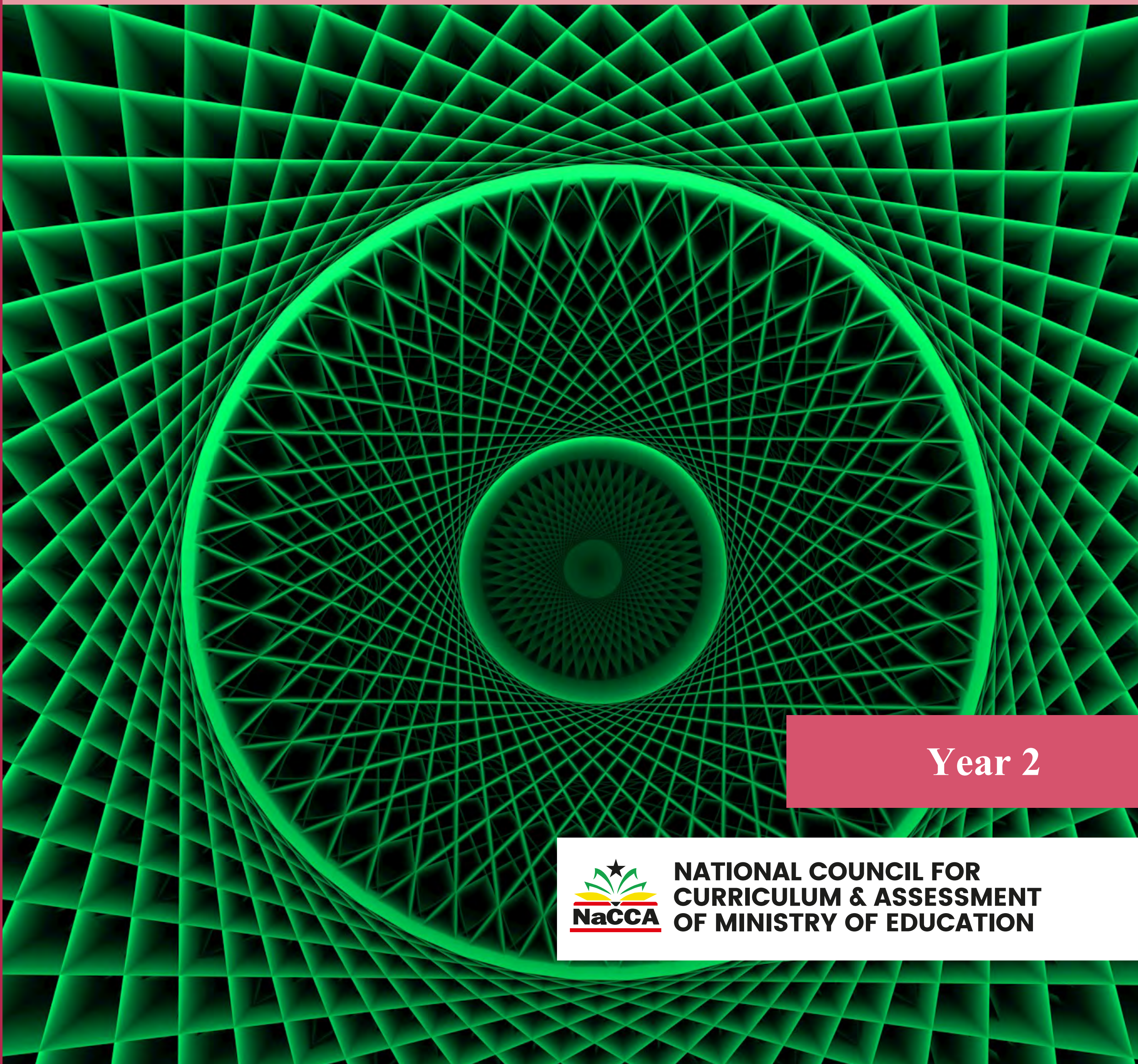
MINISTRY OF EDUCATION

ADDITIONAL MATHEMATICS for Senior High Schools

TEACHER MANUAL



Year 2

NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

MINISTRY OF EDUCATION

REPUBLIC OF GHANA

Additional Mathematics

for Senior High Schools

Teacher Manual

Year Two

NaCCA
NATIONAL COUNCIL FOR
CURRICULUM & ASSESSMENT
OF MINISTRY OF EDUCATION

ADDITIONAL MATHEMATICS TEACHER MANUAL

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Introduction

The National Council for Curriculum and Assessment (NaCCA) has developed a new Senior High School (SHS) curriculum which aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, competencies, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, further their education and enter the world of work.

This is the first time that Ghana has developed an SHS Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

This Teacher Manual for Mathematics is a single reference document which covers all aspects of the content, pedagogy, teaching and learning resources and assessment required to effectively teach Year Two of the new curriculum. It contains information for all 24 weeks of Year Two including the nine key assessments required for the Student Transcript Portal (STP).

Thank you for your continued efforts in teaching our children to become responsible citizens.

It is our belief that, if implemented effectively, this new curriculum will go a long way to transforming our Senior High Schools and developing Ghana so that we become a proud, prosperous and values-driven nation where our people are our greatest national asset.

Acknowledgements

Special thanks to Professor Samuel Ofori Bekoe, Director-General of the National Council for Curriculum and Assessment (NaCCA) and all who contributed to the successful writing of the Teacher Manuals for the new Senior High School (SHS) curriculum.

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SECTION 1: SETS AND BINOMIAL EXPANSIONS

Strand: Modelling with Algebra

Sub-Strand: Application of algebra

Learning Outcome: *Investigate De Morgan's law on sets algebraically and graphically, formulate and solve real life problems up to three sets.*

Content Standard: Demonstrate the ability to use sets properties and theories to solve real-life problems and apply binomial theorem to approximate numbers to a given index.

Hint

Portfolio Assessment: Assign portfolio assessment in Week 2. Take time to explain to learners what they are supposed to do as they embark on their portfolio task. Refer to Appendix A at the end of Section 1 and the Teacher Assessment Manual and Toolkit for information on how to go about the portfolio.

INTRODUCTION AND SECTION SUMMARY

In the realm of set theory, establishing De Morgan's laws forms a cornerstone for understanding relationships between sets and their complements. These laws state that the complement of the union of two sets equals the intersection of their complements, and similarly, the complement of the intersection of two sets equals the union of their complements. This logical framework not only enhances mathematical reasoning but also extends its applicability to fields like computer science, where it informs database operations, Boolean logic, and circuit design. Applying De Morgan's laws involves using them to simplify set expressions and derive new relationships, essential for optimizing logical conditions in programming, analysing data in statistics, and formulating proofs in theoretical computer science. These applications underscore the practical use of set theory in diverse technological and scientific disciplines. Expanding binomial expressions utilize the binomial theorem to express the powers of binomials as a series, facilitating calculations in calculus. This technique is indispensable for approximating complex functions in physics, economics, and engineering, thereby supporting numerical methods crucial for simulations and optimizations. Applying binomial expansion to approximate exponential numbers bridges theoretical mathematics with practical applications in modelling exponential growth in biology, population dynamics, and financial forecasting. This interdisciplinary linkage highlights how foundational mathematical concepts provide powerful tools for problem-solving and decision-making across various fields.

The weeks covered by the section are

Week 1

1. *Establishing De Morgan's laws of set theory*
2. *Applying De Morgan's laws of set theory*
3. *Applying laws of set theory*

Week 2

1. *Expanding binomial expressions*
2. *Applying binomial expansion to approximate exponential numbers*

SUMMARY OF PEDAGOGICAL EXEMPLARS

Teaching mathematics effectively requires a dynamic approach that caters for diverse learning needs, fostering understanding, application, and extension of mathematical concepts. On teaching strategies and differentiation, teachers should make use of combination of instructional methods, including interactive approaches, collaborative activities, and hands-on exercises, to engage learners actively. This section, therefore, requires hands-on activities where learners are encouraged to adopt varied instructional strategies, including differentiation, to accommodate the multiple learning styles and abilities within the classroom. Learners should be offered varied and appropriate opportunities to work in teams to find solutions to assigned tasks. Hence, a mixture of Inquiry-based learning, Graph and Venn diagram representations, Group discussions, Technology integrations, Real-world connections, Cooperative learning, Problem and project based learning and Scaffolded instruction, will be appropriate. Opportunities should be given to the gifted and talented learners, by including strategies for mathematical induction where learners prove the De Morgan's laws of Theory and deduce the factorial method for binomial expansion.

ASSESSMENT SUMMARY

Different forms of assessments should be carried out to ascertain learners' performance on the concepts that will be taught under this section. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP). The following mandatory assessments would be conducted and recorded for each learner

Week 1: Portfolio Task: By the end of Week 1, teachers should have assigned each learner their Portfolio task.

Week 2: Individual Class Exercise: By this Week teachers should be ready to administer and record Individual Class Exercise.

Note

For additional information on how to effectively administer these assessment modes, refer to the Appendices.

WEEK 1

Learning Indicators

1. *Establish De Morgan's laws of set theory and use it to solve related problem*
2. *Describe the set theory as a foundation for many subfields of mathematics, and create and model set problems in the areas pertaining to industry, commerce, sports, etc*

In Year 1, learners were guided to identify the regions of a three-set Venn diagram. In this week, learners will apply their understanding from that lesson and other relevant lessons to solve real world problems which can be modelled with sets.

FOCAL AREA 1: ESTABLISHING DE MORGAN'S LAWS OF SET THEORY

Let the learner consider a Venn diagram, such as the one below,

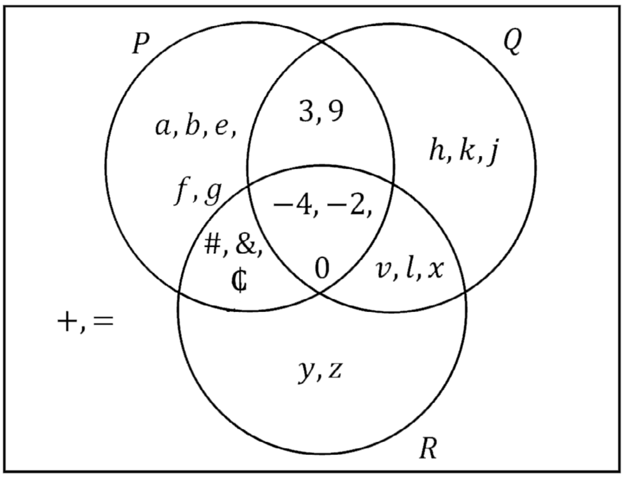


Figure 1.1

The elements of the various regions can be written as such;

$U = \{a, b, e, f, g, 3, 9, -4, -2, 0, \#, \&, ¢, h, k\,j, v, l, x, y, z, +, =\}$

$P = \{a, b, e, f, g, 3, 9, -4, -2, 0, \#, \&, ¢\}$

$P' = \{h, k\,j, v, l, x, y, z, +, =\}$

$Q = \{h, k\,j, 3, 9, -4, -2, 0, v, l, x\}$

$Q' = \{a, b, e, f, g, \#, \&, ¢, y, z, +, =\}$

$R = \{y, z, , \#, \&, ¢, -4, -2, 0, v, l, x\}$

$R' = \{a, b, e, f, g, 3, 9, h, k, j, +, =\}$

$P \cup Q = \{a, b, e, f, g, 3, 9, -4, -2, 0, \#, \&, ¢, h, k\,j, v, l, x\}$

$(P \cup Q)' = \{y, z, +, =\}$

$P' \cap Q' = \{y, z, +, =\}$

$P \cup Q \cup R = \{a, b, e, f, g, 3, 9, -4, -2, 0, \#, \&, ¢, h, k\, j, v, l, x, y, z\}$

$P \cap Q \cap R = \{-4, -2, 0\}$

$(P \cap Q \cap R)' = \{a, b, e, f, g, 3, 9, \#, \&, ¢, h, k\, j, v, l, x, y, z, +, =\}$

$P' \cup Q' \cup R' = \{a, b, e, f, g, 3, 9, \#, \&, ¢, h, k\, j, v, l, x, y, z, +, =\}$

$(P \cup Q \cup R)' = \{+, =\}$

$P' \cap Q' \cap R' = \{+, =\}$

It can be observed that;

i. $(P \cup Q)' = P' \cap Q'$

ii. $(P \cap Q)' = P' \cup Q'$

iii. $(P \cup Q \cup R)' = P' \cap Q' \cap R'$

iv. $(P \cap Q \cap R)' = P' \cup Q' \cup R'$

This is the De Morgan's laws of set theory.

De Morgan's laws also state that

i. $P - (Q \cup R) = (P - Q) \cap (P - R)$,

ii. $P - (Q \cap R) = (P - Q) \cup (P - R)$

iii. $P \cap (Q - R) = (P \cap Q) - (P \cap R)$

FOCAL AREA 2: APPLYING DE MORGAN'S LAWS OF SET THEORY

Example 1

Using Venn diagram, shade the regions represented by

i. $(A \cup B) \cap C'$

ii. $(A \cup B)' \cap C'$

iii. $(A \cap B)' \cup C'$

Solution

i. $(A \cup B) \cap C' = (A \cap C') \cup (B \cap C') = (A \cup B) - C$

Elements in A together with B excluding elements in C also

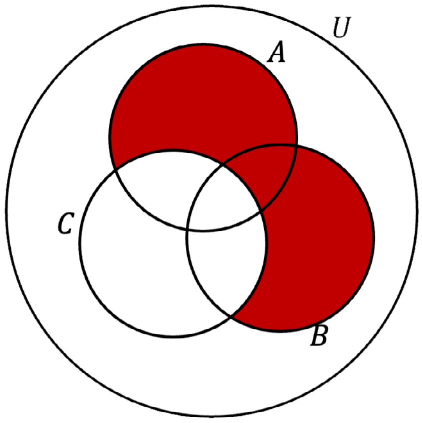


Figure 1.2

ii. $(A \cup B)' \cap C' = A' \cap B' \cap C' = (A \cup B \cup C)'$

Elements that are in the complement set of all the three subsets

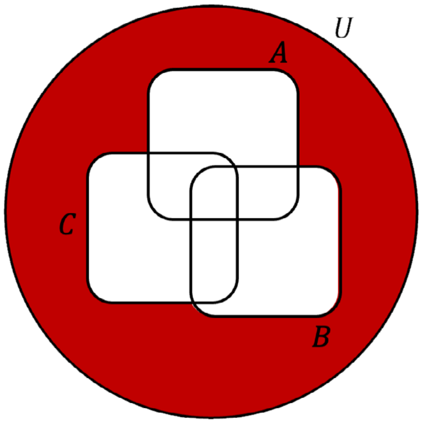


Figure 1.3

iii. $(A \cap B)' \cup C' = A' \cup B' \cup C' = (A \cap B \cap C)'$

Set of elements in all other regions except those common to all three subsets

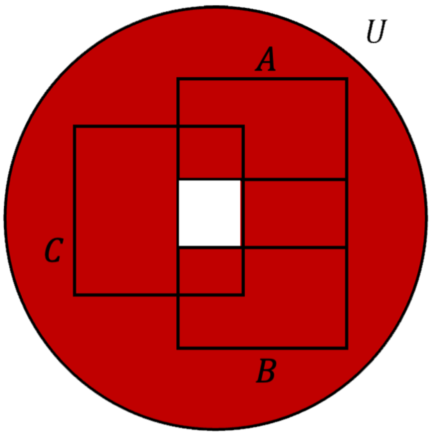


Figure 1.4

Example 2

Rewrite the following

i. $(A' \cup B')'$

ii. $(A' \cap B')'$

Solution

i. $(A' \cup B')' = (A')' \cap (B')' = A \cap B$

ii. $(A' \cap B')' = (A')' \cup (B')' = A \cup B$

FOCAL AREA 3: APPLYING LAWS OF SET THEORY

1. Set theory is used in almost every discipline including engineering, business, medical and related health sciences, along with the natural sciences. In business operations, it can be applied at every level where intersecting and non-intersecting sets are identified. For example, the sets for warehouse operations and sales operations are both intersected by the inventory set. To improve the cost of goods sold, the solution might be found by examining where inventory intersects both sales and warehouse operations.
2. Set theory can assist in planning and operations. Every element of business can be grouped into at least one set such as accounting, management, operations, production and sales. Within those sets are other sets. In operations, for example, there are sets of warehouse operations, sales operations and administrative operations. In some cases, sets intersect -- as sales operations can intersect the operations set and the sales set.
3. In kitchen: Sets of similar utensils are kept separately.
4. School bags: Sets of notebooks and textbooks are kept separately in the divisions in the school bags.
5. Music Playlist: Most of us have a different kind of playlists of songs present in our smartphones and computers. Afro-beat songs are often separated from high-life or any other genre. Hence, playlists also form the example of sets.

Example 3

The sets L, M and N are subsets of a universal set consisting of the first 10 lower-case letters of the alphabet, if $L = \{a, b, c\}$, $M = \{b, c, a, e\}$ and $N = \{a, d, e, f\}$. Determine the members of the following sets

i. $M \cup N$

ii. $L \cup N$

iii. L'

iv. $L \cap M \cap N'$

v. $(L \cup M \cup N)'$

vi. $M \cap N$

Solution

$U = \{a, b, c, d, e, f, g, h, i, j\}$

$L = \{a, b, c\}$

$L' = \{d, e, f, g, h, i, j\}$

$M = \{b, c, a, e\}$

$M' = \{d, f, g, h, i, j\}$

$N = \{a, d, e, f\}$

$N' = \{b, c, g, h, i, j\}$

i. $M \cup N = \{b, c, a, e, d, f\}$

ii. $L \cup N = \{a, b, c, d, e, f\}$

iii. $L' = \{d, e, f, g, h, i, j\}$

iv. $L \cap M \cap N' = (L' \cup M')' \cap N'$

$L' \cup M' = \{d, e, f, g, h, i, j\} \cup \{d, f, g, h, i, j\}$

$= \{d, e, f, g, h, i, j\}$

$L \cap M \cap N' = \{b, c\} \cap \{b, c, g, h, i, j\} = \{b, c\}$

v. $(L \cup M \cup N)' = L' \cap M' \cap N' = \{g, h, i, j\}$

Example 4

A sample of 100 Students' Representative Council voting delegate revealed the following concerning three candidates; Akayuure, Manukre and Odonti, who were running for the SRC Chairman, Secretary and Treasurer respectively. 14 delegates preferred both Akayuure and Manukre 49 preferred Akayuure or Manukre but not Odonti. 21 preferred Manukre but not Odonti or Akayuure. 61 preferred Manukre or Odonti but not Akayuure. 32 preferred Odonti but not Akayuure or Manukre. 7 preferred Akayuure and Odonti but not Manukre. With the aid of a Venn diagram, determine the number of voters that were in favour of all the three candidates. Assume that every member of the SRC voted for at least one candidate. Determine the candidate that went unopposed if a rule of 50% majority were used in such a decision.

Solution

Let $n(U) = \textit{Total number of delegates} = 100$

$n(A) = \textit{Number of delegates who prefered Akayuure}$

$n(M) = \textit{Number of delegates who prefered Manukre}$

$n(D) = \textit{Number of delegates who prefered Odonti}$

$n(A \cap M \cap D) = p$

$n(A \cap M) = 14$

$n(A \cup M \cap D') = 49$

$n(A' \cap M \cap D') = n((A \cup D)' \cap M) = 21$

$n(A' \cap M \cup D) = 61$

$n((A \cup M)' \cap D) = 32$

$n(A \cap M' \cap D) = 7$

$n((A \cup M \cup D)') = 0$

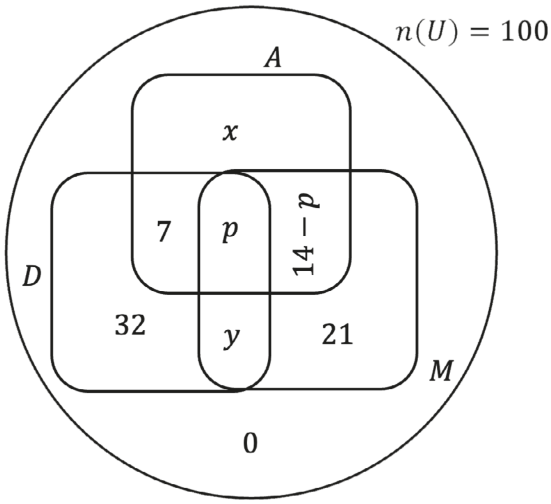


Figure 1.5

$n(A \cup M \cap D') = x + 14 - p + 21 = 49$

$x - p = 14$

$n(A' \cap M \cup D) = 21 + y + 32 = 61$

$y = 8$

$100 = x + 14 - p + 21 + y + 32 + 7 + p + 0$

$100 = 49 + 8 + 39 + p$

$p = 100 - 96 = 4$

$\therefore$ 4 voters were in favour of all the three candidates

$$100 = x + 14 - p + 21 + y + 32 + 7 + p + 0$$
$$= 74 + x + y$$
$$= 74 + x + 8$$
$$x = 100 - 82 = 18$$
$$n(A) = 7 + x + p + 14 - p = 21 + x$$
$$= 21 + 18 = 39$$

$\Rightarrow$ Percentage of voters who favoured Akayuure was $\frac{39}{100} \times 100\% = 39\%$

$$n(M) = 21 + 14 - p + p + y$$
$$= 35 + y$$
$$= 35 + 8 = 43$$

$\Rightarrow$ Percentage of voters who favoured Manukre was $\frac{43}{100} \times 100\% = 43\%$

$$n(D) = 32 + 7 + p + y$$
$$= 39 + 4 + 8$$
$$= 51$$

$\Rightarrow$ Percentage of voters who favoured Odonti was $\frac{51}{100} \times 100\% = 51\%$

$\therefore$ Manukre and Odonti went unopposed

Learning Tasks

1. Learners in pairs establish the De Morgan's Law and apply it to solve real life problems involving three sets.
2. Learners are tasked in pairs/groups to apply De Morgan's laws of set theory to real life situations.
3. Learners in individual groups apply the laws of set theory to real life situations.

PEDAGOGICAL EXEMPLARS

The lesson for the week is aimed at the learner's ability to establish the De-Morgan's laws of set theory and use it to solve real life problems; and explain the set theory as the foundation for many subfields in Mathematics, and create and model sets problems related to areas of industry, commerce and sports.

The following pedagogical strategies have been suggested for facilitators to take learners through.

1. **Review previous learning**
2. **Through Peer-to-Peer Learning**, engage the learners to work in pairs or small groups to review and explain the concepts of intersection, union, and complement of sets to one another.
3. The facilitator guides learners to utilize visual aids and representations, such as Venn diagrams, to help learners review and reinforce their understanding of set operations. Encourage learners to draw and manipulate Venn diagrams to illustrate various set operations and relationships. Also, connect the abstract concepts of set operations to real-world examples and analogies that students can relate to. For instance, use examples from everyday life, such as the intersection of students who eat Ga kenkey and those who eat rice at the dining hall, or the union of teachers who wear African print and those who wear Lacoste shirts on campus.
4. **Experiential Learning**: Guide learners to work collaboratively in pairs and groups to actively engage in investigating and establishing De Morgan's Laws and other laws of set algebra. Facilitate their use of graphical representations, such as Venn diagrams, and algebraic methods (reasoning) to explore the complement, union, and intersection of sets.
5. **Using Talk for Learning Approaches in Collaborative Groups**: Provide the platform for learners to investigate and establish the real-world applications of the laws of set algebra, including De Morgan's Laws, through the use of Venn diagrams and algebraic analysis. Encourage learners to work in collaborative groups, where they can brainstorm, discuss, and explain their understanding of these concepts using strategies such as think-pair-share/square and debates.
6. **Experiential Learning**: Guide learners to work collaboratively in pairs and groups to create and solve set problems in real-life situations by applying the laws of set algebra and other properties of sets. Provide them with authentic scenarios or contexts where they can apply their knowledge and skills.
7. **Using Talk for Learning Approaches**: Guide learners to brainstorm using strategies like think-pair-share/square, debates, and discussions to explain and talk about the relevance of learning set properties and algebra, and how they are useful in our daily lives. Encourage them to explore real-world applications and discuss the practical implications of these concepts.
8. **Using Collaborative Method of Learning**: Guide learners to engage in an inter-group competition, where members from one group create real-life set problems for the members of another group to investigate, prove, and/or answer.

KEY ASSESSMENTS

Assessment level 1: Recall

De Morgan's law for the complement of the intersection of two sets A and B states that $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's law is true.

Assessment level 2: Skills of conceptual understanding

1. Work out the values of p and m and determine which set of objects is represented by the Venn diagram below?

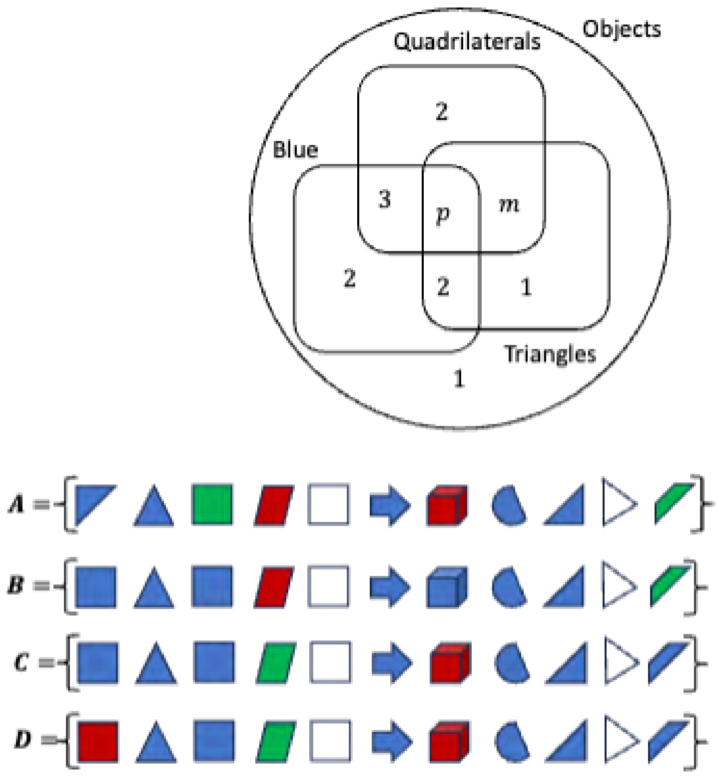


Figure 1.6

2. Find $(A \cup B)' \cap C$ for the sets A, B and C depicted below.

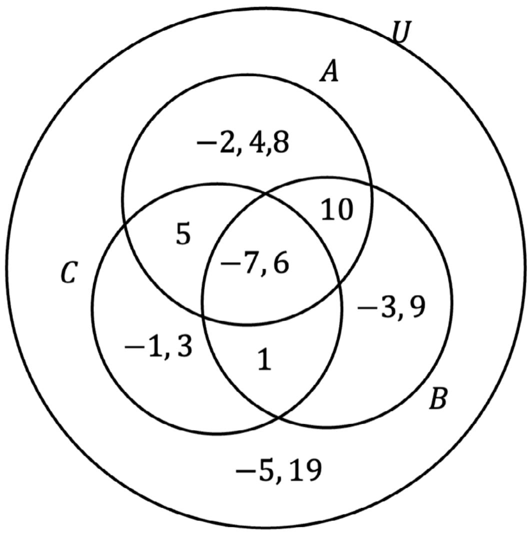


Figure 1.7

Assessment level 3: Strategic Reasoning

1. Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions

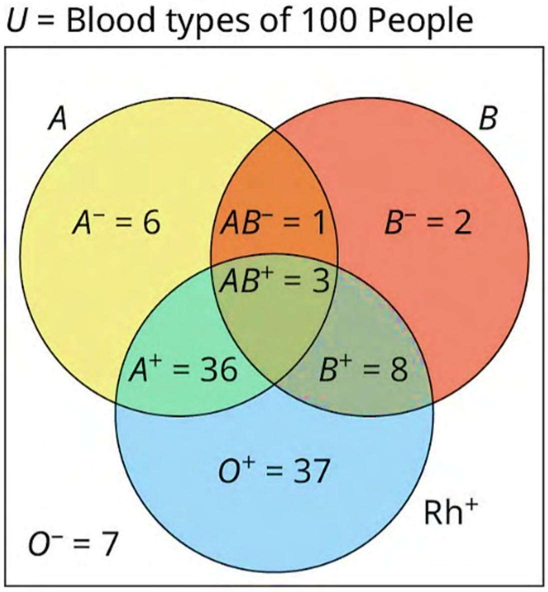


Figure 1.8

a. How many people with a type A blood factor donated blood?

b. How many people who donated blood did not have a type B blood factor?

c. Mr. Luki has blood type B+. If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Mr. Luki can accept?

WEEK 2

Learning Indicator: *Use the expansion for $(1 - x)^n$ or $(1 + x)^n$ to approximate exponential number*

FOCAL AREA 1: EXPANDING BINOMIAL EXPRESSIONS

Factorial Method

In week 3 of the year one teacher manual, the concept of combination was introduced and defined as $^{n}C_{r} = n\frac{!}{(n-r)!r!}$ where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$

The limitation of the Pascal and combination methods of binomial expansions require an alternative approach. Pascal's method, as discussed in week 3 of the year one teacher manual is only suitable for small positive integer exponents and the combination method is only applicable to binomial expressions with positive powers.

The factorial method provides a means to expand binomial expressions with powers without these limitations. Binomial expressions with negative powers or even fractional exponents can still be expanded with the factorial method

It was discussed earlier that given the binomial algebraic expression $(p + q)^n$, the coefficients of the terms of the expanded form are

$^{n}C_{0}, {}^{n}C_{1}, {}^{n}C_{2}, {}^{n}C_{3}, \ldots, {}^{n}C_{n-2}, {}^{n}C_{n-1}$ and $^{n}C_{n}$ and they translate to

$1, n, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!}, \ldots, \frac{n(n-1)}{2!}, n, 1$ respectively. These coefficients can be used to write out a general expansion for $(p + q)^n$ beyond the limitations of the Pascal's and combination methods.

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{3!}p^{n-3}q^3 + \frac{n(n-1)(n-2)(n-3)}{4!}p^{n-4}q^4$$

Example 1

Using the factorial method, expand the following binomials to the term that contains x^4.

i. $(1 - 2x)^{-3}$

ii. $(3 + x)^{\frac{2}{3}}$

Solution

i. $(1 - 2x)^{-3} = 1^{-3} - 3(1^{-3-1})(-2x) + \frac{-3(-4)}{2}(1^{-3-2})(-2x)^2 +$

$$\frac{-3(-3-1)(-3-2)}{3!}(1^{-3-3})(-2x)^3 + \frac{-3(-3-1)(-3-2)(-3-3)}{4!}(1^{-3-4})(-2x)^4 + \ldots$$

$$= 1 + 6x + 24x^2 + 80x^3 + 240x^4 + \ldots$$

ii. $(3+x)^{\frac{2}{3}} = 3^{\frac{2}{3}} + \frac{2}{3}(3)^{(\frac{2}{3}-1)}(x) + \frac{\frac{2}{3}(\frac{2}{3}-1)}{2}\left(3^{(\frac{2}{3}-2)}\right)x^2 + \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{3!}\left(3^{(\frac{2}{3}-3)}\right)x^3 + \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)(\frac{2}{3}-3}{4!}\left(3^{(\frac{2}{3}-4)}\right)$
$x^4 + \ldots$

$= 3^{\frac{2}{3}} + \frac{2}{3}\left(3^{-\frac{1}{3}}\right)x - \frac{1}{9}\left(3^{-\frac{4}{3}}\right)x^2 + \frac{4}{81}\left(3^{-\frac{7}{3}}\right)x^3 - \frac{7}{243}\left(3^{-\frac{10}{3}}\right)x^4 + \ldots$

Guide the learner to recognise that it is nearly impossible for all the terms of the expansion of binomials with negative or fractional exponents to be written hence the expansion is truncated after some terms. This truncation reduces the precision of the expansion and thus allows for binomial expansions to be used for approximating exponential numbers.

FOCAL AREA 2: APPLYING BINOMIAL EXPANSION TO APPROXIMATE EXPONENTIAL NUMBERS

Example 2

Expand $(1 - 2x)^8$ and use the expansion to approximate $(0.98)^8$ correct to 3 decimal places

Solution

$(1-2x)^8 = 1 - \frac{8(2x) + (8)(7)(-2x)^2}{2} + \frac{(8)(7)(6)(-2x)^3}{3!} + \ldots + (-2x)^8$

$= 1 - 16x + 112x^2 - 448x^3 + \ldots + 256x^8$

$(0.98)^8 = (1 - 0.02)^8$

By comparing $(1 - 2x)^8$ and $(1 - 0.02)^8$,

$2x = 0.02$ and thus $x = 0.01$

$(0.98)^8 = (1 - 2(0.01))^8 \approx 1 - 16(0.01) + 112(0.01)^2 + \ldots$

≈ 0.8512

≈ 0.851 (3 *dp*)

The number of terms from the expansion that should be used depends on the sum of terms which produces the specified level of precision. For example, the sum of the first two terms alone, $(1 - 16(0.01))$ results in 0.84 which is less than the preferred three decimal places hence more terms need to be added.

Example 3

Expand $(x + 3)^{12}$ in ascending powers of x up to the fifth term and use your expansion to approximate 7.56^{12}

Solution

The terms of $(x + 3)^{12}$ must be rewritten such that the term the contains x would be the second term and thus, the terms of the expansion would be in increasing powers of x

$(x + 3)^{12} = (3 + x)^{12}$

$$= 3^{12} + 12(3^{11})x + \frac{12(11)(3^{10})x^2}{2} + \frac{12(11)(10)(3^9)x^3}{3!} + \frac{12(11)(10)(9)(3^8)x^4}{4!} + \dots$$

$$= 531441 + 2125764x + 3897234x^2 + 4330260x^3 + 3247695x^4 + \dots$$

$$7.56^{12} = (1.56 + 6)^{12}$$

$$= [2(0.78 + 3)]^{12}$$

$$= 2^{12}(0.78 + 3)^{12}$$

By comparing $(0.78 + 3)^{12}$ to $(x + 3)^{12}$, $x = 0.78$

$$\Rightarrow (0.78 + 3)^{12} = 3^{12} + 12(3^{11})(0.78) + \frac{12(11)(3^{10})(0.78)^2}{2} + \frac{12(11)(10)(3^9)(0.78)^3}{3!} + \frac{12(11)(10)(9)(3^8)(0.78)^4}{4!} + \dots$$

$$\approx 7817683.7520792$$

$$7.56^{12} = 2^{12}(0.78 + 3)^{12} \approx 2^{12}(7817683.7520792)$$

$$\approx 32021232648.516403$$

It must be noted that the actual value of 7.56^{12} is 34,854,715,807.867203 which is about 2,833,483,159.3508 more than our approximated value. This error is called a truncation error as it is caused by the "cutting off" of some of the terms of the expansion of $(0.78 + 3)^{12}$

Learning Tasks

1. Learners are provided with task sheets to expand binomial expressions using the factorial method.
2. Learners are offered a set of standard practice problems that require applying the binomial expansion to approximate exponential numbers.

PEDAGOGICAL EXEMPLARS

The purpose for the week's lessons is aid learners expand binomials using the factorial method and apply binomial expansion to approximate exponential numbers. These are the suggested pedagogical strategies for the facilitators to take learners through.

1. **Review previous learning:** The facilitator engages learners in peer teaching and collaborative review sessions by dividing the class into small groups and assigning each group a specific aspect of binomial expansion to review and present to the rest of the class. The facilitator can consider incorporating various review techniques, such as practice problems, worked examples, and self-assessment quizzes, to cater to different learning styles and reinforce the concepts from multiple angles. Remember to provide regular feedback and support throughout the review process, addressing any lingering doubts or misconceptions students may have.

2. **Using Talk for Learning Approaches** such as Building on what others say, managing talk-for Learning, Structuring Talk for Learning, and Collaborative learning approaches, learners adopt the factorial method of binomial expressions to determine the terms, coefficient, and exponent of a given term in an expansion. The Facilitator must provide frequent opportunities for practice, assessment, and feedback.
3. **Using talk-for-learning approaches, the facilitator** introduces the concept of binomial expansion and its connection to exponential functions using visual representations. Utilize graphical methods, such as creating tables or graphs, to illustrate how the binomial expansion approximates exponential growth or decay for different values of the exponent.
4. **Experiential Learning**: The Facilitator provides learners with scaffolded examples that gradually increase in complexity. Start with simple cases where the binomial expansion is used to approximate small exponents, and then progressively move to more challenging examples involving larger exponents or decimal values. Guide students through each step, explaining the reasoning behind the algebraic manipulations and approximations involved in the binomial expansion.
5. **Experiential learning**: Guide learners using collaboration (pair or group) to participate in hands-on activities and explorations that allow them to discover the patterns and relationships between binomial expansion and exponential approximations using the factorial method. For example, have them use graphing calculators or online tools to plot the binomial expansion and compare it with the corresponding exponential function.
6. **Using incorporated collaborative learning strategies,** such as think-pair-share or group work, the Facilitator guides learners to explore real-world applications and problem-solving scenarios that require the use of binomial expansion to approximate exponential numbers. Examples could include population growth models and compound interest calculations. The facilitator must encourage learners to reflect on their learning, identify areas of confusion, and ask questions that can further enhance their understanding and mastery of applying binomial expansion to approximate exponential numbers.

KEY ASSESSMENTS

Assessment Level 1: Recall

Find the coefficient of x^5 in the binomial expansion of $(2 + 3x)^9$.

Assessment Level 2: Skills of conceptual understanding

1. In the binomial expansion of $(1 + kx)^6$, where k is constant, the coefficient of x^3 in twice as large as the coefficient of x^2. Find the value of k.

2. The coefficient of x^4 in the expansion of $(a + 2x)^6$ is 1500. Work out the two possible values of a

Assessment Level 3: Strategic Reasoning

1. Find the sum of the coefficients in the expansion of $(1 + x)^{10}$

2. Evaluate $(0.99)^{15}$ to four decimal places

3. Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^9$

Hint

- ***Portfolio Assessment:*** *Assign portfolio assessment in Week 1. Take time to explain to learners what they are supposed to do as they embark on their portfolio task. Refer to Appendix A at the end of Section 1 and the Teacher Assessment Manual and Toolkit for information on how to go about the portfolio.*
- *Assign individual class exercise and record learners scores. The scores should be ready for onward submission to the STP latest by the end of Week 4.*

Section 1 Review

The first two weeks of the year two Teacher Manual took learners through the following learning indicators and focal areas

1. Establishing De Morgan's Laws of Set Theory
2. Applying De Morgan's Laws of Set Theory
3. Applying Laws of Set Theory
4. Expanding Binomial Expressions using the Factorial Method
5. Applications of Binomial Expressions to Approximate Exponential Numbers.

Pedagogical strategies such as visual learning, collaborative discussions, peer-to-peer learning, project and problem-based learning etc. were all utilized during the teaching and learning processes with recourse to differentiation. Assessment strategies ranged from comprehensive exercises and tests covering all reviewed topics, portfolio assessments including samples of work from each focal area, and self-assessment reflection on learning progress and areas for improvement.

APPENDIX A: SAMPLE PORTFOLIO

Investigating Scientific Personalities

1. *Task: Investigate one renowned Ghanaian Mathematician, Physicist, Engineer or Scientist such as Francis Allotey (Physicist and Mathematician), Apostle Kwadwo Safo (Engineer), Thomas Mensah (Engineer), and Ebenezer Laing (Biologist). Write a brief report about their contributions to their field, their educational background, and their impact on society. Your report should include*

i. *A brief biography (max. 150 words)*

ii. *Major achievements and contributions (50 to 100 words)*

iii. *Their impact on Ghanaian society or global contributions in their field (50 to 100 words)*

Hint

Learners are to include their class exercises, class tests, quizzes, hom ework, mid-semester and end-of-first-semester examination scripts, etc. as part of artefacts to submit for their portfolio.

Timeline: *22 weeks*

2. *Marking Scheme and Rubrics for Assessment*

Examples of Ghanaian Personalities to Choose From

i. Mathematicians

Professor Francis Kofi Ampenyin Allotey (Renowned for the "Allotey Formalism")

ii. Physicists

Professor Edward S. Ayensu (International expert on environmental and sustainable development)

iii. Engineers

Dr. Thomas Mensah (Pioneer in Fiber Optics technology)

Criteria	Excellent (5 Marks)	Very Good (3 Marks)	Good (2 Mark)	Needs Improvement (1 or 0 Marks)
Biography	Detailed, clear, and engaging early life, education, and career coverage.	Covers early life, education, and career with minor omissions.	Brief or incomplete biography with unclear details.	No meaningful biography was provided.
Early Life and Background	Clearly highlights family, early life, and influences.	Covers early life and influences but lacks depth.	Mentions early life but omits key details.	No attempt or minimal information.
Educational Background	Comprehensive summary of education with institutions and degrees.	Mentions education but with minor omissions.	Provides incomplete or unclear details.	No mention or incorrect information.
Professional Background	Clear details of career path and roles connected to contributions.	Describes career path but lacks depth or connections.	Provides vague or incomplete career details.	No mention or irrelevant information.
Major Achievements and Contributions	Clearly explains major accomplishments with examples.	Identifies achievements but lacks detail.	Brief or unclear description of achievements.	No mention or incorrect information.

Marking scheme

Learner's works	Score
Scientific write-up	25 marks
Assignments/Exercises	10 marks
Projects/Case studies	10 marks
Quizzes and Tests	10 marks
Mid-semester and End-of-semester Papers	5 marks
Total marks	**60 marks**

3. *How to Administer Tasks as a Teacher*

Monitor personality write-up progress and provide guidance where needed, particularly on the accuracy of information being provided, etc.

4. *Feedback*

Encourage learners to emulate the determination and disciplined life of the scientific personalities they wrote about, etc.

