $\boldsymbol{n^r}$ since the repetitions are allowed

$n = 4$, $r = 3$

$4^3 = 64$

Example 14.6

A password can be made up of any four-digit combination.

a. How many different passwords are possible?

b. How many are possible if all the digits are odd?

Solution

a. Since the password can be made from any of the four digits, it means repetitions are allowed:
$10 \times 10 \times 10 \times 10 = 10^4 = 10\,000$

b. If all the digits must be odd we are limited to only 5 digits each time, so we have: $5 \times 5 \times 5 \times 5 = 5^4 = 625$

Example 14.7

A school wants to award prizes for 1st, 2nd, 3rd and 4th in a class of 20.

How many possible ways can the prizes be awarded, assuming no two students tie?

Solution

This is choosing 4 students from 20 and the order of those 4 matters, so we have:

$20_{P_4} = 116\,280$

The number of ways is **116 280.**

Example 14.8

We are going to use the letters {a, b, c, d, e, f, g, h} to form a 5-character "password" with no repeated characters.

How many different passwords are possible?

Solution

We are choosing 5 letters from 8 and order matters in a password, so we have:

$8_{P_5} = 8 \times 7 \times 6 \times 5 \times 4 \; = \; 6\,720$ ways

Example 14.9

Kofi, Adzo, Afiba, Haruna, Ayitey, Ghartey and Mercy form the Executives of the Reading Club. They are to choose from amongst themselves a Chairperson, Secretary and Treasurer. No person can hold more than one position. How many different outcomes are possible?

Solution

We are choosing 3 people from 7 and order matters as to the position, so we have:

$7_{P_3} \; = \; 210$ ways

Example 14.10

a. 6 prefects are given 5 special desks at which to work. How many ways can the desks be allocated?

b. In how many ways can the letters of the word 'RECTANGLE' be arranged?

c. How many four-digit numbers can be formed with digits 5, 7, 8 and 9 with no digit repeated?

Solution

a. We have 6 prefects and 5 desks and which desk they are given matters, so we have: $6_{P_5} = \frac{6!}{)6-5(!} \; = \; 6 \times 5 \times 4 \times 3 \times 2 \; = \; 720$ ways

b. There are 9 letters in the word RECTANGLE but the this includes 2 × E's which are indistinguishable from each other, so the number of arrangements is:

$$\frac{9!}{2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} \; = \; \frac{362880}{2} \; = \; 181\,440 \text{ ways}$$

c. $4 \times 3 \times 2 \times 1 \; = \; 24$ ways

SOLVING PROBLEMS INVOLVING COMBINATIONS

Activity 14.2: Revision on Combinations

In pairs or small groups, investigate the difference between combinations and permutations and how and why the formulas differ.

Combinations allow us to choose items from a group where the order of selection is not important. Unlike permutations, where the arrangement matters, in combinations, it does not matter how we arrange the items we select.

Let's solve more problems with the use of combinations.

Example 14.11

15 students vied for 3 slots in a quiz team.

In how many ways can the three slots be filled?

Solution

In this example, order does not matter and there are 15 students from which three will be chosen:

$$15_{C_3} = \frac{15!}{(15-3)!3!} = 455 \text{ ways}$$

Example 14.12

There are 9 men and 11 women in a farming cooperative group.

How many committees of 5 men and 7 women can be formed?

Solution

Order does not matter.

There are 9 men, choosing 5 men = $9_{C_5} = 126$

There are 11 women, choosing 7 women = $11_{C_7} = 330$

$$9_{C_5} \times 11_{C_7} = 126 \times 330 = 41\ 580 \text{ ways}$$

Example 14.13

Nhyira decides to form a band. She needs a bass player, 2 guitarists, a keyboard player and a drummer. She invites applications and gets 7 bass players, 6 guitarists, 5 keyboard players and 4 drummers. Assuming each person applies only once, in how many ways can Nhyira put the band together?

