MINISTRY OF EDUCATION

MATHEMATICS ASSOCIATION OF GHANA

Mathematics

for Senior High Schools

Year 2

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FOREWORD

Ghana's new Senior High School Curriculum aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, knowledge, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, progress to further studies and enter the world of work. This is the first time that Ghana has developed a Senior High School Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

The Ministry of Education is proud to have overseen the production of these Learner Materials which can be used in class and for self-study and revision. These materials have been developed through a partnership between the Ghana Education Service, teacher unions (Ghana National Association of Teachers-GNAT, National Association of Graduate Teacher -NAGRAT and the Pre-Tertiary Teachers Association of Ghana- PRETAG) and National Subject Associations. These materials are informative and of high quality because they have been written by teachers for teachers with the expert backing of each subject association.

I believe that, if used appropriately, these materials will go a long way to transforming our Senior High Schools and developing Ghana so that we become a proud, prosperous and values-driven nation where our people are our greatest national asset.

Haruna Iddrisu MP

Minister for Education

SECTION

1 NUMBER SETS

NUMBERS FOR EVERYDAY LIFE

Real Number and Numeration System

INTRODUCTION

In this section, you will learn more about the subsets of the real number system and basic operations on them. Specifically, you will study surds, exponents (indices), logarithms and modular arithmetic. These are fundamental concepts in mathematics which have various applications in everyday life. Surds have applications in physics, number theory and finance. Indices and logarithms are applied in banking, population growth models and engineering. Modular arithmetic is used to design calendar systems and duty rosters. By the end of this section, you will be able to evaluate the relationships between the laws and properties of surds, indices and logarithms and apply them to solve problems in everyday life.

KEY IDEAS

- **Applications of logarithms:** Logarithms have several applications in real life.
- **Concept of logarithms:** Logarithm is the inverse of indices. That is, logarithm can be used to reverse indices.
- **Indices in which the power is an integer:** If the power of an indicial expression is an integer, three situations arise; the integer may be positive, zero or negative**.**
- **Keywords:** Pure surds, mixed surds, compound surds, radical, radicand, rationalisation, conjugate, conjugate pair, exponents, indices, logarithm, base, augment.
- **Laws of logarithms:** Like indices, there are laws, which guide us to perform operations on logarithms.
- **Operations on surds**: This involves the addition, subtraction, multiplication and division of surds.

- **Order of operations involving indices:** when several operations are involved in a computation, there are rules that will help you to do the computation**.**
- **Perfect squares:** A perfect square is a number whose square root is a natural number. In other words, when you square a natural number, you get a perfect square**.**
- **Rationalisation**: This is the process of making the denominator of a given surd rational.
- **Rules of indices:** There are rules that will guide you to perform operations on indices.
- **Simplification of surds:** This is the process of expressing a surd in its simplest form.
- **Surd or radical:** The square root of a number, which is not a perfect square, gives rise to a surd, also known as a radical. These are irrational numbers.

INTRODUCTION AND SIMPLIFICATION OF SURDS

Do you remember the set of perfect squares you learnt in JHS? Write down at least six perfect squares and show them to a classmate. The set of perfect squares $P = \{1, 4, 9, 16, 25, 36, 49 \ldots\}$. When we find the square root of a number, which is **not** a perfect square, we get a surd. Examples of surds include $\sqrt{2}$, $\sqrt{3}$, $-2\sqrt{5}$ and $\frac{2-\sqrt{2}}{3}$.

Perform the activity below with in pairs or in a small group. This activity will help you to generate and simplify surds.

Activity 1.1: Generating Surds Using the Geodot or Graph Paper

Materials needed: Geodot or graph paper, pencils and rulers.

Step 1: Take each small square of four dots on the geodot as a unit square. If you are using a graph paper, take a one-centimetre square as a unit square.

Draw a unit square on your graph or geodot paper as shown below.

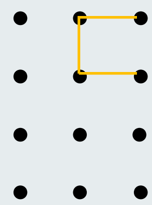


Figure 1.1: Drawing unit square on geodot

Step 2: Draw a two-unit square on the graph or geodot paper. Count the number of unit squares in the square you have drawn. If you counted four-unit squares, you are correct.

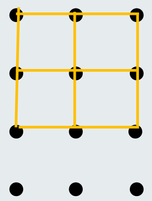


Figure 1.2: Drawiacng two-unit squares on geodot

Step 3: Draw a three-unit square on your graph or geodot paper and count the number of unit squares in it.

Step 4: If you draw a six-unit square, how many unit squares will you find in it? What are your observations? Share your observations with your group.

Step 5: Now, draw a rectangle on your geodot or graph paper and count the number of unit squares in the rectangle you have drawn. Is the number you obtained a perfect square? What conclusion can you draw from this activity? Share your ideas with your group.

You can see that the first four steps of this activity assist you to generate the set of perfect squares. The fifth step gives you an example of a non-perfect square.

If you find the square root of non-perfect squares, you will get surds or radicals like $\sqrt{2}$, $\sqrt{3}$ and so on. Note that because 1, 4 and 9 are perfect squares, $\sqrt{1} = \sqrt{1 \times 1} = 1$, $\sqrt{4} = \sqrt{2 \times 2} = 2$ and $\sqrt{9} = \sqrt{3 \times 3} = 3$.

Activity 1.2: Simplifying Surds Using the Geodot or Graph Paper

Materials needed: Geodot or graph paper, pencils and rulers.

To simplify $\sqrt{12}$, using the geodot, draw a rectangle with 12 dots as shown in the diagram below. What is the largest possible square you can draw within the space of the 12 dots? How many unit squares are in the square you have drawn?

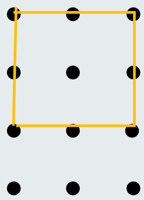


Figure 1.3: Drawing 4-unit squares on geodot

The square in the geodot has 4-unit squares within it. Thus, you can write $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$.

Similarly, to simplify $\sqrt{24}$, express 24 as a product of two numbers, one of which must be a perfect square. That is, $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2 \times \sqrt{6} = 2\sqrt{6}$.

Also, $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$.

Activity 1.3: Finding the Square Root of Perfect Squares Using Prime Factors

The square root of a perfect square can be found by breaking the number down into its prime factors. If the number has an even number of prime factors, we can find the square root easily as shown in the examples below.

Example 1.1

$\sqrt{169} = \sqrt{13 \times 13} = 13$.

Example 1.2

$\sqrt{36} = \sqrt{2 \times 2 \times 3 \times 3} = \sqrt{2 \times 2} \times \sqrt{3 \times 3} = 2 \times 3 = 6$.

Example 1.3

$\sqrt{100} = \sqrt{2 \times 2 \times 5 \times 5} = \sqrt{2 \times 2} \times \sqrt{5 \times 5} = 2 \times 5 = 10$.

There is yet another method for finding the square root of perfect squares. You will be exposed to this method in the following activity.

Activity 1.4: Finding the Square Root of Perfect Squares Using Repeated Subtraction

To use this method to find the square root of a perfect square, list the set of odd numbers and follow the steps below.

Step 1: Subtract the first odd number from the perfect square.

Step 2: Subtract the second odd number from the result of step 1.

Step 3: Subtract the third odd number from the result of step 2.

Step 4: Do this until you get zero.

Step 5: The number of times we subtract an odd number to get zero is the square root of the perfect square.

Example 1.4

Find the square root of 36, using the repeated subtraction method.

1. $36 - 1 = 35$
2. $35 - 3 = 32$
3. $32 - 5 = 27$
4. $27 - 7 = 20$
5. $20 - 9 = 11$
6. $11 - 11 = 0$

Since the subtraction was done six times to get zero, the square root of 36 is 6.

Example 1.5

Find the square root of 49, using the repeated subtraction method.

1. $49 - 1 = 48$
2. $48 - 3 = 45$
3. $45 - 5 = 40$
4. $40 - 7 = 33$
5. $33 - 9 = 24$
6. $24 - 11 = 13$
7. $13 - 13 = 0$

Because it took seven steps to get zero, $\sqrt{49} = 7$.

OPERATIONS WITH SURDS AND THEIR APPLICATIONS

In this lesson, you are going to perform the basic operations on surds. That is, you are going to learn how to add, subtract, multiply and divide surds.

Activity 1.5: Addition and subtraction of surds

You can perform this activity alone or with a partner.

Materials needed: cardboard, board markers and a pair of scissors.

Step 1: Cut out about 20 pieces from the cardboard, making sure the cards you cut out have almost the same shape and size.

Step 2: Write $\sqrt{3}$ on ten of the cards and $\sqrt{6}$ on the remaining pieces.

Step 3: Take some of the cards with $\sqrt{6}$ on them. Ask your partner to take some of the cards with $\sqrt{6}$ on them. How many cards do you have? How will you represent that on paper? How many pieces does your partner have? How will you write what your partner has on paper?

Now, how many cards do you and your partner have altogether? How will you represent the operation you have just performed in mathematical symbols? For example, it could be something like:

$$\begin{bmatrix} \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{bmatrix} + \begin{bmatrix} \sqrt{6} & \sqrt{6} \\ \sqrt{6} & \sqrt{6} \\ \sqrt{6} & \sqrt{6} \end{bmatrix} = 3\sqrt{6} + 6\sqrt{6} = 9\sqrt{6}$$

Step 4: Take some of the $\sqrt{3}$ cards. How many cards did you pick? Give some of your cards to your partner. How many do you have left? How will you represent the operation you have just performed in mathematical symbols?

Step 5: Take some of the cards with $\sqrt{3}$ on them. Let your friend take some of the cards with $\sqrt{6}$ on them. How many cards do you have in total? How will you represent the operation you have just performed on a piece of paper? Show it to your friend.

In Activity 1.5, you have learnt that we can add or subtract like surds (surds with the same radicand) by counting forwards or backwards. However, unlike surds cannot be added or subtracted.

In general, if x and y are rational numbers and z is a non-perfect square,

a $x\sqrt{z} + y\sqrt{z} = (x + y)\sqrt{z}$

b $x\sqrt{z} - y\sqrt{z} = (x - y)\sqrt{z}$

c $x\sqrt{a} \pm y\sqrt{b} = x\sqrt{a} \pm y\sqrt{b}$

Example 1.8

Simplify the following:

1. $3\sqrt{6} + 6\sqrt{6}$
2. $12\sqrt{3} + 6\sqrt{3}$
3. $3\sqrt{6} - 6\sqrt{6}$

Solution

1. $3\sqrt{6} + 6\sqrt{6} = (3 + 6)\sqrt{6} = 9\sqrt{6}$
2. $12\sqrt{3} + 6\sqrt{3} = (12 + 6)\sqrt{3} = 18\sqrt{3}$
3. $3\sqrt{6} - 6\sqrt{6} = (3 - 6)\sqrt{6} = -3\sqrt{6}$

Example 1.9

Simplify $3\sqrt{8} + 2\sqrt{18} - 6\sqrt{2}$

Solution

$3\sqrt{8} + 2\sqrt{18} - 6\sqrt{2}$

$= 3\sqrt{4 \times 2} + 2\sqrt{9 \times 2} - 6\sqrt{2}$

$= 3\sqrt{4} \times \sqrt{2} + 2\sqrt{9} \times \sqrt{2} - 6\sqrt{2}$

$= 3 \times 2 \times \sqrt{2} + 2 \times 3 \times \sqrt{2} - 6\sqrt{2}$

$= 6\sqrt{2} + 6\sqrt{2} - 6\sqrt{2}$

$= 6\sqrt{2}$

Example 1.10

Simplify $2\sqrt{24} + 3\sqrt{32} - 9\sqrt{2}$

Solution

$2\sqrt{24} + 3\sqrt{32} - 9\sqrt{2}$

$= 2\sqrt{4 \times 6} + 3\sqrt{16 \times 2} - 9\sqrt{2}$

$= 2\sqrt{4} \times \sqrt{6} + 3\sqrt{16} \times \sqrt{2} - 9\sqrt{2}$

$= 2 \times 2 \times \sqrt{6} + 3 \times 4 \times \sqrt{2} - 9\sqrt{2}$

$= 4\sqrt{6} + 12\sqrt{2} - 9\sqrt{2}$

$= 4\sqrt{6} + 3\sqrt{2}$

Activity 1.6: Multiplication of surds

Whole class activity

Materials needed: tables and chairs.

Step 1: Place three tables and three chairs in front of the class. Arrange the tables and chairs in such a way that if someone is facing the board, the chair will be on the left and the table will be on the right. Leave about a stride between the first table and chair, the second table and chair and the third table and chair.

Step 2: Two classmates to sit on the chairs and two others to squat under the first two tables.

Step 3: Ask the friends sitting on the chairs to walk together to sit on the third chair and let the friends squatting under the table walk together to squat under the third table.

This activity demonstrates that when multiplying two surds the numbers outside the square root sign multiply each other and the numbers under the square root sign multiply each other.

For example, $\mathbf{a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}}$.

Note that two surds can be multiplied whether they are like or not.

Example 1.11

Express the following in the simplest form:

1. $2\sqrt{3} \times 5\sqrt{2} = 2 \times 5\sqrt{3 \times 2} = 10\sqrt{6}$
2. $6\sqrt{2} \times 2\sqrt{2} = 6 \times 2\sqrt{2 \times 2} = 12\sqrt{4} = 12 \times 2 = 24$

Division of Surds

Division of surds is similar to multiplication of surds.

For example, $\mathbf{\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}}$.

Example 1.12

Simplify the following, leaving your answer in surd form:

1. $\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$
2. $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$
3. $\frac{\sqrt{16}}{\sqrt{9}} = \sqrt{\frac{16}{9}} = \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3} = 1\frac{1}{3}$

Rationalising Monomial Denominators

Mathematicians do not like to see an irrational number as a denominator. For example, mathematicians are uncomfortable with surds like $\frac{1}{\sqrt{2}}$, $\frac{3}{\sqrt{7}}$ and $\frac{5+\sqrt{3}}{3\sqrt{5}}$. Whenever this happens, they remove the irrational number from the denominator. This is done in a process known as rationalisation. That is, the surd expression is manipulated until the irrational denominator becomes rational.

To rationalise a surd with an irrational monomial denominator, multiply both the numerator and denominator by the same irrational expression. When you simplify this expression, you will get an expression whose denominator is a rational number.

The number you must multiply numerator and denominator by is known as the conjugate of the irrational number you want to eliminate from the denominator. The conjugate is the surd, which multiplies a given surd to make it a rational number. For example, $3\sqrt{2} \times \sqrt{2} = 3 \times \sqrt{2} \times \sqrt{2} = 3 \times 2 = 6$. Thus, the best conjugate of $3\sqrt{2}$ is $\sqrt{2}$ even though $3\sqrt{2}$ is also a conjugate. This is because $3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$.

Example 1.6

Rationalise $\frac{1}{\sqrt{2}}$.

Solution

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Example 1.7

Rationalise $\frac{3}{\sqrt{7}}$.

Solution

$$\frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

BASIC CONCEPTS AND PROPERTIES OF EXPONENTS

If 2 is multiplied by itself six times, you can write $2 \times 2 \times 2 \times 2 \times 2 \times 2$. If 2 is multiplied by itself 100 times, imagine how tedious it would be to write it out. Consider the time it will take and the space it will occupy. To simplify this, when 2 is multiplied by itself 100 times, mathematicians express this as 2^{100}, read as 'two to the power 100'. This abbreviation makes writing it far less tedious and time consuming. In the expression 2^{100}, 2 is known as the base and 100 is the power or exponent.

Powers of integers

The number to which we raise a base can be called an index (plural indices), a power or an exponent.

Positive exponents: A positive exponent, n, means that the base is multiplied by itself n times. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ and $-3^4 = -3 \times -3 \times -3 \times -3 = 81$.

Zero exponents: Any number raised to the exponent of zero is always 1. For example, $6^0 = 1$, $-4^0 = 1$ and $\left(\frac{3}{4}\right)^0 = 1$.

Negative exponents: A negative exponent, n, means that the reciprocal of the base is raised to the absolute value of the exponent. For example, $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, $-6^{-2} = \frac{1}{-6^2} = \frac{1}{-6 \times -6} = \frac{1}{36}$.

Scientific calculators can be used to verify answers, but if the base is negative, be sure to put it in brackets before raising to the exponent.

Order of Operations with Exponents

Even though BODMAS is usually interpreted as *bracket, of, division, multiplication, addition and subtraction*, *of* can also be thought of as *order*. *Order* represents roots and exponents. Thus, BODMAS implies that, moving from left to right, you have to expand the brackets first before coming to the order (roots and exponents), followed by division/ multiplication, and then addition/subtraction. That is, if mixture of these operations is found in the same expression.

For example, $3 - \sqrt{16} \div 2 = 3 - 4 \div 2 = 3 - 2 = 1$.

Or, $1 - 2 + 3 \times 4 \div 5^2 = 1 - 2 + 3 \times 4 \div 25$

$$= 1 - 2 + 3 \times \frac{4}{25}$$

$$= 1 - 2 + \frac{12}{25}$$

$$= -1 + \frac{12}{25}$$

$$= -\frac{13}{25}$$

Rules of Exponents

There are important exponent rules to remember when simplifying expressions involving indices or exponents.

They are the product rule, the quotient rule and the power of a power rule.

The Product Rule

$6^3 \times 6^5 = (6 \times 6 \times 6) \times (6 \times 6 \times 6 \times 6 \times 6) = 6^8 = 6^{3+5}$.

In general terms, $\boldsymbol{m^a \times m^b = m^{a+b}}$.

This is the product rule which tells us that when multiplying terms with the same base you add the exponents.

For example, $3^2 \times 3^6 = 3^{6+2} = 3^8$ and $7^{-4} \times 7^6 = 7^{-4+6} = 7^2$

The Quotient Rule

$5^7 \div 5^4 = \dfrac{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5 \times 5}{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5}} = 5^3 = 5^{7-4}$

In general terms, $\boldsymbol{m^a \div m^b = m^{a-b}}$.

This is the quotient rule which tells us that when dividing terms with the same base you subtract the exponents.

For example, $5^3 \div 5^6 = 5^{3-6} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ and $9^{-4} \div 9^6 = 9^{-4+6} = 9^2 = 81$.

The Power of a Power Rule

$(2^3)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3 = 2^{3+3+3+3} = 2^{3 \times 4} = 2^{12}$.

In general terms, $\boldsymbol{(a^m)^n = a^{mn}}$.

For example, $(4^3)^2 = 4^{3 \times 2} = 4^6$ and $(3^{-1})^4 = 3^{-4} = \frac{1}{3^{-4}} = \frac{1}{3^4} = \frac{1}{81}$.

Power of a Product

In general terms, $\boldsymbol{(ab)^m = a^m \times b^m}$

For example: $(2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = 36$

Exponential Equations

An **exponential equation** is an equation in which the **variable appears in the exponent**.

Form:

$a^x = b$

or

$a^{f(x)} = g(x)$

These equations require **logarithms**, **common bases**, or **trial and error** to solve.

Worked Examples

Example 1

Solve:

$2^x = 8$

Solution

Write 8 as a power of 2:

$8 = 2^3 \Rightarrow 2^x = 2^3 \Rightarrow x = 3$

Example 2

Solve:

$5^{x+1} = 125$

Solution

$125 = 5^3 \Rightarrow 5^{x+1} = 5^3 \Rightarrow x + 1 = 3 \Rightarrow x = 2$

Example 3

Solve:

$3^x = \frac{1}{27}$

Solution

$\frac{1}{27} = 3^{-3} \Rightarrow 3^x = 3^{-3} \Rightarrow x = -3$

The following examples will help you to apply the rules of indices.

Example 1.13

Evaluate the following:

1. $81^{\frac{3}{4}}$
2. $8^{\frac{2}{3}}$
3. $64^{\frac{-1}{2}} \times 216^{\frac{2}{3}}$
4. $\left(\frac{27}{216}\right)^{\frac{1}{3}}$

Solution

1. $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^{4 \times \frac{3}{4}} = 3^3 = 27.$
2. $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3 \times \frac{2}{3}} = 2^2 = 4$
3. $64^{\frac{-1}{2}} \times 216^{\frac{2}{3}} = (2^6)^{\frac{-1}{2}} \times (6^3)^{\frac{2}{3}}$

$$= 2^{6 \times \frac{-1}{2}} \times 6^{3 \times \frac{2}{3}}$$

$$= 2^{-3} \times 6^2$$

$$= \frac{1}{2^3} \times 36$$

$$= \frac{1}{8} \times 36$$

$$= \frac{36}{8} = 4.5$$

4. $\left(\frac{27}{216}\right)^{\frac{1}{3}} = \left(\frac{3^3}{6^3}\right)^{\frac{1}{3}}$

$$= \left[\left(\frac{3}{6}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[\left(\frac{1}{2}\right)^3\right]^{\frac{1}{3}}$$

$$= \left(\frac{1}{2}\right)^{3 \times \frac{1}{3}}$$

$$= \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

Real-Life Applications of Indices (Exponents)

Exponents are used in **population growth, compound interest, computing, medicine**, and **radioactive decay**.

Example 1: Compound Interest (Banking)

You invest GH¢1,000 at an annual interest rate of 10% compounded yearly. What will the value be after 3 years?

Formula:

$$A = P(1 + r)^t$$

Where:

- $P = 1000$
- $r = 0.10$
- $t = 3$

Solution

$$\begin{aligned} A &= 1000(1 + 0.10)^3 \\ &= 1000(1.1)^3 \\ &= 1000(1.331) \\ &= GH¢1331 \end{aligned}$$

Example 2: Bacteria Growth (Biology)

A bacteria population doubles every hour. If you start with 200 bacteria, how many will there be after 4 hours?

Formula:

$$\begin{aligned} \text{Population} &= 200 \times 2^4 \\ &= 200 \times 16 \\ &= 3200 \end{aligned}$$

Example 3: Computer Memory (ICT)

Each time you upgrade your memory card, its capacity doubles. If your original card is 8 GB, what will be the capacity after 3 upgrades?

$\text{Capacity} = 8 \times 2^3 = 8 \times 8 = 64\ GB$

Example 4: Radioactive Decay (Physics)

A radioactive substance decays such that half remains every 5 hours. If you start with 80 grams, how much remains after 15 hours?

Solution

15 hours = 3 half-lives

Remaining $= 80 \times \left(\frac{1}{2}\right)^3 = 80 \times \frac{1}{8} = 10\ grams$

Example 5: Sound Intensity (Science/Decibels)

Every 10 dB increase in sound intensity is 10 times more powerful. If a sound is 30 dB more intense than a baseline, how many times more powerful is it?

Solution

$10^{\frac{30}{10}} = 10^3 = 1000\ times$

CONCEPT OF LOGARITHMS

Just as subtraction can be used to reverse addition and division can be used to reverse multiplication, there is a concept that can be used to reverse indices. This concept is called logarithm. Thus, indices and logarithms are closely related. The concept of logarithms is an important tool in many fields. For example, banks use logarithms to calculate the compound interest of their customers; biologists use it to determine the rate at which the population of organisms like bacteria grow in a given medium; chemists use it to find the pH of substances and geologists use it to measure the magnitude of an earthquake.

Here you will learn the rules which govern the concept of logarithms.

Laws of Logarithms

When you see $\log_a y$, a is the base and y is the augment. This is read as logarithm of y to the base a. In short, you can say, log$y$ base a. Logarithms to base 10 are known as common logarithms. For common logarithms, sometimes the base is omitted. Thus, log$y$ = $\log_{10} y$. Common logarithms were very important in computations before the calculator was invented.

1. Since logarithms reverse indices, $\boldsymbol{a^x = y \iff \log_a y = x}$.
2. $\log_a(\boldsymbol{x \times y}) = \log_a \boldsymbol{x} + \log_a \boldsymbol{y}$. You can use this law, known as the addition -product law, if, and only if, the augment is a product.

3. $\log_a(x \div y) = \log_a x - \log_a y$. You can apply this law, known as the subtraction – quotient law, if, and only if, the augment is a quotient.
4. $\log_a y^x = x\log_a y$. You can apply this law, known as the exponent law, if, and only if, the augment has an exponent or power.
5. $\log_a a = 1$. If the augment is the same as the base the answer is always 1
   i. as $a^1 = a$.
6. $\log_a 1 = 0$. No matter the base, the logarithm of one is always zero as $a^0 = 1$
7. $\log_a y = \frac{\log_x y}{\log_x a}$. This law is used to change the base of a logarithm.

Applying the Laws of Logarithms

Apply the laws of logarithms to solve these problems.

Example 1.14

Given that $\log_2 3 = 1.58$ and $\log_2 5 = 2.32$, find the value of:

1. $\log_2 15$
2. $\log_2 45$
3. $\log_2\left(\frac{5}{3}\right)$.

Solution

1. $\log_2 15 = \log_2(3 \times 5) = \log_2 3 + \log_2 5$
   $= 1.58 + 2.32$
   $= 3.90$
2. $\log_2 45 = \log_2(9 \times 5) = \log_2 9 + \log_2 5 \;= \log_2 3^2 + \log_2 5$
   $= 2\log_2 3 + \log_2 5$
   $= 2(1.58) + 2.32$
   $= 3.16 + 2.32$
   $= 5.48$
3. $\log_2\left(\frac{5}{3}\right) = \log_2 5 - \log_2 3$
   $= 2.32 - 1.58$
   $= 0.74$

Example 1.15

Solve for x : $\log_2(x - 3) = 4$

Solution

$\log_2(x - 3) = 4$

$\Longrightarrow x - 3 = 2^4$

$\Longrightarrow x - 3 = 16$

$\Longrightarrow x = 16 + 3 = 19$

Example 1.16

Simplify: $\log_2 8 + \log_2 4$

Solution

$\log_2 8 + \log_2 4$

$= \log_2 2^3 + \log_2 2^2$

$= 3\log_2 2 + 2\log_2 2$

$= 3(1) + 2(1)$

$= 3 + 2$

$= 5$

Alternatively, $\log_2 8 + \log_2 4 = \log_2(8 \times 4)$

$= \log_2 32 = \log_2 2^5 = 5 \log_2 2$

$= 5 \times 1 = 5.$

Example 1.17

Find the value of x given that $\log_3 x + 4\log_x 3 = 5$.

Solution

$\log_3 x + 4\log_x 3 = 5$

$\Longrightarrow \log_3 x + \log_x 3^4 = 5$

$\Longrightarrow \log_3 x + \log_x 81 = 5$

$\Longrightarrow \log_3 x + \frac{\log_3 81}{\log_3 x} = 5$

$\Longrightarrow \log_3 x + \frac{\log_3 3^4}{\log_3 x} = 5$

$\Longrightarrow \log_3 x + \frac{4 \log_3 3}{\log_3 x} = 5$

$\Longrightarrow \log_3 x + \frac{4}{\log_3 x} = 5$

At this point, let $\log_3 x = y$.

$\Longrightarrow y + \frac{4}{y} = 5$

$\Longrightarrow y(y) + \frac{4}{y} \times y = 5y$

$\Longrightarrow y^2 + 4 = 5y$

$\Longrightarrow y^2 - 5y + 4 = 0$

$\Longrightarrow y^2 - y - 4y + 4 = 0$

$\Longrightarrow y(y - 1) - 4(y - 1) = 0$

$\Longrightarrow (y - 1)(y - 4) = 0$

$\Longrightarrow y - 1 = 0$ or $y - 4 = 0$

$\therefore y = 1$ or $y = 4$

$\Longrightarrow \log_3 x = 1$ or $\log_3 x = 4$

$\Longrightarrow \times = 3^1$ or $x = 3^4$

$\Longrightarrow \times = 3$ or $x = 81$.

Time for a quick recap of some of the key concepts of indices and logarithms.

Indices or Exponents or Powers:

1. These are mathematical operations that involve raising a number to a specific power known as the index/exponent. This power or exponent or index represents the number of times a base is multiplied by itself.
2. For example, the expression a^b, where ‘***a***’ is the base and ‘***b***’ is the exponent, means we multiply a by itself b number of times.

Logarithms:

1. Logarithms are the inverse operation of indices.
2. The logarithm of a number with respect to a given base tells us what exponent we need to raise the base to in order to obtain that number. For example, if $b^x = a$, then the logarithm is expressed as $\log_b a = x$ and is read as “the logarithm of a to the base b is x” meaning the logarithm of a to base b is the exponent (x) to which we raise the base (b) to get the number (a).

Let us consider some key relationships between indices and logarithms.

1. Logarithms are the inverse of indices.
2. The laws of logarithms are derived from the laws of indices.

3. The logarithmic function “cancels out” or reverses the effect of exponents in expressions or equations. It allows us to find the unknown exponent when we know the base and the result.
4. Logarithms are useful in solving exponential equations, analysing growth rates and working with data that follow exponential patterns.
5. Indices and logarithms are two sides of the same mathematical coin. They complement each other and play essential roles in various fields including science, engineering and finance.

Let us explore areas in which indices and logarithms are applied.

In Banking and Finance

Indices can be used:

1. to calculate interest rates on investments or loans.
   For example, the formula for Compound Interest, $A = P\left(1 + \frac{r}{n}\right)^{\text{nt}}$
2. in inflation rates to determine the change in prices of items.
3. in analysing stock market prices.

Logarithms can be used:

1. to determine the time it takes an investment to reach a specific value/amount.
2. in assessing financial risk and uncertainty.

In computer science and IT

1. Logarithmic procedures can be used to compress data.
2. Indices are used to calculate compression ratios.
3. Logarithms are used in calculations to secure online transactions. E.g., SSL
4. Google’s search procedures use logarithmic scaling to rank web pages.

Science and Engineering

1. In Physics, indices are used to calculate forces, energies, and velocities
   a. (e.g., kinetic energy $= 1\frac{}{2}mv^2$)
2. Sound intensity is measured in decibels(dB) using logarithmic scales.
3. In Chemistry, pH levels of acids and bases are measured using logarithms.
4. Logarithms are used in signal processing in the compression/decompression in audio and image processing.
5. In epidemic modelling, indices are used to calculate the spread and growth rates of diseases (e.g., exponential growth)
6. The decay of radioactive substances follows an exponential pattern.

7. Half-life (the time for half of a substance to decay) is determined using logarithms.

In our everyday life

1. Logarithmic scales are used to measure sound frequencies in music.
2. In photography, logarithmic exposure compensation adjusts image brightness.

This shows how useful and applicable indices and logarithms are. Explore the internet to look for other areas where they can be applied.

Applications of Common Logarithms

Before calculators were invented, common logarithms were used to make computations because it is easy to work in base 10.

Let us consider some questions and see how to solve them using the concept of common logarithms.

Example 1.18: Compound interest

Mrs. Agbenyegah invested GH¢15 000.00 at 12% interest monthly, how much will she have after 3years? Use the formula: $Amount(A) = P\left(1 + \frac{r}{n}\right)^{nt}$

Solution

$Amount(A) = P(1 + r\,\_\_\,n)^{nt}$

where $P =$ GH¢15 000, $r = 0.12$, $n = 12$, $t = 3$

$A = 15\,000 1 + \frac{0.12}{12}{}^{12\times3}$

$A = 15\,000(1.01)^{36}$

$A = 15{,}000(1.430768784)$

$A \approx$ GH¢ 21461.53

Therefore, Mrs. Agbenyegah will have a total amount of GH¢ 21461.53 after 3years.

Example 1.19: Sound (decibels)

A sound wave has an intensity of 0.01 watts/m². What is its decibel level?

Use the formula: Decibel (dB) level $= 10 \times \log_{10}\left(\frac{I}{I0}\right)$

Where I is the intensity of sound and $I0$ is the reference intensity, typically 10^{-12} W/m

Solution

Decibel Level (L) $= 10 \times \log_{10}\left(\frac{I}{I0}\right)$

$$L = 10\log\left(\frac{0.01}{10^{-12}}\right)$$

$$= 10\left(\log_{10}0.01 - \log_{10}10^{-12}\right)$$

$$= 10\left(\log_{10}10^{-2} - \log_{10}10^{-12}\right)$$

$$= 10\left(-2\log_{10}10 - (-12)\log_{10}10\right)$$

$$= 10(-2 + 12) = 10(10)$$

$$= 100\ decibels(dB)$$

Therefore, the decibel level of the sound wave is **100dB.**

Example 1.20: Sound

A data compression algorithm reduces a file size from 100MB to 10MB.

What is the compression ratio?

Use the formula: Common Ratio $= \log\left(\frac{Original\ Size}{Compressed\ Size}\right)$

Solution

Compression ratio $= \log\left(\frac{100}{10}\right) = \log(100) - \log(10)$

$$= \log_{10}10^2 - \log_{10}10$$

$$= 2\log_{10}10 - \log_{10}10$$

$$= \log_{10}10(2 - 1)$$

$$\approx 1$$

Alternately, Compression ratio $= \log\left(\frac{100}{10}\right) = \log(100) - \log(10)$

$= \log_{10}10^2 - \log_{10}10$

$= 2\log_{10}10 - \log_{10}10$ since $\log_{10}10 = 1$

$= 2(1) - 1$

≈ 1

Or, another alternative, Compression ratio $= \log\left(\frac{100}{10}\right) = \log(10) = \log_{10}10 = 1$

Therefore, the compression ratio is approximately 1

Example 1.21: Biology and Medicine

In a particular country, a population of bacteria grows from 100 to 1000 cells in 3 hours. What is the growth rate of the bacteria?

Use the formula: $r = \dfrac{\log\left(\frac{P(t)}{Po}\right)}{t}$, where:

r is the growth rate (as a decimal)

P(t) is the population at time t

Po is the initial population

t is time (in hours)

Solution

Growth rate $(r) = \dfrac{\log\left(\frac{P(t)}{Po}\right)}{t}$

$r = \dfrac{\log\left(\frac{1000}{100}\right)}{3}$

$r = \dfrac{\log 10}{3} = \dfrac{1}{3} \approx 0.3333$

Therefore, the growth rate (r) is approximately 0.3333 per hour**.**

MODULAR ARITHMETIC

In primary school, you were taught sorting and the divisibility rules of integers. Hopefully you can recall that the divisibility rule of integers is a method used to determine whether one integer is divisible by another without performing the full division.

Some of these rules are;

1. **Divisibility by 2**: A number is divisible by 2 if its last digit is even (0, 2, 4, 6, or 8).
2. **Divisibility by 3**: A number is divisible by 3 if the sum of its digits is divisible by 3. For example (0, 3, 21, 63, 84,102, 258,1095, 8631, etc.)
3. **Divisibility by 5**: A number is divisible by 5 if its last digit is 0 or 5.
4. **Divisibility by 10**: A number is divisible by 10 if its last digit is 0.
5. **Divisibility by 4**: A number is divisible by 4 if the number formed by its last two digits is divisible by 4. E.g. (112, 224, 1092, 9908, etc.)
6. **Divisibility by 9**: A number is divisible by 9 if the sum of its digits is divisible by 9. For example (9, 27, 72, 81, 117, 333, 225, 3141, 4203, 7110, 8001, 9032031, etc.)

Divisibility rules works perfectly because one integer divides another without a remainder and they are a quick way to determine whether an integer is divisible by another. Do you know any other divisibility rules? Discuss them with a classmate.

But what happens when one integer is **not** divisible by another? This results in a **remainder**.

For example, dividing 8 by 3. By the divisibility rule, a number can only be divisible by 3 if the sum of its digits is 3. Therefore, we know that this is not divisible and there will be remainder.

$\frac{8}{3} = 2\frac{2}{3}$. This mean that 8 divides 3, 2 times with a remainder of 2.

This concept is closely related to **modular arithmetic**, which deals with the remainder when one integer is divided by another.

Introduction and Basic Operations with Modulo Arithmetic

With the help of clocks/watches (analogue and digital), perform this activity.

Activity 1.7: Clock arithmetic

For this activity you need both an analogue clock/watch and a digital clock/watch.

Figure 1.4: A clock

Step 1: Identify the various sections or digits on the surface of the clock/watch.

Step 2: Set the time to 12:00 or 00:00 on the analogue clock and 00:00 on the digital clock/watch.

Step 3: Turn the clock forward by 13 hours.

Step 4: Identify the numbers obtained on both the analogue and digital clock/ watch and write them down.

Step 5: Repeat steps 2 to 4 using other number of hours.

Step 6: Discuss your findings/ observations with a classmate.

You will have observed that +13hrs is 1:00 on the analogue clock but 13:00 on the digital clock.

If you went forward by 21hrs from the starting point you have 9:00 on the analogue clock but 21:00 on the digital clock. This is because, the numbers wrap round a fixed number, 12:00 (the starting point), on the analogue clock.

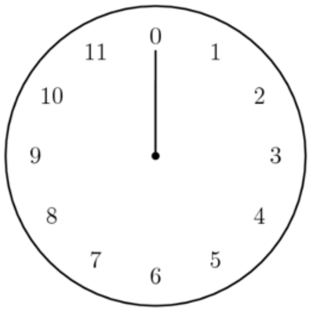


Figure 1.5: A clock

In modular arithmetic, numbers "wrap around" upon reaching a given fixed quantity (this given quantity is known as the modulus) to leave a remainder. Other examples are the days of the week, calendar months, etc.

Modular arithmetic, often referred to as remainder arithmetic, is a mathematical system where an integer is expressed by its remainder after being divided by another integer.

The modulo of any integer is determined by dividing that integer by a specified modulus and identifying the remainder as the result. The term "**mod**" is used as a notation for modulo. That is, "a mod b" represents the remainder when **a** is divided by **b**.

For example, 31(mod 7) signifies the remainder obtained when 31 is divided by 7.
$31 \ (\text{mod } 7) = 31 - 7 = 24 - 7 = 17 - 7 = 10 - 7 = 3$

9 (mod 4) signifies the remainder obtained when 9 divided by 4. Thus,

$9 \ (mod\ 4) = 9 - 4 = 5 - 4 = 1$. Therefore, $9(mod\ 4) = 1$ and can also be written as:
$9 = 1(mod\ 4)$

The remainder can be obtained either by:

1. repeatedly subtracting the modulo number from the given number until subtracting any further will result in negative numbers.

   For example, $7(mod\ 4) = 7 - 4 = 3$. If we try and subtract 4 again, we will have a negative number $\therefore 7(mod\ 4) = 3\ or\ 7 = 3(mod\ 4)$

2. dividing the given number by the modulo number and finding the remainder. For example, if $\frac{d}{c} = A\frac{b}{c}$, where A is the integer part, **c** is the divisor and **b** is the remainder, so the value of b is the answer.

   For example, $7(mod\ 4) = \frac{7}{4} = 1\frac{3}{4}$. The 3 is the remainder.

   $\Longrightarrow 7(mod\ 4) = 3$

Examples 1.22

1. 7 (mod 4) = 3 (remainder after dividing 7 by 4)
2. 2 (mod 10) =2 (remainder after dividing 2 by 10)
3. 9 (mod 3) = 0 (remainder after dividing 9 by 3)
4. 81 (mod 5) =1(remainder after dividing 81 by 5)

Hints in finding the modulo:

1. If the dividend is less than the divisor (modulo number), then the dividend remains the answer. For example:

   a. 3 (mod 7) = 3

   b. 1 (mod 5) = 1

2. Modulo 5 of all numbers ending with any of these numbers; 0, 1, 2, 3, 4 is the last or ending digit. For example:

   a. $1\ (mod\ 5) = 1$

   b. $12\ (mod\ 5) = 2$

   c. $170\ (mod\ 5) = 0$

   d. $1094\ (mod\ 5) = 4$ etc

3. Modulo 5 of all numbers ending with any of these numbers; 5, 6, 7, 8, 9 is found by subtracting 5 from the last digit. For example:

   a. $19(mod\ 5) = 9 – 5 = 4$. Therefore, 19(mod 5) = 4

   b. $36\ (mod\ 5) = 6 - 5 = 1$

Integers for a given Modulo

Let's take for example, the division of any integer by 3:

$\frac{1}{3}$ *has a aremainder of* 1

$\frac{2}{3}$ *has a aremainder of* 2

$\frac{3}{3}$ *has a aremainder of* 0

$\frac{4}{3}$ *has a aremainder of* 1

It is seen that the possible remainders when any integer divided by 3 are 0, 1 and 2. The divisor (3), is called the modulus and the arithmetic is said to be in modulo 3. Modulo 3 can therefore take one of the values {0, 1, 2}. Likewise, modulo 5 can take one of the values {0, 1, 2, 3, 4},

Modulo 6 can take one of the values {0, 1, 2, 3, 4, 5}

Modulo 9 can take one of the values {0, 1, 2, …, 8}

In general, modulo n can take one of the values $\{0, \ldots, n - 1\}$

You can try this with several other integers and discuss your observations with your classmates and teacher.

Modulo of Negative Numbers

The modulo of any negative number is determined by adding the modulo to the number successively until a positive number, which is the answer, is reached. This implies that we repeatedly add the modulus to the negative number until it gives an answer which is positive.

For example, $-5\ (mod\ 4) = -5 + 4 = -1 + 4 = 3$

Since 3 is the first positive number, we end the process that is the answer.

Therefore, $-5\ (mod\ 4) = 3$

Example 1.23

Simplify $-8\ (mod\ 3)$

Solution

$$-8(mod\ 3) = -8 + 3 = -5$$
$$= -5 + 3 = -2$$
$$= -2 + 3 = 1$$
$$\therefore -8(mod\ 3) = 1$$

Example 1.24

What is the value of -33 in modulo 11?

Solution

$$\begin{aligned} -33\ (mod\ 11) &= -33 + 11 = -22 \\ &= -22 + 11 = -11 \\ &= -11 + 11 = 0 \end{aligned}$$

$\therefore -33\ (mod\ 11) = 0$

Equivalent or Congruent Modulo

In modular arithmetic, two integers **a** and **b** are said to be **equal modulo** n, written as a ≡ b(mod n) if they leave the same remainder under the same mod **n**. In other words, a and b are equivalent if their difference is a multiple of **n**.

For example:

1. $30\ (mod\ 7) = 2$ and $23(mod\ 7) = 2$. Therefore, 30 and 23 are said to be equivalent and written as: $30\ (mod\ 7) \equiv 23(mod\ 7) = 2$ or $30 \equiv 23(mod\ 7)$
2. If $16\ (mod\ 5) = 1$ and $91(mod\ 5) = 1$

   Then $16 \equiv 91(mod\ 5)$

Basic Operations with Modulo

The basic operations are addition (+), subtraction (−), multiplication (×) and division (÷). The act of performing these operations is known as simplification of modular arithmetic.

This simplification of modular arithmetic is done by;

1. performing the operation first. (+,−,× ,÷)
2. converting the number obtained to the given modulo.

Addition

The sum of two or more numbers in a given modulo is found by adding the numbers first before converting to the given modulo.

Example 1.25

1. $4 + 5\ (\text{mod}\ 5) = 9\ (mod\ 5) = 4$

2. $8 + 12\ (mod\ 12) = 20\ (mod 12) = 8$
3. $-3 + 31 + 4\ (mod\ 9) = 32\ (mod\ 9) = 5$
4. $(5\ mod\ 3) + (2\ mod\ 3) = (5 + 2)(\ mod\ 3) = 7\ (mod\ 3) = 1$

Subtraction

The difference of two or more numbers in a given modulo is found by first finding the difference between the numbers before converting to the given modulo.

Example 1.26

1. $99\ (mod\ 5) - 34\ (mod\ 5) = (99 - 34)(mod 5) = 65\ (mod\ 5) = 0$
2. $(7\ mod\ 4) - (2\ mod\ 4) = (7 - 2)(\ mod\ 4) = 5\ (mod\ 4) = 1$
3. $12 - 14\ (mod\ 6) = -2\ (mod\ 6) = 4$

Multiplication

The product of two or more numbers in a given modulo is obtained by multiplying the numbers first before converting to the given modulo.

Example 1.27

1. $(3\ mod\ 5) \times (2\ mod\ 5) = (3 \times 2)(mod\ 5) = 6(mod\ 5) = 1$
2. $12 \times 12\ (mod\ 4) = 144\ (mod\ 4) = 0$
3. $6 \times 4\ (mod\ 7) = 24\ (mod\ 7) = 3$

Let us investigate the following properties of operation involving modulo arithmetic.

Properties of Modular Arithmetic

Commutative Property

Commutativity in operations involving modulo (e.g., addition, multiplication under mod)

Addition and multiplication modulo a number are **commutative**:

- $(a + b) mod\ n\ =\ (b + a) mod\ n$
- $(a \times b) mod\ n\ =\ (b \times a) mod\ n$

So **commutativity applies to modular addition and multiplication**.

Note!

But the statement:

"a mod b ≠ b mod a"

is **not** about commutativity. That's just comparing two different operations.

Let's look at an example:

- $7 mod\, 4 \;=\; 3$
- $4 mod\, 7 \;=\; 4$

Clearly,

$7 mod\, 4 \;\neq\; 4 mod\, 7$

So, **modulo itself (i.e., a mod b)** is **not commutative**.

Associative Property

This property is applicable to both addition and multiplication in modulo arithmetic. That is:

$(a + b) + c(mod\ n) = a + (b + c)(mod\ n)$

$(a \times b) \times c(mod\ n) = a \times (b \times c)(mod\ n)$

Let us verify:

For example: $(2 + 3) + 4(mod\ 5) = 2 + (3 + 4)(mod\ 5)$

Solving the Left-Hand Side (LHS) and Right- Hand Side (RHS) concurrently,

$5 + 4(mod\ 5) = 2 + 7(mod\ 5)$

$9\ (mod\ 5) = 9\ (mod\ 5)$

$4 = 4$

Since $(LHS) = (RHS) = 4$, the additive property holds.

Similarly

$(2 \times 3) \times 4\ (mod\ 5) = 2 \times (3 \times 4)(mod\ 5)$

$6 \times 4\ (mod\ 5) = 2 \times 12\ (mod\ 5)$

$24\ (mod\ 5) = 24\ (mod\ 5)$

$4 = 4$

Since $(LHS) = (RHS) = 4$, the multiplicative property also holds.

Therefore, associative property holds for modular arithmetic.

Identity

This refers to the existence of an additive and multiplicative identity in modular arithmetic such that, when combined with any number under a specific operation, leaves that number unchanged.

1. The additive identity in modular arithmetic is **0**

   That is, $a + 0 \ (mod\ n) = 0 + a(mod\ n) = a\ (mod\ n)$

   For example: $3 + 0(mod\ 2) = 0 + 3(mod\ 2) = 3(mod\ 2) = 1$

2. The multiplicative identity in modular arithmetic is **1**

   That is, $a\,.\,1(mod\ n) = 1\,.\,a(mod\ n) = a\ (mod\ n)$

   For example: $5 \times 1(mod\ 3) = 1 \times 5(mod\ 3) = 5(mod\ 3) = 2$

Inverse

This refers to the existence of additive and multiplicative inverses in modular arithmetic such that, when combined with any number under a specific operation, yields the identity element.

1. The **additive inverse** of modular arithmetic is the value that, when added to a number, gives a result of $0\ (mod\ n)$. That is, $a + b = 0\ (mod\ n)$, This means that **b** is the value that, when added to **a**, results in the identity element 0.

Example 1.28

a) $3 + 1\ (mod\ 4) = 0\ (mod\ 4) = 0$, 1 is the additive inverse of 3 in *mod* 4

b) $5 + 2\ (mod\ 7) = 7(mod\ 7) = 0$, 2 is the additive inverse of 5 in *mod* 7

2. The **multiplicative inverse** is the value that, when multiplied by a number, gives a result of $1\ (mod\ n)$. That is $a\,.\,b = 1\ (mod\ n)$

   This means that **b** is the value that, when multiplied by **a**, gives a result that is congruent to 1 modulo n

Example 1.29

a) $3 \times 5 = 1\ (mod\ 7)$, therefore, 5 is the multiplicative inverse of 3 in mod 7

b) $4 \times 3 = 1\ (mod\ 11)$, so 3 is the multiplicative inverse of 4 in mod 11

Explore the internet for more examples, solve them and discuss them with your classmates. How can we find the modulo number if a number and its remainder are given or known but the modulo number is unknown? For example, $24\ (mod\ x) = 3$

Let us go through the activity below to determine the missing value.

Activity 1.8: Finding the modulo number in a given equation

Determine the value of × in 24 $(mod\ x) = 3$

Step 1: Find the difference between the numbers

Thus, $24 - 3 = 21$

Step 2: find all the factors of the value obtained

Factors of $21 = \{1, 3, 7, 21\}$

Step 3: Substitute each of the factors in the given equation.

$when\ x = 1, 24\ (mod\ 1) = 0, x \neq 1$

$when\ x= 3, 24\ (mod\ 3) = 0, x\neq 3$

$when\ x = 7, 24\ (mod\ 7) = 3, x = 7$

$when\ x = 21, 24\ (mod\ 21) = 3, x = 21$

Step 4: identify the ones that make the statement true (ie, satisfy the statement), as the value of the variable.

$x = 7\ or\ 21$ because 7 and 21 both satisfy the equation.

You can search for other methods on the internet.

Now, let's try another example.

Example 1.30

Find p, if $52 = 3\ (mod\ p)$ where $0 \leq p \leq 10$

Solution

$52 = 3(mod\ p)$

$52 - 3 = 49$

Factors of $49 = \{1, 7, 49\}$

$when\ p = 1, 52\ (mod\ 1) = 0, p \neq 1$

$when\ p = 7, 52\ (mod\ 7) = 3, p = 7$

$when\ p = 49, 52(mod\ 49) = 3, p = 49$

7 and 49 both satisfy the equation. However, the question states that $0 \leq p \leq 10$,

Therefore, $p = 7$

Example 1.31

You have 36 cookies and want to share them equally among your friends. If you have a remainder of 4 cookies after sharing, how many friends do you have?

Solution

Let n represent number of friends

$\Longrightarrow 36 = 4\ (mod\ n)$

$36 - 4 = 32$

Factors of $32 = \{1, 2, 4, 8, 16, 32\}$

When $n = 1$, $36\ (mod\ 1) = 0$, $n \neq 1$

When $n = 2$, $36\ (mod\ 2) = 0$, $n \neq 2$

When $n = 4$, $36\ (mod\ 4) = 0$, $n \neq 4$

When $n = 8$, $36\ (mod\ 8) = 4$, $n = 8$

When $n = 16$, $36\ (mod\ 16) = 4$, $n = 16$

When $n = 32$, $36\ (mod\ 32) = 4$, $n = 32$

$n = 8\ or\ 16\ or\ 32$

Therefore, the number of friends is either 8 or 16 or 32

Finding the Number (unknown value) Under a Given Modulo

How do we determine the number which is being operated on by a given modulo? For example, $7x = 1\ (mod\ 2)$. Work through this activity to help you find the value of x.

Activity 1.9: Finding the unknown value in a given equation

Find the possible value(s) of x such that $7x = 1$ (mod 2)

Step 1: List all the possible remainders of the given modulo. They are $\{0,1\}$

Step 2: Substitute each of the remainder values into the given equation.

$when\ x = 0,\ 7(0)\ (mod\ 2) = 0\ (mod\ 2) = 0,\ x \neq 0$

$when\ x = 1,\ 7(1)\ (mod\ 2) = 7(mod\ 2) = 1,\ x = 1$

Step 3: Identify the value(s) of the variable (x) that makes the statement true.

Thus $x = 1$ makes the statement true.

Example 1.32

Find the value(s) of x in the given equation $5x = 2(mod\ 4)$.

Solution

$5x = 2\ (mod\ 4)$ or $5x\ (mod\ 4) = 2$

In $mod(4)$, x can take on values from 0 to $n - 1$, that is $\{0,1,2,3\}$

Substituting:

$when\ x = 0,\ 5(0) mod\ 4 = 0\ (mod\ 4) = 0\ ,\ x \neq 0$

$when\ x = 1,\ 5(1) mod\ 4 = 5\ (mod\ 4) = 1\ ,\ x \neq 1$

$when\ x = 2,\ 5(2) mod\ 4 = 10\ (mod\ 4) = 2\ ,\ x = 2$

$when\ x = 3,\ 5(3) mod\ 4 = 15\ (mod\ 4) = 3\ ,\ x \neq 3$

$\therefore x = 2$

Example 1.33

Given that $2y + 1 = 3\ (mod\ 7)$, what is the value of y?

Solution

$2y + 1 = 3\ (mod\ 7)$

In $mod(7)$, y can take on values of $\{0,1,2,3,4,5,6\}$

By substitution, let's investigate:

When $y = 0,\ 2(0) + 1\ (mod\ 7) = 1\ (mod\ 7) = 1,\ y \neq 0$

When $y = 1,\ 2(1) + 1\ (mod\ 7) = 3\ (mod\ 7) = 3,\ y = 1$

When $y = 2,\ 2(2) + 1\ (mod\ 7) = 5\ (mod\ 7) = 5,\ y \neq 2$

When $y = 3,\ 2(3) + 1\ (mod\ 7) = 7\ (mod\ 7) = 0,\ y \neq 3$

When $y = 4,\ 2(4) + 1\ (mod\ 7) = 9\ (mod\ 7) = 2,\ y \neq 4$

When $y = 5,\ 2(5) + 1\ (mod\ 7) = 11\ (mod\ 7) = 4\ ,\ y \neq 5$

When $y = 6,\ 2(6) + 1\ (mod\ 7) = 13\ (mod\ 7) = 6\ ,\ y \neq 6$

$\therefore y = 1$

Search for other methods that can be used to solve the above examples and look for more examples from the extended reading materials provided below or from the internet.

Modular arithmetic is applicable in several areas and even in our everyday lives.

Let us consider some of these areas;

Applications of Modular Arithmetic

1. In our daily lives, modular arithmetic can be used to determine;

   i. Time: Modular arithmetic can be used to determine what time it is after a number of hours. The 12-hour clock system is essentially modular arithmetic using modulo 12 (for example, 10 hours after 3pm is 1 am).

   ii. Market days in our communities: Market days in our communities rotate in a cyclic manner. For instance, in Ho, market days are every five days counting from the recent market day. Since there are only 7 days in a week, the market days wrap around it.

   The figure below shows how cyclic the days of the week are.

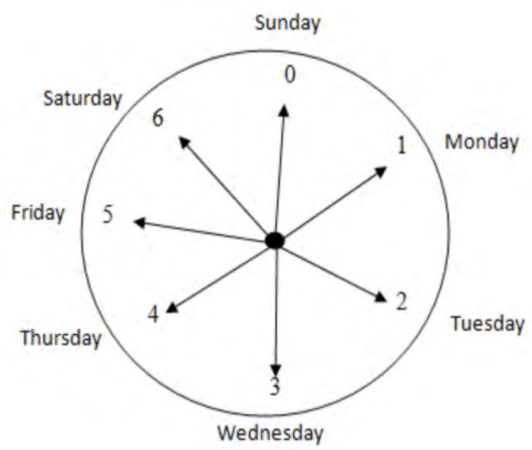


Figure 1.6: Days of the week clock

For example:

a) If the recent market day is Saturday, then the next market day will be Wednesday.

b) If today is Thursday (represented by the number 4), what day will it be in 11 days?