Strand: Modelling with Algebra

Sub-Strand: Application of algebra

Learning Outcomes

1. *Model sequence recursively and explicitly, and establish the relationship between the two forms, as well as solve real life problems involving linear and exponential sequences and series*
2. *Formulate and derive appropriate strategies to solve quadratic inequalities*
3. *Graph systems of given inequality and identify the region that provides the feasible solution and apply it to real life situations*

Content Standard: *Demonstrate the ability to apply algebraic processes and reasoning to model and solve real-life situations involving sequences, and linear programming and use appropriate techniques to solve quadratic inequalities, as well as resolve rational functions*

INTRODUCTION AND SECTION SUMMARY

Mathematical concepts like sums of sequences, convergence of series, recursive sequences, and arithmetic/geometric means are intricately connected and integral to various disciplines. Sums of sequences, whether arithmetic (AP) or geometric (GP), compute cumulative values using specific formulas

$$S_n = \frac{n}{2}(a_1 + a_n) \text{ for AP and } S_n = \frac{n}{2}[2a_1 + (n-1)d] \text{ for GP},$$

illustrating how quantities accumulate over time or iterations. Convergence and divergence of series, crucial in calculus, analyze whether infinite sequence sums approach a finite value (convergence) or diverge to infinity or oscillate (divergence). This understanding underpins concepts in limits and integration. Recursive sequences, which define each term based on the preceding term, find applications in computer science for algorithms and in dynamic modeling within economics and biology. Arithmetic and geometric means provide statistical insights into central tendencies and growth rates, directly relevant to fields like finance and data analysis. Quadratic inequalities with quadratic terms expand algebraic techniques, vital in physics (e.g., projectile motion) and economics (e.g., optimization problems). Systems of quadratic inequalities extend this to solving simultaneous constraints, applicable in engineering design and resource allocation problems. Real-life applications of quadratic inequalities involve optimizing outcomes under constraints, exemplified in profit maximization or resource allocation scenarios, highlighting their broad interdisciplinary impact across science, technology, and economics.

The weeks covered by the section are:

Week 3

1. Sums of sequences
2. Convergence and divergence of series
3. Recursive sequences
4. Arithmetic and geometric means of sequences

Week 4

1. Solving quadratic inequalities
2. Solving systems of quadratic inequalities
3. Solving real-life problems involving quadratic inequalities

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical works to find solutions to questions either algebraically or graphically involving Arithmetic and Geometric Progressions. Learners should be offered the opportunity to work in teams to find solutions to assigned tasks. Hence, Experiential learning activities, Project based learning, Problem-based learning and Mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be encouraged to participate fully in investigations as well as presentation of findings. However, make considerations and accommodations for the different groups. That is, offer below average/approaching proficiency learners the opportunity to make oral presentations and use graphical methods in determining the area and perimeter of shapes. Then, extend activities for the above average/highly proficient learners to using formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

The various concepts to be covered under this section should be assessed using various forms of assessments modes to ascertain learners' performance. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP). The following mandatory assessments would be conducted and recorded for each learner:

Week 3: Group Project: Learners must be assigned group project that will span at least four to six weeks.

Note

For additional information on the internal assessments, refer to the Hints and Reminders sign-posted at the end of the various weeks.

WEEK 3

Learning Indicators

1. *Generate the terms of a recurrence sequence and find an explicit formula for the sum of the sequence*
2. *Use recursive and explicit formulae of sequences to model situations and translate between the two forms*
3. *Determine the arithmetic and geometric means of linear and exponential sequences and apply linear and exponential sequences to solve real life problems*

In week 6 of the year 1 teacher manual, learners identified number patterns and classified some into arithmetic and geometric sequences. While sequences are list of numbers with defined patterns, a series is the sum of the terms of a sequence. This week's lessons are dedicated to finding the sum of sequences and analysing the convergence or divergence of series.

FOCAL AREA 1: SUMS OF SEQUENCES

Example 1

Indicate the first three terms in the series?

i. $\sum_{n=1}^{7}(4n - 5)$

ii. $\sum_{k=-2}^{9}(2)^{k}$

iii. $\sum_{a=2}^{11}\left(\frac{1}{2}(4)^{a-2}\right)$

Solution

To find the first three terms, replace n with 1, 2 and 3

i. $\sum_{n=1}^{10}(4n - 5)$

When $n = 1$,

$4n - 5 = 4(1) - 5 = -1$

When $n = 2$,

$4n - 5 = 4(2) - 5 = 3$

When $n = 3$,

$4n - 5 = 4(3) - 5 = 7$

$\therefore$ The first three terms are -1, 3 and 7

ii. $\sum_{k=-2}^{9}(2^k)$

When $k = -2$,

$2^k = 2^{-2} = \frac{1}{4}$

When $k = -1$,

$2^k = 2^{-1} = \frac{1}{2}$

When $k = 0$,

$2^k = 2^0 = 1$

$\therefore$ The first three terms are $\frac{1}{4}, \frac{1}{2}$ and 1

iii. $\left(\sum_{a=2}^{11}\frac{1}{2}(4)^{a-2}\right)$

When $a = 2$,

$\frac{1}{2}(4)^{a-2} = \frac{1}{2}(4)^{2-2} = \frac{1}{2}$

When $a = 3$,

$\frac{1}{2}(4)^{a-2} = \frac{1}{2}(4)^{3-2} = 2$

When $a = 4$,

$\frac{1}{2}(4)^{a-2} = \frac{1}{2}(4)^{4-2} = 8$

$\therefore$ The first three terms are $\frac{1}{2}$, 2 and 8

Method of undetermined coefficients

The method of undetermined coefficients can be used to find the sum of the first n terms of other series, by equating a given series to an identical series of the form $A + Bn + C n^2 + \ldots$ and then determining the values of the constants A, B, C, …

Example 2

Find the sum of the first n natural numbers.

Solution

For $n = 1, 2, 3, 4, \ldots$

The sum of the first n natural number can be written as

$\sum_{r=1}^{n}(r) = 1 + 2 + 3 + \ldots + n$

An identical series (with undetermined constants) is $A + Bn + Cn^2 + \ldots$

Let $\sum_{r=1}^{n}(r) = A + Bn + Cn^2$

$\Rightarrow 1 + 2 + 3 + \ldots + n = A + Bn + Cn^2 + \ldots$ (1)

To do a proper comparison of both sides of the equation and subsequently obtaining values for the undetermined coefficients, the left-hand side needs to be written to look like the right-hand side. This can be achieved by adding the next term, i.e. $(n + 1)$ to the sum of natural numbers thus

$\sum_{r=1}^{n+1}(r) = 1 + 2 + 3 + \ldots + n + (n + 1)$

Consequently, the identical series will change to $A + B(n + 1) + C(n + 1)^2$ and thus

$1 + 2 + 3 + \ldots + n + (n + 1) = A + B(n + 1) + C(n + 1)^2$

$1 + 2 + 3 + \ldots + n + (n + 1) = A + Bn + B + Cn^2 + 2Cn + C$

$1 + 2 + 3 + \ldots + n + (n + 1) = A + B + C + Bn + 2Cn + Cn^2$ (2)

(2) – (1)

$n + 1 = 2Cn + B + C$

Comparing coefficients, we have

$2C = 1$ and $B + C = 1$

$\Rightarrow C = \frac{1}{2}$ and $B = 1 - C = 1 - \frac{1}{2} = \frac{1}{2}$

$1 + 2 + 3 + \ldots + n = A + \frac{1}{2}n + \frac{1}{2}n^2 + \ldots$

When $n = 4$,

$\sum_{r=1}^{4} r = 1 + 2 + 3 + 4 = A + \frac{1}{2}(4) + \frac{1}{2}(4)^2$

$10 = A + 2 + 8$

$A = 10 - 10 = 0$

$\therefore \sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n = \frac{1}{2}n + \frac{1}{2}n^2 = \frac{n(n + 1)}{2}$

Sum of AP

It was discussed in week 6 of year 1 that sequences for which a common difference in the consecutive terms, d can be found are referred to as arithmetic or linear sequences. Arithmetic sequences have the general rule for the *nth* term, U_n as

$U_n = a(n - 1)d$ where 'a' represents the first term and 'n' is the position of the term.

Earlier, a formula was derived through the method of undetermined coefficients for finding the sum of the first n natural numbers as $\frac{n(n + 1)}{2}$. A more practical and convenient formula could be derived to find the sum of linear sequences particularly with common differences other than 1 (as in the case of natural numbers).

Consider the sum of the first 10 positive odd numbers i.e.

$S_{10} = 1 + 3 + 5 + 7 + \ldots + 17 + 19$ which can still be written as

$S_{10} = 19 + 17 + 15 + 13 + ..+3 + 1$

The two equations can be added to obtain

$2S_{10} = 20 + 20 + 20 + 20 + ..+20 + 20$

Since the number of terms of the series is 10,

$20 + 20 + 20 + 20 + ..+20 + 20 = 10(20)$ and thus

$2S_{10} = 10(20)$ and

$$S_{10} = \frac{10(20)}{2}$$

20 can be split into 1 + 19 where 1 is the first term and 19 is the last term in the series

$$S_{10} = \frac{10(20)}{2} = \frac{10(1 + 19)}{2}$$

It can be generalised that the sum of the first n terms represented by S_n of an arithmetic sequence with first term, a, common difference, d the last term, l or u_n can be found using

$S_n = \frac{n(a + l)}{2}$ which is consistent with the rule found through the method of undetermined coefficients for finding the sum of the first n natural numbers.

Since $l = u_n = a + (n - 1)d$,

$$S_n = \frac{n(a + a + (n - 1)d)}{2} = \frac{n}{2}(2a + (n - 1)d)$$

Example 3

Evaluate the sum of the first 7 terms of the arithmetic sequence in which $u_n = 2n + 6$.

Solution

We would begin by evaluating the first term

$u_1 = 2(1) + 6 = 8$

Then we find the last (7th) term as such $u_7 = 2(7) + 6 = 20$

The sum of the first 7 terms, $S_7 = 7\frac{(8 + 20)}{2} = 98$.

Example 4

Some learners are to be arranged in a 105-capacity classroom where the host expects that the difference in the number of seats between any two consecutive rows is 4 seats. If there can be only 7 rows in all, how many seats should be in the second row?

Solution

Number of rows, $n = 7$

Capacity of the room, $S_7 = 105$

Difference in number of seat, $d = 4$

Let *number of seats in the* 1*st row* $= a$

$$S_n = \frac{n(2a + (n-1)d)}{2}$$

$$S_7 = \frac{7(2a + (7-1)4)}{2}$$

$$105 = \frac{7(2a + 24)}{2}$$

$$210 = 14a + 168$$

$$14a = 42$$

$$a = 3$$

Number of seats in the 2nd row, $u_2 = a + d = 3 + 4 = 7$

Sum of GP

A geometric progression with first term, a, common ratio, r and the position of terms represented by n can be written as

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$$

$$rS_n - S_n = -a + ar^n$$

$$(r-1)S_n = ar^n - a$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$$S_n = \frac{a(r^n - 1)}{r - 1} = -\frac{a(1 - r^n)}{-(1 - r)}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

Example 5

Find the sum of the series

a) $3 + 6 + 12 + \ldots + 3072$

b) $\sum_{i=2}^{10} \left(\frac{1}{2}\right)^i$

Solution

a) $3 + 6 + 12 + \ldots + 3072$

The series can be written as $3 + 3(2) + 3(2)^2 + \ldots + 3(2)^{10}$ and thus, it can be observed that the first term, $a = 3$, the common ration, $r = 2$

The last term, $3(2)^{10}$ is the $11th$ term as $10 + 1 = 11$

$$S_{11} = \frac{3(2^{11} - 1)}{2 - 1} = 6141$$

b) $\sum_{i=2}^{10} \left(\frac{1}{2}\right)^i$

Since we start the series from $i = 2$ and end at $i = 10$, we would be summing 9 terms with $\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$ being the first, $\left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$ being the second and so on. $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$ is the last term

The common ratio if $\frac{1}{2} = 0.5$ and thus,

$$S_9 = \frac{0.25(0.5^9 - 1)}{0.5 - 1} = 0.4990234375$$

FOCAL AREA 2: CONVERGENCE AND DIVERGENCE OF SERIES

Convergence of series

Consider the series $\sum_{r=1}^{\infty} \left(\frac{1}{2^r}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$

Adding the terms in the series gives:

$S_1 = \frac{1}{2}$;

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$;

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$;

$S_4 = \frac{15}{16}$;

$S_5 = \frac{31}{32}$; $S_6 = \frac{63}{64}$; $S_7 = \frac{127}{128}$; and so on.

It is obvious the sum of the terms in the series, (S_n) is always less than 1 but gets closer to 1 as we take more and more terms.

It is reasonable to claim that $\sum_{r=1}^{\infty} \left(\frac{1}{2^r}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 1$

Sum to infinity of a GP

The formula for finding the sum to infinity of a GP is $S_\infty = \frac{a}{1-r}$

From the series above, the first term, $a = \frac{1}{2}$ and common ratio $r = \frac{1}{2} < 1$.

The formula can be used to verify that $\sum_{r=1}^{\infty}\left(\frac{1}{2^r}\right) = S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Note

S_1, S_2, S_3, S_4…are the partial sums of the series. If there is a number L such that $S_n = \sum_{r=1}^{\infty} S_n = L$. The number L is called the sum of the infinite series. If there is no such number, then the series is said to diverge.

Figure 2.1 shows the graph of a sequence with the position of the terms, n on the horizontal axis and the terms, U_n plotted on the vertical axis. The points i.e., (n, U_n) forms of the path of a curve whose height seems to get closer to 4. We can say that the values of U_n converges to one value: 4 and thus the sequence represented by the graph is a convergent sequence.

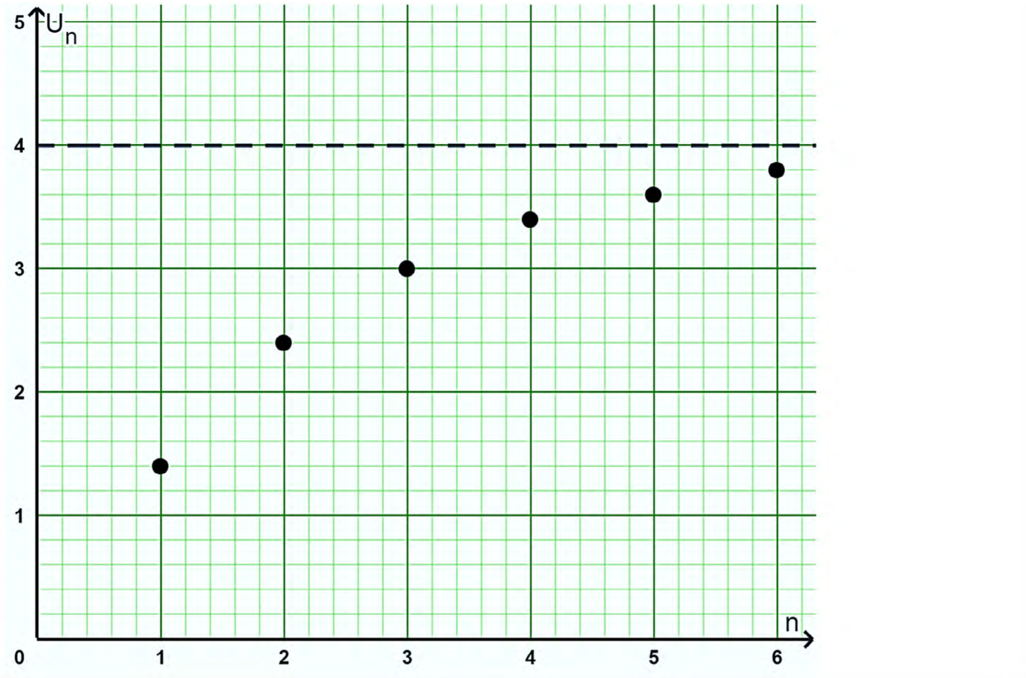


Figure 2.1

Divergence of series

The graph shown in Figure 2.2 however shows a diverging sequence as a single real number cannot be stated as the height that the terms of the sequence approaches. It seems to be rising without bounds.

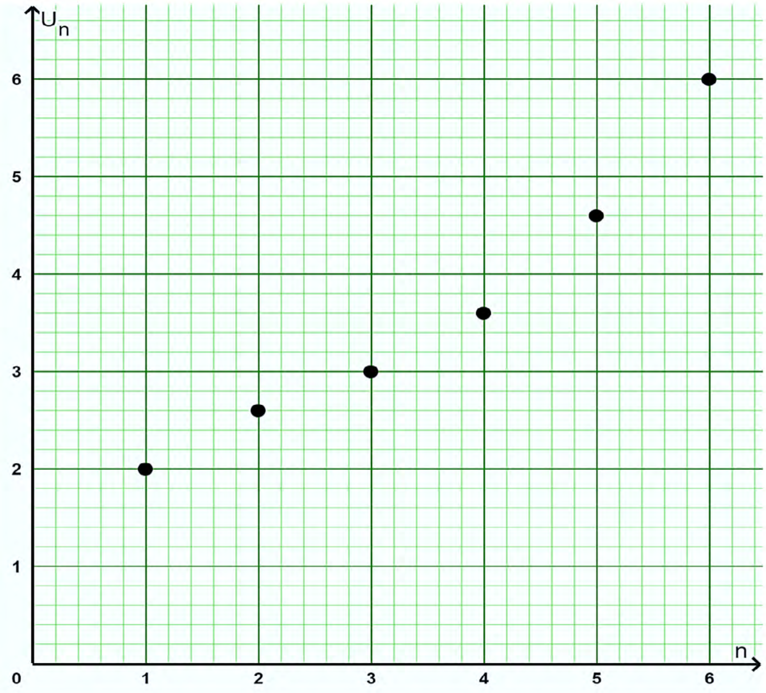


Figure 2.2

FOCAL AREA 3: RECURSIVE SEQUENCES

Example 6

Write the recurring decimals as a series. Identify the common ratio and the first and an explicit formula for the nth term.

a) 0.3333…

b) 0.54545454…

c) 3.212121…

Solution

a) $0.3333... = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$

$= 0.3 + 0.3(0.1) + 0.3(0.01) + 0.3(0.001) + \ldots$

$= 0.3 + 0.3(0.1) + 0.3(0.1)^2 + 0.3(0.1)^3 + \ldots$

The first term is 0.3, and the common ratio between each successive term is 0.1. The formula for the *nth* term will be $U_n = 0.3\left(\frac{1}{10}\right)^{n-1}$

b) $0.54545454\ldots = 0.54 + 0.0054 + 0.000054 + 0.00000054 + \ldots$

$= 0.54 + 0.54(0.01) + 0.54(0.0001) + 0.54(0.000001) + \ldots$

$= 0.54 + 0.54(0.01) + 0.54(0.01)^2 + 0.54(0.01)^3 + \ldots$

The first term is 0.54, and the common ratio between each successive term is 0.01.

The formula for the *nth* term will be $U_n = 0.54\left(\frac{1}{100}\right)^{n-1}$

c) $3.212121\ldots = 3 + 0.21 + 0.0021 + 0.000021 + 0.00000021 + \ldots$

$= 3 + 0.21 + 0.21(0.01) + 0.21(0.01)^2 + 0.21(0.01)^3 + \ldots$

From the second term (0.21), a geometric sequence with first term 0.21, and the common ratio 0.01 can be observed. The formula for the *nth* term will be $U_n = 0.21\left(\frac{1}{100}\right)^{n-1}$ and thus, $3.212121\ldots = 3 + 0.21\left(\frac{1}{100}\right)^{n-1}$ for $n = 1, 2, 3, \ldots$

Example 7

Find the recurrence relation and the initial conditions for the sequence: 1, 5, 17, 53, 161, 485, …

Solution

The recurrence formula is $a_n = 3a_{n-1} + 2$, and the initial condition is $a_0 = 1$

Example 8

A recursive rule for an AP is $a_1 = -3$; $a_n = a_{n-1} + 7$. What is the iterative rule for this sequence?

Solution

$a_1 = -3$

$a_n = a_{n-1} + 7$

$a_2 = a_1 + 7$

Common difference, $d = a_2 - a_1 = 7$

$\therefore a_n = a_1 + (n-1)d$

$= -3 + (n-1)7$

$= -3 + 7n - 7$

$= 7n - 10$

Example 9

Find the next two terms in (a_n), $n \geq 0$ for the sequence 3, 5, 11, 21, 43, 85, … the recurrence relation and initial conditions for the sequence

Solution

$a_0 = 3$

$a_1 = 2(3) - 1 = 5$

$a_2 = 2(5) + 1 = 11$

$a_3 = 2(11) - 1 = 21$

From observation, the operational sign before 1 alternates. We could use $(-1)^n$ to obtain the alternating difference since $(-1)^n$ for even values of n yields 1 while $(-1)^n$ for odd values of n yields -1

$\therefore a_n = 2a_{n-1} + (-1)^n, n \in \mathbb{Z}, n \geq 0$

FOCAL AREA 4: ARITHMETIC AND GEOMETRIC MEANS OF SEQUENCES

The mean or average of any two numbers, say p and q is obtained by finding a half of the sum of the two numbers thus: *Mean*, $m = \frac{1}{2}(p + q)$.

It must be noted that the difference between p and m is equal to the difference / distance between p and q i.e., $m - p = q - m$. It thus follows that p, m, q form an arithmetic sequence as there is a common difference and that for any arithmetic sequence with p and q as terms / elements, some means namely, $m_1, m_2, m_3, \ldots, m_k$ can be found between p and q such that $p, m_1, m_2, m_3, \ldots, m_k, q$ remains an arithmetic sequence as $m_1 - p = m_2 - m_1 = \ldots = m_k - q$ and the means are equally spaced

Example 10

Suppose cook wants to increase the amount of water in a pot of porridge being prepared from 100 *ml* to 300 *ml* in five equal steps such that the porridge gradually becomes lighter and does not have lumps. The amount of water (in *ml*) that she will need to pour can be form an arithmetic sequence. How many arithmetic means must be inserted between 100 and 300 to give the progression of amount of water, and what are the means?

Solution

We know two terms of the sequence as 100 and 300 so we are left with three more terms (arithmetic means) to complete the sequence of five terms

We can start by finding the mean of the least and maximum doses thus

$m_1 = \frac{1}{2}(300 + 100) = 200\ ml$

We have the sequence as 100, 200, 300

We now find the mean of 100 *ml* and 200 *ml*

$m_2 = \frac{1}{2}(100 + 200) = 150\ ml$ and the mean of 200 *ml* and 300 *ml*

$m_3 = \frac{1}{2}(200 + 300) = 250\ ml$

We have the requested sequence as 100, 150, 200, 250, 300

Alternatively, we could find the common difference between successive terms of the requested sequence thus

$\frac{300 - 100}{4} = 50\ ml$

We then add on the common difference to the terms till we obtain the 5th term: 300 *ml* 100, 150, 200, 250, 300

Geometric mean

The geometric mean of a list of *n* numbers is the *nth* root of the product of the numbers. It can be inferred that the geometric mean of two numbers will involve the square root of the product of the two numbers, while the geometric mean of three numbers will be the result of finding the cubic root of the product of the tree numbers and so on.

Just as Σ is used to represent sum, $\prod$ is often used to denote product hence the geometric mean of $a_1, a_2, a_3, \ldots, a_n$ can be represented as $\left(\left(\prod a\right)\right)^{\frac{1}{n}}$.

The geometric mean of two numbers, say l and b, is the length of the square whose area is equal to the area of the rectangle with sides of length, l and b. For example, for a rectangle with sides of length 25 *units* and 4 *units* has the same area, i.e., 100 *sq. units* as a square of side 10 *units* ($\sqrt{25 \times 4} = \sqrt{100}$). Geometric mean of l and b is given by $\sqrt{l \times b}$ as confirmed

The geometric mean is suitable for reporting average inflation, percentage change and growth rate since data types are expressed as fractions and hence are exponential in nature.

Example 11

Find the geometric mean of 10, 25, 5, 30

Solution

$10 \times 25 \times 5 \times 30 = 37{,}500$

Since there are four numbers, the geometric mean is $37{,}500^{\frac{1}{4}} = 13.9158$

Learning Tasks

1. Learners work in small groups to find the sums of sequences.
2. Learners in pairs find the sums of Arithmetic Progression and Geometric Progression.
3. Learners work individually to solve convergence of series graphically.
4. Learners work in groups to write the recurring decimals as a series.
5. Learners work in pairs to solve Arithmetic and Geometric means of sequences.
6. Learners work together to establish and solve real-life applications of Arithmetic Progressions and Geometric Progressions.

PEDAGOGICAL EXEMPLARS

These are suggested activities the teacher can take learners through

1. **Review previous learning:** Initiate Talk-for-Learning Approaches (building on what others say, managing Talk for Learning, Structured Talk for Learning); Experiential and Collaborative learning approaches to support revised facts about Arithmetic and Geometric sequences from Week 6 of the Year One Teacher Manual. Starting with whole group discussion and transiting through small convenient groups (e.g., mixed ability/mixed gender) or stations/centres, then pair work (e.g., using think-pair-share) and finally individual (e.g., using give-one take-one) learners recall, assess and debate each other on what they know about sequences including how to find the nth term, means of sequences and solve problems related to sequence and series in different contexts.

   For instance, Learners discuss, debate, assess and challenge colleagues on

   i) examples of geometric and arithmetic sequences (both mechanical and real-life examples)

   ii) how arithmetic and geometric sequences are similar;

   iii) how arithmetic and geometric sequences are different;

   iv) examples of sequences that are neither geometric nor arithmetic and explain how to solve such sequences.

2. **Using Talk for Learning Approaches** such as Building on what others say, managing talk-for Learning, Structuring Talk for Learning, and Collaborative learning approaches, The facilitator starts by providing each group with real-life examples involving G.P and A.P that learners can relate to, such as calculating the total distance travelled by an object with a specific pattern of motion and visual aids like diagrams, patterns, and graphs that can be used in explaining the concept of sums of sequences. For instance, you can use a visual representation of a staircase

pattern to introduce the concept of the sum of an arithmetic sequence or a geometric pattern to illustrate the sum of a geometric sequence. Furthermore, the facilitator guide learners to break down the concept of solving real-life situations involving G.P and A.P into smaller, manageable steps. Start with simpler sequences, such as arithmetic or geometric sequences, and gradually progress to more complex ones. Guide learners through the process of finding the nth term, the sum of the first n terms, and eventually the sum of an infinite sequence.

3. **Project-based learning:** In project-based learning, put learners into convenient groups and assign them to investigate the graphical solution of series convergence types (such as geometric or alternating series) using graphing software such as GeoGebra. Ask learners to plot partial sums and observe trends toward convergence or divergence. The project should require learners to analyse and interpret their graphical results, identifying convergence criteria. Additionally, they should document their findings in a report or presentation, explaining their methodology, observations, and conclusions about series convergence.
4. **Problem-based learning**: In convenient groups (ability, mixed-ability, mixed gender, etc.), guide learners to use the method of undetermined coefficients to find the sum of the first n terms of a series, by equating it to an identical series of the form $A + Bn + Cn^2 + \ldots$ and then determining the values of the constants A, B, C, $\ldots$
5. **Experiential Learning:** Learners are put into convenient groups (ability, mixed-ability, mixed gender, etc.) to engage in hands-on activities to investigate, establish, and use the formulas $S_n = \frac{n}{2}(a_1 + a_n) and\ S_n = \frac{n}{2}[2a_1 + (n-1)d]$ to find the sum of terms in an arithmetic progression (AP) and solve related problems. Encourage learners to discuss and observe patterns in the APs they have created, focusing on the relationships between the terms, the number of terms, the first term, and the common difference.
6. **Problem-based learning:** Put learners into convenient pairs (ability, mixed-ability, mixed gender, etc.) and present learners with real-world scenarios involving recurring decimals, such as financial calculations or measurements to analyse. Task learners to investigate patterns in recurring decimals, identify the repeating sequence/patterns and formulate corresponding infinite series. Encourage learners to apply mathematical concepts of fractions and infinite series, collaborating to explore different recurring decimals and their series representations. Groups create posters or presentations to illustrate their findings, including examples and explanations of how the recurring decimal converts into a series.
7. **Experiential Learning**: In experiential learning for Arithmetic and Geometric means of sequences, guide learners in convenient pairs (mixed-ability, mixed gender, etc.) to engage in hands-on activities like measuring distances or calculating growth

rates in natural phenomena. They should collect data, analyse sequences, and apply formulas to calculate arithmetic and geometric means of given sequences. By exploring real-world examples, such as population growth or financial trends, learners intuitively grasp the concepts of average values in sequences. Learners discuss their solutions and reasoning with the class, demonstrating how they applied the formulas and interpreted the results in the context of sequences.

KEY ASSESSMENTS

Assessment level 3: Strategic Reasoning

1. Show that the sum of n consecutive odd integers beginning with 1 equals n^2
2. Compute the sum of all integers between 100 and 800 that are divisible by 3
3. Evaluate $\sum_{r=1}^{n}\frac{2}{(2r+1)(2r-1)}$

Hint

split into partial fractions and solve

Assessment Level 3: Strategic Reasoning

1. A boy agrees to work at the rate of one cedi the first day, two cedis the second day, four cedis the third day, eight cedis the fourth day, etc. how much would he receive at the end of 12 days?
2. A sum of *GH¢* 400.00 is invented today at 6% per year. To what will it accumulate in five years if interest is compounded
   - **a)** annually
   - **b)** semi-annually
   - **c)** quarterly

Reminder

Assign group project work to learners at the end of Week 3. For further information on how to go about the project work, refer to the Teacher Assessment Manual and Toolkit.

WEEK 4

Learning Indicators

1. *Determine the maximum/minimum values within the given constraints*
2. *Solve real life problems involving linear inequalities*
3. *Solve quadratic inequalities involving real life problems*

FOCAL AREA 1: DETERMINING MAXIMUM / MINIMUM VALUES

Example 12

Evaluate the expression $l = 3x + 4y$ for the given feasible region to determine the point at which 'l' has a maximum value and the point at which 'l' has a minimum value.

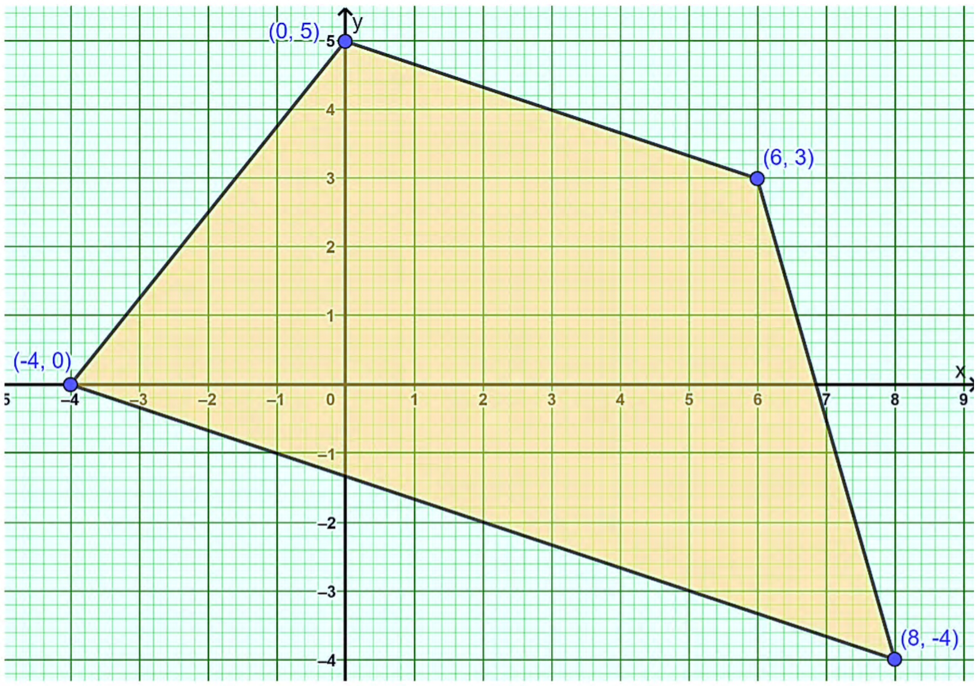


Figure 2.3

Solution

From Figure 2.3, the coordinates of the vertices of the feasible region are $(-4, 0)$, $(0, 5)$, $(6, 3)$ and $(8, -4)$

At $(-4, 0)$,

$l = 3x + 4y$

$\Rightarrow l = 3(-4) + 4(0)$

$\Rightarrow l = -12 + 0$

$\Rightarrow l = -12$

At (0, 5),

$l = 3x + 4y$

$\Rightarrow l = 3(0) + 4(5)$

$\Rightarrow l = 0 + 20$

$\Rightarrow l = 20$

At (6, 3),

$l = 3x + 4y$

$\Rightarrow l = 3(6) + 4(3)$

$\Rightarrow l = 18 + 12$

$\Rightarrow l = 30$

At (8, −4)

$l = 3x + 4y$

$\Rightarrow l = 3(8) + 4(-4)$

$\Rightarrow l = 24 - 16$

$\Rightarrow l = 8$

The maximum value of 'l' occurred at the vertex (6, 3).

The minimum value of 'l' occurred at the vertex (− 4, 0)

Example 13

For the following graphed region and the expression $z = 5x + 7y - 1$, find a point where 'z' has a maximum value and a point where 'z' has a minimum value.

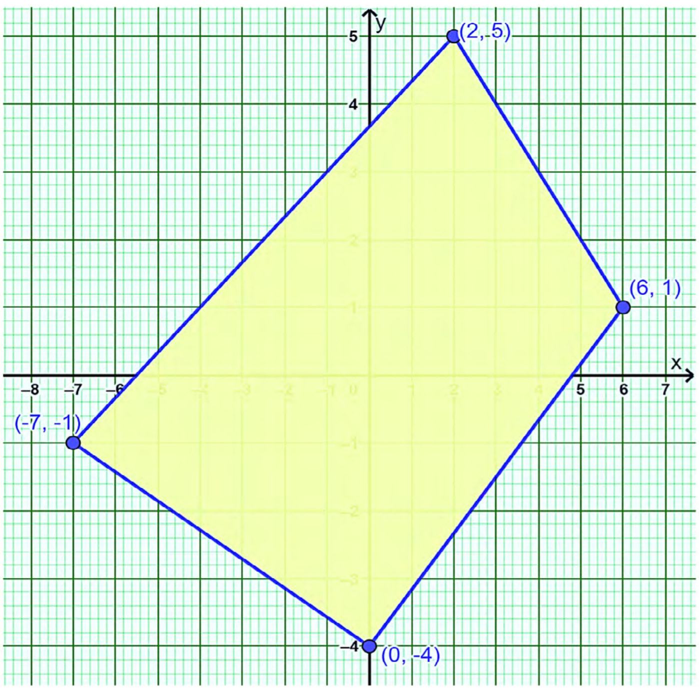


Figure 2.4

$z = 5x + 7y - 1$

At $(2, 5)$,

$z = 5(2) + 7(5) - 1 = 44$

At $(6, 1)$,

$z = 5(6) + 7(1) - 1 = 36$

At $(0, -4)$,

$z = 5(0) + 7(-4) - 1 = -29$

At $(-7, -1)$,

$z = 5(-7) + 7(-1) - 1 = -43$

The minimum value of z is -43 and it occurs at $(-7, -1)$ while the maximum value of z is 44 and it occurs at $(2, 5)$

FOCAL AREA 2: SOLVING REAL LIFE PROBLEMS INVOLVING SYSTEMS OF LINEAR INEQUALITIES

Example 14

A company that produces football jerseys makes two sets of jerseys for Accra Hearts of Oak: the traditional rainbow jerseys for home matches and the white jerseys with rainbow for away matches. To produce each jersey, two types of material – nylon and cotton – are used.

The company has 450 units of nylon in stock and 300 units of cotton. The traditional rainbow jersey requires 6 units of nylon and 3 units of cotton. The white ‘away’ jersey requires 5 units of nylon and 5 units of cotton. Each white ‘away’ jersey that is made realises a profit of *GH¢* 12.00 for the company, whereas each rainbow jersey realises a profit of *GH¢* 15.00. For the nylon and cotton that the company currently has in stock, how many jerseys should the company make to maximise their profit?

Solution

Let 'x' represent the number of rainbow jerseys

Let 'y' represent the number of white jerseys

	Units required per rainbow jersey, x	Units required per white jersey, y	Units Available
Nylon	6	5	450
Cotton	3	5	300
Profit per jersey	*GH¢* 12.00	*GH¢* 15.00	

The number of rainbow jerseys and/or white jerseys that are produced must be either zero or greater than zero. Therefore, the constraint *is* $x \geq 0$ and $y \geq 0$, respectively.

The total number of units of nylon and/or cotton required to make both types of jerseys cannot exceed 450. Therefore, the constraints are $6x + 5y \leq 450$ and $3x + 5y \leq 300$ respectively

The system of linear inequalities obtained is thus

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 6x + 5y &\leq 450 \\ 3x + 5y &\leq 300 \end{aligned}\right\}$$

The equation to identify the profit $P = 12x + 15y$

Figure 2.5 shows the graph of the system of inequalities hereby derived and indicates the feasible region.

The vertices of the solution region are located at (0, 60), (50, 30), (75, 0) and (0, 0)

At (0, 60),

$P = 12(0) + 15(60) = GH¢\ 900.00$

At (50, 30),

$P = 12(50) + 15(30) = GH¢\ 1{,}050.00$

At (75, 0),

$P = 12(75) + 15(0) = GH¢\ 900.00$

At (0, 0),

$P = 12(0) + 15(0) = 0.00$

The maximum profit that can be made by the company is *GH¢* 1,050.00 and it can be realised if the company produces 50 rainbow and 30 white jerseys respectively are made

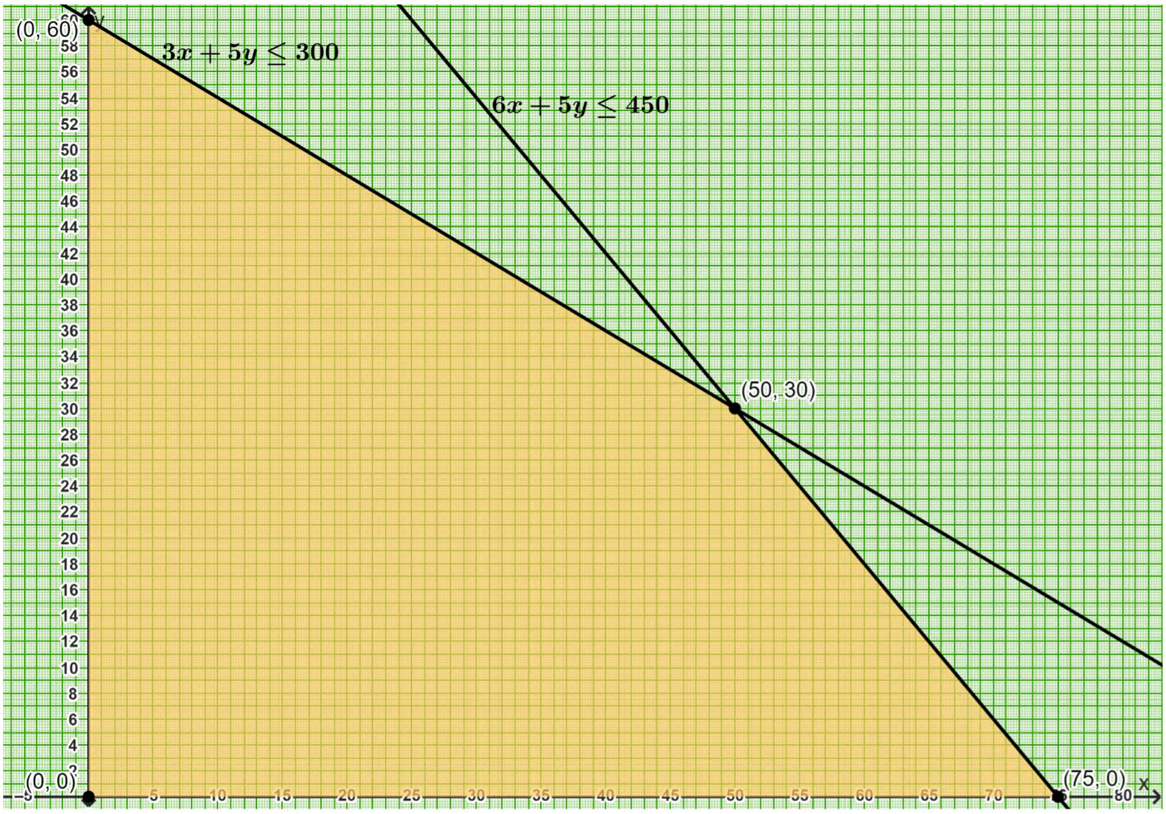


Figure 2.5

Example 15

A cocoa processing factory can provide its customers with chocolate, cocoa paste and cocoa butter by processing either of two cocoa bean varieties – Forastero or Criollo. The cocoa beans arrive at the company in railroad trucks. Each railroad truck of Forastero cocoa beans can be processed into 3 tons of chocolate, 3 tons of paste and I ton of butter. Each railroad truck of Criollo cocoa beans can process 1 ton of chocolate, 4 tons of paste and 3 tons of butter. The factory received an order for 7 tons of chocolate, 19 tons of cocoa paste and 8 tons of cocoa butter. The cost to purchase and process a carload of Forastero beans is *GH¢* 7,000.00 while the cost for Criollo beans is *GH¢* 6000.00. If the company wants to fill the order at a minimum cost, how many carloads of each variety must be bought?