Solution

There are 7 bass players, selecting 1 person = 7_{C_1}

There are 6 guitarists selecting 2-person = 6_{C_2}

There are 5 keyboard players selecting 1 person = 5_{C_1}

There are 4 drummers selecting 1 person = 4_{C_1}

$7_{C_1} \times 6_{C_2} \times 5_{C_1} \times 4_{C_1} = 7 \times 15 \times 5 \times 4 = 2\ 100$ ways

Example 14.14

How many ways can you form a 3-person committee from 5 men and 8 women:

a. with no restrictions

b. the committee must have 1 man and 2 women

c. the committee must have only 1 woman?

Solution

a. With no restrictions:

Total number of people = 5 + 8 = 13

3 are to be selected to form the committee = 13_{C_3} = 286 ways

b. The committee must have 1 man and 2 women

Ways of selecting 1 man from 5 men = 5_{C_1}=5

Ways of selecting 2 women from 8 women = 8_{C_2}= 28

Total number of ways = 5 × 28 = 140 ways

c. The committee has only 1 woman

If the committee has only 1 woman, then there must be two men.

Ways of selecting 1 woman from 8 women = 8_{C_1}= 8

Ways of selecting 2 men from 5 men = 5_{C_2}= 10

Total number of ways = 8 × 10 = 80 ways

Example 14.15

Out of 4 mathematicians and 8 statisticians, a committee consisting of 2 mathematicians and 4 statisticians is to be formed.

In how many ways can this be done if:

a. any mathematician and statistician can be chosen.

b. one particular statistician must be chosen.

Solution

a. Number of ways of choosing 2 mathematicians from 4 mathematicians.

$4_{C_2} = 6$

Number of ways of choosing 4 Statisticians from 8 Statisticians

$8_{C_4} = 70$

Number of ways of choosing the committee $= 6 \times 70 = 420$ ways

b. If one particular statistician must be on the committee, the person can be selected by $1_{C_1} = 1$ way (no surprise there!)

We are then left with 7 statisticians to choose 3 =

$7_{C_3} = 35$

Number of ways of selecting 2 mathematicians from 4 mathematicians

$4_{C_2} = 6$

Total number of ways $= 1 \times 35 \times 6 = 210$ ways.

Example 14.16

6 people are to be chosen for a new committee from 8 males and 8 females. How many different ways can the committee be chosen if:

a. there are no restrictions on who is chosen

b. there must be equal males and females on the committee

c. the current chairperson must be re-elected to the committee, but no other restrictions

d. there must be *at least* 4 females on the committee.

Solution

a. Total members $= 8 + 8 = 16$

6 persons are to be selected to form the committee $= 16_{C_6} = 8\,008$ ways

b. Equal males and females on the committee

Ways of selecting 3 males from 8males = $8_{C_3} = 56$

Ways of selecting 3 females from 8 females = $8_{C_3} = 56$

Total number of ways = 56 × 56 = 3 136 ways

c. If the current chairperson must be re-elected, the person can be selected by $1_{C_1} = 1$

We can then select 5 from the 15 people left $15_{C_5} = 3\,003$

Total number of ways = 1 × 3003 = 3 003 ways

d. At least 4 females must be on the committee means, the number of females can be, 4, 5, 6

Females	Males	Number of ways
4	2	$8_{C_4} \times 8_{C_2} = 70 \times 28 = 1960$
5	1	$8_{C_5} \times 8_{C_1} = 56 \times 8 = 448$
6	0	$8_{C_6} \times 8_{C_0} = 28 \times 1 = 28$

The total number of ways is the sum of the ways obtained in the table:

1960 + 448 + 28 = 2436 ways

Example 14.17

A group consists of 4 girls and 7 boys.

In how many ways can a team of 5 members be chosen if the team has:

a. no girls

b. at least one boy and one girl

c. at least three girls?