Solution

Thursday $(4) + 11\ days = 15$

$\Rightarrow 15(mod\ 7) = 1,$

From the cycle, 1 represents Monday. Therefore, 11days after Thursday is Monday.

c) The calendar is also designed using the concept of modular arithmetic (i.e. days of the week, months and year)

2. **Finding the remainder**: for example, 27 *mod* 4 = 3 (i.e. the remainder obtained when 27 is divided by 4)
3. **In Computer Science**: It is widely applied in various computer science areas like pseudo random number generation, finite field arithmetic, and generating permutations.
4. **Error Detection and Correction**: It's used in checksums (adding digits and using the remainder for error detection) and cyclic redundancy checks (CRC) for data transmission reliability.

There are other areas you can look out for. Do search for them are discuss them with your classmates.

EXTENDED READING

1. *Aki – Ola Series: Core Mathematics for Senior High Schools in West Africa*, Millennium edition 5 (Pages 33– 40)
2. Baffour A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 398 - 408)

8. Given that $\sqrt{5} = 2.236$, evaluate, correct to one decimal place, $2\sqrt{5}\,(4 - 3/\sqrt{5})$.

9. The length of the hypotenuse, AC, of a right-angled triangle ABC is $12\sqrt{5}$ cm. The length of the side AB is $8\sqrt{3}$cm. Find the length of the side BC, leaving your answer in surd form.

10. Simplify the following:

a. $125^{\frac{-1}{3}}$

b. $\dfrac{8^{\frac{1}{3}} \times 4^{\frac{1}{2}}}{32^{\frac{1}{5}} \times 16^{\frac{1}{2}}}$

11. Find the value of x if $2^{x+2} \times 8^{x} = 1$.

12. Simplify: $\dfrac{3^{5} \times 3\ 2 \times 3^{-9}}{3^{-7} \times 3\ 2}$

13. Find the value of y:

a. $\log_{y} 8 = \dfrac{1}{3}$

b. $\log_{27} 3^{2y} = 2$

14. Without the use of four-figure tables or calculators, evaluate:

a. $\log_{6} 36 + \log_{6} 6\sqrt{6} - \log_{6} 216$

b. $\dfrac{\log_{36} 6}{\log_{49} 7}$

c. $\dfrac{\log_{3} 27 - \log_{3} 3\sqrt{3}}{\log_{3} \frac{1}{\sqrt{3}} - \log_{3} \sqrt{3}}$

15. Find the remainder when 100 is divided by 7

16. What is the remainder when 123456789 is divided by 9?

17. Find the modular multiplicative inverse of 5 (mod 6).

18. Show that 17 and 29 are congruent under modulo 4.

19. If today is Tuesday (represented by the number 3), what day will it be in 15 days?

20. Akosua can only access her results online 14 hours after it has been uploaded on their school's website. If the results were uploaded at exactly 13:00, at what time will she be able to access it?

21. You have 19 cookies and want to share them equally among your friends. If you have a remainder of 4 cookies after sharing, how many friends do you have?

SECTION

2 EQUATIONS AND INEQUALITIES

ALGEBRAIC REASONING

Applications of Expressions, Equations and Inequalities

INTRODUCTION

In this section, you will explore solving **simultaneous linear equations** in two variables using three key methods: **elimination, substitution** and the **graphical method**. You will begin by analysing two linear equations, learning how to eliminate one variable to solve the system algebraically. Substitution, where one equation is rearranged and substituted into the other, will also be practised. You will then solve these equations graphically by plotting and identifying the point where the lines intersect. You will also apply your understanding to **real-life problems**, modelling these scenarios as simultaneous equations and solving them. The section emphasises practical applications, ensuring you can interpret your solutions and relate them to everyday contexts.

KEY IDEAS

- **Elimination method:** Add or subtract equations after manipulating coefficients to eliminate one variable, then solve for the remaining variable.
- **Formulate equations:** Translate the problem's context into two linear equations involving the identified variables.
- **Intersection point:** The point where the two lines intersect. In simultaneous equations this is the solution, if it exists.
- **Linear equation**: is a mathematical equation that shows a straight-line relationship between two variables
- **Simultaneous equation**: is a set of two or more equations that share the same variables and are solved together.
- **Substitution method:** One equation is manipulated to define one variable in terms of the other. This new definition of the variable is then substituted into the second equation and solved.

REVIEW OF PREVIOUS KNOWLEDGE

In year one, you learned how to solve linear equations with one variable. Let's revisit some examples as a quick review. This will lay the groundwork for better understanding of the concept of solving simultaneous equations.

Example 2.1

Solve $4x + 9 = 33$

Solution

Given: $4x + 9 = 33$

Subtract 9 from both sides to isolate the term with x:

$4x + 9 - 9 = 33 - 9$

$4x + 0 = 24$

$4x = 24$

Divide both sides by 4 to solve for x:

$x = \frac{24}{4}$

$x = 6$

Therefore, the solution is $x = 6$.

Example 2.2

Solve $7y - 12 = 2y + 8$

Solution

Given: $7y - 12 = 2y + 8$

Subtract 2y from both sides to get all y terms on one side:

$7y - 2y - 12 = 2y - 2y + 8$

$5y - 12 = 8$

Add 12 to both sides to isolate the term with y:

$5y - 12 + 12 = 8 + 12$

$5y = 20$

Divide both sides by 5 to solve for y:

$y = \frac{20}{5}$

$y = 4$

Therefore, the solution is $y = 4$.

Example 2.3

Solve $\frac{3y}{4} + \frac{1}{8} = 3$

Solution

Given: $\frac{3y}{4} + \frac{1}{8} = 3$

Multiply the entire equation by 8 (LCM of 4 and 8) to remove the fractions:

$8\left(\frac{3y}{4}\right) + 8\left(\frac{1}{8}\right) = 8 \times 3$

$6y + 1 = 24$

Isolate y by subtracting 1 from both sides:

$6y + 1 - 1 = 24 - 1$

$6y = 23$

Divide both sides by 6 to solve for y:

$y = \frac{23}{6}$

So, the solution is $y = \frac{23}{6}$.

Now that we've completed the revision, let's explore how to solve simultaneous equations using the elimination method.

SOLVING SIMULTANEOUS EQUATIONS USING THE ELIMINATION METHOD

To grasp the elimination method for simultaneous equations, keep the following conditions in mind.

1. When solving simultaneous equations, you are looking for a solution which is true for both equations simultaneously, ie at the same time.
2. When we solve simultaneous linear equations with two unknowns, our goal is to find the value of each unknown that satisfies both equations at the same time.
3. Ensure that the coefficient (the number in front) of one of the unknowns is the same in both equations. This will help us eliminate that unknown.
4. We can eliminate this matching unknown by either adding or subtracting the two equations.

Perform the following activities on how to solve simultaneous equation using the Elimination Method

Activity 2.1: Solving simultaneous equation

Solve $x + y = 5$ and $x - y = 1$ simultaneously.

Solution

Let:

Yellow shapes represent positive numbers.

Red shapes represent negative numbers.

x represent circular shapes.

y represent square shapes.

Constants be represented by triangular shapes.

If the variable is x and has a positive value, represent it with a yellow circle. If x has a negative value, use a red circle.

For the variable y, use a yellow square for positive values and a red square for negative values.

For constants, represent positive values with a triangular shape and negative values with a red triangular shape

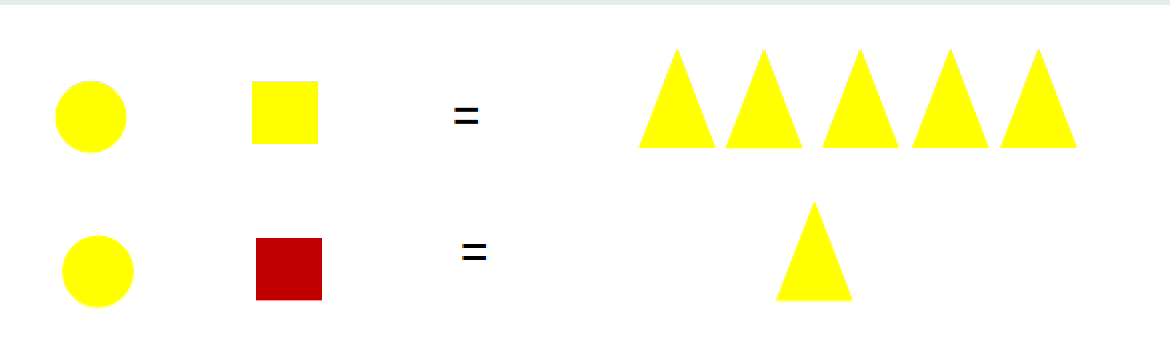
Using the elimination method:

Step 1: Align the equations for clarity

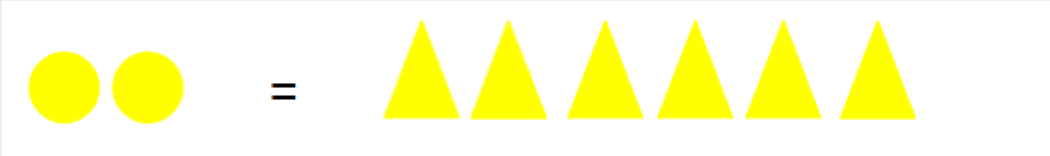
$x + y = 5$ (Equation 1)

$x - y = 1$ (Equation 2)

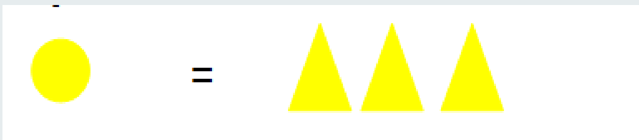
Step 2: Use the specified shapes to represent the different parameters in the given equations: a yellow circle for a positive x, a yellow square for a positive y, a red square for a negative y, 5 yellow triangular shapes for positive constant in equation 1, and a yellow triangular shape for the constant in equation 2.



Step 3: Add the shapes to equations to eliminate y shape



Step 4: You will now share the 6 constants numbers between the two x variables.



$$x = 3$$

Step 5: Substitute the value of $x = 3$ (yellow circles) into either of the original equations to find y (yellow triangles). Using equation 1.

Now, you have $3 + y = 5$

Step 6: Now taking one (circle) and one (triangle) from each side until no circles are left at the left hand side will lead to:

$$y = 2$$

Therefore, the solution is: $x = 3$, $y = 2$

Confirm that this is correct, but substituting into equation 2:

$x - y = 1$

$3 - 2 = 1$ is true, therefore, the solution is correct.

Activity 2.2: Solving simultaneous equation

Solve the simultaneous equations, $2x + 3y = 12$ and $4x - 3y = 6$

Solution

Using the elimination method, follow these steps:

Step 1: Align the equations for clarity

$2x + 3y = 12$ (Equation 1)

$4x - 3y = 6$ (Equation 2)

Step 2: Look for a variable with the same magnitude coefficient. If the two coefficients have the same sign subtract the equations, if they are different, add them. In this case, we add the two equations to eliminate y:

$(2x + 3y) + (4x - 3y) = 12 + 6$

$2x + 4x = 18$

$6x = 18$

Step 3: Now, divide both sides by 6 to find x

$x = \frac{18}{6}$

$x = 3$

Step 4: Substitute the value of $x = 3$ into either of the original equations to find y. We will use Equation 1:

$2x + 3y = 12$

$2(3) + 3y = 12$

$6 + 3y = 12$

Step 5: Isolate y by subtracting 6 from both sides to solve for y

$6 - 6 + 3y = 12 - 6$

$3y = 6$

Step 6: Divide both sides by 3

$y = \frac{6}{3}$

$y = 2$

The solution to the simultaneous equations is: $x = 3$, $y = 2$

To confirm that this is correct, substitute the values we have found into the equation we did not use. In this case, equation 2:

$4(3) - 3(2) = 12 - 6 = 6$, as this is correct, we can be confident in our solution.

Activity 2.3: Solving simultaneous equation

Solve the simultaneous equations, $2x + 3y = 12$ and $4x - y = 5$

Solution

Using the elimination method, follow these steps:

Step 1: Align the equations for clarity

$2x + 3y = 12$ (Equation 1)

$4x - y = 5$ (Equation 2)

Step 2: Match the coefficients

To eliminate one variable, you can multiply (equation 2) by 3 to make the coefficients of y in both equations the same magnitude.

(Note, you could also have chosen to multiply (Equation 1) by 2 to make the coefficients of x the same, either one would give you the same answers.)

$3(4x - y) = 3(5)$

This gives you:

$12x - 3y = 15$ (Equation 3)

Now, we have:

$2x + 3y = 12$, (Equation 1) = The equation we have not touched yet

$12x - 3y = 15$, (Equation 3) = The manipulated equation 2

Step 3: As the y coefficients have different signs; you will add Equation 1 and Equation 3 to eliminate y:

$(2x + 3y) + (12x - 3y) = 12 + 15$

$14x = 27$

Step 4: Now, divide both sides by 14 to find x:

$x = \frac{27}{14}$

Step 5: Now, substitute $x = \frac{27}{14}$ back into one of the original equations to find y. We will use Equation 1:

$$2\left(\frac{27}{14}\right) + 3y = 12$$

$$\frac{54}{14} + 3y = 12$$

$$\frac{54}{14} - \frac{54}{14} + 3y = 12 - \frac{54}{14}$$

$$3y = 12 - \frac{54}{14}$$

$$3y = \frac{57}{7}\ (\textit{Divide both sides by } 3)$$

$$y = \left(\frac{57}{7}\right) \div 3$$

$$y = \left(\frac{57}{7}\right) \times \frac{1}{3} = \frac{57}{21}$$

$$y = \frac{19}{7}$$

The solution to the simultaneous equations is: $x = \frac{27}{14}$, $y = \frac{19}{7}$

To confirm that this is correct, substitute the values we have found into the equation we did not use. In this case, equation 2:

$4\left(\frac{27}{14}\right) - \frac{19}{7} = \frac{54}{7} - \frac{19}{7} = \frac{35}{7} = 5$, as this is correct, we can be confident in our solution.

For further practice, visit YouTube and search for tutorials on solving simultaneous equations using the elimination method. Watching these videos will help you better understand the steps and techniques involved. Make sure to practise what you learn.

Now that you have learned how to solve simultaneous equations using the elimination method, you can apply this knowledge to solve simultaneous equations using the substitution method with ease. Let's move on to solving simultaneous linear equations in two variables using the substitution method.

SOLVING SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES USING THE SUBSTITUTION METHOD

The substitution method involves rearranging one of the equations for one variable, making it the subject of the equation, and then substituting that expression into the other equation. This transforms the system of equations into a single equation with one variable, making it easy to solve. Once we have the value for one variable, we can substitute this in to find the other unknown variable.

Follow these steps to solve simultaneous linear equations with two variables using the substitution method:

1. Rearrange one equation to solve and have one variable as the subject.
2. Substitute the expression for that variable into the other equation.
3. Simplify and solve for the remaining variable.
4. Substitute back to find the other variable.
5. Verify your solution by substituting both values back into the original equation. This step is optional, but it is good practice.

Let's carry out the following tasks on solving simultaneous equations using the substitution method:

Activity 2.4: Solving simultaneous equation

Solve the simultaneous equation $2x + y = 7$ and $3x - y = 4$.

Solution

Step 1: Choose an equation to rearrange to solve for one of the variables:

Choosing Equation 1: $2x + y = 7$

Step 2: Solve for one variable (y):

$y = 7 - 2x$

Step 3: Substitute the new expression for y into Equation 2:

$3x - (7 - 2x) = 4$

Step 4: Simplify and solve for x:

$3x - 7 + 2x = 4$

$5x - 7 = 4$

$5x = 11$

$x = \frac{11}{5}$

Step 5: Substitute x back into one of the original equations to find y. Using Equation 1:

$2(\frac{11}{5}) + y = 7$

$\frac{22}{5} + y = 7$

Subtract $\frac{22}{5}$ from both sides of the equation to isolate y:

$\frac{22}{5} - \frac{22}{5} + y = 7 - \frac{22}{5}$

$y = \frac{13}{5}$

The solution to the simultaneous equations is: $x = \frac{11}{5}$, $y = \frac{13}{5}$

Verification

Equation 1: $2\left(\frac{11}{5}\right) + \frac{13}{5} = \frac{22}{5} + \frac{13}{5} = 7$ (True)

Equation 2: $3\left(\frac{11}{5}\right) - \frac{13}{5} = \frac{33}{5} - \frac{13}{5} = 4$ (True)

Activity 2.5: Solving simultaneous equation

Solve the simultaneous equation $2x + 3y = 12$ and $x - y = 1$

Solution

$2x + 3y = 12$ (Equation 1)

$x - y = 1$ (Equation 2)

Step 1: Choose an equation to solve for one variable. Choosing equation 2:

$x - y = 1$

Step 2: Solve for one variable (x)

$x = y + 1$

Step 3: Substitute the new expression for x into equation 1:

$2(y + 1) + 3y = 12$

Step 4: Simplify and solve for y:

$$2y + 2 + 3y = 12$$
$$5y + 2 = 12$$
$$5y = 10$$
$$y = 2$$

Step 5: Substitute y = 2 back into one of the original equations to find x. Choosing equation 2:

$$x - 2 = 1$$
$$x = 3$$

Verify by plugging (x, y) into both original equations:

Equation 1: $2(3) + 3(2) = 6 + 6 = 12$ (True)

Equation 2: 3 - 2 = 1 (True)

SOLVING LINEAR EQUATIONS IN TWO VARIABLES USING THE GRAPHICAL METHOD

The graphical method involves plotting linear equations on a coordinate plane to visually identify the solution to simultaneous equations. Each equation represents a straight line and the point where the lines intersect is the solution to the system of equations.

Follow the steps below to solve simultaneous equation by using graphical method:

Step 1: Write the equations in slope-intercept form:

a. Convert each equation into the form $y = mx + c$, where:

m is the slope (gradient) of the line.

c is the y-intercept (where the line crosses the y-axis).

Step 2: Plot the lines:

a For each equation, choose at least two values of x, and calculate the corresponding values of y.

b Plot these points on a coordinate plane and draw the line through them, extending the lines across the whole of your graph paper.

Step 3: Find the intersection point:

a After plotting both lines, observe where they intersect.

b The coordinates of this intersection point represent the solution to the system of equations.

Interpreting Graphs of Linear Equations in Two Variables

1. If two lines intersect at a single point, the system has one unique solution. This point is the common solution for both equations.
2. If two lines are parallel, they never intersect, meaning the system has no solution (inconsistent system).
3. If two lines overlap (i.e., they are the same line), the system has infinitely many solutions, as all points on the line satisfy both equations.

Note that when two lines intersect, it shows where two conditions or relationships balance or agree.

Now that you have a basic understanding of how to solve simultaneous equations graphically, let's engage in the following activities to deepen our grasp of these concepts:

Activity 2.6: Graphing simultaneous equation

Solve $-2\boldsymbol{x} + \boldsymbol{y} = 1$ and $\boldsymbol{x} + \boldsymbol{y} = 4$

Solution

Step 1: Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

For equation 1: $-2x + y = 1$

Solve for y: $y = 2x + 1$

For equation 2: $x + y = 4$

Solve for y: $y = -x + 4$

Step 2: Create a table of values for each equation.

Choose values of x to calculate corresponding values of y for each equation.

For Equation 1: $y = 2x + 1$

Table 2.1: *x and y coordinates for equ. 1*

x	y
0	1
1	3
2	5

For Equation 2: $y = -x + 4$

Table 2.2: *x and y coordinates for equ. 2*

x	y
0	4
1	3
2	2

Step 3: Plot the points on a graph

Using the tables from Step 2, plot the points on a graph:

For: $y = 2x + 1$, plot the points (0, 1), (1, 3) and (2, 5).

For: $y = -x + 4$, plot the points (0, 4), (1, 3) and (2, 2).

Step 4: Draw the lines

Draw a straight line through the points for each equation, extending your lines across the whole of the graph paper. The lines represent the two equations as shown below:

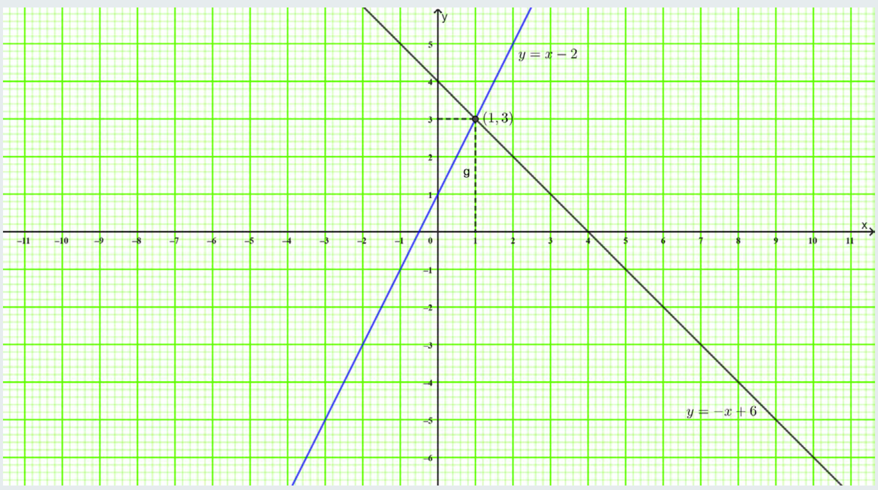


Figure 2.1: Graph of a simultaneous equation

Step 5: Find the point of intersection

The solution to the system of equations is the point where the two lines intersect. In this case, the lines intersect at the point (1, 3).

Step 6: Interpret the solution

The coordinates of the intersection point (1, 3) represent the solution to the system of equations. This means that:

$x = 1$ and $y = 3$

These values satisfy both equations simultaneously.

Activity 2.7: Graphing simultaneous equation

Solve the simultaneous equation $2x + y = 6$ and $x - y = 1$ by using the graphical method

Solution:

Step 1: Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

For equation 1: $2x + y = 6$

Solve for y: $y = -2x + 6$

For equation 2: $x - y = 1$

Solve for y: $y = x - 1$

Step 2: Create a table of values for each equation.

Choose values of x to calculate corresponding values of y for each equation.

For $y = -2x + 6$

Table 2.3: *x and y coordinates for equ. 1*

x	y
0	6
1	4
2	2

For $y = x - 1$

Table 2.4: *x and y coordinates for equ. 2*

x	y
0	-1
1	0
2	1

Step 3: Plot the points on a graph

Using the tables from Step 2, plot the points on a graph:

For $y = -2x + 6$, plot the points (0, 6), (1, 4) and (2, 2).

For $y = x - 1$, plot the points (0, −1), (1, 0) and (2, 1).

Step 4: Draw the lines

Draw a straight line through the points for each equation. The lines represent the two equations as shown below:

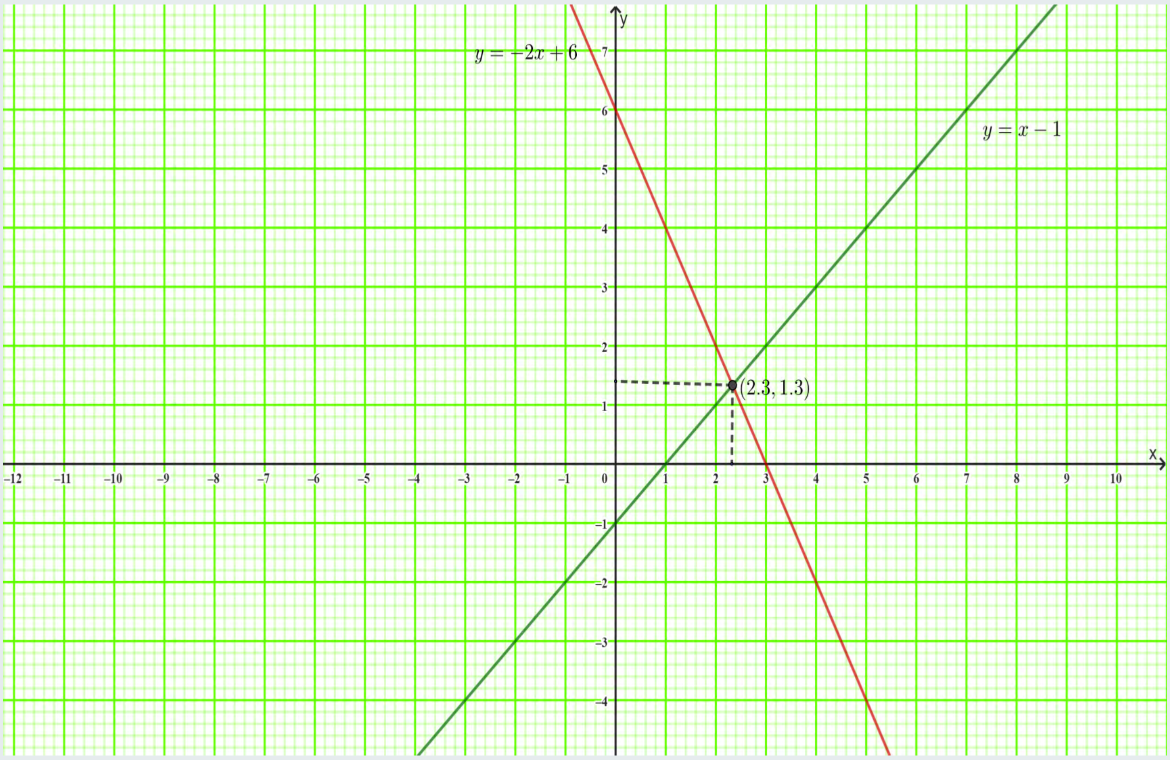


Figure 2.2: Graph of a simultaneous equation

Step 5: Find the point of intersection

The solution to the system of equations is the point where the two lines intersect. In this case, the lines intersect at the point (2.3, 1.3).

Step 6: Interpret the solution

The coordinates of the intersection point (2.3, 1.3) represent the solution to the system of equations. This means that:

$x = 2.3$ and $y = 1.3$

These values satisfy both equations simultaneously.

Activity 2.8: Graphing simultaneous equation

Study the graph below carefully and use it to answer the questions that follow.

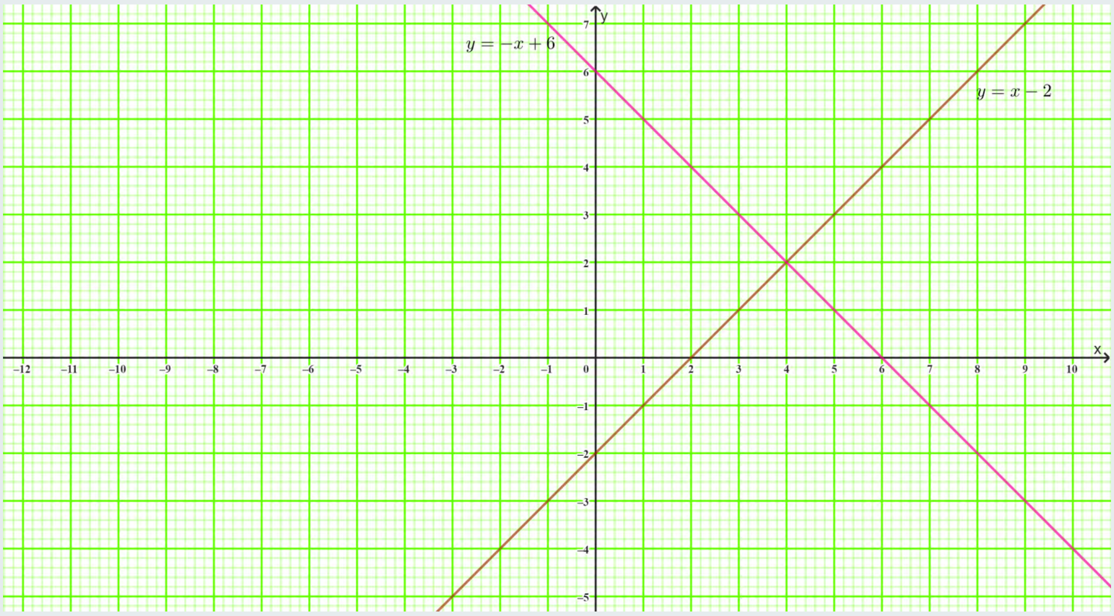


Figure 2.3: Graph of a simultaneous equation

1. What is the gradient of the line with the equation $x - y = 2$?
2. Which of the two lines has a negative gradient?
3. What is the coordinate of the point of intersection of the two lines?
4. Show that the point of intersection is truly the solution to the two systems of equations.

Solution

1. $x - y = 2$

   Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

   $y = x - 2$

   Gradient $= 1$ as the coefficient of the x term in this form tells us the gradient.

2. The line $x + y = 6$ has a negative gradient, as $y = -x + 6$. The negative coefficient of x indicates the negative gradient.
3. The coordinate of the point of intersection of the two lines is (4, 2)
4. (4, 2) means $x = 4$, $y = 2$

   For $x - y = 2$

   $4 - 2 = 2$ True

   For $x + y = 6$

   $4 + 2 = 6$ True

REAL-LIFE PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

To effectively tackle real-life problems involving simultaneous equations, use the following strategies:

1. Read the problem carefully to identify the quantities involved and what needs to be found.
2. Assign variables to the unknowns in the problem
3. Translate the problem into mathematical equations based on the relationships described in the problem.
4. Use methods such as substitution or elimination or graphical to solve the simultaneous equations.
5. Once you have found the values of your variables, interpret them in the context of the original problem.
6. For good practice and to confirm you have the correct solution, substitute your values back into the original equations to verify they satisfy both equations.

Activity 2.9: Real-life problem

Sarah buys 3 apples and 2 bananas for Gh¢7.50. John buys 2 apples and 3 bananas for Gh¢7.00. How much does one apple and one banana cost?

Solution

You need to represent the situation with a system of simultaneous linear equations.

Let:

x represent the cost of one apple.

y represent the cost of one banana

Step 1: Formulate the system of equations from the problem

Sarah buys 3 apples and 2 bananas for Gh¢7.50

$3x + 2y = 7.50$ (Equation 1)

John buys 2 apples and 3 bananas for Gh¢7.00.

$2x + 3y = 7$ (Equation 2)

Step 2: To eliminate x, we will multiply the first equation by 2 and the second equation by 3, making the coefficients of x equal to 6 in both equations.

Multiply Equation 1 by 2:

$2(3x + 2y) = 2(7.50)$

$6x + 4y = 15$ (Equation 3)

Multiply Equation 2 by 3:

$2x + 3y = 7$

$3(2x + 3y) = 3(7)$

$6x + 9y = 21$ (Equation 4)

Step 3: The coefficients of x are the same sign, so we subtract Equation 3 from Equation 4 to eliminate x.

$(6x + 9y) - (6x + 4y) = 21 - 15$

$6x - 6x + 9y - 4y = 6$

$5y = 6$

Step 4: Divide both sides by 6 to find y:

$y = \frac{6}{5} = 1.20$

The cost of one banana is Gh¢1.20

Step 5: Now that you have the cost of a banana, we can substitute $y = 1.20$ into one of the original equations to find x, the cost of an apple. Using equation 1:

$3x + 2(1.20) = 7.50$

$3x + 2.40 = 7.50$

Step 6: Subtract 2.40 from both sides of the equation to isolate x:

$3x + 2.40 - 2.40 = 7.50 - 2.40$

$3x = 5.10$

Step 7: Divide both sides by 3 to find x

$x = \frac{5.10}{3} = 1.70$

The cost of one apple is Gh¢1.70.

Activity 2.10: Real-life problem

A market woman in Accra sells plantains and tomatoes. She sells 2 plantains and 3 tomatoes for Gh¢12 and she sell 1 plantain and 1 tomato for Gh¢5. How much does 1 plantain cost and how much does 1 tomato cost?

Solution

Let:

x = cost of 1 plantain

y = cost of 1 tomato

$2x + 3y = 12$ (2 plantains and 3 tomatoes for GH¢12): Equation 1

$x + y = 5$ (1 plantain and 1 tomato for Gh¢5): Equation 2

Step 1: Rearrange equation 2 for x to be the subject:

$x = 5 - y$

Step 2: Substitute x into Equation 1:

$2(5 - y) + 3y = 12$

Step 3: Simplify:

$10 - 2y + 3y = 12$

$10 + y = 12$

$y = 2$

The cost of one tomato is Gh¢2.00

Step 4: Substitute y back into any of the original equation. Using equation 2:

$x + 2 = 5$

$x = 3$

The cost of one plantain is Gh¢3.00

Step 5: Confirm your values work and you have the correct solution.

Activity 2.11: Real-life problem

A farmer in the Northern Region sells maize and soybeans. He sells 2 bags of maize and 1 bag of soybeans for Gh¢150. If he sells 1 bag of maize for Gh¢40, how much does 1 bag of soybeans cost?

Solution

$2x + y = 150$ (2 bags of maize and 1 bag of soybeans for Gh¢150): Equation 1

$x = 40$ (1 bag of maize costs Gh¢40): Equation 2

Step 1: Substitute $x = 40$ into Equation 1:

$2(40) + y = 150$

Step 2: Simplify:

$80 + y = 150$

Step 3: Solve for y:

$y = 150 - 80$

$y = 70$

$x = 40$ (cost of 1 bag of maize)

$y = 70$ (cost of 1 bag of soybeans)

Therefore, 1 bag of soybeans costs Gh¢70.

Activity 2.12: Real-life problem

In the local market of **Chinderi,** a town in the **Oti Region**, the price of a pineapple is Gh¢2.00 and the price of a mango is Gh¢3.00. **Nana Okoegye Abraham**, the chief of Chinderi, went to the market and bought a total of 10 fruits, spending Gh¢24.00.

How many pineapples and how many mangoes did he buy?

Solution

Let:

x represent the number of pineapples

y represent the number of mangoes

$x + y = 10$ (total fruits): Equation 1

$2x + 3y = 24$ (total cost): Equation 2

Step 1: Rearrange Equation 1 for x

$x = 10 - y$

Step 2: Substitute x into Equation 2:

$2(10 - y) + 3y = 24$

Step 3: Simplify:

$20 - 2y + 3y = 24$

$20 + y = 24$

Step 4: Solve for y:

$y = 4$

Step 5: Substitute y back into Equation 1:

$x + 4 = 10$

$x = 6$

$x = 6$ (number of pineapples)

$y = 4$ (number of mangoes)

Therefore, Nana Okoegye Abraham bought 6 pineapples and 4 mangoes.

Using Software Demos to Graph Simultaneous Equations

Objective

You will learn how to use the Demos software on your phone to graph simultaneous equations with two variables. This activity will help you visualise solutions to equations and understand their graphical representations. Follow these steps to install and use the app with the guidance of your teacher.

Step 1: Download and Install Demos Software

For Android Users:

1. Open the **Google Play Store** on your Android phone.
2. In the search bar, type **"Demos Graphing Calculator"** and select the app from the results.
3. Tap **Install** to download the app.
4. Once installed, open the app and allow any necessary permissions.

For Apple Users (iOS):

1. Open the **App Store** on your iPhone or iPad.
2. In the search field, type **"Demos Graphing Calculator"** and select the app from the results.
3. Tap **Get** and, if prompted, confirm with your Apple ID or passcode.
4. Once installed, open the app and allow any necessary permissions.

Step 2: Prepare for Graphing

With your teacher's guidance, you will:

1. Open the Demos app on your phone.
2. Enter the simultaneous equations as instructed by your teacher.
3. Use the app's graphing features to view the intersection points and understand solutions visually.

Note: Remember to bring your phone to class fully charged or have it ready at home during designated learning time. If you have questions about downloading, installing or using the software, ask your teacher for help.

EXTENDED READING

1. *Akrong Series: Core mathematics for Senior High Schools* New International Edition (Pages 196 – 201)
2. Aki – Ola series. Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 93– 102)
3. Baffour Asamoah, B. A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 438 - 458)

REVIEW QUESTIONS 2

1. Solve these simultaneous equations using the elimination method.

a. $5x + 2y = 14$

$3x + y = 7$

b. $2x + 3y = 12$

$x - 4y = -6$

c. $6x + y = 10$

$4x - y = 10$

d. $7x + 3y = 20$

$2x - 3y = -5$

2. Solve these simultaneous equations using the substitution method.

a. $x + 3y = 9$

$2x - y = 4$

b. $2x + y = 5$

$3x - 4y = 10$

c. $x - 2y = 7$

$4x + y = 3$

d. $5x - y = 11$

$x + y = 7$

3. Solve these pairs of simultaneous equations graphically

a. $2x + 5y = 9$ and $2x + 3y = 7$

b. $3x + 4y = 23$ and $2x - 4y = 2$

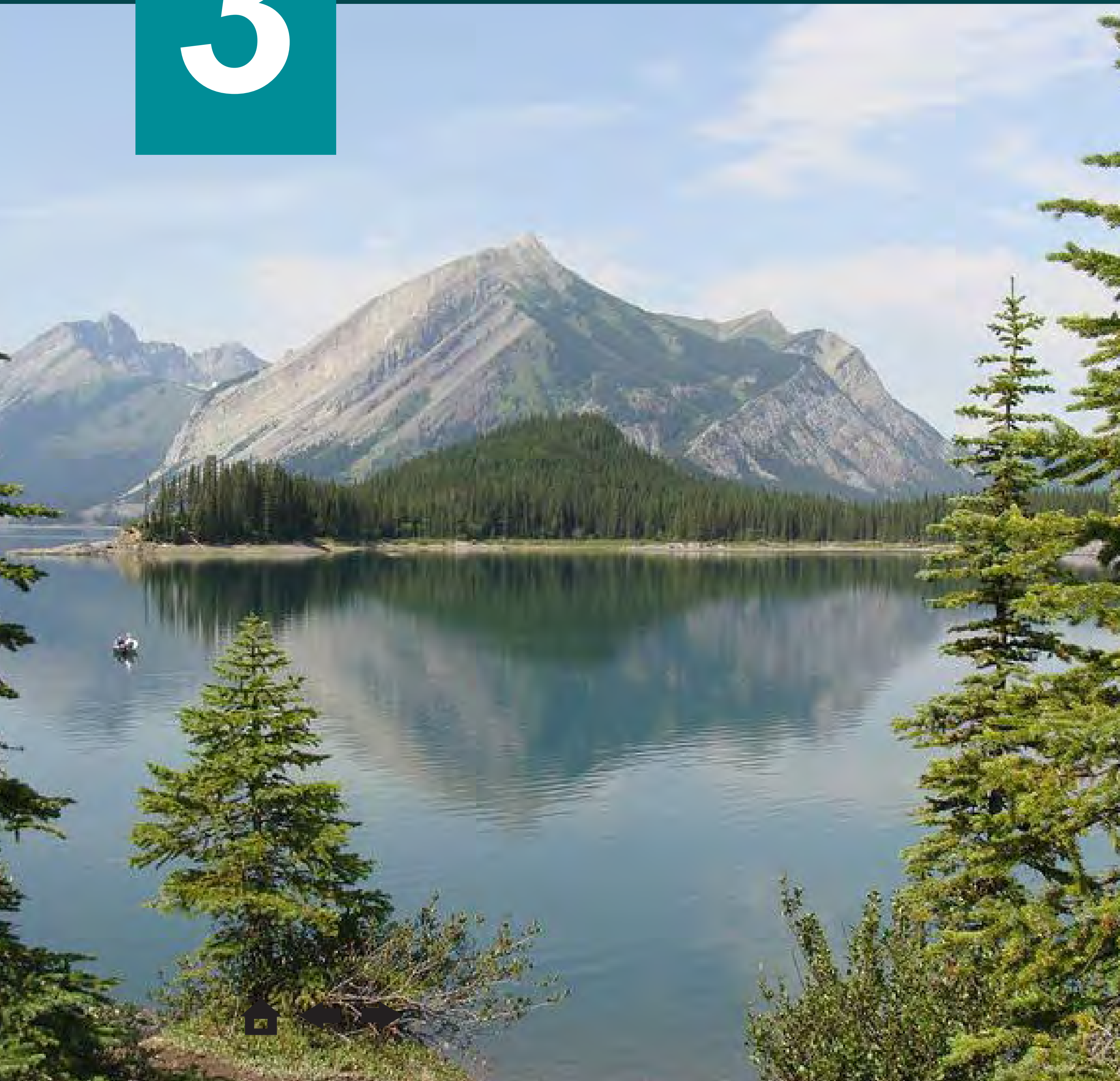
c. $7x + 2y = 33$ and $4x + 2y = 24$

4. A farmer sells two types of crops: maize and beans. The farmer sells maize at Gh¢ 4.00 per bowl and beans at Gh¢ 6.00 per bowl. If he sold a total of 30 bowls for Gh¢150.00, how many bags of maize and beans did he sell?

5. Ama and Kofi run a small food stall in Accra, selling two types of dishes: rice with chicken for Gh¢ 10 and jollof rice for Gh¢ 8. On a busy Saturday, they sold a total of 50 dishes and made Gh¢ 420 from their sales.

SECTION

3 RIGID MOTION



GEOMETRY AROUND US

Spatial Sense

INTRODUCTION

Rigid motions, such as translations and reflections, are fundamental ideas in geometry and transformations. Translation means shifting a shape without turning or changing its size, while reflection flips a shape over a line to create a mirror image. These transformations help in understanding symmetry and patterns in figures. Rotation, another type of rigid motion, involves turning a shape around a fixed point called the centre of rotation. This helps to analyse the orientation of shapes and predict their positions after rotation. Enlargement is a transformation that makes a shape bigger, or smaller, while keeping its form and proportions. This is important for scaling objects in areas like architecture, engineering and art. Knowing how to enlarge shapes ensures accurate measurements and representations in these fields.

KEY IDEAS

- **Enlargement**: This involves increasing or decreasing the size of a shape by a certain scale factor while maintaining its proportions.
- **Image:** The object after a transformation is called the image. After a translation the image remains congruent and the orientation remains the same, but the position changes as determined by the translating vector. After a reflection or a rotation, the images will be congruent to the original but will have been reflected or rotated from the original. After an enlargement the image will be similar to the original.
- **Mirror line**: A line over which the reflection occurs. Every point on the object has a corresponding point on the other side of the mirror line at an equal distance.
- **Reflection**: This is a transformation that "flips" an object across a mirror line to create a mirror image.

- **Rotation**: This is the circular movement of an object around a point. The angle of rotation, the centre of rotation and the direction (clockwise or counter clockwise) are key aspects.
- **Rotational symmetry**: An object or shape is said to have a rotational symmetry if it can be rotated about a central point and still look the same.
- **Scale factor**: A number that indicates how much bigger or smaller the image will be compared with the original shape.
- **Transformation**: This changes the size and/or the position of a shape. There are four types of transformation: translation, reflection, rotation and enlargement
- **Translating vector**: This is a directed line segment with magnitude and direction that describes how far and in which direction each point of the object moves.
- **Translation**: A mathematical translation is a geometric transformation that moves a shape or object from one position to another on a plane without changing its size, shape or orientation.

TRANSFORMATIONS

A transformation is a function that shifts or alters a shape in some manner to create a new shape known as the image. The original shape can also be referred to as the preimage. The points on the preimage serve as the inputs for the transformation, while the points on the image are the outputs. There are four types of transformation: translation, reflection, rotation and enlargement.

TRANSLATION BY A TRANSLATION VECTOR

A vector is a quantity that has both direction and magnitude (size). On a coordinate plane, it is shown as an arrow that goes from one point to another.

A translation vector, often written as $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ shows how far and in which direction each point of a shape will move. For example, if $\vec{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, it means every point on the shape is shifted one unit to the left and three units up. This means the shape stays the same in size and orientation, but it slides to a new position on the grid without turning or changing its size.

Rigid motion is a type of transformation where the size and shape of a figure remain unchanged. The only thing that changes is the figure's position, but it does not alter in any other way.

In this section, you need to apply the knowledge gained in vectors from year 1.

Performing Translations: Study the diagram below.

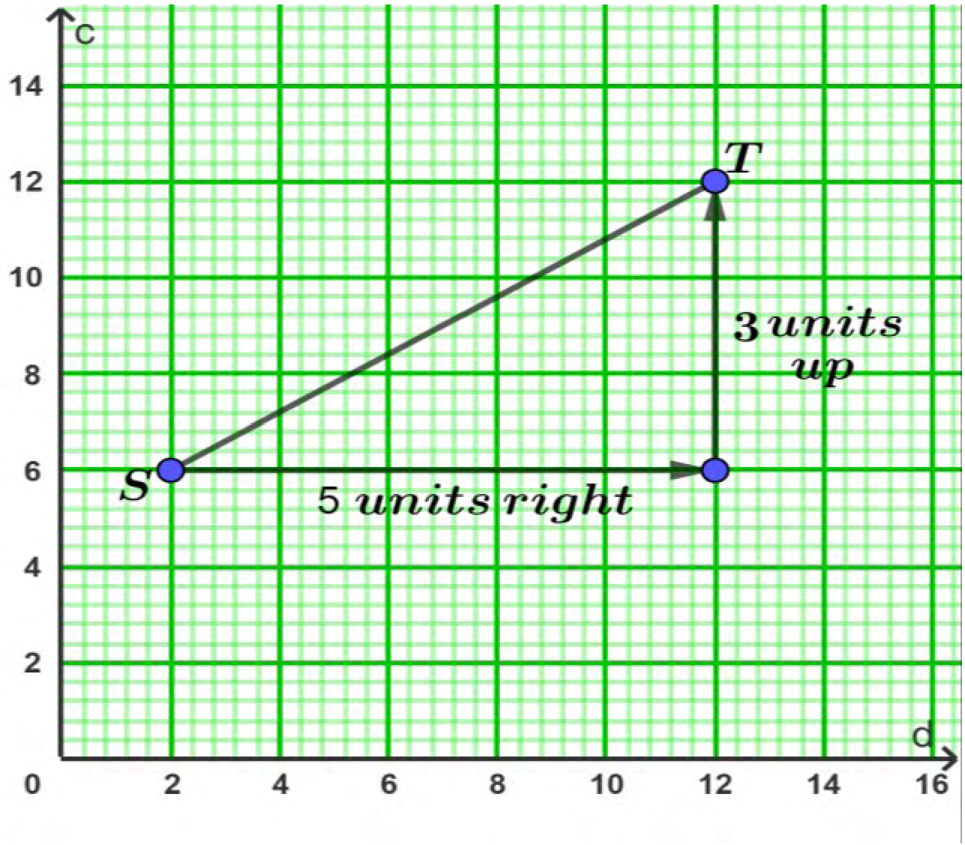


Fig 3.1: The diagram shows a vector

The diagram shows a vector. The starting point of the vector is S and the ending point is T. The vector is called $\overrightarrow{ST}$ and is read as "vector ST." The horizontal part of $\overrightarrow{ST}$ is 5 and the vertical part is 3. When we combine the horizontal and vertical parts, we get what is called the component form of the vector. So, the component form of ST is $\begin{pmatrix}5\\3\end{pmatrix}$.

Activity 3.1: Identifying translation vectors

Study the graphs below and write down the components of given vectors. Compare your answers with your classmates.

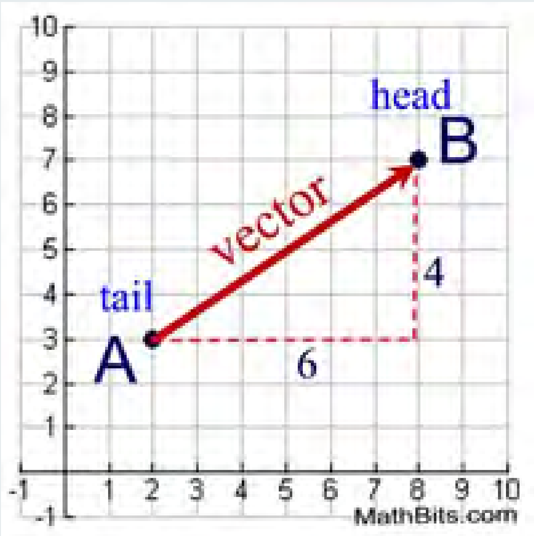


Figure 3.2

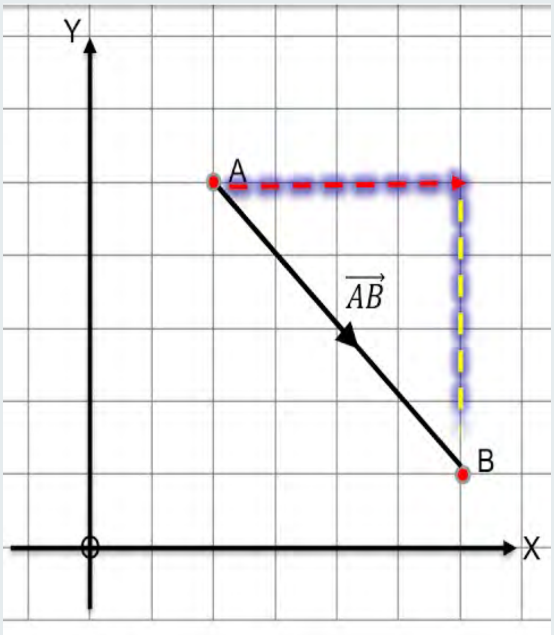


Figure 3.3

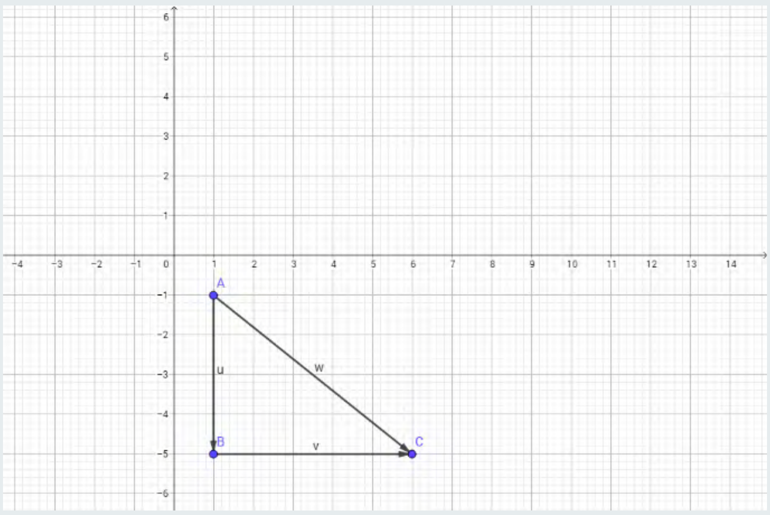


Figure 3.4

Translating a Figure Using a Vector: Study the following examples to enhance your understanding of translation.

Example 3.1

The vertices of $\triangle ABC$ are A (0, 3), B (2, 4) and C (1, 0). Translate $\triangle ABC$ using the vector $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

Solution

First, graph △ABC. Use $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ to move each vertex 5 units right and 1 unit down. Draw △A′B′C′, labelling the image vertices. Notice that the vectors drawn from preimage vertices to image vertices are parallel.

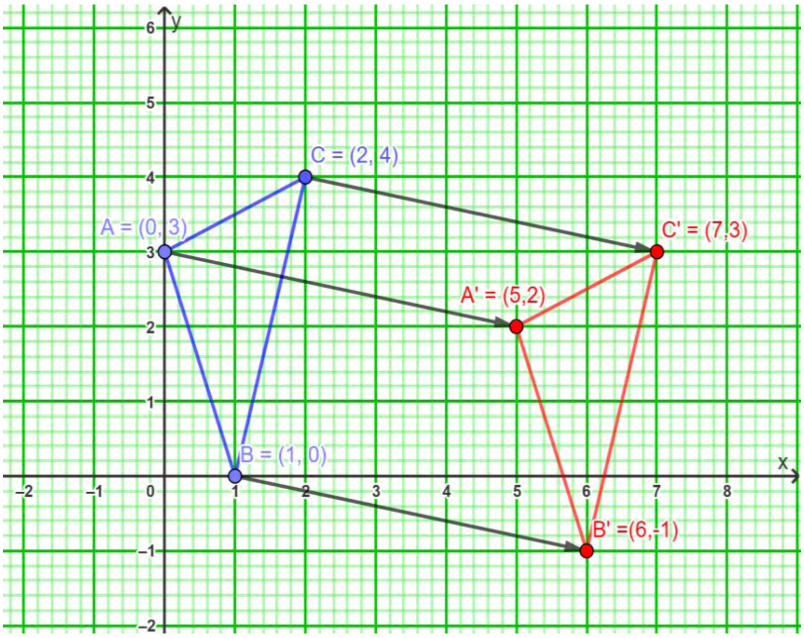


Figure 3.5: Graph of △ABC translated by $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

As a general rule, you can represent a translation along the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ using a rule, which is expressed as $(x, y) \rightarrow (x + a, y + b)$.

Applying this to example 1, to go from A to A′, you move 5 units right and 1 unit down, so you move along the vector $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and the same applies for B to B' and C to C'. So, a rule for the translation is $(x, y) \rightarrow (x + 5, y - 1)$.

Activity 3.2: Translating given coordinates using translation vectors

The vertices of △PQR are P(1, 5), Q(3, 6), and R(2, 1). Translate △PQR using the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$. Write down the rule for the translation. Compare the results with your classmates.

Example 3.2

Graph quadrilateral ABCD with vertices A (−1, 2), B (−1, 5), C (4, 6), and D (4, 2) and its image after the translation $(x, y) \rightarrow (x + 3, y - 1)$.

Solution

Graph quadrilateral ABCD. To find the coordinates of the vertices of the image, add 3 to the x-coordinates and subtract 1 from the y-coordinates of the vertices of the preimage. Then graph the image, as shown.

(x, y) → (x + 3, y − 1)

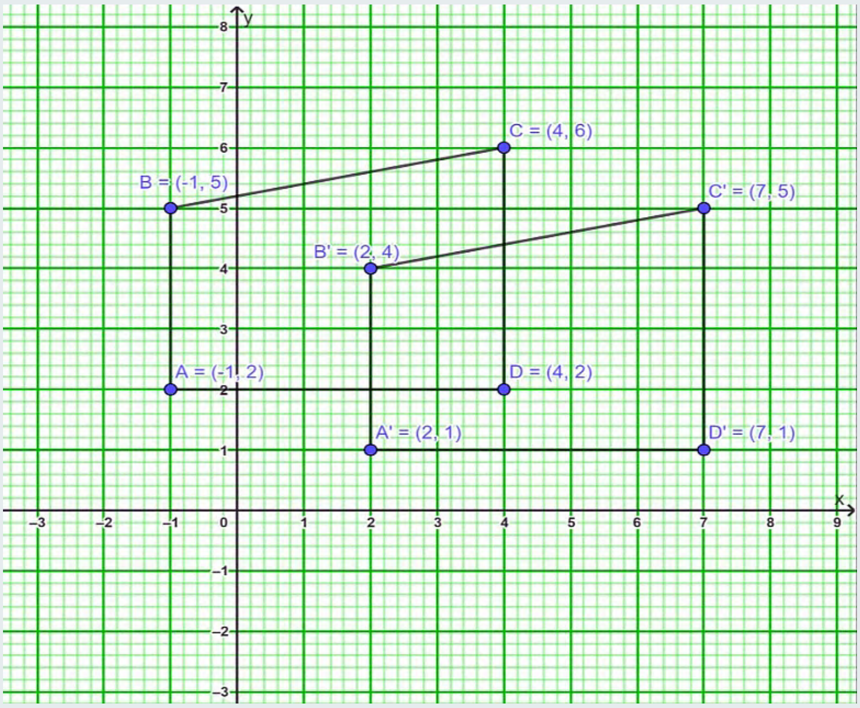


Figure. 3.6: Graph of quadrilateral ABCD translated by $(x, y) \to (x + 3, y - 1)$

A (−1, 2) → A′ (2, 1)

B (−1, 5) → B′ (2, 4)

C (4, 6) → C′ (7, 5)

D (4, 2) → D′ (7, 1)

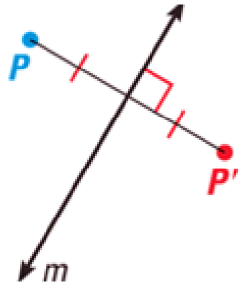
Activity 3.3: Translating given coordinates using translation vectors

The vertices of quadrilateral WXYZ are W (0, 2), X (3, 5), Y (5, 3), and Z (2, 0). Working in pairs, translate quadrilateral WXYZ using the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

REFLECTIONS OF IMAGES

Reflection is a transformation that uses a line as a "mirror" to reflect an image. The diagram below illustrates reflection of images.