Solution

Varieties of cocoa beans		Totals
Carloads of Forastero (*x*)	Carloads of Criollo (*y*)	
3	1	7
3	4	19
1	3	8

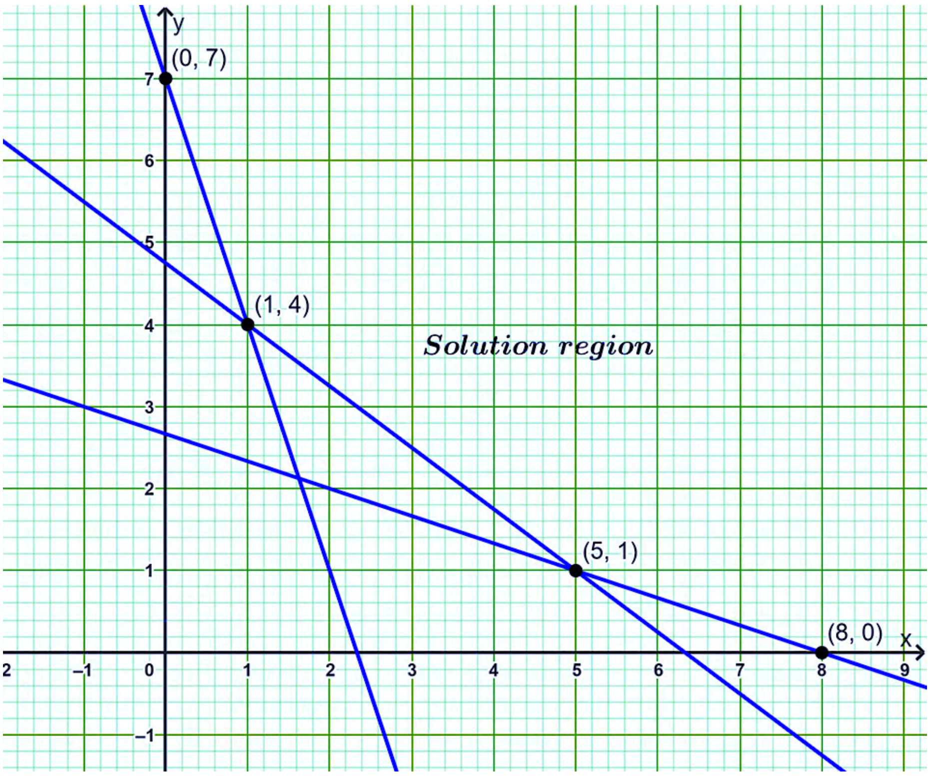


Figure 2.6

$Total\ cost = 7{,}000x + 6{,}000y$

$x = 8,\ y = 0,$

$Total\ cost = 7{,}000(8) + 6{,}000(0) = GH¢\ 56{,}000.00$

$x = 5,\ y = 1,$

$Total\ cost = 7{,}000(5) + 6{,}000(1) = GH¢\ 41{,}000.00$

$x = 1,\ y = 4,$

$Total\ cost = 7{,}000(1) + 6{,}000(4) = GH¢\ 31{,}000.00$

$x = 0,\ y = 7,$

$Total\ cost = 7{,}000(0) + 6{,}000(7) = GH¢\ 42{,}000.00$

1 carload of Forastero and 4 carloads of Criollo need to be bought to obtain the minimum cost for filling the order which is $GH¢\ 31{,}000.00$

FOCAL AREA 3: SOLVING QUADRATIC INEQUALITIES

We discussed quadratic functions and their graphs in week 8 of the year one teacher manual. The general quadratic function is of the form $P(x) = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$. By extension, the general quadratic inequality would be of the forms,

$ax^2 + bx + c < d$,

$ax^2 + bx + c > d$,

$ax^2 + bx + c \leq d$ or

$ax^2 + bx + c \geq d$, where d is a constant

To solve a quadratic inequality of any of the forms above, we would first seek to get the value of d, the expression on one side of the inequality to zero, we factorise, then we use algebraic reasoning or study the graph of an illustration of the inequality to determine the solution region. Just like linear inequalities, we expect solution regions or ranges but not values for the independent variable.

Example 16

Solve $x^2 - 3x - 4 < 0$

Solution

We would first attempt to factorise $x^2 - 3x - 4$ as the right-hand side is already zero.

$x^2 - 3x - 4 = (x-4)(x+1)$

$(x-4)(x+1) < 0$

We would now find critical values. The critical values provide the extremities for the solution range.

$x - 4 = 0 \Rightarrow x = 4$

$x + 1 = 0 \Rightarrow x = -1$

The possible solution regions are thus $x < -1$, $-1 < x < 4$ and $x > 4$ as shown on the number line. We can choose values for x in the possible solution regions and use them as test values

	Test Values	$x - 4$	$x + 1$	$(x - 4)(x + 1)$	Sign
$x < -1$	-2	-6	-1	6	Positive
$-1 < x < 4$	0	-4	1	-4	Negative
$x > 4$	5	1	6	6	Positive

The original statement, $x^2 -3x-4 < 0$ holds for only values of x for which the product of the factors is negative (less than zero). From the table, the only range that meets this criterion is $-1 < x < 4$ and hence, that is the solution to the quadratic inequality.

Another way to find the solution is to graph the quadratic function and determine the solution region from the graph. Lessons in week 8 of the year one teacher manual discussed graphing of parabolic functions.

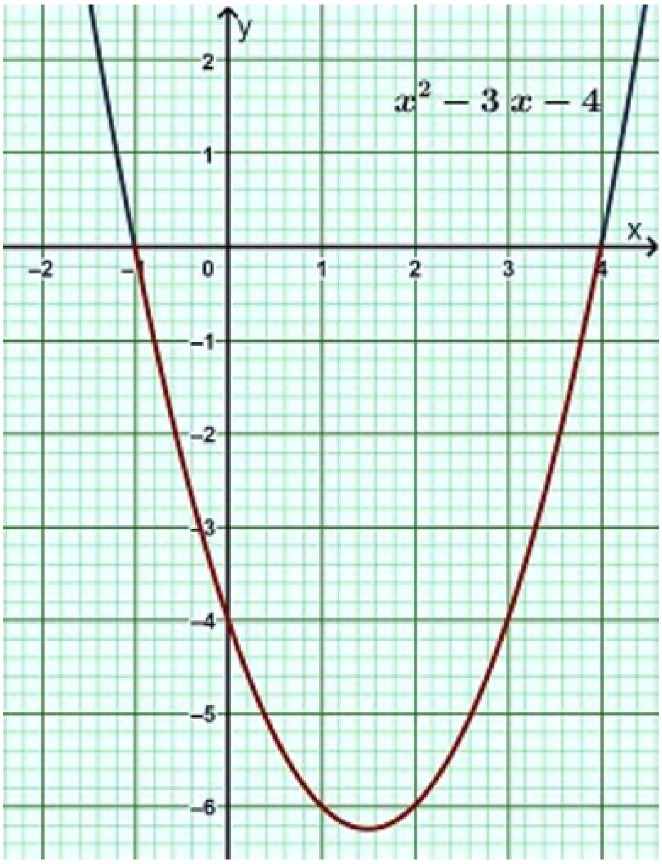


Figure 2.7

It can be observed from Figure 2.7 that the part of the graph of $x^2 - 3x - 4$ that yield negative values (less than zero) for y and thus lies below the $x - axis$ is in the interval $-1 < x < 4$.

FOCAL AREA 4: SOLVING SYSTEMS OF QUADRATIC INEQUALITIES

Example 17

Graph the system of quadratic inequalities.

$y < -x^2 + 3$ and $y \geq x^2 + 2x - 3$

Solution

The graph for the two inequalities and the possible solution regions are shown in Figure 2.8.

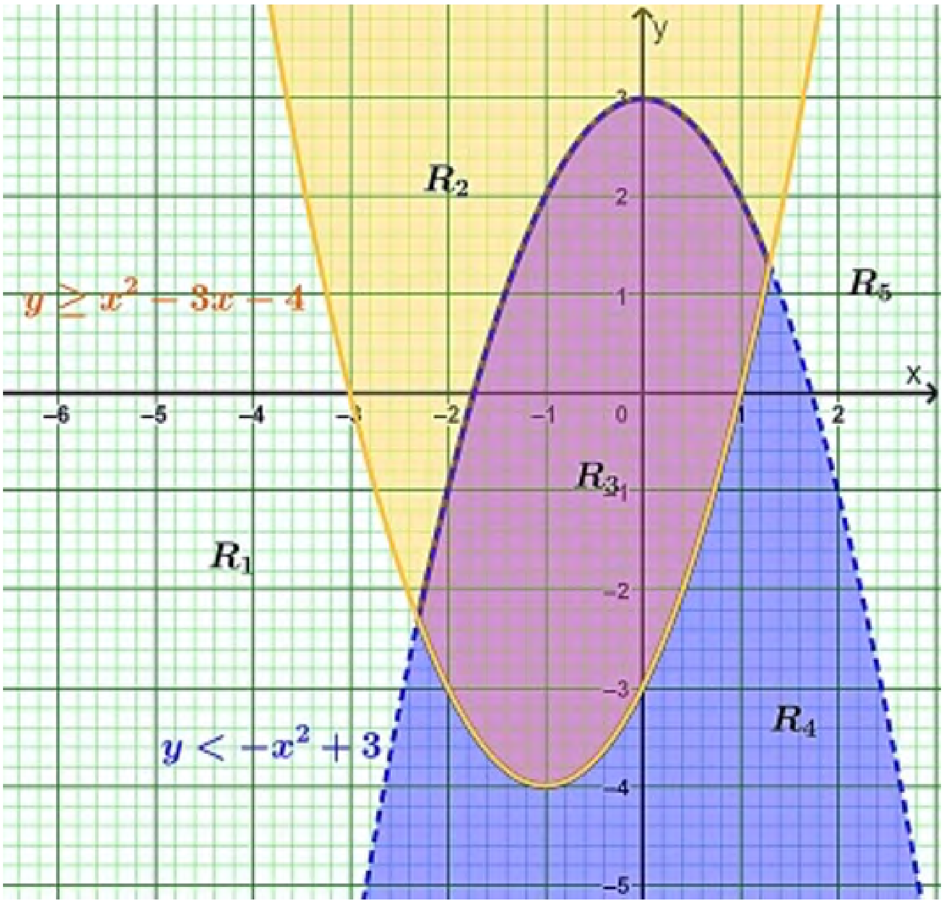


Figure 2.8

R_3 is the only solution region common to both inequalities. This implies that all points in that region, including but not exclusive to (0, 0), (−1.4, 0.6) (−1, 1), (0.5, 2), (1, 1) and (1, 1.6) are in the solution set for the system.

FOCAL AREA 5: SOLVING REAL-LIFE PROBLEMS INVOLVING QUADRATIC INEQUALITIES

Example 18

Ama is instructed to make a garden plot which has an area less than 18 m^2. The length should be 3 m longer than the width.

a. How would you represent the width of the garden plot?

b. What would be the mathematical sentence?

c. What are the possible dimensions of the garden plot?

Solution

Let the length of the plot be l and the width be w.

"The length should be 3 m longer than the width" implies that $l = w + 3$.

$Area\ of\ the\ plot = lw < 18$

$\Rightarrow (w + 3)w < 18$

$w^2 + 3w < 18$

$\Rightarrow w^2 + 3w - 18 < 0$

$(w-3)(w+6) < 0$

$w - 3 = 0 \Rightarrow w = 3$

$w + 6 = 0 \Rightarrow w = -6$

	$w - 3$	$w + 6$	$(w - 3)(w + 6)$
$w < -6$	Negative	Negative	Positive
$-6 < w < 3$	Negative	Positive	Negative
$w > 3$	Positive	Positive	Positive

The solution range for the inequality is $-6 < w < 3$

The width of the garden must be between 0 and 3 m. Lengths can never be negative, so we are ignoring the -6 m solution part. And the length is simply 3 m longer than the width. The length can be between 3 m and 6 m.

Learning Tasks

1. Learners to work out quadratic inequalities by using factorisation, and then use illustrations, algebraic reasoning or study the graph of an illustration of the inequality to determine the solution region.

2. Learners to solve systems (finding the solution sets) of quadratic inequalities algebraically or graphically.
3. Learners to work out/ solve real-life problems involving quadratic inequalities.

PEDAGOGICAL EXEMPLARS

Teacher should consider the following activities

1. **Review Previous Learning:** Put learners into convenient groups (ability, mixed-ability, mixed gender, etc.) to review and solve linear inequalities using various approaches, including graphing, illustrations, and algebraic reasoning.
2. **Research-based learning**
   a. Divide learners into small groups or pairs and assign each group or pair a specific aspect of quadratic inequalities to research, such as
      - Graphing quadratic inequalities in two variables
      - Solving quadratic inequalities in one variable using graphs and number lines
      - Solving quadratic inequalities in one variable using algebraic methods
      - Real-life applications of quadratic inequalities

   b. Engage learners in researching their assigned topic, using various resources (textbooks, online materials, etc.) and make a presentation to share their findings with the class, including examples, graphs, illustrations, and step-by-step solutions.

   c. After the presentations, provide learners with a set of quadratic inequalities problems, both in one variable and two variables to solve using the methods and approaches they learned from the presentations and their research. Learners present their solutions to the class, explaining their thought processes, and justifying their approaches.

   d. Encourage learners to use graphs, illustrations (number lines or coordinate planes), and algebraic reasoning to solve the quadratic inequalities.
3. **Think-Pair-Share and Brainstorming/Project-based Learning**
   a. Individually, learners brainstorm and list down real-life situations that can be modelled using quadratic inequalities.

   b. Learners pair up and share their ideas, discussing and expanding on each other's examples and develop a mathematical model for the selected situations, representing them as quadratic inequalities.

c. Using graphing software (e.g., GeoGebra) or Excel spreadsheets, learners in groups sketch the graphs of their quadratic inequalities and determine their solutions.

d. Groups prepare a presentation to share their real-life context, mathematical model, graphical representation, and solutions with the class. After each presentation, learners ask questions, provide feedback, and engage in discussions about the modelling process, the accuracy of the solutions, and the real-life implications.

e. Throughout the activity, the teacher facilitates the discussions, asking probing questions, and providing guidance and support as needed.

KEY ASSESSMENTS

Assessment level 2: Skills of conceptual understanding

1. Find the solutions to $x^2 - 3x - 10 \leq 0$ by sketching an appropriate graph
2. Solve $x^2 - x - 12 > 0$ by factorising and sketching the graph.

Assessment Level 3: Strategic Reasoning

A company determines that the cost in cedis C, of producing x bags of rice is $C = 100x + 250$. The revenue R, in cedis, from selling all of the bags of rice is $R = x^2 + 750$. How many bags of rice should the company produce and sell if the company wishes to earn a profit at least *GH¢* 8,000.00

Reminder

Learners' score individual class exercise should be ready at the end of Week 4 for submission into the STP.

Section 2 Review

The third and fourth weeks of the year two Teacher Manual guided learners through the following learning indicators and focal areas

1. Find the sums of sequences.
2. Establish the sums of Arithmetic Progression and Geometric Progression.
3. Represent convergence of series in graphs.
4. Write the recurring decimals as a series.
5. Solve Arithmetic and Geometric means of sequences.
6. Establish and solve real-life applications of Arithmetic Progressions and Geometric Progressions.

7. Determining maximum / minimum values
8. Solving real life problems involving systems of linear inequalities
9. Solving quadratic inequalities
10. Solving systems of quadratic inequalities
11. Solving real-life problems involving quadratic inequalities

Collaborative learning, peer-to-peer learning, research-based learning etc., which took care of learning differences were some of the activities used during the learning to ensure that the learning indicators were achieved. Class exercises, home-based assignments, research and project works, etc. were utilized as assessment strategies.

SECTION 3: POLYNOMIAL FUNCTIONS

Strand: Modelling with Algebra

Sub-Strand: Application of algebra

Learning Outcome: *Apply conventional and personal strategies to find the factors and zeroes of polynomial functions with degrees higher than 2 and sketch their graphs by hand and/or using technology*

Content Standard: *Demonstrate the ability to use varying strategies to find the perform operations on polynomial functions and sketch their graphs by hand and/or using technology*

INTRODUCTION AND SECTION SUMMARY

Polynomial functions are vital in advanced mathematics, connecting algebra and calculus. They are widely used in multiple fields, from simple mathematical modelling to complex scientific applications. Section 3 of the Teacher's Manual provides an in-depth exploration of these functions, equipping learners with crucial skills for analysis and problem-solving.

This section covers several key aspects of polynomial functions. It introduces various methods for finding polynomial roots, including factoring techniques and specialized theorems like the rational root theorem. The manual also delves into graphing higher-degree polynomials, helping teachers and students visualize and understand the complex behaviours of these functions. A significant concept covered is Descartes' rule of signs, a powerful tool for predicting the number of positive and negative roots a polynomial might possess.

Furthermore, the section examines fundamental mathematical principles governing polynomials. It explores the fundamental theorem of algebra, which provides insights into complex roots, and the complex conjugates theorem, which reveals how certain roots appear in pairs. The manual also covers linear and quadratic factor theorems, enhancing understanding of polynomial factorization and root characteristics.

Additionally, polynomial functions are linked to economics, where they model supply and demand curves; physics uses them to describe motion and energy relationships. Engineers apply polynomials in designing road curves and bridge structures. Computer graphics rely on polynomials for creating smooth shapes and animations. In finance, polynomials help calculate compound interest and optimize investment strategies. Acoustics and sound engineering utilize polynomials for signal processing.

The week covered by the section is:

Week 5

1. Finding factors and zeroes of polynomial functions
2. Graphing polynomial functions with degrees higher than 2
3. Applying Descartes' rule of signs theorem
4. Applying the fundamental theorem of algebra
5. Exploring Complex conjugates theorem
6. Examining and Linear and quadratic factor theorems

SUMMARY OF PEDAGOGICAL EXEMPLARS

Throughout the section, hands-on activities and real-life examples will be used to enhance understanding and application. Learners should engage in a variety of teaching and learning strategies as the section aims to equip learners with a solid foundation in polynomial functions and their related theorems, preparing them for advanced topics and real-world problem-solving.

The pedagogical approaches include hands-on activities where physical objects or models are used to demonstrate polynomial functions and their related problems. Also, visual representations such as diagrams, and graphs to illustrate polynomial functions with degrees higher than 2 which will help learners visualize and understand complex concepts. Furthermore, collaborative learning which encourages group work and discussions where learners can share their understanding, solve problems together, and learn from each other's perspectives as well as teamwork and deeper comprehension on complex conjugates and linear and quadratic theorems. Again, problem-based learning which should present learners with real-life problems requiring the application Descartes' rule of signs theorem and the fundamental theorem of algebra. And to ensure differentiated instruction, provide different learning materials and tasks based on learners' abilities and preferences. Offer additional challenges for advanced learners and additional support for those who need it. Finally, break down complex concepts into smaller, more manageable parts. Start with simpler concepts and gradually introduce more complex ideas as learners' progress.

These pedagogical approaches emphasize active learning, technology integration, and collaboration, ensuring students gain a deep understanding of polynomial functions and their theorems.

ASSESSMENT SUMMARY

The various concepts to be covered under this section should be assessed using various forms of assessments modes to ascertain learners' performance. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP).

WEEK 5

Learning Indicators

1. *Find the factors and zeros of a polynomial function using conventional and personal strategy*
2. *Sketch the graph of a polynomial function with degrees higher than 2 hand and by using technology (e.g., GeoGebra, Demos, PhET Simulations, and Geometer's Sketch Pad) where appropriate*

FOCAL AREA 1: FINDING FACTORS AND ZEROES OF POLYNOMIAL FUNCTIONS

Factorising Polynomials

Terms are factors of a polynomial if, when they are multiplied, they equal that polynomial:

For example, $x^2 + 2x - 15 = (x - 3)(x + 5)$ and so, $(x - 3)$ and $(x + 5)$ are factors of the polynomial $x^2 + 2x - 15$.

Before discussing how to solve polynomial equations, discuss the zero-product property with learners.

Zero-Product Property

This property of real numbers states that if the product of two real numbers equals zero, then at least one of the numbers is zero. Consider two real numbers, p and q,

If $p \times q = 0$, then either

$p = 0$ and $q \neq 0$ or

$p \neq 0$ and $q = 0$ or

$p = 0$ and $q = 0$

Note that p and q could represent mathematical expressions

Solving a Polynomial Equation

By rearranging the terms to have zero on one side: e.g.,

$x^2 + 2x = 15$

$\Rightarrow x^2 + 2x - 15 = 0$

$\therefore (x + 5)(x - 3) = 0$

At this point, we apply the zero-product property thus

$x + 5 = 0$ or $x - 3 = 0$;

so $x = -5$ or $x = 3$

Like all equations, an identity must be achieved i.e., the value of both sides of the equation should be equivalent) for values of the variables which are solutions to the equation. For the current example, substituting -5 or 3 for x in the equation should result in both sides being equal.

For the left-hand side, taking the value of x to be -5 results in $x^2 + 2x = (-5)^2 + 2(-5) = 15$ which is equal to the right-hand side confirming that indeed $x = -5$ is a solution to the quadratic equation. A similar test can be done for $x = 3$.

Setting the factors of a polynomial expression to ***zero*** gives the s***olution(s)*** to the equation when the polynomial expression equals zero. Another name for the "solutions of a polynomial equation" is the ***roots*** or **zeroes** of the polynomial.

If the polynomial expression, written as a function, say $P(x)$ if graphed, then the solution to $P(x) = 0$ is the same as the x – *intercept* of the graph. It is the point of intersection between the $y = P(x)$ and $y = 0$. This interpretation is stemmed from the understanding that the system $\begin{cases} y = P(x) \\ y = 0 \end{cases}$ can be solved by substitution, i.e., $P(x) = 0$: the equation which we seek to solve.

Figure 3.1 shows the graph of $y = x^3 + 3x^2 - x - 3$ and $y = x^2 - 9x + 18$ and the points of intersection: $A(-3, 0)$, $B(-1, 0)$ and $C(1, 0)$ and $D(3, 0)$ and $E(6, 0)$ between the x – *axis* and the respective graphs. It can thus be concluded that the solution, roots or zeroes of $y = x^3 + 3x^2 - x - 3$ are $x = -3$, $x = -1$ and $x = 1$ while those for $y = x^2 - 9x + 18$ are $x = 3$ and $x = 6$.

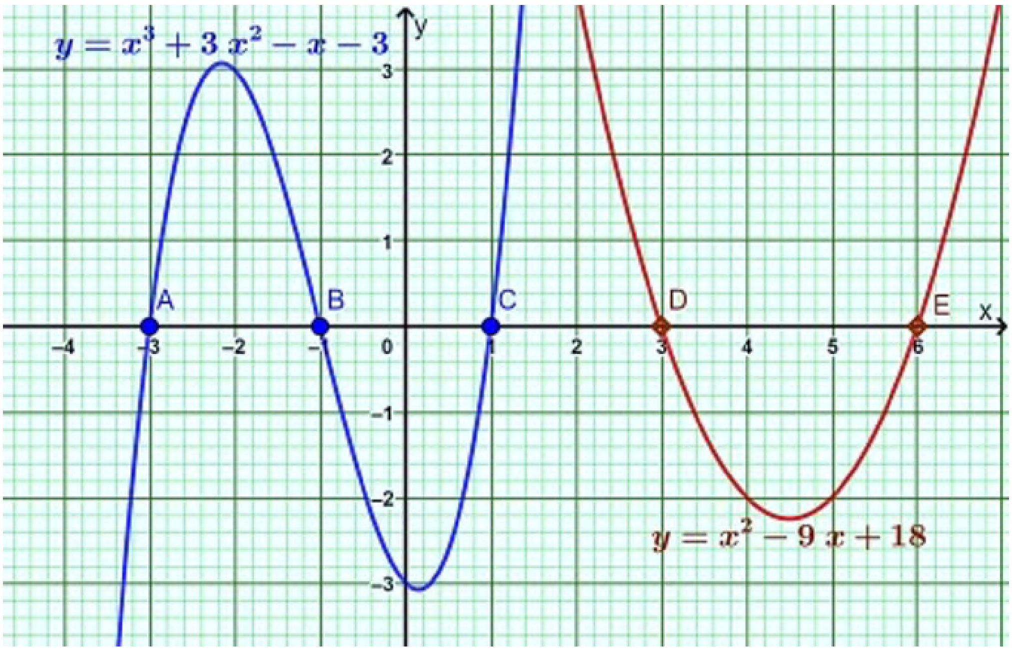


Figure 3.1

Rational Zeros Theorem

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ...+a_1 + a_0$ has integer coefficients, then every rational zero of $P(x)$ is of the form $\frac{p}{q}$; where p is a factor of the constant coefficient, a_0 and q is a factor of the leading coefficient a_n

Example 1

List all possible rational zeros given by the rational zeroes theorem of

$P(x) = 6x^4 + 7x^3 - 4$

Solution

By comparing $6x^4 + 7x^3 - 4$ to the general polynomial expression of the same degree (4), i.e. $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, $a_4 = 6$ and $a_0 = -4$

Factors of $a_0 = -4 = \pm1, \pm2$ and ±4

Factors of $a_n = a_4 = 6 = \pm1, \pm2, \pm3, \pm6$

Possible $\frac{p}{q} = \frac{\pm1}{\pm1}, \frac{\pm1}{\pm2}, \frac{\pm1}{\pm3}, \frac{\pm1}{\pm6}, \frac{\pm2}{\pm1}, \frac{\pm2}{\pm2}, \frac{\pm2}{\pm3}, \frac{\pm2}{\pm6}, \frac{\pm4}{\pm1}, \frac{\pm4}{\pm2}, \frac{\pm4}{\pm3}, \frac{\pm4}{\pm6}$

By simplifying the fractions and eliminating duplicates, we get the following list of possible values for $\frac{p}{q} = \pm1, \pm2, \pm4, \frac{\pm1}{\pm2}, \frac{\pm1}{\pm3}, \frac{\pm2}{\pm3}, \frac{\pm4}{\pm3}, \frac{\pm1}{\pm6}$

Example 2

Find all real zeros of the polynomial $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$.

Solution

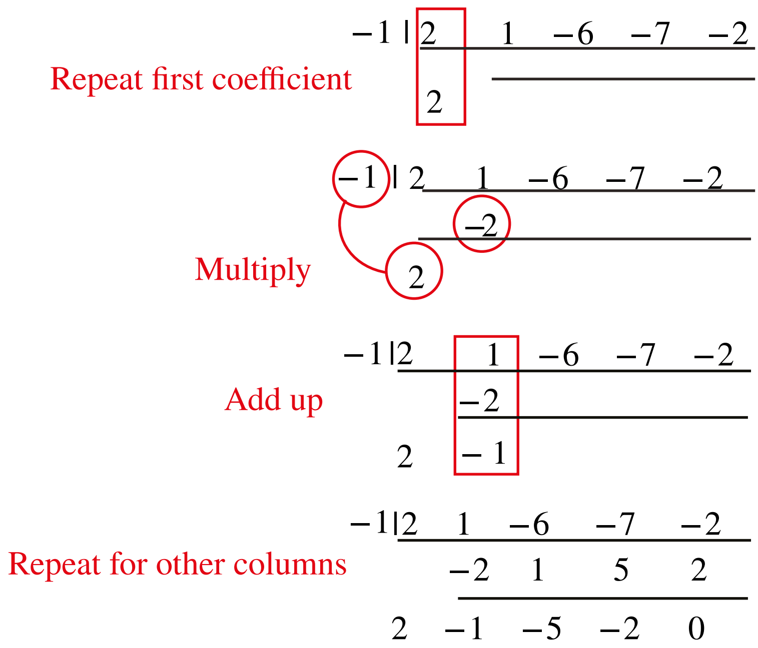
Applying the rational zeros theorem, we have $\pm1, \pm2, , \frac{\pm1}{\pm2}$ as the possible rational zeros.

We can now check to find which of the possible roots are really roots. Note that the roots relate to factors of the polynomial. This implies that the root, ±1 relates to $x - 1$ and $x + 1$. We can apply the remainder theorem of polynomial functions as discussed in week 10 of the year one teacher manual to identify which of these factors is a factor and hence which of the two rational zeroes is a zero/root.

$P(1) = 2(1)^4 + (1)^3 - 6(1)^2 - 7(1) - 2 = -12 \neq 0$ but

$P(-1) = 2(-1)^4 + (-1)^3 - 6(-1)^2 - 7(-1) - 2 = 0$ and thus, $x + 1$ is confirmed to be a factor

We now find the other factor by dividing $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$ by $x + 1$.



Since the remainder is zero, 1 is a zero

This also tells us that $P(x)$ factors as

$2x^4 + x^3 - 6x^2 - 7x - 2 = (x + 1)(2x^3 - x^2 - 5x - 2)$

Applying the rational zeros theorem coupled with division again for $2x^3 - x^2 - 5x - 2$ and again for the quadratic expression which will be obtain as a factor gives $P(x)$ as:

$2x^4 + x^3 - 6x^2 - 7x - 2 = (x + 1)^2(x - 2)(2x + 1)$

Thus, the zeros of $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$ are $(-1, 0)$, $(2, 0)$ and$\left(-\frac{1}{2}, 0\right)$

FOCAL AREA 2: SKETCHING POLYNOMIAL FUNCTIONS WITH DEGREES HIGHER THAN 2

The zeroes of a polynomial functions can be used to sketch polynomial function of higher degrees without having to apply the principles of differentiation.

For example, to sketch $y = x^3 + 3x^2 - x - 3$

1. we can apply the rational zeroes theorem to obtain the zeroes and subsequently the roots as $x = -3$, $x = -1$ and $x = 1$
2. we then test algebraically to determine the behaviour of the curve between consecutive roots thus

   The intervals of x between the roots are $x < -3$, $-3 < x < -1x$, $1 < x < 1$ and $x > 1$. We can choose values for x in these intervals and use them as test values

	$x < -3$ or $-\infty < x < -3$	$-3 < x < -1$	$1 < x < 1$	$x > 1$
Test values	-5	-2	0	3
y	-48	3	-3	48
Sign	Negative	Positive	Negative	Positive
Behaviour	Below the $x - axis$	Above the $x - axis$	Below the $x - axis$	Above the $x - axis$

3. To obtain a more accurate sketch, we can find the $y - intercept$ by finding the value of y that corresponds to $x = 0$ i.e., the $y - axis$. That value is -3

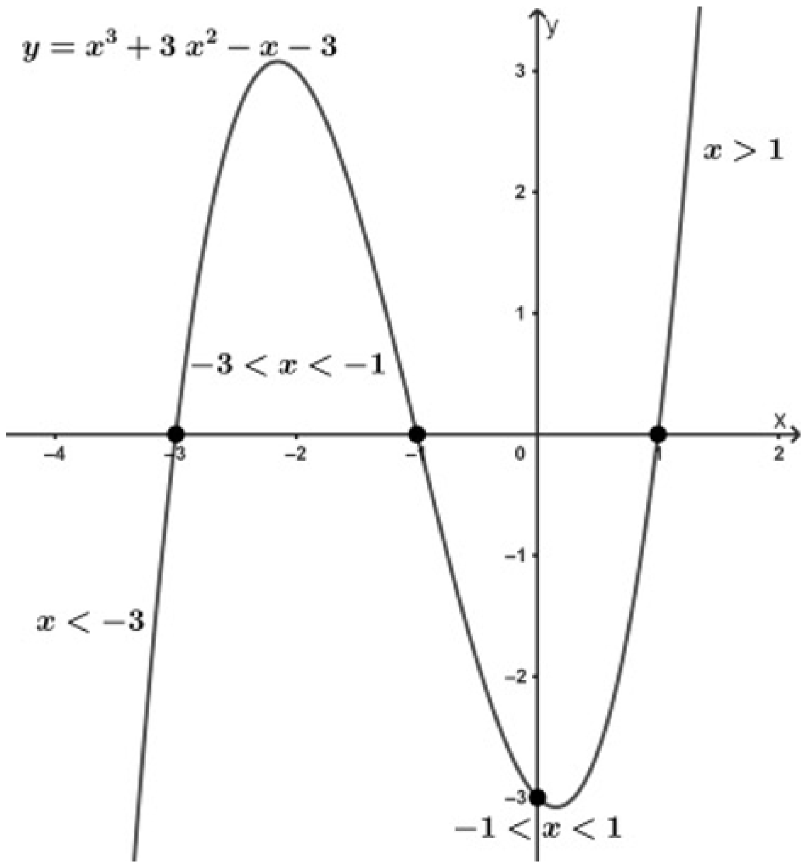


Figure 3.2

FOCAL AREA 3: APPLYING DESCARTES' RULE OF SIGNS THEOREM

Let $P(x)$ be a polynomial with real coefficients

1. The number of positive real zeros of $P(x)$ is either equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ is either equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Missing terms (those with 0 coefficients) are counted as no change in sign and can be ignored.

Example 3

Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 6x^3 + 17x^2 - 31x - 12$ can have. Then determine the possible total number of real zeros

Solution

$P(x) = 6x^3 + 17x^2 - 31x - 12$

$P(x)$ has one positive real zero, as there is only one sign change going across the equation. Therefore, $P(x)$ has one positive root.

Also; $P(-x) = -6x^3 + 17x^2 + 31x - 12$.

When we input $-x$ into the equation (see above), we can see two sign changes going across the equations. Therefore, $P(-x)$ with its two variations in sign could have 2 or 0 negative roots.

Combining the findings, $P(x)$ has either one or three real zeros

Positive	Negative	Real
1	0	1
1	2	3

FOCAL AREA 4: APPLYING THE FUNDAMENTAL THEOREM OF ALGEBRA

The fundamental theorem states that

1. Every polynomial equation with a degree higher than zero has at least one root in the set of complex numbers.
2. A polynomial equation of the form $P(x) = 0$ of degree 'n' with complex coefficients has exactly 'n' Roots in the set of complex numbers.

In the real number system, the square root function is defined only on non-negative numbers. It is why the domain of a radical function, say $g(x) = \sqrt{p(x)}$ is the set of real numbers for which $p(x) \geq 0$ as discussed in week 7 of the year one teacher manual.

Consider the equation, $x^2 - 16 = 0$

$x^2 - 16 = 0$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Again, consider the equation, $x^2 + 16 = 0$

$x^2 + 16 = 0$

$\Rightarrow x^2 = -16$

Since the value of x would be the square root of -16, a value which is not in the real number system, there is the need to introduce an imaginary number say i which will be defined as

$i = \sqrt{-1}$

$\Rightarrow i^2 = -1$

Then $\sqrt{-16}$ can be rewritten as $\sqrt{16} \times \sqrt{-1} = \pm 4i$.

An imaginary number is defined as ki were k is a real number

Example 4

Investigate whether if a polynomial has '*n*' complex roots, will its graph necessarily have '*n*' $x - intercepts$?

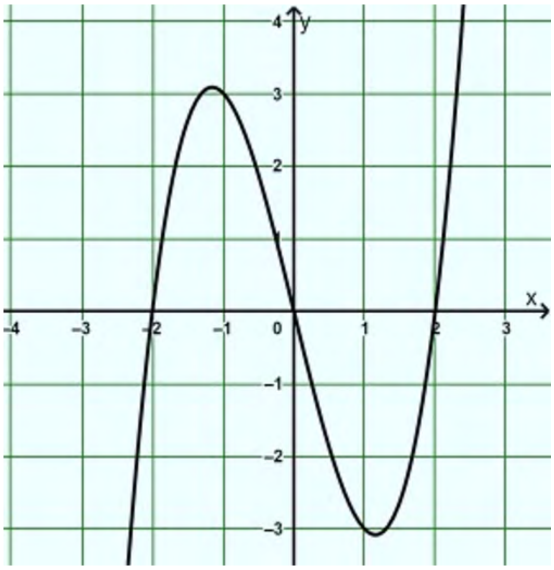


Figure 3.3

For the graph of $y = (x - 1)^3 + 3(x - 1) - (x - 1) - 3$ in Figure 3, the degree, $n = 3$. By observation, the roots are $x = -2$, $x = 0$ and $x = 2$. We can also see from the graph that there are three $x - intercepts$.

The graph of $y = x^{\,3} - 4x$ in Figure 3.4 however, though still with degree, $n = 3$, has only one real $x - intercept$ or root at $x = -1$. The other 2 roots must have *imaginary* components.

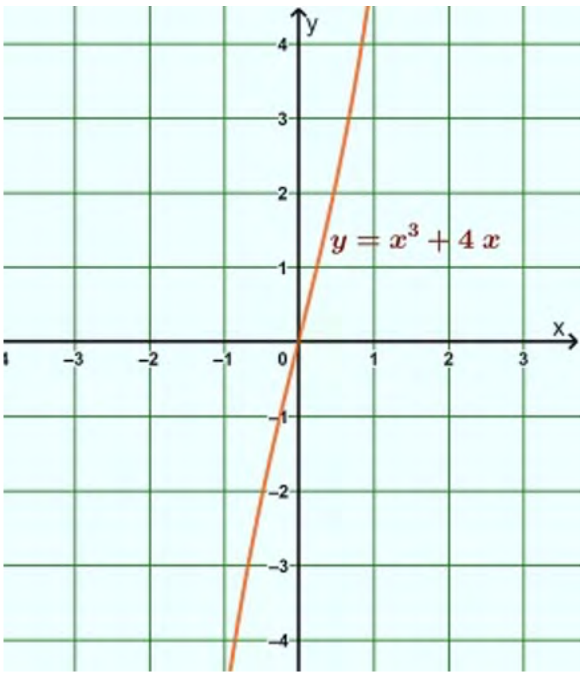


Figure 3.4

Conclusion: Just because a polynomial has '*n*' roots does not mean they are all Real!

Note 2

Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 + a_0$, $(n \geq 1, a_n \neq 0)$ with complex coefficients has at least one complex zero, hence for every polynomial $P(x)$, there is a complex number c_1 such that $P(c_1) = 0$. From the factor theorem, this tells us that $x - c_1$ is a factor of $P(x)$. Thus, we can write $P(x) = x{-}c1 \times Qx$, where $Q(x)$ has degree $n - 1$.

Example 5

Factorise the polynomial $P(x) = 4x^5 - 324x$ completely and find all its zeros. State the multiplicity of each zero.

Solution

$P(x) = 4x(x - 3)(x + 3)(x^2 + 9)$

$= 4x(x - 3)(x + 3)(x^2 - (-9))$

$= 4x(x - 3)(x + 3)(x - 3i)(x + 3i)$

Therefore, the zeros of P are 0, 3, -3, $3i$ and $-3i$. Since each factor occurs only once, all the zeros are of multiplicity 1, and the total number of zeros is five.

FOCAL AREA 5: COMPLEX CONJUGATES THEOREM

The theorem states that roots / zeros that are not real are complex with an imaginary component. Complex roots with imaginary components always exist in conjugate pairs.

It follows that if $a+bi$, $b \neq 0$ is a zero of a polynomial function, then its conjugate, $a - bi$, is also a zero of the function.

Example 6

Find all the roots of $f(x) = x^3 - 5x^2 - 7x + 51$. If one root is $4 - i$.

Solution

Applying the Complex Conjugate Theorem, and Descartes' rules we have the

factor $[x - (4 - i)]$, and $[xc - (4 + i)]$.

The product of factors gives $[x - (4 - i)]. [x - (4 + i)] = x^2 - 8x + 17$

Since the product of the two non-real factors is $x^2 - 8x + 17$, then the third factor (that gives us the negative real root) is the quotient of $P(x)$ divided by $x^2 - 8x + 17$, which is $x = -3$

Therefore, the roots of $P(x)$ are -3, $-4i$ and $4i$.

FOCAL AREA 6: LINEAR AND QUADRATIC FACTOR THEOREMS

Example 7

Given the polynomial $P(x) = x^4 + 2x^2 - 63$,

a. Factor P into linear and irreducible quadratic factors with real coefficients.

b. Factor P completely into linear factors with complex coefficients.

Solution

a. $P(x) = x^4 + 2x^2 - 63$

$$= x^4 + 2x^2 + 1^2 - 63 - 1^2$$
$$= (x^2 + 1)^2 - 64$$
$$= [(x^2 + 1) - 8][(x^2 + 1) + 8]$$
$$= (x^2 - 7)(x^2 + 9)$$
$$= (x - \sqrt{7})(x + \sqrt{7})(x^2 + 9)$$

b. $P(x) = (x - \sqrt{7})(x + \sqrt{7})(x - 3i)(x + 3i)$

Learning Tasks

Teacher to ask learners to perform the following to check for understanding:

1. Find factors and zeroes of polynomial functions.
2. Use graphs to solve polynomial functions with degrees higher than 2.
3. Establish and apply Descartes' rule of signs theorem.
4. Use graphs to establish and apply the fundamental theorem of algebra.
5. Explore the Complex conjugates theorem to solve real-world problems.
6. Explore the Linear and quadratic factor theorems.

PEDAGOGICAL EXEMPLARS

1. **Review previous knowledge:** In a whole class discussion, review with learners how to explore by hand and use ICT tools like GeoGebra to identify the solutions or roots of polynomial equations.
2. **Whole Class discussions and demonstrations:** Lead the class to investigate and identify the possible rational zeros of a polynomial function. Using the Rational Zeros Theorem, learners determine the zeros, factors, and roots of the polynomial, and verify their findings with a graph.

3. **Group and pair activities:** Using think-pair share and problem-based learning, engage learners in graphing polynomial functions with degrees higher than 2. They collaborate to explore, share ideas, and solve problems, fostering deep understanding of polynomial graph behaviours.
4. **Problem-based group learning:** In small and manageable groups, engage learners to discuss and explore how to apply Descartes' Rule of Signs, exploring how it helps predict the nature of the roots of a polynomial function.
5. **Problem-based group learning:** Using mixed-ability groups, guide learners to work together to apply the Fundamental Theorem of Algebra graphically. They will investigate and identify the zeros, factors, and roots of a polynomial function.
6. **Think-pair activities:** Using think-pair share/square, learners explore the Complex Conjugate Theorem to solve problems involving the zeros, factors, and roots of polynomial functions and present to the class.
7. **Whole class discussions and presentations:** Lead the class to establish and prove that any polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors, all with real coefficients.
8. **Individual tasks:** resent learners with individual worksheets to complete.