Solution

a. No girl means the number of girls = 0 and number of boys = 5

$4_{C_0} \times 7_{C_5} = 1 \text{ x } 21 = 21$ ways

b. at least one boy and one girl, means:

Boys	Girls	Number of ways
1	4	$7_{C_1} \times 4_{C_4} = 7 \text{ x } 1 = 7$
2	3	$7_{C_2} \times 4_{C_3} = 21 \text{ x } 4 = 84$
3	2	$7_{C_3} \times 4_{C_2} = 35 \text{ x } 6 = 210$
4	1	$7_{C_4} \times 4_{C_1} = 35 \text{ x } 4 = 140$

The total number of ways is the sum of the ways obtained in the table:
$7 + 84 + 210 + 140 = 441$ ways

c. at least three girls, means:

Girls	Boys	Number of ways
3	2	$4_{C_3} \times 7_{C_2} = 4 \text{ x } 21 = 84$
4	1	$4_{C_4} \times 7_{C_1} = 1 \text{ x } 7 = 7$

The total number of ways is the sum of the ways obtained in the table:
$7 + 84 = 91$ ways

Example 14.18

A mathematics paper has 13 questions from which a candidate has to answer any 10 questions.

How many different sets of questions can be chosen?

Solution

$13_{C_{10}} = 286$ ways

EXTENDED READING

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REVIEW QUESTIONS

1. A password consists of three digits, 0 through 9, followed by three letters from an alphabet having 26 letters. If repetition of the digits is allowed, but repetition of the letters is not allowed, determine the number of different passwords that can be made.
2. How many three different course meals can be served from a menu that has 5 choices for drinks, 7 choices for vegetables, and 4 choices for desserts?
3. 4 medical doctors are to be selected from a group of 8 to undertake medical outreach. In how many ways can this be done if:
   a. any of the medical doctors can be selected
   a. two particular doctors must be part of the team
   4. In how many ways can 3 boys and 5 girls be chosen from 20 boys and 15 girls to represent a school for a competition?
5. A committee of 5 is to be formed from 7 men and 8 women. Determine the number of ways if:
   a. only 1 man will be on the committee
   a. 2 women will be on the committee
   b. at least 4 men will be on the committee
   c. at most 3 men will be on the committee
6. A bag contains 5 red, 4 black and 3 yellow balls. In how many ways can the following be selected?
   a. 3 red balls
   a. 3 black and 1 red
   b. 2 red and 2 yellow
7. When three dice are rolled with two coins, how many outcomes are possible?
8. Kwame has 5 shirts, 4 pairs of shoes and 8 pairs of trousers. How many outfits can Kwame choose from a shirt, pair of shoes and pair of trousers.
9. The digits 1, 2, 3, 4 and 5 are to be used in 3-digit ID Cards. How many different cards are possible if repetitions are permitted?

10. There are 25 people in a class. Any 5 of these can be chosen for a quiz team. How many ways can they choose this team?

11. An examination paper is divided into two sections, A and B. Section A contains 5 questions and section B contains 8 questions. The candidate is to attempt all the questions in section A and choose 5 from section B. In how many ways can the candidate choose his questions?

12. A committee of 6 is to be formed from 15 board members. How many different ways this can be done if it must:
    a. include 2 particular board members
    b. exclude the chairperson, the secretary and the treasurer?

13. How many 3-digit numbers can be formed from the digits 2, 3, 5, 6, 8 and 9 which are divisible by 5 and none of the digits is repeated?

14. There are three places A, B and C such that 4 roads connect A and B and 5 roads connect B and C. In how many ways can one travel from A to B?

15. The Mathematics Club will select a president, a vice president and a treasurer for the club.
    If there are 20 members in the club, how many different selections of a president, a vice president and a treasurer are possible if each club member can be selected to only one position?

16. A candidate taking the WASSCE Additional Mathematics exam must answer 9 out of 15 questions.
    a. How many choices are available to the candidate if there are no restrictions on the questions chosen?
    b. How many choices are available to the candidate if they must answer the first 5 questions?
    c. How many choices are available to the candidate if they must answer at least 4 of the first five questions?

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GLOSSARY

A sector of a circle - A pie-shaped part of a circle made of the arc along with its two radii.

Adjoint – this is formed by taking the **cofactors** of each element, arranging them in a matrix, and then **transposing** this cofactor matrix.