1. The "mirror" line, *m,* is called the **Line of Reflection.**
2. P and P′ are the same perpendicular distance from the line of reflection.



3. The line connecting P and P′ is perpendicular to the line of reflection.

Rules of Reflection

To perform a reflection in a coordinate plane, you need to follow some basic rules that relate to a given line of reflection.

Reflections across the x–Axis

Rule: $(x, y) \rightarrow (x, -y)$

Description: When a point is reflected across the x-axis, the x-coordinate remains the same, while the y-coordinate changes sign.

Reflections across the y–Axis

Rule: $(x, y) \rightarrow (-x, y)$

Description: When a point is reflected across the y-axis, the y-coordinate remains the same, while the x-coordinate changes sign.

Reflections across the line $y = x$

Rule: $(x, y) \rightarrow (y, x)$

Description: When a point is reflected across the line $y = x$, the coordinates are swapped.

Reflections across the line $y = -x$

Rule: $(x, y) \rightarrow (-y, -x)$

Description: When a point is reflected across the line $y = -x$, the coordinates are swapped and their signs are changed.

Reflections across a vertical line $x = a$

Rule: (x, y) → (2a−x, y)

Description: When a point is reflected across a vertical line $x = $ a, the y-coordinate remains the same, while the x-coordinate is transformed by the formula 2a − x.

Reflections across a horizontal line $y = b$

Rule: (x, y) → (x, 2b−y)

Description: When a point is reflected across a horizontal line y = b, the x-coordinate remains the same, while the y-coordinate is transformed by the formula 2b − y.

Example 3.3

The vertices of ΔLMN are L (−3, 3), M (1, 2), and N (−2, 1). Plot the coordinates on a graph and join the points to form ΔLMN. Draw the image ΔL'M'N' of ΔLMN using the y-axis as the mirror line.

Solution

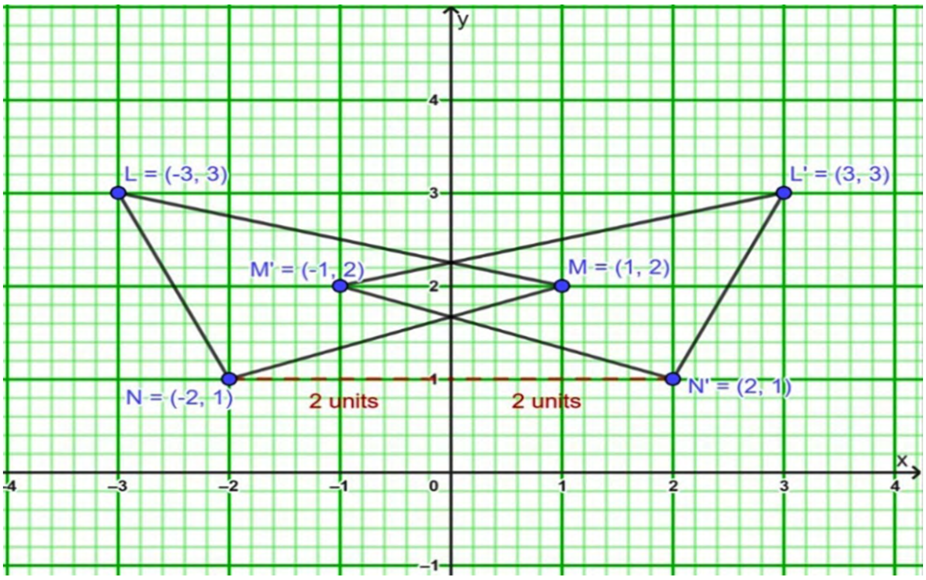


Figure 3.7: Graph of ΔLMN reflected in the y-axis

From the graph, we can identify the coordinates as;

(a, b) → (−a, b)

L (−3, 3) → L′(3, 3)

M (1, 2) → M′(−1, 2)

N(−2, 1) → N′(2, 1)

Activity 3.4: Identifying reflections of objects in a mirror line

Working in pairs, study the diagrams below and describe the type of reflections in each. Remember that all reflections must be defined by their 'mirror line', so be sure to state correctly the line of reflection.

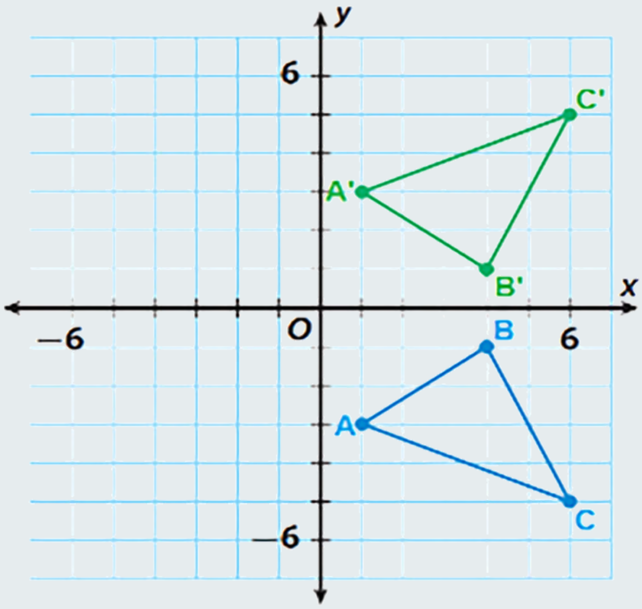


Figure 3.8: Graph of a reflected object

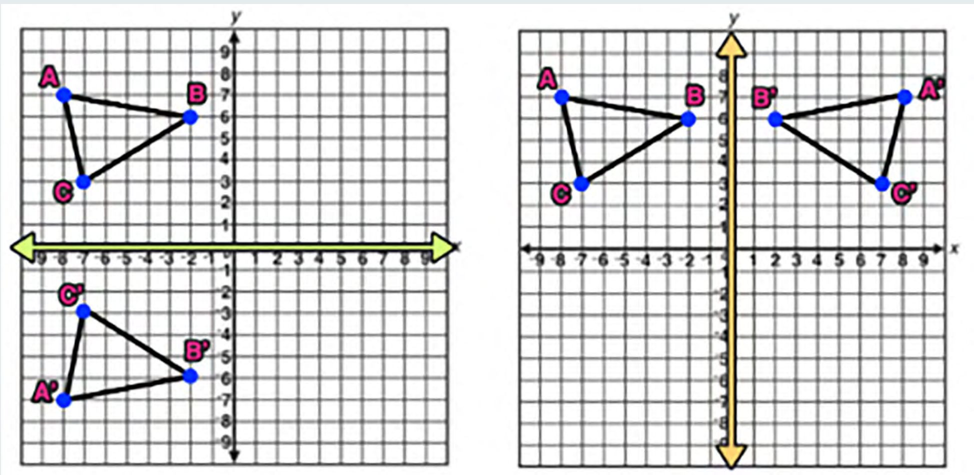


Figure 3.9: Graph of a reflected object

Activity 3.5: Reflecting objects on a mirror line

Using the diagrams below reflect ΔABC in the *x* axis and parallelogram ABCD in the y axis. Compare your answers with your classmates.

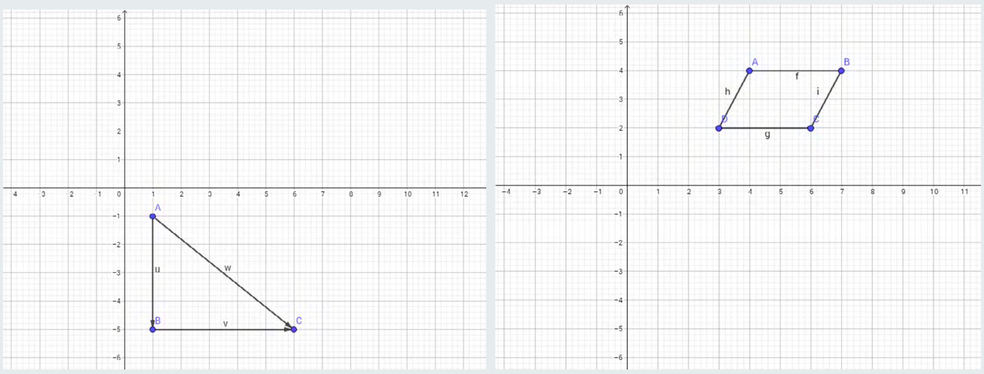


Figure 3.10: Reflecting objects on a mirror line

Example 3.4

Graph ΔABC with vertices A (1, 3), B (4, 4), and C (3, 1). Reflect ΔABC in the line y = −*x*.

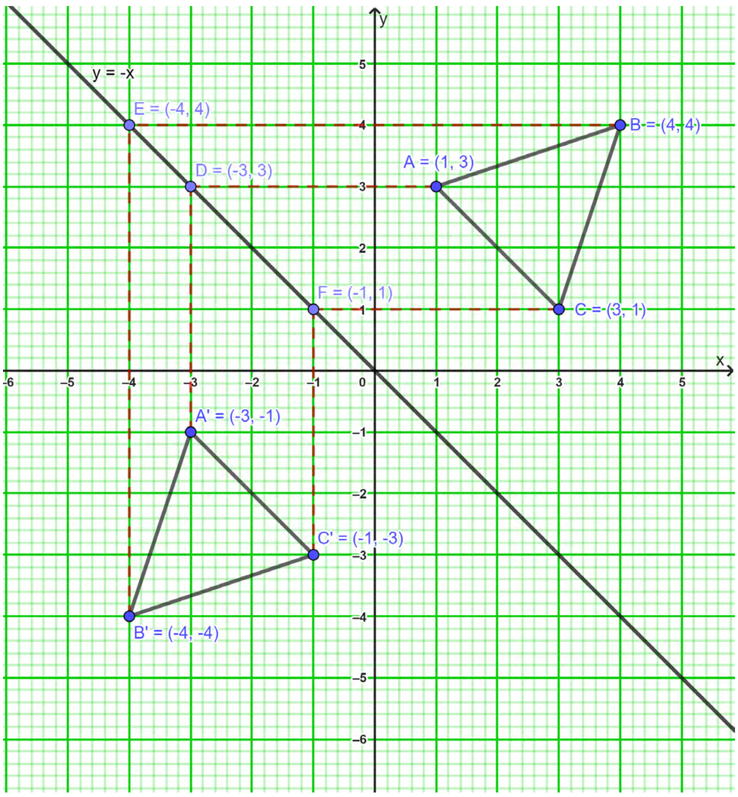


Figure 3.11: Graph of ΔABC in the line y = −x

From the graph, we can see how ΔABC is reflected in the mirror line *y* = −*x*. Study the dash lines carefully to understand how each point is reflected.

Rotation of Images of Plane Shape

A rotation is when a shape is turned around a fixed point, called the centre of rotation, by a certain angle and direction, usually measured in degrees. This movement keeps the shape the same size and proportion but changes its orientation. During a rotation, each point on the shape follows a circular path around the centre. The angle of rotation shows how far the shape turns, and the direction (clockwise or anti-clockwise) dictates the direction in which it moves. Common rotation angles are 90°, 180°, and 270°, though any angle can be used.

Real-life Examples of Rotation

In real life, we see rotation in things like a spinning wheel, a ceiling fan, or the hands of a clock. Below are some examples you may find around you

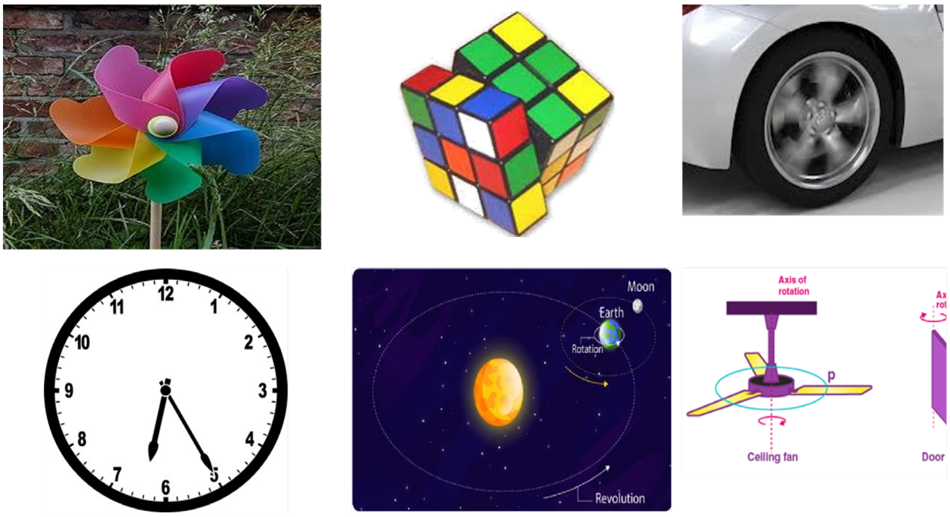


Figure 3.12: Real-life Examples of Rotation

Rotations can be illustrated using graphs as illustrated below.

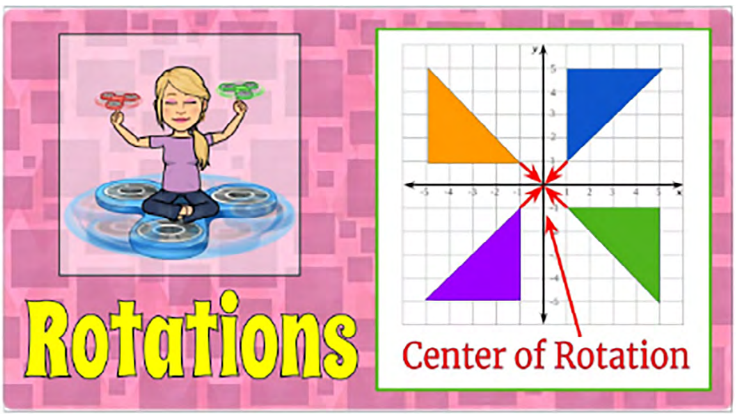


Figure 3.13: Rotation in a coordinate plane

Activity 3.6: Investigating real-life examples of rotation

Working in small groups, look for objects around the school or outside the school that demonstrate rotation. Write the list of those objects and describe how they rotate (clockwise or counter clockwise) to your classmates. If possible, take videos of how these objects rotate and share these.

Coordinate Rules for Counterclockwise (anticlockwise) Rotations about the Origin

Study the diagrams below to see how rotations of 90^0, 180^0, 270^0, and 360^0 about the origin are done.

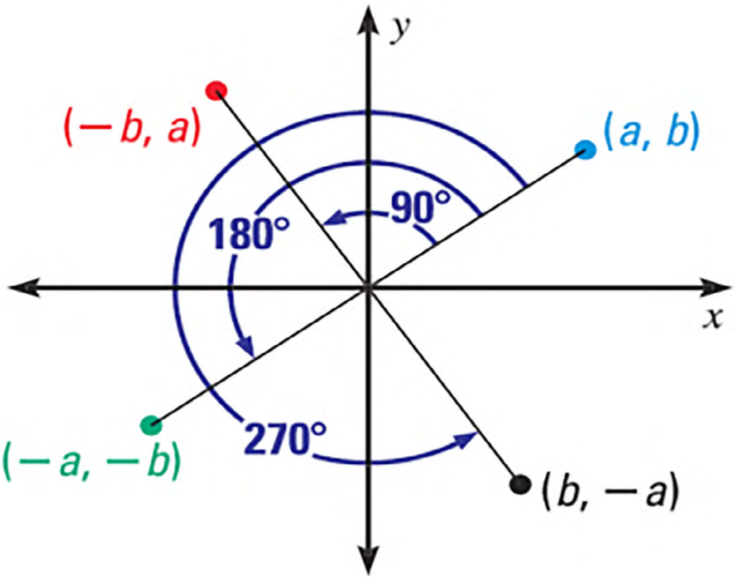


Figure 3.14: Rotations 90°, 180°, 270°, and 360° about the origin

From the above diagram, you can make the following conclusions:

90° Anticlockwise Rotation (or 270° Clockwise Rotation)

Rule: $(x, y) \rightarrow (-y, x)$

Description: When a point is rotated 90° anticlockwise about the origin, the x-coordinate becomes the negative y-coordinate, and the y-coordinate becomes the x-coordinate.

180° Rotation (Anticlockwise or Clockwise)

Rule: $(x, y) \rightarrow (-x, -y)$

Description: When a point is rotated 180° about the origin, both the x- and y-coordinates change signs.

270° Anticlockwise Rotation (or 90° Clockwise Rotation)

Rule: $(x, y) \rightarrow (y, -x)$

Description: When a point is rotated 270° anticlockwise (or 90° clockwise) about the origin, the x-coordinate becomes the y-coordinate, and the y-coordinate becomes the negative x-coordinate.

Example 3.5

If E (−3, 2), F (−3, 4), G (1, 4), and H (2, 2). Find the image matrix for a 270° anticlockwise rotation about the origin.

Solution

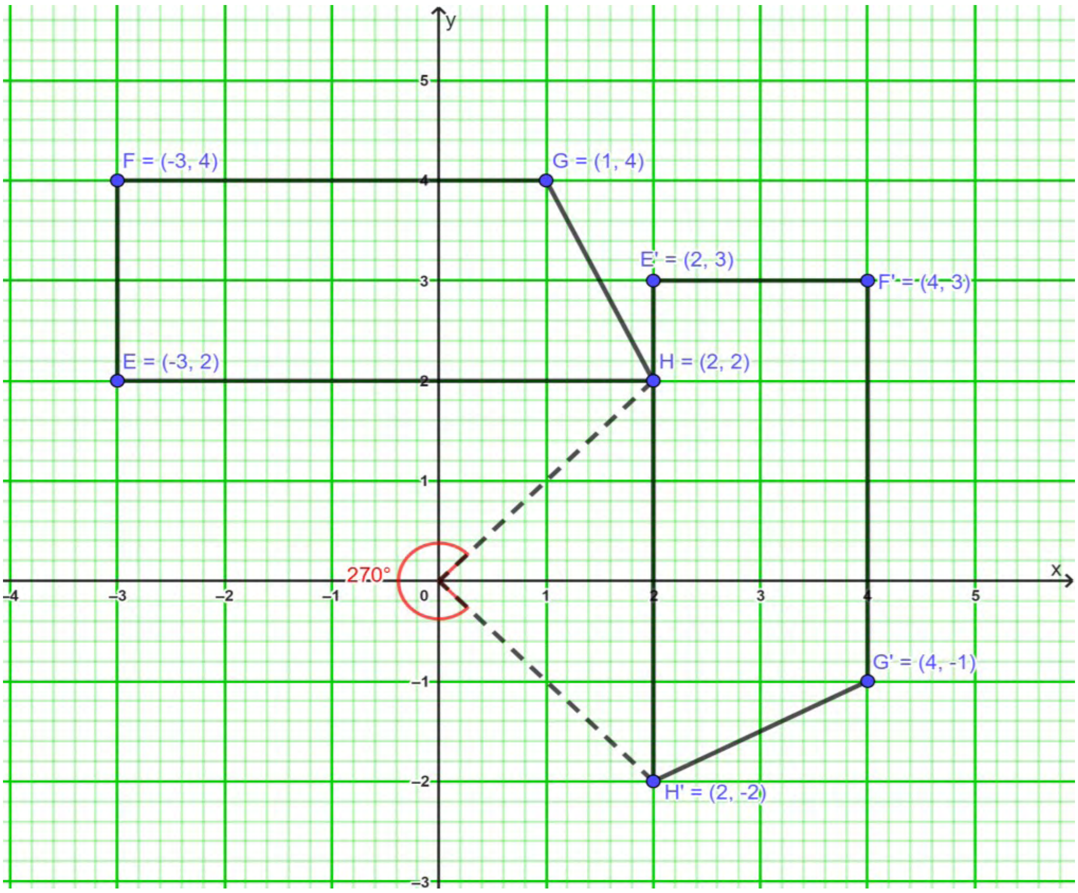


Figure 3.15: Graph of EFGH rotated 270° anticlockwise about the origin

(a, b) → (b, −a)

E(−3, 2) → E′(2, 3)

F(−3, 4) → F′(4, 3)

G(1, 4) → G′(4, −1)

H(2, 2) → H′(2, −2)

Example 3.6

Graph △JKL with vertices J(3, 0), K(4,3) L(6,0) and its image after a 90° clockwise rotation about the origin.

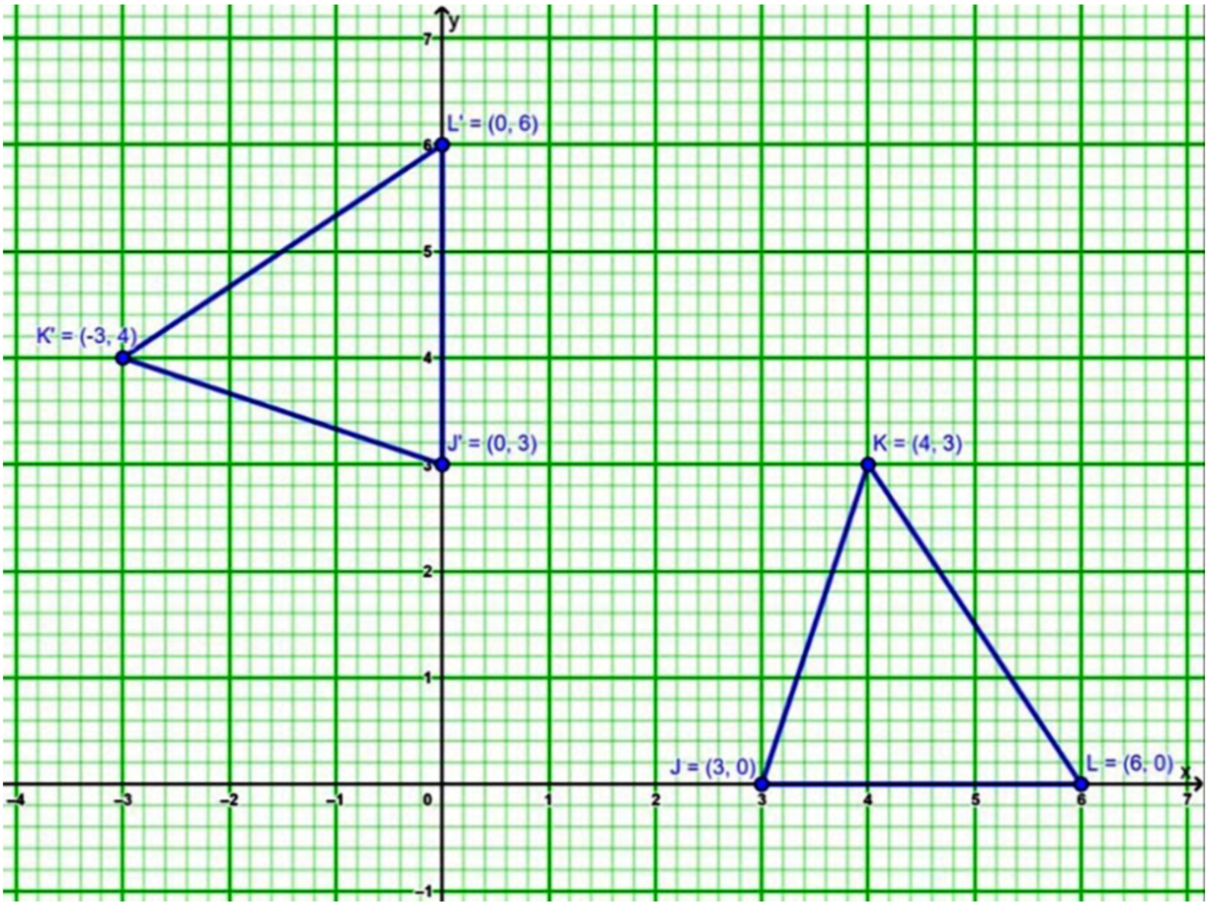


Figure 3.16: Graph of △JKL 90° through clockwise rotation about the origin

Example 3.7

Graph △ABC with vertices A(0, 3), B(-4,0) and C (0, -1) and its image after a 270° clockwise rotation about the point P(2,0).

Solution

Remember that a 270° clockwise rotation is the same as a 90° anti clockwise rotation.

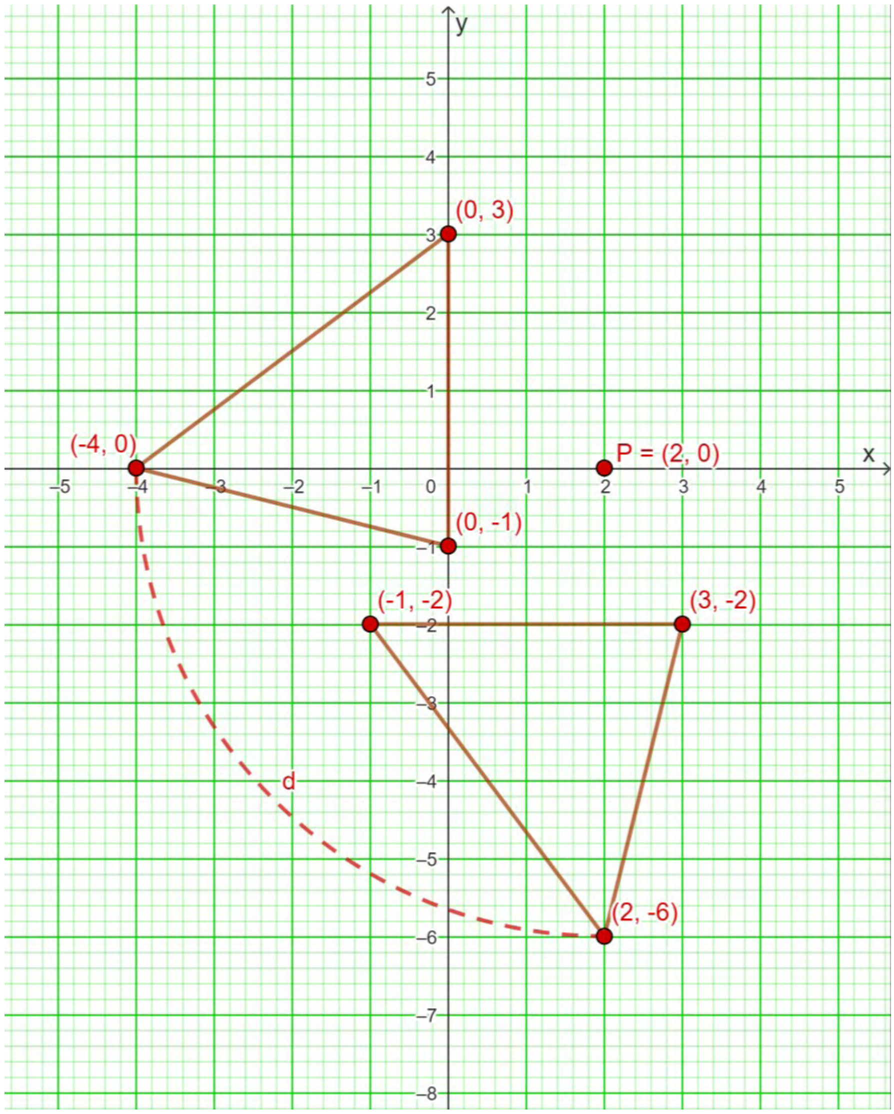


Figure 3.17: Graph of △ABC with 270° clockwise rotation about the point P (2,0)

Activity 3.7: Rotation of objects in a coordinate plane

By looking at the graph below, describe how to locate point A′ if point A is rotated 270^0 clockwise around the origin.

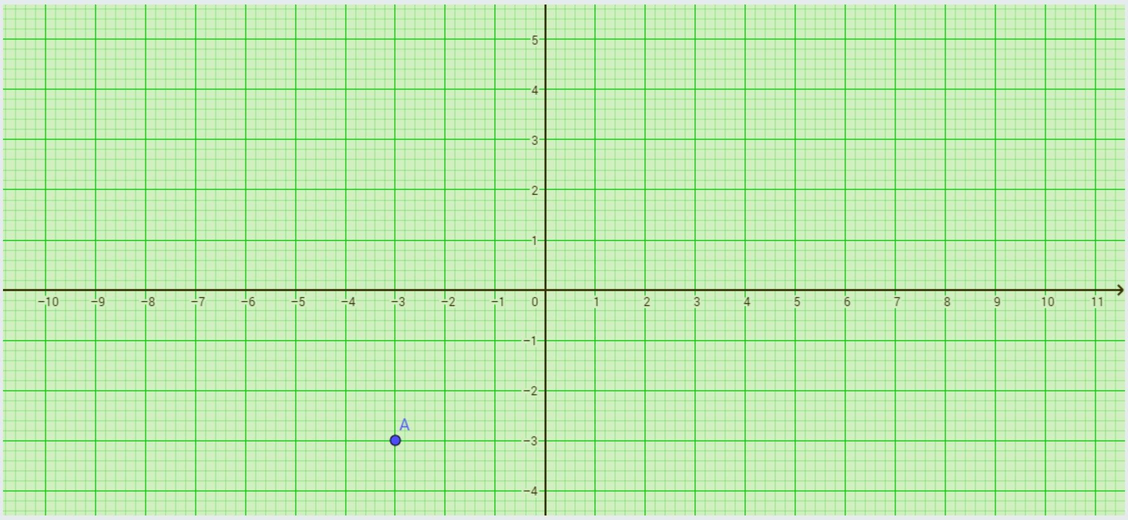


Figure 3.18: Locating point A′ if point A is rotated 270° clockwise around the origin

Activity 3.8: Rotation of objects in a coordinate plane

Using the graph below

1. Rotate point A 90^0 counterclockwise around the origin (0, 0). Plot the new position and label it A′.
2. Repeat the rotation of A by $180^0$and then 270^0 counterclockwise around the origin. Plot these points and label them A′′ and A′′′ respectively.
3. Record the coordinates of each point after rotation.
4. Describe any patterns you observe in the coordinates after each rotation.

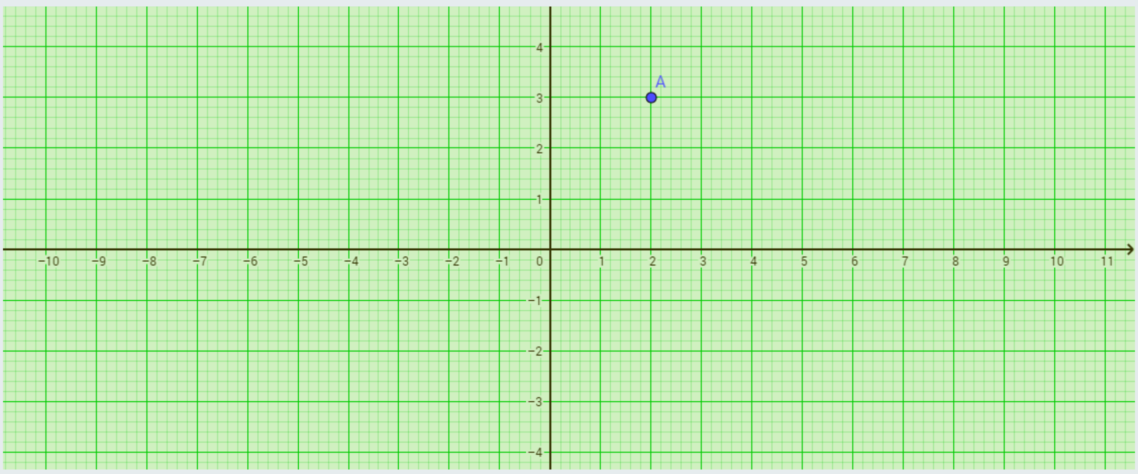


Figure. 3.19: Locating the image of A

Rotational Symmetry: A figure has rotational symmetry if it looks the same after being rotated a certain angle. Examples of some objects that exhibit rotational symmetry are: Gye Nyame, Fawohodie, and Eban

Activity 3.9: Exploring Rotational Symmetry with Adinkra Symbols

Materials Needed

- Printed sheets of various Adinkra symbols (examples, Gye Nyame, Fawohodie, Eban)
- Scissors
- Markers
- Blank paper for drawing

Steps:

1. Cut out an Adinkra symbol, for example, Gye nyame (representing the supremacy of God), or Fawohodie (symbolising independence) or Eban (symbolising security and safety) or any other Ghanaian symbol.
2. Place the symbol on a blank sheet of paper.
3. Rotate the symbol by 90°, 180° and 270° and mark each position with a light pencil outline.
4. Note the angle(s) that make(s) the symbol look the same as the original position.
5. Share your observations with your classmates. Discuss with them, why some symbols look the same at multiple rotations while others only match after a full 360° turn.
6. Connect the idea of rotational symmetry to Ghanaian art and culture.
7. Discuss with your classmates how symmetry adds beauty and balance in design.

Activity 3.10: Exploring rotational symmetry

Work in small groups, in pairs or individually to create your own symbols or patterns that have rotational symmetry. Show them to your classmates or teacher and explain the rotational symmetry incorporated in each symbol or pattern.

Activity 3.11: Rotational symmetry of a geometric figure

The figure below shows the rotational symmetry of a rectangle. By looking at the illustrations, identify a rectangular object in your immediate environment and discuss its rotational symmetry with your classmates.

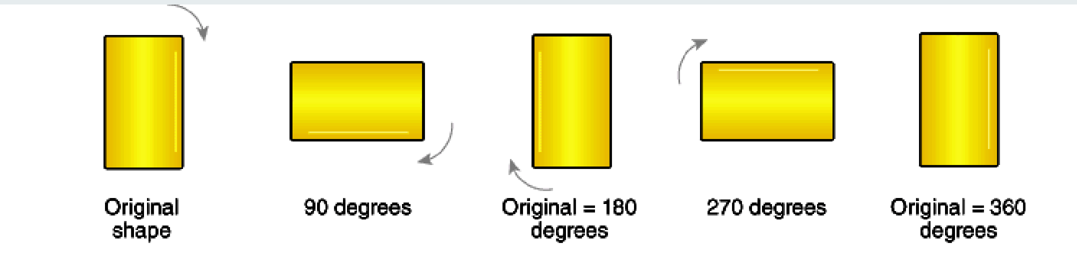


Figure 3.20: Order of rotational symmetry

The above example shows the rotation of a rectangle 90° each time. The rectangle has the rotational symmetry of order 2 because when it is rotated twice, we get the original shape at 180° and again, when it is rotated twice, the original form is obtained at 360°. So, the order of rotational symmetry of the rectangle is 2.

Enlargement of Plane Shapes

Enlargement (or **dilation**) is a type of transformation that changes the size of a shape while maintaining its overall geometry. Enlargement can either increase the size of a shape (magnification) or decrease it (reduction), depending on the scale factor. Unlike other rigid motions such as translation, rotation and reflection, which maintain the size and shape of the figure, enlargement changes the size but keeps the shape similar.

In an enlargement, a plane shape is scaled by a specific factor relative to a fixed point called the centre of enlargement. This scale factor, often denoted as "***k***", determines how much the shape is enlarged or reduced. If $k > 1$, the shape is enlarged; if $k < 1$, the shape is reduced in size. The transformation ensures that all points on the original shape (the pre-image) move radially outward or inward from the centre of enlargement by a distance proportional to k.

The mathematical rules governing enlargement involve multiplying the coordinates of each point in the original shape by the scale factor k, resulting in a new shape (the image) that is similar to the original but proportionally larger or smaller.

Look at the diagram:

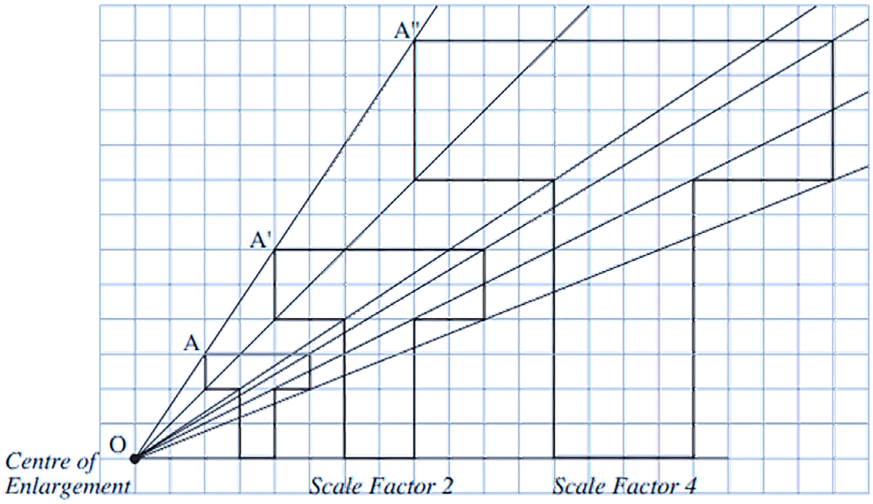


Figure 3.21: Enlargement of Shape A

The example shows how the original, A, was enlarged with scale factors 2 and 4. A line from the centre of enlargement passes through the corresponding vertex of each image.

Note

The distances, OA′ and OA′′, are related to OA by:

$OA' = 2 \times OA$

$OA'' = 4 \times OA$

The same is true of all the other distances between O and corresponding points on the images.

Activity 3.12: Enlarging plane shapes using graph paper

Materials Needed

- Graph paper
- Ruler
- Pencil
- Coloured markers

Steps

1. Draw a triangle or rectangle, on graph paper. Ensure the shape has coordinates that are easy to identify by placing vertices at grid intersections.

2. Identify the centre of enlargement at the origin (0,0).
3. Multiply each coordinate of the original shape by the scale factor to get the new enlarged coordinates. For example, if a vertex is at (2, 3) in the original shape and the scale factor is 2, the new coordinate would be (4, 6) when the centre of enlargement is at the origin.
4. Plot these new coordinates and join them to form the enlarged shape.
5. Measure the sides of the original and enlarged shapes. You will observe that each side of the enlarged shape is exactly twice (or the scale factor times) the corresponding side of the original shape.
6. Discuss with your classmates, how the angles remain the same and that the shapes are similar.
7. Experiment with different scale factors (e.g., 3, 0.5) to see how the shape changes with enlargement and reduction.

Example 3.8

Plot the coordinates A (2, 1) B (1, 3) C (3, 2) of △ABC. Perform an enlargement of △ABC using a scale factor of 2 and a centre of the enlargement at the origin to form △A′B′C′.

Compare the coordinates, side lengths and angle measures of △ABC and △A′B′C′.

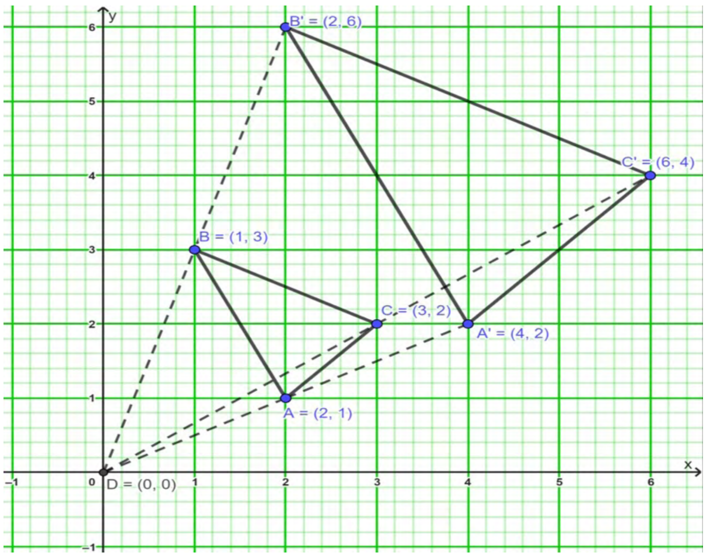


Figure 3.22: Enlargement of △ABC

Sometimes, the scale factor is a negative number. When this happens, the figure rotates 180°. So, when $k > 0$, a dilation with a scale factor of $-k$ is the same as the composition of a dilation with a scale factor of k followed by a rotation of 180° about the centre of dilation. Using the coordinate rules for a dilation centred at (0, 0) and a rotation of 180°, you can think of the notation as $(x, y) \rightarrow (kx, ky) \rightarrow (-kx, -ky)$.

Example 3.9

Graph △FGH with vertices F(−4, −2), G(−2, 4), and H(−2, −2) and its image after a dilation centred at (0, 0) with a scale factor of $\frac{-1}{2}$.

Solution

Use the coordinate rule for a dilation with centre (0, 0) and $k = \frac{-1}{2}$ to find the coordinates of the vertices of the image.

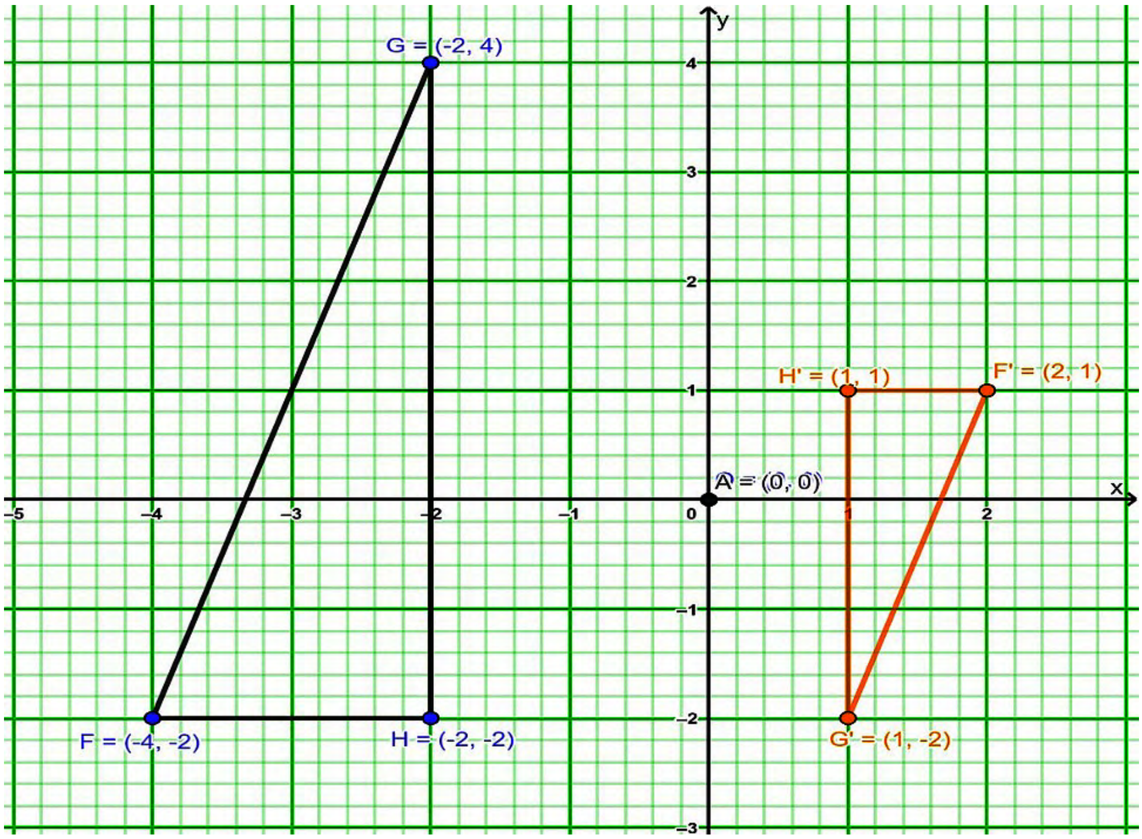


Figure 3.23: Graph of △FGH dilated at centre (0, 0) and scale factor $\frac{-1}{2}$.

Then graph △FGH and its image.

$(x, y) \rightarrow (-\frac{1}{2}x, -\frac{1}{2}y)$

F(−4, −2) → F′(2, 1)

G(−2, 4) → G′(1, −2)

H(−2, −2) → H′(1, 1)

Example 3.10

Graph △FGH with vertices F(−4, −2), G(−2, 4), and H(−2, −2) and its image after a dilation centred at (0, 0) with a scale factor of $\frac{1}{2}$.

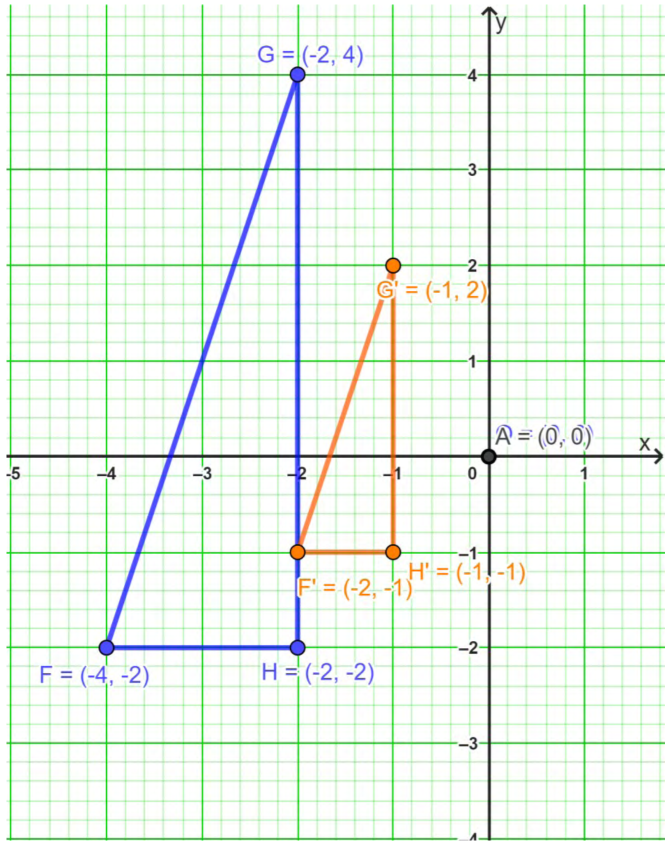


Figure 3.24: Graph of △FGH and its image

Graph △FGH and its image.

$(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$

F(−4, −2) → F′(-2, -1)

G(−2, 4) → G′(-1, 2)

H(−2, −2) → H′(-1,-1)

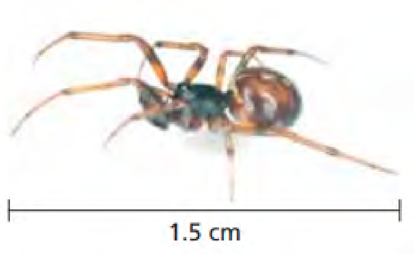
Looking at Examples 2 and 3, it should be noted that the scale factor had different effects on the image of △FGH. The negative scale factor is affected differently from the positive scale factor as it shows a rotation about the origin as well as the dilation.

Applications

Example 3.11

You are using a magnifying glass that shows the image of an object that is six times the object's actual size. Determine the length of the image of the spider seen through the magnifying glass.

Solution



$$\frac{\textit{image length}}{\textit{actual length}} = k$$

$$\frac{x}{1.5} = 6$$

$$x = 9$$

The image length through the magnifying glass is 9 centimetres.

Example 3.12

In Accra, a construction company is building a new community park with a traditional Ghanaian Kente-inspired mosaic on the main walkway. The original Kente design on a cloth measures 50cm × 80cm. To make it suitable for the walkway, the company needs to enlarge the design by a scale factor of 5.

1. Calculate the dimensions of the enlarged Kente mosaic that will be laid on the walkway.
2. Find the area of the original design and the area of the enlarged mosaic.
3. How many times greater is the area of the enlarged mosaic compared with the original design?
4. If the cost of laying the mosaic is GH¢20 per square metre, calculate the total cost to lay the enlarged Kente mosaic on the walkway.

Solution

1. Original dimensions = 50cm × 80cm

   Enlarged dimensions = 50cm × 5 = 250cm

   = 80cm × 5 = 400cm

   Dimensions of enlarged mosaic = 250cm × 400cm
2. Original length in metres = 0.5m

   Original width in metres = 0.8m

   Original area = 0.5m × 0.8m = 0.4m^2

   New length in metres = 2.5m

New width in metres = 4 m

Enlarged Area = 2.5m × 4m = $10m^2$

3. Scale factor, k = $\frac{\textit{Area of enlarged mosaic}}{\textit{Area of original mosaic}} = \frac{10m^2}{0.4m^2}$ = 25 (Note that this is the length scale factor squared)

4. Total cost to lay the enlarged Kente mosaic on the walkway = Area of enlarged mosaic × GH¢20 = $10m^2$ × GH¢20 = GH¢200.00.

EXTENDED READING

Watch this video on translations, reflections and rotations by clicking on the link: https://youtu.be/YD3HIMUae_4. Or search through internet for tutorials on translations, reflections, rotations and enlargements.

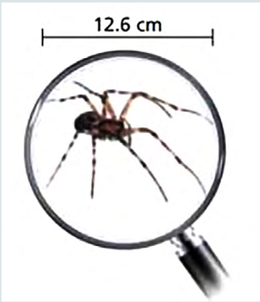
10. Graph ΔABC with vertices A(−4, 4), B(−1, 7), and C(-1,4) and its image after a 270^0 clockwise rotation about the origin.

11. Graph △PQR and its image after an enlargement centred at C with scale factor k.

a. P(−2, −1), Q(−1, 0), R(0, −1); C(0, 0), k = 4

b. P(5, −5), Q(10, −5), R(10, 5); C(0, 0), k = 0.4

12. The image of a spider seen through the magnifying glass is 4 times its actual length as shown in the diagram.



Find the actual length of the spider.

SECTION

4 DATA COLLECTION, ORGANISATION AND REPRESENTATION

MAKING SENSE OF AND USING DATA

Statistical Reasoning and its Application in Real Life

INTRODUCTION

In today's information age, data collection and analysis are crucial. Researchers use surveys, interviews and observations to gather numerical, categorical and ordinal data. Effective organisation and visualisation through graphs, charts and tables all help to facilitate understanding.

Mastering data analysis and interpretation enable you to develop informed decision-making and problem-solving skills. You can apply data concepts to real-world projects, building a foundation for statistics, economics and data science studies.

KEY IDEAS

- **A data collection instrument** is a tool or method used to gather, record and measure data from respondents, participants or sources. Its purpose is to collect accurate, reliable and relevant data for research, analysis, or decision–making.
- **Data Analysis** is the process of extracting insights, patterns and meaning from data to inform business decisions, solve problems or answer questions.
- **Data** is information that can be used for analysis, decision–making, or communication
- **Data organisation** refers to the process of structuring, categorising and arranging data in a logical and systematic way to facilitate efficient storage, retrieval, analysis and visualisation.
- **Data representation** refers to the process of presenting data in a clear, concise and meaningful way to facilitate understanding, analysis and decision-making.
- **Measures of Dispersion**, also known as variability or spread, describes how spread out or scattered data points are from the central tendency (mean, median, mode).

DATA COLLECTION INSTRUMENTS

Data collection instruments play a crucial role in gathering accurate and relevant information from diverse sources. These tools are vital for successful research projects, experiments and surveys. In year one we discovered that data collection instruments fall into two primary categories: quantitative and qualitative. Quantitative instruments, such as surveys and questionnaires, gather numerical data for statistical analysis. Conversely, qualitative instruments, including interviews, focus groups and observations, provide rich insights into individuals' thoughts, behaviours and experiences. This session will focus on developing effective data collection tools to gather high -quality data from participants.

Some Data collection instruments are as follows;

1. A questionnaire
2. An interview guide
3. Observation guide
4. A Survey

Activity 4.1: Designing and validating a questionnaire

Working individually or in pairs, read through the steps to design a good questionnaire. Consider what topics you would want to investigate and how you would create a questionnaire to support this. Look at the sample questionnaire. Could you improve it? How would you go about this? Are there any areas which could cause upset to any communities? Could there be any bias creeping into the question style which may influence responses? Such things must always be considered when constructing data collection instruments.

Steps to develop a *Simple Questionnaire* and validate it to collect data

Step 1: Define the research objective and scope

a Clearly say what the main purpose of your research is.

b. Mention how long the study will take and point out any challenges you might face.

Step 2: Identify the target population and sample size

a. Clearly explain the group of people you are studying.

b. Try to choose a sample that truly reflects the whole group you are interested in.

Step 3: Determine the questionnaire format

a Online,

b Paper-based,

c Mixed

Step 4: Decide on the question types

a Closed-ended questions – require specific options for answers, for example, yes/no, multiple choice, rating scale.

b Open-ended questions – allow respondents to answer in their own words and give more detailed responses.

Step 5: Develop clear, concise and logical questions. Use simple language and avoid jargon

Step 6: Organise the questionnaire

a Give a brief introduction explaining the purpose of the questionnaire, remembering to assure respondents that their responses will be confidential.

b Think about question order. Begin with the simple questions, keeping the more challenging ones for later.

c Keep the questionnaire short and to the point. It should not take too long to be completed.

Step 7: Pilot-test the questionnaire (small group of about 5-10 respondents), requesting feedback on it.

Step 8: Revise and refine the questionnaire based on the feedback.

Step 9: Distribute the questionnaire.

a Decide how you will distribute. For example, in person, via email or through social media.

b Provide clear instructions on how to complete and return the questionnaire.

A Sample Questionnaire

Climate Resilience Questionnaire for Senior High Students

Introduction

Thank you for participating in this survey. Your responses will help us understand your perceptions and experiences of climate change and resilience. All answers will be kept confidential.