KEY ASSESSMENTS

Assessment Level 2: Skills of conceptual understanding

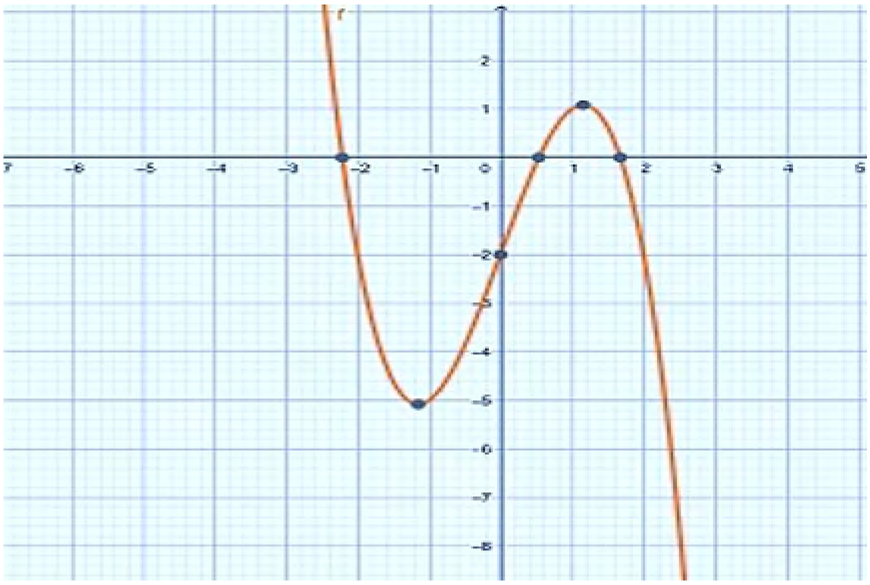


Figure 3.5

1. The graph shows a polynomial function. Study it carefully and answer the questions that follow.
   a. Write down the zeros of the polynomial.
   b. Identify the roots or solution for the polynomial.

  c. State the factors of the polynomial.

  d. Write the polynomial function for the graph.

2. Find the maximum number of positive and negative real zeros of the polynomial function $f(x) = 2x^4 + x^3 - 6x^2 - 7x + 1$.

3. Find all zeros of the polynomial $P(x) = x^4 + x^3 + 3x^2 - 5x$

4. Verify that $(x - 3)$ is a factor of $P(x) = 2x^3 - 3x^2 - 4x - 5$, and write $P(x)$ as the product of $(x - 3)$ and the reduced polynomial $Q(x)$

Assessment level 3: Strategic Reasoning

A software company produces a computer game. The company has determined that its profit P, in Ghanaian cedis, from the manufacture and sale of x games is given by $P(x) = 0.25x^3 - 2x^2 - 60x$, where $0 \leq x \leq 30$. What is the maximum loss, to the nearest hundred Ghanaian cedis, the company can expect from the sale of its game?

Section 3 Review

This section covered essential concepts in polynomial functions. Learners learnt to find factors and zeros, graph polynomials with degrees higher than 2, and apply Descartes' Rule of Signs to predict the number of positive and negative roots. The Fundamental Theorem of Algebra is used to establish that every polynomial equation has at least one complex root. The Complex Conjugates Theorem is explored to understand the occurrence of complex roots in pairs. Finally, the Linear and Quadratic Factor Theorems help students factorize polynomials into linear and irreducible quadratic factors, providing a comprehensive understanding and application of polynomial behaviour and properties.

SECTION 4: CIRCLES AND LOCI

Strand: Geometric Reasoning and Measurement

Sub-Strand: Spatial sense

Learning Outcomes

1. *Deduce the equation of a circle and find its centre and radius.*
2. *Determine the equation of a locus under a given condition.*

Content Standard: Demonstrate understanding of loci and their applications.

INTRODUCTION AND SECTION SUMMARY

Exploring the equation of a circle and its properties delves into the fundamental geometry defined by the set of points equidistant from a central point (the centre) in a plane. This section shall look into the equation $(x-h)^2+(y-k)^2=r^2$ which characterizes the circle, where (h, k) represents the centre coordinates and r denotes the radius. This equation is pivotal in geometry and calculus, serving as a basis for understanding geometric transformations and spatial relationships. We shall derive the equation of a circle using various methods, such as completing the square or using distance formulas, showcasing its connection to algebra and analytical geometry as the application of these techniques enables precise placement of circles in Cartesian coordinates and facilitates geometric proofs and constructions in mathematics. There will be further exploration into deriving circle equations from given points, crucial for solving geometric problems and designing spatial layouts in architecture and engineering. The latter part of the section shall be dedicated to understanding tangent and normal lines to a circle as these concepts extend into calculus, providing insights into rates of change and optimization in physics and engineering applications. Learners will have the opportunity to deduce relationships among loci under specific conditions connects to broader mathematical concepts such as conic sections and transformations, fostering connections to physics (e.g., orbital mechanics) and computer graphics (e.g., modelling curves).

The weeks covered by the section are:

Week 6

1. Exploring the properties of a circle and its parts.

2. Deriving the equation of a circle.

Week 7

1. Deriving the equation of a circle from the endpoints of a diameter.
2. Tangents and Normals.
3. Deducing relation of various loci under given conditions.

SUMMARY OF PEDAGOGICAL EXEMPLARS

The Section 4 of the Year 2 Teacher Manual emphasizes interactive, experiential learning strategies to teach circles effectively. Learners will actively participate in constructing circles using traditional tools such as rulers, pencils, and compasses, supplemented by modern ICT tools like GeoGebra. Activities are designed to cater to diverse learning styles and abilities, promoting engagement through collaborative team projects and mixed-group settings. Practical exercises include identifying and constructing circles in real-world contexts, fostering a tangible understanding of their applications. By tackling real-life circle problems collaboratively, learners enhance their critical thinking and problem-solving skills. Learners will delve into the components, properties, and equations of circles through hands-on exploration. Tasks involving tangents and normals which provide practical insights into geometric relationships and their relevance in fields such as architecture, engineering, and navigation will be treated. Additionally, learners will deduce loci under specific conditions, connecting theoretical concepts to practical scenarios.

ASSESSMENT SUMMARY

The various concepts to be covered under this section should be assessed using various forms of assessments modes to ascertain learners' performance. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP). The following mandatory assessments would be conducted and recorded for each learner:

Week 6: *Mid-semester Examination:* *Teachers are supposed to administer mid-semester examination covering all the concepts taught from Week 1.*

Note

For additional information on the internal assessments, refer to the Hints and Reminders sign-posted at the end of the various weeks.

WEEK 6

Learning Indicators

1. *Explore the equation a circle and its properties using technological tools, e.g. GeoGebra, Geometer's sketchpad, paper cutting, Geoboard and other realia*
2. *Apply knowledge of the distance between two points and the Pythagoras theorem to describe a circle in the algebraic form*

FOCAL AREA 1: EXPLORING THE PROPERTIES OF A CIRCLE AND ITS PARTS

It is important to describe a circle as a path/line created by a set of points which are all of equal distance from a fixed point. It is expected that no three points on a circle are collinear (are on the same straight line). A circle is described as radially symmetrical as any diameter constructed in a circle is a line of symmetry i.e., every diameter divides the circle into two equal halves. Circles are such that they are all identical except for perhaps, the magnitude of the radius

A circle can be constructed either by hand (using a pair of compasses and a ruler) or with the aid of a graphing/construction software such as GeoGebra if either of the following properties are provided

1. The location of the centre and the magnitude of the radius
2. The location of the centre and the location of a point on the circumference of the circle
3. The locations of three points on the circumference of the circle

Parts of a circle

Figure 4.1 shows a circle whose centre is located at $A = (-6, -2)$ and with B, C, D, E and F as points on the circumference. The circle touches the $x - axis$ at $(-6, 0)$ so we can get another point on the circle in that point of intersection. We can deduce the radius of the circle by finding $|AB|$ or $|AC|$ as we can easily read the coordinates of B and C or by finding the distance between $(-6, 0)$ as it also lies on the circumference, and A, the centre of the circle. Either distance will result in 2 *units* and that is the radius of the circle.

The part of the circle which lies along the circumference between any two points on the circumference for example, C and D, B and F or C and E is an arc and can be named using the points on the ends as in "arc *CD*", "arc *BF*" or "arc *CE*" respectively.

The portion of the area inside a circle which is bounded by a chord and an arc is called a segment of the circle while the area bounded by two radii and an arc is called a sector.

Figure 4.1 shows a segment from the circle, bounded by chord *CD* and the circumference as well as a sector, *BAF*

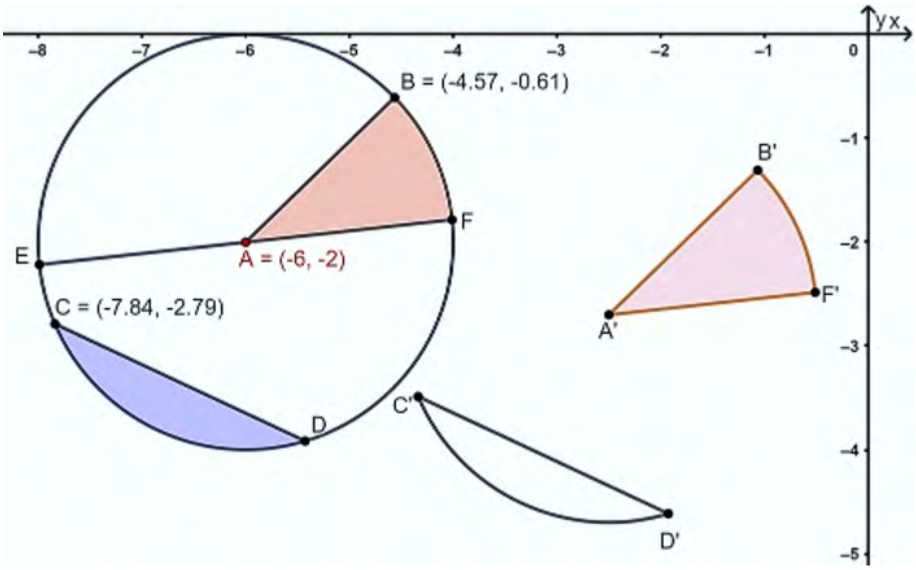


Figure 4.1

FOCAL AREA 2: DERIVING THE EQUATION OF A CIRCLE

The equation of a circle could be in the standard or general form. To derive the standard form of the equation of a circle,

1. Learners can construct a circle and label the centre as $O(h, k)$ or using any constants of their choice and indicate any point, for example, $P(x, y)$on the circumference of the circle

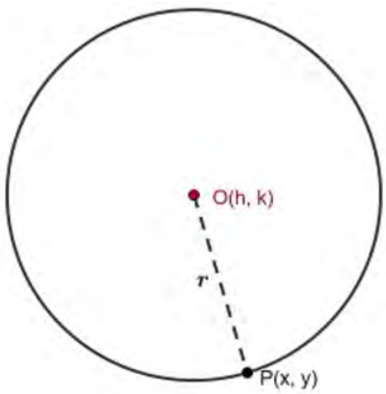


Figure 4.2

2. Learners can then apply their understanding of finding the distance between two points to state the equation of distance between O and P, which is the radius, r of the circle.

   $|OP| = r = \sqrt{(x - h)^2 - (y - k)^2}$

   $r^2 = (x - h)^2 - (y - k)^2$

if the centre of the circle is located at the origin i.e., (0, 0), $h = 0$ and $k = 0$ and we have the equation of the circle to be

$r^2 = (x - h)^2 - (y - k)^2$

$r^2 = (x - 0)^2 - (y - 0)^2$

$r^2 = x^2 + y^2$

Example 1

Write the standard form of the equation of a circle with centre (2,4) and radius of 10.

Solution

$(x - 2)^2 + (y - 4)^2 = 10^2$

Example 2

A goat is tethered by a rope of length 2 metres away from a peg. If the peg is at the point (3, 2). Find an equation to describe the boundary of the area the goat can go round the peg.

Solution

The boundary of the area will be a circle hence we can deduce that the radius will be 2 *metres* and the centre of the circle located at (3, 2)

The equation of the circle and thus the boundary will be $(x - 3)^2 + (y - 2)^2 = 4$

The general equation of a circle

An expansion of the standard equation of a circle provides its general equation.

$r^2 = (x - h)^2 - (y - k)^2$

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$

Hence, the general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$,

Where g replaces $-h, f$ replaces $-k$ and $c = h^2 + k^2 - r^2$ since h, k *and* r are numbers.

It is important to note the following about the general equation of a circle

1. The degree of the equation is two

2. The coefficients of the x^2 and y^2 terms are equal

Example 3

Write the general equation of a circle with a centre (2,4) and a radius of 10.

Solution

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Since the location of the centre is (2, 4), $g = -h = -2$ and $f = -k = -4$

$c = h^2 + k^2 - r^2 = 2^2 + 4^2 - 10^2 = -80$

$x^2 + y^2 + 2(-2)x + 2(-4)y + c = 0$

$x^2 + y^2 - 4x - 8y - 80 = 0$

Example 4

The point (4,6) is on a circle whose centre is (1,2). Write a general equation of the circle.

Solution

$r = \sqrt{(4-1)^2 + (6-2)^2}$

$r = 5$

Using $r = 5$ and centre (1,2), we can write the standard equation of the circle and then expand to get the general equation

$(x-1)^2 + (y-2)^2 = 5^2$

$x^2 - 2x + 1 + y^2 - 4y + 4 = 25$

$x^2 + y^2 - 2x - 4y - 17 = 0$

Example 5

Find the centre and the radius of the circle with the equation

$x^2 + y^2 - 14x + 16y - 12 = 0$

Solution

Using completing the square method

$x^2 + y^2 - 14x + 16y - 12 = 0$

$x^2 - 14x + y^2 + 16y - 12 = 0$

$x^2 - 14x + \left(\frac{14}{2}\right)^2 + y^2 + 16y + \left(\frac{16}{2}\right)^2 - 12 - \left(\frac{14}{2}\right)^2 - \left(\frac{16}{2}\right)^2 = 0$

$x^2 - 14x + 7^2 + y^2 + 16y + 8^2 - 12 - 7^2 - 8^2 = 0$

$(x-7)^2 + (y+8)^2 - 12 - 7^2 - 8^2 = 0$

$(x-7)^2 + (y+8)^2 = 125$

Comparing the equation obtained to the standard equation reveals that $h = 7$, $k = -8$ and $r^2 = 125 \Rightarrow r = 5\sqrt{5}$

Therefore, the centre is located at $(7, -8)$, and the radius is $5\sqrt{5}$ *units*

Alternative method

We can right away compare $x^2 + y^2 - 14x + 16y - 12 = 0$ to the general equation of a circle i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ and obtain

$2g = -14$ and $2f = 16$

$\Rightarrow g = -7$ and $f = 8$

Since the centre of the circle is $(-g, -f)$, the centre of this circle is $(7, -8)$

The radius of the circle is found thus

$r^2 = g^2 + f^2 - c$

$r^2 = 7^2 + (-8)^2 + 12$

$r = \sqrt{125} = 5\sqrt{5}$ *units*

Learning Tasks

Teacher to ask learners to perform the following tasks to check for understanding. Please cater for differentiation.

1. Explore the properties of a Circle and its parts.
2. Derive the equation, standard equation and general equation of a Circle.
3. Use completing of squares or alternative method to find the centre and radius of Circles.

PEDAGOGICAL EXEMPLARS

Teacher should consider the following activities:

1. **Reviewing Previous Knowledge:** Begin by assessing learners' prior knowledge of circles from Junior High School. Discuss common references to circles and address any misconceptions that may arise.
2. **Experiential Learning:** Facilitate hands-on experiences where learners, in mixed-ability and gender groups, use technological tools like GeoGebra or traditional drawing instruments to construct circles. Task them with identifying and describing key properties such as the centre, radius, diameter, chord, sector, arc, and segment.
3. **Problem-Based Learning:** Divide learners into small, manageable groups and assign them the task of describing the geometric construction of a circle and deriving its algebraic equation. Encourage discussion within groups and share their findings with the class, highlighting the diversity of circles constructed and equations derived.
4. **Whole Class Discussions and Demonstrations:** Convene a whole-class discussion to collectively brainstorm and explore the derived equations of circles. Engage

learners in demonstrating how these equations are derived, encouraging them to articulate their reasoning and methods.

5. **Group and Pair Activities:** Utilize strategies like think-pair-square/share to deepen understanding. Task learners with discussing and deriving the standard form of the circle's equation collaboratively. Provide practical problems for them to model and solve, applying their knowledge to determine the centre and radius of circles through methods such as completing the square or alternative algebraic techniques.
6. **Individual tasks:** Present learners with individual worksheets to complete. You may also ask learners to pose some questions for their friends to solve. Take time to deal with common misconceptions that learners are likely to have.

KEY ASSESSMENTS

Assessment Level 2: Skills of conceptual understanding

Identify the locations of the centre and find the radius of the circles with equations

a. $x^2 + y^2 - 4x - 6y + 9$

b. $x^2 + y^2 - 4x + 10y + 20 = 0$

Assessment Level 3: Strategic Reasoning

A circle can be drawn on a piece of paper by fastening one end of a string with a thumbtack, pulling the string taut with a pencil and tracing a curve. Explain why this method works.

WEEK 7

Learning Indicators

1. *Apply knowledge of properties of lines to derive the equations of a circle, a tangent and normal to a circle*
2. *Deduce relations of various loci under given conditions*

FOCAL AREA 1: DERIVING THE EQUATIONS OF A CIRCLE FROM GIVEN POINTS

Deriving the equation of a circle from the endpoints of a diameter

Learners are reminded of the circle theorem that a diameter subtends an angle of 90° at the circumference of a circle and the relationship between the gradients of two perpendicular lines i.e., if m_1 and m_2 are the gradients of two perpendicular lines, $m_1 m_2 = -1$.

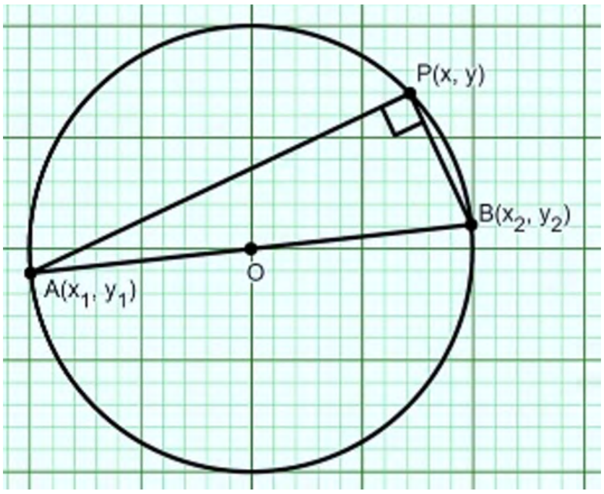


Figure 4.3

From Figure 4.3, $A(x_1, y_1)$ and $B(x_2, y_2)$ are points on the circumference of the circle such that the line segment, $|AB|$ passes through the centre, O and thus, $|AB|$ is a diameter to the circle. Since the point $P(x, y)$, representing an arbitrary point is located on the circumference, AP is perpendicular to BP and hence the product of the gradient of $|AP|$, say m_1 and the gradient of $|BP|$, say m_2 is -1.

Let the gradient of $|AP| = m_1 = \frac{y - y_1}{x - x_1}$ and

gradient of $|BP| = m_2 = \frac{y - y_2}{x - x_2}$

$m_1 \times m_2 = -1$

$\frac{y-y_1}{x-x_1} \bullet \frac{y-y_2}{x-x_2} = -1$

$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

$(y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$

The equation obtained is sufficient but can be simplified further by expansion to get the general form

$x^2 - xx_1 - xx_2 + y^2 - yy_1 - yy_2 + x_1x_2 + y_1y_2 = 0$

$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$

Since x_1, x_2, y_1 and y_2 are constants, the values of $(x_1 + x_2)$, $(y_1 + y_2)$ and $x_1x_2 + y_1y_2$ will be constants too. They can therefore be represented by $2g$, $2f$ and c respectively thereby obtaining

$x^2 + y^2 + 2gx + 2fy + c = 0$

Note that, this is only an attempt to prove that $(y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$ is a variant of the equation of a circle. When learners need to find the equation, they can simplify the equation they obtain after equating the product of the expressions they obtain as the gradients of the perpendicular chords.

A learner might also wish to find the coordinates of the centre of the circle, find the magnitude of the radius and substitute the results into the standard form of the equation of a circle. It must be noted that the centre of a circle is located at the midpoint of a diameter.

From Figure 4.3, $O = midpoint\ of\ |AB| = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

We can let $h = \frac{x_1 + x_2}{2}$ and $k = \frac{y_1 + y_2}{2}$

$Radius,\ r = \sqrt{(h - x_1)^2 + (k - y_1)^2}$ or $r = \sqrt{(h - x_2)^2 + (k - y_2)^2}$

The results can then be substituted into $(x - h)^2 + (y - k)^2 = r^2$

Example 1

Find the equation of a circle through the ends (5,7) and (1,3) of its diameter and find the centre and radius.

Solution

Equation of the circle

Using $(y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$,

$(x - 5)(x - 1) + (y - 7)(y - 3) = 0$

$x^2 + y^2 - 6x - 10y + 26 = 0$

Centre

$2g = -6$ and $2f = -10$, Therefore the centre of the circle is (3,5)

Radius

$r^2 = 3^2 + 5^2 - 26$

$r = \sqrt{8}$

Alternatively,

$$\textit{Centre} = \textit{Midpoint of the diameter} = \left(\frac{5+1}{2}, \frac{7+3}{2}\right)$$

$$= (3, 5)$$

$\textit{Radius},\ r = \sqrt{(5-3)^2 + (7-5)^2} = \sqrt{4+4} = \sqrt{8}$

Using $(x-h)^2 + (y-k)^2 = r^2$

$(x-3)^2 + (y-5)^2 = 8$

Equation of a Circle given three points

Every point that lies on the circumference of a circle satisfies the equation of the circle. If the coordinates of a point that lie on the circumference of a circle are substituted into the equation, there should be an identity.

To find the equation of a circle given the coordinates of three points that lie on the circumference, the pairs of coordinates can be substituted into either the general equation or standard equation of a circle resulting equations simultaneously to obtain the values of the constants g, f and c or h, k and r respectively.

Example 2

Find the equation of the circle passing through the points; (1,3), (−1,5) and (− 1,1).

Solution

Using the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$,

For (1, 3),

$1^2 + 3^2 + 2g(1) + 2f(3) + c = 0$

$2g + 6f + c = -10$

For (−1, 5),

$2g - 10f - c = 26$ and

For (−1, 1),

$2g - 2f - c = 2$

Solving the system simultaneously,

$g = 1, f = -3, c = 6$

Therefore, the general equation of the circle is $x^2 + y^2 + 2x - 6y + 6 = 0$

Using the standard equation $(x - h)^2 + (y - k)^2 = r^2$,

For (1, 3),

$(1 - h)^2 + (3 - k)^2 = r^2$

$h^2 + k^2 - 2h - 6k + 10 = r^2$

For (−1, 5),

$(-1 - h)^2 + (5 - k)^2 = r^2$

$h^2 + k^2 + 2h - 10k + 26 = r^2$ and

For (−1, 1),

$(-1 - h)^2 + (1 - k)^2 = r^2$

$h^2 + k^2 + 2h - 2k + 2 = r^2$

Solving the equations simultaneously,

$h = -1, k = 3, r = 2$

Therefore, the standard equation of the circle is $(x + 1)^2 + (y - 3)^2 = 4$

Alternatively,

We can find the point of intersection of the perpendicular bisectors of at least two line segments that join the points on the circumference on the circle. And a property of circle confirms that, that point of intersection is the centre of the circle. The procedure is shown in Figure 4.4.

We find the midpoint of $|PQ|$, M_{PQ}

$M_{PQ} = \left(\frac{-1+1}{2}, \frac{5+3}{2}\right) = (0, 4)$

Then we find Gradient of $|PQ| = \frac{5-3}{-1-1} = -1$

$\Rightarrow$ Gradient of the perpendicular bisector of $|PQ|$, $l_1 = 1$

$\therefore$ the equation of l_1 will be $(y - 4) = -1(x - 0)$ or $y = x + 4$

We can also find the midpoint of $|QR|$, M_{QR}

$M_{QR} = \left(\frac{-1+1}{2}, \frac{1+3}{2}\right) = (0, 2)$

The we find Gradient of $|QR| = \frac{1-3}{-1-1} = 1$

$\Rightarrow$ Gradient of the perpendicular bisector of $|QR|$ $l_2 = -1$

$\therefore$ the equation of l_2 will be $(y - 2) = -1(x - 0)$ or $y = 2 - x$

l_1 and l_2 can be solved simultaneously to obtain the point of intersection as $(-1, 3)$

The centre of the circle is thus located at $C(-1, 3)$.

The radius can be found thus: $|CR| = \sqrt{(-1 - (-1))^2 + (3 - 1)^2} = 2$ *units* and hence the equation of the circle is $(x + 1)^2 + (y - 3)^2 = 4$

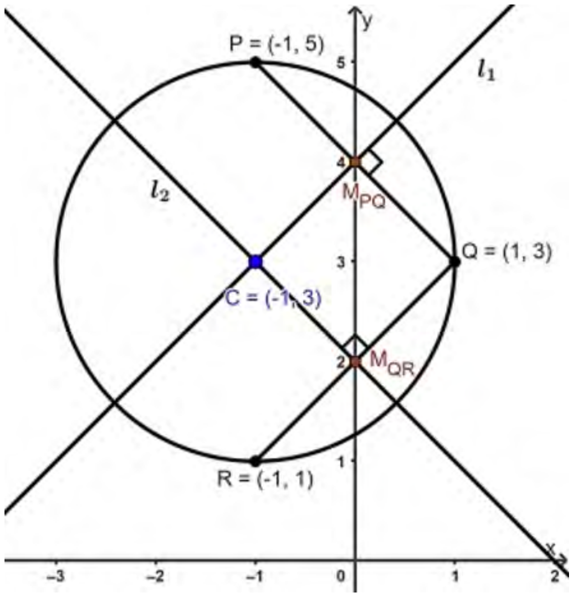


Figure 4.4

FOCAL AREA 2: TANGENTS AND NORMALS

A tangent to a circle is a straight line that touches the circumference of a circle at one point only. According to circle theorems, a tangent is always perpendicular to the radius at the point of contact (point of tangency).

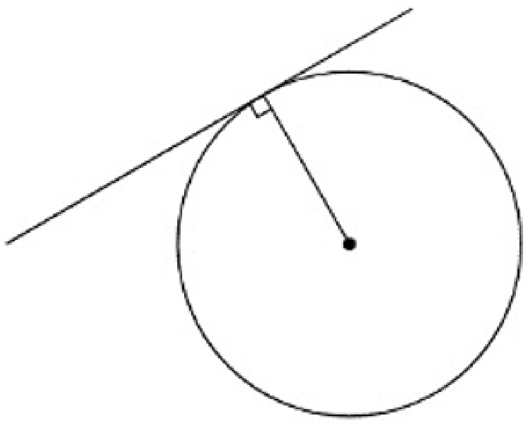


Figure 4.5

Example 3

Find the equation of the tangent through (3,4) and on the circle $x^2 + y^2 = 25$

Solution

Learners verify that the point lies on the circle.

$3^2 + 4^2 = 25$

Therefore, (3,4) lies on the circle.

From the equation, learners deduce that the circle has its centre at the origin, (0,0) and the radius is 5.

Coordinates to be used to find the slope of the radius of the circle are (0,0) and (3,4)

The slope of the radius $= \frac{4-0}{3-0} = \frac{4}{3}$

Since the tangent is perpendicular to the radius, the radius of the tangent is $-\frac{3}{4}$

The equation of the tangent is given as $y = mx + c$, (3,4)

Substituting $m = -\frac{3}{4}$, $x = 3$ and $y = 4$ into $y = mx + c$ results in

$4 = -\frac{3}{4}(3) + c$

$c = \frac{25}{4}$

Therefore, the equation of the tangent is $y = -\frac{3}{4}x + \frac{25}{4}$

Length of a tangent

There are situations when we are interested in finding the length of a portion of a tangent between a point outside of a circle and a point on the circle. Figure 4.6 shows a circle centred at $C(a, b)$ with a point on the circle, being the point of tangency, T and an external point, $P(x, y)$. The radius of the circle is represented by r, the distance between the centre and the external point represented by d and the length of the portion of interest of the tangent represented by l.

Circle theorems indicate that the radius, $|CT|$ is perpendicular to the tangent, $|PT|$ and thus, a right triangle, ΔCPT is formed.

From the Pythagoras theorem regarding the lengths of the sides of a right triangle, $d^2 = r^2 + l^2$ and consequently, $l = \sqrt{d^2 - r^2}$.

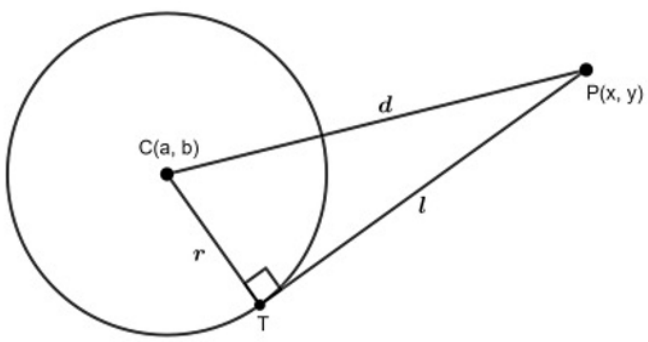


Figure 4.6

Example 4

Find the length of the tangent from the point (2,8) to the circle $x^2 + y^2 + 4x - 10y + 20 = 0$.

Solution

Learners collaborate in their groups to find the centre and radius of the circle as $(-2,5)$ and 3, respectively.

Learners use $(-2,5)$ and (2,8) to find the length of the hypotenuse as 5.

Learners apply Pythagoras theorem to find the length of the tangent as 4.

Normal to a circle

The normal to a circle at a point is the line perpendicular to the tangent and passing through the point of tangency. Since the normal and tangent lines are perpendicular to each other, the products of their gradient is -1 and the equation of the normal is of the form $y = mx + c$.

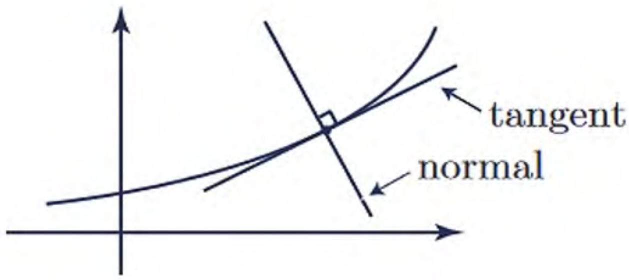


Figure 4.7

Example 5

Find the equation of the normal to the circle $x^2 + y^2 = 5$ that passes through (6, 2).

Solution

The circle has its origin at the centre; therefore, the coordinates at the centre is (0,0).

The gradient of the normal/radius is $\frac{2-0}{6-0} = \frac{1}{3}$.

To find c,

$y = mx + c$

$2 = \frac{1}{3}(6) + c$

$c = 0$

Therefore, the equation of the normal is $y = \frac{1}{3}x$

FOCAL AREA 3: DEDUCING RELATION OF VARIOUS LOCI UNDER GIVEN CONDITIONS

A locus is the set of points of that meet a criterion.

Condition 1: Locus of points equidistant from the centre

A circle is a locus because all points on the circumference of a circle are of equal distance from the centre of the circle. The criterion in this example is "of equal distance from the centre". As discussed earlier, all points on the circumference of a circle satisfy the equation of the circle. This is so because they meet the criterion.

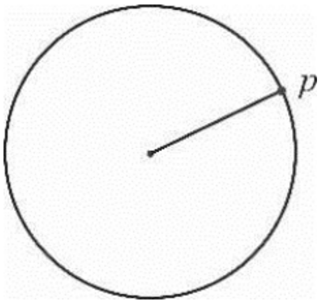


Figure 4.8

P represents any point on the circumference of the circle. We can also say that P represents any point which is r units away from the centre of the circle. We therefore expect that the equation for this locus will be of the form, $(x - h)^2 + (y - k)^2 = r^2$ where h and k represent the x and y coordinates of the centre respectively or $x^2 + y^2 + 2gx + 2fy + c = 0$ where $-g$ and $-f$ are the x and y coordinates of the centre of the circle as discussed earlier

Condition 2: Locus equidistant from two fixed points

The locus of points equidistant from two fixed points is the perpendicular bisector of the line segment between the two fixed points. Since it is also a straight line, we expect that its equation will be of the form $y = mx + c$. It passes through the midpoint of the line segment and its gradient is the negative reciprocal of the gradient of the line segment between the two fixed points. Figure 4.9 shows the locus of points equidistant from A and B which is represented by the perpendicular bisector

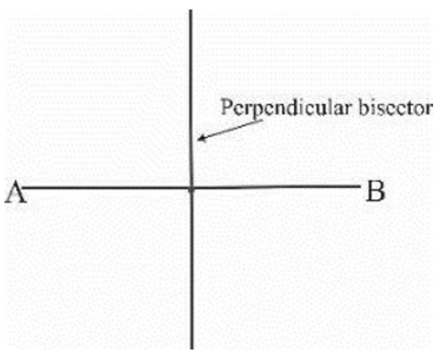


Figure 4.9

Condition 3: Locus from two fixed lines

The locus of points equidistant from two fixed lines is an angle bisector, and for any point lying on the angle bisector, the perpendicular distance is equidistant from the two lines.

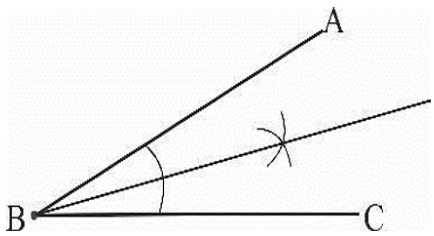


Figure 4.10

Condition 4: Locus equidistant from a line

The locus of points equidistant from a line, *AB* is a parallel line.

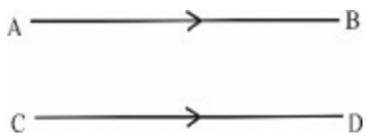


Figure 4.11

Example 6

Find the equation of the locus of a moving point $P(x, y)$, which is always at a distance of 3 units from a fixed point $Q(1,2)$.

Solution

$(x - 1)^2 - (y - 2)^2 = 3$

$x^2 + y^2 - 2x + 4y - 12 = 0$

Example 7

Find the equation of a locus of a point that is equidistant from the points $A(-2,0)$ and $B(3,2)$.

Solution

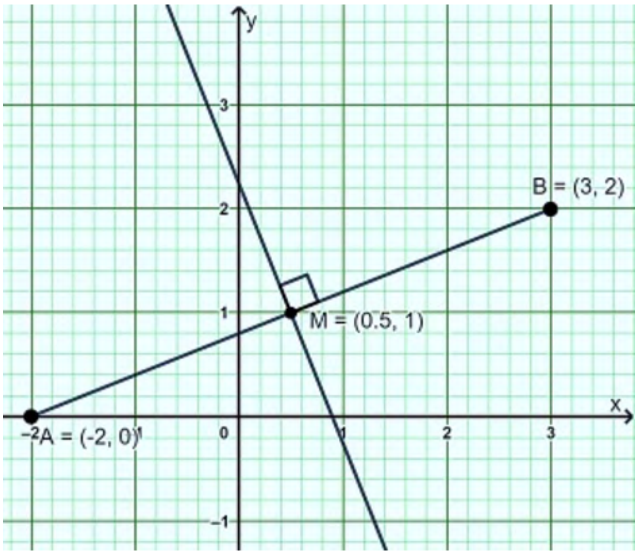


Figure 4.12

Figure 4.12 shows the points $A = (-2, 0)$ and $B(3, 2)$. One of the points that is equidistant from A and B is the midpoint of the line segment $|AB|$. In this example, the midpoint is $M = (0.5, 1)$ hence the required locus is the perpendicular bisector of $|AB|$.

Let the point on the locus be $P(x, y)$.

$$\sqrt{(x+2)^2 + (y-0)^2} = \sqrt{(x-3)^2 + (y-2)^2}$$

$$10x + 4y - 9 = 0$$

Example 8

If $A(2, 0)$ and $B\ (0, -2)$ are two fixed points and point P moves with a ratio so that $|AP|:|BP| = 1:3$. Find the equation of the locus of point

Solution

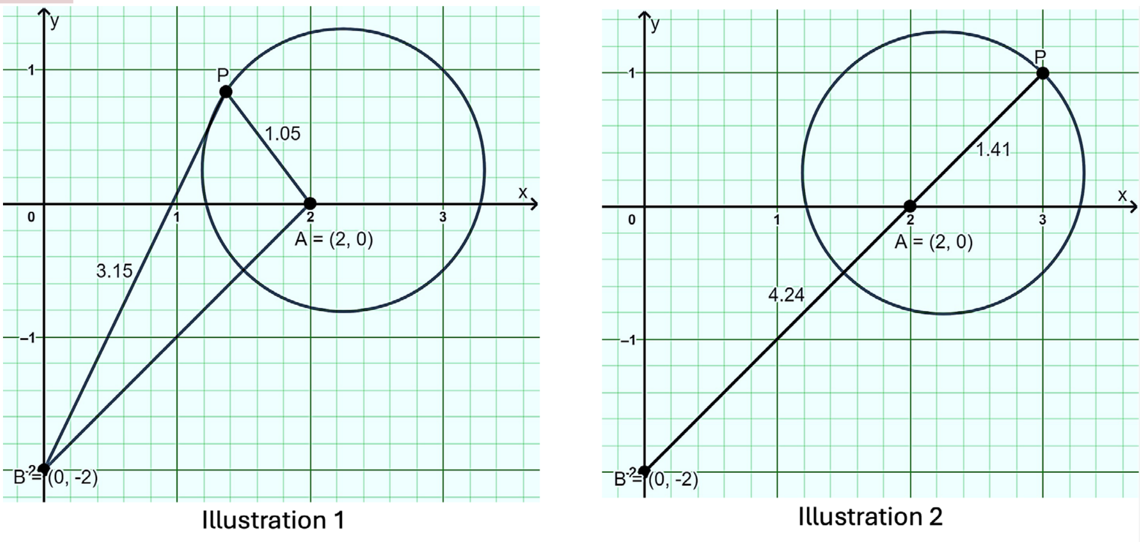


Figure 4.13

Illustrations 1 and 2 in Figure 4.13 shows different positions of P however, $|AP|$ and $|BP|$ in both diagrams are in the ratio, 1:3

Let $P = (x, y)$

$\frac{|AP|}{|BP|} = 1\frac{}{3}$

$3|AP| = |BP|$

$3\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-0)^2 + (y+2)^2}$

$\left(3\sqrt{(x-2)^2 + (y-0)^2}\right)^2 = \sqrt{(x-0)^2 + (y+2)^2}^2$

$9[(x-2)^2 + (y-0)^2] = [(x-0)^2 + (y+2)^2]$

$8x^2 + 8y^2 - 36x - 4y + 32 = 0$

Example 9

A point $P(x, y)$ moves so that AP and BP are perpendicular. Given that $A(1,2)$ and $B(2,4$) find the equation of the locus P.

Solution

The product of slopes of perpendicular lines is -1, therefore $\left(\frac{y-2}{x-1}\right)\left(\frac{y-4}{2-x}\right) = -1$

The equation of the locus P is

$x^2 + y^2 - 3x - 6y + 10 = 0$

Example 10

What is the locus of point $P(x, y)$ that is always 2 units from the line $x = 4$?

Solution

$A(4, y)$, $P(x, y)$ and the constant distance $(d) = 2$

$|AP| = \sqrt{(x-4)^2 + (y-y)^2} = 2$

$x^2 - 8x + 12 = 0$

$x = 2 \text{ } or \text{ } x = 6$

Example 11

$A(4,5)$ and $B(-2,7)$ are given points. Find the equation of the line such that $2PA = PB$.

Solution

$P(x, y)$, $A(4,5)$ and $B(-2,7)$

Given condition is $2PA = PB$

$\therefore 4PA^2 = PB^2$

$4[(x-4)^2 + (y-5)^2] = (x+2)^2 + (y-7)^2$

The equation of the locus is

$3x^2 + 3y^2 - 36x - 26y + 111 = 0$

Learning Tasks

Learners are expected to be able to perform the following tasks after going through the activities for the week

1. Derive the equation of a circle given its diameter's endpoints or three points.
2. Discover the equation of a tangent to a given circle
3. Find the length of a tangent to a given circle from an external point.
4. Apply their knowledge of Pythagoras theorem and find the distance between two points to find the length of the tangent.
5. Establish an equation of a normal to a given circle.
6. Deduce the equation of a locus under given conditions

PEDAGOGICAL EXEMPLARS

Teacher to consider the following themes when administering the suggested pedagogical approaches

1. **Reviewing Previous concepts:** Review learners' previous knowledge of equations i.e. standard and general of a circle from Week 5.
2. **Group and pair activities:** In mixed-ability groups, learners brainstorm and derive various ways to find the equation of the circle given the endpoints of its diameter and share their findings with the whole class. Learners extend their knowledge and understanding to find the centre and the radius. Learners apply the concept of midpoint to find the radius of the circle and the equation of the circle using the idea of the distance between two points.
3. **Whole Class discussions and demonstrations:** Lead a whole class discussion on the formula for establishing the equation of a circle given three points using the general equation of the circle or the equation of a circle in standard form.
4. **Problem-based group learning:** Learners in their collaborative groups recollect what a tangent is and the theorem between the tangent and the radius of a circle. In their collaborative groups, discuss and share ideas on how to find the equation of a tangent to a given circle.
5. **Group activities:** In convenient groups, discuss how to find the length of a tangent to a given circle from an external point. Learners in their groups create practical

problems, and other groups solve and present their solutions to the class. In their groups, share ideas on how to solve a practical question. Group leaders lead their members to sketch a diagram to illustrate the length of a tangent to a given circle from an extended point. In their groups, brainstorm and apply their knowledge of Pythagoras theorem and find the distance between two points to find the length of the tangent.