Arbitrary constant represents the infinite number of antiderivatives of a function, as differentiation removes constant terms.

Axioms - these are foundational statements or principles that are accepted as true without proof. They form the basis from which other truths are derived.

In probability theory, axioms serve as the building blocks for defining probabilities and establishing the rules that govern how probabilities are calculated and combined.

Bivariate data is data which has only two variables or characteristics.

Chain rule: A formula for computing the derivative of the composition of two or more function. If $f(x) = g[h(x)]$, then $f^{I}(x) = g^{I}[h(x)] \times h^{I}(x)$.

Circle - a path traced by all points in a plane which are equidistant from a fixed point in the plane.

Circumference - The length of a complete circle. It is also known as the perimeter of a circle.

Combinations: Selections of items where order does not matter

Complex conjugate – Two complex numbers whose real parts have the same magnitude but opposite in sign, likewise their imaginary parts.

Complex number – any number which can be written in the form a + bi, where a and b are real numbers and i is called the imaginary unit which satisfies $i^2 = -1$.

Compound Interest is interest calculated on the principal and the interest accumulated over the period. The compound interest formula is given below.

Convergence of series – This is the property of an infinite series where the sum of its terms approaches a finite limit as the number of terms increases indefinitely.

Correlation Coefficient is a numerical value that measures the strength and direction of the linear relationship between two variables.

Correlation is a measure of the nature and strength of the relationship between two or more variables.

Critical Point: A point on the graph of a function where the first derivative is zero or undefined, potentially indicating a maximum, minimum or saddle point.

De Morgan's Laws: They are rules used to relate intersections and unions through complements

Decibels (Db) is a logarithmic unit used to measure sound level.

Dependent events – are events where the occurrence of one event affects the probability of the other event. That is, the outcome of one event influences the likelihood of the other event happening.

Depreciation is the "using up" or "the reduction in value" or "reduction in performance" of an item.

Derivative: The measure of the rate of change of a function. The derivative of a function $f(x)$ is denoted by $f'(x)$ or $\frac{dy}{dx}$.

Descartes' Rule of Signs – this is a rule that identifies the number of possible positive and negative roots of a polynomial equation.

Differentiation measures the rate at which a function changes with respect to a change in its input.

Differentiation: The process of finding the derivative of a function.

Divergence of series – This is the property of an infinite series where the sum of its terms does **not** approach a finite limit as the number of terms increases indefinitely.

Dot product - is an operation that takes two vectors and produces a scalar (a single number).

Drone is an unmanned aerial vehicle (UAV) that can be controlled remotely or programmed to fly autonomously.

Exponential Decay is a decrease directly proportional to the current size.

Exponential function (e^x): A function where the rate of growth is proportional to the current value, often to model population growth or chemical reactions.

Exponential Growth is a growth rate directly proportional to the current size

Factorial: The product of all positive integers up to a given number.

For example, 4 factorial = $4! = 4 \times 3 \times 2 \times 1 = 24$

Factors –values of the variable that make the polynomial function equal to zero.

First Derivative ($f'(x)$): The derivative of a function, representing the rate of change or the gradient of the tangent to the function at a given point.

Fundamental Counting Rules: These rules help determine the total number of ways to perform sequences of events.

Half-angles refer to angles that are half of a given angle. In trigonometry, half-angle formulas are used to express trigonometric functions of half an angle in terms of the trigonometric functions of the original angle.

Identity is an equation that is true for all values of the variable(s) within its domain.

Implicit Differentiation: A method to find the derivative of a function given implicitly rather than explicitly.

Independent events - are events where the occurrence of one event does not affect the probability of the other event. The outcome of one event does not change the likelihood of the other event occurring.

Infinite Series – A sum of infinite sequence

Integration is the process of finding the integral of a function. It represents the accumulation of quantities, such as areas under curves or the total of infinitesimal changes.

Limits is a fundamental concept in calculus that describes the value a function has as the input (or variable) approaches a certain value.