Section 1: *Demographics*

1. Age: ______________________________
2. Gender:
   a. Male
   b. Female
3. Grade level: ______________________________
4. School location (urban/rural): ______________________

Section 2: *Climate Change Awareness*

1. How often do you hear about climate change in the news?
   a. Daily
   b. Weekly
   c. Rarely
   d. Never
2. What do you think is the main cause of climate change? (Select one)
   a. Human activities
   b. Natural processes
   c. Both
   d. Unsure
3. Have you learned about climate change in school?
   a Yes
   b No

Section 3: *Climate Resilience Knowledge*

1. What does climate resilience mean to you? (Open-ended)
2. Which of the following climate-related hazards have you experienced or learned about? (Select all that apply)
   a. Floods
   b. Droughts
   c. Heatwaves
   d. Storms
   e. Other (please specify) ______________________
3. How do you think individuals can contribute to climate resilience? (Select all that apply)
   a. Reducing energy consumption
   b. Conserving water
   c. Planting trees
   d. Reducing waste
   e. Other (please specify) ______________________

Section 4: *Climate-Related Experiences*

1. Have you or your family experienced any climate-related impacts (e.g., flooding, drought)?
   a. Yes
   b. No
2. How concerned are you about climate change affecting your community?
   a. Very concerned
   b. Somewhat concerned
   c. Not very concerned
   d. Not at all concerned
3. Have you participated in any climate-related activities (e.g., tree planting, clean-ups)?
   a. Yes
   b. No

Section 5: *Attitudes and Behaviours*

1. How important is addressing climate change to you? (Scale: 1-5, where 1 is "not important at all" and 5 is "very important")
2. Do you think your actions can make a difference in addressing climate change?
   a. Yes
   b. No
   c. Unsure
3. Which climate-friendly behaviours do you practice regularly? (Select all that apply)
   a. Recycling
   b. Using public transport
   c. Reducing meat consumption
   d. Using energy-efficient appliances
   e. Other (please specify) ______________________

Section 6: *Conclusion*

Thank you for taking the time to complete this survey! Your input is valuable.

Activity 4.2: Designing an interview guide

Working individually or in pairs, read through the steps to design an interview guide. Once again, consider what topics you would want to investigate in this way. Look at the sample guide. Think about what will work well and what you think could be done better.

Steps to develop an interview guide:

Step 1: Define the research objective
Identify the specific goal and outcome of the interview

Step 2: Determine the interview type

a Structured interview (Standard questions)

b Semi–structured interview (Flexible questions)

c Unstructured interview (Open–ended conversation)

d Focus group interview (Group discussion)

Step 3: Identify the target population

a Define the demographic characteristics

b Determine the sample size

Step 4: Develop the interview protocol

a Create an outline of topics and questions

b Ensure questions are clear, concise, relevant, non-leading and open-ended for qualitative data

Step 5: Select the interview method

a Face to face interview

b Telephone interview

c Video conference interview

d Online interview (E.g. Email, chat)

Step 6: Prepare the interview Questions

a Background questions (Demographics)

b Introductory questions (Context)

c Main questions (research focus)

d Probing questions (follow–up)

e Closing questions (Final thoughts)

Step 7: Pilot – Test the Interview

a Conduct a trial interview

b Refine questions and protocol

Step 8: Ensure validity and reliability

a Validate questions through expert review

b Ensure consistency in questioning

Step 9: Plan for data analysis

a Determine data analysis methods

b Develop a coding scheme

Step 10: Obtain informed consent

a Explain the research purpose

b Ensure confidentiality

c Obtain participant consent

A Sample Interview Guide

Here is a sample interview guide on mathematics as a core subject:

Introduction

a Introduce yourself

b Explain the purpose of the interview

c Ensure confidentiality

Section 1: *Mathematics Background*

a What sparked your interest in mathematics?

b How long have you been studying/teaching mathematics?

c What area of mathematics do you specialise in (e.g., algebra, geometry)?

Section 2: *Teaching/Learning Mathematics*

a What makes mathematics challenging?

b How do you make mathematics engaging and fun?

c What teaching methods do you use to explain complex concepts?

d How do you assess understanding in mathematics?

e What role does technology play in mathematics education?

Section 3: *Importance and Applications*

a Why is mathematics a core subject?

b How does mathematics impact everyday life?

c Can you share examples of real-world applications of mathematics?

d How does mathematics relate to other subjects (e.g., science, engineering)?

e What career opportunities require strong mathematics skills?

Section 4: *Challenges and Future*

a What challenges do you face teaching/learning mathematics?

b How can mathematics education be improved?

c What are your thoughts on mathematics curriculum development?

d How can we increase student interest in mathematics?

e What's the future of mathematics education?

Conclusion

a Thank you for your time

b Any final thoughts/questions

Activity 4.3: Designing an observation guide

Working individually or in pairs, read through the steps to design an observation guide. Consider what topics you would want to investigate in this way. Look at the sample guide. How could it be improved? Which areas do you like?

Steps to develop an observation guide:

Step 1: Define the Research Objective

a Clearly articulate the research question or hypothesis

b Identify the specific behaviours or phenomena to observe

Step 2: Choose the Observation Method

a Participant observation (active participation)

b Non-participant observation (passive observation)

c Structured observation (pre-defined checklist)

d Unstructured observation (open-ended notes)

Step 3: Select the Observation Site

a Natural setting (e.g., classroom, workplace)

b Controlled setting (e.g., laboratory)

c Public or private space

Step 4: Develop an Observation Guide

a Identify key behaviours or events to observe

b Create a checklist or coding scheme

c Establish clear definitions and criteria

Step 5: Train Observers

a Ensure inter-rater reliability

b Provide observer training and guidelines

c Establish consistency in observation

Step 6: Conduct the Observation

a Record observations using, for example, field notes, audio or video recordings, photographs

b Maintain observer neutrality

c Ensure confidentiality

Step 7: Record and Transcribe Data

a Transcribe field notes or recordings

b Code and categorise data

c Use data management software

Step 8: Analyse Data

a Content analysis

b Thematic analysis

c Coding

d Data visualisation

Step 9: Validate Findings

a Member checking (validate with participants)

b Peer debriefing (discuss with classmates)

c Triangulation (use multiple methods)

Step 10: Report Findings

a Write a detailed report

b Include methodology, results and conclusions

c Use visual aids (e.g., tables, figures)

A Sample Observation Guide

Here is a sample observation guide for data collection for an inspection of a lesson in a classroom:

Observer's Name: ____

Date: ____

Time: ____

Location: _____________________________________

Subject/Group: _________________________________

1. *Classroom Environment (20 points)*
    - Organisation and layout (5)
    - Is the room well-organised?
    - Are materials easily accessible?
    - Technology and resources (5)
    - Are technology tools available?
    - Are educational resources adequate?
    - Safety and cleanliness (5)
    - Are hazardous materials handled properly?
    - Is the room clean and well-maintained?
    - Is the lighting sufficient?
    - Is the temperature comfortable?
2. *Teacher Behaviour (30 points)*

    a Instructional methods (10)
    - Are methods engaging and effective?
    - Are different learning styles accommodated?

    b Communication (10)
    - Is communication clear and concise?
    - Does the teacher listen actively?

    c Classroom management (5)
    - Are transitions smooth?
    - Are behavioural expectations clear?

    d Feedback and assessment (5)
    - Is feedback constructive?
    - Are assessments aligned with objectives?

3. *Student Behaviour (30 points)*

a Engagement and participation (10)
- Are students actively engaged?
- Do students participate willingly?

b Motivation and interest (10)
- Are students motivated?
- Is the lesson relevant?

c Collaboration and teamwork (5)
- Do students work together effectively?
- Do students respect each other's ideas?

d Self-directed learning (5)
- Are students encouraged to take ownership?
- Do students set goals?

4. *Lesson Planning (20 points)*

a Clear objectives (5)
- Are objectives clearly stated?
- Are objectives aligned with curriculum?

b Relevant materials (5)
- Are materials relevant and engaging?
- Are materials adequate?

c Effective lesson pacing (5)
- Is pacing appropriate?
- Are transitions smooth?

d Assessment and evaluation (5)
- Is assessment aligned with objectives?
- Is feedback constructive?

Additional Notes:

....

Observer's Comments:

....

Activity 4.4: A Survey

Working individually or in pairs, read through the steps to design a survey. Note the similarities with a questionnaire. Consider what topics you think would suit this type of data collection. Look at the sample survey. Does this seem appropriate? How could it be improved?

Steps to develop a survey:

Step 1: Define the Goal

Clearly state what you want to achieve with the survey.

Step 2: Identify the Audience

Determine who will participate in the survey.

Step 3: Choose a Method

Decide how you will collect data (e.g., online, phone, in-person).

Step 4: Develop Questions

Create clear, concise questions that will get you the information you need.

Step 5: Test the Survey

Pilot-test with a small group to ensure the questions are understood and work as you expected.

Step 6: Collect Data

Gather responses from the identified audience.

Step 7: Analyse Data

Examine and summarise the results.

Step 8: Interpret Results

Draw conclusions based on the data analysis.

Step 9: Report Findings

Share the results with stakeholders.

Step 10: Evaluate and Improve

Assess the survey process and make adjustments for future surveys.

A Sample Survey

Here is a sample survey to investigate student subject choice:

Student Subject Choice Survey

Introduction:

Thank you for participating in this survey. Your responses will help us understand your interests and preferences in subject choices. All answers will be kept confidential.

Section 1: *Demographics*

1. Age: ______________________________
2. Grade Level: ______________________________
3. Current subjects: ______________________________

Section 2: *Subject Interests*

1. Which subjects do you enjoy learning the most? (Select up to 3)
   a. Mathematics
   b. Science
   c. English
   d. History
   e. Foreign Language
   f. Arts
   g. Other (please specify) ______________________
2. What factors influence your subject choices? (Select all that apply)
   a. Interest in the subject
   b. Career prospects
   c. Teacher influence
   d. Parental advice
   e. Peer recommendations
   f. Other (please specify) ______________________

Section 3: *Subject Difficulty*

1. How challenging do you find each subject? (Scale: 1-5, where 1 is "easy" and 5 is "difficult")
   a. Mathematics: _______
   b. Science: _______
   c. English: _______
   d. History: _______
   e. Foreign Language: _______
   f. Arts: _______

Section 4: *Career Aspirations*

1. What career do you aspire to? ______________________________
2. How relevant are your current subjects to your career goals? (Scale: 1-5, where 1 is "not relevant" and 5 is "very relevant")________________________

Section 5: *Open-Ended Questions*

1. What motivates you to learn a particular subject?
2. Are there any subjects you wish were offered in your school? If yes, please specify.

Conclusion:

Thank you for taking the time to complete this survey. Your input is valuable.

Application of Digital Technology in Data Collection

In designing these tools, computer application software such as Microsoft Word, Excel, Notepad, etc. can help to make the creating of these data collections easier and faster.

Figure 4.1: Sample for data collection

Some ideas for topic areas which you might like to consider for your own data collection instruments:

1. Science and Technology

   For example, climate change, renewable energy, artificial intelligence or plastic pollution
2. Health and Medicine

   For example, mental health, nutrition, vaccines, addiction or pandemics
3. Social Sciences

   For example, education systems, bullying, gender equality or cultural diversity
4. Environmental Studies

   For example, sustainable agriculture, deforestation or water conservation
5. Economics and Business

   For example, globalisation, consumer behaviour or labour markets
6. History and Politics

   For example, democracy, elections, colonialism or human rights
7. Arts and Literature

   For example, art movements, film studies, literary analysis or music therapy
8. Technology and Society

   For example, social media, cybersecurity or the digital divide.

DATA ORGANISATION AND REPRESENTATION

Data organisation and representation are crucial for making information easily understandable, analysable and interpretable. Effective organisation involves structuring data logically and coherently. Visualisation methods, such as charts, graphs, tables and diagrams, facilitate clear presentation. Various representation methods suit different data types and purposes.

Each method has unique advantages. For example, bar graphs highlight categorical differences, while line graphs track temporal changes. By selecting the appropriate representation, data can be communicated effectively, enabling informed decision-making and insightful analysis. Proper data organisation and representation transform complex information into actionable knowledge.

Pictorial Representation of Data

Pictorial representation of data, or data visualisation, graphically displays information to facilitate understanding and insight. Its purpose is to simplify complex data, enhance comprehension and communicate information effectively. Pictorial representation includes bar charts, pie charts, line graphs, scatter plots and histogram.

Real–world applications span business, healthcare, education, finance, etc. Consideration for Effective data visualisation includes clarity, accuracy, simplicity, color scheme, labelling etc.

Cumulative Frequency Curve (Ogive)

An ogive, or cumulative frequency curve, visually displays a dataset's cumulative frequency distribution. This graphical representation helps analyse the distribution and identify key statistics like median, quartiles and percentiles.

The ogive provides a clear picture of how cumulative totals accumulate, facilitating understanding of the dataset's distribution and statistical measures.

Activity 4.5: Constructing a cumulative frequency curve (or Ogive)

Working in small groups follow the steps below to create a cumulative frequency curve:

Step 1: Gather data: collect the raw data you want to represent with a cumulative frequency curve

Step 2: Put your data into a frequency distribution table showing class intervals and frequencies

Step 3: Calculate cumulative frequencies by adding each class's frequency to previous totals.

Step 4: Plot cumulative frequencies against each class interval's *upper boundary*.

Step 5: Connect plotted points with a smooth curve to form the ogive.

Step 6: Give your graph a title and label your axes: Make sure to properly label the x-axis and the y-axis (cumulative frequency).

Example 4.1:

Draw an ogive for the distribution of test scores for twenty students in music class

Table 4.1: *Frequency Distribution Table*

Test Score in class intervals	Frequency
$0 < x \leq 20$	4
$20 < x \leq 40$	6
$40 < x \leq 60$	5
$60 < x \leq 80$	3
$80 < x \leq 100$	2

Solution

Table 4.2: *Cumulative Frequency Distribution Table*

Test Score	Frequency	Upper boundary	Cumulative Frequency
$0 < x \leq 20$	4	20	4
$20 < x \leq 40$	6	40	10 (4+6)
$40 < x \leq 60$	5	60	15(4+6+5)
$60 < x \leq 80$	3	80	18(4+6+5+3)
$80 < x \leq 100$	2	90	20(4+6+5+3+2)

Plot the following points from the cumulative frequency table above on the graph sheet; (20, 4), (40, 10), (60, 15), (80, 18) and (100, 20).

Scale: On the x-axis, 2 cm : 20 units and 1 cm : 2 units on y-axis

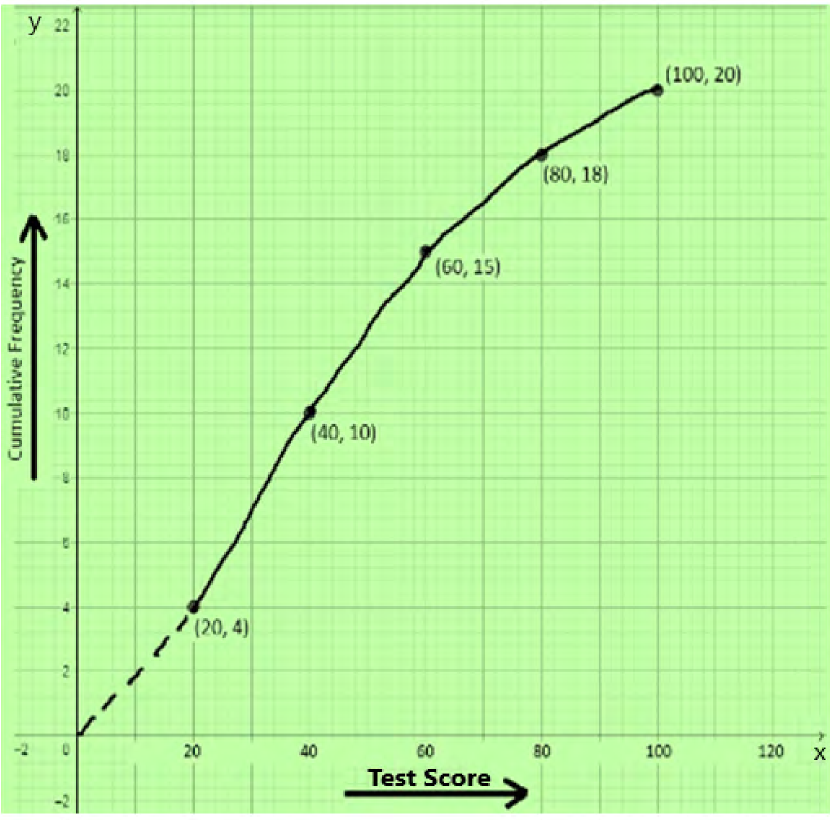


Figure 4.2: A cumulative frequency curve to show the results from a music test

Example 4.2

Construct a cumulative frequency distribution graph, also known as an ogive, using the data on the heights of 50 students, displaying the cumulative frequency against the corresponding height values.

Table 4.3: *Frequency Table*

Height (cm)	Frequency
$140 < x \leq 145$	8
$145 < x \leq 150$	12
$150 < x \leq 155$	18
$155 < x \leq 160$	10
$160 < x \leq 165$	2

Solution

Calculate the cumulative frequency for each class interval:

Table 4.4: *Cumulative Frequency Table*

Height (cm)	Frequency	Upper boundary	Cumulative freq.
$140 < x \leq 145$	8	145	8
$145 < x \leq 150$	12	150	20
$150 < x \leq 155$	18	155	38
$155 < x \leq 160$	10	160	48
$160 < x \leq 165$	2	165	50

Plot your cumulative frequency curve, remembering to use the upper boundary.

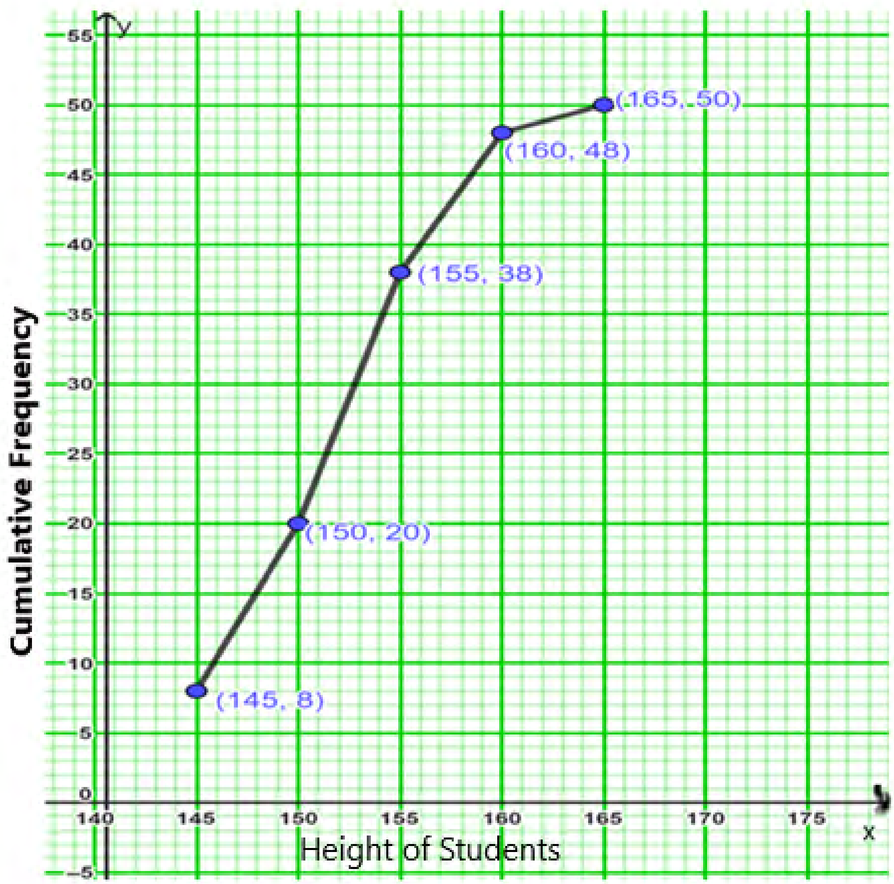


Figure 4.3: A cumulative frequency curve to show the heights of students

Types of Ogives

The graphs above demonstrate the more common 'less than' ogive curves, which shows cumulative frequency. However, there is also another type – the 'more than ogive', which shows the cumulative relative frequency.

***Less than* Ogive (or cumulative frequency Ogive)** displays the cumulative frequency of values less than or equal to a specific value.

For example, a school wants to know how many students are shorter than 165 cm. We would plot the cumulative frequency against the upper boundary of each class interval as we have done above.

***More than* Ogive (or Cumulative Relative frequency Ogive)** displays the cumulative frequency of values greater than or equal to a specific value.

For example, a company wants to know how many employees earn more than GHC 200,000.00 per year. We would plot the cumulative frequency against the lower boundary of each interval. Therefore, if the class intervals are 0 – 10, 11 – 20, 31- 40 etc, the 'more than' ogive will plot the cumulative frequency against 0, 11, 31 etc

Activity 4.6: Constructing a More than Ogive

Working in small groups follow the steps below to create a more than ogive:

Step 1: Calculate Cumulative Frequency:

- Determine the cumulative frequency for each class interval by successively adding the frequencies, starting from the highest-class interval.

Step 2: Identify Lower Class boundaries

- Use the lower boundary of each class interval for plotting.

Step 3: Plot Points:

- Plot the cumulative frequency values against the lower-class boundaries on a graph.

Step 4: Draw the Curve:

- Connect the plotted points with a smooth curve.

Example 4.3:

Graph the two ogives for the following frequency distribution of the weekly wages of the given number of workers at Serene Hotel. Use your curves to find the median wage.

Table 4.5: *Frequency Distribution Table*

Weekly wages	Number of workers
$0< x \leq 20$	4
$20< x \leq 40$	5
$40< x \leq 60$	6
$60< x \leq 80$	3

Solution

Table 4.6: *Cumulative Frequency Table*

Weekly wages	Number of workers	C.F. (Less than)	C.F. (More than)
$0 < x \leq 20$	4	4	18 (total)
$20 < x \leq 40$	5	9 (4+5)	14 (18-4)
$40 < x \leq 60$	6	15 (9+6)	9 (14-5)
$60 < x \leq 80$	3	18 (15+3)	3 (9-6)

For plotting the *less than* curve, use the points (20,4), (40,9), (60,15) and (80,18). These are joined, freehand, to obtain the less than ogive.

For plotting the *more than* curve, use the points (0,18), (20,14), (40,9) and (60,3). These are joined, freehand, to obtain the more than ogive.

These are shown in the graph below.

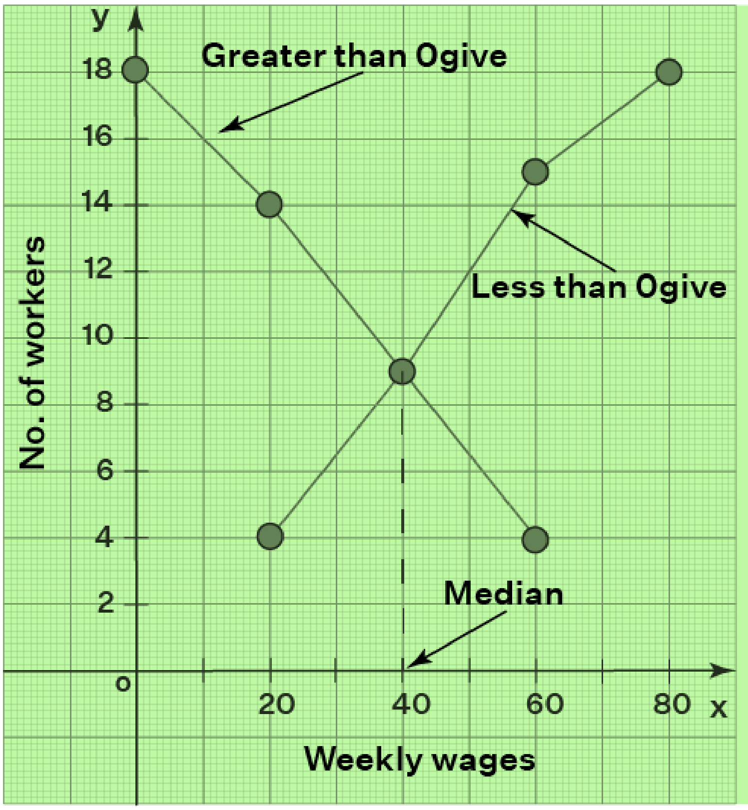


Figure 4.4: A cumulative frequency curve showing the weekly wages of hotel workers

The median: A perpendicular line on the x-axis is drawn from the point of intersection of these curves. This perpendicular line meets the x-axis at a certain point. This determines the median. In this case, the median is 40.

Waffle Charts

Waffle charts are a visually appealing and effective way to represent data proportions and percentages. They consist of a grid of small squares or "waffles", each representing a specific portion of the whole, typically 1% or another fixed unit of measure. This makes waffle charts particularly useful for comparing parts of a whole in a clear and concise manner.

Waffle charts are an excellent alternative to pie charts and bar graphs when you want to emphasise the composition of different categories within a dataset. They are easy to read and understand, making them a popular choice for presentations and reports where visual clarity is crucial.

Activity 4.7: Constructing a Waffle Chart

Working in pairs follow the steps below to create a waffle chart.

Step 1: Gather the data you want to represent.

Step 2: Convert the data into percentages if it is not already in this form.

Step 3: Design a 10x10 grid for 100 squares, or adjust the grid size based on your data.

Step 4: Fill the squares according to the data proportions using distinct colours for each category.

Step 5: Add labels and a legend (or key) to help interpret the chart.

What are the benefits of waffle charts?

- Waffle charts facilitate easy comparison of categorical data
- They provide a clear, visual representation of data making it easy to understand.
- A waffle chart is ideal for displaying large datasets in a compact space.
- They break down complex data into simple visual chunks.

Example 4.4

The following data illustrate the three-year enrollment growth in a Senior High School in Keta. Represent it using a waffle chart.

Table 4.7: *Enrollment Data*

Year	Enrollment Increase (%)
2018	25
2019	45
2020	78

Solution

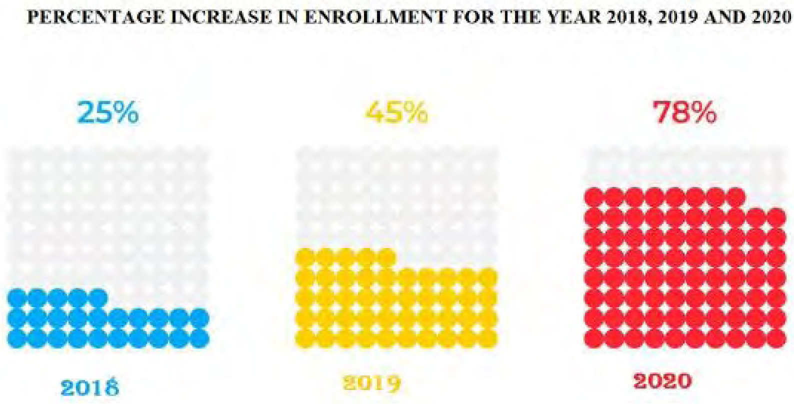


Figure 4.5: Waffle chart showing enrollment growth

Now think about data that you consider would suit being represented in a waffle chart.

Box Plots

A box and whisker plot (or box plot) is a graph that displays the data distribution by using five numbers. Those five numbers are the minimum, first (lower) quartile, median, third (upper) quartile and maximum.

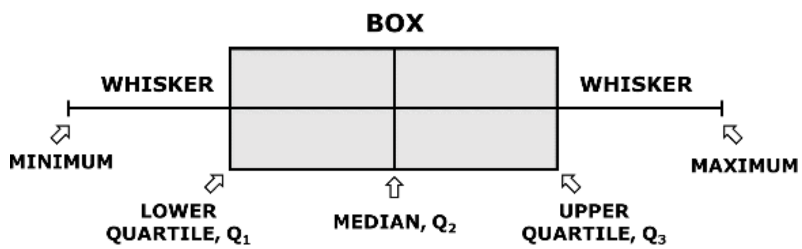


Figure 4.6: A box and whisker plot

In a box and whisker plot: The left and right sides of the 'box' are the lower and upper quartiles. The box covers the interquartile interval, where 50% of the data is found.

- The vertical line that splits the box in two is the median. Sometimes, the mean is also indicated by a dot or a cross on the box plot.
- The whiskers are the two lines outside the box, which go from the minimum data value to the lower quartile (the start of the box) and then from the upper quartile (the end of the box) to the maximum data value.

Activity 4.8: Creating a box and whisker plot

Working individually create a box-and-whisker plot to visualise the distribution of Dziifa's 20 dice rolls, which yielded the following results: 6, 3, 3, 6, 3, 5, 6, 1, 4, 6, 3, 5, 5, 2, 2, 2, 2, 3, 2, 3.

Follow these steps to do this:

Step 1: The first thing is to order the data from smallest to largest: 1 2 2 2 2 2 3 3 3 3 3 3 4 5 5 5 6 6 6 6

Step 2: Calculate the median value.

Since the number of data points is even, we have:

$$Me = \frac{x_{10} + x_{11}}{2} = \frac{3 + 3}{2} = 3$$

Step 3: Next calculate the lower quartile (Q1). Remember that this is the 'median' of the lower half of the data. Q1 = 2

Step 4: Calculate the upper quartile (Q3). Remember that this is the 'median' of the upper half of the data. Q3 = 5

Step 6: Find the minimum and maximum data values. Here, minimum = 1 and maximum = 6

Step 7: Choose an appropriate scale for your data.

Step 8: Draw a box from the lower quartile value 2 to the value 3, which is the median, and put a vertical line through the median.

Step 9: Then, draw the box from the median to the upper quartile.

Step 10: Then draw your "whiskers". Those are the lines that extend parallel with the scale from the box. The whisker goes from the lower quartile to the minimum value, 1, and from the upper quartile to the maximum value, 6.

And this is what your box plot should look like:

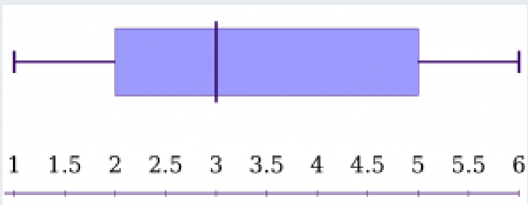
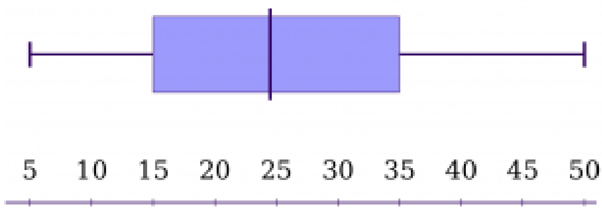


Figure 4.7: A box plot of Dziifa's 20 dice rolls

Now let us do an example where we need to interpret a box plot.

Example 4.5

Find the range, the interquartile range and the median of the data from the box plot below.



Solution

The minimum value of the given data is 5 and the maximum is 50

∴ The range is R = 50 – 5 = 45.

The lower quartile is 15 and the upper quartile is 35.

∴ The interquartile range is IQR = 35 – 15 = 20

The median is line in the centre of the box which is 25.

Now think about data that you consider would suit being represented in a box plot and why it would work well.

Activity 4.9: Model and solve real-life problems involving data

Embracing our immediate environment, we can uncover valuable data on pressing community concerns. By harnessing this data and leveraging the data presentation techniques we have acquired, we can uncover insights and drive meaningful discussions on these issues. Consider the following mini-project, designed to bridge the gap between theory and practice, enabling you to apply your data collection, analysis and visualisation skills to authentic, community-focused initiatives.

Can you undertake a similar project addressing concerns within your community?

REAL-LIFE PROJECT ON DATA COLLECTION, ORGANISATION AND PRESENTATION

Survey Title: Student Preferences and Habits

Objective: To collect, organise and present data on student preferences and habits regarding extracurricular activities, study habits and social media usage

Introduction for Respondents:

1. Please answer honestly.
2. Responses are confidential.
3. Ask questions if unsure.

***Section 1**: Demographic Information*

1. Age:
   - o Under 15
   - o 15-17
   - o 18-20
   - o Over 20
2. Gender:
   - o Male
   - o Female
3. Grade Level:
   - o SHS 1
   - o SHS 2
   - o SHS 3

***Section 2**: Extracurricular Activities*

1. What extracurricular activities do you participate in? (Check all that apply)
   - o Sports
   - o Music
   - o Drama
   - o Clubs
   - o Other (please specify)
2. How many hours per week do you dedicate to extracurricular activities?

Section 3: *Study Habits*

1. How many hours per day do you spend studying?
2. What study methods do you prefer? (Check all that apply)
   - o Online resources
   - o Physical textbooks
   - o Notes
   - o Study groups
   - o Other (please specify)

Section 4: *Social Media Usage*

1. Which social media platforms do you use? (Check all that apply)
   - o Facebook
   - o Instagram
   - o Twitter
   - o TikTok
   - o Other (please specify)
2. How many hours per day do you spend on social media?

Process

1. *Data Collection:*
   - o Distribute surveys to 50 students
   - o Collect responses through online forms or paper surveys
   - o Record data in a spreadsheet (e.g., Google Sheets, Excel)
2. *Data Organisation:*
   - o Categorise data by section (extracurricular activities, study habits, social media usage)
   - o Calculate frequencies, means and percentages
   - o Create tables and charts to visualise data
3. *Data Presentation:*
   - o Create a bar chart to display popular extracurricular activities
   - o Use a pie chart to show study method preferences

- Create a histogram to illustrate social media usage hours
- Write a brief report summarising key findings

ANALYSE AND INTERPRET DATA USING MEASURES OF DISPERSION

Activity 4.10: Investigating the Disadvantages of the Mean

Six students each from 3 Arts 1, 3 Arts 2 and 3 Arts 3 in ANNASS took a Core Mathematics examination. The results, in percentages, were as follows:

3 Arts 1: 100, 80, 70, 30, 20 and 0

3 Arts 2: 60, 55, 55, 45, 45 and 40

3 Arts 3: 53, 52, 51, 49, 48 and 47

Working in a group, find the mean mark for each of the three classes. Discuss the results within your group. Can you draw any conclusions from your results?

This activity shows that the mean does not tell us anything about the spread of the data. In each class, the mean mark is 50% yet the performances are very different. In 3 Arts 1, the mean hides the fact someone got 100% and another person got 0%.

Mathematicians have found ways of measuring the spread, or dispersion, of the data, not merely the central tendency. With measures of dispersion, mathematicians can determine how close together or how far apart the values under discussion are from each other. In this way, we can conclude whether the data is closely clustered around the mean, or widely scattered.

The simplest measure of dispersion is the **range**.

Activity 4.11: Finding the range

Still within your groups, find the range for each of the three classes in activity 10 by finding the difference between the highest and lowest marks in each class. Discuss the outcome of this activity with your group. Can you interpret the results?

For 3 Arts 1, 3 Arts 2 and 3 Arts 3, the ranges are 100%, 20% and 6% respectively. Thus, you can appreciate the fact that the wider the range, the more scattered the data.

Let us know look at another measure of dispersion, the **mean absolute deviation.**

Activity 4.12: Finding the mean deviation

If $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the means for 3 Arts 1, 3 Arts 2 and 3 Arts 3 respectively, copy and complete the table below.

Table 4.8: *Means and mean deviations for 3 Arts 1, 3 Arts 2 and 3 Arts 3*

Marks in percentages			Deviations from the mean		
3 Arts 1	3 Arts 2	3 Arts 3	3 Arts 1 $x-\bar{x}_1$	3 Arts 2 $x-\bar{x}_2$	3 Arts 3 $x-\bar{x}_3$
100	60	53	100 – 50 = 50	60 – 50 = 10	53 – 50 = 3
80	55	52	80 – 50 =	55 – 50 =	52 – 50 =
70	55	51	70 – 50 =	55 – 50 =	51 – 50 =
30	45	49	30 – 50 =	45 – 50 =	49 – 50 =
20	45	48	20 – 50 =	45 – 50 =	48 – 50 =
0	40	47	0 – 50 =	40 – 50 =	47 – 50 =
			$\sum (x-\bar{x}_1) =$	$\sum (x-\bar{x}_2) =$	$\sum (x-\bar{x}_3) =$

From this you should have discovered that in each class the sum of deviations from the mean, $\bar{x}$, is always zero. For this reason, this activity looks like an exercise in futility. However, activity 13 will build upon this activity to derive another measure of dispersion.

Activity 4.13: Finding the mean absolute deviation

The absolute value of 10, denoted by $|10| = 10$. Also, $|-10| = 10$.

Therefore, when finding the absolute value, choose the number and neglect the sign. With this knowledge, copy and complete the table below. You can do this alone or with some friends.

Table 4.9: *Mean absolute deviation for 3 Arts 1, 3 Arts 2 and 3 Arts 3*

3 Arts 1 $x-\bar{x}_1$	3 Arts 2 $x-\bar{x}_2$	3 Arts 3 $x-\bar{x}_3$	3 Arts 1 $\|x-\bar{x}_1\|$	3 Arts 2 $\|x-\bar{x}_2\|$	3 Arts 3 $\|x-\bar{x}_3\|$
50	10	3	50		
30	5	2	30		
20	5	1	20		
−20	−5	−1	20		
−30	−5	−2	30		
−50	−10	−3	50		
			$\sum\|x-x_1\| = 200$	$\sum\|x-x_2\| =$	$\sum\|x-x_3\| =$

After completing the table, you can appreciate the fact that there is a relationship between the sum of the absolute deviations from the mean and how scattered the data is. That is, the smaller the value of $\sum|x-\bar{x}|$, the more clustered the data is about the mean. When you divide the sum of the absolute values of the deviations from the mean by the number of values, you will get the measure of dispersion known as the mean absolute deviation. The mean absolute deviations for 3 Arts 1, 3 Arts 2 and 3 Arts 3 are $\frac{200}{6} = 33.3\%$, $\frac{40}{6} = 6.7\%$ and $\frac{12}{6} = 2.0\%$ respectively. Thus, the smaller the mean absolute deviation, the more clustered the data about the mean.

Let us now look at another measure of dispersion known as the **variance**. Activity 14 will assist you to find the variance of a given set of data.

Activity 4.14: Finding the variance of a given set of data

Copy and complete the table below, working alone or with a small group. Here, you must find the square of the absolute deviations from the mean. When you find the sum of the squares of the deviations from the mean and divide this result by the number of values, you will get the measure of dispersion called the **variance**.

Table 4.10: *Finding variance*

3 Art 1 $x-\bar{x}_1$	3 Art 2 $x-\bar{x}_2$	3 Art 3 $x-\bar{x}_3$	3 Art 1 $(x-\bar{x}_1)^2$	3 Art 2 $(x-\bar{x}_1)^2$	3 Art 3 $(x-\bar{x}_1)^2$
50	10	3			
30	5	2			
20	5	1			
−20	−5	−1			
−30	−5	−2			
−50	−10	−3			
			$\sum(x-\bar{x}_1)^2$	$\sum(x-\bar{x}_2)^2$	$\sum(x-\bar{x}_3)^2$

The variance is given by $v = \sum\frac{(x-\bar{x})^2}{n}$. Like all the measures of dispersion, the smaller the variance the more clustered that data about the mean.

The square root of the variance is called the **standard deviation**. The standard deviation is an important measure in banking, the social and physical sciences as well as engineering.

Example 4.6

The ages, in years, of six boys playing football on a park are 12, 14, 16, 18 and 10.

1. Calculate the:
   a. range;
   b. mean absolute deviation;
   c. variance;
   d. standard deviation.
2. What conclusion can you draw from your answers?

Solution

Table 4.11: *Calculating variance and standard deviation*

x	$x - \bar{x}$	$\lvert x - \bar{x} \rvert$	$(x - x)^2$	x^2
12	−2	2	4	144
14	0	0	0	196
16	2	2	4	256
18	4	4	16	324
10	−4	4	16	100
$\sum x = 70$		$\sum \lvert x - \bar{x} \rvert = 12$	$\sum (x - \bar{x})^2 = 40$	$\sum x^2 = 1020$

1. **a** Range = highest value – lowest value = 18 − 10 = 8 years

b The mean absolute deviation $= \frac{\sum \lvert x - \bar{x} \rvert}{n} = \frac{12}{5} = 2.4$ years

c The variance $= \frac{\sum (x - \bar{x})^2}{n} = \frac{40}{5} = 8$

Alternatively, we can use another formula to find the variance.

The variance $= \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{1020}{5} - \left(\frac{70}{5}\right)^2$

$= 204 - 14^2 = 204 - 196 = 8$

d The standard deviation is the square root of the variance. Hence, the standard deviation is $\sqrt{8} = 2.828 \approx 3$ years.

2. One standard deviation of approximately 3 years suggests that most of the values should be about ± 3 years away from the mean. Therefore, with a mean of 14 years and a standard deviation of 3 years shows that many of the data points will be from about 11 to 17 years.

Note, with large datasets and the data being symmetrical about the mean about 68% of data points will fall within 1 standard deviation either side of the mean and 95% of data points within 2 standard deviations. When a data set is small or skewed this will not be as definitive.

Example 4.7

The ages, in years, of the students in 2 Agric. are as follows: 19, 15, 18, 16, 15, 17 and 16.

Find the:

a. median

b. interquartile range

c. quartile deviation.

Solution

a. To find the median, pick the central value after arranging the values of the data in ascending or descending order of magnitude: 15, 15, 16, 16, 17, 18, 19. Thus, the median is 16 years.

b. The interquartile range = upper quartile − lower quartile. Note that the upper and lower quartiles are the values of the data that are associated with the $\frac{3\sum f}{4}th$ and the $\frac{\sum f}{4}th$ positions. Because the quartiles depend on the position of the values of the data, it is expedient to get natural numbers. For this reason, if there is an odd number of values, mathematicians add one to the total frequency before dividing it by 2 or 4. As you can see, the given data has seven values. Thus, we add one to the total frequency before we find the lower and upper quartiles. Thus, the values associated with the second and sixth positions are the lower and upper quartiles respectively. Thus, 15 years and 18 years are the lower and upper quartiles respectively. Since the difference between the upper and lower quartiles is 3, 3 years is the interquartile range.

c. The quartile deviation, also known as the semi-interquartile range, is obtained when the interquartile range is divided by 2. Thus, the quartile deviation is 1.5 years.

EXTENDED READING

1. *Research Methods for High School Students* by Kathleen M. Harris
2. *Data Collection and Analysis: A Handbook for High School Students* by Michael J. Craw
3. *Survey Research Methods for High School Students* by Floyd J. Fowler
4. *Qualitative Research Methods: A Guide for High School Students* by Robert K. Yin
5. *Statistics and Data Analysis for High School Students* by Michael R. Middleton
6. Use a search engine of your choice to get more information on the measures of dispersion. There are several videos on You Tube that will assist you to master this focal area.

b) Constructing a Greater Than Ogive:

2. A high school is analysing the distribution of students' participation in various extracurricular activities. The data for a total of 100 students is as follows:

   a. Sports: 35 students
   b. Music: 25 students
   c. Science: 18 students
   d. Art: 13 students
   e. Drama: 9 students

   Create a waffle chart to visually represent the distribution of students' participation in these extracurricular activities.

3. A statistics teacher collects the final exam scores of 15 students in a class. The scores are as follows: 72, 85, 90, 65, 78, 80, 92, 88, 76, 84, 91, 87, 69, 95, 82. Calculate the minimum, first quartile (Q1), median (Q2), third quartile (Q3), and maximum of the exam scores. Using this information draw a box plot to represent the distribution of the exam scores.

Review Questions Section 4C

1. Which of the following measures is not a measure of dispersion?
   a. mean absolute deviation
   b. median
   c. range
   d. standard deviation

2. The mean and the standard deviation are closely related, true or false?

3. The table below illustrates the marks obtained by 40 students in St. Mary's Boys' SHS in a class test marked out of 10.

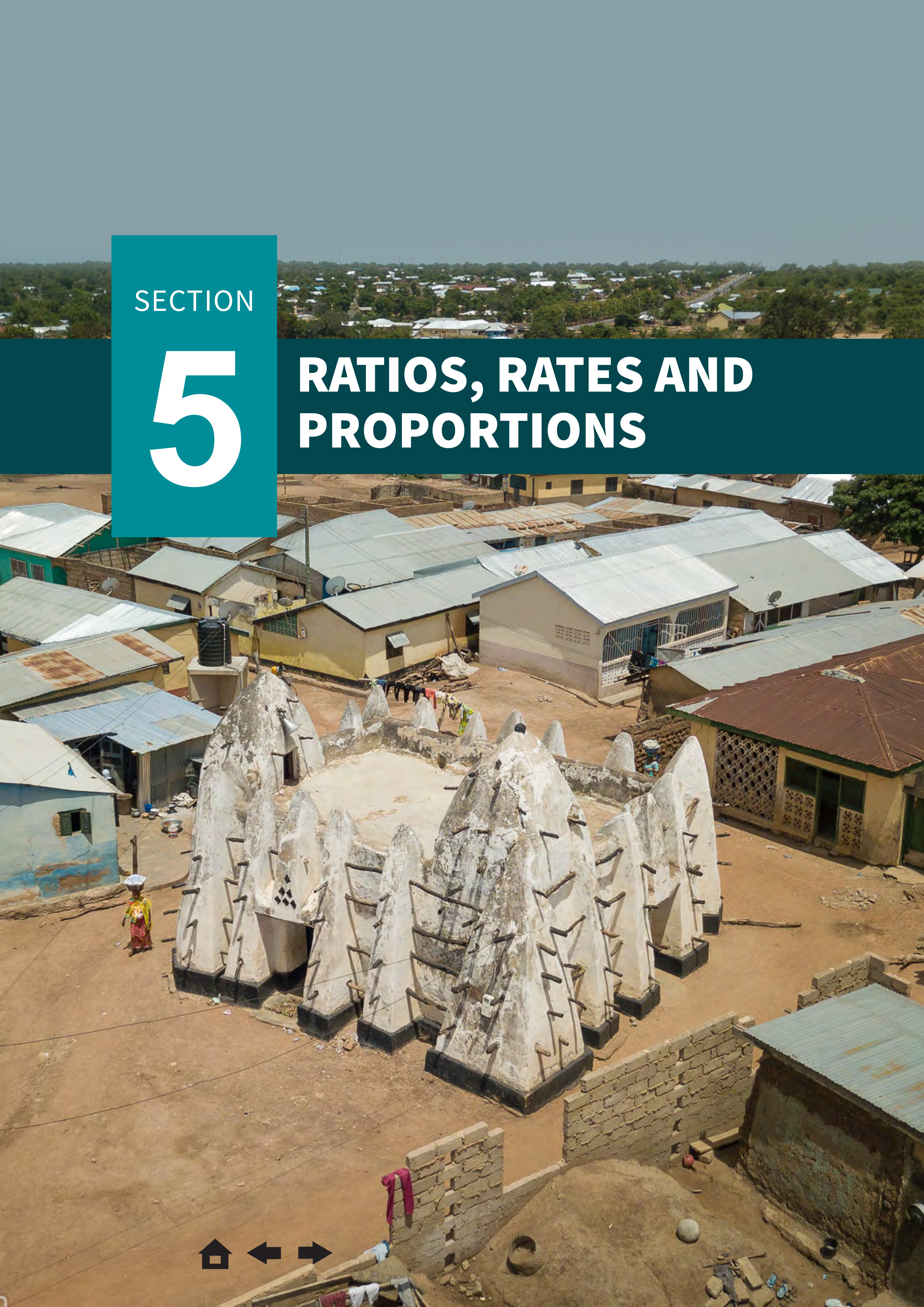
   Table 4.13: *Frequency Table*

Marks	1	2	3	4	5
Frequency	6	14	12	5	3

   Find the quartile deviation for the distribution below and interpret your answer.

SECTION

5 RATIOS, RATES AND PROPORTIONS



NUMBERS FOR EVERYDAY LIFE

Proportional Reasoning

INTRODUCTION

This section covers the concepts of **ratio**, **rate** and **proportion** as methods for comparing quantities. A **ratio** compares two quantities of the same type, while a **rate** compares quantities of different types, for example, speed or price per litre of fuel. **Proportion** then shows the equivalence of two ratios or rates, highlighting how relationships scale in practical situations. Through these concepts you will gain essential skills for real-world problem-solving and quantitative analysis.

KEY IDEAS

- **Distance-time graphs** represent movement; their slope indicates speed and flat sections indicate rest.
- **Proportion**: Proportion is an equation that shows that two ratios or rates are related by a constant.
- **Rate**: A rate compares two quantities of different types, such as distance over time (e.g., miles per hour) or price per unit.
- **Ratio**: A ratio compares two quantities of the same type, indicating how much of one quantity there is relative to the other.
- **Scaling Ratios**: Ratios can be scaled up or down to create equivalent ratios, helping to set up and solve proportions in various scenarios.
- **Speed** is calculated as the rate of change of distance over time
- **Unit Rate**: The unit rate is the rate for one unit of a given quantity, for example, kilometres per hour or cost per item.

CONCEPTS OF RATIOS AND RATES

Ratio versus Rate

You have encountered the concepts of ratios and rates in Junior High School Mathematics. Now, we will explore how these concepts apply to our everyday lives.

Ratios

A ratio shows how many times one number contains another. For example, if there are 10 boys and 30 girls in a classroom, the ratio of boys to girls is expressed as $10 : 30 = 1 : 3$ (read as 1 to 3), meaning for every boy, there are three girls. The two main types of ratios we will discuss are part-to-part and part-to-whole ratios.

Part-to-Part Ratio

A part-to-part ratio compares the amounts of two distinct components within a mixture.

For example, consider the preparation of a popular Ghanaian dish like jollof rice. If you are making jollof rice and you use 2 cups of rice and 1 cup of tomato paste, the ratio of rice to tomato paste is a part-to-part ratio. This means you are comparing the amount of rice to the amount of tomato paste directly, which can be expressed as 2:1. This ratio indicates that for every 2 cups of rice, you need 1 cup of tomato paste.

Part-to-Whole Ratio

On the other hand, a part-to-whole ratio compares one component to the total amount of all components in a mixture. Let's take another example involving a popular Ghanaian beverage, sobolo (hibiscus tea).

For example**,** when preparing sobolo, if you use 3 cups of dried hibiscus flowers, 1 cup of sugar, and 4 cups of water, the total amount of the mixture is $3 + 1 + 4 = 8$ cups. The ratio of dried hibiscus flowers to the total mixture is a part-to-whole ratio. In this case, it can be expressed as 3:8. This means that out of the total 8 cups of sobolo, 3 cups are made up of dried hibiscus flowers.

To establish a ratio, we compare two quantities with the same unit of measure. The ratio can be written as a:b or $\frac{a}{b}$ and is read as "a to b."

We can create a ratio that compares the number of males to the number of females in your class, school, town, district, region or country. You can express it as:

Number of males : Number of females

Example 5.1

In a class containing 25 boys and 15 girls, find the ratio of:

1. Boys to girls
2. Girls to boys

Solution

1. Ratio of boys to girls = 25: 15 = 5:3
2. Ratio of girls to boys15: 25 = 3:5

Now undertake the following activities to deepen your understanding of ratios.

Activity 5.1: Understanding part-to-part ratio by mixing part of gari and part of sugar

Materials:

- Gari
- Sugar
- Measuring cups
- A bowl or container

Procedure:

Step 1: Measure the gari: Use a measuring cup to measure out 2 cups of gari.

Step 2: Measure the sugar: Use a measuring cup to measure out 1 cup of sugar.

Step 3: Record the measurements: Write down the measurements of gari and sugar in your notebook.

Step 4: Understand the part-to-part ratio: Recognise that the ratio of gari to sugar is 2:1. This means that for every 2 parts of gari, there is 1 part of sugar.

Step 5: Mix the gari and sugar in a bowl or container.

Step 6: Observe and record: Observe the mixture and record your findings.

Step 7: Simplify the ratio (if necessary): If the ratio is not in its simplest form, simplify it by dividing both numbers by their greatest common divisor.

Share your findings with the class, including the measurements, ratio and observations.

How does the ratio of gari to sugar affect the taste and texture of the mixture?

Activity 5.2: Finding the ratio of teaching staff to non-teaching staff and non-teaching staff to teaching staff in your school

Materials:

Pen/pencil, Paper/notebook and access to school administration office.

Procedure:

Step 1: Visit the school administration office: Politely ask for the number of teaching and non-teaching staff in the school.

Step 2: Record the numbers: Write down the numbers provided by the administration.

Step 3: Find the ratio of teaching staff to non-teaching staff: For example, if there are 50 teachers and 30 non-teachers, the ratio is 50 : 30.

Step 4: Find the ratio of non-teaching staff to teaching staff: From the example above, this would be 30 : 50.

Step 5: Simplify the ratios (if necessary): If the ratios are not in their simplest form, divide both numbers by their highest common denominator. In our example, this would make 5:3 and 3:5 respectively.

Step 6: Record and present your findings

Share your ratios with classmates and compare results.

Discuss the meaning of the ratios and how they relate to school operations and planning.

Activity 5.3: Measure the height of your desk and chair and find the ratio of desk height to chair height

Materials:

Ruler or measuring tape, desk, chair, pencil or pen, paper

Procedure:

Step 1: Measure desk height from floor to top surface.

Step 2: Measure chair height from floor to top of seat.

Step 3: Record measurements.

Step 4: Find the ratio of desk height to chair height.

Step 5: Simplify the ratio if necessary.

Step 6: Share findings with a partner or the class.

Example 5.2

In your school football team, there are 11 players and 4 substitutes. What is the ratio of:

1. Players to substitutes.
2. Substitutes to players.

Solution

a. Number of players = 11

b. Number of substitutes = 4

1. Players to substitutes

   Step 1: Write the ratio format: Players : Substitutes = 11:4

   Step 2: Check for simplification: The numbers 11 and 4 have no common factor other than 1.

   Step 3: Final ratio: Players to substitutes = 11 : 4.

   Therefore, the ratio of players to substitutes is **11 : 4**.

2. Substitutes to players

   Step 1: Write the ratio format: Substitutes : Players = 4:11.

   Step 2: Check for simplification: The numbers 4 and 11 have no common factor other than 1.

   Step 3: Final ratio: Substitutes to players = 4 : 11.

   Therefore, the ratio of substitutes to players is **4 : 11.**

Example 5.3

In a community of 1200 people, there are 700 adults and the rest are children. Find the ratio of adults to children.

Solution

Given: total population= 1,200 and the population of adults = 700

This implies that the population of children = Total population – Adults population
1 200 – 7,00 = 500 children

Therefore, the ratio of adults to children is 700: 500 = 7: 5

Example 5.4

A gold-buying company called **Day Break** in Prestea has a workforce of 120 employees. 30 of these employees are female. Find the ratio of male employees to female employees.

Solution

Step 1: Calculate the number of male employees

Total employees = 120, Female employees = 30

Male employees = Total employees – Female employees = 120 – 30 = 90

Step 2: Write the ratio

Male employees : Female employees = 90:30

Step 3: Simplify the ratio

Find the highest common factor (HCF) of 90 and 30, which is 30.

Divide both terms of the ratio by 30:

90 ÷ 30 = 3, 30 ÷ 30 = 1

Therefore, the ratio of male employees to female employees is **3 : 1**.

Rates

A **rate** compares two quantities of different types or units. It describes how one quantity changes in relation to another, such as the speed of a vehicle and the distance it travels.

Example 5.5

A train covers a distance of 150 km in 2 hours.

Calculate its speed.

Solution

Step 1: Recall the formula for speed

Speed (or rate) $= \frac{\textit{Total distance traveled}}{\textit{Total time taken}}$

Step 2: Substitute the given values into the formula:

Distance = 150 km

Time = 2 hours

Speed (or rate) $= \frac{\textit{Total distance traveled}}{\textit{Time taken}} = \frac{150km}{2\ hours}$

Step 3: Perform the division

Speed = 75km/h

Therefore, the speed of the train is **75 km/h**

Example 5.6

A bus travels from Accra to Kumasi, covering a distance of 250 km in 5 hours. Determine its rate.