6. **Group and pair activities:** Learners in collaborative groups, discuss the equation of a normal to a given circle. Through interactions in learners' collaborative groups, learners discover that a normal is a line perpendicular to the tangent. Present groups of learners with some problems to solve on practical examples to consolidate the concept. Learners in their groups create questions for other groups to solve and present their solutions to the class.
7. **Group activities and presentations** :In collaborative groups, learners deduce the equation of a locus under given conditions and share their thoughts with the whole class. Learners to consider these points in the deductions and presentations
   - **a.** discuss the meaning of locus.
   - **b.** deduce that a set of points that satisfy a given condition is a circle.
   - **c.** establish locus as a geometric path that can be written algebraically.
8. **Whole Class discussions and demonstrations:** Lead a whole class discussion on how to construct loci under the following given conditions.

   Condition 1: Locus equidistant from the centre.

   Condition 2: Locus equidistant from two fixed points.

   Condition 3: Locus from two fixed lines

   Condition 4: Locus equidistant from a line.

   Condition 5: The equation of a locus from a fixed point

   Present learners with individual worksheets to take home, which involves creating and solving questions on other conditions, and submit later.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Find the equation of the circle that passes through the point located at $(-3, 5)$ and has a radius of 3 *units*
2. Deduce the equation of the circle which passes through the points $(3, 3)$, $(6, 0)$ and $(1, -2)$ and determine its radius

Assessment Level 3: Strategic reasoning

The Sissala local government of Ghana wishes to construct a market place between two towns: Tumu and Gwollu, which are approximately 35 kilometres apart. The market should be within 15 kilometres of Tumu and less than 25 kilometres of Gwollu. On a graph sheet and with a scale of $2cm{:}5\ cm$, shade the region where the market can be built to meet the criterion

SECTION 4 REVIEW

In this Section, learners were able to treat and discuss the following indicators and focal areas with expedience to sub-focal areas;

1. Explore the properties of a Circle and its parts.
2. Derive the equation, standard equation and general equation of a Circle.
3. Use the completing of squares or alternative methods to find the centre and radius of Circles.
4. Derive the equation of a circle given the endpoints of its diameter or given three points.
5. Discover the equation of a tangent to a given circle.
6. Find the length of a tangent to a given circle from an external point.
7. Apply their knowledge of Pythagoras theorem and find the distance between two points to find the length of the tangent.
8. Establish an equation of a normal to a given circle.
9. Deduce the equation of a locus under given conditions.

Also, engaging pedagogical approaches such as whole class discussions and demonstrations, group and pair activities, problem-based group learning, etc. were utilized to help learners. Furthermore, group presentations, project work presentations, individual take-home assignments etc. were used as a form of formative and summative assessments.

APPENDIX B

Structure of Mid-Semester Examination

a. *Cover contents from weeks 1-5*

- **i.** *Section A- Multiple Choice (15 questions)*
- **ii.** *Section B- (5 short answer questions, all to be answered)*
- **iii.** *Section C- Real-life Application (4 questions, 1 to be answered).*

b. *Time: 1 hour 30 minutes*

c. *Total Score: 100 marks*

d. *Table of test specification*

Table of Test Specification

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
1	1. Establishing De Morgan's laws of set theory 2. Applying De Morgan's laws of set theory 3. Applying laws of set theory	Multiple Choice	2	1	1		4
		Fill-in/short answer		1			1
		Real-life Application					
2	1. Expanding binomial expressions 2. Applying binomial expansion to approximate exponential numbers	Multiple Choice	1	1	1		3
		Fill-in/short answer		1			1
		Real-life Application		1			1
3	1. Sums of sequences 2. Convergence and divergence of series 3. Recursive sequences 4. Arithmetic and geometric means of sequences	Multiple Choice	1	1	1		3
		Fill-in/short answer		1			1
		Real-life Application		1	1		2

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
4	1. Solving quadratic inequalities	Multiple Choice	1	1			2
	2. Solving real-life problems involving linear inequalities	Fill-in/short answer		1			1
	3. Solving real-life problems involving quadratic inequalities	*Real-life Application*		1			1
5	1. Finding factors and zeroes of polynomial functions	Multiple Choice	1	1	1		3
	2. Graphing polynomial functions with degrees higher than 2	Fill-in/short answer		1			1
	3. Applying Descartes' rule of signs theorem 4. Applying the fundamental theorem of algebra 5. Complex conjugates theor*em* 6. *Linear and quadratic factor theorems*	*Real-life Application*					
	Total		6	13	5	0	24

SECTION 5: VECTORS

Strand: Geometric Reasoning and Measurement

Sub-Strand: Spatial sense

Learning Outcome: *Find trigonometric values using compound, multiple and half angles and prove the sine and cosine rule.*

Content Standard: *Demonstrate knowledge and understanding of spatial sense in relation to related problems.*

INTRODUCTION AND SECTION SUMMARY

Week 13 lessons in section 5 of the year one teacher manual discussed straight lines, their properties and division of same and week 14 lessons in section 6 introduced learners to vectors and operations on same while week 15 in section 7 of the manual introduced learners to trigonometric functions. The lessons in this section seek to help learners apply the knowledge obtained in those lessons to divide line segments and establish the sine and cosine rules.

The weeks covered by the section are:

Week 8

1. *Transposing vectors*
2. *Dividing a line or vector in a given ratio*
3. *Finding and applying the dot product of vectors*
4. *Establishing and Applying the Sine and the Cosine Rule*
5. *Projection of one vector on a given vector*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities in applying operations on vectors to division of straight lines, establishing and applying some trigonometric rules and finding areas of triangles. Learners should be given the opportunity to work in groups through experiential learning activities and mixed-ability groupings. All learners, irrespective of their learning abilities, should be assisted to fully take part in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations and use visual aids to deepen understanding. More proficient learners should be provided with tasks involving more applications of operations on vectors in real-life situations and other fields of study such as physics (forces, resolution of forces, etc.) to challenge them and maintain their interest and participation in lessons.

ASSESSMENT SUMMARY

The various concepts to be covered under this section should be assessed using various forms of assessments modes to ascertain learners' performance. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP).

WEEK 8

Learning Indicators

1. *Apply the knowledge of operations of vectors to solve simple geometric problems, including the position vector of a point that divides a vector internally and externally in a given ratio*
2. *Derive the rule for scalar (dot) product and use it to solve problems relating to angles between two vectors*
3. *Use vectors to establish the sine and cosine rules and solve problems involving areas of polygons*
4. *Determine the projection of one vector on a given vector*

FOCAL AREA 1: TRANSPOSING VECTORS

The transpose vector, V^t of a given a vector, V of order $n \times 1$ such as $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ is the vector of order $1 \times n$ having the same elements i.e., $V^t = [v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_n]$

For example, the transpose of a vector $W = [-2 \quad 3 \quad 1]$ represented by $W^t = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

FOCAL AREA 2: DIVIDING A LINE OR VECTOR IN A GIVEN RATIO

To find the position vector $\overrightarrow{OD}$ that divides a given vector, $\overrightarrow{AB}$ or line, $|AB|$ respectively in a given ratio, $m{:}n$, study the diagram below carefully

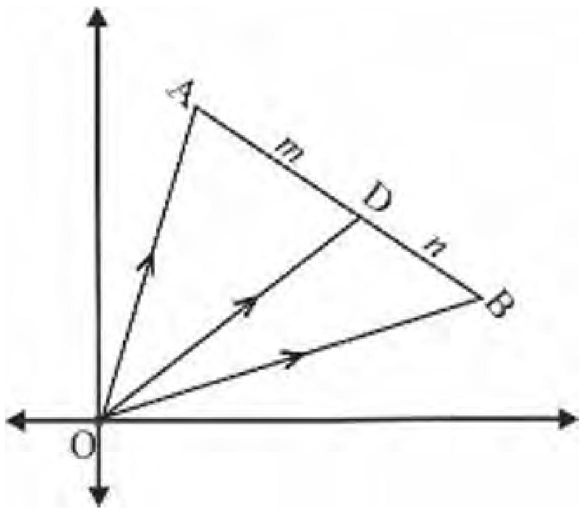


Figure 5.1

D is a point on $\overrightarrow{AB}$ such that $|\overrightarrow{AD}|:|\overrightarrow{DB}| = m:n$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \overrightarrow{OA} + |\overrightarrow{AD}| \frac{\overrightarrow{AB}}{\overrightarrow{AB}}$

Note that the equation is unchanged since $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ is a unit vector and thus, $\overrightarrow{AD}\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ produces the vector in the direction of $\overrightarrow{AD}$

$\overrightarrow{OD} = \overrightarrow{OA} + \frac{|AD|}{|AB|} \bullet \overrightarrow{AB}$...equation (1)

But $|\overrightarrow{AD}|:|\overrightarrow{DB}| = m:n$

$\Rightarrow |AB| = m + n$ and $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ as $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OA} = \boldsymbol{a}$

So $\overrightarrow{OD} = \boldsymbol{d} = \boldsymbol{a} + \frac{m}{m+n}(\boldsymbol{b} - \boldsymbol{a})$

$\boldsymbol{d} = \frac{n\boldsymbol{a} + m\boldsymbol{b}}{m+n}$.. equation (2)

Note

The vector $\overrightarrow{AD}$ is equal to the product of its magnitude and the unit vector in the same direction as $\overrightarrow{AD}$ i.e

$|AD| \bullet \frac{\overrightarrow{AB}}{\overrightarrow{AB}}$

Equation (2) gives the required position vector whether the line or vector is divided internally or externally. In the internal case, both m and n are positive. However, in the external case, any one of the ratios is taken as negative.

Example 1

P and Q are points on the position vectors $2i - 3j$ and $2i - j$, respectively, on a vector $\overrightarrow{PQ}$. T is a point on $\overrightarrow{PQ}$ such that $|PT|:|TQ| = 2:3$. Find the position vector of the point T.

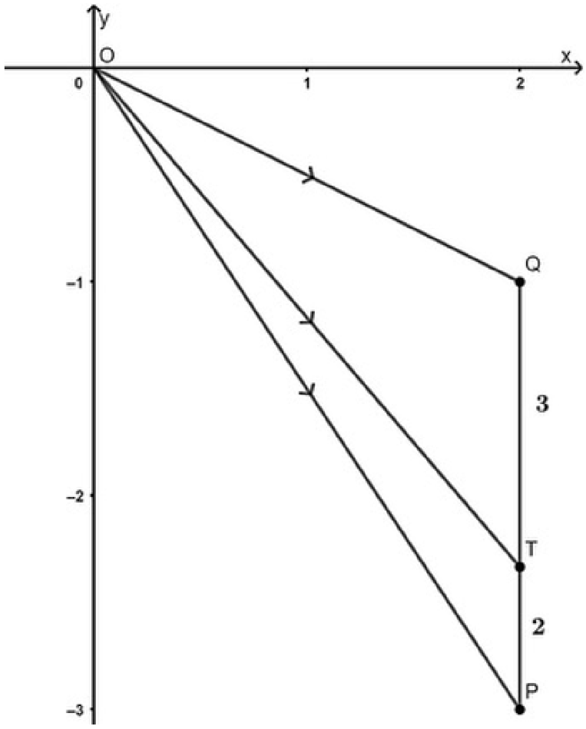


Figure 5.2

Solution

Finding position vector $\overrightarrow{OT}$

$$\overrightarrow{OT} = \frac{2\overrightarrow{OQ} + 3\overrightarrow{OP}}{2+3}$$

$$= \frac{2(2i - j) + 3(2i - 3j)}{5}$$

$$= \frac{10i - 11j}{5}$$

$$= 2i - 2.2j$$

FOCAL AREA 3: FINDING AND APPLYING THE DOT PRODUCT OF VECTORS

The dot product of two vectors is the sum of the products of all corresponding entries. For example, if $\boldsymbol{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ then the corresponding pairs of entries are (x_1, x_2) and (y_1, y_2) hence, the dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$, represented by $\boldsymbol{u} \bullet \boldsymbol{v}$ and read as "$\boldsymbol{u}$ dot $\boldsymbol{v}$" is such that

$\boldsymbol{u} \bullet \boldsymbol{v} = (x_1 \times x_2) + (y_1 \times y_2)$ to produce

$\boldsymbol{u} \bullet \boldsymbol{v} = x_1 x_2 + y_1 y_2$

$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ and its transpose $\boldsymbol{v}^t = [v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_n]$ can be multiplied using dot product to produce

$$\boldsymbol{v} \bullet \boldsymbol{v}^t = (v_1 \times v_1) + (v_2 \times v_2) + (v_3 \times v_3) + \ldots + (v_n \times v_n)$$
$$= (v_1)^2 + (v_2)^2 + (v_3)^2 + \ldots + (v_n)^2$$
$$= \boldsymbol{v} \bullet \boldsymbol{v}$$

Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$ and the angle between them as shown in Figure 5.3, $\boldsymbol{a} \bullet \boldsymbol{b}$ is the product of the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the cosine of the angle between the vectors. This theorem will be proven later

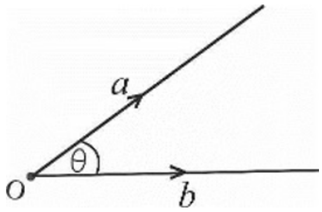


Figure 5.3

Symbolically

$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta)$

Therefore

$\cos(\theta) = \dfrac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

Note

1. The dot product of any two vectors is a scalar and not a vector hence the derived name: scalar product
2. Dot / scalar multiplication of vectors is commutative, i.e., $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$

Properties of the scalar (dot) product

Property 1

If $\boldsymbol{a} = [p \quad q]$ and $\boldsymbol{b} = [r \quad s]$ then $\boldsymbol{a} \bullet \boldsymbol{b} = pr + qs$

Also

$$\cos(\theta) = \frac{pr + qs}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Hence,

$$\theta = \cos^{-1}\left(\frac{pr + qs}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$

Putting $p = r$ and $q = s$ in equation (1) gives

$$\boldsymbol{a} \bullet \boldsymbol{a} = p^2 + q^2$$

$$p^2 + q^2 = |\boldsymbol{a}|^2$$

$$\therefore \boldsymbol{a} \bullet \boldsymbol{a} = |\boldsymbol{a}|^2$$

The scalar product of a vector with itself equals the square of its magnitude.

Property 2: Parallelism property

When two vectors **a** and **b** have the same direction, then the angle between them, $\theta = 0$ and $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(0)$ and

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$$

Property 3

If two vectors a and b are perpendicular, then $\theta = 90°$ *and* $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(90°)$ *so* that $\boldsymbol{a} \bullet \boldsymbol{b} = 0$

Example 2

Given that $\boldsymbol{r} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{t} = m\boldsymbol{i} - 6\boldsymbol{j}$ are perpendicular. Find the value of the constant m.

Solution

Thus $\boldsymbol{r}$ and $\boldsymbol{t}$ are perpendicular, so $\boldsymbol{r} \bullet \boldsymbol{t} = 0$.

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} m \\ -6 \end{pmatrix} = 0$$

$$3m - 24 = 0$$

$$m = 8$$

Other properties

4. Multiplication by constant: $p\boldsymbol{a} \bullet q\boldsymbol{b} = pq|\boldsymbol{a}||\boldsymbol{b}|$ where p and q are constants, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors

5. The Dot product follows the distributive law: $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{b} \bullet \boldsymbol{c}$ where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors

Example 3

Find the angle between the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{j}$

Solution

$$\boldsymbol{a} \bullet \boldsymbol{b} = (2\boldsymbol{i} + 3\boldsymbol{j}) \bullet (4\boldsymbol{i} - \boldsymbol{j})$$
$$= 8 - 3$$
$$= 5$$
$$|\boldsymbol{a}| = \sqrt{4 + 9} = \sqrt{13}$$
$$|\boldsymbol{b}| = \sqrt{16 + 1} = \sqrt{17}$$
$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{13} \times \sqrt{17}}\right)$$
$$\theta = \cos^{-1}(0.3363)$$
$$\theta = 70.4°$$

The angle between the vectors $2\boldsymbol{i} + 3\boldsymbol{j}$ and $4\boldsymbol{i} - \boldsymbol{j}$ if 70.4°

Example 4

The vector $\boldsymbol{n} = \begin{pmatrix} 1650 \\ 3200 \end{pmatrix}$ gives the numbers of units of two types of baking pans produced by a company. The vector, $\boldsymbol{p} = \begin{pmatrix} 15.25 \\ 10.50 \end{pmatrix}$ gives the prices (in Ghanaian cedis) of the two types of pans

a) Find the dot product, $\boldsymbol{n} \bullet \boldsymbol{p}$ and interpret the result in relation to the problem

b) Identify the vector operation used to increase the prices by 5%

Solution

a) $\boldsymbol{n} = \begin{pmatrix} 1650 \\ 3200 \end{pmatrix}, \boldsymbol{p} = \begin{pmatrix} 15.25 \\ 10.50 \end{pmatrix}$

$$\boldsymbol{n} \bullet \boldsymbol{p} = \begin{pmatrix} 1650 \\ 3200 \end{pmatrix} \bullet \begin{pmatrix} 15.25 \\ 10.50 \end{pmatrix}$$
$$= GH¢\ 58{,}762.50$$

$\boldsymbol{n} \bullet \boldsymbol{p}$ represents the total selling prices for all the baking pans produced by the company. It therefore would cost *GH¢* 58,762.50 to buy all the 4,850 pans (gotten from 1,650 + 3,200) produced by the company

b) To increase the price of either of the types of pans by 5%, the original price would have to be multiplied by 105% (which is a constant scalar). It would therefore require scalar multiplication of the prices vector, i.e., $\boldsymbol{p}$ to perform an adjustment in prices

FOCAL AREA 4: ESTABLISHING AND APPLYING THE SINE AND THE COSINE RULE

Refer to triangle *PQR* with the sides p, q and r, respectively and use the diagram to establish the cosine rule given as $p^2 = q^2 + r^2 - 2qrcos\theta$, where $\theta \leq P$

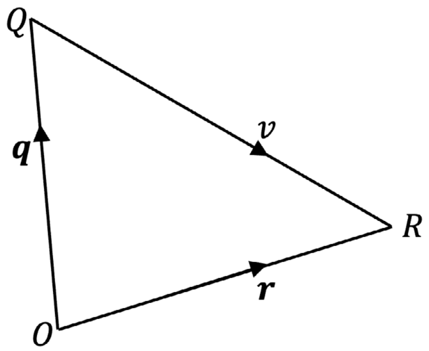


Figure 5.4

From Figure 5.4

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \boldsymbol{r} - \boldsymbol{q}$

$\overrightarrow{QR} \bullet \overrightarrow{QR} = (\boldsymbol{r} - \boldsymbol{q}) \bullet (\boldsymbol{r} - \boldsymbol{q})$

$= \boldsymbol{r} \bullet \boldsymbol{r} + \boldsymbol{q} \bullet \boldsymbol{q} - 2\boldsymbol{q} \bullet \boldsymbol{r}$

But $\boldsymbol{r} \bullet \boldsymbol{r} = |\boldsymbol{r}|^2$, $\boldsymbol{q} \bullet \boldsymbol{q} = |\boldsymbol{q}|^2$ and $\boldsymbol{q} \bullet \boldsymbol{r} = |\boldsymbol{q}||\boldsymbol{r}|\cos(\widehat{O})$

$\overrightarrow{QR} \bullet \overrightarrow{QR} = |\boldsymbol{q}|^2 + |\boldsymbol{r}|^2 - 2|\boldsymbol{q}||\boldsymbol{r}|\cos(\widehat{O})$

Thus $|v|^2 = |\boldsymbol{q}|^2 |\boldsymbol{r}|^2 - 2|\boldsymbol{q}||\boldsymbol{r}|\cos(\widehat{O})$

$\therefore |v|^2 = |\boldsymbol{q}|^2 + |\boldsymbol{r}|^2 - 2|\boldsymbol{q}||\boldsymbol{r}|\cos(\widehat{O})$

Also $|\boldsymbol{q}|^2 = |\boldsymbol{r}|^2 + |v|^2 - 2|\boldsymbol{r}||v|cos\widehat{Q}$

And $|\boldsymbol{r}|^2 = |\boldsymbol{q}|^2 + |v|^2 - 2|\boldsymbol{q}||v|cos\widehat{R}$

Where $\widehat{O}$, $\widehat{Q}$ and $\widehat{R}$ are angles at points P, Q and R.

Area of a triangle using the Sine rule

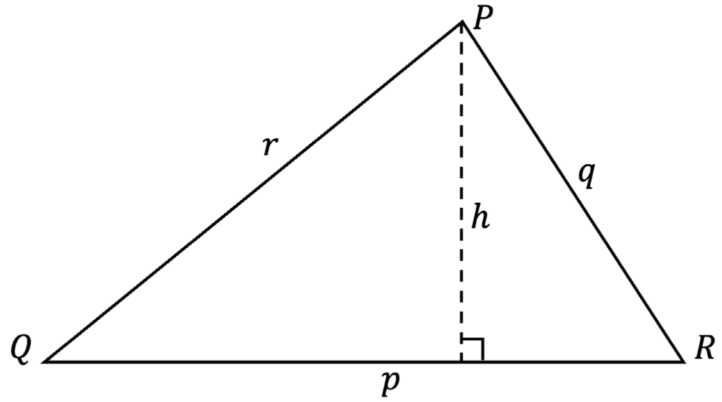


Figure 5.5

Figure 5.5 shows triangle PQR with an altitude of length, h

Area of triangle PQR is given by

$$Area = \frac{1}{2} \times |PQ| \times h$$
$$= \frac{1}{2} r \times h$$

But $h = |PR|\sin(R) = |q|\sin(R)$

So

$$Area = \frac{1}{2}|p||q|\sin(R)$$

Thus, the area of a triangle equals a half of the product of the magnitudes of the lengths of two sides (or two vectors which bound the triangle) and the sine of the angle between the two sides (or vectors)

Example 5

$A(1,-2)$, $B(3,0)$ and $C(1,2)$ are vertices of triangle ABC

a. Express $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ as column vectors

b. Use scalar multiplication to calculate angle ABC

c. Find the area of triangle ABC

Solution

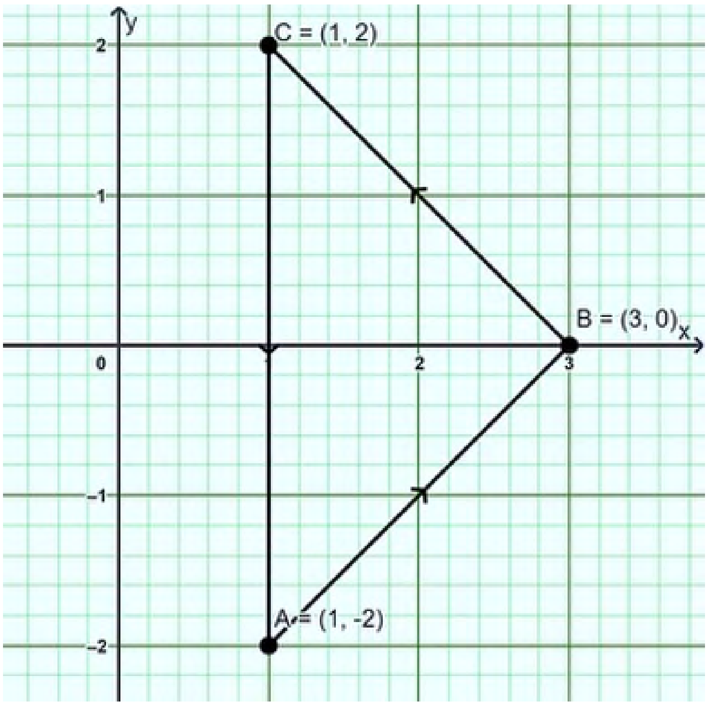


Figure 5.6

a. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}3\\0\end{pmatrix} - \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$

$\overrightarrow{CA} = \begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\-4\end{pmatrix}$

b. $\cos(\angle ABC) = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{BC}}{|\overrightarrow{BC}||\overrightarrow{AB}|}$

$|\overrightarrow{AB}| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$|\overrightarrow{BC}| = 2\sqrt{2}$

$\cos(\angle ABC) = \dfrac{(2\boldsymbol{i} + 2\boldsymbol{j}) \bullet (-2\boldsymbol{i} + 2\boldsymbol{j})}{2\sqrt{2} \times 2\sqrt{2}}$

$\cos(\angle ABC) = -\dfrac{4 + 4}{8}$

$\cos(\angle ABC) = 0$

$\angle ABC = \cos^{-1}(0)$

$\therefore < ABC = 90°$

c. Since ΔABC is a right isosceles triangle, $\angle CAB = \angle BCA = 45°$

Area of ABC equals $\frac{1}{2}|AC||AB|\sin(\angle CAB)$ or $\frac{1}{2}|AB||BC|\sin(\angle ABC)$ or $\frac{1}{2}|BC||AC|\sin(\angle BCA)$ i.e., From the graph in Figure 5.6, $\overrightarrow{AC}$ lies on the vertical line which joins A to C. It takes 4 steps to move from A to C and vice versa hence $|AC| = 4$ *units*.

Alternatively,

$$|AC| = |\overrightarrow{AC}|$$

$$= \sqrt{0^2 + (-4)^2}$$

$$= \sqrt{16} = 4 \text{ units}$$

This implies that the area of triangle ABC is

$\frac{1}{2}(4)(2\sqrt{2})\sin(45°)$ or $\frac{1}{2}(2\sqrt{2}) \times (2\sqrt{2})(2\sqrt{2})\sin(90°)$ or $\frac{1}{2}(2\sqrt{2})(4)\sin(45°)$

Therefore, the area of triangle ABC is 4 *sq units*. This result can be confirmed from the graph by determining the number of squares of the grid bounded by the triangle

FOCAL AREA 5: PROJECTION OF ONE VECTOR ON A GIVEN VECTOR

It has been established that addition combines two vectors to create a resultant vector. In some situations, we would be interested in finding the components of a vector i.e., parts of the vector in different directions. Vector projections just happen to be the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection.

For example, if a child is pulling the handle of a cart at a 55° angle as shown in Figure 5.7, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (in the horizontal direction)

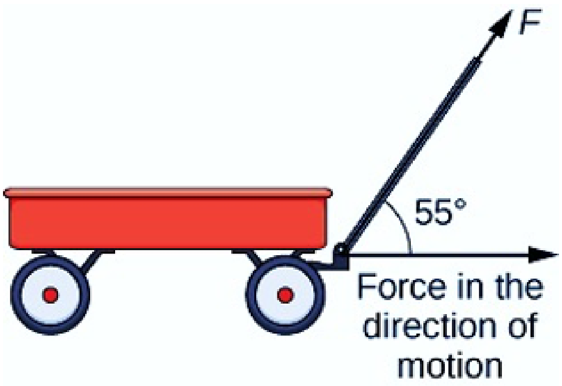


Figure 5.7

Figure 5.8 shows a diagram of the unit vector, $\hat{\boldsymbol{u}}$ and the position vector, $\overrightarrow{OA}$. Vector $\boldsymbol{a}$ makes an angle θ with the unit vector $\hat{\boldsymbol{u}}$. We seek to make a projection of $\overrightarrow{OA}$ in the direction of $\hat{\boldsymbol{u}}$ denoted by $\overrightarrow{OP}$

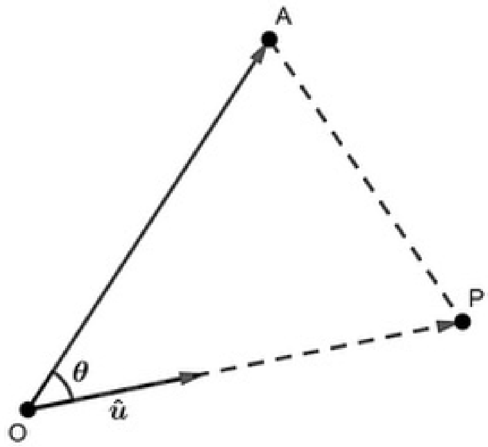


Figure 5.8

$\boldsymbol{a} \bullet \hat{\boldsymbol{u}} = |\boldsymbol{a}||\boldsymbol{u}|\cos(\theta)$, but $|\hat{\boldsymbol{u}}| = 1$

$= |\boldsymbol{a}|cos(\theta)$

$= |\overrightarrow{OP}|$

Example 6

Find the projection of a vector $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$ in the direction of vector $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$.

Solution

Find $|\boldsymbol{b}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\boldsymbol{b} = |\boldsymbol{b}|\hat{\boldsymbol{b}} = \sqrt{5}\hat{\boldsymbol{b}}$

Now, the projection of $\boldsymbol{a}$ in the direction of $\hat{\boldsymbol{b}}$ is given by $\boldsymbol{a} \bullet \hat{\boldsymbol{b}}$

$\boldsymbol{a} \bullet \hat{\boldsymbol{b}} = (2\boldsymbol{i} + 3\boldsymbol{j}) \bullet \frac{1}{\sqrt{5}}(\boldsymbol{i} + 2\boldsymbol{j})$

$= \frac{2 + 6}{\sqrt{5}} = \frac{8\sqrt{5}}{5}$

So, the projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\frac{8\sqrt{5}}{5}$ units long.

Learning Tasks

Learners are expected to perform the following tasks as part of the activities for the lessons in the week

1. Learners deduce methods to divide line segments in given ratios by constructing and relating to diagrams,
2. derive the scalar (dot) product and solve problems relating to angles between two vectors.
3. discuss and establish the properties of the scalar (dot) product

4. discuss their understanding of vectors and use the idea to establish the cosine and sine rules and
5. discuss how to determine the area of a triangle

PEDAGOGICAL EXEMPLARS

1. **Reviewing Previous concepts:** Review learners' previous knowledge of straight lines, their properties and division of same in week 13 lessons in section 5, vectors and operations on same in week 14 lessons in section 6 and trigonometric functions in week 15, section 7 of the year one teacher manual.

   Learners should be given the opportunity to share with the larger group their personalised learning from the previous lessons. Opportunity should be given to learners to choose how they would like to express themselves i.e., writing on the board or talking out loud. Facilitators should direct the discussion using strategies such a building on what others say or the use of probing questions

2. **Group and pair activities:** In mixed-ability groups, learners are given physical objects (like sticks or rulers) to represent vectors. They are the tasked to physically move these objects to transpose vectors.

3. **Whole Class discussions and demonstrations:** Lead a whole class discussion on the formula for establishing the equation of a circle given three points using the general equation of the circle or the equation of a circle in standard form.

   Provide learners the opportunity to also share some personal strategies to deducing the equation of a circle. Learners with high proficiency can be given the opportunity to lead the larger group to deduce procedures to finding the equation of a circle given the location of three points on the circle

4. **Problem-based group learning:** Learners in their collaborative groups are assigned problems that require calculating dot products to solve, discussing their methods.

   Challenge learners with problems that require projecting vectors onto different axes and planes

   Learning tasks should be broken into smaller tasks for learners who may be struggling. Some learners can be assigned peer tutoring roles to assist learners who may be struggling.

5. **Group activities:** Learners in groups collaborate and draw pairs of vectors and measure the angles between them. Learners then discuss how their answers relate to the scalar dot product.

6. Use dynamic geometry software to demonstrate how the sine and cosine rules are derived and used where available. In classroom settings where these tools are unavailable, learners should be guided through activities (drawing different triangles

and investigating the relationships between the sides and the interior angles) to establish the sine and cosine laws

KEY ASSESSMENTS

Assessment level 1: Recall

If $\boldsymbol{p} = \langle 7, 0, 2\rangle$, $\boldsymbol{q} = \langle -2, 2, -2\rangle$, and $\boldsymbol{r} = \langle 0, 2, -3\rangle$ find $(\boldsymbol{r} \bullet \boldsymbol{p})\boldsymbol{q}$

Assessment Level 2: Skills and conceptual Understanding

Resolve $\boldsymbol{u}$ into the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, where $\boldsymbol{u}_1$ is parallel to $\boldsymbol{v}$ and $\boldsymbol{u}_2$ is perpendicular to $\boldsymbol{v}$.

a) $\boldsymbol{u} = \langle 3, 1\rangle$, $\boldsymbol{v} = \langle 6, 1\rangle$

b) $\boldsymbol{u} = \langle 8, 6\rangle$, $\boldsymbol{v} = \langle 20, 20\rangle$

Assessment Level 3: Strategic Reasoning

From an origin O, the points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $2\boldsymbol{b}$ respectively. The points O, A and B are not collinear. The midpoint of AB is M, and the point of trisection of AC nearer to A is T. Draw a diagram to show O, A, B, C, M and T. Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, the position vectors M and T. Use your results to prove that O, M and T are collinear and find the ratio in which M divides O

Assessment Level 4: Extended Thinking and Reasoning

Naa Shorkor sells kenkey by the road in the evenings. She prepares the balls of kenkey herself but buys the fried fishes from a friend. One evening, she made a total sale of *GH¢* 880.00 for 150 balls of kenkey and a number of fishes. If she sells a ball of kenkey for *GH¢* 4.00 and a fish for *GH¢* 5.00,

1. write out the vectors, $\boldsymbol{a}$ and $\boldsymbol{p}$ where $\boldsymbol{a}$ represents the number of balls of kenkey and number of fishes sold and $\boldsymbol{p}$ represents the prices of a ball of kenkey and a fish

2. If her friend said that she supplied 60 fishes to Naa Shorkor, determine whether her friend spoke the truth

Section 5 Review

This section discussed all the lessons to be taught in week 8. This section discussed the dot product of vectors and the applications for proving the cosine rule and finding the area of triangles. It is encouraged that portions of the sections which require a review of concepts be administered to learners as presentation tasks. Leaners be provided with adequate learning materials such as graphing utilities, graph books or grid boards to enable them to investigate operations of vectors in a plane.

SECTION 6: MATRICES

Strand: Modelling with Algebra

Sub-Strand: Application of Algebra

Learning Outcome: *Multiply matrices, determine the inverse of a 2 x 2 matrix, find the determinant up to a 3x 3 matrix and represent matrices in linear transformations*

Content Standard: Demonstrate the ability to carry out matrix operations, determine the inverse of a linear transformation and represent real life situations in matrix forms

INTRODUCTION AND SECTION SUMMARY

Matrices play an important role in various fields of science. They provide a strong foundation for advanced mathematical concepts and practical applications. In year one we studied the operations (addition, subtraction and multiplication) of matrices. It is expected that learners' previous knowledge in Matrices will be used as a foundation for this section. In this section we will explore multiplication of matrices, inverse of 2 by 2 matrices, determinant of matrices up to 3 by 3 matrices and represent matrices in linear transformation. As learners explore these concepts, they will realise that matrices are like patterns and they will unlock these patterns and discover their real-world application. By the end of this section learners will be equipped with understanding to solve problems in matrices and apply its knowledge to solve real-life problems.

The weeks covered by the section are:

Week 9

1. *Revision of types of matrices and matrix algebra*
2. *Determinant of* 3×3 *matrices*
3. *Inverse of a* 2×2 *matrices*

Week 10

1. *Using matrices to solve systems of linear equations*
2. *Using matrices to model and solve real-life problems*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on working with 3×3 matrices. Learners should be given the opportunity to work in groups composed with recourse to crosscutting issues, and as individuals translate real-life problems into matrix forms and solve them. Therefore, collaborative learning, experiential learning and initiate talk for learning, project-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, considerations and accommodations should be made for the different groups. Learning tasks of varying degrees should be administered to learners based on their abilities. Learners who have the ability, should be challenged to solve more application problems which require operations on 3×3 matrices through personal strategies and / or with the aid of computer software where available. They should also be given the opportunity to tutor learners who may be struggling.

ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests and homework assignments, can be used to evaluate learners understanding of concepts and their ability to solve problems. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for STP.

WEEK 9

Learning Indicators

1. *Distinguish between 'singular' and 'non-singular' square matrices (2 × 2 and 3 × 3) and evaluate determinants*
2. *Multiply an m × n matrix by an n × 1 matrix*
3. *Find the inverse of a matrix using linear transformation*

Introduction

In week 12 of the year one teacher manual, learners discussed the determinant of 2×2 matrices and performed arithmetic operations on 2×2 matrices. In this section learners will revise briefly what they learnt in the previous year and extend their understanding of matrices. Learners will discuss finding the determinant of a 3×3 matrices, finding the inverse of 2×2 and 3×3 matrices, solving systems of linear equations and applying matrix operations to solving real-life problems.

FOCAL AREA 1: REVISION OF TYPES OF MATRICES AND MATRIX ALGEBRA

Square matrix: This is a matrix that has the same number of rows and columns (for example, 2×2, 3×3, 4×4, etc.).

Example 1

An example of a 2×2 square matrix is $\begin{pmatrix} 1 & 4 \\ 8 & 5 \end{pmatrix}$

An example of a 4×4 square matrix is $\begin{pmatrix} 4 & 2 & 0.5 & 6 \\ 0.1 & 8 & 7 & 0.8 \\ 12 & 10 & 0.9 & 5 \\ 48 & 8 & 0.2 & 10 \end{pmatrix}$

Diagonal matrix: This is a square matrix with non-zero entries only on its main diagonal (from top left to bottom right).

Example 2

An example of a diagonal matrix is $\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

Identity Matrix: The multiplicative identity matrix is a special square matrix with ones (1s) on its main diagonal and zeros (0s) elsewhere. The identity matrix of a 2×2 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and the identity matrix of a 3 × 3 matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It denoted by I. The identity matrices play a crucial role in solving matrix equations.

Triangular matrix: A triangular matrix is a square matrix in which elements below and/or above the diagonal are all zeros. There are two types of triangular matrices that is Upper triangular and lower triangular matrix

An upper triangular matrix is a square matrix whose elements below the main diagonal are zero.

Example 3

$$\begin{pmatrix} 3.5 & -5 & 2 \\ 0 & 8 & 11 \\ 0 & 0 & 5 \end{pmatrix}$$

A lower triangular matrix is a square matrix whose all elements above the main diagonal are zero.

Example 4

$$\begin{pmatrix} -9 & 0 & 0 \\ 7 & 13 & 0 \\ 12 & 4 & 8 \end{pmatrix}$$

Matrix Algebra

Example 5

Given that $A = \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -2 \\ 6 & 1 \end{pmatrix}$,

Evaluate the following;

a) $A + B$

b) $B - A$

c) AB

d) $2A$

Solution

a) $A + B = \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & -3 \\ -1 & -1 \end{pmatrix}$

b) $B - A = \begin{pmatrix} -4 & -2 \\ 6 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix}$

$= \begin{pmatrix} -7 & -7 \\ 13 & 3 \end{pmatrix}$

c) $AB = \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix} \times \begin{pmatrix} -4 & -2 \\ 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} 18 & -1 \\ 16 & 12 \end{pmatrix}$

d) $2A = 2 \times \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix}$

$= \begin{pmatrix} 6 & 10 \\ -14 & -4 \end{pmatrix}$

FOCAL AREA 2: DETERMINANT OF 3 × 3 MATRICES

The determinant of a 2 × 2 matrix is given by $(ad - bc)$, however because of the increased entries in a 3 × 3 matrix the calculation is different. As already stated in the introduction of the section summary operation on matrices are in patterns, therefore learners have to pay attention to the pattern used to calculate the determinant of a 3 × 3 matrix. In order to evaluate the determinant of a 3 × 3 matrix we would have to define minors and cofactors of a square matrix.

Minor of a square matrix

The minor is the value of the determinant of the matrix that results from crossing out the row and column of the element under consideration. Therefore, if we let $A = (a_{ij})$ be a matrix of order $m \times m$. Then, the minor of element (a_{ij}) (denoted by (A_{ij}) is the determinant of the $(m - 1) \times (m - 1)$ matrix obtained after removing row i and column j from A.

Taking a 3 × 3 square matrix is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The minor of A can be found by crossing out the row and column of a_{11}, a_{12}, a_{13}

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Illustration 1

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Illustration 2

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Illustration 3

Figure 24

From Figure 24 the minor of a_{11} is $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$, a_{12} is $\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$ and a_{13} is $\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

Note that each element in the square matrix has its own minor.

Example 6

Given $B = \begin{pmatrix} -9 & 5 & 11 \\ 7 & 13 & 10 \\ 12 & 4 & 8 \end{pmatrix}$, determine the minor of B_{12}

Solution

The minor of B_{12} is the determinant of matrix formed by crossing out row one and column two from B

The minor of $B_{12} = \begin{vmatrix} 7 & 10 \\ 12 & 8 \end{vmatrix}$

$= (7 \times 8) - (10 \times 12)$

$= -64$

Example 7

Given $D = \begin{pmatrix} 2 & -5 & 1 \\ 4 & 6 & -7 \\ 3 & 9 & 8 \end{pmatrix}$, determine the minor of D_{23}

Solution

The minor of D_{23} is the determinant of matrix formed by crossing out row two and column three from D

The minor of $D_{23} = \begin{vmatrix} 2 & -5 \\ 3 & 9 \end{vmatrix}$

$= (2 \times 9) - (-5 \times 3)$

$= 33$

Cofactors of a square matrix

Each element in a square matrix has its own cofactor. The cofactor is the product of the element's

place sign and minor. The place sign of C_{ij} is given by $(-1)^{i+j}$. Therefore, the cofactor of an element, C_{ij} in a square matrix is given $(-1)^{i+j} \bullet M_{ij}$. Each element in the square matrix has its own cofactor.

Example 8

Given $B = \begin{pmatrix} -9 & 5 & 11 \\ 7 & 13 & 10 \\ 12 & 4 & 8 \end{pmatrix}$, determine the cofactor of 4 and -9

Solution

The minor of 4 is $= \begin{vmatrix} -9 & 11 \\ 7 & 10 \end{vmatrix}$

$= (-9 \times 10) - (11 \times 7)$

$= -167$

The place sign of 3 is $(-1)^{3+2} = -1$

$\therefore$ the cofactor of 4 is $(-1)(-167) = 167$

The cofactor if -9 is $= (-1)^{1+1}\begin{vmatrix} 13 & 10 \\ 4 & 8 \end{vmatrix}$

$= (1)[(13 \times 8) - (10 \times 4)]$

$= 64$

$\therefore$ the cofactor of -9 is 64

Now that we have defined the minor and cofactor of square matrices, we can evaluate the determinant of a matrix by using the theorem of expanding by cofactors. The theorem states that the value of a determinant can be found by expanding by cofactors of any row or column. Note that no matter which row or column we choose, we will always get the same value for the determinant. Therefore, it is advisable to use the row or column which has most of its entries to be zeros to ease calculations.