Linear Factor Theorem – this is a theorem that states a polynomial function of degree n has a linear factor x - a if and only if a is a root of the polynomial equation.

Locus - A locus (plural = loci) is a set of points which satisfies a given condition or criterion

Loudness of Sound is the intensity or the amount of energy in a sound wave.

Maximum Point: A point on the graph where the function reaches a local peak and the second derivative is less than zero ($f''(x) < 0$).

Minimum Point: A point on the graph where the function reaches a local low and the second derivative is greater than zero ($f''(x) > 0$).

Mutually exclusive events –are events that cannot occur at the same time. If one event occurs, the other cannot.

Natural logarithm (ln): The inverse function of the exponential function, returning the power to which the base number *e* must be raised to produce a given value.

Normal - A straight line which is perpendicular to the tangent

Optimisation– This is the process of finding the best solution or outcome from a set of possible choices, typically by maximising or minimising a specific objective function under given constraints.

Permutations: Arrangements of items where order matters

Polynomial function – these are terms containing the sum of a finite number of terms, each term is the product of a constant and one or more variables raised to a nonnegative integer power.

Probability - is a branch of mathematics that deals with the likelihood of different outcomes in uncertain or random situations. It quantifies uncertainty by assigning a numerical value between 0 and 1 to the likelihood of an event occurring, where 0 means the event will never occur and 1 means the event will certainly occur.

Product rule: A formula for finding the derivative of the product of two functions. If $u(x)$ and $v(x)$ are functions, then $(uv)^I = u^I v + u v^I$

Projecting a vector – determining how much of one vector lies along the direction of another.

Projection of a vector - the projection of a vector *a* onto another vector *b* is a vector that represents how much of *a* lies in the direction of *b*.

Pythagorean Identities are fundamental trigonometric identities derived from the Pythagorean theorem. They express relationships between the sine, cosine and tangent of an angle.

Quadratic Factor Theorem – If a polynomial function of degree n has a quadratic factor $x^2 + bx + c$ then the quadratic equation $x^2 + bx + c = 0$ has real roots, and vice versa.

Quotient rule: A formula for finding the derivative of two functions. If $u(x)$ and $v(x)$ are functions, then $(\frac{u}{v})^I = \frac{u^I v - u v^I}{v^2}$.

Ratio - it is a comparison between two quantities that shows how many times one quantity is contained in another or how one quantity relates to another. It is expressed in the form: $\frac{a}{b}$ or $a{:}b$.

Saddle Point: A critical point where the function flattens, but it is neither a maximum nor a minimum.

Scatter plot (or scatter graph) is a graph which is used to visually show the relationship between variables of bivariate data.

Scrap Value is the value of the item at the end of its useful life.

Second Derivative (f′′(x)): The derivative of the first derivative, used to determine the concavity of a function and classify critical points as maxima, minima or saddle points.

Segment - The area between a chord and an arc of a circle

Tangent - A straight line or plane that touches a curve or curved surface at a point, but if extended does not cross it at that point.

Term: A single part of an expression in the expansion, typically of the form $^nC_r\ a^{n-r} b^r$

The perpendicular height – it is the shortest distance from a point (such as the vertex of a shape) to a line or plane.

Transposing vectors – changing the orientation of a vector from a column vector to a row vector or vice versa.

Univariate data is data with only one variable or a single characteristic.

Variable is a characteristic or measurement that can be determined for a population.

Verification involves checking or proving that a given statement, equation or result is true using logical reasoning, computations or supporting evidence.

Zeros – these are the values of the variables that make the polynomial function equal to zero.

This book is intended to be used for the Year Two Additional Mathematics Senior High School (SHS) Curriculum. It contains information and activities to support teachers to deliver the curriculum in the classroom as well as additional exercises to support learners' selfstudy and revision. Learners can use the review questions to assess their understanding and explore concepts and additional content in their own time using the extended reading list provided.

All materials can be accessed electronically from the Ministry of Education's Curriculum Microsite.

[Scan the QR Code to visit the website]

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