Solution

The rate (or speed) of the car is written as $\frac{\textit{Total distance traveled}}{\textit{Total time taken}} = \frac{250km}{5hours} = 50km/h$

Undertake the activity below to deepen your understanding of rates.

Activity 5.4: Understanding rates by calculating transport fares

Materials:

- A list of transport fares for different routes to your school
- Calculators
- Paper and pencils

Procedure:

Step 1: Choose a route: For example, Korkorse to Nchumuruman SHS

Step 2: Find the transport fare: Look up the transport fare for the chosen route.

Step 3: Calculate the rate: Calculate the rate of the transport fare per kilometre. For example, if the fare is GH¢5 for a 10km journey, the rate would be $\frac{Gh¢5}{10km}$ = 0.5 which means Gh¢0.50 per kilometre travelled.

Step 4: Record the rate: Record the rate on a piece of paper.

Step 5: Compare rates: Compare the rates for different routes with your classmates and discuss any observations.

Discuss how understanding rates can help in real-life situations, such as planning a trip or budgeting for transportation.

Example 5.7

A farm has 28 animals and the ratio of cows to goats is 4:3.

How many cows and goats are there?

Solution

Method 1:

Since the total ratio is 4 + 3 = 7, we partition the 28 animals into 7 groups of 4 as follows:

Figure 5.1: 4 groups of 4 cows

Figure 5.2: 3 groups of 4 goats

We observe from the figure above that the four groups of cows account for a total of 16 cows, while the three groups of goats make up a total of 12 goats.

As a result, the farm has 16 cows and 12 goats. Check, 16:12 = 4:3

Method 2:

Given that the ratio of cows to goats is 4:3, we can continuously group the animals in sets of fours for cows and sets of threes for goats until all the animals are grouped. The table below illustrates this process.

Table 5.1: *Ratios*

Step/group	Cows	Goats	Ratio	Total
1st	4	3	4:3	7
2nd	8	6	8:6 = 4:3	14
3rd	12	9	12:9 = 4:3	21
4th	**16**	**12**	16:12: = 4:3	**28**

We will see that on the 4th step there are 16 cows and 12 goats making a total of 28 animals.

Method 3:

Let x be the number of parts in the ratio.

Given the total number of animals to be 28, we set up the equation: $4x + 3x = 28$.

Solve for x:

$7x = 28$,

$\therefore x = 4$

Calculating for the number of cows,

We have $4x = 4(4) = 16$,

Calculating for the number of goats, we have $3x = 3(4) = 12$.

Therefore, there are 16 cows and 12 goats in the farm.

CONNECTION BETWEEN RATIOS AND RATES

If you have a ratio and you want to use it in a different situation or with a different group, we can use proportions to create *equivalent ratios* and work out any unknown amounts in the same way. Proportions can be seen as an equation of two equivalent ratios or rates.

Let us now engage in activities involving **setting up and solving proportions** to apply a given ratio or rate to a new scenario.

Example 5.8

A family uses 4 cups of groundnut paste to make groundnut soup for 8 servings.

How much groundnut paste would they need for 20 servings?

Solution

Step 1: Identify the known ratio:

4 cups of groundnut paste → 8 servings.

This can be expressed as a ratio: $\frac{4\ cups}{8\ servings}$

Step 2: Set up a Proportion**:**

We want to find the amount of groundnut paste (x) for 20 servings:

$$\frac{4\ cups}{8\ servings} = \frac{x cups}{20\ servings}$$

Step 3: Solve the Proportion:

Using cross-multiplication:

$4 \times 20 = 8 \times x$

$80 = 8x$

Divide both sides by 8 to isolate x:

$x = \frac{80}{8}$

$x = 10$

Therefore, the family would need **10 cups of groundnut paste** to make soup for 20 servings.

Example 5.9

If a pot of waakye for 4 people needs 2 cups of beans, how many cups of beans are required for 10 people?

Solution

We set up the proportion as $2cups \rightarrow 4$ people and $xcups \rightarrow 10$ people

$\Rightarrow \frac{2cups}{4\ people} = \frac{xcups}{10\ people}$

Now, solve for x:

$x = \frac{2 \times 10}{4} = \frac{20}{4}$

$x = 5$

So, 5 cups of beans are required for 10 people.

Example 5.10

On a map of Korkorse, 1 inch represents 10 kilometres.

How many kilometres are represented by 4 inches on the map?

Solution

Step 1: Identify the scale of the map:

The scale of the map is 1 inch represents 10 kilometres.

Step 2: Identify the number of inches to be converted:

We need to find the distance represented by 4 inches on the map.

Step 3: Set up the proportion:

Since 1 inch represents 10 kilometres, we can set up a proportion to find the distance represented by 4 inches:

$$\frac{1\ inch}{10\ km} = \frac{4\ inches}{\times\ km}$$

Step 4: Cross-multiply and solve for x

$1 \times \times = 4 \times 10$

$x = 40$

Therefore, the distance represented by 4 inches on the map is 40 kilometres.

Example 5.11

If a pack of sachet water (30 sachets) costs Gh¢5.00, how much would it cost for 100 sachets?

Solution

Step 1: Identify the known rate:

Gh¢5.00 → 30 sachets

This can be written as a rate: $\frac{\text{Gh¢5}}{30\ sachets}$

Step 2: Set Up a Proportion:

We want to find the cost (x) for 100 sachets:

$$\frac{\text{Gh¢5}}{30\ sachets} = \frac{\text{Gh¢x}}{100\ sachets}$$

Step 3: Using cross-multiplication solve for x:

$5 \times 100 = 30 \times x$

$500 = 30x$

Divide both sides by 30 to isolate x:

$x = \frac{500}{30}$

$x = 16.67$

Therefore, the cost of 100 sachets of water would be **Gh¢16.67**.

Example 5.12

A model of the Kantanka Onantefo SUV is built to a scale of 1:18.

If the actual length of the SUV is 4.5 metres, how long is the model SUV?

Solution

Step 1: Understand the scale ratio:

The scale of the model is **1:18**, which means 1 unit of length on the model represents 18 units of the same length on the actual vehicle.

Step 2: Write down the given information:

Actual length of the SUV = **4.5 metres**

Scale = **1:18**

Step 3: Set up a proportion:

The ratio of the model's length to the actual length is equal to the scale:

$$\frac{\text{Model lenth}}{\text{Actual length}} = \frac{1}{18}$$

Let the model length be × (in metres). Substituting the actual length:

$$\frac{x}{4.5} = \frac{1}{18}$$

Step 4: Solve for ***x***: Cross-multiply to eliminate the fraction:

$x \times 18 = 4.5 \times 1$

$18x = 4.5$

Divide both sides by 18 to isolate x

$x = \frac{4.5}{18} = 0.25$

Therefore, the model SUV is **0.25 metres (= 25 cm)** long

APPLICATION OF PROPORTIONS

Now let's explore proportions which involve comparing two ratios or rates in depth. For example, when calculating the speed of a vehicle, we typically compare the distance covered with the time it takes. Equivalent proportions are essentially the same, even though they may appear slightly different in form.

You can determine if proportions are equivalent by verifying if their ratios are equal. Examples of this can be observed in geometric shapes like similar triangles.

Ratios of Angles in Triangles

When discussing ratios in angles, we often refer to the relationships between the angles of a triangle. The ratios can help us understand how the angles compare with each other and can be useful in solving problems involving triangles.

Understanding the application of ratios to angles in triangles is crucial as it lays the foundation for various mathematical concepts and real-world applications.

Pythagoras' Theorem

This fundamental concept in geometry states that, in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

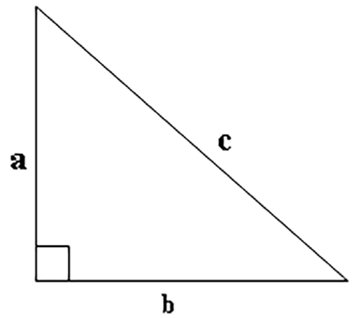


Figure 5.3: Representation of Pythagoras's theorem

Mathematical Representation of Pythagoras's theorem from the figure above is:

$a^2 + b^2 = c^2$

Where:

a and **b** are the lengths of the two sides that are adjacent to the right angle

c is the length of the hypotenuse (the side opposite the right angle and the longest side)

For example: In a right-angled triangle with sides of length 3, 4 and 5:

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

$25 = 25$

This demonstrates that Pythagoras' Theorem holds true for this triangle and will hold true for all right-angled triangles.

Remember too that all that lovely trigonometry is all about trigonometric ratios. Where, in a right-angled triangle the following always holds true with these ratio:

$$sin\theta = \frac{Opposite}{Hypotenuse}$$

$$cos\theta = \frac{Adjacent}{Hypotenuse}$$

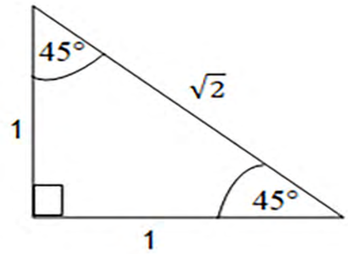
$$tan\theta = \frac{Opposite}{Adjacent}$$

Example 5.13

If the interior angles of a triangle are 45°, 45° and 90°:

1. What is the ratio of its interior angles?
2. What is the ratio of its side lengths?

Solution



1. The ratio of the interior angles is: 45 : 45 : 90, which simplifies to 1 : 1 : 2.
2. The ratio of its side lengths is: 1 : 1 : $\sqrt{2}$, as the triangle must be isosceles so we can make:
   $a = b = 1$ and $c = \sqrt{1^2 + 1^2} = \sqrt{2}$

Everyday uses of proportions

Proportions help us solve problems in everyday life when one part of a relationship is unknown.

Let us explore the practical applications of proportions in our daily lives through the following examples.

Example 5. 14

A kente cloth weaver wants to create a traditional kente design by combining red and yellow reels.

If they use 20 reels of red, how many threads of yellow should they add to maintain a 4:3 ratio of red to yellow?

Solution

Step 1: Write down the ratio: Red : Yellow = 4:3

Step 2: Identify the number of red threads: Number of red threads = 20

Step 3: Set up a proportion:

Since the ratio of red to yellow is 4:3, we can set up a proportion with the number of red threads:

$\frac{20}{x} = \frac{4}{3}$

Step 4: Solve for x:

To solve for x, we can cross-multiply:

$20 \times 3 = 4 \times x$

$60 = 4x$

Now, divide both sides by 4:

$x = \frac{60}{4}$

$x = 15$

Therefore, the kente cloth weaver should add 15 reels of yellow to maintain a 4:3 ratio of red to yellow.

Alternative Method

Table 5.2: *Ratios*

Parts (Kente)	Red reels	Yellow reels	Ratio	Total
1st	4	3	4:3	7
2nd	8	6	8:6	14
3rd	12	9	12:9	21
4th	16	12	16:12	28
5th	20	15	20:15	35

We notice from the table above that for every 20 of red reels there is a corresponding 15 of yellow thread. Thus, 4 : 3 = 20 : 15, so 15 yellow reels are needed.

Example 5.15

A caterer preparing for the Homowo festival wants to make traditional kpokpoi in a 4:1 ratio of cornmeal to palm nut soup.

If they have 16 cups of cornmeal, how many cups of palm nut soup do they need to keep the ratio?

Solution

Step 1: Write down the ratio: Cornmeal : Palm nut soup = 4:1

Step 2: Identify the number of cups of cornmeal: Number of cups of cornmeal = 16

Step 3: Set up a proportion:

Since the ratio of cornmeal to palm nut soup is 4:1, we can set up a proportion.

$\frac{16}{x} = \frac{4}{1}$

Step 4: Solve for x:

Cross-multiply:

$16 \times 1 = 4 \times x$

$16 = 4x$

Now, divide both sides by 4:

$x = \frac{16}{4}$

$x = 4$

Therefore, the caterer needs 4 cups of palm nut soup to keep the 4:1 ratio of cornmeal to palm nut soup.

DISTANCE-TIME GRAPHS

Distance-time graphs, also known as travel graphs, are graphical representations that illustrate the relationship between the distance travelled by an object or person and the time taken to cover that distance.

A distance-time graph typically displays:

1. Distance (in units such as metres, kilometres, miles, etc.) on the vertical axis (y-axis)
2. Time (in units such as seconds, minutes, hours, etc.) on the horizontal axis (x-axis)

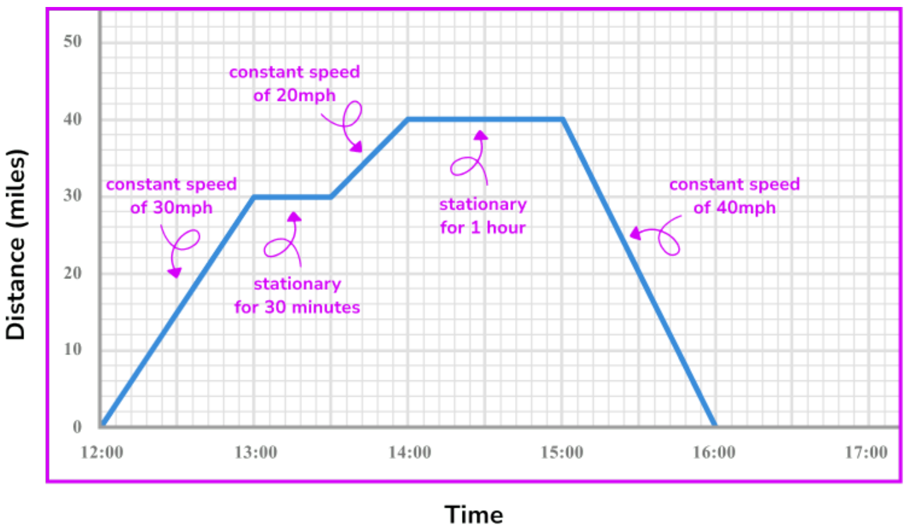


Figure 5.4: Features of a distance-time graph

The implications of the gradient of a distance time graph

1. In a distance-time graph, the gradient (or slope) of the line indicates the speed or velocity of the object.
2. It illustrates the rate at which distance changes in relation to time.
3. Mathematically, $gradient = \frac{rise}{run} = \frac{change\ in\ distance}{change\ in\ time}$
4. A steeper gradient suggests that the object covers more distance in a shorter amount of time, indicating a higher speed as illustrated by the ***30mph line*** in the figure above.
5. A shallower gradient means the object covers less distance over a longer period, reflecting a slower speed as illustrated by the ***20mph line*** above.
6. A horizontal line (with zero gradient) shows no change in distance over time, meaning the object is at rest as illustrated by the ***flat lines*** above.
7. Note that from 15:00 to 16:00 hours the object is returning, at a constant speed of 40mph, to the starting point. This means that the distance represented on the y-axis is the distance from a fixed start point.
8. Straight lines represent constant speed or velocity. If the lines are curved, concave or convex, this represents acceleration or deceleration respectively, as shown below.

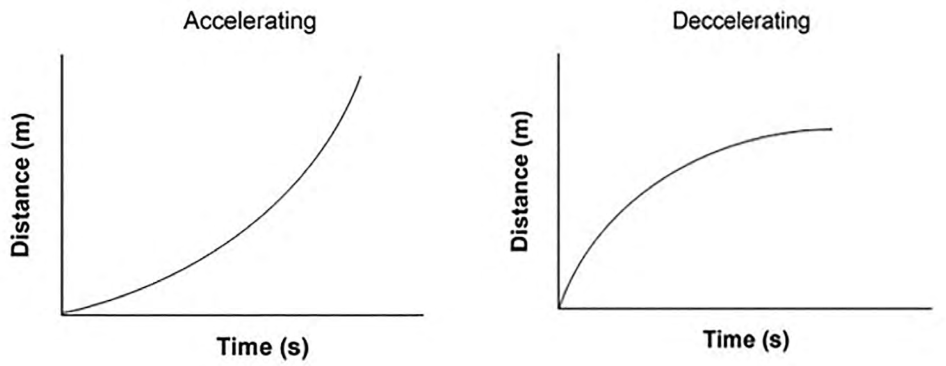


Figure 5.5: Acceleration and deceleration

Working in pairs, do the following activity to understand how to plot a distance time graph.

Activity 5.5: Understanding how to plot a distance-time graph and calculate average speed.

Materials:

- Graph paper
- Pencil
- Ruler
- Calculator

Instructions:

1. Draw your axes:

The x-axis represents time in hours and should go from 0 to 3 hours.

The y-axis represents distance in kilometres and should go from 0 to 150 kms.

2. Plot the points: On the graph paper, plot the points (0, 0) and (3, 150) to represent the distance travelled by a bus over time.

3. Draw the line: Draw a straight line through the two points to represent the distance-time graph. This shows a steady speed.

4. Calculate the average speed: Use the formula:

$$Average\ speed = \frac{Total\ distance}{Total\ time}$$

In this case, the total distance is 150 km and the total time is 3 hours.

5. Record the average speed: Write down the average speed on a piece of paper.

6. Analyse the graph: Look at the distance-time graph and describe how it appears.

It should be a straight line with a positive gradient showing a steady speed

Example 5.16

A cyclist covers a distance of 36 km in 1.5 hours.

What is the average speed and how would this appear on a distance-time graph?

Solution

Step 1: Write down the given information: Distance covered = 36 km

Time taken = 1.5 hours

Step 2: Recall the formula for average speed: $Average\ Speed = \frac{Total\ Distance}{Total\ Time}$

Step 3: Substitute the values into the formula: $Average\ Speed = \frac{36\ km}{1.5\ hours}$

Therefore, the average speed = 24 km/h

On a distance-time graph, this would appear as a straight line with a gradient (or slope) representing the average speed of 24 km/h. The line would start at the origin (0 km, 0 hours) and end at the point (36 km, 1.5 hours).

Example 5.17

On a distance-time graph, a line goes from (0 hours, 0 km) to (4 hours, 200 km).

What is the speed of the object?

Solution

Calculate Speed:

$$\text{Speed} = \frac{Total\ distance}{Total\ time}$$

$$\text{Speed} = \frac{200\ km}{4\ hours} = 50\ \text{km/h}.$$

Therefore, the speed of the object is 50 km/h.

Example 5.18

A swimmer covers a distance of 1 500 metres in 25 minutes.

What was their speed in m/s?

Solution

Step 1: Write down the given information:

Distance covered = 1 500 metres

Time taken = 25 minutes

Step 2: Convert the time from minutes to seconds:

Since there are 60 seconds in a minute, multiply the time in minutes by 60:

25 minutes × 60 = 1 500 seconds

Step 3: Recall the formula for speed:

$$\text{Speed} = \frac{Distance}{Time}$$

Step 4: Substitute the values into the formula:

$$\text{Speed} = \frac{1{,}500\ metres}{1{,}500\ seconds}$$

Speed = 1 m/s

Therefore, the swimmer's speed is 1m/s

APPLICATION OF RATIOS, RATES AND PROPORTIONS TO FINANCIAL MATHEMATICS

Ratios, rates and proportions are fundamental tools in financial mathematics, **health, sharing of properties, sports, utility bills**, **Currency Exchange**, etc.

Let us examine how these concepts are applied in real-life scenarios.

Everyday Applications

Example 5.19

A nutritionist recommends a ratio of 50% carbohydrates, 25% protein and 25% fat in a diet plan. If a person aims to consume 2 200 worth of calories per day, how many calories should come from carbohydrates?

Solution

To find the number of calories from carbohydrates:

Total daily calories = 2 200

Percentage of carbohydrates = 50% = 0.5

Calories from carbohydrates = Total daily calories × Percentage of carbohydrates

= 2200 × 0.5

= 1100

So, the person should allocate 1100 calories from carbohydrates.

Example 5.20

A different dietitian suggests a ratio of 60% carbohydrates, 20% protein and 20% fat for a healthy eating plan. If someone is budgeting 2 500 for their daily caloric intake, how many calories should come from protein?

Solution

Step 1: Write down the given information

Total daily calories = 2 500

Percentage of protein = 20% = 0.2

Step 2: Recall the formula to find the calories from protein:

Calories from protein = Total daily calories × Percentage of protein

Step 3: Substitute the values into the formula:

Calories from protein = 2 500 × 0.2

Step 4: Calculate the calories from protein:

Calories from protein = 500

Therefore, the amount of calories from protein should be 500.

Example 5.21

A health coach advises a ratio of 45% carbohydrates, 35% protein and 20% fat in a meal plan.

If an individual is planning to use1800 calories each day, how many calories should come from fat?

Solution

Step 1: Write down the given information:

Total daily calories = 1800

Percentage of fat = 20% = 0.2

Step 2: Recall the formula to find the calories from fat:

Calories from fat = Total daily calories × Percentage of fat

Step 3: Substitute the values into the formula:

Calories from fat = 1800 × 0.2

Step 4: Calculate the calories from fat:

Calories from fat = 360

Therefore, the amount of calories from fat should be 360.

Blood Pressure and Heart Rate Ratios

Example 5.22

A healthy resting blood pressure is typically within the range of 90 – 120 mmHg (systolic) and 60 – 80 mmHg (diastolic).

If someone's resting systolic blood pressure is 110 mmHg, what is a desirable range for their heart rate?

Generally, a good rule of thumb is that diastolic pressure should be around two-thirds of heart rate plus + 2

Solution

Diastolic Pressure $= \frac{2}{3} \times \textit{Heart Rate} + 2$

Step 1: Determine the desirable diastolic pressure range:

The typical diastolic pressure range is **60 – 80 mmHg**

Step 2: Rearrange the formula to solve for heart rate:

Heart rate $= \dfrac{\text{Diastolic Pressure} - 2}{\frac{2}{3}}$

Heart Rate=1.5 × (Diastolic Pressure – 2)

Step 3: Calculate the heart rate for each extreme value in the diastolic range:

For a diastolic pressure of 60 mmHg:

Heart Rate = 1.5 × (60 – 2) = 1.5 × 58 = 87*bpm*

For a diastolic pressure of 80 mmHg:

Heart Rate = 1.5 × (80 – 2) = 1.5 × 78 = 117 *bpm*

Therefore, the desirable range for heart rate is between about 87 bpm to 117 bpm

Calculating Body Mass Index (BMI)

Body Mass Index (BMI) is a widely used measurement to assess an individual's weight status in relation to their height. The formula is $BMI = \frac{weight(kg)}{height(m)^2}$.

After calculating the BMI, we compare it to the BMI Categories

– *Underweight*: < 18.5

– *Normal weight*: 18.5 – 24.9

– *Overweight*: 25 – 29.9

– *Obese*: ≥ 30

Note, these must be taken with caution as there are many more factors at play here.

Example 5.23

Niyil weighs 85 kg and has a height of 1.8 metres.

Calculate his Body Mass Index (BMI)

Solution

$$BMI = \frac{weight(kg)}{height(m)^2}$$

Weight = 85 kg

Height = 1.8 m

$$BMI = \frac{85}{(1.8)^2} = \frac{85}{3.24} \approx 26.23$$

Niyil's BMI is approximately 26.23

Example 5.24

Yamba Bernice weighs 60 kg and has a height of 1.5 metres.

Calculate her Body Mass Index (BMI)

Solution

$$BMI = \frac{weight(kg)}{height(m)^2}$$

Weight = 60 kg

Height = 1.5 m

$$BMI = \frac{60}{(1.5)^2} = \frac{60}{2.25} \approx 26.67$$

Yamba Bernice's BMI is approximately 26.67

Undertake the activity below on Calculating BMI and knowing the category it falls

Activity 5.6: Calculating Your BMI and Category

Step 1: Measure Your Weight and Height:

- Measure your weight in kilograms (kg) using a weight scale.
- Measure your height in metres (m) using a height measuring tape or stadiometre.

Step 2: Record Your Measurements:

- Write down your weight in kg: _____
- Write down your height in m: _____

Step 3: Calculate Your BMI:

- Use the formula: $\text{BMI} = \frac{weight\ (in\ kg)}{height\ (in\ meters)^2}$
- Substitute your values: BMI = _____ kg / (_____ m)²
- Calculate your BMI: _____

Step 4: Determine Your BMI Category:

- Use the BMI category chart below:
- Underweight: BMI < 18.5
- Normal weight: BMI = 18.5-24.9
- Overweight: BMI = 25-29.9
- Obese: BMI ≥ 30
- Determine which category you fall into: _____

Step 5: Record Your Results:

- Write down your BMI and category: BMI = ____, Category:

However, please note that there are a lot of caveats around this so please do take your category with caution.

Example 5.25

A patient weighing 50 kg is prescribed amoxicillin at a dose of 10 mg/kg/day, to be given in 2 divided doses. How many milligrams of amoxicillin should be given per dose?

Solution

Determine the total daily dose: The prescription states 10 mg/kg/day.

For a patient weighing 50 kg: Total daily dose $= 10mg/kg/day \times 50kg$

$= 500mg/day.$

Divide the total daily dose by the number of doses per day: The medication is to be given in 2 divided doses:

$$\text{Dose per administration} = \frac{\text{Total daily dose}}{Numner\ ofdoses\ per\ day}$$

$$= \frac{500}{2}$$

$$= 250\text{mg}$$

The patient should receive 250 mg of amoxicillin per dose**.**

Example 5.26

A man is 175 cm tall and weighs 82 kg. Calculate his BMI and determine which BMI category he falls into.

Solution

The formula is $BM\text{I} = \frac{weight(kg)}{height(m)^2}$

Convert height to metres

$175\,\text{cm} = 1.75\,\text{m}$

$$BM\text{I} = \frac{82}{(1.75)^2} = \frac{82}{3.0625} \approx 26.8$$

Therefore. the man's BMI is 26.8 kg/m².

Comparing with BMI Categories

$-\ Underweight: < 18.5$

$-\ Normal\ weight: 18.5 - 24.9$

$-\ Overweight: 25 - 29.9$

$-\ Obese: \geq 30$

The man's BMI places him in the **Overweight** category.

Division of Assets Among Heirs

Example 5.27

A will states that a property valued at Gh¢ 250 000 should be divided equally among 5 siblings. How much will each sibling receive?

Solution

Total value of property = Gh¢ 250 000

Number of siblings = 5

Amount per sibling = Total value of property ÷ Number of siblings

= Gh¢ 250 000 ÷ 5 = Gh¢ 50 000

Each sibling will receive Gh¢ 50 000.

Partnership Profit Sharing

Example 5.28

Two partners agree to share profits in a 4:1 ratio.

If the total profit for the year is GH¢75 000, how much will each partner receive?

Solution

Calculate the total parts in the ratio: 4 + 1 = 5 parts

Then, divide the total profit by the total parts:

Total profit = Gh¢75,000

Total parts = 5

Profit per part = Total profit ÷ Total parts = Gh¢75 000 ÷ 5 = Gh¢15 000

Now, multiply the profit per part by each partner's parts:

Partner 1 (4 parts): 4 × Gh¢15 000 = Gh¢60 000

Partner 2 (1 part): 1 × Gh¢15 000 = Gh¢15 000

So, Partner 1 will receive Gh¢60 000 and Partner 2 will receive Gh¢15 000.

Example 5.29

Three partners agree to share profits in a 5:3:2 ratio.

If the total profit for the year is Gh¢90 000, how much will each partner receive?

Solution

5 + 3 + 2 = 10 parts in total

Divide the total profit by the total parts:

Total profit = Gh¢90 000

Total parts = 10

Profit per part = Total profit ÷ Total parts = Gh¢90 000 ÷ 10 = Gh¢9 000

Now, multiply the profit per part by each partner's parts:

Partner 1 (5 parts): 5 × Gh¢9 000 = Gh¢45 000

Partner 2 (3 parts): 3 × Gh¢9 000 = Gh¢27 000

Partner 3 (2 parts): 2 × Gh¢9 000 = Gh¢18 000

So, Partner 1 will receive Gh¢45 000, Partner 2 will receive Gh¢27 000 and Partner 3 will receive Gh¢18 000

Land Area Distribution

Example 5.30

A farmer has 200 hectares of land and wants to distribute it among his 3 daughters in a 2:3:5 ratio. How much land will each daughter receive?

Solution

2 + 3 + 5 = 10 parts in total

Divide the total land by the total parts:

Total land = 200 hectares

Total parts = 10

Land per part = Total land ÷ Total parts = 200 hectares ÷ 10 = 20 hectares

Now, multiply the land per part by each daughter's parts:

Daughter 1 (2 parts): 2 × 20 hectares = 40 hectares

Daughter 2 (3 parts): 3 × 20 hectares = 60 hectares

Daughter 3 (5 parts): 5 × 20 hectares = 100 hectares

So, Daughter 1 will receive 40 hectares, Daughter 2 will receive 60 hectares and Daughter 3 will receive 100 hectares.

Investment Portfolio Allocation

Example 5.31

An investor has a portfolio of Gh¢200 000 and wants to allocate it with 70% in stocks and 30% in bonds.

How much should be invested in each asset class?

Solution

Given:

- Total Portfolio = Gh¢200 000
- Stocks = 70%
- Bonds = 30%

Step 1: Calculate the amount for stocks:

Amount in stocks $= 70\% \times 200\ 000$

$$= \frac{70}{100} \times 200\ 000$$

$$= Gh¢140\ 000$$

Step 2: Calculate the amount for bonds:

Amount in bonds $= 30\% \times 200\ 000$

$$= \frac{30}{100} \times 200\ 000$$

$$= Gh¢60\ 000$$

Therefore, the amount that should be invested in **Stocks** is Gh¢140 000 and **Bonds** is Gh¢60 000

Shooting Percentages in Sports

Example 5.32

A soccer player takes 20 penalty kicks during a season and successfully scores 15 of them.

What is their scoring percentage for the season?

Solution

Successful penalty kicks = 15

Total penalty kicks attempted = 20

$$\text{Scoring percentage} = \frac{\textit{Successful penalty kicks}}{\textit{Total penalty kicks attempted}} \times 100$$

$$= \frac{15}{20} \times 100$$

$$= 0.75 \times 100$$

$$= 75\%$$

The player's scoring percentage for the season is 75%

Currency Exchange

Currency Exchange refers to the process of converting one country's currency into another country's currency. This is typically done based on the current exchange rate, which is the value of one currency in relation to another.

Example 5.33

The exchange rate between the Ghanaian cedi (Gh¢) and the US dollar (USD) is 1 USD = Gh¢. 12.50 How much would $100 USD is in cedis?

Solution

To find the equivalent amount in Ghanaian cedis, multiply the amount in US dollars by the exchange rate:

$100 USD × 12.50 Gh¢/USD = Gh¢1 250

So, $100 USD is equivalent to Gh¢1 250

Commission Rates for Transactions

Example 5.34

When exchanging currency, there might be a commission fee. If you're exchanging €200 (EUR) and the commission rate is 3%, what is the total cost in EUR?

Solution

To find the total cost, calculate the commission fee and add it to the original amount:

$\text{Commission rate} = 3\% = \frac{3}{100} = 0.03$

Original amount = €200

Commission fee = Original amount × Commission rate = €200 × 0.03 = €6

Total cost = Original amount + Commission fee = €200 + €6 = €206

Therefore, the total cost is €206.

Purchasing Power Comparisons

Purchasing power refers to the amount of goods and services that can be bought with a currency in a particular place. Even if exchange rates are similar, the cost of living can differ from one country to another.

For example: In Ghana, Gh¢10 might be enough to buy a loaf of bread, while in the US, the same loaf could cost $2 USD, which is equivalent to Gh¢20.40 at the current the exchange rate. Although the exchange rate makes the cedi appear stronger, the lower cost of living in Ghana means that $2 USD would go further in Ghana than in the US.

Currency Pair Ratios in Forex Trading

Forex traders predict how currency exchange rates will change in the future. They often use currency pair ratios to compare the value of two currencies.

For example, the EUR/USD pair represents the Euro against the US Dollar. If the ratio is 1.20, it means 1 Euro costs 1.20 US Dollars. Traders analyse different factors to anticipate whether this ratio will rise (indicating a stronger Euro against the Dollar) or fall (indicating a stronger Dollar against the Euro), which influences their potential gains or losses.

Utility Bills

Activity 5.7: Electricity consumption project

Objective:

Calculate and report the average daily electricity consumption of your family for a month.

Follow the instructions to carry out electricity consumption project:

Step 1: Gather Information: Collect your family's electricity bills for the past month.

Step 2: Record Data: Note down the total kilowatt-hours (kWh) consumed for the month.

Step 3: Calculate Daily Average: Divide the total kWh consumed by the number of days in the month to find the average daily consumption.

Step 4: Additional Observations: Take note of any factors that might have affected your family's electricity consumption, such as changes in weather, number of people at home, or new appliances.

Step 5: Report Findings: Prepare a short report or presentation to share with the class, including:

1. Total monthly electricity consumption (kWh)
2. Average daily electricity consumption (kWh)
3. Any notable observations or factors affecting consumption

This project aims to help you understand your family's electricity consumption patterns and encourage responsible energy usage.

Drug Dosage Calculations

Drug dosage calculations are crucial in healthcare to ensure patients receive the correct amount of medication.

Common dosage calculation formulas

$$Dosage = \frac{(Prescribed\ dose \times Patient\ weight)}{Standard\ dose}$$

$$Drip\ rate\ (mL/hr) = \frac{(Volume \times Drip\ factor)}{Time}$$

$$\text{Dose per kilogram: Dose (mg/kg)} = \frac{\text{(Desired dose} \times \text{Weight in kg)}}{\text{Dose on hand}}$$

$$\text{Dose per pound: Dose (mg/lb)} = \frac{\text{(Desired dose} \times \text{Weight in lb)}}{\text{Dose on hand}}$$

$$\text{Volume to be administered: Volume (mL)} = \frac{\text{(Dose} \times \text{Volume of solution)}}{\text{Strength of solution}}$$

In real-world healthcare settings, accurate dosage calculations play a critical role in patient care.

Specifically:

1. Nurses rely on these calculations daily to administer medications safely and effectively.
2. Pharmacists verify dosage calculations to ensure accuracy and prevent medication errors.

3. Doctors prescribe medications based on these calculations, taking into account individual patient factors and medication ratios.

EXTENDED READING

1. *Akrong Series: Core mathematics for Senior High Schools* New International Edition (P ages 319 - 325)
2. *Aki – Ola series: Core Mathematics for Senior High Schools in West Africa*, Millennium edition 5 (Pages 300– 303)
3. Baffour A. (2015). *Baffour BA series: Core mathematics. Accra*: Mega Heights, (Pages 327 - 362)

14. What does a concave-upward curve on a distance-time graph typically signify?

15. What event is represented when two or more lines meet at a single point on a distance-time graph?

16. What shape or feature on a distance-time graph indicates that an object is slowing down?

17. What physical quantity is represented by the area beneath the curve on a distance-time graph?

18. Describe the shape of a distance-time graph for an object that travels away from a reference point, reverses direction and returns to the starting point.

19. How would you represent a trotro that travels at a steady speed from Accra to Tema, stops for a break and then continues at a faster speed to Kumasi on a distance-time graph?

20. A bus accelerates from rest to 25 m/s over 15 seconds. What is its acceleration, and how would this be represented on a distance-time graph?

21. A distance-time graph shows a line from (0s, 0m) to (4s, 40m), then a horizontal line to (8s, 40m). What was the object's average speed over the entire 8 seconds?

22. A motorbike travels 25 km east in 20 minutes, then returns 15 km west in 15 minutes. Calculate the total distance traveled and the displacement. How would this journey appear on a distance-time graph?

23. If the ratio of the sides of a right-angled triangle is 15: 20: x, where × is the hypotenuse. Find the missing length x and use these values to establish the interior angles of the triangle, to the nearest degree.

24. A basketball player takes 8 free throws during a game. They successfully make 5 of them. What is their success rate (win-loss ratio) expressed as a decimal for free throws during this game?

25. Player × has an efficiency rating of 102, while Player Y has a rating of 95. Based on this rating system, which player is considered statistically more successful?

26. The exchange rate between the US dollar (USD) and the Japanese Yen (JPY) is 1 USD = 150 JPY. You have $300 USD and want to convert it to Yen. How many Yen will you receive?

SECTION

6 PATTERNS AND RELATIONS INVOLVING SEQUENCES AND SERIES

ALGEBRAIC REASONING

Patterns and Relationships

INTRODUCTION

A teacher was tired and wanted to rest. To occupy the class, he asked them to add the first 100 natural numbers. The teacher was surprised that in less than no time, one learner had found the answer. This happened at a time when scientific calculators and computers had not been invented. Use a search engine of your choice to find the name of this brilliant learner. Can you find the sum of an arithmetic progression in less than one minute using a short cut? What is the secret?

In this section, you will learn the secret behind this mathematical short cut. First, we will investigate how to find a formula that will help us generate all the terms of both arithmetic and geometric progressions. This formula is usually called the general or *nth* term of a sequence. Next, you will deduce a formula that will help you to find the sum of the first n terms of linear and exponential sequences.

In finance, sequences and series are used to calculate compound interest as well as interest rates. When designing structures, engineers employ sequences and series. In computer science, they are used to improve algorithms that facilitate searching and sorting items. In the physical and social sciences, they are used to investigate population growth of humans and bacteria.

KEY IDEAS

- **Exponential sequence**: If the ratio of any two consecutive terms of a sequence is a constant, it is an *exponential sequence* or *geometric progression*.
- **Linear sequence**: If the difference between any two consecutive terms of a sequence is a constant, it is an *arithmetic progression* or *linear sequence*.
- **Sequence**: A set of numbers in which order is important is known as a sequence. For example, the set of natural numbers is a sequence, or the set of even numbers, or odd numbers are sequences.
- **Series**: When you replace the commas of a sequence with addition signs, the sequence becomes a series.

PATTERNS AND RELATIONS INVOLVING SEQUENCE AND SERIES PART 1

Activity 6.1: Investigating natural and artificial patterns

In small groups, discuss where you will find patterns on the school campus. Take your classmates to the place where the patterns are and show the patterns to them. Can you find patterns in nature? Are there natural patterns on campus? Floor tiles, bathroom tiles and clothing are good examples of patterns created by humans. Games like chess and draught are played on boards with patterns. The leaves of plants, the teeth and the feathers of some animals have patterns designed by the omnipotent creator, God Almighty.

Activity 6.2: Generating a sequence with equilateral triangles

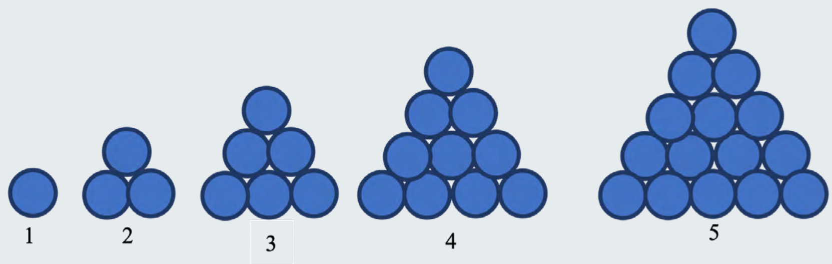


Figure. 6.1: Triangular numbers

Gather some plastic bottle tops and arrange them on a cardboard as shown in Figure 6.1, noting the number of bottle tops after each step.

How many bottle tops are there in the first arrangement? How many are there in the second arrangement? How many are there in the third and so on? Write the sequence generated on a piece of paper and show it to a classmate. Do you realise that each step creates an equilateral triangle? This is because every side of each triangle has the same number of bottle tops.

In the first arrangement, there is only one bottle top. In the second arrangement, there are three bottle tops. In the third arrangement, there are six. The fourth and fifth arrangements have 10 and 15 bottle tops respectively. Thus, the sequence is 1, 3, 6, 10, 15 … These are known as *triangular numbers*.

What type of sequence is this? Is it geometric or arithmetic? It is in fact, neither. There is no constant difference or ratio.

Activity 6.3: Generating a sequence with squares

Take a graph paper. Alone or in pairs, carry out, the following activity, step by step.

Step 1: Draw a one-centimetre square on the graph paper and count the number of one-millimetre squares you have in it.

Step 2: Draw a two-centimetre square on the graph paper and count the number of small squares you have in it.

Step 3: Draw a three-centimetre square on the graph paper and count the number of squares you have in it.

Step 4: Continue this until you have a sequence with six terms. Compare your sequence with that of your classmates. Can you predict the number of one-millimetre squares you will find in an eighteen-centimetre square?

What type of sequence is this? Is it linear, exponential, or neither? Explain how you know.

Activity 6.4: Generating a sequence with matchsticks

Step 1: Take a match box. Use three of the matchsticks to form a triangle.

Step 2: Add 3 sticks to one side of this triangle to form a square on one side. Thus, you have a square attached to the triangle. Note the number of sticks in this arrangement.

Step 3: Add 3 more sticks to the square to form a second square. Note the number of matchsticks in the new arrangement.

Step 4: Add another set of sticks to form a third square and note the number of matchsticks in this arrangement. Write down the sequence generated and show it to a classmate.

Can you predict the number of matchsticks after forming the 20^{th} rectangle?

The sequences you have generated so far are man-made sequences. But some sequences occur in nature? For example, the number of new leaves on a germinating plant follow a sequence. This is a sequence known as the Fibonacci sequence and it is found in many places in nature.

Activity 6.5: Generating the sequence of Fibonacci

Perform this activity with in pairs or individually.

Step 1: Start with the first two terms of this sequence as 0 and 1.

Step 2: Add the first two terms to get the third term.

Step 3: Add the second and third terms to get the fourth term.

Step 4: Add the third and fourth terms to get the fifth term.

Step 5: Add the fourth and fifth terms to get the sixth term.

Thus, you have the sequence, 0, 1, 1, 2, 3, 5, 8, 13, 21 …

These are the first terms of Fibonacci's sequence. Generate the first 20 terms of Fibonacci's sequence.

Activity 6.6: Finding the general term of a linear sequence or arithmetic progression

Step 1: Generate a sequence of numbers, if the first term is 3 and the common difference is 5.

This means that, since the first term is 3, add the constant difference, 5, to get the second term. Then add 5 to the second term to get the third term. Next, add 5 to the third term to get the fourth term and so on. The sequence generated is as follows: 3, 8, 13, 18, 23, 28 …

Step 2: Let the first term of an arithmetic progression or linear sequence be a and let the common difference be d. Write down the first 6 terms of this sequence. Can you predict the 10^{th} and 100^{th} terms? What will the *nth* term be?

Table 6.1 displays the terms of the sequence whose first term is a and common difference is d.

Table 6.1: *Finding the general term of a linear sequence*

Position	Term	Simplified term
1^{st}	a	$a + 0d$
2^{nd}	$a + d$	$a + 1d$
3^{rd}	$(a + d) + d$	$a + 2d$
4^{th}	$(a + d + d) + d$	$a + 3d$
5^{th}	$(a + d + d + d) + d$	$a + 4d$
⋮	⋮	⋮
10^{th}	$(a + d + d + d + d + d + d + d + d) + d$	$a + 9d$
100^{th}	$(a + d + d + d + d + d + d + \ldots) + d$	$a + 99d$
⋮	⋮	⋮
n^{th}	$(a + d + d + d + d + d + d + \ldots) + d$	$a + d(n - 1)$

From Table 6.1, you can realise that the general term, U_n, is given by $\boldsymbol{U_n = a + (n - 1)d}$.

With this formula, you can determine the position of a given term or the term itself.

Example 6.1

Find the 18^{th} and 36^{th} terms of the following sequences:

i. 4, 7, 10, …

ii. 13, 7, 6, …

Solution

i. First you have to determine whether the sequence is linear or exponential in order to use the correct formula.

$10 - 7 = 7 - 4 = 3$. Hence, there is a common difference, $d = 3$, so it is a linear sequence.

The first term is 4. So $a = 4$.

We want the 18^{th} term, so $n = 18$.

Putting all these values into the formula for the general term:

$U_n = a + (n - 1)d$

$\Longrightarrow U_{18} = 4 + (18 - 1)3 = 4 + (17 \times 3) = 4 + 51 = 55$

Thus, the 18^{th} term of this sequence is 55.

ii. Similarly, the 36^{th} term, $U_{36} = 4 + (36 - 1)3 = 4 + 105 = 109$.

Example 6.2

List the set of odd numbers up to the sixth term, starting from one. Find a formula for this sequence. Hence, calculate the 50^{th} term.

Solution

The set of odd numbers, D = {1, 3, 5, 7, 9, 11 …}.

The first term, $a = 1$and the common difference, $d = 2$.

Consequently, the general term

$U_n = a + (n - 1)d = 1 + 2(n - 1) = 1 + 2n - 2 = 2n - 1.$

Hence, the 50^{th} term, $U_{50} = 2(50) - 1 = 100 - 1 = 99$.

Example 6.3

Kofi generated a linear sequence such that the 6^{th} and 9^{th} terms were 19 and 28 respectively. Determine the first term as well as the common difference.

Solution

For any linear sequence, any term is given by the relation $U_n = a + (n - 1)d$.

Thus, $U_6 = a + (6 - 1)d = a + 5d$. Similarly, the 9^{th} term,

$U_9 = a + (9 - 1)d = a + 8d$.

Thus, you have generated two equations, $a + 5d = 19$ and $a + 8d = 28$. Subtracting the first equation from the second, you will get

$(a + 8d) - (a + 5d) = 28 - 19$

$\Longrightarrow a + 8d - a - 5d = 9$

$\Longrightarrow 3d = 9$

$\therefore d = 3.$

Putting $d = 3$ in $a + 5d = 19$ or $a + 8d = 28$, $a + 5(3) = 19 \Longrightarrow a = 19 - 15 = 4$.

Thus, the first term is 4 and the common difference is 3.

Now, you are going to unravel the secret behind the mathematical short cut encountered in the introduction: how to find the sum of the first 100 natural numbers in seconds.

Activity 6.7: Finding the sum of the first 10 natural numbers

Step 1: On the first line in your exercise book, write the first 10 natural numbers with an addition sign separating the terms, starting from the lowest to the highest.

Step 2: On the next line in your exercise book, write the first 10 natural numbers with an addition sign separating the terms, this time, starting from the highest to the lowest as shown in the Table 6.2.

Step 3: Add the number on top to the number below it and write the sum beneath each pair of numbers.

Table 6.2: *Finding sum of the first 10 natural numbers*

1	+ 2	+ 3	+ 4	+ 5	+ 6	+ 7	+ 8	+ 9	+ 10
10	+ 9	+ 8	+ 7	+ 6	+ 5	+ 4	+ 3	+ 2	+ 1
11	**+ 11**	**+ 11**	**+ 11**	**+ 11**	**+ 11**	**+ 11**	**+ 11**	**+ 11**	**+ 11**

Since repeated addition is the same as multiplication, adding 10 elevens is the same as 11 times 10, which is equal to 110. However, by reversing and adding the terms of both sequences, you have doubled the sum. Hence, dividing 110 by 2 gives you the sum of the first 10 numbers. Hence, the sum of the first 10 natural numbers is 55. We can then do a similar task to find the sum of the first 100, or even 1000, natural numbers.

Activity 6.8: Finding the sum of the first 12 even numbers

Repeat the steps in activity 6.7, using the first 12 even numbers. See Table 6.3.

Table 6.3: *Finding the sum of the first 12 even numbers*

2	+ 4	+ 6	+ 8	+ 10	+ 12	+ 14	+ 16	+ 18	+ 20	+ 22	+24
24	+ 22	+ 20	+ 18	+ 16	+ 14	+ 12	+ 10	+ 8	+ 6	+ 4	+ 2
26	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**	**+ 26**

Since repeated addition is the same as multiplication, adding 26 to itself 12 times is the same as 26 times 12, which is equal to 312. However, by reversing and adding the terms of both sequences, you have doubled the sum. Hence, dividing 312 by 2 gives you the sum of the first 12 even numbers. Hence, the sum of the first 12 even numbers is 156.

Activity 6.9: Generating a formula for the sum of the first *n* terms of a linear sequence

To generate a linear sequence whose first term is a and common difference is d, add the common difference to the first term a to get the second term. To get the third term add d to the second term. The fourth term is obtained by adding the common difference to the third term. See table 6.2.

Table 6.4: *Formula for the sum of the first **n** terms*

Position	Terms	Terms in reverse order	Sum of the terms
1st	a	$a + (n - 1)d$	$2a + (n - 1)d$
2nd	$a + d$	$a + (n - 2)d$	$2a + (n - 1)d$
3rd	$a + 2d$	$a + (n - 3)d$	$2a + (n - 1)d$
4th	$a + 3d$	$a + (n - 4)d$	$2a + (n - 1)d$
5th	$a + 4d$	$a + (n - 5)d$	$2a + (n - 1)d$
6th	$a + 5d$	$a + (n - 6)d$	$2a + (n - 1)d$
7th	$a + 6d$	$a + (n - 7)d$	$2a + (n - 1)d$
8th	$a + 7d$	$a + (n - 8)d$	$2a + (n - 1)d$
⋮	⋮	⋮	⋮
n^{th}	$a + (n - 1)d$	a	$2a + (n - 1)d$

From the last column of the table 6.2, the sum of each pair of corresponding terms is $2a + (n - 1)d$. However, there are n terms. Consequently, the sum of the two sequences is $n[2a + (n - 1)d]$. Because the reversal doubled the sum, you have to divide by 2 to get S_n, which is the sum of the first n terms of any linear sequence.

Thus, the sum of the first n terms of any linear sequence is given by:

$$\boldsymbol{S_n = \frac{n}{2}\left[2a + (n - 1)d\right]},$$

where a, d and n are the first term, common difference and position of the term respectively.

Furthermore, $S_n = \frac{n}{2}[2a + (n - 1)d] = S_n = \frac{n}{2}[a + a + (n - 1)d]$. But $a + (n - 1)d$ is the general or last term. Hence:

$$\boldsymbol{S_n = \frac{n}{2}[\textit{first term} + \textit{last term}] = \frac{n}{2}\left[a + l\right]},$$

where l is the last term. This is the secret behind the mathematical trick you encountered in summing the first 100 natural numbers.

Example 6.4

Find the sum of the first 20 terms of the following linear sequences:

i. 6, 9, 12 …

ii. 9, 3, −3 …

Solution

i. For any linear sequence, the sum of the first n terms is given by: $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$.

$d = 12 - 9 = 9 - 6 = 3$ = common difference and first term, $a = 6$.

Therefore, the sum of the first 20 terms, S_{20}, is given by

$S_{20} = \frac{20}{2}[2(6) + (20-1)3] = 10[12 + (19 \times 3)] = 10(12 + 57) = 690.$

ii. $3 - 9 = -3 - 3 = -6$ = common difference, d and first term is 9.

Therefore, the sum of the first 20 terms, S_{20}, is given by:

$$\begin{aligned} S_{20} &= \frac{20}{2}[2(9) + (20-1)(-6)] \\ &= 10[18 + (19 \times -6)] \\ &= 10(18 - 114) \\ &= -960. \end{aligned}$$

Alternatively, you can use the second formula to find the sum of the first 20 terms of the given sequences.

That is, $S_n = \frac{n}{2}[\textit{first term} + \textit{last term}] = \frac{n}{2}[a + l]$.

i. There are 20 terms, so the last term $U_{20} = 6 + 3(20 - 1) = 63$.

Thus, $S_{20} = \frac{20}{2}[6 + 63] = 10(69) = 690.$

ii. There are 20 terms, so the last term $U_{20} = 9 + (-6)(20 - 1) = 9 - 114 = -105$.

Thus, $S_{20} = \frac{20}{2}[9 + (-105)] = 10(-96) = -960.$

Example 6.5

The sum of the first n terms of an arithmetic progression is 375.

If the first term is 4 and the common difference is 3, find the number of terms of this sequence.

Solution

The sum of the first n terms of any arithmetic progression is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$.

The first term, $a = 4$ and the common difference, $d = 3$. The sum of the first n terms is 375. Thus, you have,

$S_n = \frac{n}{2}[2(4) + (n - 1)3] = 375$

$\Longrightarrow \frac{n}{2}[8 + 3n - 3] = 375$

$\Longrightarrow n[3n + 5] = 750$

$\Longrightarrow 3n^2 + 5n - 750 = 0$

$\Longrightarrow 3n^2 + 50n - 45n - 750 = 0$

$\Longrightarrow n(3n + 50) - 15(3n + 50) = 0$

$\Longrightarrow (n - 15)(3n + 50) = 0$

$\therefore n - 15 = 0$ or $3n + 50 = 0$

Thus, $n = 15$ and $n = -\frac{50}{3}$.

Since n cannot be negative, $n = 15$.

That is, the number of terms is 15.

PATTERNS AND RELATIONS INVOLVING SEQUENCE AND SERIES II

Activity 6.10: Finding the general term, or *nth* term, of an exponential sequence (or Geometric Progression, or GP)

Step 1: Generate a sequence of numbers, where the first term is 3 and the common ratio is 5. That is, multiply the first term by the common ratio to get the second term. Then multiply the second term by 5 to get the third term and so on. The sequence generated is as follows: 3, 15, 75, 375, 1775 …

Step 2: Let the first term of a geometric progression or exponential sequence be a and let the common ratio be r. Write down the first 5 terms of this sequence. Can you predict the 10^{th} and 100^{th} terms? What will the nth term be?

In a tabular form, we have:

Table 6.5: *Finding nth term of an exponential sequence*

Position	Term	Simplified term
1^{st}	a	ar^0
2^{nd}	$a \times r$	ar^1
3^{rd}	$ar \times r$	ar^2
4^{th}	$ar^2 \times r$	ar^3
5^{th}	$ar^3 \times r$	ar^4
$\vdots$	$\vdots$	$\vdots$
10^{th}	$ar^8 \times r$	ar^9
100^{th}	$ar^{98} \times r$	ar^{99}
$\vdots$	$\vdots$	$\vdots$
n^{th}	ar^{n-1}	ar^{n-1}

From Table 6.5, you can realise that the general term, U_n, is given by $\boldsymbol{U_n = ar^{n-1}}$.

With this formula, or relation, you can determine the position of a given term or the term itself in a given exponential sequence.

Example 6.6

Find the 12^{th} and 18^{th} terms of the following sequences:

i. 1, 2, 4, 8, …

ii. 9, 4.5, 2.25, 1.125, …

Solution

i. First you have to determine whether the sequence is linear or exponential in order to use the correct formula. $8 \div 4 = 4 \div 2 = 2 \div 1 = 2$.

Hence, the common ratio is $r = 2$. The first term is 1.

For the 12^{th} term, $n = 12$.

Substituting all these values into the formula for the general term, you have:

$U_n = ar^{n-1} \Longrightarrow U_{12} = (1)(2^{12-1}) = 2^{11} = 2048.$

Thus, the 12^{th} term of this exponential sequence is 2048.

Similarly, the 18^{th} term, $U_{18} = (1)(2^{18-1}) = 2^{17} = 131072.$

ii. First you have to determine whether the sequence is linear or exponential in order to use the correct formula. $4.5 \div 9 = 2.25 \div 4.5 = 1.125 \div 2.25 = \frac{1}{2}$.

Hence, the common ratio is $r = \frac{1}{2}$. The first term is 9.