In general, to evaluate the determinant of a matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ by expanding cofactors.

Choose say the second row then $|A| = a_{21}(C_{21}) + a_{22}(C_{22}) + a_{23}(C_{23})$

$= a_{21}(-1)^{2+1}(M_{21}) + a_{22}(-1)^{2+2}(M_{22}) + a_{23}(-1)^{2+3}(M_{23})$

Example 9

Evaluate the determinant of $D = \begin{pmatrix} 2 & -5 & 1 \\ 4 & 6 & -7 \\ 3 & 9 & 8 \end{pmatrix}$ by expanding cofactors.

Solution

Finding the determinant of D by say row 1

$$|D| = \begin{pmatrix} 2 & -5 & 1 \\ 4 & 6 & -7 \\ 3 & 9 & 8 \end{pmatrix}$$

$$= 2(C_{11}) + (-5)(C_{12}) + 1(C_{13})$$

$$= 2(-1)^{1+1}(M_{11}) + (-5)(-1)^{1+2}(M_{12}) + 1(-1)^{1+3}(M_{13})$$

$$= (2)\begin{vmatrix} 6 & -7 \\ 9 & 8 \end{vmatrix} + (5)\begin{vmatrix} 4 & -7 \\ 3 & 8 \end{vmatrix} + (1)\begin{vmatrix} 4 & 6 \\ 3 & 9 \end{vmatrix}$$

$$= (2)[(6 \times 8) - (-7 \times 9)] + (5)[(4 \times 8) - (-7 \times 3)] + (1)[(4 \times 9) - (6 \times 3)]$$

$$= 505$$

Find the determinant by choosing different rows to establish that no matter the row chosen the determinant will always be the same

Using the Sarrus rule to find the determinant

An alternative method to evaluate determinant of matrices is the Sarrus rule. To find the determinant of say $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ extend the matrix by writing the first and second columns to the write hand side of A

i.e., $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

Add the product of the diagonals $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

Subtract the product of the opposite diagonals $-a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$

Put the two formulae together to find the determinant. That is

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

Example 10

Evaluate the determinant of $D = \begin{pmatrix} 2 & -5 & 1 \\ 4 & 6 & -7 \\ 3 & 9 & 8 \end{pmatrix}$

Solution

Using the Sarrus rule

$$\begin{pmatrix} 2 & -5 & 1 \\ 4 & 6 & -7 \\ 3 & 9 & 8 \end{pmatrix}\begin{matrix} 2 & -5 \\ 4 & 6 \\ 3 & 9 \end{matrix}$$

$\det(D) = (2)(6)(8) + (-5)(-7)(3) + (1)(4)(9) - (-5)(4)(8) - (2)(-7)(9) - (1)(6)(3)$

$= 505$

Singular and Non-Singular Matrices

A square matrix is said to be a singular matrix if its determinant is zero and it has no inverse. Whereas a non-singular matrix is a square matrix whose determinant is a non-zero and it has an inverse.

Difference between singular and non-singular matrix

	Non-singular	Singular
A is	invertible	not invertible
Columns	independent	dependent
Rows	independent	dependent
$det(A)$	$\neq 0$	$= 0$
$Ax = 0$	one solution $x = 0$	infinitely many solutions or
$Ax = b$	one solution	no solution

To determine whether a square matrix is singular or non-singular find the determinant of the given matrix. If it is zero then the matrix is singular and if it is not zero then it is a non-singular matrix.

Example 11

Find the value of x if $B = \begin{pmatrix} x & -8 \\ 4 & 2 \end{pmatrix}$ is a singular matrix

Solution

If B is a singular matrix then $ad - bc = 0$

$\Rightarrow (x)(2) - (-8)(4) = 0$

$2x + 32 = 0$

$x = 16$

Transpose of a matrix

If a matrix A is of order $m \times n$, then the transpose of the matrix A denoted by A^T is of order $n \times m$. The transpose of a matrix is found by interchanging its rows into columns or columns into rows. For example, given $A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -7 \end{pmatrix}$, then $(A^T) = \begin{pmatrix} 2 & 4 \\ -3 & 5 \\ 1 & -7 \end{pmatrix}$.

Example 12

Given that $X = \begin{pmatrix} 21 & 9 & 14 \\ -3 & -8 & 5 \\ 1 & 4 & -7 \end{pmatrix}$, find X^T.

Solution

$$X^T = \begin{pmatrix} 21 & -3 & 1 \\ 9 & -8 & 4 \\ 14 & 5 & -7 \end{pmatrix}$$

Example 13

What is the transpose of a 4×3 matrix?

Solution

The transpose of a 4×3 is a 3×4 matrix.

Example 14

Given that $A = \begin{pmatrix} 5 & -3 & 11 \\ 4 & 52 & -17 \end{pmatrix}$, Find $(A^T)^T$ and state your observation.

Solution

If $A = \begin{pmatrix} 5 & -3 & 11 \\ 4 & 52 & -17 \end{pmatrix}$ then $(A^T) = \begin{pmatrix} 5 & 4 \\ -3 & 52 \\ 11 & -17 \end{pmatrix}$

Therefore $(A^T)^T = \begin{pmatrix} 5 & -3 & 11 \\ 4 & 52 & -17 \end{pmatrix}$

It is observed that applying the transpose to a given matrix twice in succession results in the original matrix

i.e., Given a matrix A, then $(\boldsymbol{A}^T)^T = \boldsymbol{A}$

Addition Property of Transpose

Given $A = \begin{pmatrix} 7 & 2 \\ 3 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix}$, find $(A + B)^T$ and $A^T + B^T$. What is the relationship between $(A + B)^T$ and $A^T + B^T$.

$$\left(A + B\right) = \begin{pmatrix} 7 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 \\ 4 & -9 \end{pmatrix}$$

$$(A + B)^T = \begin{pmatrix} 4 & 4 \\ 6 & -9 \end{pmatrix}$$

$A^T = \begin{pmatrix} 7 & 3 \\ 2 & -4 \end{pmatrix}$ and $B^T = \begin{pmatrix} -3 & 1 \\ 4 & -5 \end{pmatrix}$

$\therefore A^T + B^T = \begin{pmatrix} 7 & 3 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ 4 & -5 \end{pmatrix}$

$= \begin{pmatrix} 4 & 4 \\ 6 & -9 \end{pmatrix}$

The sum of $(A + B)^T$ and $A^T + B^T$ are the same.

This shows that adding the transpose of two individual matrices say X and Y i.e., $X^T + Y^T$ is equal to the transpose of an addition of the two matrices $(X + Y)^T$. That is

$(\boldsymbol{X} + \boldsymbol{Y})^T = \boldsymbol{X}^T + \boldsymbol{Y}^T$.

FOCAL AREA 3: INVERSES OF 2 × 2 MATRICES

Before the inverse of a matrix is found it is important to define the adjoint of a matrix.

Adjoint of a Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the adjoint of matrix A, written as A^*, is $A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

For example, the adjoint of $D = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}$ is $D^* = \begin{pmatrix} 6 & 2 \\ -5 & 3 \end{pmatrix}$

Inverse of a 2 × 2 matrix

The inverse of a matrix can be found if the matrix is a non-singular matrix. The inverse of a matrix is another matrix, which when multiplied with the given matrix gives the multiplicative identity. That is given a 2 × 2 matrix, say A, then its inverse is another 2 × 2 matrix denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{\text{det}A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 15

Find the inverse of matrix $A = \begin{pmatrix} 3 & 4 \\ -5 & -6 \end{pmatrix}$

Solution

$A^{-1} = \frac{1}{etA}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$A^{-1} = \frac{1}{3–6 - (-5)(4)}\begin{pmatrix} -6 & -4 \\ 5 & 3 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} -6 & -4 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -2 \\ \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

Learning Tasks

1. Learners identify and create problems involving operation on matrices
2. Learners write the transpose of the matrix and verify the transpose properties
3. Learners explore matrix multiplication, multiplying $m \times n$ matrix by an $n \times 1$ matrix
4. Learners create a square matrix identify the minors and co-factors matrix
5. Learners determine the adjoint matrix of given matrices

PEDAGOGICAL EXEMPLAR

The lessons in this week seek to help learners to equip learners understanding of some properties of matrices namely, determinants and inverses. They are meant to form a sequel of the week 12 lessons of the year one teacher manual and thus, ample opportunity should be given to learners to review the previous lessons. The following pedagogical approaches are suggested for facilitators to take learners through.

Through collaborative and experiential learning,

1. Learners work in pairs; one of the pairs identifies a matrix they know whiles the other partially writes an example (learners' reverse roles).
2. Learners work in pairs: one pair identifies and creates a problem involving operation on a matrix and tasks the other pair to solve, justify, prove or investigate the context (learners' reverse roles). The facilitator should check the appropriateness of the problems posed by learners. The pairs should consist of a blend of learners with carrying proficiency. Learners who may be struggling should be paired by those who are more proficient.
3. Learners revise Equality of matrices, Scalar multiple of a matrix, Addition of two matrices, Multiplication of two matrices and Properties of Matrices (commutative, Associative and Distributive law). This can be done using a diagnostic test from a workbook at the beginning of the lesson. The facilitator can allow learners to work in small groups or as individuals. Solutions to the test should be discussed and misconceptions corrected before the mains lessons for the week are begun
4. Learners work with a partner: one of the pair identifies and create a 2x2 and 3x3 matrix whiles the other pair writes the transpose of the matrix and verify the transpose properties. Learners change roles.

5. Learners work with a partner: One pair identifies and creates a 2x2 matrix while the other pair investigates whether the matrix is singular or non-singular. Learners change roles and summarise their findings
6. Learners should be guided to use the scientific calculator and / or other relevant technological tools to aid learning where applicable

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Find the determinant of $N = \begin{pmatrix} 8 & 4 & -1 \\ 0 & 1 & 3 \\ 5 & 4 & 8 \end{pmatrix}$
2. Determine whether $M = \begin{pmatrix} -3 & 2 & 5 \\ 6 & 1 & -4 \\ 7 & 0 & 6 \end{pmatrix}$ is singular or not.
3. Determine the value of a so that the $\begin{pmatrix} 7 & 3 \\ -2 & a \end{pmatrix}$ is singular.
4. Find the inverse of $\begin{pmatrix} 3 & 2 \\ 4 & 8 \end{pmatrix}$
5. Given $\begin{pmatrix} 1 & 0 \\ 2 & x \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 6 & 3 \end{pmatrix}$, find the value of x.

Assessment Level 3: Strategic Reasoning

1. Find the determinant of $\begin{pmatrix} x & 0 & 0 \\ -1 & -5x & 0 \\ 2 & 1 & x \end{pmatrix}$ and hence, the solution set of

$$\begin{vmatrix} x & 0 & 0 \\ -1 & -5x & 0 \\ 2 & 1 & x \end{vmatrix} = -80$$

WEEK 10

Learning Indicators

1. *Transform systems of linear equations into a matrix form and state the matrix representing a linear transformation*
2. *Model and solve problems based on real life situations using matrices*

Introduction

The lessons in this week are dedicated to solving systems of linear equations and modelling and solving real-life problems using matrices. Learners will be guided to translate real-life problems into matrix form and solve same

FOCAL AREA 1: USING MATRICES TO SOLVE SYSTEMS OF LINEAR EQUATIONS

Solving systems of linear equations $AX = B$, using the matrix method follows the same principle as solving systems of equations. The matrix method for solving systems of linear equations is more helpful when the number of equations is many. In this section we will use both the matrix method and Cramer's rule to solve systems of linear equations.

Equations involving a 2 × 2 matrix

Solving equations involving 2×2 matrices involve applying all the concepts learnt in previous sections in matrices.

Example 1

Find the values of x and y in $\begin{pmatrix} -2x+3 & -4 \\ 3 & 3y+4 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ 3 & 15 \end{pmatrix}$

Solution

$$\begin{pmatrix} -2x+3 & -4 \\ 3 & 3y+4 \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ 3 & 15 \end{pmatrix}$$

$$-2x + 3 = -7$$

$$x = 5$$

And $3y + 4 = 15$

$$y = \frac{11}{3}$$

In Example 1, the entries of the matrices on the left hand and the right hand were compared to each other. Since the matrices are equal the entries with the variables were equated and a solution was found.

Example 2

Given that $\begin{pmatrix} 2 & 3 \\ 4 & x \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, find x.

Solution

$$\begin{pmatrix} (2 \times 1) + (3 \times 2) \\ (4 \times 1) + (2 \times x) \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 4 + 2x \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$x = \frac{7}{2}$$

In Example 2, the concept of multiplication of matrices were applied before equating the entry with the variable to its corresponding entry.

Using Cramer's rule for involving 2 × 2 matrix

With Cramer's rule you will need to apply the determinant of a matrix to find the solution of the system $AX = B$.

Given two linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, rewrite the equations in the form $AX = B$.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Note

A is the coefficient matrix, X is the variable and matrix B is the constant matrix. We have to find the determinants so that $\det(A) = a_1b_2 - b_1a_2$,

$$\det(A_x) = \begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix} = c_1b_2 - b_1c_2 \text{, and}$$

$$\det(A_y) = \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} = a_1c_2 - c_1a_2$$

Therefore $x = \frac{\det(A_x)}{\det(A)}$, and $y = \frac{\det(A_y)}{\det(A)}$

Example 3

Solve the for x and y in the equation $3x + y = 5$ and $x + y = 1$.

Solution

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$\det(A) = 3 - 1 = 2$

To find $\det A_x = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix}$

$= 4$

To find $\det A_y = \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}$

$= -2$

$x = \dfrac{\det(A_x)}{\det A}$ and $y = \dfrac{\det(A_y)}{\det A}$

$= \dfrac{4}{2}$ $\quad = -\dfrac{2}{2}$

$= 2$ $\quad = -1$

Equations involving 3 × 3 matrix

Using Cramer's rule for involving 3×3 matrix

With the Cramer's rule you will need to apply determinant of a 3×3 matrix to find the solution of the system $AX = B$.

Given the system for 3 equations, say $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, rewrite the equations in the form $AX = B$.

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Then find $\det(A)$.

Use $\begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}$ to find $\det(A_x)$, $\begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}$ to find $\det(A_y)$ and $\begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$ to find $\det(A_z)$. Here to $x = \dfrac{\det(A_x)}{\det(A)}$, $y = \dfrac{\det(A_y)}{\det(A)}$ and $z = \dfrac{\det(A_z)}{\det(A)}$

Example 4

Solve for x, y and z in the equations $2x - y + 2z = -5$, $x + 3y + 4z = 1$ and $x + 2y + 2z = -2$.

Solution

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix} = -8$$

$$\det(A_x) = \begin{vmatrix} -5 & -1 & 2 \\ 1 & 3 & 4 \\ -2 & 2 & 2 \end{vmatrix} = 36$$

$$\det(A_y) = \begin{vmatrix} 2 & -5 & 2 \\ 1 & 1 & 4 \\ 1 & -2 & 2 \end{vmatrix} = 4$$

$$\det(A_z) = \begin{vmatrix} 2 & -1 & -5 \\ 1 & 3 & 1 \\ 1 & 2 & -2 \end{vmatrix} = -14$$

Therefore

$x = \frac{\det(A_x)}{\det A} = \frac{36}{-8}$ $\quad$ $y = \frac{\det(A_y)}{\det A} = \frac{4}{-8}$ $\quad$ $z = \frac{\det(A_z)}{\det A} = -\frac{14}{-8}$

$x = -4.5$ $\quad$ $y = -0.5$ $\quad$ $z = 1.75$

FOCAL AREA 2: USING MATRICES TO MODEL AND SOLVE REAL-LIFE PROBLEMS

Input-Output Analysis

It is useful to use an inverse matrix to solve a system of equations when it is necessary to solve repeatedly a system of equations with the same coefficient matrix but different constant matrices. Input–output analysis is one such application of this method.

In an economy, some of the output of an industry is used by the industry to produce its own product. For example, an electric company like the Volta River Authority (VRA) uses water from the Akosombo dam and electricity to produce electricity, and a Ghana Water Company Limited (GWCL) uses water and electricity to produce drinking water.

Input–output analysis attempts to determine the necessary output of industries to satisfy each other's demands plus the demands of consumers.

Consider the Ghanaian manufacturing industrial economy to be predominantly dependent on electricity and water. Suppose that the production of *GH¢* 1.00 worth of electricity requires *GH¢* 0.05 worth of water bought from GWCL and *GH¢* 0.02 worth of electricity. Suppose also that *GH¢* 0.08 worth of water and *GH¢* 0.10 worth of electricity is needed to produce *GH¢* 1.00 worth of water. The input–output matrix, A is

Input Requirements

Input − Output matrix, A = **From** $\quad$ *Electricity Water* $\quad$ $\begin{matrix} \textit{Electricity} & \textit{Water} \\ \end{matrix}$ $\begin{bmatrix} 0.02 & 0.05 \\ 0.10 & 0.08 \end{bmatrix}$

The amount of electricity or water required by consumers other than VRA and GWCL is called the final demand on the economy and represented as a column matrix. Suppose that for our example, Ahenasa's textile company needs *GH¢* 2 *million* worth of electricity and *GH¢* 3 *million* worth of water for production, then the final demand matrix is

$D = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ where $d_V = 2$ and $d_G = 3$ for reference's sake

We can represent the total output of each utility company, i.e., VRA and GWCL, (in millions of Ghanaian cedis) as follows

$x = total\ output\ of\ VRA$

$y = total\ output\ of\ GWCL$

The objective of input-output analysis is to determine the values of x and y that will satisfy the amount the consumer demands. To find these values, consider VRA. The amount of output from VRA left from the consumption by GWCL for the consumer

$d_V = x - (amount\ of\ electricity\ used\ by\ the\ two\ industries)$

To find the amount of electricity used by both companies in the industry, refer to the input–output matrix. Production of x million Ghanaian cedis worth of electricity takes $0.02x$ of electricity and the production of y million Ghanaian cedis worth of water requires $0.05y$ of electricity. Thus,

$Amount\ of\ electricity\ used\ by\ the\ industries = 0.02x + 0.05y$

$d_V = x - (0.02x + 0.05y)$

$\Rightarrow 2 = 0.98x - 0.05y$

A similar analysis could be made for the input and output for GWCL to obtain

$d_G = y - (0.10x + 0.08y)$

$\Rightarrow 3 = -0.10x + 0.92y$

And a system of two linear equations with two variables can be formed as such

$$\left.\begin{aligned} 0.98x - 0.05y &= 2 \\ -0.10x + 0.92y &= 3 \end{aligned}\right\}$$

An alternative is to use matrix and algebra of same to obtain the system and solve same

If $X = total\ output\ of\ the\ two\ industries\ of\ the\ economy$, *then*

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$AX = \begin{bmatrix} 0.02 & 0.05 \\ 0.10 & 0.08 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

AX represents the cedi amounts of products used in production for the two industries. Thus, the amount available for consumer demand is $X - AX$. As a matrix equation, we can write

$X - AX = D$

$(I - A)X = D$, where I is a 2×2 identity matrix

Assuming that the inverse of $I - A$ exists and is represented by $(I - A)^{-1}$,

$X = (I - A)^{-1} D$

For our immediate problem,

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.02 & 0.05 \\ 0.10 & 0.08 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98 & 0.05 \\ 0.10 & 0.92 \end{bmatrix}$$

Since $\begin{vmatrix} 0.98 & 0.05 \\ 0.10 & 0.92 \end{vmatrix} \neq 0$, $(I - A)$ is invertible and

$$(I - A)^{-1} = \begin{pmatrix} \frac{4600}{4483} & -\frac{250}{4483} \\ -\frac{500}{4483} & \frac{4900}{4483} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{4600}{4483} & -\frac{250}{4483} \\ -\frac{500}{4483} & \frac{4900}{4483} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1.8848985054651 \\ 3.0559892928842 \end{pmatrix} \approx \begin{pmatrix} 1.88 \\ 3.06 \end{pmatrix}$$

The solution suggests that *GH¢* 1.88 *million* worth of electricity and *GH¢* 3.06 *million* worth of water needs to be produced by VRA and GWCL to meets the demands of production and other consumers

Example 5

A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: A, B, and C. For the experiment it's important that the subject consumes exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	A	B	C
Potassium / *mg*	50	75	10
Protein / *g*	5	10	3
Vitamin D / units	90	100	50

Solution

Let a, b and c represent the number of ounces of A, B and C foods to be eaten daily respectively. The following system can be derived from the table

$$\begin{cases} 50a + 75b + 10c = 500 \\ 5a + 10b + 3c = 75 \\ 90a + 100b + 50c = 1150 \end{cases}$$ which can further be simplified to

$$\begin{cases} 10a + 15b + 2c = 100 \\ 5a + 10b + 3c = 75 \\ 9a + 10b + 5c = 115 \end{cases}$$

The system can be written in matrix form as

$$\begin{pmatrix} 10a & 15b & 2c \\ 5a & 10b & 3c \\ 9a & 10b & 5c \end{pmatrix} = \begin{pmatrix} 100 \\ 75 \\ 115 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 15 & 2 \\ 5 & 10 & 3 \\ 9 & 10 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 100 \\ 75 \\ 115 \end{pmatrix}$$

The coefficient matrix, $A = \begin{pmatrix} 10 & 15 & 2 \\ 5 & 10 & 3 \\ 9 & 10 & 5 \end{pmatrix}$, the variable matrix, $S = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

and $B = \begin{pmatrix} 100 \\ 75 \\ 115 \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 10 & 15 & 2 \\ 5 & 10 & 3 \\ 9 & 10 & 5 \end{vmatrix} = 150$$

$$\det(A_a) = \begin{vmatrix} 100 & 15 & 2 \\ 75 & 10 & 3 \\ 115 & 10 & 5 \end{vmatrix} = 750$$

$$\det(A_b) = \begin{vmatrix} 10 & 100 & 2 \\ 5 & 75 & 3 \\ 9 & 115 & 5 \end{vmatrix} = 300$$

$$\det(A_c) = \begin{vmatrix} 10 & 15 & 100 \\ 5 & 10 & 75 \\ 9 & 10 & 115 \end{vmatrix} = 1500$$

Therefore

$a = \dfrac{\det(A_a)}{\det(A)} = \dfrac{750}{150}$ $\quad$ $b = \dfrac{\det(A_b)}{\det A} = \dfrac{300}{150}$ $\quad$ $c = \dfrac{\det(A_c)}{\det A} = \dfrac{1500}{150}$

$a = 5$ $\quad$ $b = 2$ $\quad$ $c = 10$

Implying that 5 ounces of A, 2 ounces of B and 10 ounces of C need to be eaten in a day to fulfil the daily nutritional requirements

Learning Tasks

1. Learners create a simultaneous linear equation while the other pair transforms it into a matrix and vice versa. Learners change roles and summarise their findings
2. Learners translate information from a real-life problem into matrix for, perform operations on the matrices obtained and interpret results in the context of the problem

PEDAGOGICAL EXEMPLARS

The following pedagogical approaches are suggested for facilitators to take learners through;

Collaborative, experiential and problem-based learning.

1. Task learners as individuals or in pairs (in the case where some learners seem to be struggling) to solve systems of linear equations using Cramer's rule.

   Facilitators are encouraged to break the procedure into simpler tasks for learners. Learners can be tasked to first write out the coefficient matrix, find cofactors, then determinants and so forth. Assistance should be given to learners who may be struggling with some of the tasks. Struggling learners should be complimented on the fulfilment of the little or more tasks that they are able to perform

   Learners should be guided to use the scientific calculator and / or other relevant technological tools to aid learning

2. In mixed ability groups, considering gender, where applicable and other cross-cutting issues, task learners translate several real-life problems into matrix forms, perform matrix operations and interpret result. Facilitators are encouraged to formulate problems within learners' context, learning area and society.

3. Learners should be tasked to present solutions to problems to the larger group. Roles should be distributed based on learners' ability and preference. Learners should be given the opportunity to decide on how to present evidence of learning. While some learners should be challenged to use ICT tools for their presentation, oral presentation with or without visualisations on carboard, flipboards etc. should also be accepted

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Use Cramer's rule to find the values for x and y in the equations $2x + y = 10$ and $3x - 2y = 29$.
2. Use Cramer's rule to find the values for x, y and z in the equations

   $-x + 2y - 3z = 1$, $2x + z = 0$ and $3x - 4y - 4z = 2$

Assessment Level 3: Strategic reasoning

A simplified economy has three major industries: mining, manufacturing, and transportation. The input-output matrix for this economy is

		Input requirement of mining	manufacturing	transportation
From	*mining*	0.15	0.23	0.11
	manufacturing	0.08	0.10	0.05
	transportation	0.16	0.11	0.07

Set up, but do not solve, a matrix equation that when solved will determine the gross output needed to satisfy consumer demand for *GH¢* 50 million worth of mining, *GH¢* 32 million worth of manufacturing, and *GH¢* 8 million worth of transportation.

Reminder

Learners should be ready to submit their Group Project Work by the end of Week 10. Please ensure to score it promptly and submit the score on the STP.

Section 6 Review

This section reviews all the lessons taught for weeks 9 and 10. This section built on lessons in the year one additional mathematics teacher manual to introduce learners to finding inverses of matrices, determinant of 3×3 matrices and applications of matrices for solving real-life problems. Pedagogical exemplars which promote the achievement of the stated learning indicators and by extension, outcomes and content standard and still take into consideration cross-cutting issues were suggested. Assessment strategies that assess learning on varying levels of DoK are promoted

SECTION 7: CORRELATION

Strand: Handling Data

Sub-Strand: Organising, representing and interpreting data

Learning Outcome: *Describe the nature and strength of relationship between two given variables using scatter diagrams and correlation coefficients*

Content Standard: Demonstrate understanding of the nature and strength of relationship between two given variables

INTRODUCTION AND SECTION SUMMARY

Some fundamental skills in statistics are the ability to organise, represent and analyse data. These abilities are essential life skills. There are different types of data in statistics. In this section we will discuss univariate and bivariate data and correlation. Understanding these concepts can provide insights into data distribution and relationships. The ability to visualise and analyse data will help one understand outcomes of a particular phenomenon. Using practical examples and interactive activities will help learners understand and appreciate these concepts and make the learning of statistics practical.

The weeks covered by the section are:

Week 11

1. *Univariate and Bivariate data*
2. *Scatter plots*

Week 12

1. *Analyse scatter plots*
2. *Spearman's rank correlation*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners collaborate to plot graphs and interpret the relationship between the variables. Therefore, collaborative learning, experiential learning and initiate talk for learning, problem-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. Then, extend activities for some learners to use formulas and computer applications to solve problems.

ASSESSMENT SUMMARY

The various concepts to be covered under this section should be assessed using various forms of assessments modes to ascertain learners' performance. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP). The following mandatory assessments would be conducted and recorded for each learner:

***Week 12**: End of Semester Examination: Teachers are supposed to administer end of semester examination covering all the concepts taught from Week 1.*

Note

For additional information on how to go about the end of semester exams, please refer to Appendix C at the end of Section 7.

WEEK 11

Learning Indicators

1. *Distinguish between univariate and bivariate data and give examples and explain the concept of correlation*
2. *Construct a scatter plot using given dataset and use it to describe the relationship between two variables*

Introduction

In week 11 we will distinguish between univariate and bivariate data, state examples of the data and discuss correlation. We will use scatter plots to describe the relationship between two variables. Remember that we discussed measures of central tendencies and dispersion in year 1.

FOCAL AREAS 1: DISTINGUISHING BETWEEN UNIVARIATE AND BIVARIATE DATA AND THE CONCEPT OF CORRELATION

Univariate and bivariate data

Univariate data means one variable (one type of data). It involves the measurement of a single attribute in a dataset. Univariate data are mostly analysed with descriptive analysis. In the instance of univariate data the measures of central tendencies give information about the central value in the data and the measures of dispersion give information about the spread of the data. Examples of univariate data are ages of learners, heights of learners, travel time etc. With univariate data you can make plots like bar graphs, pie charts and histograms.

Bivariate data means two variables (two types of data). With bivariate data we have **two** sets of related data we want to **compare**. The analysis of bivariate data focuses on understanding the relationship, or association, between these two variables. Bivariate data can be analysed using correlation and regression. Visually bivariate data can be represented with scatter plots and line graphs. Examples of bivariate data age of learners and their height, price of goods and quantities of goods, hours learners spend studying and their test scores. Bivariate data can be used to make comparison.

Table 1: *Difference between univariate and bivariate data*

Univariate data	Bivariate data
Involves a single variable	Involves two variables
Does not deal with causes or relationships	Deals with causes and relationships

Univariate data	Bivariate data
The purpose is to describe	The purpose is to explain
It does not contain any dependent variable	It contains only one dependent variable

Concept of correlation

Correlation is a statistical measure that is used to describe the nature and strength of the relationship between two variables. A scatter plot is used to illustrate the relationship between the data. The relationship can be a positive correlation, negative correlation or no correlation.

Positive correlation: As one variable increases, the other variable also increases.

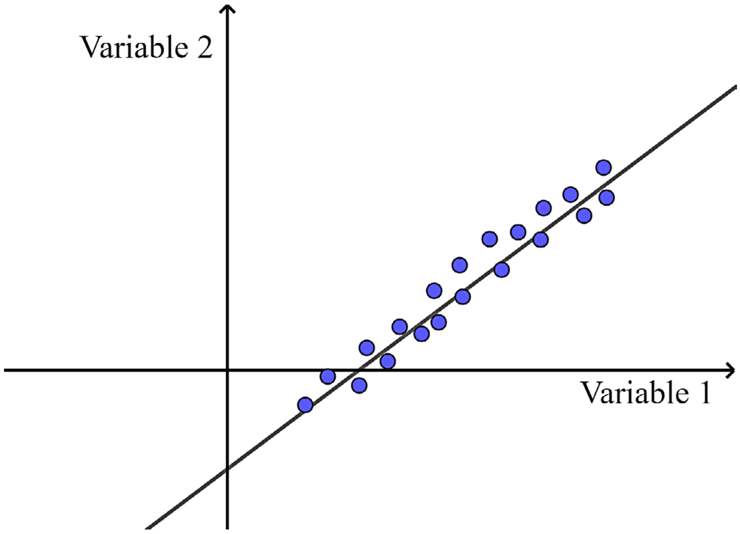


Figure 7.1

Negative Correlation: As one variable increases, the other variable decreases.

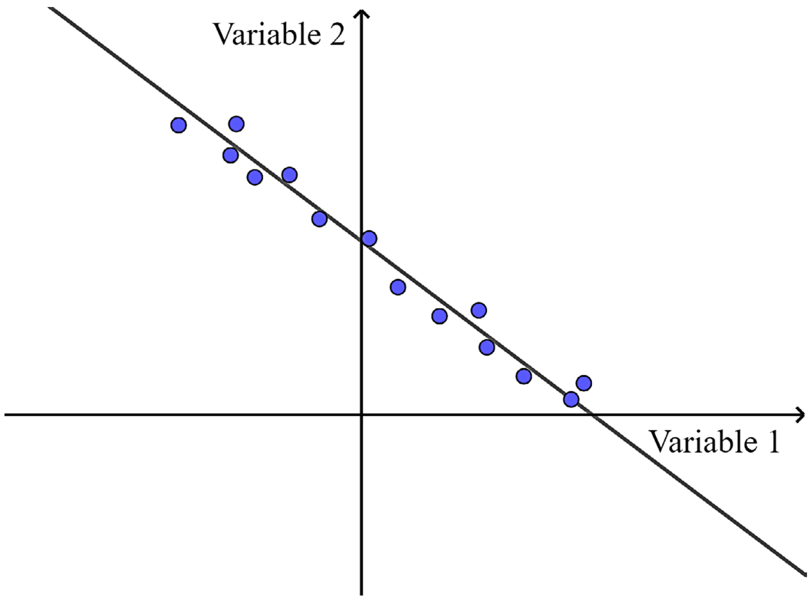


Figure 7.2

No Correlation: No apparent relationship between the variables.

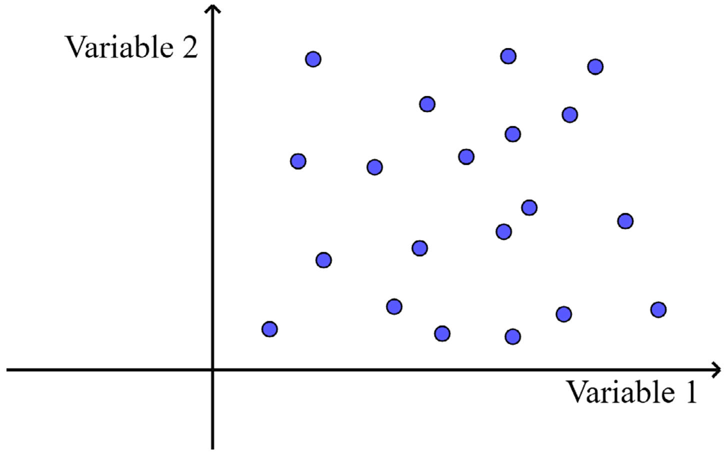


Figure 7.3

For example, correlation can be used to determine how performance relates to the number of hours used to study and also to make decisions on the likelihood of rains in the dry season, likelihood of disease outbreak in a community, number of people in the house and amount of water used. To interpret correlation, values between -1 to 1 are used. Values close to -1 or 1 represent a strong correlation and values close to 0 indicate negligible correlation.

Size of Correlation	Interpretation
Between ± 0.90 and ± 1.00	Very high positive (negative) correlation
Between ± 0.70 and ± 0.90	High positive (negative) correlation
Between ± 0.50 and ± 0.70	Moderate positive (negative) correlation
Between ± 0.30 and ± 0.50	Low positive (negative) correlation
Between ± 0.00 and ± 0.30	Negligible correlation

FOCAL AREA 2: CONSTRUCTING A SCATTER PLOT USING GIVEN DATASET

Representation of bivariate data in a scatter plot

A *scatter plot* is a type of graph that is used to represent the relationship between two variables. It is the simplest way to represent bivariate data. It is also known as a scatter diagram. Scatter plots are used to identify relationships, show trends, spotting patterns and perform correlation analysis.

To draw scatter plots the independent and the dependent variable must be identified

Independent variable: This is the variable that is being manipulated in an experiment. It is the input. When drawing a scatter plot the independent variable is plotted on the x axis.

Dependent variable: This is the variable that is being measured. It is the outcome. The dependent variable is plotted on the y axis on a scatter plot.

For example, a teacher wants to find the effect of a teaching strategy on the performance of learners. With this the **independent variable** is the **teaching strategy** and the **performance of learners** is the **dependent variable**.

Scatter plots are used in situation where a large data set is given, when each comprises a pair of values and the given data is in a numeric form.

Example 1

Determine the independent and dependent variables from the following scenarios

1. The effect of the hours of studying on test scores.
2. The wages earned from the number of days worked.

Solution

1. Hours of studying is independent and test scores is dependent
2. Number of days worked is independent and the wages earned is dependent.

Constructing a scatter plot manually

To construct a scatter plot

1. Identify the independent and dependent variables
2. Assign the independent variable to the x axis and the dependent variable to the y axis
3. Plot the points on the (x, y) axis
4. Label the axes and include a chart title.

Constructing a scatter plot using excel

1. Identify the independent and dependent variables, arrange the values so that independent variable is on the left and the dependent variable is on the right.
2. Go to insert tab, in the chart group click on scatter and select the scatter plot.
3. Use the chart tools to add titles on the axes and to give an overall title.

Example 2

The following data represents the scores of learners in geography and economics. Construct the data on a scatter plot.

Geography	Economics
37	44
60	75
20	60
70	25
40	58
55	69
65	43
90	56
72	38
53	29
23	44
15	34
71	58
56	45
43	78
73	64
55	43
65	67
80	45
45	71

Solution

Using Excel

	Geography	Economics
1	Geography	Economics
2	37	44
3	60	75
4	20	60
5	70	25
6	40	58
7	55	69
8	65	43
9	90	56
10	72	38
11	53	29
12	23	44
13	15	34
14	71	58
15	56	45
16	43	78
17	73	64
18	55	43
19	65	67
20	80	45
21	45	71

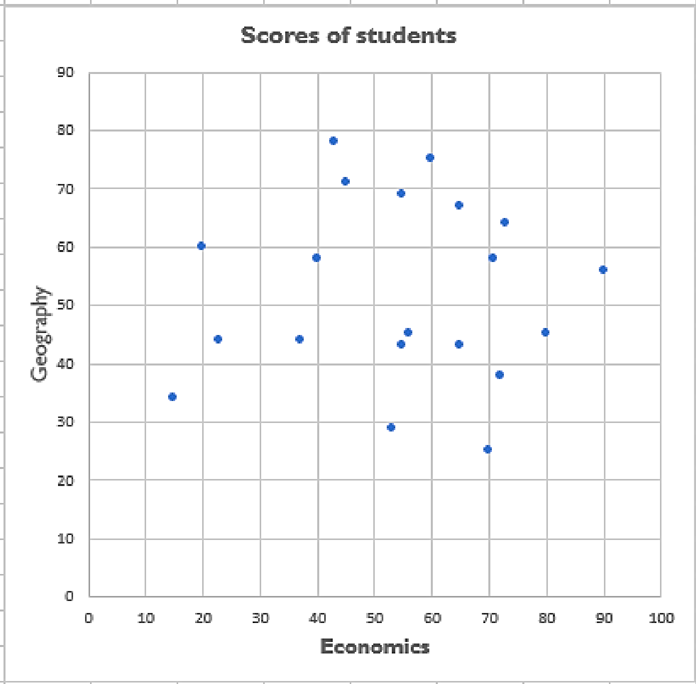


Figure 7.4:

Excel output

Constructing by Hand

A scatter plot to show students geography test score against their economics test score

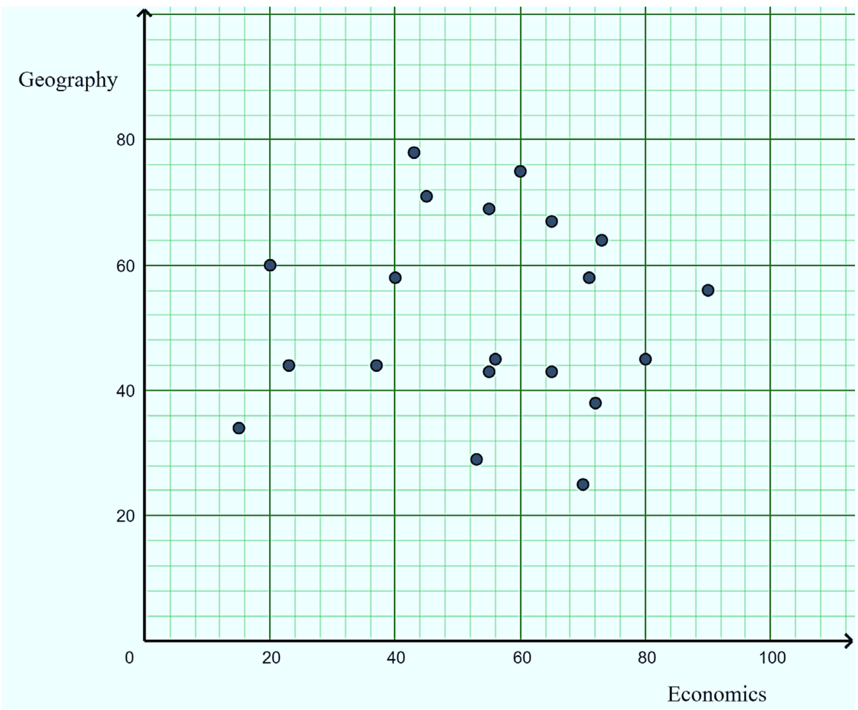


Figure 7.5

FOCAL AREA 3: DESCRIBING THE RELATIONSHIP THE RELATIONSHIP BETWEEN TWO VARIABLES

The scatter plot is a graphical representation of bivariate data. Another way to describe the relationship between two variables is the use of functions. For our lessons, linear functions (which represent straight lines) will be used to approximate the relationship. We can only approximate the relationship since the data points may not necessarily lie on a straight line.

We will seek to find the line which best approximates the data points. This line is called the line of best fit.

Therefore, it must be noted, that our approximation of the relationship (line of best fit) is only appropriate for predicting data points within a given interval. Furthermore, points which are relatively far from the line of best fit are called outliers. These points affect the accuracy of the line of best fit for predictions.

We will obtain this line through observation. In the third year, learners will discover how to use the least squared method to obtain a more accurate function to represent such relationships.

Example 3

The scatter plot below shows the relationship between the ages of and the heights of some students.

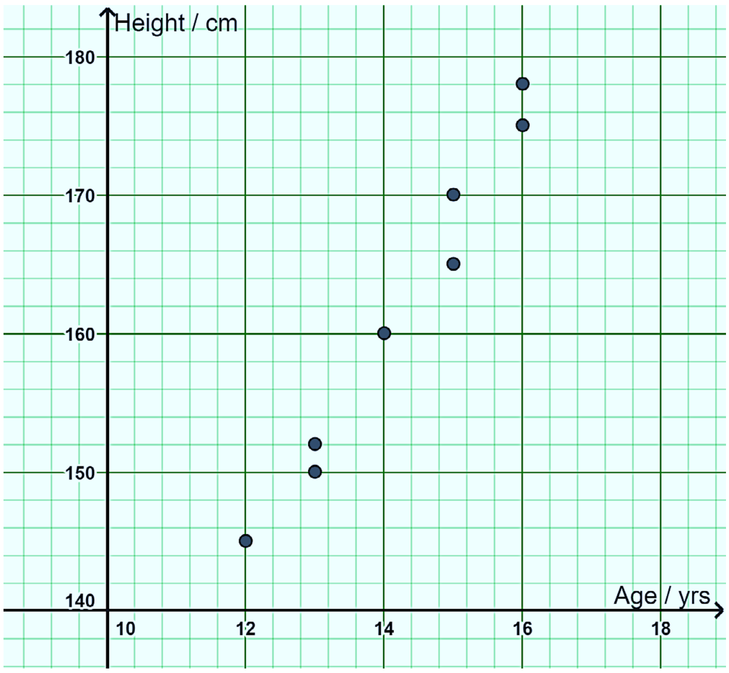


Figure 7.6

The line of best fit is that which will pass through most number of data points or at least, is closest to most number of points. For this example, the line that passes through (13, 152) and (14, 160) look closest to most of the points as shown in Figure 7.7.

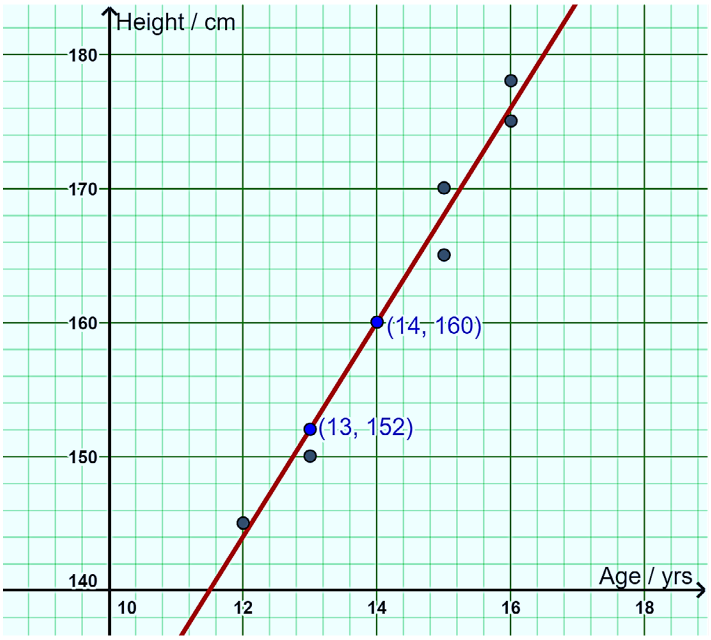


Figure 7.7

The gradient of the line can be found thus: $\frac{160 - 152}{14 - 13} = 8$

With gradient obtained and either of the points: (13, 152) or (14, 160), we can find the equation of the line of best fit. We'll choose (13, 152) and hence, the equation of the line of best fit can be obtained as such

$y - 152 = 8(x - 13)$

$y = 8x + 48$

In this example, the line of best fit obtained is most appropriate for predicting the height of students whose weight is between 13 and 14 years, but can be extended for being within the range of the age data, so between 12 and 16 years. This is known as interpolation. Anything outside of this range would be extrapolating and is liable to be unreliable.

Example 4

The graph shows a scatter plot showing the relationship between the number of hours that 6 students exercise and their respective weights.

1. Use a linear function to model the relationship between the number of hours used for exercise and the weights
2. Estimate the weight of a student who exercises for 3.5 hours a week

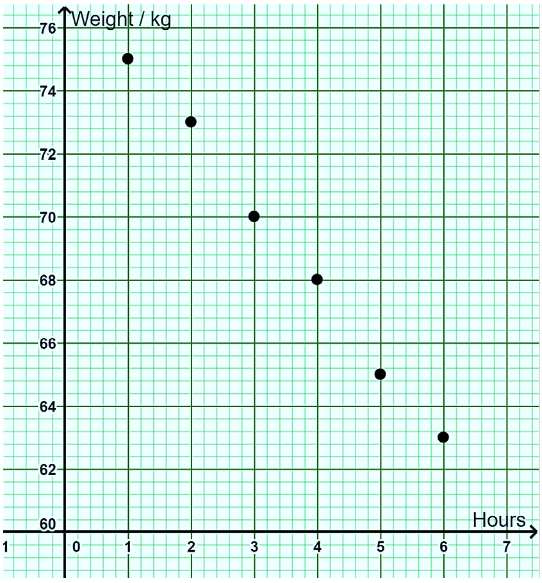


Figure 7.8

Solution

1. As in the previous, two points on the scatter plots is sufficient to find a linear function (line of best fit) to model the relationship.

   Interestingly, the points on the scatter plot seem to be lie on a straight line and hence any two points chosen will yield a model which provides predictions with very small margins of errors.

   It is better however to chose points which are closer to the point when the hour is 3.5 as a prediction would be made for that. (3, 70) and (4, 68) meet this description.

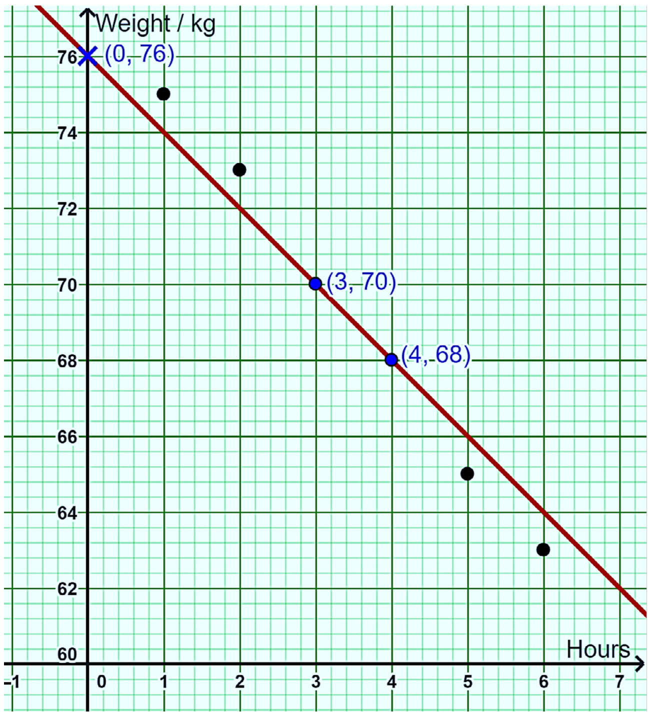


Figure 7.9

Let the linear function be

$W(H) = aH + b$ where,

W represents the weight of a student who exercises for H hours,

a represents the gradient and

b represents the weight of a student who does not exercise at all

$a = \frac{70 - 68}{3 - 4} = -2$

$\Rightarrow W(H) = -2H + b$

Since (3, 70) lies on the line and thus satisfy the function i.e., $W(3) = 70\ kg$,

$70 = -2(3) + b$

$b = 76$

$\therefore W(H) = -2H + 76$

Note that the value of b can be obtained by reading the intercept of the vertical axis where possible. In the current example, the line drawn / contructed through (3, 70) and (4, 68) cuts the weight axis at (0, 76) and thus, the value of b is 76

2. $W(3.5) = -2(3.5) + 76$

$= 69\ kg$

A student who exercises for 3.5 hours a week is likely to weigh 69 kg

Learning Tasks

1. Learners distinguish between univariate or bivariate data
2. Learners recollect how to plot coordinates in the $x - y$ plane
3. Learners plot points on a scatter diagram given a simple dataset
4. Learners create bivariate data and construct a scatter plot to illustrate the relationship between the data
5. Learners construct scatter diagrams from given datasets

PEDAGOGICAL EXEMPLARS

The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Collaborative / Exploratory Learning**: Learners in groups discuss univariate and bivariate data.
   a. Begin with clear, concise explanations of univariate and bivariate data. Use simple, relatable examples (e.g., heights of learners in the class)

- **b.** Utilise charts, graphs and visual representations to help learners grasp the concepts
- **c.** Introduce concepts with interactive sessions that include Q&A sessions to keep learners engaged
- **d.** Pair below-average learners with slightly more proficient peers to provide additional support and encouragement
- **e.** Encourage group discussions and problem-solving activities to deepen understanding
- **f.** Regularly change groups to ensure all learners have the opportunity to work with different peers and benefit from varied perspectives

2. Experiential Learning: Learners plot scatter diagrams and describe the nature of relationship between the variables

- **a.** Utilise charts, graphs and visual representations to help learners grasp the concepts
- **b.** Provide datasets and have learners construct scatter diagrams. Use real-life data to make the activity relevant
- **c.** Work through several examples together, providing step-by-step instructions on plotting points and drawing scatter diagrams

  Ensure that instructions are clear and repeated if necessary
- **d.** Differentiate instruction by offering various levels of challenge within the same activity
- **e.** Offer flexible grouping strategies to allow for peer learning and mentorship

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. The scatterplot compares the number of hours of sleep 9 adults had the previous night, and the length of time taken to solve a puzzle. The line of best fit has also been drawn.

Use the line of best fit to predict the length of time it would take an adult to solve the puzzle if they had 7 hours sleep

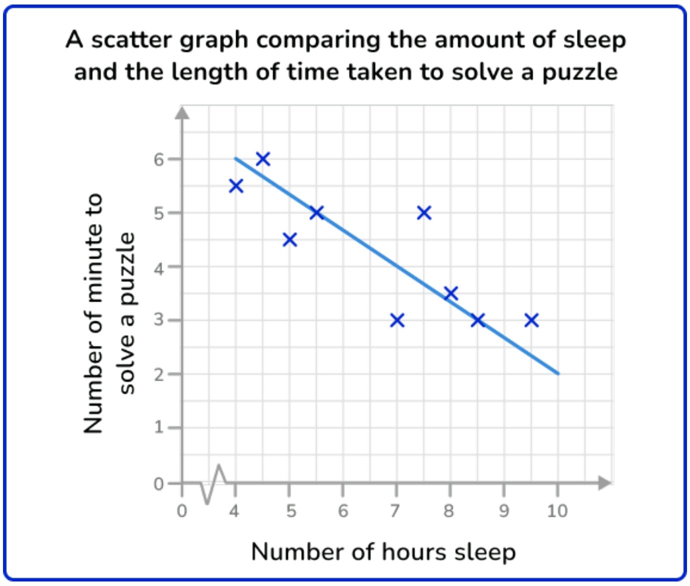


Figure 7.10

2. The Body Mass Index (BMI) of a person is an estimate of body fat based on height, h and weight, w. It doesn't measure body fat directly, but instead uses an equation to make an approximation. BMI can help determine whether a person is at an unhealthy or healthy weight. The formula for calculating a person's BMI is given by $BMI = w\frac{}{h^2}$. The following table shows some values heights recorded with their corresponding weights

Height	77	75	76	70	70	73	74	74	73
Weight	230	220	212	190	201	245	218	260	196

Draw a scatter diagram to visualise the data and describe the relationship between the weight and the height

3. Draw a scatter plot for the given data that shows the number of games played and scores obtained in each instance

Number of games	3	5	2	6	7	1	2	7	1	7
Scores	80	90	75	80	90	50	65	85	40	100

WEEK 12

Learning Indicators

1. *Analyse and describe visual data in a scatter plot by interpreting the relationship between given bivariate datasets*
2. *Describe the Spearman's rank correlation coefficient and interpret the result within a given situation*

Introduction

In week 11, learners discussed the types of data, correlation, scatter plots and lines of best fit. We described the types of relationship between datasets. This week we will plot scatter diagrams of datasets, analyse and interpret the nature of the data. Visualising scatter plots helps to identify patterns, trends and correlation. In this week's lessons, our focus will be on the correlation. It will involve interpreting the relationship between bivariate data. Understanding the relationship between the data will help us draw conclusions of a given data.

FOCAL AREA 1: ANALYSING AND DESCRIBING VISUAL DATA IN A SCATTER PLOT BY INTERPRETING THE RELATIONSHIP BETWEEN GIVEN BIVARIATE DATASETS

As discussed in the previous week, the correlation between bivariate data could be positive, negative or have no correlation. With the use of examples, we can plot scatter diagrams by hand or with the use of a technological tool and analyse the visual data. The focus will be on whether the relationship between the data is positive linear, negative linear, non-linear and no association.

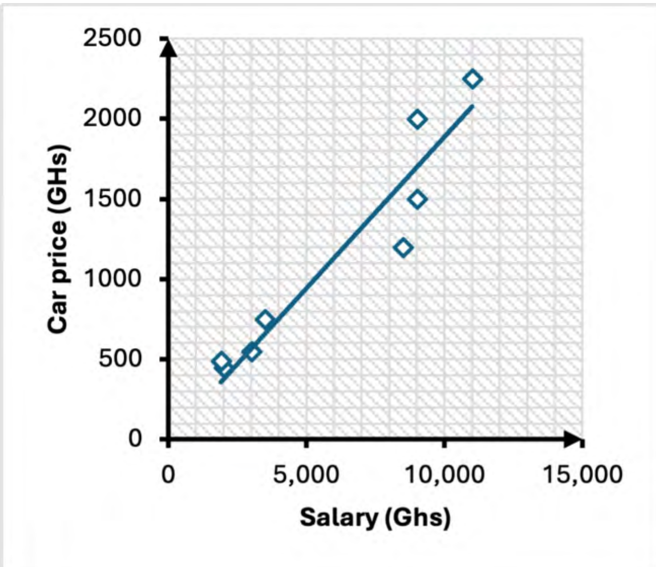


Figure 7.11

The scatter diagram above shows positive linear relationship between salary earned by government workers in Ghana and the price of car they own. That is as the salary increases there is a greater likelihood to buy a more expensive car.

Example 1

The table below represents the scores of 10 students in Mathematics and English.

Mathematics	9	6	7	2	5	1	8	3	10	4
English	8	7	8	4	7	3	9	5	10	6

Draw a scatter diagram to represent the data and determine the type of correlation between the scores.

Solution

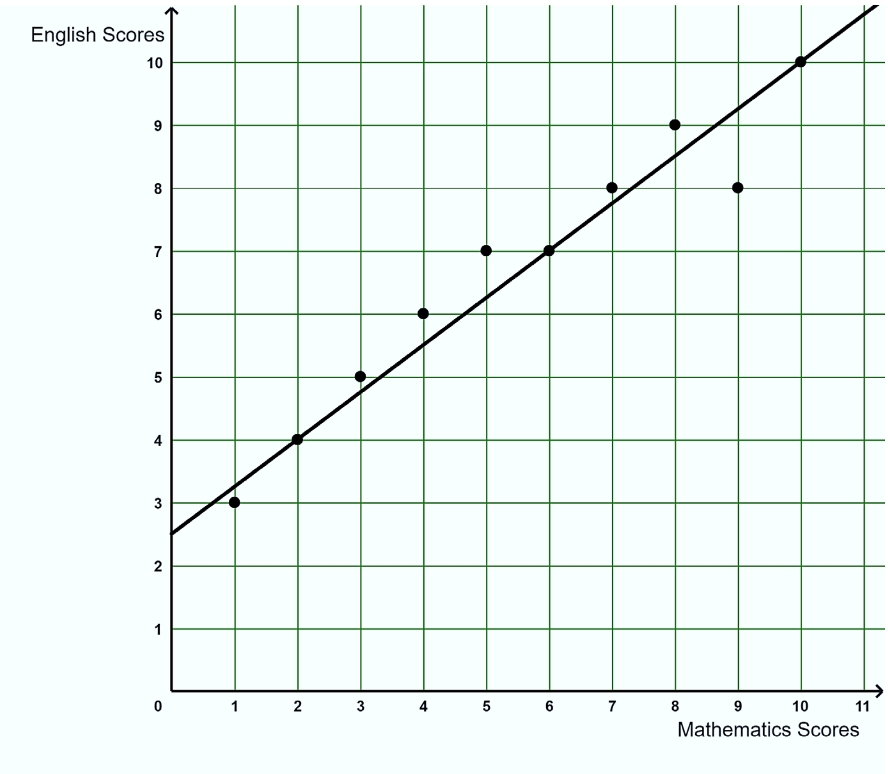


Figure 7.12

It can be observed that the line of best fit is an increasing linear function and thus has a positive gradient. Also, the points are located close to the line of best fit. Therefore, it can be concluded that the correlation is a strong positive correlation.

Example 2

Learners are exploring the relationship between scores on their mid-semester exam and final semester exam in mathematics. Here are some of the scores reported by the class

Name	Mid-Semester score	Semester Score
Learner 1	88	84
Learner 2	71	80
Learner 3	75	77
Learner 4	94	95
Learner 5	68	73
Learner 6	55	60
Learner 7	50	62
Learner 8	53	65
Learner 9	58	70
Learner 10	65	73

Solution

Figure 7.13

The mid-semester scores and exams scores are positively correlated, meaning that the better the learners performed in the mid-semester, the better they performed in the end of semester exams.

FOCAL AREA 2: DESCRIBING THE SPEARMAN'S RANK CORRELATION COEFFICIENT AND INTERPRET THE RESULT WITHIN A GIVEN SITUATION

The **Spearman's rank correlation** is a correlation analysis used to find the relationship between ranked variables. It tells how well the relationship between two variables can be described using a monotonic function, that is, it checks if the order (or rank) of data points is consistent between two variables.

Condition for use of Spearman's rank correlation: When both variables are in ordinal scale or rank order form or when the data set can be converted to ordinal form. The data must be at least ordinal and the scores on one variable must be monotonically related to the other variable.

Apply the formula $r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$ to find the Spearman's rank correlation coefficient.

Where n is the sample size and Σd^2 is the sum of the squares of the difference in ranks.

Example 3

The following are score obtained by 5 students in Physics and Biology

Physics	98	67	93	90	83
Biology	91	65	92	93	85

Illustrate the relationship between the scores with a scatter plot, calculate the Spearman's rank correlation and describe the correlation between the scores.

Solution

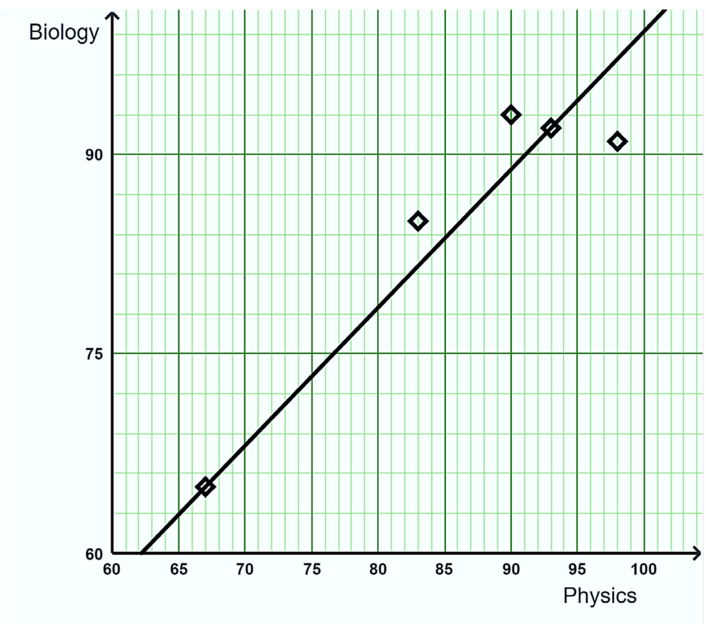


Figure 7.14

We can observe a positive correlation since the line of best fit is an increasing straight line (it has a positive gradient) and thus we expect a positive value for Spearman's rank correlation

Physics	Biology	Physics Rank	Biology Rank	Difference	Squared difference
98	91	1	3	-2	4
67	65	5	5	0	0
93	92	2	2	0	0
90	93	3	1	2	4
83	85	4	4	0	0

$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$

$= 1 - \frac{6(8)}{5(5^2 - 1)}$

$= 0.6$

Since this value is positive, it indicates a positive correlation. Therefore, there is a positive correlation between the rankings of learners in Physics andBiology.

Example 4

The table below shows the scores given by two teachers at an art competition in a school.

Names	Esinu	Aba	Hamdiatu	Salifu	Nhyira	Boateng	Dzifa	Deladem	Kwame
Teacher 1	15	9	13	12	17	6	14	9	11
Teacher 2	11	12	7	10	13	5	16	6	14

a. Describe the correlation between the scores of the two teachers with the aid of a scatter plot

b. Calculate the Spearman's rank correlation coefficient.

Solution

a.

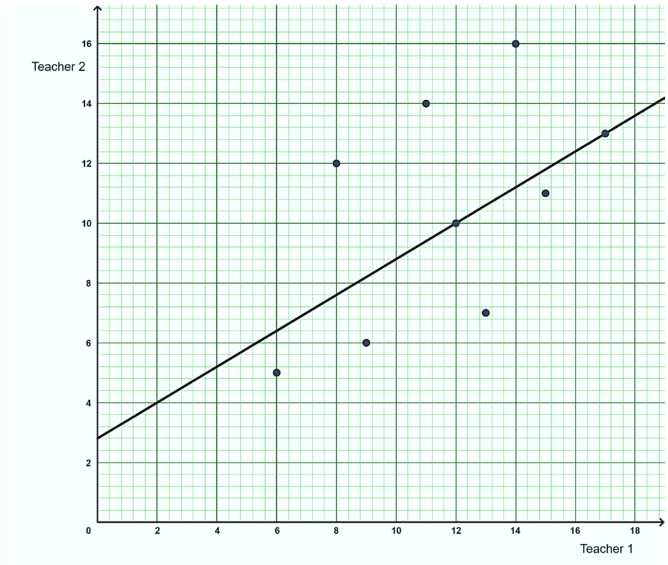


Figure 7.15

The line of best fit shows that the scores given by teacher 1 increases as the scores given by teacher 2 increases. We therefore expect a positive correlation value for Spearman's rank

b.

Teacher 1	Rank	Teacher 2	Rank	d	d^2
15	2	11	5	−3	9
8	8	12	4	4	16
13	4	7	7	−3	9
12	5	10	6	−1	1
17	1	13	3	−2	4
6	9	5	9	0	0
14	3	16	1	2	4
9	7	6	8	−1	1
11	6	14	2	4	16

$$r = 1 - \frac{6(\sum d^2)}{n(n^2 - 1)}$$

$$= 1 - \frac{6(60)}{9(9^2 - 1)}$$

$$= 0.5$$

The value obtained shows that correlation between the scores of the two teachers is moderate.

Learning Tasks

1. Learners describe the nature of relationship between weight and height using scatter plots.
2. Learners calculate and interpret Spearman's Rank correlation.
3. Learners draw conclusions based on the value of Spearman's rank correlation.

PEDAGOGICAL EXEMPLARS

Experiential / Collaborative Learning

1. Learners in groups calculate and interpret Spearman's Rank correlation.
2. Learners in groups discuss the type of data suitable for the use of Spearman's rank correlation.
3. Learners solve other examples and draw conclusions based on the value of Spearman's rank correlation.
   a. Begin with a simple explanation of correlation using everyday examples (e.g., the relationship between hours studied and test scores). Use visuals and concrete examples to illustrate positive and negative correlations.
   b. Provide learners with small, straightforward datasets. Guide them step-by-step through ranking the data.
   c. Use a simplified formula for Spearman's rank correlation coefficient and guide learners through each step, focusing on understanding rather than memorisation.
   d. Provide guided practice sessions where learners work through problems with support, gradually increasing independence.
   e. Discuss the interpretation of the correlation in simple terms, emphasising the idea of strong, weak, positive and negative correlations.
   f. Discuss various scenarios and have learners interpret the results in context.
   g. Facilitate discussions on interpreting results in various complex scenarios, including potential anomalies and outliers.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

Calculate the Spearman's rank correlation coefficient for the data and interpret the result.

x	35	54	80	95	73	73	35	91	83	81
y	40	60	75	90	70	75	38	95	75	71

Assessment Level 4: Extended Critical Thinking and Reasoning

1. The data below shows the time taken for people of different weights to complete a task.

Weight (kg)	55	60	65	70	75	80	85	90	95	100
Time (mins)	14	17	16	20	22	28	26	34	27	45

a. Draw a scatter plot to represent the given data.

b. Describe the correlation between the time taken to finish the task and weight of the individual.

c. Describe the relationship shown in the scatter plot.

d. Will it be appropriate to use the scatter plot to estimate the time it would take for an individual who weighs 40kg to complete a task? State a reason for your answer.

2. The data below show the number of hours learners spent learning for an examination and their scores at the end of the examination.

Hours spent	1	2	4	5	6	8	10	12	14	16	18	25
Score	20	28	23	85	32	63	52	60	58	68	69	80

a. Draw a scatter plot to represent the given data.

b. Describe the correlation between the hours spent learning and the score obtained.

c. Describe the relationship shown between the hours spent learning and the score obtained.

d. What can you say about the score of the learner who spent 5 hours learning?

Note

- *The Recommended Mode of Assessment for Week 12 is **End of Semester Examination.***
- *Refer to Appendix D at the end of Section 7 for further information on how to go about the end of semester examination.*

Section 7 Review

This section reviews all the lessons taught for weeks 11 and 12. Types of data (univariate and bivariate) were discussed. Support is provided to teachers to guide learners to develop their ability to represent data graphically by constructing scatter plots manually on grid sheet or with the aid of MS Excel and describe the relationship between bivariate data. It needs emphasising that all learners should be given the required support to experience learning for themselves. The use of scientific calculators and / or any other relevant technological tools should be promoted during learning. Assessment strategies should be flexible to allow learners to decide on their preferred modes of showing evidence of learning.

APPENDIX D

Structure of End of Semester Examination

a) *Cover content from weeks 1-12*

b) *Take into consideration DoK levels*

  i. *Section A- Multiple Choice (30 questions)*

  ii. *Section B- (5 short answer questions, all to be answered)*

  iii. *Section C- Real-life Application (4 questions, 2 to be selected).*

c) *Time: 2 hours.*

d) *Total Score: 100 marks.*

Table of specification, ToS. (See Appendix D in PLC Session 12 for sample ToS)

Table of Test Specification

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
1	1. Establishing De Morgan's laws of set theory 2. Applying De Morgan's laws of set theory 3. Applying laws of set theory	Multiple Choice	2	1	1		4
		Short answer					
		Real-life Application			1		1
2	1. Expanding binomial expressions 2. Applying binomial expansion to approximate exponential numbers	Multiple Choice	1	1	1		3
		Short answer		1			1
		Application					

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
3	1. Sums of sequences 2. Convergence and divergence of series 3. Recursive sequences 4. Arithmetic and geometric means of sequences	Multiple Choice	1	1	1		3
		Short answer		1			1
		Real-life Application		1			1
4	1. Solving quadratic inequalities 2. Solving real-life problems involving linear inequalities 3. Solving real-life problems involving quadratic inequalities	Multiple Choice	1	1			2
		Short answer					
		Real-life Application		1			1
5	1. Finding factors and zeroes of polynomial functions 2. Graphing polynomial functions with degrees higher than 2 3. Applying Descartes' rule of signs theorem 4. Applying the fundamental theorem of algebra 5. Complex conjugates theorem 6. Linear and quadratic factor theorems	Multiple Choice	1	1	1		3
		Short answer					
		Real-life Application					

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
6	1. Exploring the properties of a circle and its parts. 2. Deriving the equation of a circle.	Multiple Choice	1				1
		Short answer		1			1
		Real-life Application					
7	1. Deriving the equation of a circle from the endpoints of a diameter. 2. Tangents and Normal 3. Deducing the relation of various loci under given conditions.	Multiple Choice		1	1		2
		Short answer		1			1
		Real-life Application					
8	1. Transposing vectors 2. Dividing a line or vector in a given ratio 3. Finding and applying the dot product of vectors 4. Establishing and Applying the Sine and the Cosine Rule 5. Projection of one vector on a given vector	Multiple Choice	1	1	1		3
		Short answer					
		Real-life Application					
9	1. Revision of types of matrices and matrix algebra 2. Determinant of 3×3 matrices 3. Inverse of a 2×2 matrices	Multiple Choice	1	1	1		3
		Short answer		1			1
		Real-life Application					

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
10	1. Using matrices to solve systems of linear equations 2. Using matrices to model and solve real-life problems	Multiple Choice			1		1
		Short answer					
		Real-life Application					
11	1. Univariate and Bivariate data 2. Scatter plot	Multiple Choice	1	1	1		3
		Short answer					
		Real-life Application					
12	1. Analyse scatter plot 2. Spearman's rank correlation	Multiple Choice	1	1			2
		Short answer					
		Real-life Application			1		
	Total		11	17	11		39

SECTION 8: INDICES AND LOGARITHMS

Strand: Modelling with Algebra

Sub-Strand: Application of Algebra

Learning Outcome: *Apply indices and logarithms to solve real life problems, including logarithms with different bases, and sketch and interpret logarithmic functions.*

Content Standard: *Demonstrate understanding of the laws and properties of indices and apply the ideas to solve problems.*

INTRODUCTION AND SECTION SUMMARY

As nature would have it and by human design, most phenomena in everyday life like population growth and decay, depreciation of assets, the loudness of sound and many more cannot be accurately modelled with linear functions. Simply put, the outcomes of these situation are not so straightforward. A pair of values for the independent and dependent quantities are not necessarily in simple ratios (representing gradient or slope). Logarithmic functions as well as its inverse (exponential functions) as introduced to learners in week 5 lessons in section 2 of the year one teacher manual provide suitable alternative models and have a wide range of applications in real-life. The lessons in this section seek to reinforce learners' understanding of these functions but also make more vivid to learners the practical importance of mathematics in the real world.

The weeks covered by the section are:

Week 13

1. *Application of the laws of logarithms*
2. *Graphing logarithmic functions*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities, discussions on whole class and small group basis and presentations where learners engage in practical activities in transforming exponential functions into logarithmic and linear functions and otherwise and plotting graphs. Learners should be given the opportunity to work in groups through experiential learning activities and mixed-ability groupings. All learners, irrespective of their learning abilities, should be assisted to fully take part in investigations and presentation of findings. However, make considerations and accommodations for the different groups.

That is, offer approaching proficiency learners the opportunity to make oral presentations and use visual aids to deepen understanding. Where possible, graphing utilities, scientific calculators and any other relevant technological tools should be incorporated into instruction.

ASSESSMENT SUMMARY

Various forms of assessments should be carried out to ascertain learners' performance on the concepts that will be taught under this section. Teachers are entreated to administer these assessments and record them for onward submission into the Student Transcript Portal (STP).

WEEK 13

Learning Indicators

1. *Review indices and apply the idea to create and solve real life problems involving indices*
2. *Draw graphs of logarithmic functions using an appropriate technology and by hand, and interpret them*

FOCAL AREA 1: APPLICATION OF THE LAWS OF INDICES AND LOGARITHMS

Compound interest

Interest is money paid for money used. It can be paid by financial institutions to customers on savings or investments made or it could be paid by individuals or entities to financial institutions on loans which have been taken. The amount that was deposited/invested or borrowed as the case might be is called the principal. The interest to be paid is usually stated in terms of percentages which is fair as the principal amount would vary. The interest is usually paid at the end of an agreed term (monthly, quarterly, etc.). Oftentimes, the interest, stated as a percentage, is given annually.

The simple interest, I on a principal amount P, for a time in years t, at an interest rate per year r, is given by $I = Prt$, and the amount A after the interest has been added can be expressed by $I = P + Prt = P(1 + rt)$

Compound interest on the other hand, is paid periodically over the term of the loan which results in a new principal at the end of each interval of time. For example, if Kwame decides to buy some 91-day treasury bills through a bank, he gets to decide what transaction is to made on his behalf at the end of the ninety-one days.

He gets the option to "roll-over" his investment. With this option, the sum of his original principal and the interest becomes the new principal for the next investment period (the next ninety-one days). This transaction is repeated for the future investment periods of ninety-one days until when Kwame terminates the investment.

With compound interest we work out the interest for the first period, add it to the total, and then calculate the interest for the next period, and so on ...

Assume that for Kwame's investment, his initial buy was *GH¢* 1,000.00 and the annual rate was 23%. It would imply that the interest rate for the investment period i.e., ninety-one days ($\frac{1}{4}$ of a year) would be $\frac{23}{4}\% = 5.75\%$. Figure 8.1 shows an illustration of the computations of his interests and the new investment value.

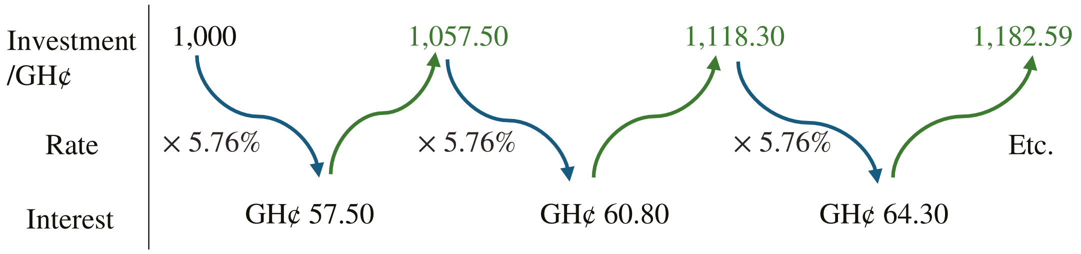


Figure 8.1

The type of interest obtained after the whole investment duration (in Kwame's case, *GH¢* (1,182.59 – 1,000) = *GH¢* 182.59) is called the compound interest and it is given by

$A = P\left(1 + \frac{r}{k}\right)^{kt}$ where

A is the balance after t years of compounding,

P is the starting balance of the account (the original principal),

r is the annual interest rate,

k is the number of compounding periods in one year,

$\frac{r}{k}$ as used in the formula caters for the simple interest rate for individual intervals for which interests will be computed and

the exponent, kt gives the total number of computations

It must be noted that the formula being discussed is an indicial equation as the exponent, kt is variable/changes. Also, $1 + \frac{r}{k} > 1$ so raising a number larger than one to an increasing power is by definition, exponential growth. When needed, the formula can be converted to a logarithmic equation.

Example 1

Korkor invests *GH¢* 5,000.00 in 91-day treasury bills at 23% per annum for five years. How much interest is she due if the money is compounded for 3 years?

Solution

The principal amount, P = *GH¢* 5,000.00

There will be four compounding periods in a year since ninety-one days is equivalent to 3 months, i.e., a quarter of a year and thus, $k = 4$

The annual rate, $r = 23\,\% = 0.23$ while t, the number of years for the investment is 3

$$\textit{The worth of the investment, } A = 5000\left(1 + \frac{0.23}{4}\right)^{4(3)}$$

$$= GH\text{¢}\ 9{,}779.90$$

The interest is thus the difference of the principal and the future value of the investment i.e.,

GH¢ (9,779.90 – 5,000)

GH¢ 4,779.90

Example 2

Hamid currently has *GH¢* 15,000.00 at his disposal for which he has no budget for yet. He decides to lend it out to his friend, Senyo, with the intention of retrieving an estimate of *GH¢* 35,000.00. Although they are friends, Hamid for want of some level of insurance, decides that the loan deal be renewed quarterly at an annual rate of 30%. Determine the number of years that Hamid will have to wait to realise his estimated amount at the end of the period.

Solution

Principal Amount = *GH¢* 15,000.00

Estimated future value = *GH¢* 35,000.00

Lending rate = 30 % = 0.30

Number of transactions per year = 4

The problem qualifies to be a compound interest application problem since after every transaction, the new debt is the sum of the interest to be paid and the previous amount borrowed

Using the formula, $A = P\left(1 + \frac{r}{k}\right)^{kt}$

$$35{,}000 = 15000\left(1 + \frac{0.3}{4}\right)^{4(t)}$$

$$2.3333 = (1.075)^{4t}$$

$$\log(2.3333) = \log((1.075)^{4t})$$

$$4t = \frac{\log(2.3333)}{\log(1.075)}$$

$$t = \frac{\log(2.3333)}{4\log(1.075)}$$

$$t = 2.92891 \approx 3$$

Hamid would have to wait for three years to realise his *GH¢* 35,000.00

Loudness

The loudness L, of a sound (in decibels) perceived by the human ear depends on the ratio of the intensity, I of the sound to the threshold, l_0 of hearing for the average human ear

$L = 10\log\left(\frac{I}{l_0}\right)$

Example 3

Find the loudness of a sound that has an intensity 10,000 times the threshold of hearing for the average human ear.

Solution

Let the threshold of hearing be l_0

Intensity, $I = 10{,}000\, l_0$

Loundness, $l = 10\log\left(\frac{I}{l_0}\right)$

$= 10\log\left(\frac{10{,}000\, l_0}{l_0}\right)$

$= 10\log(10{,}000)$

$= 40$ *decibels*

Depreciation

Example 4

A machine purchased for business use loses value over a period of years. The value of the machine at the end of its useful life is called its "scrap value". By one method of depreciation (where it is assumed a constant percentage of the value depreciates annually), the scrap value S, is given by $S = C(1 - r)^n$,

Where C is the original cost, n is the useful life in years and r is the constant percent of depreciation. Find the scrap value of a machine costing *GH¢* 30,000.00, having a useful life of twelve years and a constant annual rate of depreciation of 15%

Solution

$C =$ *GH¢* 30,000.00

$n = 12$

$r = 15\%$

$S = 30000(1 - 0.15)^{12}$

$=$ *GH¢* 4,267.25

FOCAL AREA 2: MODELLING WITH LOGARITHMIC FUNCTIONS

Any relation of a non-linear form $y = ax^n$, where x and y are variables and a being constant can be reduced to a linear form $y = mx + c$, by first taking the logarithm of both sides of the relation, then applying the addition and the power rule, rearrange and compare with $y = mx + c$

Example 5

The table presents the value of two related variables a and b, as obtained in an experiment

a	1	2	3	4	5
b	3	10	51	190	760

If it is true that a and b are connected by a scientific law given by $b = kc^a$, where c and k are constants,

1. Plot a suitable graph and verify that the law is valid.

2. Use your graph to estimate

a. the values of c and k correct to one decimal place

b. the value of b when $a = 3.5$

Solution

a	1	2	3	4	5
b	3	10	51	190	760

1. We seek a linear function which is an approximation to the exponential function, $b = kc^a$. The definition of logarithm, which depicts the relationship between logarithms and indices as discussed in week 5 of the year one teacher manual provides a means to convert $b = kc^a$ to a linear function approximation.

We now take the logarithm to the base ten of both sides of $b = kc^a$ and use the properties of logarithms to expand the resulting expression as shown below

$\log(b) = \log(kc^a)$

$\log(b) = \log(k) + \log(c^a)$

$\log(b) = \log(k) + a\log(c)$

The resulting equation is of the form $y = mx + c$ and by comparison, $y = \log(b)$, $x = a$, $m = \log(c)$ and $c = \log(k)$

We would need to compute log(b) and for the easy plotting of the points, approximate the values to about two decimal places

a	1	2	3	4	5
$log(b)$	0.48	1.00	1.71	2.28	2.88

The linear function which sufficiently approximates the exponential function is that whose graph passes through or closest to most of the plotted points. It would result in the least error margin should it be used to approximate values of a or b

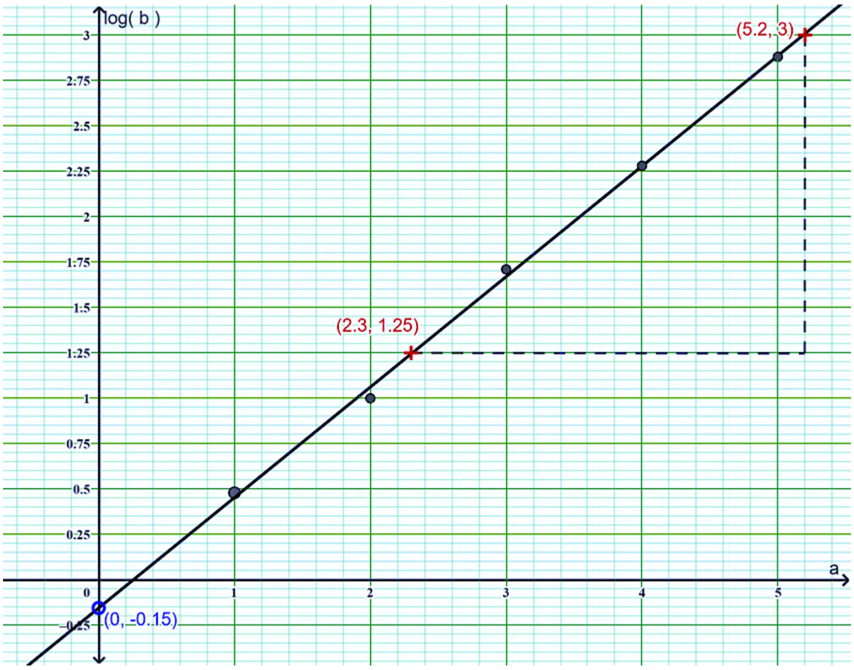


Figure 8.2

2. From Figure 8.2, we can identify two points on the straight line namely, (5.2, 3) and (2.3, 1.25). these two points can then be used to calculate the slope of the line thus

   a. $Gradient,\ m = \log(c) = \frac{3 - 1.25}{5.2 - 2.3} = \frac{1.75}{2.9} = 0.60345$

      $c = 10^{0.60345} = 4.01282 \approx 4.0\ (1\ dp)$

   b. Also, the $y -$ intercept, $c = \log(k) = -0.15$

      $\Rightarrow k = 10^{-0.15} = 0.70795 \approx 0.7\ (1\ dp)$

Example 6

The Kwame Nkrumah Mausoleum located in Accra, is a catacomb that houses the remain of the great national and Pan-African leader. The shape of the mausoleum, shown in Figure 8.3 is said to portray a tree cut at its trunk and implies the unfinished work and vision of Dr. Kwame Nkrumah before his demise. Another representation offered is an upside-down sword which in the Akan culture is a symbol of peace. It can also be viewed as an uprooted tree to signify the unfinished incomplete work of Dr. Nkrumah to totally unite Africa. The edge of the blade of the sword or the root of the trunk (as the interpretation might be), traced with a red curve of illustration 2 in Figure 8.3 can be modelled by a logarithmic equation as the shape resembles a logarithmic graph. We can assume the function is of the form $h(d) = p\log_b(d) + q$ where the $h(d)$ is the height of the catacomb d *metres* away from the center line, shown with a blue green line in illustration 2 of Figure 8.3.