For the 12^{th} term, $n = 12$.

Substituting all these values into the formula for the general term, you have:

$U_n = ar^{n-1} \Longrightarrow U_{12} = (9)(0.5^{12-1}) = \frac{9}{2048}$

Thus, the 12^{th} term of this exponential sequence is $\frac{9}{2048}$.

Similarly, the 18^{th} term, $U_{18} = (9)(0.5^{18-1}) = \frac{9}{131072}$.

Example 6.7

Consider the following sequence: $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots, 32$.

a. Find a general formula for this sequence.

b. How many terms are there in this sequence?

Solution

This is particularly tricky with a lot of indices work, so go through this in a small group, explaining each step to each other.

First, you have to determine the type of sequence in order to use the correct formula.

Since $\frac{1}{8} \div \frac{1}{16} = \frac{1}{16} \div \frac{1}{32} = 2$, the common ratio is 2. Therefore, this is an exponential sequence.

The first term is $\frac{1}{32}$. Putting these values into the equation we have:

$U_n = a\,r^{n-1} = \left(\frac{1}{32}\right) 2^{n-1}$

$= \left(\frac{1}{2^5}\right)(2^n)(2^{-1}) = (2^{-5})(2^n)(2^{-1}) = (2^{-5-1})(2^n) = (2^{-6})(2^n) = 2^{-6+n} = 2^{n-6}$

Thus, the general term, U_n of the sequence is given by $U_n = 2^{n-6} = \frac{2^n}{2^6} = \frac{2^n}{64}$

Since, the last term is 32 and the general term can be used to find any term, you have the equation $\frac{2^n}{64} = 32$.

Now, $\frac{2^n}{64} = 32$

$\Longrightarrow 2^n = 32 \times 64 = 2^5 \times 2^6 = 2^{5+6} = 2^{11}$

$\Longrightarrow 2^n = 2^{11}$

$n = 11$

Thus, there are 11 terms in this sequence.

Example 6.8

The 6th and 9th terms of a geometric progression are 3 and 81 respectively. Determine the first term as well as the common ratio.

Solution

For any exponential sequence, any term is given by the relation $U_n = ar^{n-1}$.

Thus, $U_6 = ar^{6-1} = ar^5 = 3$. Similarly, the 9th term, $U_9 = ar^8 = 81$.

Now, $\frac{U_9}{U_6} = \frac{ar^8}{ar^5} = \frac{81}{3}$

$\frac{ar^8}{ar^5} = \frac{81}{3}$

$\frac{r^8}{r^5} = 27$

$r^{8-5} = 3^3$

$r^3 = 3^3$

$r = 3$

Now, put $r = 3$ in the equation $ar^5 = 3$, $a(3^5) = 3 \Longrightarrow 243a = 3 \Longrightarrow a = \frac{3}{243} = \frac{1}{81}$.

Thus, the first term is $\frac{1}{81}$ and the common ratio is 3.

Activity 6.11: Deriving the formula for the sum of the first n terms of geometric progression.

Step 1:

Generate an exponential sequence in which the first term is a and the common ratio is r.

$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots, ar^{n-2}, ar^{n-1}$.

When the terms of this sequence are added, you will get the sum of the first n terms of an exponential sequence.

Thus, $S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots + ar^{n-2} + ar^{n-1}$.

The sum of the terms of a sequence is also known as a series.

Let us multiply the series S_n by r :

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots + ar^{n-2} + ar^{n-1}$$
$$\Rightarrow rS_n = ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots + ar^{n-2} + ar^{n-1} + ar^n.$$

By now you will have realised that the only difference between the series S_n and rS_n below it are the first term a and the last term ar^n. Thus, when you subtract S_n from rS_n or vice versa, you will get $S_n - rS_n = a - ar^n$ or $rS_n - S_n = ar^n - a$.

Thus, $S_n(1 - r) = a(1 - r^n)$.

Thus, the sum of the first n terms of an exponential sequence, S_n is given by:

$S_n = \frac{a(1-r^n)}{1-r}$, we tend to use this if $r < 1$ and $S_n = \frac{a(r^n-1)}{r-1}$, and use this if $r > 1$.

But either work all the time.

Example 6.9

Find the sum of the first 12 terms of the following geometric progressions or exponential sequence:

i. 6, 3, 1.5, 0.75 …

ii. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3} \cdots$

Solution

i. For any exponential sequence, the sum of the first n terms is given by: $S_n = \frac{a(1-r^n)}{1-r}$, where $r < 1$ and $S_n = \frac{a(r^n-1)}{r-1}$, where $r > 1$.

Now, $1.5 \div 3 = 3 \div 6 = 0.5 =$ common ratio, r. Since $r < 1$, we will use the formula $S_n = \frac{a(1-r^n)}{1-r}$, where $r < 1$ (but the other formula works too!)

Now, the first term $a = 6$ and $n = 12$ since you want to find the sum of the first 12 terms.

Thus, $S_{12} = \frac{6(1-0.5^{12})}{1-0.5} = 12(1-0.5^{12}) = 11.997$ (3 decimal places).

ii. $\frac{1}{9} \div \frac{1}{27} = \frac{1}{3} \div \frac{1}{9} = 3 =$ common ratio. Since r, the common ratio, is $3 > 1$, we will use the formula $S_n = \frac{a(r^n-1)}{r-1}$ (but the other one would work too).

Therefore, $S_{12} = \frac{\frac{1}{27}(3^{12}-1)}{3-1} = \frac{3^{12}-1}{54} = \frac{531441-1}{54} = \frac{531440}{54} = 9841.48$ (to 2 decimal places).

If the common ratio, $r < 1$, then you can find the sum of the series as the number of terms n approaches infinity. The sum of an infinite series as n approaches infinity ($n \to \infty$) is given by the formula $S_\infty = \frac{a}{1-r}$.

Example 6.10

Find the sum of the geometric progression, $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ as the number of terms approaches infinity.

Solution

The common ratio, $r = \frac{1}{3}$ and the first term $a = 1$.

Thus, the sum to infinity, $S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1\frac{1}{2}$.

REAL LIFE PROBLEMS INVOLVING SEQUENCES AND SERIES

We will now look at how the concept of sequences and series are applied in various fields.

Banking: Compound interest

When an amount of money is deposited at a bank, the bank owes the customer interest because the bank uses the customer's money to make a profit. The interest on the deposited amount depends on the interest rate of the bank. If the bank's rate of interest is 10% and you deposit GH¢100.00 on the first of January 2025, on the first of January, 2026, GH¢10.00 will be deposited into your account as the interest. Thus, the amount in your account will be GH¢110.00. Table 6.6 will give you the amount of money in the account at the beginning of every year.

Table 6.6: *Calculating compound interest*

Year	Amount	Simplified amount	Amount
1	100	$100 \times \left(\frac{110}{100}\right)^0$	GH¢100.00
2	$100 \times \frac{110}{100}$	$100 \times \left(\frac{110}{100}\right)^1$	GH¢110.00

Year	Amount	Simplified amount	Amount
3	$100 \times \frac{110}{100} \times \frac{110}{100}$	$100 \times \left(\frac{110}{100}\right)^2$	GH¢121.00
4	$100 \times \frac{110}{100} \times \frac{110}{100} \times \frac{110}{100}$	$100 \times \left(\frac{110}{100}\right)^3$	GH¢133.10
5	$100 \times \frac{110}{100} \times \frac{110}{100} \times \frac{110}{100} \times \frac{110}{100}$	$100 \times \left(\frac{110}{100}\right)^4$	GH¢146.41
⋮	⋮	⋮	
nth		$100 \times \left(\frac{110}{100}\right)^{n-1}$	GH¢$100(1.1)^{n-1}$

Thus, at the beginning of the n^{th} year the amount $U_n = 100 \times \left(\frac{110}{100}\right)^{n-1} = 100(1.1)^{n-1}$.

By now, you will have noticed that the sequence generated in table 6.4 is a geometric progression whose first term, $a = 100$ and the constant ratio, $r = \frac{110}{100} = 1.1$. In this way, sequences and series can be used to calculate the compound interest for customers who have deposits at the bank. Thus, you can use the concept of sequences and series to calculate the compound interest without generating the sequence.

Example 6.9

Mr. Mann deposited GH¢1800.00 in a bank whose interest rate was 9% per annum. Calculate the money in his account after the:

a. 3rd year;

b. 6th year;

c. 12th year.

Solution

At the end of the n^{th} year, the amount of money, U_n, in his account is given by $U_n = ar^n$.

Thus, $U_n = 1800\left(\frac{100+9}{100}\right)^n = 1800\left(\frac{109}{100}\right)^n = 1800(1.09)^n$

a. $U_3 = 1800(1.09)^3 = 2331.0522 \approx$ GH¢2331.05

b. $U_6 = 1800(1.09)^6 = 3018.7802 \approx$ GH¢3018.78

c. $U_{12} = 1800(1.09)^{12} = 5062.7966 \approx$ GH¢5062.80

Example 6.10

When Mr. Ackah started his teaching career his monthly salary was GH¢500.00. It was agreed that every year GH¢100.00 will be added to his salary.

a. Calculate the salary earned after 10 years in the teaching service.

b. How long will it take him to earn GH¢5000.00 a month?

Solution

Forming the sequence for this situation, you will have: 500, 600, 700, …

Thus, you have a linear sequence in which the first term, $a = 500$ and common difference, $d = 100$.

a. The general term, $U_n = a + (n - 1)d$ and the salary after 10 years means that the $n = 10$.

$\therefore U_{10} = 500 + (10 - 1)(100) = 500 + 900 = 1400.$

Thus, after 10 years of teaching his salary was GH¢1400.00.

b. Now, the last or general term, $U_n = 5000$.

Thus, $500 + (n - 1)100 = 5000$

$\Longrightarrow (n - 1)100 = 5000 - 500 = 4500$

$\Longrightarrow n - 1 = 45$

$\Longrightarrow n = 46$

Thus, it will take him 46 years to earn a salary of GH¢5000.00 per month

Example 6.11

Madam Anvo decided to save 12% of her monthly salary every month for her pension. If her salary was GH¢8000.00, calculate the amount saved for her pension after 15 years.

Solution

First, you have to find the amount saved for her pension, which is 12% of the salary.

Thus, the amount saved per month is $\frac{12}{100} \times 8000 = 0.12 \times 8000 =$ GH¢960.00.

This amount can be used to form a sequence: 960, 1920, 2880, 3840, …

Obviously, this is a linear sequence, where the first term is 960 and the common difference is 960.

For this reason,

$U_{15} = 960 + (15 - 1)960 = 960 + 14(960) = 960 + 13440 =$ GH¢14 400.00.

Consequently, after 15 years, Madam Anvo would have saved GH¢14 400.00.

Social sciences

In the social sciences like geography and economics, sequences and series are used to anticipate or predict the population of regions or countries. This helps social scientists and politicians to plan for the future.

Example 6.12

The population growth of Africa's most populous nation, Nigeria, is approximately 3% per year. If the population of Nigeria in 2024 is 240 million, calculate the population of Nigeria in 2030 to the nearest million.

Solution

The population growth will obey an exponential sequence.

The first term $a = 240\ 000\ 000$ and the common ratio $r = \frac{100 + 3}{100} = \frac{103}{100} = 1.03$.

Thus, the general term, P_n, for the yearly population of Nigeria is given by

$P_n = a\,r^n = 240\ 000\ 000(1.03)^n$.

Since the difference between 2024 and 2030 is 6 years, the value of n is 6.

Thus, $P_6 = 240,\ 000,\ 000(1.03)^6 = 286\ 572\ 551.16696 \cong 287 \textit{million}$ (to the nearest million)

Physical sciences

In the physical sciences like biology, it can be important to determine the rate at which living things grow. The chemist might also be interested in the rate at which a radioactive substance decay. The botanist may be interested in the rate at which a particular tree grows. In all such situations, this cannot be done without sequences and series.

Example 6.13

A biochemist realised that the growth rate of bacteria in a medium is 5% per day.

If there were about 1200 bacteria at the beginning of the experiment, how many bacteria were there after 10 days?

Solution

The population growth of the bacteria in the medium follows an exponential sequence in which the first term is 1200 and the common ratio is 1.05.

That is, $a = 1200$, $r = 1.05$, $n = 10$.

Putting all these values into the relation, $P_n = ar^n = 1200(1.05)^n$.

$\Longrightarrow P_6 = 1200(1.05)^6 = 1608.11476875$. Thus, there will be approximately 1 600 bacteria in the medium after 10 days.

Example 6.14

Researchers sent out by the Forestry Commission calculated that a particular tree grows by 0.9 metres every year.

If the tree was 2.7 metres tall this year, how long will it take the tree to reach a height of 36 metres?

Solution

You can generate the sequence to know the type of sequence you are dealing with.

2.7, (2.7 + 0.9), (2.7 + 0.9 + 0.9), (2.7 + 0.9 + 0.9 + 0.9), …= 2.7, 3.6, 4.5, 5.4, …

It is clear that the sequence is linear with the first term, $a = 2.7$ and the common difference is 0.9.

Thus, you have to use the relation $U_n = a + (n - 1)d$.

$\Longrightarrow 2.7 + (n - 1)0.9 = 36$

$\Longrightarrow (n - 1)0.9 = 36 - 2.7 = 33.3$

$n - 1 = \frac{33.3}{0.9} = 37$

$\therefore n = 37 + 1 = 38$

Thus, it will take the tree 38 years to reach a height of 36 metres.

Social Services

Hotels and supermarkets sometimes display items in attractive geometric shapes to attract customers. In such a case, there is the need to apply sequences and series to calculate the number of tins in a display.

Example 6.15

Yaba displays the tins of milo in her shop to form a pyramid. There are 15 tins in the first or bottom row of this pyramid and just one tin on the topmost row. If each successive row has 2 fewer tins than the row below it, calculate the total number of milo tins in the display.

Solution

The sequence in the question is given by 15, 13, 11, 9… 1.

Thus, the first term, $a = 15$ and the common difference, $d = 2$. The total number of tins in the display is the same as the sum of the terms of this sequence. You will notice that this is a linear series in which the constant difference is -2 and $n = 8$. Thus, the sum, $S_n = \frac{n}{2}(a + l)$, where a, l *and* n are the first term, last term and number of terms respectively.

$\therefore S_8 = \frac{8}{2}(15 + 1) = 4(16) = 64$. This means that there are 64 tins in the display.

Auditing

Auditors may be interested in the value at which an institution disposed of its worn-out assets. In such a case, calculating the rate of depreciation of a vehicle or machine can help them to know whether the institution got value for money. Sequences and series are important tools when calculating the rate of depreciation of an item.

Example 6.16

The auditors calculated that the school bus depreciates by 9% of its initial value every year. If the value of the bus was GH¢144 000.00 in 2023, calculate the value of the bus in 2028?

Solution

Depreciation of items normally follows a geometric progression in which the first term is the initial value of the bus (cost price) and the rate of depreciation is 12%. Thus, the common ratio is $100\% - 12\% = 0.88 = r$. Therefore, you have to use the general term of a geometric progression, $U_n = ar^n$, where $n = 2028 - 2023 = 5$.

$U_5 = 144\ 000 \times 0.88^5 = 75993.\ 396 \approx$ GH¢75, 993.40

Thus, in 2028, the value of the bus will be approximately GH¢75 993.40.

EXTENDED READING

1. Akrong Series: *Core mathematics for Senior High Schools* New International Edition (Pages 802 - 808)
2. Aki – Ola series: *Core Mathematics for Senior High Schools in West Africa*, Millennium edition 5 (Pages 300– 303)
3. Baffour A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 701 - 717)

REVIEW QUESTIONS

1. The fifth term of an arithmetic progression is 18 and the eleventh term is 36. Find the sum of the first term and the common difference.

A. 9

B. 8

C. 7

D. 6

2. Find the sum of the odd numbers from 1 up to 99.

A. 5000

B. 2500

C. 2050

D. 1050

3. If $1, 2x + 1, 5, 3x + 4, 9, \dots$ is a linear sequence, find the:

a. value of the constant x ;

b. first 8 terms of the sequence;

c. 15^{th} term;

d. sum of the first 21 terms of the sequence.

4. The sum of the first 6 terms of an arithmetic progression is 21 and the sum of the first 10 terms is 55. Find the sum of the first twenty terms of this sequence.

5. Determine whether the following sets of numbers form linear or exponential sequences.

i. $\frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

ii. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

iii. $3, 1, \frac{1}{3}, \frac{1}{9}, \dots$

6. Write down the first six terms of the geometric progression in which the first term is 2 and the common ratio is $\frac{1}{3}$.

SECTION

7 SURFACE AREAS AND VOLUMES

GEOMETRY AROUND US

Measurement

INTRODUCTION

Being able to solve problems involving surface area and volume in both SI (metric) and imperial units is important in fields like engineering, construction and daily life. In this section, we will review common units used for surface area measurements. In the SI system, we use square metres (m^2), while in the imperial system, square feet (ft^2) or square inches (in^2) are used. Understanding these units and how to convert them is essential for accurate calculations.

We will also focus on solving real-world problems, like determining the amount of paint needed to cover a wall or the material required to wrap a gift. Whether we are using square metres or square feet, knowing how to convert between units is important.

For volume, we will work with SI units like cubic metre (m^3) or litres (l), and imperial units like cubic feet (ft^3) or gallons. Mastering these skills helps us solve practical problems in both school and everyday life.

KEY IDEAS

- **Imperial Units** are measurements historically used in the British Empire and still in use in some countries like the United States. Even in Ghana, carpenters use feet and inches while traders use pounds and gallons. Examples include feet (ft), pounds (lbs) and gallons (gal) for length, weight, and volume, respectively.
- **SI Unit** is the International System of Units (SI). It is the standard system of measurement used globally. It includes units like the metre (m) for length, kilogram (kg) for mass and litre (l) for volume.
- **Surface Area** is the total area of the outer surface of a 3-D (three dimensional) object. It is measured in square units. It represents the amount of material needed to cover the surface of an object, like the surface of a cube or a sphere.

- **Unit Conversion** is the process of changing a measurement from one unit to another, such as converting from centimetres to inches or from litres to gallons. This often requires multiplying or dividing by a conversion factor.
- **Volume/Capacity** is the amount of space a 3-D object occupies, measured in cubic units. Volume refers to the interior space of objects like cubes or spheres, while capacity often refers to the volume of a container, such as how much liquid it can hold.

MEASUREMENT OF SURFACE AREAS INVOLVING IMPERIAL AND SI UNITS

We often come across problems that involve measuring the "surface area" of different objects, whether it is for your science projects, construction tasks or everyday life situations.

Surface Area

"Surface area" refers to the total area of the outer surface of a 3-dimensional object, such as a cube, cylinder or sphere. It is the sum of the areas of all the faces or curved surfaces of the object. For example, the surface area of a cuboid is the sum of the areas of its six faces.

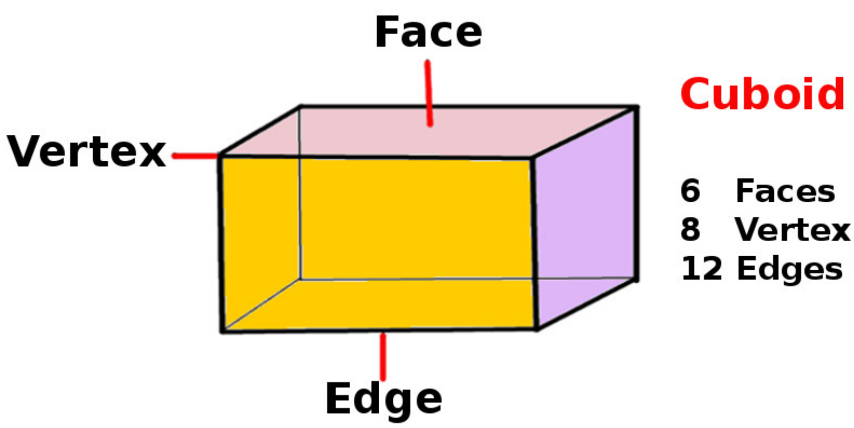


Figure 7.1: Faces, vertices and edges of a cuboid

Calculating the Surface Area of given 3D shapes

The images below illustrate the surface areas of some solid objects (3D shapes).

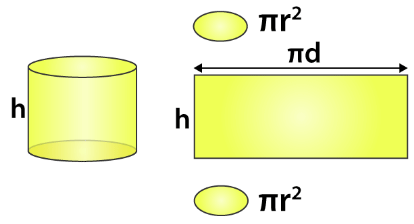


Figure 7.2: Cylinder and its net

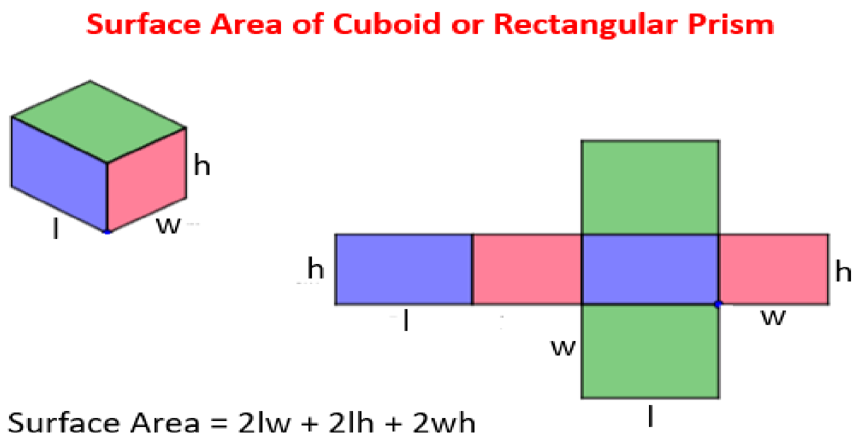


Figure 7.3: Rectangular prism and its net

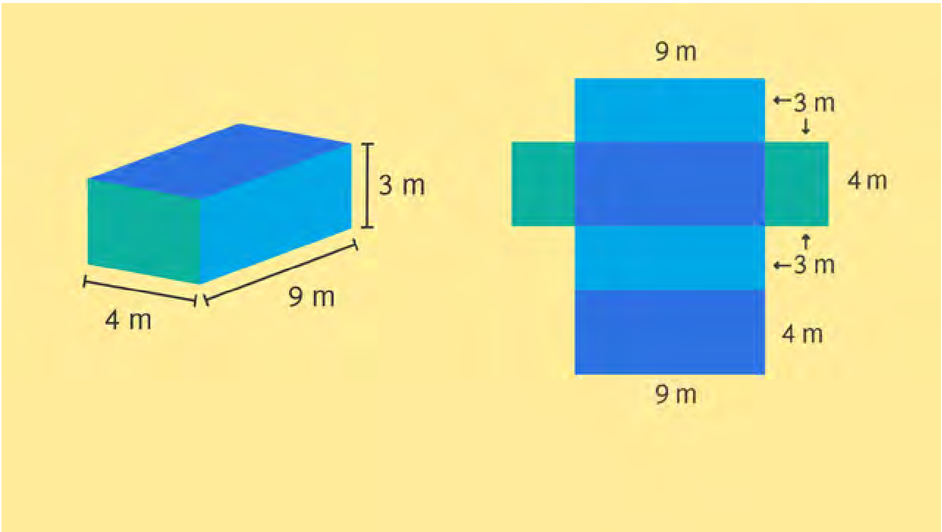


Figure 7.4: Rectangular prism and its net

Activity 7.1: Finding the surface areas of solids

Materials needed: Toilet roll, Milo tins, milk tins, match boxes, chalk boxes, marker boxes, football and any other geometric object.

Step 1: Sort out the materials into cylinders, spheres, cubes and cuboids.

Step 2: Tabulate these objects with columns headed: ***Name of object, Net and Total surface area***.

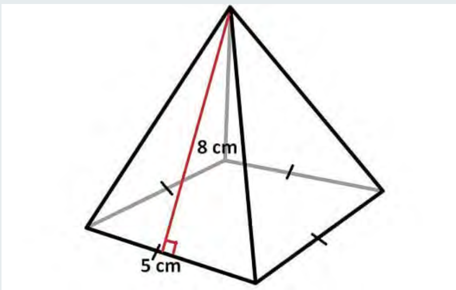
Step 3: Discuss with your classmates and establish the various nets and surface areas of the solids. Show them to your teacher for validation.

Hint: You will realise that when you take any cylinder or box and dismantle it, you obtain the net. The surface area of the cylinder or box will be the area of the net or the sum of the areas of all the faces. See figures 7.2, 7.3 and 7.4

Activity 7.2: Finding the surface area of a square pyramid

Materials needed:

Paper, ruler and a square pyramid.



Step 1: Identify the number of faces. In the above pyramid, the base is a square with side length 5 cm and each of the four walls is a triangle with base 5 cm and height 8 cm.

Step 2: Find the area of one triangle:

Area of each triangular side wall $= \frac{1}{2} \times base \times height = \frac{1}{2} \times 5 \times 8 = 20 \text{ cm}^2$

Step 3: Area of four triangles $= 4 \times 20 = 80cm^2$

Step 4: Find the area of the square base:

Area of the square base $= 5 \times 5 = 25 \text{ cm}^2$

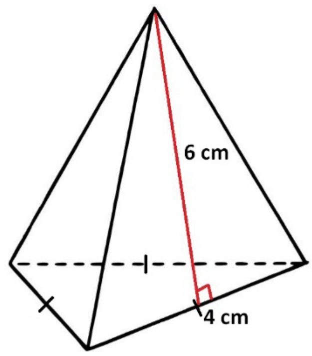
Step 5: Surface area of the pyramid = Sum of areas of all 5 faces

$= (25 + 80) \text{ cm}^2$

Therefore, the surface area of the above pyramid $= 105 \text{ cm}^2$

Example 7.1

Find the surface area of the triangular based pyramid given below.



Solution

Surface area of the pyramid = Sum of areas of all 4 faces

The base is an equilateral triangle with side length 4 cm and each wall is a triangle with base 4 cm and height 6 cm.

Let us find the area of each face.

Area of the base = $\frac{1}{2} \times 2\sqrt{3} \times 4 = 4\sqrt{3}$ cm^2

(Note that the height of the base triangle is found through Pythagoras's theorem:

$h^2 + 2^2 = 4^2 \therefore h = \sqrt{12} = 2\sqrt{3}$)

Area of each side wall = $\frac{1}{2} \times 4 \times 6 = 12$ cm^2

Area of all 3 side walls = $3 \times 12 = 36$ cm^2

Surface area of the pyramid = $(4\sqrt{3} + 36)$ cm^2

Surface area of the above pyramid = $4(\sqrt{3} + 9)$ cm^2

Real-Life Importance of Surface Area Calculation

Calculating surface area is important in various real-life situations, like packaging, manufacturing/construction/architecture.

Packaging

To wrap an object, you need to calculate the number of materials needed.

Figure 7.5: Wrapping an object by using the concept of surface area

You may also click on the link for a video on the importance of packaging: https://youtu.be/SJGpKnI-784 .

Manufacturing/Construction/Architecture

Knowing the surface area helps to establish the cost of painting and designing for functionality and aesthetics (beauty).

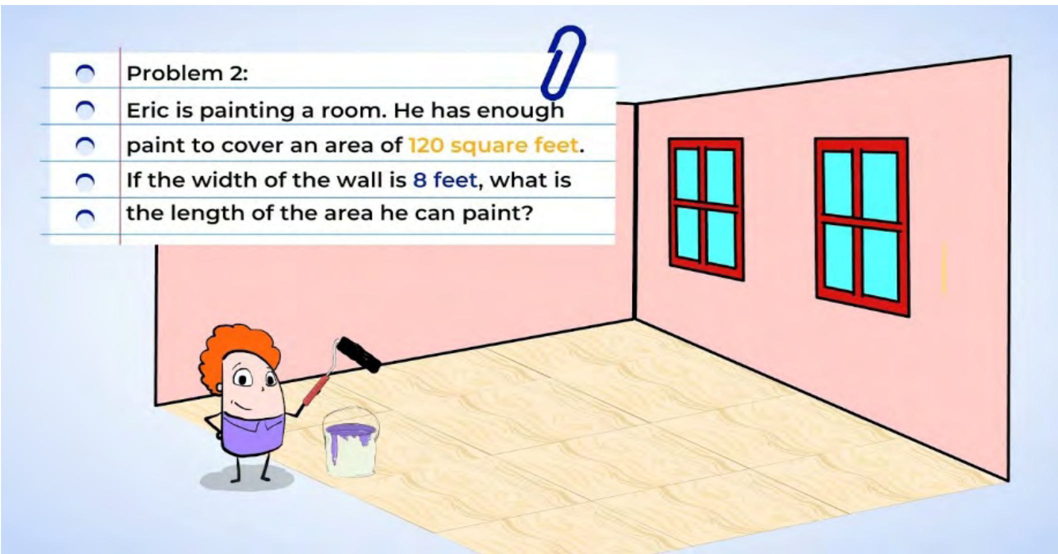


Figure 7.6: Surface area in Construction and design

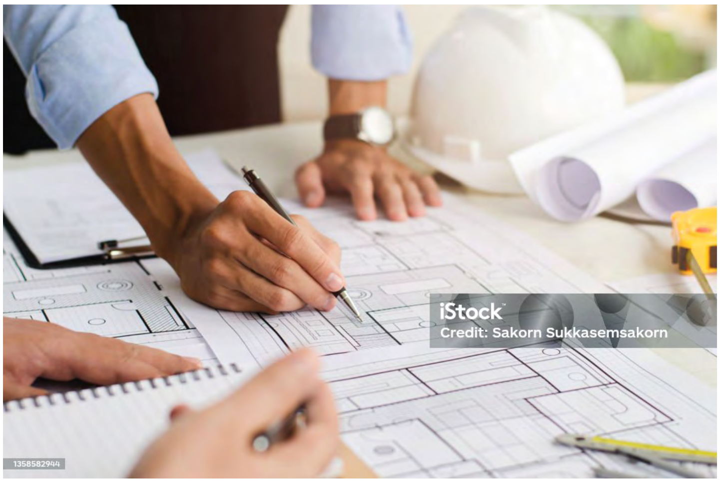


Figure 7.7: Surface Area in architecture

The above examples show how calculating surface area is important in everyday tasks, like determining the amount of paint needed for a room. Knowing the surface area helps avoid wasting resources and ensures the job is done efficiently, as well as helps in estimating costs. In industries such as construction, packaging and manufacturing, calculating surface area is critical for tasks such as material estimation, designing and cost planning.

Example 7.2

A painter needs to paint the walls and ceiling of a rectangular room. The room has a length of 8 metres, a width of 6 metres and a height of 4 metres. The door area is 2 metres by 1.5 metres, and there are two windows, each measuring 1.5 metres by 1.5 metres. The painter needs to calculate the surface area of the walls and ceiling to determine how much paint to buy.

How much area does the painter need to cover with paint?

Solution

The room has four walls:

Two walls have dimensions of 8 metres (length) by 4 metres (height).

Two walls have dimensions of 6 metre (width) by 4 metre (height).

Surface area of two 8-metre-long walls:

Area $= 2\ (8 \times 4) = 2 \times 32 = 64$

Surface area of two 6-metre-wide walls:

Area $= 2\ (6 \times 4) = 2 \times 24 = 48$

The ceiling has an area of:

Area = $8 \times 6 = 48$

Door area = $2 \times 1.5 = 3$

Each window has an area of $1.5 \times 1.5 = 2.25$.

Total area of two windows = $2 \times 2.25 = 4.5$.

Total area of walls and ceiling:

$64 + 48 + 48 = 160$

Areas of the door and windows = $(3 + 4.5) = 7.5$

The painter needs to cover $(160 - 7.5)$ m^2 = 152.5m^2.

Activity 7.3: Applications of surface area in painting your classroom

Your classroom paint has faded and needs to be repainted. Your class prefect has drawn the attention of the form teacher who asked for an estimate of the amount of paint required to paint the room. You then contacted a painter who requested the surface area to be covered excluding doors and windows. Calculate the surface area that needs to be covered with paint.

Activity 7.4: Applications of surface area in a home

You are tasked with designing a small swimming pool for a family home. The pool has a rectangular shape with a length of 10 metres, a width of 5 metres and a depth of 2 metres.

Calculate the surface area to determine the quantity of tiles required to cover the pool's walls and floor.

In small groups, discuss why calculating the surface area is important in real-life situations like construction or design, especially when considering material costs.

Units of Measurement

Measurement is fundamental in science, engineering and everyday life. Different countries and industries use various systems of measurement. Two of the most widely known systems are the "Imperial system" and the "International System of Units (SI)", also known as the "Metric System".

1. **Imperial System**: This is commonly used in countries like the United States and the United Kingdom, although the metric system is becoming more common in the UK. It includes units such as pounds (lbs), feet (ft), inches (in) and gallons.
2. **SI (Metric) System**: This is the globally adopted system used by almost all countries including Ghana. It includes units such as kilograms (kg), metres (m), and litres (l).

The ability to convert between these two systems is important, especially in scientific, technological and international contexts.

Conversion between Imperial and SI (Metric) Units

We will often come across both the Imperial (British) system and SI system of measurement. This guide will help us understand how to convert between these two systems, focusing on commonly used units of measurement for length, area, mass and volume.

Study the table below:

Table 7.1: *Metric and Imperial units*

Metric Units	Imperial Units
1 centimetre	0.394 inches
1 metre	3.281 feet = 1.093 yards
1 kilometre	0.621 miles
1 gramme	0.035 ounces
1 kilogram	2.205 pounds
1 millilitre	0.034 fluid ounces
1 litre	1.057 quart = 0.264 gallons
25.4 millimetres	1 inch
0.3048 metres	1 foot = 12 inches
0.9144 metres	1 yard = 3 feet
1.60934 kilometres	1 mile = 1 760 yards

The above table illustrates the relationship between metric units and imperial units.

These are called *conversion factors* and are used to convert one unit to another.

Example 7.3

Convert 5 m^2 to cm^2

Solution

Here we are converting within the metric system.

$5\,m^2 = 5(1m \times 1m) = 5(100cm \times 100cm) = 5 \times 10\,000 = 50\,000\,cm^2$

Note: To convert square metres to square centimetres, multiply by 10 000.

Example 7.4

Convert 85 000 cm^2 to m^2

Solution

$85\,000\,cm^2 = \frac{85\,000}{10\,000} = 8.5\,m^2$

Note: To convert square centimetres to square metres, divide by 10 000.

Example 7.5

Convert 8 ft^2 to in^2.

Solution

Here we are converting within the imperial system. Remember that 1 foot = 12 inches.

$8\,ft^2 = 8(1feet \times 1\ feet) = 8(12 \times 12) = 8 \times 144 = 1\ 152\,in^2$

Note: To convert square feet to square inches, multiply by 144.

Example 7.6

Convert 310 in^2 to ft^2

Solution

$310\ in^2 = \frac{310}{144} = 2.152777\ ft^2$

Note To convert square inches to square feet, divide by 144.

Guidelines for Converting Units of measurement:

1. Identify the appropriate unit to convert (either "from" or "to").
2. Identify the conversion factor.
3. Multiply or divide using the conversion factor.

Table 7.2: *Summary of some Key Conversion Formula*

Quantity	Unit (Imperial or SI)	Unit (SI or Imperial)	Conversion Factor
Length	1 inch	2.54cm	1 in = 2.54cm
	1 foot	0.3048m	1 ft = 0.3048m
	1 yard	0.9144m	1 yd = 0.9144m
	1 mile	1.60934km	1 mile = 1.60934km
Area	1 m^2	10 000 cm^2	1m^2 = 10 000cm^2
	1 m^2	10.7639 ft^2	1 m^2 = 10.7639 ft^2
	1cm^2	0.155 in^2	1cm^2 = 0.155 in^2
	1 ft^2	144 in^2	1 ft^2 = 144 in^2
	1 ft^2	0.092903 m^2	1 ft^2 = 0.092903 m^2
	1 in^2	6.4516 cm^2	1 in^2 = 6.4516 cm^2
Mass	1 pound	0.453592kg	1 pound = 0.453592kg
	1 ton	907.1847kg	1 ton = 907.1847kg
	1 kilogram	1 000g	1kg = 1 000grams
Volume	1 gallon (US)	3.78541litres	1 gal (US) = 3.7854 l
	1 gallon (UK)	4.54609 litres	1 gal (UK) = 4.54609 l
	1 litre	1 000 cm^3	1 litre = 1 000cm^3

By understanding and practising these conversions, we can efficiently handle different units of measurements. Understanding and mastering conversions between the Imperial and SI systems is essential, especially for progress in science and mathematics and in general, everyday tasks.

Example 7.7

Convert 3 gallons (UK) to litres.

Solution

1 gallon (UK) = 4.54609 litres

3 gallons (UK) = 3 × 4.54609 litres

Therefore, 3 gallons (UK) = 13.63827 litres

Activity 7.5: Converting Imperial to SI and Vice Versa

The measurements of a rectangular room are:

length of the room = 20 feet, width = 15 feet, height = 10 feet.

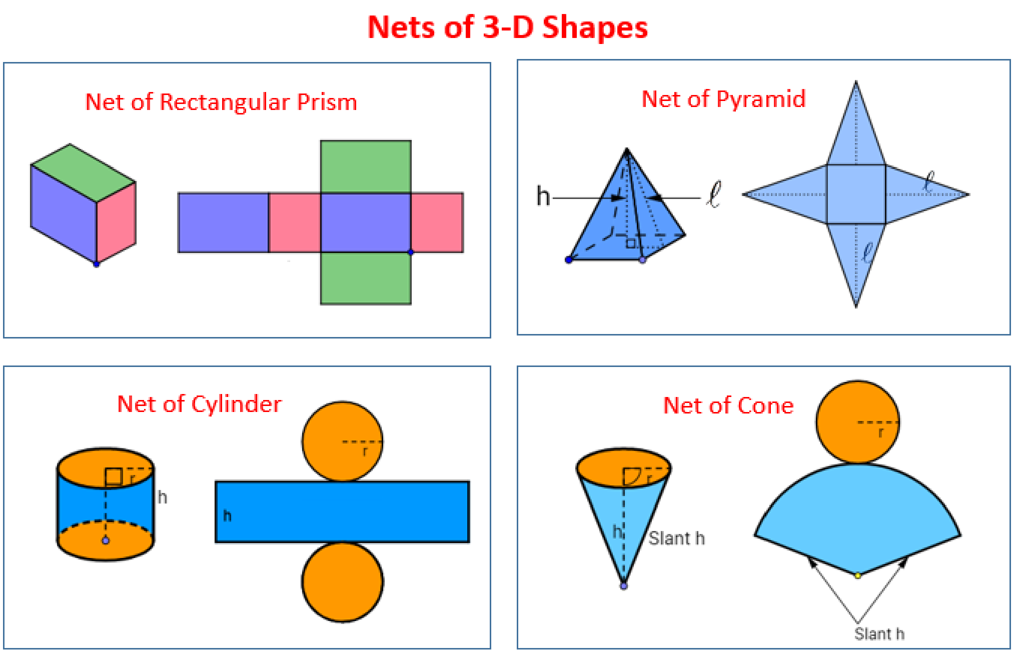
Convert these measurements into the metric system (SI units) to calculate the volume of the room in cubic metres.

Steps to do this:

1. Convert the room's length, width and height from feet to metres, remember, 1ft = 0.3048m.
2. After converting the dimensions, calculate the volume of the room in cubic metres.
3. In small groups, discuss why it is important to understand how to convert between Imperial and SI units when working on real-world problems.

Nets of 3D objects (prisms, cones, pyramids, spheres, etc.)

We will now explore nets of 3D shapes. Below we can see nets of cuboids. Imagine taking the 3D shape and dismantling it and spreading it out to form the net.



Net of a hexagonal prism

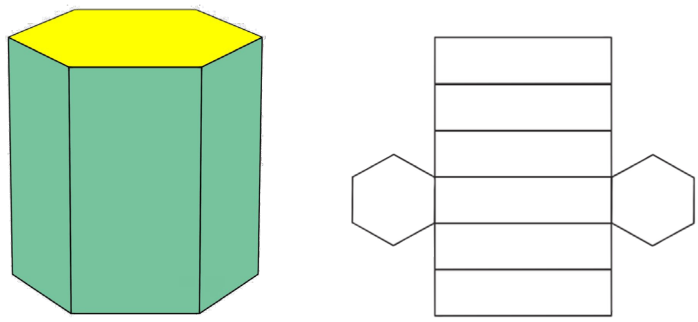


Figure 7.8: Nets of 3D shapes

Click on the link for a video on nets of solids. Then carry out the activity below: https://youtu.be/s7GrS0b3FRw.

Activity 7.6: Nets of 3D objects

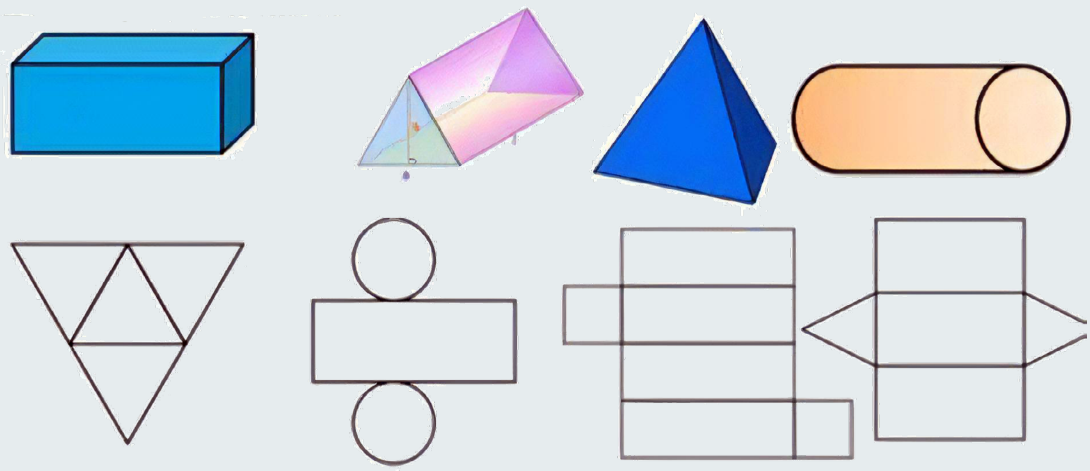


Figure 7.9: Nets of 3D shapes.

Activity 7.7: Finding nets of 3D shapes

Working in small groups, draw the nets of the 3D shapes in figure 7.10 below.

Compare your results with other groups and discuss any differences.

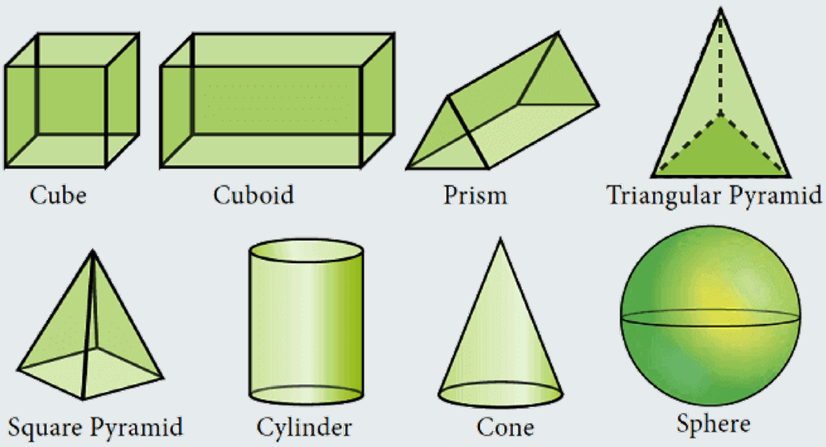


Figure 7.10: Nets of 3D objects

MEASUREMENTS OF VOLUME AND CAPACITY

All three-dimensional (3-D) objects take up a certain amount of space, this is known as their **volume**. Conversely, **capacity** refers to the maximum amount of substance or material that a 3D object can hold. Therefore, capacity is the container's volume. Volume is measured in cubic units, whereas, capacity is often measured in litres, millilitres or gallons.

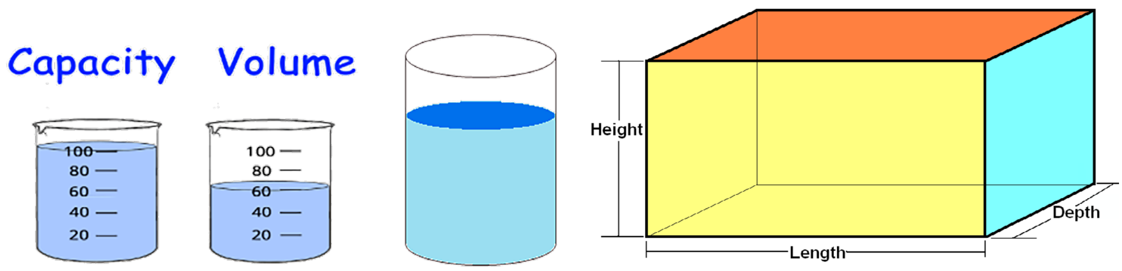


Figure 7.11: Measurement of Volume and Capacity

Click on the link for more information on volume and capacity: https://youtu.be/xSbsbz7Ovb4, or search the internet for more information on volume and capacity.

Surface Area measures the area occupied by the outer layer of a solid. Or we can think of it as the area of all the shapes/planes that make up the outside of the figures (solids). In contrast, volume measures the carrying capacity of a figure/ shape or the space enclosed within the formation.

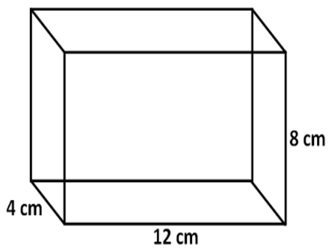
Table 7.3: *Differences between volume and capacity*

Volume	Capacity
Volume indicates the total amount of space covered by an object in three-dimensional space.	Capacity refers to the ability of something to hold, absorb or receive other substances.
Common units of measurement = cm^3, m^3	Common units of measurement = litres, gallons
Both solid and hollow objects have volume.	Only hollow objects have capacity.
Example – Cube, Cuboid, Cone and Cylinder	Example – Only hollow shapes such as a hollow cone or hollow hemisphere

Calculating volumes and capacities of 3D shapes

Example 7.8

Calculate the volume of the cuboid below.



Solution

Since the base is a rectangle and all the side walls are also rectangles, we have;

Volume of the cuboid = Base Area × Height

Area of base = $12 \times 4 = 48$ cm^2

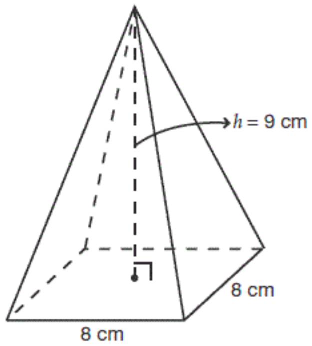
Height of the cuboid = 8 cm.

Volume of cuboid = 48×8

= 384 cm^3

Example 7.9

Find the volume of pyramid given below.



Solution

Volume of the pyramid $= \frac{1}{3} \times Base\ Area \times Height$

Area of the base $= 8 \times 8 = 64$ cm^2.

Height of the pyramid $= 9$ cm

Volume of the pyramid $= \frac{1}{3} \times 64 \times 9 = 192$ cm^3

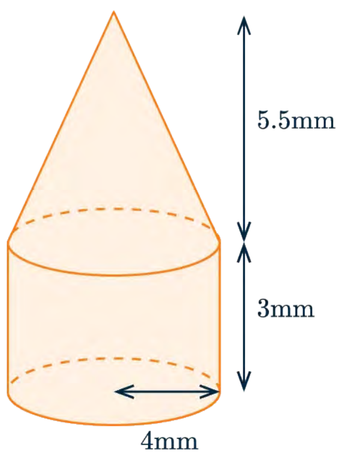
Example 7.10

Study the composite shape below.

The shape is a cone attached to the top of a cylinder.

The base of the cylinder has radius 4 mm, the height of the cylinder portion is 3 mm and the height of the cone is 5.5 mm.

Calculate the volume of the whole shape to the nearest whole number.



Solution

To work out the volume of the shape, we need to work out the two volumes separately.

Volume of cylinder $= \pi \times 4^2 \times 3 = 48\pi$

Volume of cone $= \frac{1}{3}\pi \times 4^2 \times 5.5 = \frac{88}{3}\pi$

Then, the volume of the shape is the sum of these two answers:

Volume of whole shape $= 48\pi + \frac{88}{3}\pi = (48 + \frac{88}{3})\pi = \frac{232}{3}\pi = 242.9498\text{mm}^3$ $= 243\text{mm}^3$ to the nearest whole number.

Example 7.11 (CHALLENGE)

A solid sphere and a solid hemisphere have equal total surface area. Prove that the ratio of their volumes is $3\sqrt{3} : 4$.

Solution

Let the radius of the sphere $= r_1$

Let the radius of the hemisphere $= r_2$

$\therefore$Total surface area of sphere $= 4\pi r_1^{\,2}$

$\therefore$Total surface area of hemisphere $= 3\pi r_2^{\,2}$

$4\pi r_1^{\,2} = 3\pi r_2^{\,2}$

$r_1^{\,2} = \left(\frac{3}{4}\right) r_2^{\,2}$

$r_1 = \left(\frac{\sqrt{3}}{2}\right) r_2$

Volume of sphere $= \left(\frac{4}{3}\right) \pi r_1^{\,3}$

Volume of hemisphere $= \left(\frac{2}{3}\right) \pi r_2^{\,3}$

$\left(\frac{4}{3}\right) \pi r_1^{\,3} : \left(\frac{2}{3}\right) \pi r_2^{\,3}$

$\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^3 r_2^{\,3} : \frac{2}{3} r_2^{\,3}$

$\frac{12\sqrt{3}}{24} r_2^{\,3} : \frac{2}{3} r_2^{\,3}$

$36\sqrt{3} : 48$

$3\sqrt{3} : 4$ as required.

REAL LIFE IMPORTANCE OF VOLUME AND CAPACITY

Volume and capacity have significant real-life importance across various fields and everyday activities. Here are a few key areas where these concepts are commonly applied:

1. **Cooking and Food Preparation**: When measuring ingredients like liquids (for example, water, milk, oil) or solids (for example, flour, sugar), volume measurements are crucial. For example, recipes may call for "1 cup of flour" or "500 millilitres of water". Containers, such as pots, cups or measuring spoons, are designed to hold a certain capacity. The capacity of these items determines how much they can hold, affecting how much food can be prepared at once.

Figure 7.11: Measurement of capacity in food preparation

2. **Healthcare**: Medical professionals use volume measurements to administer medications in liquid form (for example, syringes often measure in millilitres) or to monitor the volume of fluids in the body, such as blood volume in patients or patients are prescribed a teaspoon of liquid medication.

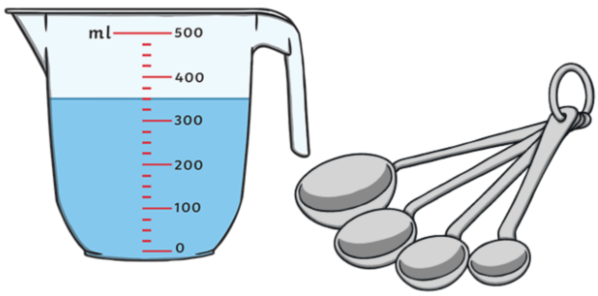


Figure 7.12: Measurement of volume and capacity in healthcare

The capacity of medical devices, such as IV bags, oxygen tanks and dialysis machines are crucial in delivering appropriate amounts of fluids or gases to patients.

3. **Transportation and Shipping**: The volume of goods being transported; whether in shipping containers, trucks, or ships, affects how much cargo can be moved at once. Volume calculations ensure that space is used efficiently.

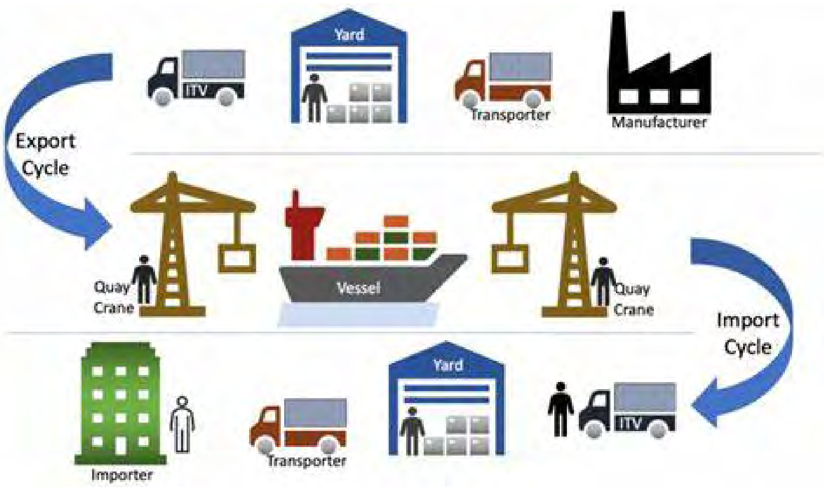


Figure 7.13: Measurement of Volume and Capacity in transportation

Vehicles, such as trucks, buses or airplanes, have capacity limits for passengers and cargo, which must be adhered to for safety and efficiency. For example, a shipping container's capacity dictates how much freight can be packed.

4. **Architecture and Construction**: Volume calculations are used in construction for determining the amount of material needed for projects (for example, how much concrete is required to fill foundations).

Figure 7.14: Measurement of Volume and Capacity in Architecture

In buildings, the capacity of rooms, elevators and facilities must be considered to ensure that they can accommodate the expected number of people or goods. For example, an auditorium's seating capacity or the capacity of a water tank for a building's plumbing system.

5. **Environmental and Energy Systems**: Volume is used to measure environmental factors like air and water pollution, where the amount of substance per unit volume (for example, mg per litre) can affect health and safety standards.

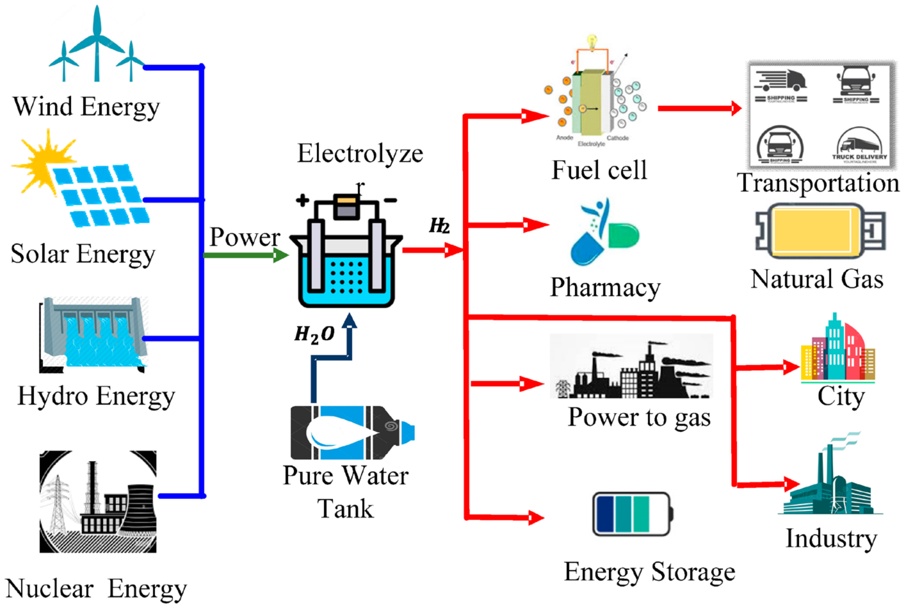


Figure 7.15: Measurement of Volume and Capacity in environment and energy systems

Energy storage systems, such as batteries or fuel tanks, rely on capacity measurements to ensure they can store enough energy for specific tasks, such as powering an electric vehicle or providing backup power during outages.

6. **Retail and Manufacturing**: In retail, products like liquids (for example, beverages or cleaning products) are sold based on volume (for example, litres, gallons).

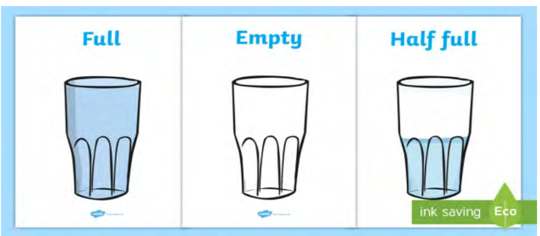


Figure 7.16: Volume and Capacity in retail and manufacturing

The capacity of storage systems, such as warehouse shelves, refrigerators and production lines, determine how much can be stored or produced, affecting inventory management and supply chain logistics.

7. **Personal and Daily Life**: Everyday tasks like filling up a gas tank, pouring a drink or using a shower all involve volume calculations. Knowing the volume of a container (like a water bottle) is important for determining how much it holds.

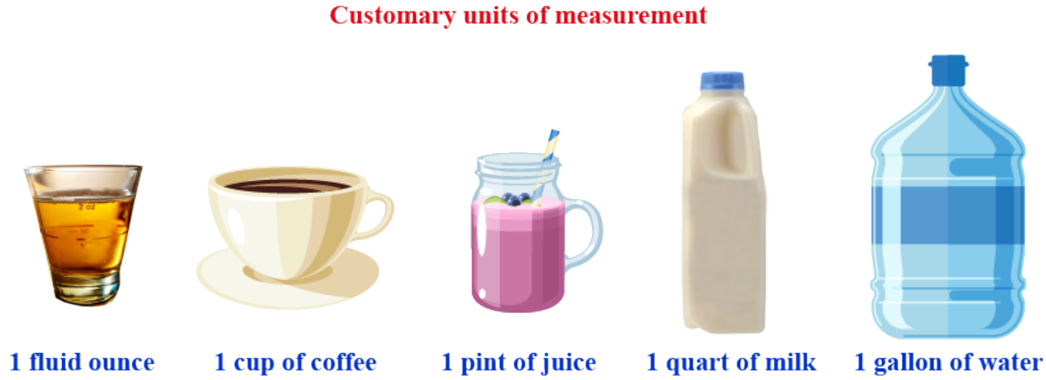


Figure 7.17: Measurement of Volume and Capacity in daily life

Understanding the capacity of a bag, suitcase or even a parking space helps in making decisions about how much can fit or how much space is available.

Therefore, we can see that volume and capacity are essential concepts that impact both the efficiency and effectiveness of everyday tasks, from managing household activities to ensuring safety and functionality in industries like healthcare, transportation, and construction.

Common Metric Units for Volume

1. **Cubic Metres (m^3):** The cubic metre is the primary metric unit for measuring large volumes in the metric system. It is often used for measuring the volume of rooms, storage areas, or large containers.
2. **Cubic Millimetre (mm^3):** The cubic millimetre is a smaller metric unit typically used in scientific fields, especially in chemistry and medicine, to measure volumes of liquids and solids in lab experiments.
3. **Litres (l):** The litre is a commonly used metric unit for measuring liquids, particularly in everyday settings such as beverages, fuel and household items. For example, liquids like milk, water and soft drinks are typically measured in litres.

4. **Millilitres (ml):** The millilitre, a smaller metric unit than the litre and it is frequently used in cooking, medicine and scientific research for measuring small volumes, such as ingredients in recipes or medication dosages.

Common Imperial Units for Volume

1. **Cubic Feet (ft^3)**: The cubic foot is used for measuring larger volumes in areas like storage, shipping, and construction. It is commonly used to describe the volume of appliances, such as refrigerators and freezers.
2. **Cubic Inches (in^3)**: The cubic inch is a smaller imperial unit often used in engineering and automotive industries to measure things like engine displacement and small components.
3. **Gallons**: Gallons are widely used in the United States for measuring larger volumes of liquids, including fuel, milk and water. It is important to note that the U.S. gallon differs from the imperial gallon used in the UK.
4. **Quarts**: A quart represents a quarter of a gallon and is often used for measuring smaller quantities of liquid, such as ingredients in recipes and products like motor oil.
5. **Pints**: A pint is half of a quart and is commonly used to measure smaller liquid volumes, such as drinks (for example, beer, milk) or food items like ice cream.
6. **Fluid Ounces**: Fluid ounces are the smallest common imperial unit for liquid volume, used for very small quantities such as drink servings, cooking ingredients and doses of medicine.

Conversion between SI and Imperial units

When converting between SI (metric) units and imperial units of volume and capacity, the following formulae and conversion factors are commonly used:

SI to Imperial Units

1 Cubic Metre (m^3) = 35.3147 Cubic Feet (ft^3)

Example 7.12

You have a container with a volume of 2 cubic metres, Convert this to cubic feet.

Solution

2 cubic metres = $2\text{m}^3 \times 35.3147\,\text{ft}^3/\text{m}^3 = 70.6294\,\text{ft}^3$

1 Litre (*l*) = 0.21997 British gallons

Example 7.13

You have a 75-litre container of water. Convert this to UK gallons.

Solution

75 litres = $75\,l \times 0.21997\,\text{gal}/l = 16.49775\,\text{gal}$

Note: 1 litre = 0.264172 US gallons

Imperial to SI Units

1 Cubic Foot (ft^3) = 0.0283168 Cubic Metres (m^3)

Example 7.14

A storage polytank has a volume of 34 cubic feet. Convert this to cubic metres.

Solution

34 cubic feet = $34\,\text{ft}^3 \times 0.0283168\,\text{m}^3/\text{ft}^3 = 0.9627712\,\text{m}^3$

1 British (Imperial) gallon is equal to 4.54609 litres.

Example 7.15

You have a 50-gallon (UK) storage tank. Convert this to litres.

Solution

50-gallon (UK) = $50\,\text{gal} \times 4.54609\ 1\,l/\text{gal} = 227.3045\,l$

Calculating Surface area and volume of 3D shapes

Study the 3D shapes in Figure 7.18 and make sure you can understand how the formulas are derived.

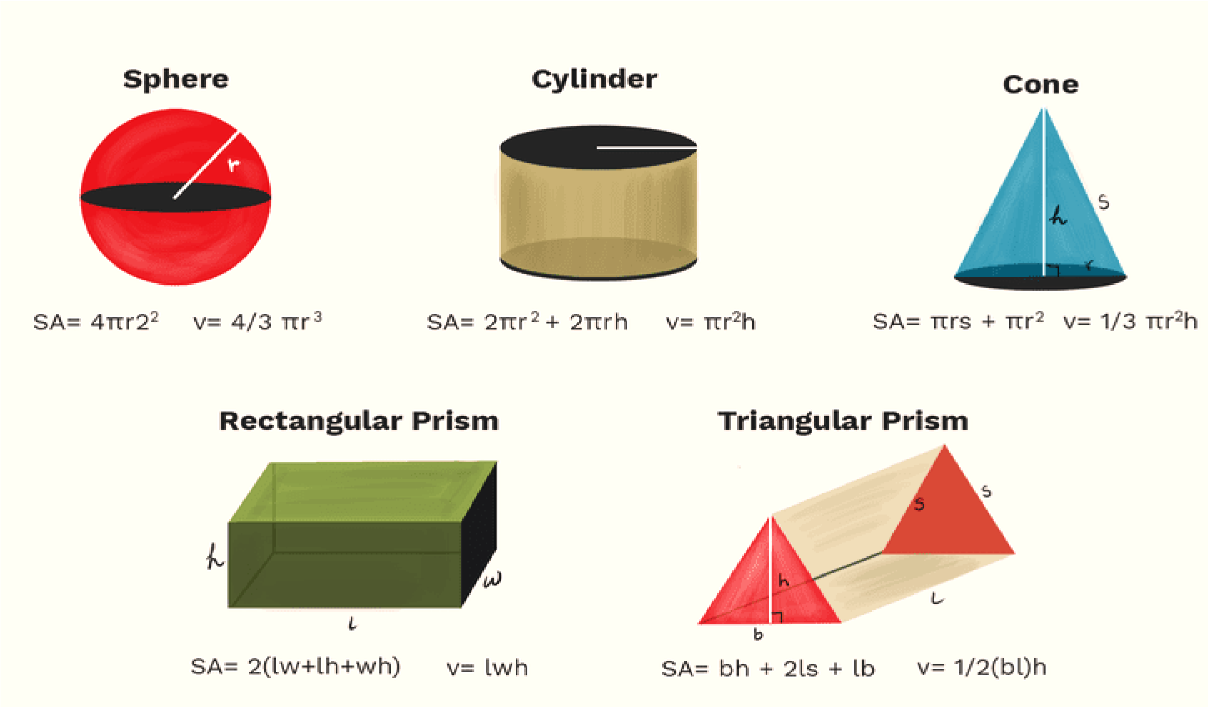


Figure 7.18: Calculating surface area and volume of 3D shapes

You may also click on the link: https://youtu.be/eBAq_caikJ4 for further clarifications on surface area and volumes of prisms.

Example 7.16

A rectangular roof measures 12 metre in length and 8 metre in width. You need to cover the entire roof with roofing tiles. Each roofing tile covers an area of 0.5 square metre. How many tiles are required to cover the entire roof?

Solution

Area of the roof = length × width = 12 metre × 8 metre = 96 square metres.

Number of tiles = Total area ÷ Area covered by each tile = 96 square metre ÷ 0.5 square metre/tile = 192 tiles

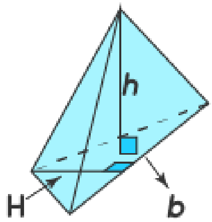
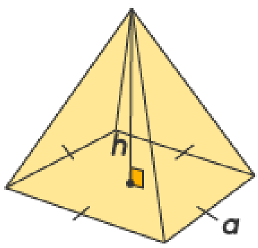
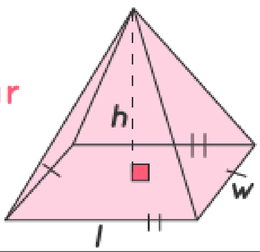
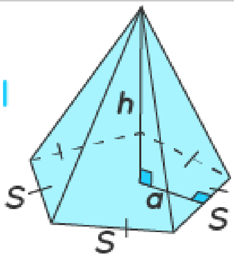
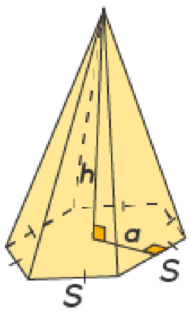
Therefore, 192 tiles are required to cover the roof.

Volumes of Pyramids

There are different types of pyramids. The name of pyramids is derived from its base shape. Different types of pyramids are shown in the table below. Their volume is always calculated by this formula:

Volume of Pyramid $= \frac{1}{3} \times$ Area of Base $\times$ Vertical Height

Table 7.4: *Volumes of pyramids*

Pyramid	Volume
Triangular Pyramid 	$V = \frac{1}{3} \times B \times h$ $= \frac{1}{3} \times \frac{1}{2} \times bH \times h$ $V = \frac{1}{6} BHh$
Square Pyramid 	$V = \frac{1}{3} \times B \times h$ $V = \frac{1}{3} a^2h$
Rectangular Pyramid 	$V = \frac{1}{3} \times B \times h$ $V = \frac{1}{3} \times lw \times h$ $V = \frac{1}{3} lwh$
Pentagonal Pyramid 	$V = \frac{1}{3} \times B \times h$ $= \frac{1}{3} \times \frac{5}{2} Sa \times h$ $V = \frac{5}{6} Sah$
Hexagonal Pyramid 	$V = \frac{1}{3} \times B \times h$ $V = \frac{1}{3} \times 3aS \times h$ $V = aSh$

REAL WORLD PROBLEMS INVOLVING VOLUME OF A 3-D OBJECT

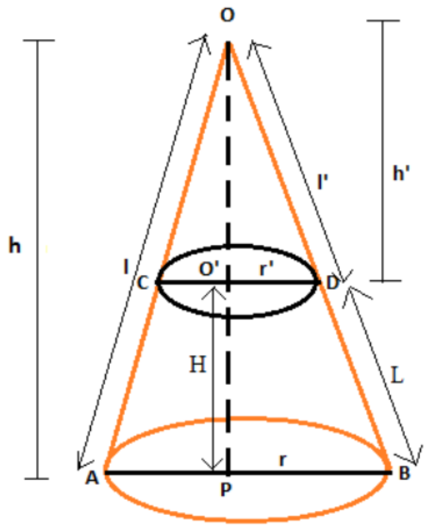
Let us look at the following example.

Example 7.17

You are designing a new water reservoir in a community garden, and the reservoir will be in the shape of a frustum of a cone (a cone with its top sliced off). The reservoir has the following dimensions: The diameter of the bottom is 10 metres and the diameter of the top is 6 metres. The height of the frustum is 8 metres. The community plans to use this reservoir to collect rainwater for irrigation.

1. Calculate the volume of the frustum of the cone to determine how much water it can hold in cubic metre.
2. Convert the volume into litres, knowing that 1 cubic metre equals 1 000 litres and explain how this helps in estimating the amount of water the reservoir can store.

Solution



1. $V_{fc} = \frac{1}{3}\pi H\left(r^2 + r r^1 + r_1^2\right)$

   Where $V_{fc} = Volume\ of\ frustum\ of\ cone$

   H = Height of frustum = 8m

   r = radius of the bottom $= \frac{10}{2} = 5m$

   $r^1 = radius\ of\ the\ top = \frac{6}{2} = 3m$

Now, substituting into the formular, we have;

$V_{fc} = \frac{1}{3}\pi 8(5^2 + 5 \times 3 + 3^2)$

$V_{fc} = \frac{1}{3}\pi 8(25 + 15 + 9)$

$V_{fc} = \frac{1}{3}\pi 8(49) = \frac{392}{3}\pi$

$V_{fc} = 410.5024\ m^3$

2. If $1m^3 = 1000 litres$

Then, $410.5024\,m^3 = 410.5024\,m^3 \times \frac{1000\ litres}{m^3}$ = 410 502.40 litres

The reservoir can hold approximately **410 502 litres** of water, which helps the community estimate how much rainwater it can collect for irrigation. Since a typical watering system for a garden may require a few hundred litres of water per session, this volume is quite substantial and would allow the garden to be watered over many sessions before needing to refill.

Example 7.18

Amina is interested in knowing how much her family spends on water for showers. Water costs **GH¢60.00 for 1,000 gallons**. Her family averages **6 showers per day**. The average length of a shower is **15 minutes**. She places a bucket in her shower and turns on the water. After one minute, the bucket has **3.5 gallons** of water.

About how much money does her family spend on water for showers in a 30-day month?

Solution

Step 1: Total time of showers per day

$6\ showers \times 15\ minutes = 90\ minutes$

Step 2: Gallons used per day

$90\ minutes \times 3.5\ gallons/minute = 315\ gallons/day$

Step 3: Gallons used per month

$315 \times 30 = 9450\ gallons/month$

Step 4: Cost of 9450 gallons

Since $1000\ gallons = GH¢60.00$,

Then $\frac{9450}{1000} \times GH¢60.00 = 9.45 \times 60 = GH¢567.00$

Answer

The family spends approximately GH¢567.00 on water for showers in a 30-day month.

Example 7.19

Ama loves playing with building blocks. She built a structure with 10 cubic blocks.

If the edge of each cube is 3.5 inches, what is the volume of her structure?

Solution

The volume of a cube = $\text{Length}^3 = 3.5^3 = 42.875\text{in}^3$

There are 10 cubes in her structure.

Volume of the structure = 10 × Volume of one cube = $10 \times 42.875\ \text{in}^3 = 428.75\ \text{in}^3$ Therefore, the volume of her structure is $428.75\ \text{in}^3$.

Example 7.20

Kofi has a glass that is cylindrical in shape. The height of the glass is 20 units and the radius of the base is 5 units.

What is the capacity of the glass?

Solution

Height of the glass = 20 units and the radius = 5 units.

Volume of a cylinder = $\pi r^2 h$ cubic units.

Volume of the glass, $V = \pi r^2 h = \pi \times (5^2) \times 20 = \pi \times 500 = 1570.8$ cubic units.

Therefore, the glass's capacity is approximately 1571 cubic units.

EXTENDED READING

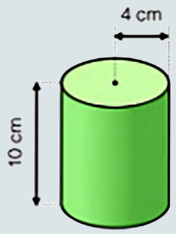
1. Aki – Ola series: Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 300– 303)
2. Akrong Series: Core mathematics for Senior High Schools New International Edition (Pages 802 - 808)
3. Baffour A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 701 - 717)
4. Click on the link to get additional information on surface area and volume:

https://youtu.be/eBAq_caikJ4	
https://youtu.be/W1Gr-nKE1b8.	

REVIEW QUESTIONS

Find the surface area of each of the following solid shapes in $cm^{2,}$ correct to 1 decimal place.

1.



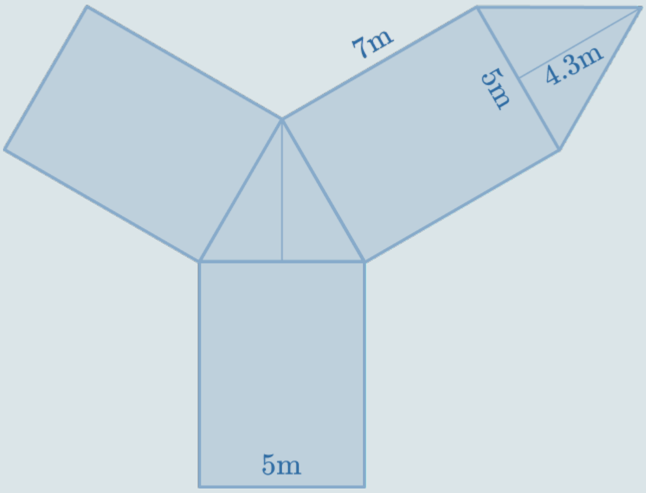
2.

3.

4. Convert 60 miles to kilometres.

5. The net of a triangular prism is given below.

Find the surface area (the triangles are equilateral triangles).



6. What is the total surface area of a cylinder whose radius is 4.5 units and height is 8 units.

7. You are building a raised rectangular garden bed that is 3 metres long, 2 metre wide and 0.5 metre deep. How much soil (in cubic metres) is required to fill the garden bed?

8. You are designing a new water reservoir in a community garden and the reservoir will be in the shape of a frustum of a cone (a cone with its top sliced off). The reservoir has the following dimensions: The diameter of the bottom is 12 metres, and the diameter of the top is 8 metres. The height

SECTION

8

WORKING WITH DATA & PROBABILITY EXPERIMENTS

MAKING SENSE OF AND USING DATA

Statistical and Probability Reasoning and their Application in Real Life

INTRODUCTION

The ability to accurately predict the winner of an election in a democracy is big business. Obviously, the ability to correctly predict the winner of an election can save political parties a lot of time and resources. Manufacturers would also like to know how ready a market is for a new product. In such a case, market research can be conducted by the manufacturers to determine how best to advertise and sell the product.

In this section, you will apply your knowledge of statistics to solve problems in your community. By designing tools to collect data and carefully analysing and interpreting the data collected, stakeholders will receive reliable statistics to make informed decisions. In this way, statistics can help you to study local and national issues. With the data collected, analysed and interpreted, you can predict and prepare for future events.

KEY IDEAS

- **Event** is any subset of the sample space.
- **Experiment** is performing an action and studying what happens.
- **Outcome** is the result of an experiment. Each experiment can have multiple possible outcomes.
- **Probability** is the possibility or likelihood that a given event will occur.
- **Random experiment** is an experiment whose outcome cannot be predicted.
- **Sample space** is the set of all the possible outcomes.

DEVELOPING AND IMPLEMENTING A MINI PROJECT ON DATA COLLECTION

A close relative of statistics is probability. Probability is an important tool in scientific prediction. With probability you can make predictions about what is most likely to happen. Manufacturers use probability to determine the lifespan or quality of their products. Probability plays an important role in approving drugs for public consumption. In probability, we will study events which are equally likely, mutually exclusive, independent or dependent.

Imagine you want to study the height of learners in your school, you will need a tape measure or a metre rule. You will also need a piece of chalk to mark the height of students on the wall before measuring the height. In such an experiment or research, the measuring instruments, or tools, are readily available on campus. In the Home Economics department, you can find a tape measure. From the Physics laboratory you can find a metre rule.

However, suppose you want to conduct an investigation into the attitude of students towards food served in the dining hall. Where can you find an instrument or tool to measure the attitude of students? In such a case, the best tool or measuring instrument is a questionnaire or interview.

As much as gathering information is important, until the data is processed very little sense can be made out of the raw data. This is where the analysis and interpretation of data comes in. Thus, tallying the data, drawing a frequency distribution table, finding the measures of central tendency or dispersion, as well as drawing charts and graphs to illustrate the data will assist stakeholders to understand the data at a glance and make informed decisions. This is the essence of statistics: gathering data, analysing the collected data and interpreting the data to make well-informed decisions.

Activity 8.1: Investigating the height of 50 students in your school

Working in small groups carry out the following activity:

Step 1: Seek the permission of a teacher, guardian and the participants whose heights you are going to study in your research or experiment. Choose the participants at random, for example, the first 50 students going to the dormitory after classes.

Step 2: Mark the height of each participant on a wall, using a piece of chalk and measure it in centimetres or metres and record it.

Step 3: Choose appropriate class intervals for your data. Tally the heights of the participants and construct a frequency distribution table for the data. Decide if you should split the data between males and females and defend your decision.

Step 4: From the frequency distribution table, find the measures of central tendency and dispersion and draw charts and graphs to illustrate the data.

Step 5: Based on the interpretation of data, summarise your observations and draw conclusions. Consider how you could improve the investigation if you were to carry it out again.

Activity 8.2: Investigating the percentage of heads or tails when a fair coin is tossed 50 times

Working in small groups carry out the following activity:

Step 1: Toss a fair coin and record the outcome, using H for head and T for tail. Do this 50 times.

Step 2: Draw a table to show how many heads and tails were obtained as well as the percentage of each outcome.

Step 3: Draw a conclusion from the results.

Activity 8.3: Measuring students' attitude towards food served in the dining hall

Working in small groups carry out the following activity:

In this activity, we cannot simply measure and record the height of something or carry out a simple experiment. As you learnt in year one, to collect qualitative data requires the use of, for example, observations, interviews or questionnaires. You can choose which method to use. Below is an example of a questionnaire.

Step 1: Decide which method you will use to gather the data. In this example, we have used a questionnaire.

Step 2: Design the tool, remembering the rules guiding the construction of questionnaires. The most important question to ask is which dining hall food students like best. However, that information alone will not be sufficient for a thorough analysis. For this reason, details like the class, program, age, gender and the house of the participants will help to make your analysis more meaningful. Also, the reason behind the choice of food might be very helpful.

Step 3: Decide how many students from each form group and gender you will ask. Distribute the questionnaires through your chosen means, be it, for example, handing out hard copies or using social media. Remember to give a date for the return deadline.

Step 4: Go through the completed questionnaires to tally the results. A spreadsheet works well for this. After sorting out, you can discover how many first-year learners like breakfast or how many girls in form 3 like jollof rice etc.

Step 5: Use tables, appropriate charts and graphs to interpret and illustrate the data collected.

Thank you for taking the time to complete this questionnaire. The information will be kept confidential.

Date: ……………………………………………………………

Tick in the appropriate bracket.

1. **Gender**: Male ( ) Female ( )
2. **Level**: Form 1 ( ) Form 2 ( ) Form 3 ( )
3. **House**: House 1 ( ) House 2 ( ) House 3 ( ) House 4 ( )
4. **How many times do you go to the dining hall every day?**
   None ( ) once ( ) twice ( ) thrice ( )
5. **What is your favourite meal in the dining hall?**
   Breakfast ( )
   Lunch ( )
   Supper ( )
6. **Whatisyourfavouritefoodinthedininghall?**……………………….....
7. **Give one reason for your choice.** ……………………………………

Figure 8.1: A sample questionnaire to measure the attitude of students towards food served in the dining hall

See the Review Questions where you will carry out your own mini-project. It is good if you choose topics to investigate which are important to you.

SIMPLE AND COMPOUND PROBABILITY EXPERIMENTS

Probability is a tool for making scientific guesses. It gives us methods with which we can make predictions or estimates as accurate as possible. This means that probability has wide applications in many fields.

Application of probability in pharmacy

Before a new drug is approved for human use, the drug goes through rigorous tests. Usually, the new drug is administered to a sample of willing participants. The positive and negative effects of the drug are noted, as well as the percentage of participants experiencing positive or negative side effects. It is for this reason that the side effects of drugs can be seen inside or outside the packaged drug, along with the instructions and dosage. For example, this could be why consumers of certain drugs are advised not to drive after taking the drug.

If the research proves the efficacy and safety of the drug beyond reasonable doubt, it is approved for public use. That is, if the positive effects outweigh the negative effects, the drug is approved. For example, a drug like chloroquine can cause severe itching in certain users while others did not experience the negative effects, so it was deemed that the positives outweigh the negatives and it is approved for human use.

Application of probability in computer science

When typing on your phone, the keyboard can offer suggestions of what you are going to type, to save your typing it. This is known as predictive text. For example, immediately you type *t*, the keyboard may suggest *the* or *thanks*. The computer will have learnt your usual style of typing and use of language and, using probability, decide what you are likely to be wanting to type and give you help in this. In this way, probability helps speed up your communications.

Application of probability in weather forecast

Have you ever seen the meteorologists giving the weather forecast on the television? Probability is an important tool in these predictions. For example, if in the last 10 years, some conditions in the clouds produced rain in the Axim area in June, then the probability that the same conditions will produce rain in the Axim area in June this year is very high.

Probability always lies between zero and one

We are going to learn how to measure the likelihood that a given event will occur. To do this we will assign it a numerical value. For example, what percentage will you assign to the probability that a particular classmate will come to school today? For those who come to school regularly, the probability will approach 100%. For those who are less reliable, the probability that they will come to school today will be less. Thus, probability is always a rational number lying between zero and one. The *lowest probability is zero*. This means that the event will *never happen*. The *highest probability is one*. This means that the event will *definitely happen*. If you calculate a probability to be below 0, or above 1 means that you have gone wrong somewhere!

Random experiments

Experiments are conducted to study the effect of a particular action. If the results of the experiment depend on chance or luck, you have a random experiment. Tossing a fair coin or die is a classic example of a random experiment. Drawing identical balls from a bag or box at random is another example of a random experiment. Predicting who will win the football match or horse race is also a random experiment. One performance of a random experiment is a trial. So, tossing a fair coin once is one trial of the random experiment.

Sample space

When a fair coin is tossed once, the result is either a head (H) or a tail (T). Thus, H and T are the only outcomes of this random experiment. The set of all possible outcomes is known as the sample space, S. Accordingly, when a fair coin is tossed once, the sample space or universal set is $S = \{H, T\}$. When a fair die is tossed once, the sample space, $S = \{1, 2, 3, 4, 5, 6\}$.

Event

Any subset of the sample space is called an event. Thus, rolling a 4 is an event when a fair die is rolled once. In that case, the event $E = \{4\}$ is the subset of the sample space. In the same way, getting a head is an event when a fair coin is tossed once. Getting a tail is another event.

Simple and compound events

If the event has a single outcome, you have a simple event. For example, *rolling a six* when a fair die is tossed once is a simple event: $E = \{6\}$. Or if you toss a fair coin once, the outcome is a simple event.

If the event contains two or more elements, you have a compound event. For example, when you toss a fair coin and a fair die together, the outcome is a compound event. In this case, the events include {H, 1}, {H, 2}, {T, 6} and so on.

Equally Likely Events

How will your class select a class prefect in such a way that every student in the class is given an equal chance of becoming the class prefect? Discuss this with your classmates.

For example, to give every student in the class an equal chance, each student could write his or her name on a piece of paper. These are collected in and put in an opaque container. The teacher could then choose a paper at random. In this way, everyone in the class will have the same chance of becoming the class prefect.

If a fair die is tossed once, getting 1, 2, 3, 4, 5 or 6 are equally likely events. This is because all of them have got an equal chance of showing up. If all the events in a random experiment have an equal chance of occurring, you have a set of equally likely events. In this case, the probability that the event E will occur, denoted by P(E), is given by $P(E) = \frac{n(E)}{n(S)}$, where S is the sample space. Frequently terms like *at random, fair, unbiased* are hints that the events under discussion are equally likely.

Example 8.1

A bag contains 26 identical balls labelled A to Z. If a ball is chosen at random from the bag, calculate the probability that it is a vowel.

Solution

Since items are being chosen at random, you are being told that you have equally likely events. Thus, the formula to use is $P(E) = \frac{n(E)}{n(S)}$. The event under consideration is E = {vowels} and the sample space S = {letters in the English alphabet}.

Thus, $n(E) = n\{a, e, i, o, u\} = 5$ and $n(S) = n\{\text{English alphabet letters}\} = 26$.

Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{5}{26}$.

Example 8.2

Identical cards in a box are numbered from 1 to 40. A card is chosen at random.

Calculate the probability that a prime number is chosen.

Solution

The sample space, $S = \{1, 2, 3\ldots 40\}$. Thus, $n(S) = 40$.

Event $E = \{\text{prime numbers}\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$.

Thus, $P(E) = \frac{n(E)}{n\,(S)} = \frac{12}{40} = \frac{2}{5} = 0.4 = 40\,\%$.

Example 8.3

A fair die was rolled once. Find the probability that a multiple of 3 was rolled.

Solution

The sample space, $S = \{1, 2, 3, 4, 5, 6\}$ and the event $C = \{\text{multiple of } 3\} = \{3, 6\}$.

Thus, $P(C) = \frac{n(C)}{n\,(S)} = \frac{2}{6} = \frac{1}{3}$.

Mutually Exclusive Events

Two events C and F are *mutually exclusive* if they *cannot occur* at the same time. For example:

1. When a fair die is tossed once, rolling a 1 and a 2 simultaneously are mutually exclusive events. That is, you can roll a 1 or a 2 but not both at the same time.
2. If there are red and blue identical balls in a bag and a single ball is picked at random, picking a red ball and picking a blue ball are mutually exclusive events.
3. The event that your mathematics teacher will come to school today and the event that they will not come to school today are mutually exclusive events. These can never happen simultaneously, ie, your teacher cannot be both in school and not in school.

You can remember that in sets, $n(C \cup F) = n(C) + n(F) - n(C \cap F)$. However, if C and F are mutually exclusive events, then C and F cannot occur at the same time.

Hence, $n(C \cap F) = 0$.

Therefore, $n(C \cup F) = n(C) + n(F)$.

$$\Longrightarrow \frac{n(C \cup F)}{n(S)} = \frac{n(C)}{n(S)} + \frac{n(F)}{n(S)}$$

$\therefore$ **P(C ∪ F) = P(C) + P(F)**.

This is the **addition law of probability** (also called the AND rule).

Independent Events

Two events C and F are *independent*, if the probability that C will happen does not affect the probability that F will happen. In such a case, both C and F can happen at the same time and the probability of this is denoted by $\boldsymbol{P(C \cap F) = P(C) \times P(F)}$.

This is the **multiplication law of probability** (also called the OR rule).

When you toss a fair coin twice, the outcomes of the first and second tosses will be independent. This is because the outcome of the first toss has no influence on the second outcome. Making sure that events are independent is important in decision-making. It is for this reason that in examinations the sitting arrangements and invigilators are employed to make the performance of the candidates as independent as possible. It is for the same reason that during a 100-metre race, competitors are not allowed to steal *start* or tracks.

The probability of an event and its complement

Another close relative of probability is sets. You will have realised that the sample space is a universal set and the events are subsets of this sample space. If U is the universal set, A is a set and A′ is the complement of A, then:

$A \cup A' = U$

$\Longrightarrow n(A) + n(A') = n(U)$

$$\Longrightarrow \frac{n(A)}{n(U)} + \frac{n(A')}{n(U)} = \frac{n(U)}{n(U)}$$

$\Longrightarrow P(A) + P(A') = 1.$

$\therefore$ **P(A) = 1– P(A′)**.

What conclusion can you draw from this? It means that if the probability of an event is known, you can easily calculate the probability of its complement. For example, the probability that it will rain today is 0.1. Therefore, the probability that it will not rain today is 1 – 0.1 = 0.9.

Example 8.4

Ato tossed a fair die once. Calculate the probability that he got either 3 or 6.

Solution

For a fair die, the sample space, S = {1, 2, 3, 4, 5, 6}. The event T = {3} and Q = {6}. Since events T and Q cannot happen at the same time, they are mutually exclusive events. Thus, you must apply the addition law of probability.

That is, $P(Q \text{ or } T) = P(Q \cup T)$

$$= P(Q) + P(T)$$
$$= \frac{n(Q)}{n(S)} + \frac{n(T)}{n(S)}$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{2}{6}$$
$$= \frac{1}{3}.$$

Thus, the probability that he got either 3 or 6 is $\frac{1}{3}$.

Example 8.5

Show that the total probability of all the possible outcomes is 1, when a fair:

1. coin is tossed once.
2. die is tossed once.

Solution

1. When a fair coin is tossed once, the possible outcomes are a head (H) or a tail (T). The probability of getting a head is half and the probability of getting a tail is also half; and half plus half is one. Thus, the total probability of all the possible outcomes in this case is one. That is,

$$P(H \cup T) = P(H) + P(T)$$
$$= \frac{n(H)}{n(S)} + \frac{n(T)}{n(S)}$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= \frac{2}{2}$$
$$= 1$$

2. Similarly, if P(1), P(2), and P(3) represent the probability of getting 1, 2 and 3 respectively, then the sum of all the possible outcomes when a fair die is tossed once is given by:
   $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1.$
   Note that all these outcomes are mutually exclusive.

Example 8.6

The probability that it will rain today is 0.3 and the probability that it will rain tomorrow is 0.6. Assume that these events are independent. Find the probability that:

1. it will *not* rain *today*;
2. it will *not* rain *tomorrow*;
3. it will *rain today and tomorrow*.

Solution

1. If the event that it will rain today is denoted by R then the event that it will not rain today is denoted by R′. Hence, $P(R') = 1- P(R) = 1 - 0.3 = 0.7$. Thus, the probability that it will not rain today is 0.7.
2. The probability that it will not rain tomorrow is given by $1 - 0.6 = 0.4$.
3. Let P(R) denote the probability that it will rain today and P(T) denote the probability that it will rain tomorrow. Since R and T are independent events: $P(R \cap T) = P(R) \times P(T) = 0.3 \times 0.6 = 0.18$. Therefore, the probability that it will today and tomorrow is 0.18.

Example 8.7

The probabilities that Arango and Kabenla will score a goal during a match are 0.18 and 0.72 respectively. Assume that these events are independent. If both of them play, what is the probability that:

1. both of them will score;
2. only one of them will score;
3. at least one of them will score.

Solution

Let $P(A) = 0.18$ be the probability that Arango will score a goal and $P(K) = 0.72$ be the probability that Kabenla will score a goal.

Then $P(A') = 1 - 0.18 = 0.82$ is the probability that Arango will not score a goal and $P(K') = 1 - 0.72 = 0.28$ is the probability that Kabenla will not score a goal.

A and K are independent events so we can apply the multiplication law of probability.

1. P (both of them will score) = $P(A \cap K) = P(A) \times P(K) = 0.18 \times 0.72 = 0.1296$.
2. There are two possibilities or events. That is, the event that Arango will score but Kabenla will not score OR the event that Arango will not score but Kabenla will score. These are mutually exclusive.

   Note that in probability OR goes with *addition* or *union* and AND goes with *multiplication* or *intersection*.

   P (only one of them will score) $= P(A \cap K') + P(A' \cap K)$

   $= P(A) \times P(K') + P(A') \times P(K) = 0.18 \times 0.28 + 0.82 \times 0.72$

   $= 0.0504 + 0.5904 = 0.6408$
3. If at least one of them will score, then there are three events which are mutually exclusive: either Arango scores and Kabenla does not score or Arango does not score but Kabenla scores or both of them score. Consequently, the probability that at least one of them will score is given by;

   P (at least one of them will score) $= P(A \cap K') + P(A' \cap K) + P(A \cap K)$

   $= P(A) \times P(K') + P(A') \times P(K) + P(A) \times P(K)$

   $= 0.18 \times 0.28 + 0.82 \times 0.72 + 0.18 \times 0.72$

   $= 0.0504 + 0.5904 + 0.1296 = 0.7704$

   Alternatively, P (none will score) + P (at least one will score) = 1.

   Thus, P (at least one will score) = 1 – P (none will score)

   $= 1 - P(A' \cap K')$

   $= 1 - [P(A') \times P(K')] = 1 - (0.82 \times 0.28) = 1 - 0.2296 = 0.7704$.

   When there are more than two events, this route is the best, the shortest.

Dependent Events

If two events *cannot* occur at the same time, they are mutually exclusive. Consequently, if two events are *not* mutually exclusive, they can happen at the same time. In that case, they could be either dependent or independent. If they are independent, the probability that one will occur will not affect the probability that the other will occur. However, if they are dependent, then the probability that the first event will occur will determine the probability that the second event will occur.

Consider this scenario. There are 39 cards in a box. Only three of them are black. In a class of nine boys and six girls, anyone who picks a black card at random is given Gh¢50.00. What conclusions can you draw if all the boys are allowed to choose a card before the girls and replacement is not allowed?

If you thought that the competition was unfair to the girls in the class, you are right. This is a classic example of a series of events which are dependent on each other.

If two events C and F are dependent on each other, then you can find the probability that C will occur given that F has already taken place. That is, if two events C and F are dependent, a condition can be placed on F while finding the probability that C will occur. This is called **conditional probability**. The probability that C will occur given that F has already taken place is denoted by P(C|F).

Now, $\mathbf{P}\left(\mathbf{C}\middle|\mathbf{F}\right) = \frac{\mathbf{P(C \cap F)}}{\mathbf{P(F)}}$. This is the formula for conditional probability.

We can rearrange this to give us another formula. This formula will help you to calculate the probability of events you will meet when drawing identical items at random without replacement. That is, $\mathbf{P(F)} \times \mathbf{P(C|F)} = \mathbf{P(C \cap F)}$.

In other words, the probability that events C and F will occur is product of the probability that F will occur and the probability that C will occur given that F has occurred.

Example 8.8

There are 6 red and 3 blue identical balls in a box. If two balls are drawn at random, one after the other without replacement, calculate the probability that:

1. both of them are red;
2. both balls have the same colour;
3. both balls have different colours.

Solution

n(red) = 6 and n(blue) = 3 and n(balls) = 9.

If $P(R_1)$ is the probability that the first ball drawn is red and $P(R_2)$ is the probability that the second ball drawn is red, then:

1. $P(\text{both are red}) = P(R_1 \cap R_2) = P(R_1) \times P(R_2|R_1) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$.

Since the balls are drawn at random without replacement, $P\left(R_1\right) = \frac{n(\text{red})}{n(\text{balls})} = \frac{6}{9}$.

After the first red ball has been taken out, there are only five red balls out of the eight balls left in the box. Thus, the probability that the second ball is red is $\frac{5}{8}$.

$P(R_2|R_1)$ represents the probability that the second ball is red, given that the first ball is red.

2. **(ii)** P(both balls have the same colour) = P(both are red) + P(both are blue)

$$= P(R_1 \cap R_2) + P(B_1 \cap B_2)$$

$$= P\left(R_1\right) \times P\left(R_2\right) + P\left(B_1\right) \times P\left(B_2\right)$$

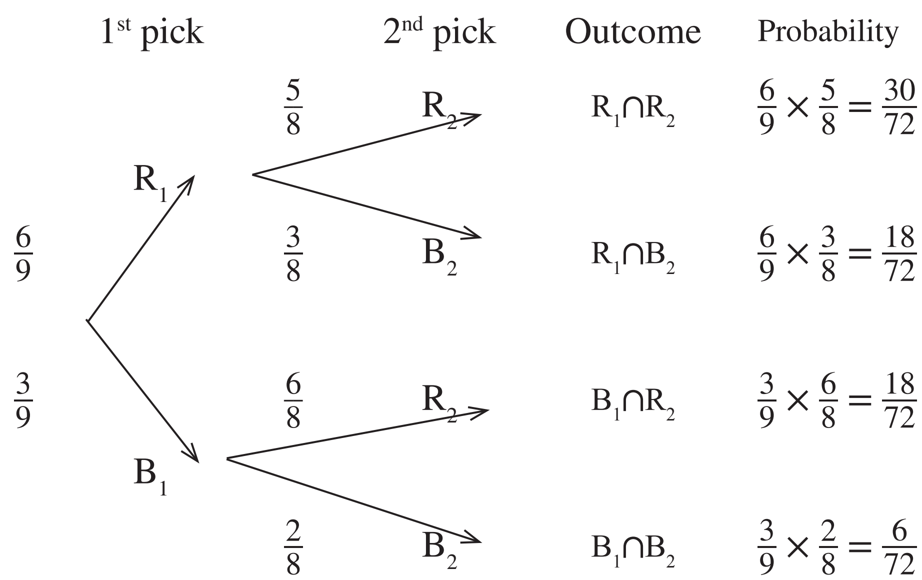
$$= \frac{6}{9} \times \frac{5}{8} + \frac{3}{9} \times \frac{2}{8} = \frac{30}{72} + \frac{6}{72} = \frac{36}{72} = \frac{1}{2}$$

3. **(iii)** P(both balls have different colours) = $P(R_1 \cap B_2) + P(B_1 \cap R_2)$

$$= P\left(R_1\right) \times P\left(B_2\right) + P\left(B_1\right) \times P\left(R_2\right)$$

$$= \frac{6}{9} \times \frac{3}{8} + \frac{3}{9} \times \frac{6}{8} = \frac{18}{72} + \frac{18}{72} = \frac{36}{72} = \frac{1}{2}$$

Tree diagrams can be used to solve this problem (see below). Visit You Tube to find a video that will help you solve this problem using tree diagrams. Here is a link you could use, but there are plenty more available, https://www.youtube.com/watch?v=PYEvSuz1Dxo



Example 8.9

Kwam tosses a fair die once. Find the probability that the outcome:

1. is an odd number, given that it is a prime number;
2. is a prime number, given that it is odd.

Solution

Let E be the event that a prime number is chosen and F be the event that an odd number is chosen. The sample space $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{2, 3, 5\}$ and $F = \{1, 3, 5\}$. Thus, $n(F \cap E) = n\{3, 5\} = 2$.

1. $P(\text{it is odd}|\text{it is a prime number}) = P\left(F|E\right) = \frac{n(F \cap E)}{n(E)} = \frac{2}{3}$.
2. $P(\text{it is a prime number}|\text{it is odd}) = P\left(E|F\right) = \frac{n(F \cap E)}{n(F)} = \frac{2}{3}$.

Example 8.10

The results of an election are shown in the table below. Use it to answer the following questions.

	NCP	ACP
Female votes	81	72
Male votes	108	99

1. If a voter is female, find the probability that she voted for ACP.
2. Find the probability that a voter voted for NCP, given that the voter is a male.

Solution

Let F be the event that a voter is a female, M be the event that the voter is a male, N be the event that the voter voted for NCP and A be the event that the voter voted for ACP. Now let us find the totals for the number of males, females and NCP and ACP voters.

	NCP	ACP	Total
Females	81	72	**153**
Males	108	99	**207**
Total	**189**	**171**	**360**

1. P (voted for ACP|voter is female) $= P\left(A \middle| F\right) = \frac{n(A \cap F)}{n(F)} = \frac{72}{153} = \frac{8}{17}$.
2. P (voted for NCP|voter is male) $= P\left(N \middle| M\right) = \frac{n(N \cap M)}{n(M)} = \frac{108}{207} = \frac{12}{23}$.

EXTENDED READING

1. Akrong Series: Core mathematics for Senior High Schools New International Edition (Pages 258 – 266)
2. Aki – Ola series: Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 612 – 641)
3. Baffour A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 539 – 568)

6. Of the 60 students in 1 Language, 18 boys read Twi and 9 of the boys do not read Twi. In addition, 20 girls read Twi and 13 girls do not read Twi.

a. If a girl is chosen at random from the class, find the probability that she does not read Twi;

b. If a student reads Twi, find the probability that the student is a boy.

7. There are 5 blue and 4 white identical balls in a box. If two balls are drawn from the box at random, one after the other, calculate the probability that both balls have different colours, if replacement:

a. is allowed;

b. is *not* allowed.

Choose one or more of the topics below to carry out your own mini-project or choose a mini-project of your own which is important to you, your school or your local region.

8. You have been tasked by a micro finance institution to investigate the number of people in your community who save with microfinance institutions on a daily basis.

Conduct research and present your work to the micro finance institution.

9. A marketing agent has tasked your club to produce a report on the different languages spoken in your area. Conduct a small survey and use it to write your report.

10. According to health experts, fizzy drinks cause health problems.

Conduct a mini-project to determine whether students in your school are at risk.

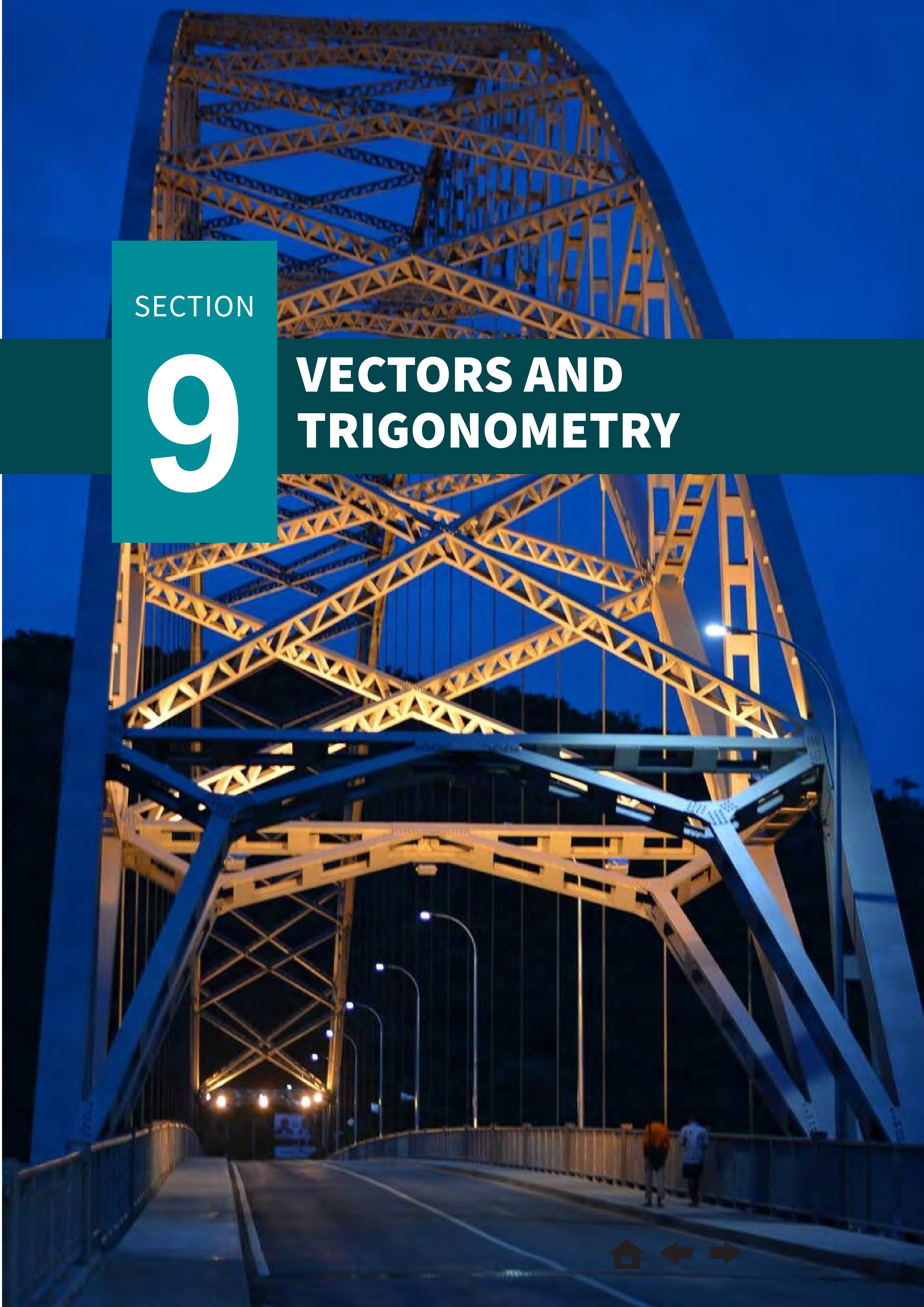
11. Measure the height and mass of 40 people in your school.

Calculate the average body mass index.

Based on the average body mass index, what conclusions would you draw? What recommendations would you give?

12. Drug abuse includes taking medication without a doctor's prescription.

Provide the school authorities with reliable data that can help them to make well-informed decisions concerning this type of drug abuse.



SECTION

9 VECTORS AND TRIGONOMETRY

GEOMETRY AROUND US

Measurement

INTRODUCTION

In this section, we will explore two key areas: vector operations and trigonometry. The focus will be on the addition, subtraction and scalar multiplication of vectors. We will uncover the properties that govern these operations, such as the commutative, associative and distributive laws and how these concepts apply to real-life situations. We will then review basic trigonometric ratios and extend our knowledge to inverse trigonometric functions which enables the calculation of angles when the ratio values are known. Finally, we will explore the real-world applications of trigonometry in various fields of mathematics

KEY IDEAS

- **Directed line segment** is a segment with an initial point and a terminal point and it is called a vector.
- **Position vector** is a vector that describes the position of a point on the Cartesian plane relative to the origin resultant vector.
- **Scalar quantity** is a quantity that has *magnitude only*. For example, mass, length, time, temperature, density, speed are scalar quantities.
- **Trigonometry** is the branch of mathematics which deals with the measurements of angles and lengths of sides of triangles and their applications.
- **Vector quantity** has both *magnitude and direction*. For examples, force, velocity, acceleration and momentum are vector quantities.

PROPERTIES AND OPERATIONS OF VECTORS

The concept of vectors

Vectors are mathematical entities with magnitude and direction, used to represent quantities in space. They can be added, scaled and multiplied using operations like vector addition, scalar multiplication, dot product and cross product. These operations enable calculations of angles, lengths and orientations between vectors, providing a powerful tool for problem-solving.

Real–life application of vectors

1. **Walking to School**: Imagine your teacher asks you to fetch a book from another classroom. If they say, *"Walk 30 metres straight ahead and then turn left for 20 metres,"* they are giving you a vector. The distance (30 metres and 20 metres) is the magnitude and the directions (straight ahead and left) define the path.
2. **Playing Football**: When you pass a ball to a teammate, the ball travels in a specific direction with a certain force. For example, kicking the ball *10 metres towards the goalpost* represents a vector because it combines the distance (10 metres) with the direction (towards the goalpost).

Vector as a Visual Representation

Vectors can be visually represented as arrows on a diagram or graph. The length of the arrow shows the **magnitude** (size) of the vector, and the direction of the arrow shows the **direction** of the vector. Let us explore this with some examples:

1. **Walking to the Market**

   Imagine you are walking to the market:

   a Start from your house and walk 3 kilometres east. Then turn and walk 4 kilometres north to reach the market.

   b Draw an arrow pointing **east** for the first part (3 km) and another arrow pointing **north** for the second part (4 km).

   c To find the direct distance to the market, draw a single arrow from the starting point to the market. This is the **resultant vector**.

2. **Flying a Paper Plane**

   When you throw a paper plane:

   a If you throw it straight, the arrow representing the vector points in the direction of the throw.

   b The longer the arrow, the harder the throw (the greater the magnitude).

   c For example, a throw 5 metres away in a north-east direction can be drawn as an arrow starting from you and pointing and ending at the plane's landing spot.

Definition of terms

In our everyday life, we encounter two main types of quantities: scalar and vector:

A **scalar** is a quantity that has only **magnitude**, without any specific direction.

Examples of scalars include:

1. Mass
2. Length
3. Time
4. Temperature
5. Density
6. Speed

These quantities only tell us how much there is, without any reference to direction.

A **vector** is a directed line segment, which has both an initial point and a terminal point. It is a quantity that has both **magnitude** (size) and **direction**.

Examples of vectors include:

1. Force
2. Velocity
3. Acceleration
4. Momentum

These quantities not only tell us how much there is but also in which direction.

Position Vector

A **position vector** is a vector that represents the position of a point or object relative to a reference point, usually the origin of a coordinate system. It indicates both the **magnitude** (distance) and the **direction** from the origin to the point.

In simpler terms, a position vector tells you where something is located in space, relative to a defined starting point (usually the origin (0,0) in a 2D plane.

If M (x, y) is any point on the Cartesian plane and O is the origin, then the point M relative to O is a position vector $\overrightarrow{OM}$. That is $\overrightarrow{OM} = \begin{pmatrix} x \\ y \end{pmatrix}$

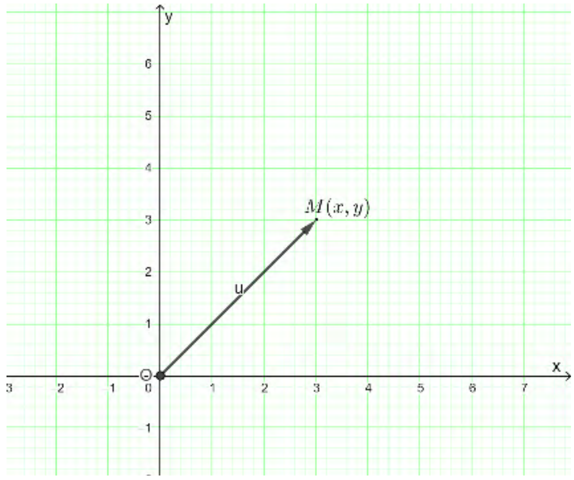


Figure 9.1: Position vector $\overrightarrow{OM}$

For example, the point M (4, 5) on the Cartesian plane will be a position vector $\overrightarrow{OM}$ $= \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

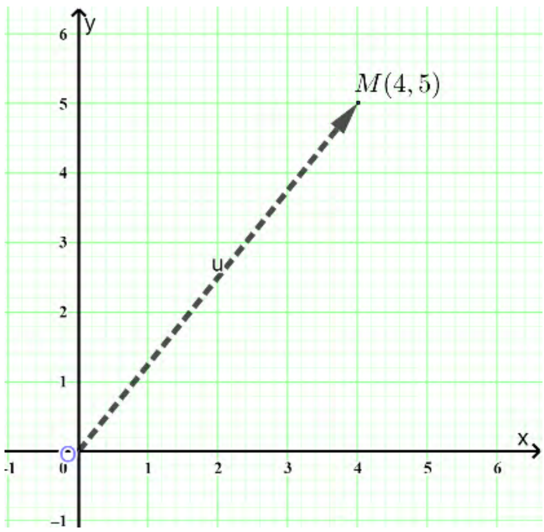


Figure 9.2: Position vector $\overrightarrow{OM} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Expressing two given points as a Vector

Suppose the two points are $A(x_1, y_1)$ and $B(x_2, y_2)$

The vector from A to B is given by:

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$

For example: If A (2, 3) and B (5,7) then:

$$\overrightarrow{AB} = (5 - 2, 7 - 3) = (3, 4)$$

This vector, $\longrightarrow AB = (3,4)$, indicates the direction and magnitude from point A to point B.

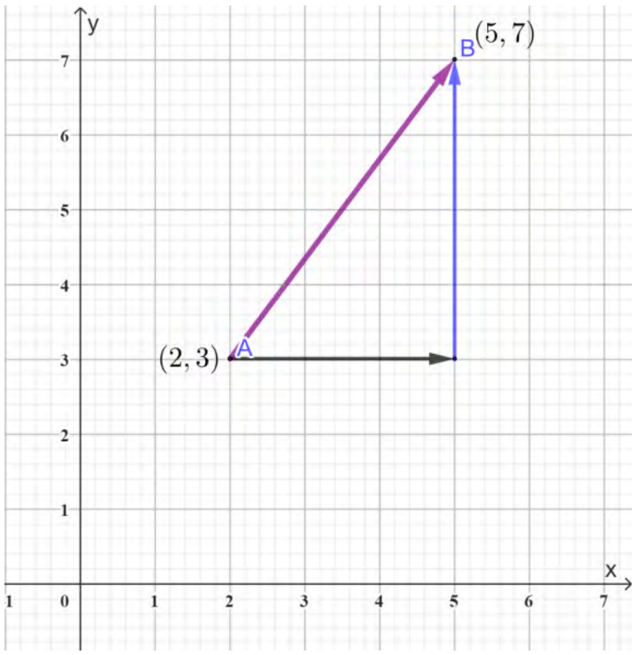


Figure 9.3: A vector from A to B

When we combine two or more vectors, like forces acting on an object or the velocity of moving objects, we are finding their total effect. This total effect is called the resultant vector. It shows the overall result when the vectors are added together. For example, if you walk 3 kilometres north and then 4 kilometres east, the resultant vector shows your final position from where you started. We can use simple maths to find the size and direction of the resultant vector, which we will learn about later.

Understanding Resultant Vectors

A resultant vector is the combined effect of two or more vectors. It shows the overall direction and size of the combined vectors. Imagine several forces acting on an object; the resultant vector represents the single force that would have the same impact.

Visualising Resultant Vectors

Activity 9.1: Using the Head-to-Tail Method to visualise vectors

Working individually, or in pairs, follow these steps to visualise resultant vectors using the Head to Tail Method:

Step 1: Begin by drawing the first vector to scale, with its direction and magnitude accurately represented.

Step 2: Position the tail of the second vector at the head (arrow tip) of the first vector.

Step 3: If there are more vectors, repeat the process by placing the tail of each new vector at the head of the previous one.

Step 4: Draw a straight line from the tail of the first vector to the head of the last vector. This line represents the resultant vector.

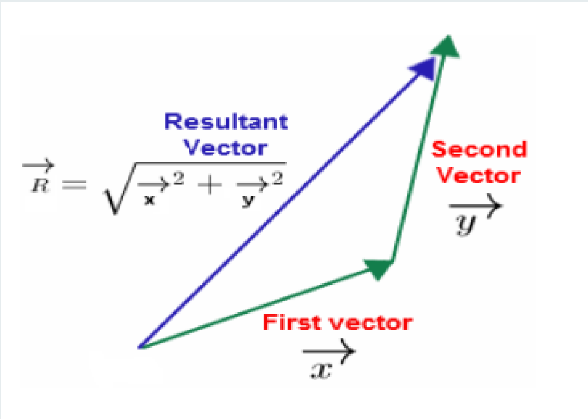


Figure 9.4: Resultant vector

Step 5: Use a ruler to measure the length (magnitude) of the resultant vector and a protractor to determine its direction relative to a reference axis.

Key Points to Remember:

- The order in which you add the vectors doesn't matter. The resultant will always be the same.
- Always use a scale and, where applicable, indicate units for clarity.

Activity 9.2: Using the Parallelogram Method to visualise vectors

Working individually, or in pairs, follow these steps to visualise resultant vectors using the Parallelogram Method:

Step 1: Begin by drawing the two vectors to scale, starting from the same origin point. Ensure their direction and magnitude are accurately represented.

Step 2: From the head of each vector, draw lines parallel to the other vector to form a parallelogram. These lines should be drawn to the same scale as the original vectors.

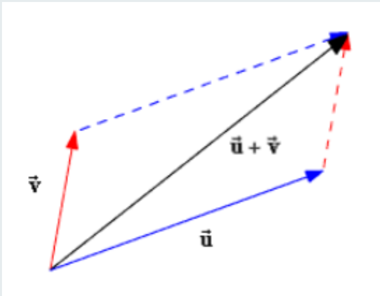


Figure 9.5: Parallelogram law

Step 3: Draw a diagonal line of the parallelogram starting from the origin point where the two vectors meet. This diagonal represents the resultant vector.

Step 4: Use a ruler to measure the length (magnitude) of the resultant vector and a protractor to determine its direction relative to a reference axis.

Calculating Resultant Vectors

If you know the magnitudes and angles of the individual vectors, you can use trigonometry to determine the magnitude and direction of the resultant vector.

Activity 9.3: Using the Head to Tail Method to find the resultant vector

In small groups work through the following steps to find the resultant vector.

Two forces act on an object. One force has a magnitude of 12 N and the other has a magnitude of 16N. The angle between the two forces is 40°.

Find the magnitude and direction of the resultant force.

Step 1: Draw the Vectors:

Draw the First Vector: Represent the 12 N force as a vector. Let's call this vector A

Draw the Second Vector: Represent the 16 N force as another vector. Let's call this vector B

Align the Vectors: Place the tail of vector B at the head of vector A, ensuring that the angle between them is 40°. Imagine sliding the tail of vector B to the tip of vector A without changing the angle between them. Note, the interior angle between them is 140°, compared with the 40° on the exterior.

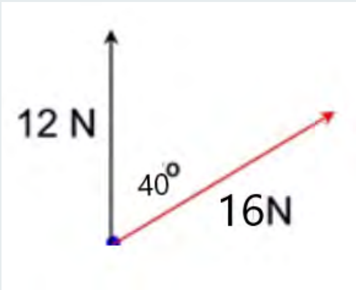


Figure 9.6: Resultant vector

Step 2: Determine the Resultant Vector:

Connect the Tail of A to the Head of B: Draw a line from the tail of vector A to the head of vector B. This line represents the resultant force vector, let's call it R.

Step 3: Calculate the Magnitude of the Resultant Force

To calculate the magnitude of the resultant force, you can use the cosine rule since you know the magnitudes of the two forces and the angle between them.

Cosine Law Formula:

$R^2 = A^2 + B^2 - 2ABcos(\theta)$

Where:

R is the magnitude of the resultant force.

A is the magnitude of force A (12 N).

B is the magnitude of force B (16 N).

θ is the angle between the two forces (140°).

Calculation:

$R^2 = 12^2 + 16^2 - 2 \times 12 \times 16\cos(140°)$

$R^2 = 144 + 256 - 384cos(140°)$

$R^2 = 400 - 384 \times (-)0.766\ldots$

$R^2 = 400 + 294.161\ldots$

$R^2 = 694.161\ldots$

$R = \sqrt{694.161\ldots} \approx 26.3N$

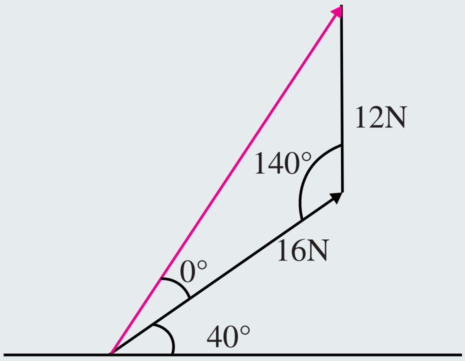


Figure 9.7: Resultant vector

Step 4: Formula for Direction (Angle)

To find the angle θ between the resultant force and A and B, use the Sine Rule or a trigonometric formula:

$$\frac{sin\theta}{12} = \frac{sin140}{26.347}$$

$$sin\theta = \frac{12sin140}{26.347}$$

$$\theta = sin^{-1}\left(\frac{12sin140}{26.347}\right) = 17.0°$$

Magnitude of Resultant Force: Approximately 26.3 N

Direction of Resultant Force: Approximately 17° above vector B.

Activity 9.4: Using the Parallelogram Method to find the resultant vector

In your small groups work out Activity 9.3 using the parallelogram method.

As you will see, they are very similar in their method to give the same solutions.

Example 9.1

There are two vectors:

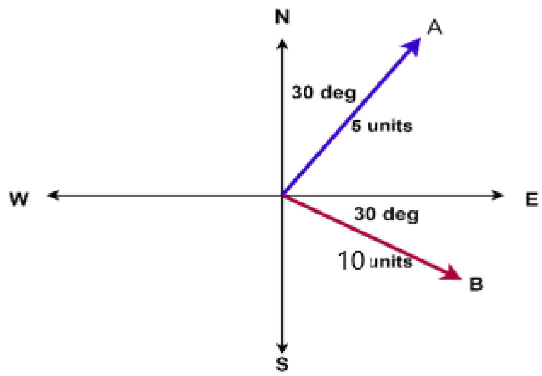
Vector A: Magnitude = 5 units, Direction = 30° from the vertical

Vector B: Magnitude = 10 units, Direction = 30° below the horizontal

Find the magnitude of the resultant vector.

Solution

To find the resultant vector, you would use the cosine rule to calculate its magnitude.



$$OA = \begin{pmatrix} 5cos60° \\ 5sin60° \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5\sqrt{3} \end{pmatrix}$$

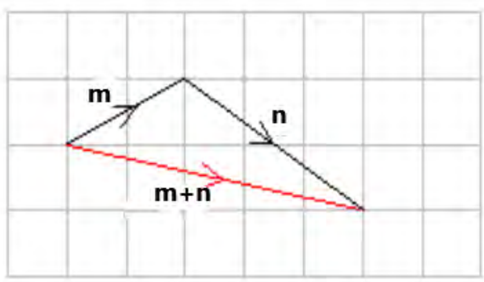
$$OB = \begin{pmatrix} 10cos30° \\ -10sin30° \end{pmatrix} = \begin{pmatrix} 5\sqrt{3} \\ -5 \end{pmatrix}$$

$$OA + OB = \begin{pmatrix} 2.5 \\ 2.5\sqrt{3} \end{pmatrix} + \begin{pmatrix} 5\sqrt{3} \\ -5 \end{pmatrix} = \begin{pmatrix} 11.16 \\ -0.67 \end{pmatrix}$$

$$\text{Magnitude} = \sqrt{(11.16)^2 + (-0.67)^2} = \sqrt{124.55 + 0.45} \approx \sqrt{125} = 11.18\ units$$

Example 9.2

Find the sum of the two given vectors ***m*** and ***n***.



Solution

Draw the vector ***m***.

Draw the 'tail' of vector ***n***, joined to the 'nose' of vector ***m***.

The vector $\boldsymbol{m} + \boldsymbol{n}$ is from the 'tail' of ***m*** to the 'nose' of ***n***.

VECTOR OPERATIONS IN TWO DIMENSIONS

Geometric representation

A vector $\boldsymbol{v} = (x, y)$ can be represented geometrically as an arrow:

1. The tail of the arrow is at the origin (0, 0).
2. The head of the arrow is at the point (x, y).
3. The length (or magnitude) of the arrow represents the size of the vector.
4. The direction of the arrow represents the direction of the vector.

Algebraic Representation

A vector in two-space can be represented algebraically as:

$\boldsymbol{v} = (x, y)$ or $\boldsymbol{v} = x\,i + y\,j$

where:

x and y are the horizontal and vertical components of the vector, respectively and i and j are the unit vectors in the x and y directions, respectively.

The relationship between geometric and algebraic representations:

1. x is the change in x-coordinate from tail to head
2. y is the change in y-coordinate from tail to head
3. Zero vector: (0, 0)
4. Unit vectors: $\hat{\imath} = (1, 0)$ *and* $\hat{\jmath} = (0, 1)$
5. General form: $\boldsymbol{a} = x\hat{\imath} + y\hat{\jmath}$

Activity 9.5: Using graph paper to illustrate the connection between geometric and algebraic representations

Working individually, or in pairs, follow the steps below to illustrate the connection between geometric and algebraic representations.

Step 1: On graph paper mark on the x and y axes.

Step 2: Select two vectors, A and B, with their components (x, y). For example, A = (3, 4) and B = (2, 1).

Step 3: Plot Vector A by starting at the origin (0, 0) and move horizontally 3 units (x-component) and then vertically 4 units (y-component). Mark the endpoint as the head of Vector A.

Step 4: Plot Vector B by starting at the origin (0, 0) and move horizontally 2 units (x-component) and then vertically 1 unit (y-component). Mark the endpoint as the head of Vector B.

Step 5: Label Vector A and Vector B with their corresponding components (x, y).

Step 6: To illustrate vector addition, place the tail of Vector B at the head of Vector A. Draw the resultant vector from the tail of Vector A to the head of Vector B.

Step 7: Verify that the geometric representation of the vectors matches their algebraic representation.

For example, Vector A = 3i + 4j, Vector B = 2i + 1j and Vector A + B = 5i + 5j

Vector addition and subtraction

Vector Addition

Geometrically:

1. Head-to-tail method: Align the tail of the second vector with the head of the first vector. The resulting vector is represented by an arrow drawn from the tail of the first vector to the head of the second.
2. Parallelogram method: Construct a parallelogram where the two vectors form adjacent sides. The diagonal of the parallelogram, starting from the same point as the vectors, represents their sum.

Note that this is the same as the Resultant Vector investigated above.

Algebraically:

For vectors $\mathbf{m} = (x_1, y_1)$ and $\mathbf{n} = (x_2, y_2)$:

$\mathbf{m} + \mathbf{n} = (x_1 + x_2, y_1 + y_2)$

Example 9.3

Given that, $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, find the sum of the vectors

Solution

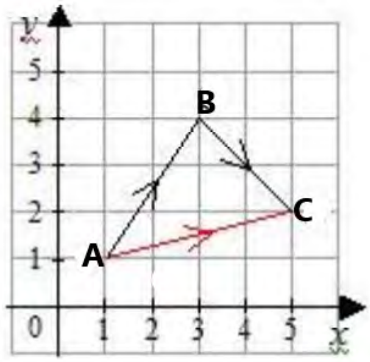
The sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is the same as the vector $\overrightarrow{AC}$.

That is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Therefore, we add the corresponding components of the vectors as:

$$\overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



Vector Subtraction

Subtraction as Vector Addition

Subtracting a vector B is equivalent to adding its negative, –B, which is the same vector pointing in the opposite direction: A – B = A+ (–B)

Steps for Geometric Representation:

1. Plot vector A.
2. Reverse the direction of vector B to obtain –B.
3. Use the head-to-tail method to add A and –B.
4. Draw the resultant vector from the tail of A to the head of –B

Example 9.4

$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Find **m** – **n**

Solution

1. Plot **m** and **n** on a Cartesian plane.
2. Reverse the direction of **n** to get $-\mathbf{n}=\begin{pmatrix} -2 \\ -1 \end{pmatrix}$.
3. Add **m** and –**n**

   $\mathbf{m} - \mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

   Or think of it as:

$$\mathbf{m} - \mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Key concepts:

1. Vector addition is *commutative*: $a + b = b + a$
2. Vector addition is *associative*: $(a + b) + c = a + (b + c)$
3. The *zero vector is the additive identity*
4. The negative of a vector: $-a = (-x, -y)$

Scalar multiplication and its effect on vector magnitude and direction

Scalar multiplication in vector mathematics involves multiplying a vector by a scalar (a real number). This operation affects both the magnitude and sometimes the direction of the vector:

Effect on Magnitude:

The magnitude of a vector scales by the absolute value of the scalar.

1. If $\boldsymbol{v}$ is a vector and $\boldsymbol{k}$ is a scalar, then the magnitude of $\boldsymbol{kv}$ is given by:

   a $|kv| = |k| \cdot |v|$
2. If $|k| > 1$, the vector's magnitude increases.
3. If $0 < |k| < 1$, the vector's magnitude decreases.
4. If $k = 0$, the vector becomes the zero vector, which has zero magnitude.

Effect on Direction

The direction of the vector changes based on the sign of the scalar:

1. If $k > 0$, the direction of the vector remains the same.
2. If $k < 0$, the direction of the vector reverses (points in the opposite direction).

Unit Vector

A unit vector is a vector with a magnitude of 1. It is primarily used to indicate the direction of a vector without considering its magnitude. Unit vectors are widely used in mathematics, physics, and engineering.

To find the unit vector $\hat{\boldsymbol{u}}$ in the direction of a given vector $\boldsymbol{v}$, divide the vector by its magnitude:

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

Example 9.5

$\boldsymbol{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Find its unit vector

Solution

Step 1: Compute the magnitude:

$|\boldsymbol{v}| = \sqrt{3^2 + 4^2} = 5$

Step 2: Divide each component by the magnitude:

$$\hat{u} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$\therefore \textit{unit vector } \hat{\boldsymbol{v}} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

Properties of scalar vectors

1. Distributive property: $(\boldsymbol{a} + \boldsymbol{b}) = k\boldsymbol{a} + k\boldsymbol{b}$
2. Associative property: $(kl) = (l\boldsymbol{a})$, where k and l are scalars
3. Vector decomposition: $a = x\boldsymbol{a}\,\hat{\imath} + y\boldsymbol{a}\,\hat{\jmath}$

Table 9.1: *Property of Vector Addition*

Property of Vector Addition	Explanation
Existence of identity	For any vector **v**, **v** + **0** = **v** Here, **0** vector is the additive identity
Existence of inverse	For any vector **v**, **v** + - **v** = **0** and thus, an additive inverse exists for every vector.
Commutativity	Addition is commutative; for any two arbitrary vectors **c** and **d**, **c** + **d** = **d** + **c**
Associativity	Addition is associative; for any three arbitrary vectors **i**, **j** and **k**, **i** + **j** + **k** = **i** + **k** + **j** i.e., the order of addition does not matter.

Example 9.6

If $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $x + y$

Solution

$x + y = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$,

$x + y = \begin{pmatrix} 3 + 2 \\ 4 + -1 \end{pmatrix}$

$x + y = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Example 9.7

Given vectors $\boldsymbol{m} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\boldsymbol{m} + \boldsymbol{w}$

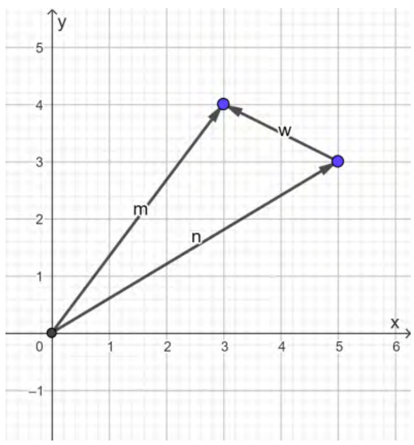
1. algebraically
2. geometrically.

Solution

1. Algebraically:

   $$\boldsymbol{m} + \boldsymbol{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 + 2 \\ 4 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

2. Geometrically:



- Draw vector **m** from the origin to point $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. From the head of ***m***, draw on the vector ***w*** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
- The resultant is vector ***n*** which is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Example 9.8

Subtract vector $\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ from vector $\boldsymbol{q} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

Solution

$$\boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5-3 \\ 7-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Example 9.9

Multiply vector $\boldsymbol{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by scalar $k = 2$.

Solution

$$k \,.\, v = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 2 \\ 2 \times 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Real-life uses of vectors

1. Vectors can be used in finding the direction in which the force is applied to move an object.
2. The concept of vectors aids in understanding how gravity uses a force of attraction on an object to work.
3. Vectors can be used in obtaining the motion of a body which is confined to a plane.
4. Vectors help in defining the force applied on a body simultaneously in the three dimensions.
5. In the field of Engineering, for a structure not to collapse, vectors are used where the force is much stronger than the structure will sustain.
6. Vectors are used in various oscillators.

TRIGONOMETRIC RATIOS

Trigonometric ratios

Trigonometry is a branch of mathematics that deals with the relationships between the sides and angles of triangles, particularly triangles with right angles (90-degree angles). The term “trigonometry” comes from the Greek words “Trigonon” meaning “triangle” and “metry” meaning “measure”. Trigonometry involves the study of Core Concepts such as Angles, Triang les and Trigonometric functions.

Right-Angled Triangle

Activity 9.6: Construction of a right -angled triangle using a ruler and a compass

Step 1: Draw a line segment from P to A of any length. Mark its midpoint with the letter C.

Step 2: Place the compass point on one end of the base say A and draw an arc.

Step 3: Move the compass point to the other end of the base, P, and draw another arc, with the same radius, intersecting the first arc.

Step 4: Draw a line from the intersection point to point C on the line PA, creating a right angle (90°).

Step 5: Label the vertices: A (base), B (intersection), and C (right angle).

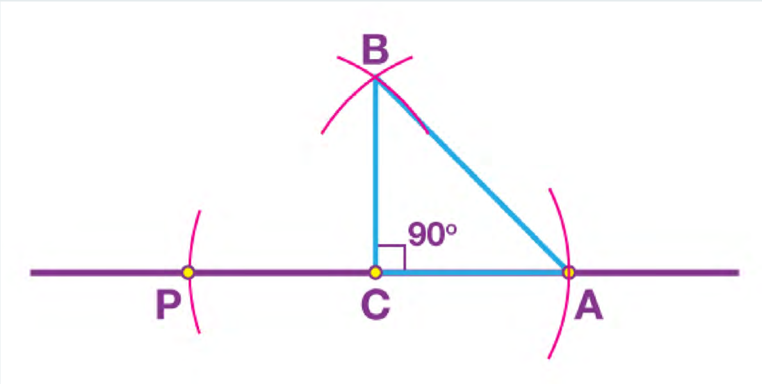


Figure 9.9: Right -angled triangle

Pythagoras' Theorem

Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle. It states that $a^2 = b^2 + c^2$ as shown below.

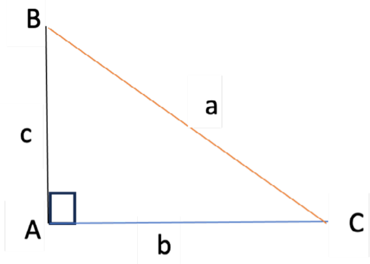


Figure 9.10: Right -angled triangle

Activity 9.7: Computing the length of the side BC in the figure below

Working in pairs, revise how to find the length of the hypotenuse, BC, in the figure below.

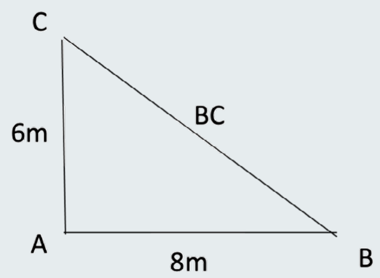


Figure 9.11: Right -angled triangle

Solution

Step 1: Using Pythagoras theorem

$a^2 = b^2 + c^2$

Step 2: Substitute values into the formula in step 1

$BC^2 = 8^2 + 6^2$

Step 3: Simplify

$BC^2 = 64 + 36 = 100$

Step 4: Find the square roots of both sides

$BC = \sqrt{100}$

Therefore, BC = 10m

Trigonometric ratios

Trigonometric ratios are mathematical relationships between the sides of a right-angled triangle, defined as the ratios of the lengths of its sides relative to a specific acute angle.

Activity 9.8: Revision of the basic trigonometric ratios

Working in pairs carry out the following activity to revise the basic trigonometric ratios.

Step 1: Sketch a right-angled triangle.

Step 2: Label all the three sides, as below.

Step 3: Determine the sine, cosine and the tangent of the acute angle, θ.

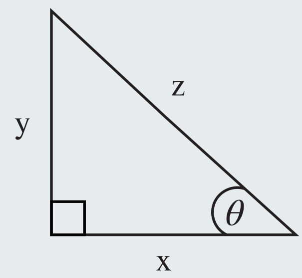


Figure 9.12: Right -angled triangle

Hopefully you recalled that the three basic trigonometric ratios based on the triangle above are:

- $sine\theta = \frac{Opposite}{Hypotenuse} = \frac{y}{z}$, written as $sin\theta$
- $cosine\theta = \frac{Adjacent}{Hypotenuse} = \frac{x}{z}$, written as $cos\theta$
- $tangent\theta = \frac{Opposite}{Adjacent} = \frac{y}{x}$, written as $tan\theta$

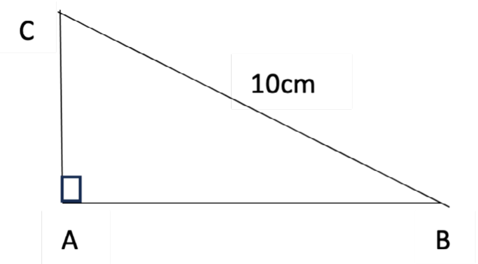
For an easy reminder of the above trigonometric ratios, remember the following:

- $\text{Sine} = \frac{\textbf{\textit{Opposite}}}{\textbf{\textit{Hypotenuse}}} = \textbf{SOH}$
- $\text{Cosine} = \frac{\textbf{\textit{Adjacent}}}{\textbf{\textit{Hypotenuse}}} = \textbf{CAH}$
- $\text{Tangent} = \frac{\textbf{Opposite}}{\textbf{Adjacent}} = \textbf{TOA}$
- (***SOH* : *CAH* : *TOA***)

Example 9.10

In a right-angled triangle, the hypotenuse is 10 *cm* and one angle is 30^0. Find the lengths of the other two sides.

Solution



$sin\ 30 = \frac{Opp}{Hyp} = \frac{|AC|}{|BC|} = \frac{|AC|}{10}$

$|AC| = 10 sin\ 30 = 10 \times 0.5 = 5\text{cm}$

By Pythagoras' theorem:

$BC^2 = AB^2 + AC^2$

$10^2 = AB^2 + 5^2$

$AB^2 = 10^2 - 5^2 = 100 - 25 = 75$

$|AB| = \sqrt{75} = 8.660 = 8.7cm$

Alternative Approach:

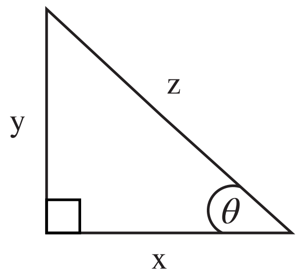
$Cos30 = \frac{|AB|}{|10|}$

$10 \times cos30 = |AB|$

$|AB| = 10 \times cos30\ = 10 \times 0.866 = 8.66 = 8.7cm$

The inverse of trigonometric ratios

The inverse trigonometric ratios are cosecant, secant and cotangent.



These are:

- $cosec\theta = \frac{1}{\sin\theta} = \frac{z}{y}$, therefore $cosec\theta = \frac{z}{y}$
- $sec\theta = \frac{1}{Cos\theta} = \frac{z}{x}$, therefore $sec\theta = \frac{z}{x}$
- $cot\theta = \frac{1}{tan\theta} = \frac{x}{y}$, therefore $cot\theta = \frac{x}{y}$

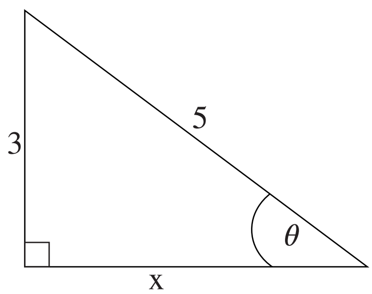
Activity 9.9: Derivation of the inverse of trigonometric ratios

In small groups, work together to see how the above inverse trigonometric ratios were derived. Explain them to each other.

Solving problems with the trigonometric ratios

Example 9.11

If $\sin\theta = \frac{3}{5}$, find $\cos\theta$, $\tan\theta$ and the value of θ in degrees.



Solution

By Pythagoras theorem

$5^2 = 3^2 + x^2$

$x^2 = 5^2 - 3^2 = 25 - 9 = 16$

$x = \sqrt{16} = 4$

$Cos\theta = \frac{4}{5}$ and $tan\theta = \frac{3}{4}$

Trigonometry in Real Life

Areas in real life where trigonometry is applicable are:

1. Building Design, Architecture and construction: Roof angles and support structures; Bridge design; etc
2. Surveying and Flight Navigation: Calculating distances and heights; GPS and triangulation; etc.
3. Music: Analysis of sound waves and frequencies
4. Oceanography and Meteorology: Analysis of wave patterns and ocean currents.

Sample equipment used in trigonometry

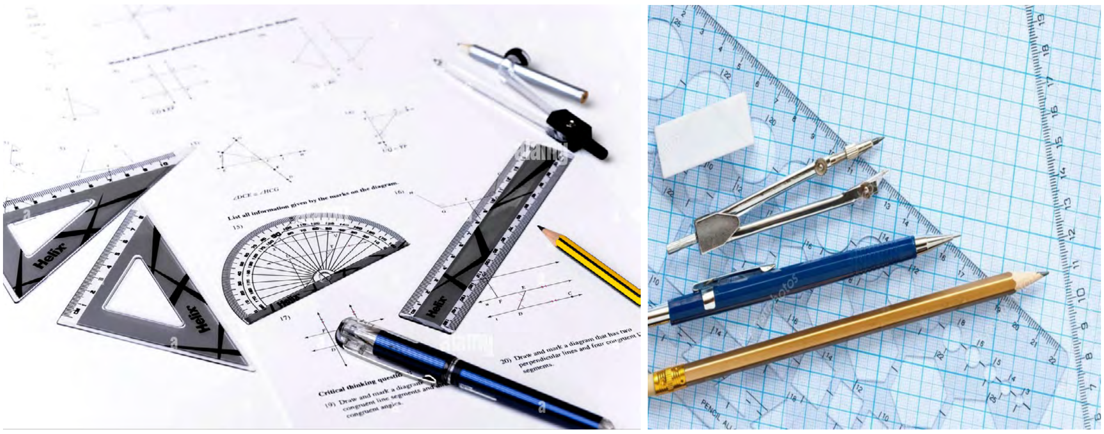


Figure 9.13: Sample equipment used in trigonometry

Angles of Elevation and Depression

Angles of Elevation of an object B from an observer at A, who is below the level of B, is the angle which AB makes above the horizontal.

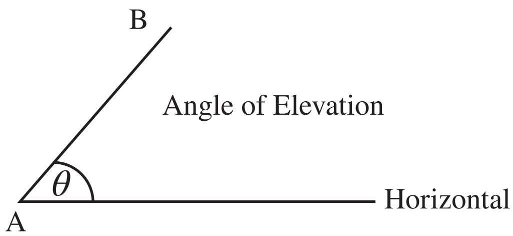


Figure 9.14: Angle of elevation

Example 9.12

A surveyor measures the angle of elevation to the top of a building to be 60^0.

If the surveyor is standing 30 metres away from the base of the building what is the height of the building?

Solution

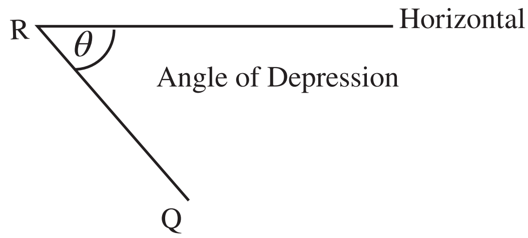
Step 1: Use the tangent ratio to relate the angle, the distance from the building and the height of the building: $\tan(60) = \frac{height}{30}$

Step 2: Solve for the height:

$\tan(60) = \frac{height}{30}$

Height $= 30 \times tan(60) = 51.96\text{m}$

Angles of Depression of an object Q from an observer at R, who is above the level of Q is the angle which RQ makes below the horizontal.



Example 9.13

A person is standing on a cliff looking out at a ship in the distance. The angle of depression to the ship is 30^0.

If the person is standing 20 metres above sea level, how far is the ship from the shore.

Solution

Use the tangent ratio to relate the angle, the height above sea level and the distance to the ship: $\tan(30^0) = \frac{20}{distance}$

Solve for the distance to the ship:

Distance $= \frac{20}{\tan(30)} = 34.64m$

Examples 9.14

A boy is standing near a tree. He looks up at the tree and wonders, "How tall is the tree?"

What method could the boy use to find the height of the tree which does not involve climbing it?

Solution

We have a right-angled triangle.

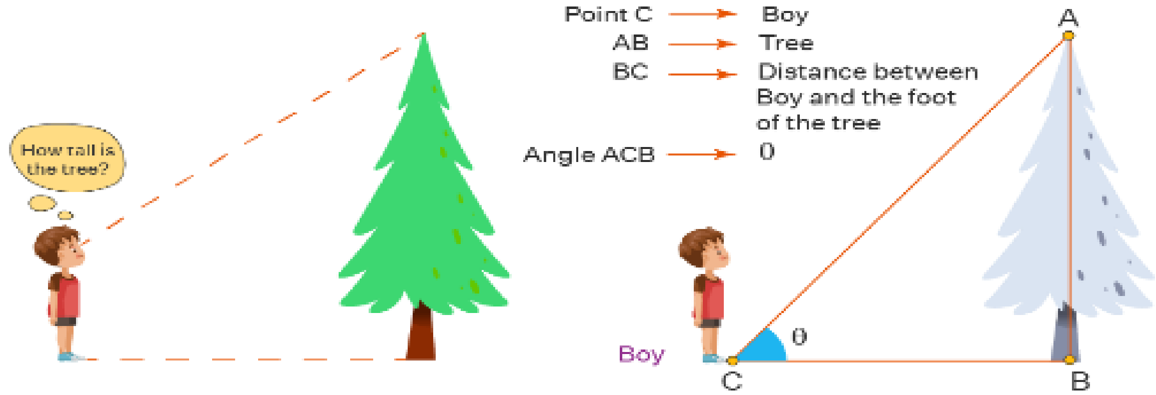


Figure 9.15: Angles of elevation and depression

In a right-angled triangle, ΔABC, $tan\ of\ angle\ \theta$ is:

the ratio of the height of the tree to the distance between boy and foot of the tree.

$$\therefore \tan\theta = \frac{\textit{Height of tree}}{\textit{Horizontal distance between boy and tree}} = \frac{AB}{CB}$$

For example, if the distance $|CB| = 30m$ and the angle formed $\theta = 45^0$, then

$$\tan\theta = \frac{AB}{CB}$$

$$\tan 45° = \frac{B}{30m}$$

$$Height,\ AB, = 30tan \times 45° = 30m$$

Example 9.15

A building designer wants to create a right angled, triangular roof with a 30° angle from the base to the sloping side. If the length of the base of the roof is 10 metres, what is the vertical height of the roof?

Solution

Step 1: Identify the given information:

The angle of the roof is 30° and the length of the base is 10 metres.

Step 2: Determine the trigonometric ratio to use:

Since we know the angle and the adjacent side (the base), we can use the tangent ratio to find the height.

Step 3: Set up the equation:

$\tan(30°) = \frac{Height}{10}$

Step 4: Solve for the height:

height $= 10 \times \tan(30°) = 5.773$ metres

Therefore, the height of the roof is 5.773m

Example 9.16

An airplane is flying at an altitude of 3000 metres. The angle of depression to an airfield on the ground is 20°.

If the plane were to start descending on the hypotenuse, what is the distance to the airfield to the nearest 10m?

Solution

Step 1: Identify the given information:

The altitude of the airplane is 3000 metres and the angle of depression is 20°.

Step 2: Determine the trigonometric ratio to use:

Since we know the angle and the opposite side (the altitude), we can use the sine ratio to find the distance of the hypotenuse.

Step 3: Set up the equation:

$\sin(20) = \frac{3000}{Distance\ to\ airfield}$

Step 4: Solve for the distance:

$Distance\ to\ airfield = \frac{3000}{\sin(20)} = 8771.4,$

Therefore, the distance the plane must fly along the hypotenuse is 8770m to the nearest 10m.

Example 9.17

A ladder leans against a brick wall. The foot of the ladder is 2m from the wall and makes a 30° angle with the ground.

How long is the ladder?

Solution

Step 1: Identify the given information:

The angle between the ground and the ladder is 60° and the distance from the ground to the wall is 1.5m.

Step 2: Determine the trigonometric ratio to use:

Since we know the angle and the adjacent side (the distance from the wall) and the ladder is the hypotenuse, we can use the cosine ratio to find the length of the ladder. .

Step 3: Set up the equation:

$$\cos(60) = \frac{1.5}{Length\ of\ Ladder}$$

Step 4: Solve for the length of ladder

$$\text{Length of ladder} = \frac{1.5}{\cos(60)} = 3m.$$

Example 9.18

A man is 20m from the base of a flagpole. His eyeline is 1.7m above the ground. The angle of elevation from his eyeline to the top of the flagpole is 40°.

How tall is the flagpole?

Solution

Step 1: Identify the given information:

The angle between the man and the top of the flagpole is 40° and the distance from the man to the base of the flagpole is 20m.

Step 2: Determine the trigonometric ratio to use:

Since we know the angle and the adjacent side (the distance from the flagpole) and the flagpole is the opposite, we can use the tangent ratio to find the height of the flagpole from the man's eyeline.

Step 3: Set up the equation:

$$\tan(40) = \frac{Height\ from\ eyeline\ to\ top\ of\ flagpole}{20}$$

Step 4: Solve for the height from eyeline to top of flagpole:
Height from eyeline to top of flagpole = 20tan(40) = 16.78*m*.

Step 5: Calculate the full height of the flagpole:
We need to know the full height of the flagpole, so we must add on the height of the man's eyeline.

Therefore, the full height of the flagpole = 16.78 + 1.7 = 18.48m

EXTENDED READING

1. www.cambridge.edu.au/GO www.cambridge.edu.au/education
2. Essential Mathematics for the Australian Curriculum Year 10 & 10A 3ed ISBN 978-1-108-77346-1 © Greenwood et al. 202

REVIEW QUESTIONS

1. Find $\overrightarrow{PQ}$ for the points $P\begin{pmatrix}-2\\3\end{pmatrix}$, $Q\begin{pmatrix}4\\7\end{pmatrix}$.

2. A is $\begin{pmatrix}2\\5\end{pmatrix}$, B is $\begin{pmatrix}6\\12\end{pmatrix}$, C is $\begin{pmatrix}-3\\10\end{pmatrix}$ and D is $\begin{pmatrix}8\\19\end{pmatrix}$. Convert in component forms the vectors: $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

3. Given P $\begin{pmatrix}-7\\1\end{pmatrix}$, Q $\begin{pmatrix}-3\\6\end{pmatrix}$, and S $\begin{pmatrix}3\\-3\end{pmatrix}$.

a. Find $\overrightarrow{PS}$ and $\overrightarrow{PQ}$.

b. Calculate the lengths of $\overrightarrow{PS}$ and $\overrightarrow{PQ}$.

4. The point A is (2, 3) and $\overrightarrow{AB} = \begin{pmatrix}4\\-9\end{pmatrix}$, what are the coordinates of point B?

5. A person standing on the ground observes a bird on top of a tree at an angle of elevation of 60°. If the person is 30 metres away from the base of the tree, how high is the bird above the ground?

6. If $sec q = \frac{17}{8}$ find:

a. the trigonometric ratios of

(i) $sin\theta$ and

(ii) $tan\theta$

b. the value of q in degrees.

7. If $tan q = \frac{15}{8}$, find the values of $cos q$ and $sin q$.

8. If $cot q = 2$, find the values of $sin q$, $cos q$ and $tan q$.

9. If $cosec q = \sqrt{5}$, find the values sin q, $cos q$, $tan q$ and q in degrees.

10. A ladder is placed against a wall. The bottom of the ladder is 6 metres from the wall, and the top of the ladder touches the wall 8 metres above the ground. How long is the ladder?

11. A boat is 100 metres from the shore. The angle of elevation from the boat to the top of a lighthouse is 30°. How tall is the lighthouse?

12. A ladder, 10 metres long, leans against a wall, making an angle of 60° with the ground. How far is the base of the ladder from the wall?

Answers to Review Questions

Answers to Review Questions for Section 1

1. C

2. **a.** $2\sqrt{6}$

b. $3\sqrt{5}$

c. $x\sqrt{2}$

d. $2\sqrt{x}$

3. $3+6\sqrt{3} = 3\left(1 + 2\sqrt{3}\right)$

4. **a.** $2\sqrt{3}$

b. $\frac{1}{6}\sqrt{6} - \frac{1}{4}\sqrt{2}$

c. $\frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{2}$

5. **a.** $5\sqrt{6}$

b. $4\sqrt{2}$

c. $3\sqrt{3} - 12\sqrt{2}$

6. $7 - \sqrt{15}$

7. **a.** $4\sqrt{3} - \sqrt{6}$

b. $\sqrt{2} - \frac{\sqrt{10}}{2}$

8. 11.9 (to 1 decimal place)

9. $4\sqrt{33}$cm

10. **a.** $\frac{1}{5}$

b. $\frac{1}{2}$

11. $\frac{-1}{2}$

12. 27

13. **a.** 512

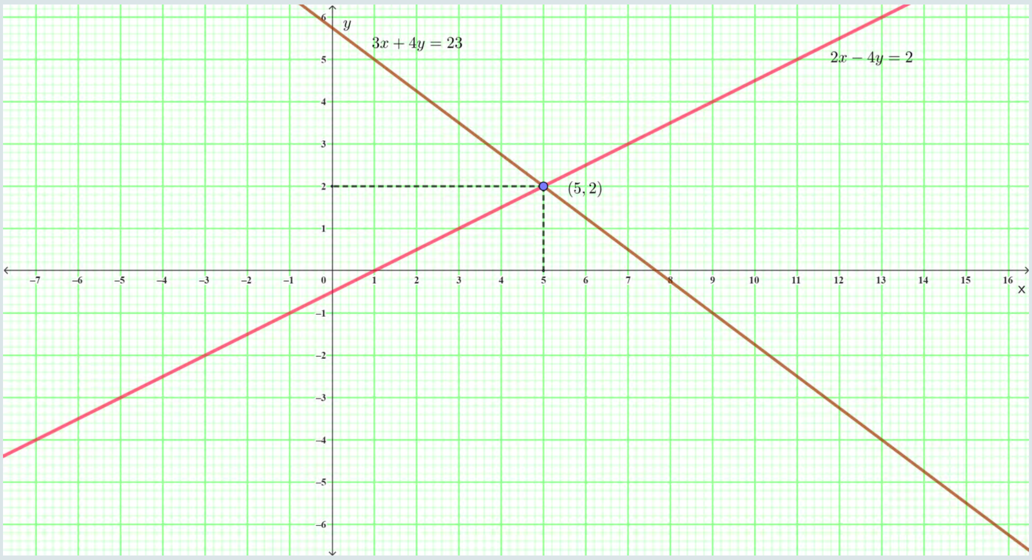
b. 3

14. **a.** 0.5

b. 1

From the graph the solution is (2, 1)

a.



From the graph the solution set is (5, 2)

b.

From the graph the solution set is (3, 6)

4. The farmer sold 15 bowls of maize and 15 bowls of beans.

5. **a.** $x + y = 50$ (Equation 1) where x = rice with chicken and y = jollof rice

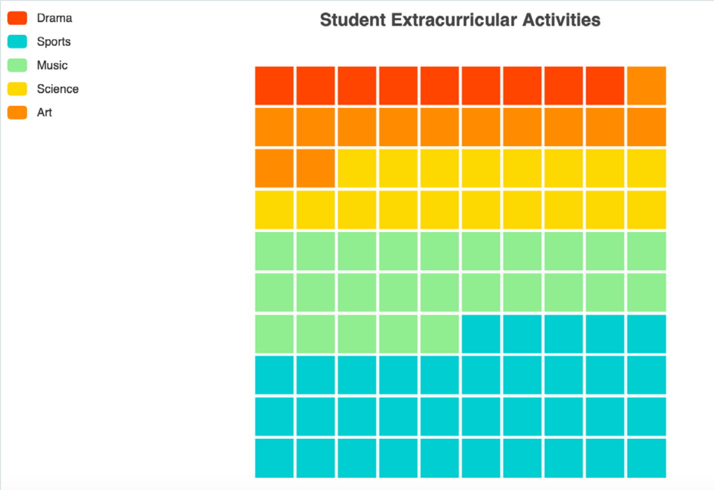
a. $10x + 8y = 420$ (Equation 2)

b. Ama and Kofi sold 10 dishes of rice with chicken **and** 40 dishes of jollof rice.

Answers to Review Questions for Section 4A

1. The primary purpose is to gather accurate and reliable data to answer research questions, test hypotheses or evaluate concepts.
2. Quantitative instruments (e.g., surveys, questionnaires) and qualitative instruments (e.g., interviews, observations).
3. A survey is a broader data collection method, while a questionnaire is a specific tool used to collect data only through questions.
4. Any or all of the following responses: A well designed instrument will be **valid** (collect data on what you have planned), **reliable** (would get the same results if carried out again), **objective** (will not be influenced by the person carrying out the collection), **simple** (a simple instrument will gain the best results as everyone will know what is required) and **relevant** (data collection is relevant to the required objective).
5. Pilot-testing involves testing the instrument with a small group to ensure validity, reliability and clarity, reducing errors and improving data quality when it is rolled out to the whole test population.
6. Through expert review, pilot-testing and ensuring the instrument measures what it intends to measure.
7. Open-ended questions allow respondents to answer freely, while closed-ended questions provide pre-defined response options.
8. Surveys, questionnaires, interviews, observations and tests.
9. Reliability is achieved through consistent questioning, clear instructions and minimising bias.
10. Potential sources of error include **sampling bias** (participants are not randomly selected from the population), **measurement error**, **non-response error** and **data entry error**.

2.



3. Minimum = 65

Q1 = 76

Q2 = 84

Q3 = 90

Maximum = 95

The box plot should have its whiskers from 65 to 76 and 90 to 95 and the box should go from 76 to 90 with a vertical line at the median of 84.

Answers to Review Questions for Section 4c

1. B median is a measure of central tendency not dispersion.

2. True. The mean is required to calculate the standard deviation and the latter looks at how spread out the data is about the mean.

3. The quartile deviation is 0.5. This shows that 50% of the students – those lying between the upper quartile and lower quartile – are 0.5 mark away from the median. Thus, the values of the data are close together. That is, the candidates have about the same academic potential.

4. A mean of 18% and a standard deviation of 5% show that in Burkina Faso the bulk of the interest rates range from 13% to 23%. That is, the interest rates are largely ±5% from the mean.

In Ghana, the interest rates largely range from 10% to 30%. Whilst in the Ivory Coast, the interest rates largely range from 4% to 44%.

17. The graph forms a peak, showing movement away, then reversal and return to the starting point.
18. A graph with a straight line for steady speed, a horizontal line for the stop, and a steeper straight line for the faster speed.
19. Acceleration is 1.67 m/s^2; the distance-time graph would show a concave-upward curve.
20. The average speed is 5 m/s.
21. Total distance = 40 km; displacement = 10 km east. The graph shows an increasing slope, followed by a gentler downward slope.
22. Missing length: $x = 25$ Angles: 37°,53°,90°
23. The success rate is 0.625.
24. Player X, with a rating of 102, is considered statistically more successful than Player Y, who has a rating of 95.
25. You will receive 45 000JPY.
26. The total cost of the transaction is £102.
27. You would get more sandwiches for your money in the UK, as one sandwich costs only £4, compared to £6 for a sandwich in Brazil when converted to GBP. These answers provide clarity on the calculations and comparisons made in each scenario.
28. You consumed a total of 600 m^3 of water for the entire month.
29. Your total electricity bill is Gh¢45.

Answers to Review Questions for Section 6

1. A
2. B
3. a. $x = 1$
   b. 1, 3, 5, 7, 9, 11, 13, 15
   c. $U_{15} = 29$
   d. $S_{21} = 441$
4. The sum of the first 20 terms is 210.

5. 126.5 m^2.

6. $112.5\ \pi = 353.4$ units^2

7. $3m^3$ of soil are required to fill the garden bed.

8. a The volume of the frustum is $\frac{3040}{3}\pi = 3\ 183m^3$.

   b 3 183 000 litres

   The reservoir can hold approximately 3 million litres of water, which helps the community estimate how much rainwater it can collect for irrigation. Since a typical watering system for a garden may require a few hundred litres of water per session, this volume is quite substantial and would allow the garden to watered over many sessions before needing to refill.

9. 352cm^2

10. $h_1 : h_2 = 5 : 7$

11. 144 cm^3

12. The sun's radius is about 100 times bigger than the earth's. Therefore, the volume is about $100^3 = 1000\ 000$ times bigger, or about a million earths will fit in the sun's volume.

Answers to Review Questions for Section 8

1. a 0.54

   b 0.7397

2. a $\frac{9}{49}$

   b $\frac{12}{49}$

   c $\frac{16}{49}$

3. a. $\frac{11}{24}$

   b. $\frac{1}{4}$

   c. $\frac{3}{4}$

4. a. $\frac{1}{4}$

   b. $\frac{1}{9}$

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GLOSSARY

3D Graph: Visual representation of multivariate data.

Angle of Depression is the angle between the horizontal and a line of sight to an object below the horizontal.

Angle of Elevation is the angle between the horizontal and a line of sight to an object above the horizontal.

Angles: Ratios can be applied to measure and analyse geometric relationships.

Arithmetic Progression (AP): A sequence in which the difference between consecutive terms is constant, called the common difference.

Bias: Systematic error or distortion in data collection or analysis.

Box and Whisker Plot: Graph showing data distribution and outliers.

Centre of Enlargement: Centre of enlargement is a fixed point from which a shape is scaled up or down during an enlargement transformation.

Centre of Rotation: Centre of rotation is the fixed point around which a shape rotates during a rotation transformation.

Common Difference: The fixed amount added to each term in an AP to get the next term.

Common Ratio: The fixed multiplier between consecutive terms in a GP.

Confidentiality: Protection of sensitive information.

Congruent: identical

Conversion: The process of changing a measurement from one unit to another (e.g., from square feet to square metres, or from gallons to litres).

Cost Estimation: The process of calculating the approximate cost of materials, labour and other resources needed for a task, such as construction or manufacturing.

Cultural Sensitivity: Respect for diverse backgrounds and experiences.

Data Protection: Measures to safeguard data privacy.

Dependent events is if successive events are affected by previous events.

Dispersion: Measure of data spread or variability.

Distance-Time Graph: Shows the relationship between distance and time; gradient indicates the speed.

Efficiency: The ability to achieve maximum productivity with minimal wasted effort or resources.

Elimination method: A technique of solving simultaneous equations by adding or subtracting the equations after manipulating coefficients to eliminate one variable.

Equally likely events is if all the events have an equal chance of being chosen.

Event is any subset of the sample space.

Experiment is performing an action to study its effects.

Financial Mathematics: Uses ratios, rates and proportions for interest, profit, or currency conversions.

Fluid ounce: imperial measure for relatively small volumes of liquids.

Formulate equations: The process of translating a real-world problem into a system of linear equations with the identified variables.

Geometric Progression (GP): A sequence where each term is obtained by multiplying the previous term by a fixed number, called the common ratio.

Health Applications: Ratios and rates applied in BMI, dosage calculations or heart rate analysis.

Independent events is if the events have no influence on each other.

Indices (Exponents or Powers): A concept indicating the number of times a number (the base) is multiplied by itself

Informed Consent: Participant agreement to data collection.

Intersection point: The point where two lines meet, representing the solution to a system of equations.

Line segment: part of a line that has two distinct endpoints.

Linear equation: A mathematical equation showing a straight-line relationship between two variables.

Logarithm Laws and Properties: The rules that govern operations with logarithms, such as the product, quotient, and power laws.

Material Estimation: The process of calculating how much material is required for a task, such as bricks for building a wall, paper for wrapping a product or paint for painting a surface.

Mutually exclusive events is if the events cannot happen simultaneously.

***nth* Term**: The formula that represents the general term of a sequence based on its position, *n*.

Ogive: Cumulative frequency graph showing data distribution.

Outcome is the result of an experiment.

Paint Coverage: The amount of surface area a specific quantity of paint can cover. Knowing this helps avoid wasting paint and ensures an efficient job.

Pattern Recognition: The ability to observe and identify recurring arrangements or relationships within a sequence.

Plane Figures: Two-dimensional shapes, such as triangles, squares, or circles, used to represent or explore patterns visually.

Probability is the possibility or likelihood that a given event will occur.

Proportion: A statement of a relationship between two ratios with a constant.

Pythagoras's Theorem: In a right triangle, $a^2+b^2=c^2$, where c is the hypotenuse

Random experiment is an experiment whose outcomes cannot be predicted.

Rate: A type of ratio comparing quantities with different units (e.g., speed, cost per item).

Ratio: A comparison of two quantities, expressed as a:b or b:a,or in a fraction, or in words.

Rationalise the Denominator: The process of eliminating surds from the denominator of a fraction by multiplying both the numerator and the denominator by an appropriate term that makes the denominator rational.

Resource Management: The process of using resources (like materials, time, or labour) effectively to avoid waste and reduce costs.

Sample space is the set of all the possible outcomes.

Scalar Multiplication is multiplying a vector by a scalar (a constant), scaling its magnitude while preserving its direction (if scalar > 0).

Sequence: An ordered set of numbers or objects arranged according to a specific rule or pattern.

Simultaneous equation: A set of two or more equations that share the same with variables and are solved together.

Speed: A rate measuring distance per time

Substitution method: A method of solving simultaneous equations by rearranging one equation for one variable and substituting this value into the other equation.

Surds: Expressions containing roots ($\sqrt{}$) that cannot be simplified to remove the root sign.

Term: Each individual element in a sequence.

Trial is one performance of a random experiment.

Vector Addition is for combining two vectors to create a resultant vector, performed by adding their respective components in two-dimensional space.

Vector Subtraction is finding the difference between two vectors and subtracting their respective components.

Waffle Diagram: Visual representation of categorical data.

This book is intended to be used for the Year Two Mathematics Senior High School (SHS) Curriculum. It contains information and activities to support teachers to deliver the curriculum in the classroom as well as additional exercises to support learners' selfstudy and revision. Learners can use the review questions to assess their understanding and explore concepts and additional content in their own time using the extended reading list provided.

All materials can be accessed electronically from the Ministry of Education's Curriculum Microsite.

[Scan the QR Code to visit the website]

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