Illustration 1

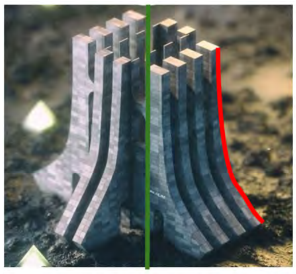


Illustration 2

Figure 8.3

Learning Tasks

Learners are expected to perform the following tasks as part of the activities for the lessons in the week

1. Model real life problems involving indices and logarithms and present answers,
2. Explain how to solve problems involving indices and logarithms and the relationship between the two concepts
3. Use any spreadsheet programme such as Microsoft Excel and ICT tools such as GeoGebra to draw graphs of logarithmic functions and explain their solution process
4. Investigate and explain real life models of logarithmic functions and talk about what happens when specific conditions are varied/imposed
5. State and prove the rules of logarithms and make deductions
6. Discuss and investigate the relationship between indices and logarithms, state the laws of logarithms and make deductions from the laws

PEDAGOGICAL EXEMPLARS

1. **Reviewing Previous concepts:** Review learners' previous knowledge of laws of indices and logarithms and exponential and logarithmic functions in week 5 lessons in section 2 of the year one teacher manual.

   Learners in small groups can be tasked to write out some laws of indices and logarithms that they remember and exchange it with other group for peer assessment. Learners can then reconcile errors that may arise and also recall the laws they might have omitted.

Learners' knowledge on graphing linear functions should also be reviewed through a whole class activity. Learners who are able can be assigned the role of demonstrating the procedure to graphing a linear function to the larger group.

2. **Experiential learning:** Facilitators should graph logarithmic functions using graphing calculators or software to help students visualize the shape and behaviour and provide the opportunity to learners in the mixed-ability groups to discuss the properties of such graphs and the behaviour and share their observations with the larger group. In the situation where graphing utilities are not available, printed graphs of logarithmic functions of models of the same can be used.

   As individuals or pairs (in the case of struggling learners), learners should be guided to approximate exponential and logarithmic function with linear functions and graph them. Learners should be provided with support to transform exponential equations into linear equations.

3. **Problem-based group learning:** Learners in their collaborative groups are assigned real-life problems that require them to model exponential functions with linear functions, graph them and use them to predict values for the dependent quantities.

   As a project, challenge learners to collect data in around them and model it using logarithmic or exponential functions. Encourage learners to explore advanced applications in science, engineering, and finance.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual Understanding

1. Consider the logarithmic equation $\log(2x + 3) = \log(x) + 1$
   a. Solve the equation using the properties of logarithms
   b. If $y_1 = \log(2x + 3)$ and $y_2 = \log(x) + 1$, then the graph of $y_1 - y_2$ is shown in Figure 8.4. explain how root as indicated in the graph confirms the solution found in part (a)

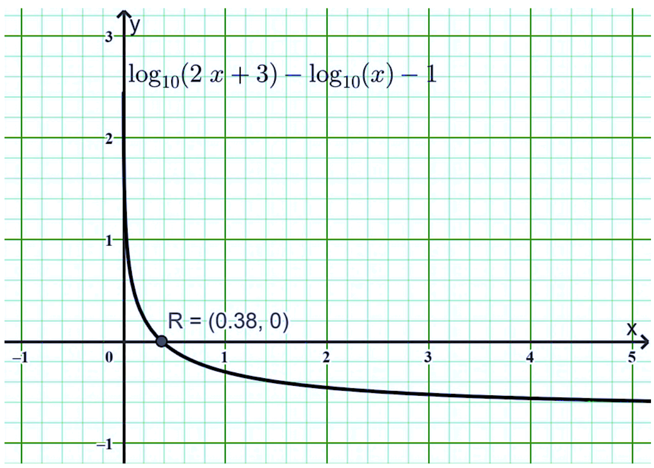


Figure 8.4

Assessment Level 3: Strategic Reasoning

Which plan is a better plan?

Plan A: Invest *GH¢* 1,000.00 at 4% compounded quarterly for three years

Plan B: Invest *GH¢* 1,000.00 at 3.9% compounded monthly for three years

Assessment Level 3: Strategic Reasoning

The magnitude of a star if defined by the equation

$M = 6 - 2.5\log\left(\frac{I}{I_0}\right)$, where I_0 is the measure of the faintest star and I is the actual intensity of the star being measured. The dimmest stars are of magnitude 6 and the brightest are of magnitude 1. Determine the ratio of the intensities between stars of magnitude 1 and 3.

Section 8 Review

This section discussed all the lessons to be taught in week 13. A review of the laws of indices and logarithms and the resulting functions and solving of indicial and logarithmic equations were reviewed. Emphasis however, was placed on modelling and solving real-life problems with logarithmic and exponential functions. It is encouraged that leaners be provided with adequate learning materials such as graphing utilities, graph books or grid boards to enable them to investigate these graphs.

SECTION 9: TRIGONOMETRIC IDENTITIES

Strand: Geometric Reasoning and Measurement

Sub-Strand: Measurement of Triangles

Learning Outcomes

1. *Find trigonometric values using compound, multiple and half angles and prove the sine and cosine rule.*
2. *Verify whether or not a given trigonometric equation is an identity, and solve trigonometric equations.*

Content Standard: *Demonstrate understanding of trigonometric identities and apply algebraic techniques to verify identities and solve trigonometric problems on them.*

INTRODUCTION AND SECTION SUMMARY

Learning about trigonometric identities, the sine and cosine rules, and solving trigonometric equations is essential in senior high school mathematics. Trigonometric identities simplify complex expressions and prove mathematical results. The sine and cosine rules are used to solve non-right-angled triangles, relating side lengths and angles. Solving trigonometric equations amounts to finding angles that satisfy given conditions, crucial for applications in physics and engineering. Using practical examples and interactive activities will help learners understand and appreciate these concepts, making trigonometry more accessible and enjoyable.

The weeks covered by the section are:

Week 14

1. *Trigonometric identities*

Week 15

1. *Deriving and applying the sine and cosine Rule*
2. *Solving Trigonometric Equations*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on trigonometry. Therefore, collaborative learning, experiential learning and initiate talk for learning, problem-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. Then, extend activities for some learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Various forms of assessments should be carried out to ascertain learners' performance on the concepts that will be taught under this section. Teachers are entreated to administer these assessments and record them for onward submission into the Student Transcript Portal (STP). The following assessment would be conducted and recorded for each learner:

Week 14: *Individual Class Exercise*

Week 15: *Individual Group Project*

Note

For additional information on how to effectively administer these assessment modes, refer to the Appendices.

WEEK 14

Learning Indicator: *Prove and apply compound angles to derive the identities for multiple angles and half angles*

Introduction

In year 1 we discussed Trigonometric functions and the applications of angles of elevation and depression in solving real life examples. In year two the concentration will be on trigonometric identities. Identities are equations that are true for all replacements of the variable(s) for which both sides are defined. The discussion will centre around fundamental identities and extend to using the sine and cosine rule to solve real-life questions. We will also discuss trigonometric equations.

FOCAL AREA 1: TRIGONOMETRIC IDENTITIES

Fundamental Identities

The fundamental identities in trigonometry are derived from the Pythagorean theorem. They are the basic identities in trigonometry. Understanding the Pythagorean identities will form a basis for studying all other identities. From year 1 we established that;

$\sin(\theta) = \frac{opposite}{hypotenuse}$, $\cos(\theta) = \frac{adjacent}{hypotenuse}$, $\tan(\theta) = \frac{opposite}{adjacent}$

$\csc(\theta) = \frac{hypotenuse}{opposite}$, $\sec(\theta) = \frac{hypotenuse}{adjacent}$, $\cot(\theta) = \frac{adjacent}{opposite}$

From the above it can be observed that $\sin(\theta)$ is the reciprocal of $\csc(\theta)$, $\cos(\theta)$ is the reciprocal of $\sec(\theta)$ and $\tan(\theta)$ is the reciprocal of $\cot(\theta)$. From this observation we can derive the reciprocal identities.

Reciprocal Identities

The reciprocal identities are;

$\sin(\theta) = \frac{1}{\csc(\theta)}$, $\cos(\theta) = \frac{1}{\sec\theta}$, $\tan(\theta) = \frac{1}{\cot\theta}$

$\csc(\theta) = \frac{1}{\sin\theta}$, $\sec(\theta) = \frac{1}{\cos\theta}$ $\cot(\theta) = \frac{1}{\tan\theta}$

Quotient Identities

The quotient identities can also be derived from the definition of trigonometry. From the definition $\sin(\theta) = o\frac{opposite}{hypotenuse}$, $\cos(\theta) = a\frac{adjacent}{hypotenuse}$

Therefore; $\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}}$

$= \dfrac{opposite}{adjacent}$ and this is the definition of $\tan(\theta)$

Similarly, $\dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{\frac{adjacent}{hypotenuse}}{\frac{opposite}{hypotenuse}}$

$= \dfrac{adjacent}{opposite}$ which is the definition of $\cot(\theta)$.

The reciprocal identities are $\tan(\theta) = \dfrac{opposite}{adjacent}$ and $\cot(\theta) = \dfrac{adjacent}{opposite}$

Pythagorean Identities

Consider the right triangle

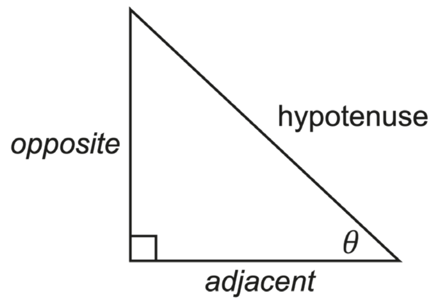


Figure 9.1

If we let $r = hypotenuse$, $x = adjacent$ and $y = opposite$

From Pythagoras theorems $r^2 = x^2 + y^2$, then we can derive 3 identities by dividing through the Pythagoras theorem with r^2, x^2 and y^2.

Dividing through by r^2

$\Rightarrow \frac{r^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2}$

$\Rightarrow 1 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$ but $\frac{x}{r} = cos\theta$ and $\frac{y}{r} = sin\theta$

$\Rightarrow cos^2\theta + sin^2\theta = 1$

Dividing through by x^2

$\Rightarrow \frac{r^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$

$\Rightarrow \left(\frac{r}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$

$\Rightarrow (sec\theta)^2 = 1 + (tan\theta)^2$

$\Rightarrow sec^2\theta = 1 + tan^2\theta$

Dividing through by y^2

$\Rightarrow \frac{r^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$

$\Rightarrow \left(\frac{r}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$

$\Rightarrow (csc\theta)^2 = (cot\theta)^2 + 1$

$\Rightarrow csc^2\theta = cot^2\theta + 1$

The Pythagorean identities are

1. $\cos^2\theta + \sin^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

Compound angles identities

The identities we have discussed above are in one variable. The other trigonometric identities that involve two variables are known as the compound identities. The compound identities are also known as sum and difference formula.

To derive the compound angles, consider a unit circle. A unit circle can be drawn such that the initial side of the angle, $(\alpha - \beta)$ lies on the positive $x - axis$ and hence intersect the circle at $(1, 0)$ while the terminal side is consequently rotated such that instead of intersecting the circle at $(\cos\alpha, \sin\alpha)$, it intersects at $(\cos(\alpha - \beta), \sin(\alpha - \beta))$, maintaining the length of the chord between the sides of angle, $(\alpha - \beta)$ as shown in Figure .

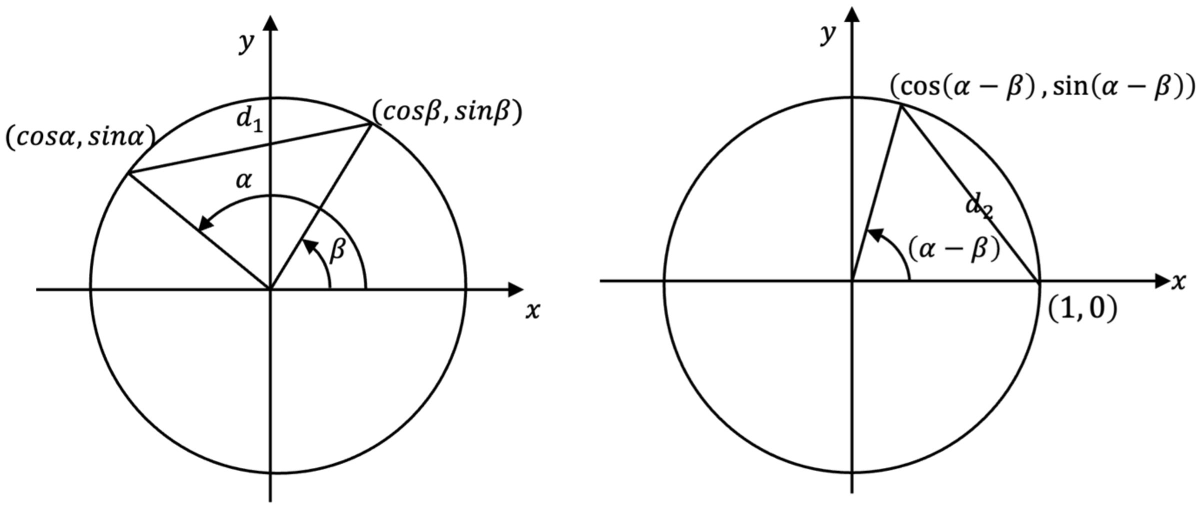


Figure 9.2

From the formula for finding the distance between two points, $|d_1| = |d_2|$

$\sqrt{(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2} = \sqrt{(cos(\alpha - \beta) - 1)^2 + (sin(\alpha - \beta) - 0)^2}$

$(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2 = (cos(\alpha - \beta) - 1)^2 + (sin(\alpha - \beta) - 0)^2$

From the left-hand side

$cos^2\alpha + cos^2\beta - 2cos\alpha cos\beta + sin^2\alpha + sin^2\beta - 2sin\alpha sin\beta$

$= (cos^2\alpha + sin^2\alpha) + (\ cos^2\beta + sin\ 2\beta) - 2(cos\alpha cos\beta + sin\alpha sin\beta)$

$= 1 + 1 - 2(cos\alpha cos\beta + sin\alpha sin\beta)$

$= 2 - 2(cos\alpha cos\beta + sin\alpha sin\beta)$

From the right-hand side

$cos^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta) + sin^2(\alpha - \beta)$

$= cos^2(\alpha - \beta) + sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1$

$= 1 - 2\cos(\alpha - \beta) + 1$

$= 2 - 2\cos(\alpha - \beta)$

Equating the left-hand side and the right-hand side

$\Rightarrow 2 - 2(cos\alpha cos\beta + sin\alpha sin\beta) = 2 - 2\cos(\alpha - \beta)$

$\therefore \cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$..................................*equation* 1

We can use *equation* (1) to prove the other compound identities

To find the identity for $cos(\alpha + \beta)$ replace β by $-\beta$ in *equation* 1, This would result in

$\cos(\alpha - (-\beta)) = cos\alpha cos(-\beta) + sin\alpha sin(-\beta)$

But $cos(-\beta) = cos\beta$ and $sin(-\beta) = -sin\beta$

$\therefore \cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$.............................. *equation* 2

Using *equation* 2, we can derive the compound identities for sine. Before we derive the compound angle identity for sine, use the compound identity for cosine to simplify cos $(90° - \theta)$.

$\cos(90° - \theta) = \cos(90°)\cos(\theta) + \sin(90°)\sin(\theta)$

Since $\cos(90°) = 0$ and $\sin(90°) = 1$

$\cos(90° - \theta) = \sin(\theta)$ and $\sin(90° - \theta) = \cos(\theta)$, these are called co-function identities.

Now

$\cos(90° - (\alpha + \beta)) = \cos(90°)\cos(\alpha + \beta) + \sin(90°) \sin(\alpha + \beta)$

Since $\cos(90°) = 0$ and $\sin(90°) = 1$, then $\cos(90° - (\alpha + \beta)) = \sin(\alpha + \beta)$.

Also, $\cos(90° - (\alpha + \beta)) = \cos(90° - \alpha - \beta)$

And it can be rewritten as $\cos((90° - \alpha) - \beta)$

$\Rightarrow \cos((90° - \alpha) - \beta) = \cos(90° - \alpha)\cos(\beta) + \sin(90° - \alpha) \sin(\beta)$

Since $\cos((90° - \alpha) - \beta) = \cos(90° - (\alpha + \beta))$ and $\cos(90° - (\alpha + \beta)) = \sin(\alpha + \beta)$,

$\cos((90° - \alpha) - \beta) = \sin(\alpha + \beta)$

$\therefore \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$..................................*equation* 3

Substitute β with $-\beta$ in *equation* 3

$\sin(\alpha + (-\beta)) = \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta)$

$\therefore \sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Since

$\cos(-\alpha) = \cos(\alpha)$ as cosine is an even function and

$\sin(-\beta) = -\sin(\beta)$ as sine is an odd function

From the quotient identities we established $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, therefore the quotient identities can be used to prove the identities for $\tan(\alpha+\beta)$and $\tan(\alpha-\beta)$.

That is

$$tan(\alpha+\beta) = \frac{sin(\alpha+\beta)}{cos(\alpha+\beta)} = \frac{sin\alpha cos\beta + cos\alpha sin\beta}{cos\alpha cos\beta - sin\alpha sin\beta}$$

Multiplying the denominator and numerator by $\frac{1}{cos\alpha cos\beta}$ gives

$$\tan(\alpha+\beta) = \frac{sin\alpha cos\beta + cos\alpha sin\beta}{cos\alpha cos\beta - sin\alpha sin\beta}\left(\frac{\frac{1}{cos\alpha cos\beta}}{\frac{1}{cos\alpha cos\beta}}\right)$$

$$\tan(\alpha+\beta) = \frac{\left(\frac{sin\alpha cos\beta}{cos\alpha cos\beta}\right) + \left(\frac{cos\alpha sin\beta}{cos\alpha cos\beta}\right)}{\left(\frac{cos\alpha cos\beta}{cos\alpha cos\beta}\right) - \left(\frac{sin\alpha sin\beta}{cos\alpha cos\beta}\right)}$$

$$\tan(\alpha+\beta) = \frac{\left(\frac{sin\alpha}{cos\alpha}\right) + \left(\frac{sin\beta}{cos\beta}\right)}{1 - \left(\frac{sin\alpha}{cos\alpha}\right)\left(\frac{sin\beta}{cos\beta}\right)}$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan\alpha\tan\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots equation\ 4$$

Replace $\boldsymbol{\beta}$ by $-\boldsymbol{\beta}$ in equation 4 to obtain the identity for $\tan(\alpha-\beta)$

That is $\tan(-\beta) = -\tan\beta$

$$\tan(\alpha+(-\beta)) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The compound identities are as follows

1. $\mathbf{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$
2. $\mathbf{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$
3. $\mathbf{\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}$
4. $\mathbf{\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}$
5. $\mathbf{\tan(\alpha+\beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan\alpha\tan\beta}}$
6. $\mathbf{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$

Application of compound angle identities

In this section will apply compound identities to solve examples. Mostly we require two special angles whose sum or difference will result into the given angle.

Example 1

Find the exact vale of sin(75°) without using calculators.

Solution

In this example we need to apply our knowledge of special angles discussed in year one. To solve the example, we need special angles add up to 75°, and these are angles are 45° and 30°.

Therefore sin(75°) can be expressed as sin(45°+30°)

$$\sin(45°+30°) = \sin(45°)\cos(30°) + \cos(30°)\sin(45°)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 2

Find the exact vale of tan (105°) without using calculators.

Solution

$$\tan(105°) = \tan(60°+45°)$$

$$= \frac{\tan(60°) + \tan(45°)}{1 - \tan(60°)\tan(45°)}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{31}}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= \frac{2(2+\sqrt{3})}{-2} = -2-\sqrt{3}$$

Multiple-angle identities

The concept of multiple-angle identity stems from the sum of angles. The identities of the compound angles (sum) can be used to derive the identities of multiple angles.

Note, multiple angles can be double and triple angles but for the purpose of this section the discussion will be limited to double angles.

To derive the double-angle identities we assume that both angles in the sum of two angles are the same. From the compound angle we derived the formula for

$\mathbf{cos}(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$,

the double angle formula is based on the analogy that β is equal to α. Therefore, the double angle for *cosine* is $\cos(\alpha + \alpha) = cos\alpha cos\alpha - sin\alpha sin\alpha$

Double angle identities

1. $\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$\mathbf{sin2\alpha = 2sin\alpha cos\alpha}$

2. $\cos(\alpha + \alpha) = cos\alpha cos\alpha - sin\alpha sin\alpha$

$\mathbf{cos2\alpha = cos^2\alpha - sin^2\alpha}$

Note the form the Pythagorean identities $cos^2\alpha$ and $sin^2\alpha$ can be expressed as $cos^2\alpha = 1 - sin^2\alpha$ and $sin^2\alpha = 1 - cos^2\alpha$ respectively. Therefore, other double angle identities can be derived for $\cos2\alpha$. That is

$\mathbf{cos2\alpha = 1 - 2sin^2\alpha}$ and

$\mathbf{cos2\alpha = 2cos^2\alpha - 1}$

3. $\tan2\alpha = \tan(\alpha + \alpha)$

$\tan2\alpha = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$

$\mathbf{tan2\alpha = \frac{2\,tan\alpha}{1 - tan^2\alpha}}$

Verifying Identities

We can apply the trigonometric identities to verify whether an equation is an identity or not. Using graphs to illustrate identity, sketch the graph of $y = \frac{cot^2(x)}{\csc(x)}$ and $y = \csc(x) - \sin(x)$.

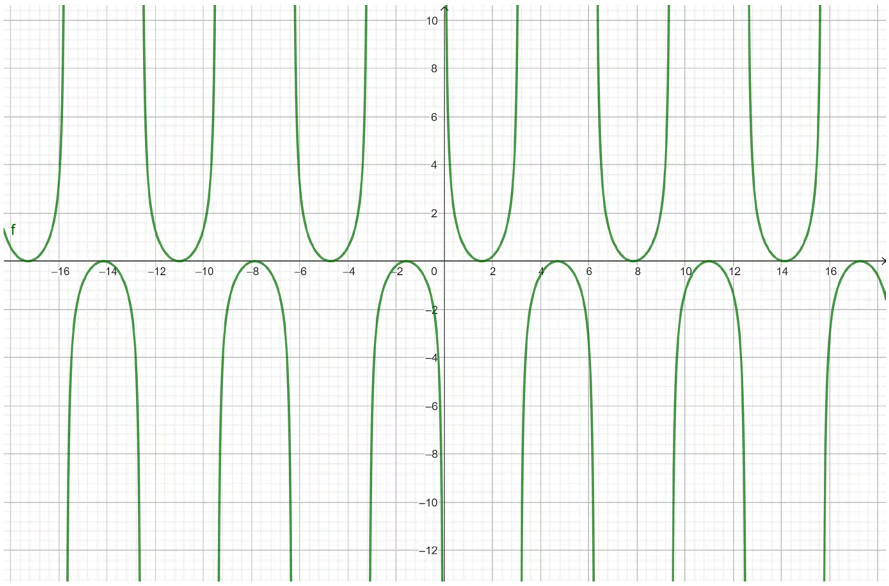


Figure 9.3

Graph of $y = \dfrac{cot^2(x)}{csc(x)}$

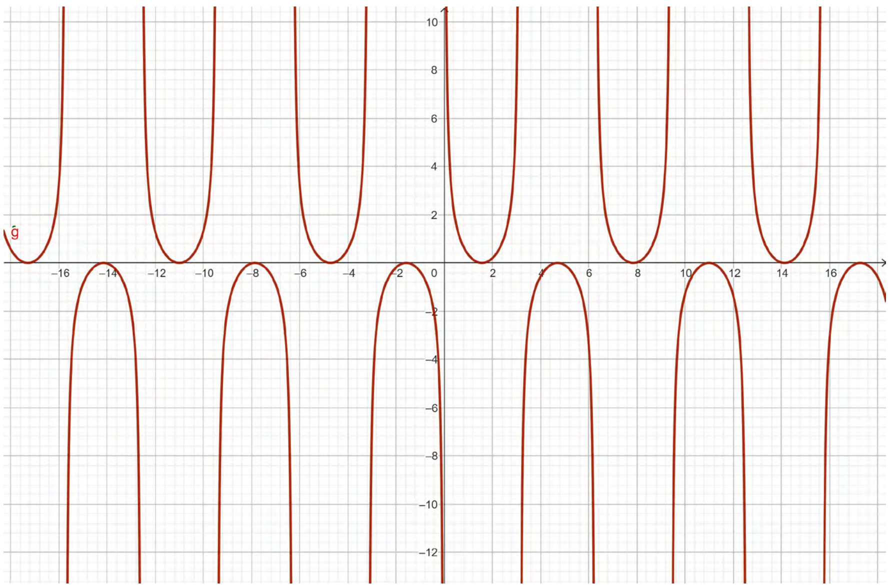


Figure 9.4

Graph of $y = \csc(x) - \sin(x)$

From the two graphs it can be concluded that $\dfrac{cot^2(x)}{csc(x)}$ is equal to $\csc(x) - \sin(x)$. This means that $\dfrac{cot^2(x)}{csc(x)} = \csc(x) - \sin\left(x\right)$, is true for all values of the input on which they are defined.

Algebraically,

$$\frac{cot^2(x)}{csc(x)} = \mathbf{csc}(x) - \sin\left(x\right),$$

$$\frac{\frac{cos^2 x}{sin^2(x)}}{\frac{\mathbf{1}}{\sin(x)}} = \frac{\mathbf{1}}{\sin(x)} - \sin(x),$$

$$\frac{cos^2(x)}{sin(x)} = \frac{1 - sin^2(x)}{\sin(x)},$$

$$\frac{cos^2(x)}{sin(x)} = \frac{cos^2(x)}{sin(x)}$$

There are no rules when verifying identities. Generally, it is advisable to work from the complex side to obtain the simple side. To verify identities, you can work from one side to obtain the other side or work from both sides to obtain different identities not given. As these are identities and not equations, we work on the left-hand side or the right-hand side of the identities separately. We do not treat them as we would an equation by manipulating across the equals sign.

Example 3:

Verifying from one side

Verify that $1 - 2\,sin^2x = 2\,cos^2x - 1$ is an identity.

Solution

Working from the left side

$$\begin{aligned} 1 - 2\,sin^2x &= 1 - 2(1 - cos^2x) \\ &= 1 - 2 + 2cos^2x \\ &= -1 + 2cos^2x \end{aligned}$$

This can be rewritten as $2cos^2x - 1$ which is the same as the left side. Therefore, it can be concluded that the equation is an identity.

Example 4:

Verifying from both sides

Verify that $\frac{secx + tanx}{secx - tanx} = \frac{1 + 2sinx + \sin^2x}{\cos^2x}$ is an identity

Solution

Left side

$$\frac{secx + tanx}{secx - tanx} = s\frac{ecx + tanx}{secx - tanx}\left(\frac{cosx}{cosx}\right)$$

$$= \frac{cosxsecx + cosxtanx}{cosxsecx - cosxtanx}$$

But $cosx = \frac{1}{secx} \Rightarrow 1 = cosxsecx$

$$\frac{secx + tanx}{secx - tanx} = \frac{1 + cosxtanx}{1 - cosxtanx}$$

$$= \frac{1 + cosx\left(\frac{sinx}{cosx}\right)}{1 - cosx\left(\frac{sinx}{cosx}\right)}$$

$= \frac{1 + sinx}{1 - sinx}$

Right side

$x^2 + 2xy + y^2 = (x + y)^2$

$\Rightarrow 1 + 2sinx + \sin^2 x = (1 + sinx)^2$

$$\frac{1 + 2sinx + \sin^2 x}{\cos^2 x} = \frac{(1 + sinx)^2}{\cos^2 x}$$

$$= \frac{(1 + sinx)^2}{1 - \sin^2 x}$$

$$= \frac{(1 + sinx)^2}{(1 + sinx)(1 - sinx)}$$

$$= \frac{1 + sinx}{1 - sinx}$$

Since both sides of the original equation can be simplified to the same expression, $\left(\frac{1 + sinx}{1 - sinx}\right)$, the equation is an identity.

Learning Tasks

1. Learners verify basic trigonometric identities using substitution.
2. Learners calculate missing side lengths or angle measures in basic right-angled triangles using trigonometric functions.
3. Learners derive compound angles from basic trigonometric ratios and share their findings in class.
4. Learners state the importance of compound-angle identities and apply their knowledge of special angles to solve practical examples without using calculators.
5. Learners create questions on compound angles for other groups to solve and present their solutions to the class.
6. Learners research on multiple-angle and half-angle identities.

PEDAGOGICAL EXEMPLARS

The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Exploratory Learning:** Learners in their mixed-ability groups or as individuals research on angle identities and share their findings in class. On group basis, roles should be shared among members based on their strengths. Facilitators should encourage collaboration and tolerance among members of the groups. Access to relevant reading materials and ICT tools should be provided where applicable

2. **Experiential, Collaborative / Problem-based Learning**
   - **a.** Learners engage in whole class activities / discussions to deduce trigonometric identities. Break down concepts into smaller, manageable steps, providing clear explanations and ample opportunities for practice. 3D models, physical object
   - **b.** Learners in their mixed-ability groups create questions on trigonometric identities for other groups to solve and present their solutions to the class. Facilitators should check the appropriateness of the questions created
   - **c.** Conduct interactive demonstrations and / or simulations using technology or manipulatives to illustrate trigonometric identities

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Verify that $\frac{\tan(t) - \cot(t)}{\sin(t)\cos(t)} = \sec^2(t) - \csc^2(t)$ is an identity
2. Verify that $\tan(x) + \cot(x) = \frac{\csc(x)}{\cos(x)}$ is an identity

Assessment Level 3: Strategic Reasoning

If you start with a trigonometric expression and express it differently or simplify it, then setting the original expression equal to the new expression yields a trigonometric identity.

For example,

$\cos(t) + \tan(t)\sin(t)$ can be expressed as $\cos(t) + \left(\frac{\sin(t)}{\cos(t)}\right)\sin(t)$

Which can further be expressed as $\frac{\cos^2(t) + \sin^2(t)}{\cos(t)}$ then $\frac{1}{\cos(t)}$ which is equivalent to sec (t). We can obtain an identity thus $\cos(t) + \tan(t)\sin(t) = \sec(t)$. Use this technique to make up your own identity, then give it to a partner to verify.

Hint

The Recommended Mode of Assessment for Week 14 is Individual Class Exercise. Ensure to conduct this assessment and record the score for onward submission into the STP by Week 16.

WEEK 15

Learning Indicators

1. *Derive the sine and cosine rules and apply them to solve problems*
2. *Use correct algebraic techniques to isolate the trigonometric functions and find the values for the angle that makes the equation true*

Introduction

In week 8, learners applied operations on vectors to derive the cosine law. In this week's lessons, we will refer to some geometric constructions and apply our knowledge of trigonometric functions to derive the sine and cosine rules and also apply them to solve problems

FOCAL AREA 1: DERIVING AND APPLYING THE SINE AND COSINE RULE

In year 1 we learnt how to solve for angles and the sides of a right triangle. In this section we will concentrate on obliques triangles. The sine and cosine rules are used to solve problems involving oblique triangles. An oblique triangle is a triangle which does not contain a right angle. Which implies that an oblique triangle contains either three acute angles or two acute angles and one obtuse angle.

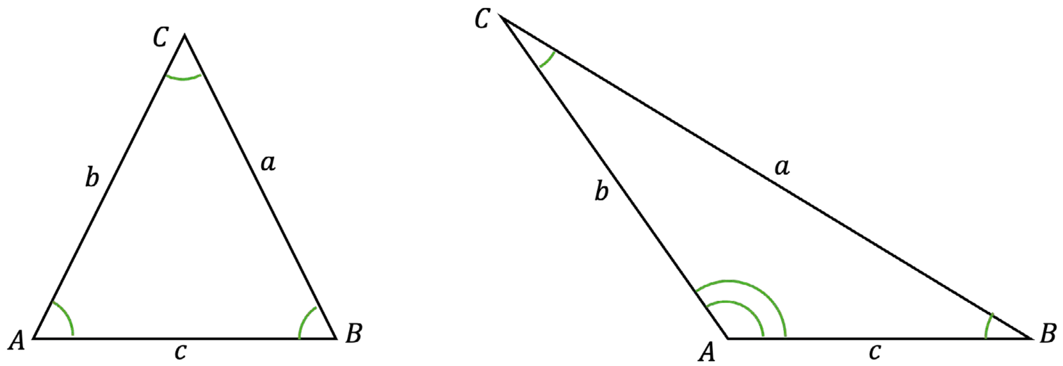


Figure 9.5

The vertices of the triangles are represented by upper case (A, B & C) and their corresponding opposite sides, are represented with lower case (a, b, c).

In order to solve an obliques triangle as least one of the sides and any two other measures should be known.

There are four cases when solving oblique triangles

1. When two angles and any angle side is given
2. When two sides and an angle opposite one of them is given

3. When three sides are given
4. When two sides and an angle included between the two sides are given.

Sine rule

The first two cases (1 and 2) can be solved using the sine rule. In any triangle ABC, the ratio of a side and the sine of the opposite angle is a constant. That is

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \text{ or } \frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

We will derive a law for the *sines* of angles using an oblique triangle (acute and obtuse)

Consider the following triangles below

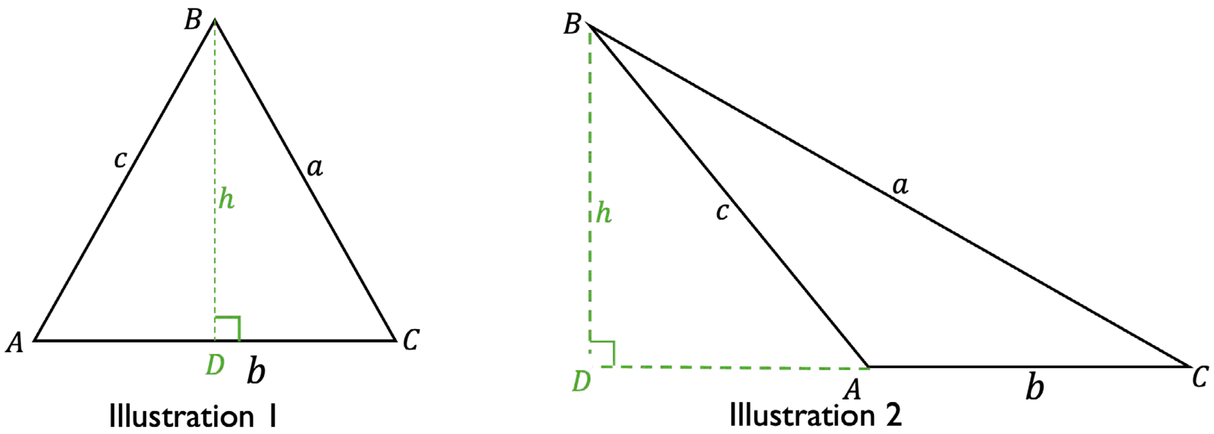


Figure 9.6

The triangle in illustration 1 represents an oblique triangle with three acute angles and the triangle in illustration 2 is also an oblique triangle with two acute angles and an obtuse triangle. In both triangles, h is the length of the perpendicular from B to D. From ΔADB in both triangles c represents the hypotenuse side and from ΔBDC in both triangles c represents the hypotenuse side.

Deriving the sine rule

In ΔADB, $\sin(A) = \frac{h}{c}$

$$\Rightarrow h = c\sin(A)$$

In ΔBDC, $\sin(C) = \frac{h}{a}$

$$\Rightarrow h = a\sin(C)$$

$$\Rightarrow a\sin(C) = c\sin(A)$$

$$\therefore \frac{a}{\sin(C)} = \frac{c}{\sin(A)}$$

Similarly, constructing the perpendicular from A or C yield $\frac{b}{\sin(C)} = \frac{c}{\sin(B)}$ and $\frac{a}{\sin(B)} = \frac{b}{\sin(A)}$.

Therefore, for any triangle, ABC, with sides a, b and c, $\frac{a}{\sin(B)} = \frac{b}{\sin(A)}$, $\frac{a}{\sin(C)} = \frac{c}{\sin(A)}$ and $\frac{b}{\sin(C)} = \frac{c}{\sin(B)}$. This implies that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ or $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$

Application of the sine rule

Example 1

Solve ΔABC if $A = 32.0°$, $B = 81.8°$ and $a = 42.9\ cm$

Solution

This is a problem where two angles and angle opposite one of them is given and hence the application of the law of *sines* is required

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

$$\frac{42.9}{\sin(32.0°)} = \frac{b}{\sin(81.8°)}$$

$$b = \frac{42.9\sin(81.8°)}{\sin(32.0°)}$$

$$b = 80.1\ cm$$

Also,

$A + B + C = 180°$ (*Sum of interior agles of a triangle*)

$$C = 180° - (A + B)$$

$$C = 180° - (32.0° + 81.8°) = 66.2°$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

$$\frac{42.9}{\sin(32.0°)} = \frac{c}{\sin(66.2°)}$$

$$c = \frac{2.9\sin(66.2°)}{\sin(32.0°)}$$

$$b = 74.1\ cm$$

Cosine Rule

The third and fourth cases can be solved using the cosine rule. Let ΔABC be any oblique triangle. Choose a coordinate system such that vertex B is at the origin and side BC is along the positive $x - axis$.

Let (x, y) be the coordinate of the vertex A of the triangle.

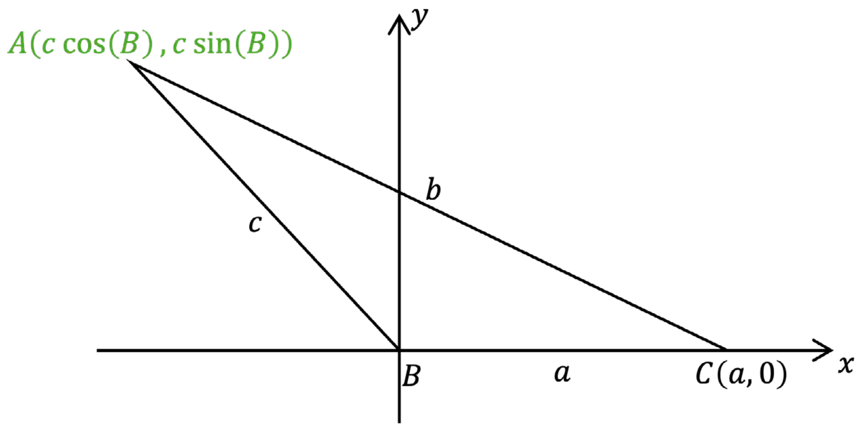


Figure 9.7

$\sin(B) = y\frac{=}{c} \Rightarrow y = c\sin(B)$

Also, $\cos(B) = x\frac{=}{c} \Rightarrow x = c\cos(B)$ hence $(x, y) = (c\cos(B), c\sin(B))$ and the coordinate of $A{:}(c\cos(B), c\sin(B))$ as shown in Illus. 8.1.5.1

C is located at $(a, 0)$

$\therefore b = \sqrt{(c\cos(B) - a)^2 + (c\sin(B))^2}$

$b^2 = c^2\cos^2(B) - 2ac\cos(B) + a^2 + c^2\sin^2(B)$

$b^2 = a^2 + c^2(\cos^2(B) + \sin^2(B)) - 2ac\cos(B)$

$b^2 = a^2 + c^2 - 2ac\cos(B) \qquad [\cos^2(B) + \sin^2(B)]$

Note that if the vertex A is located at the origin and B on the $x - axis$, then:

$a^2 = b^2 + c^2 - 2bc\cos(A)$. In the same instance if C is at the origin and A on the $x - axis$ then $c^2 = a^2 + b^2 - 2ab\cos(C)$.

It can be concluded that for a triangle, ΔABC the square of any side is the difference between the sum of the squares of the other two sides and the product of the lengths of the two sides and the *cosine* of the angle opposite it.

Therefore, the cosine rules are

1. $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos(A)}$
2. $\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos(B)}$
3. $\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos(C)}$

Example 2

Solve ΔABC if $A = 42.9°$, $b = 12.9\ cm$ and $c = 15.4\ cm$

Solution

This problem is of case 3: *SAS*

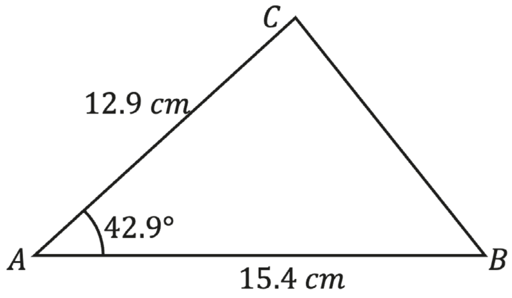


Figure 9.8

$a^2 = b^2 + c^2 - 2bccos(A)$

$a = \sqrt{12.9^2 + 15.4^2 - 2(12.9)(15.4)cos(42.9°)}$

$= 10.6\ cm$

$b^2 = a^2 + c^2 - 2accos(B)$

$2(10.6)(15.4)\cos(B) = 10.6^2 + 15.4^2 - 12.9^2$

$\cos(B) = 0.5609$

$B = 55.9°$ (*all other values for B make* $A + B > 180°$)

$C = 180° - (42.9° + 55.9°) = 81.2°$ (*C is expected to be largest since c is longest*)

FOCAL AREA 2: SOLVING TRIGONOMETRIC EQUATIONS

Solving trigonometric equations follows the same principle for solving equations using algebraic manipulations. Therefore, the algebraic manipulation for solving linear, quadratic etc. equations apply here to. We will consider linear and quadratic equations.

Linear equations

Example 3

Find the value of x in the equation $2\sin(x) - 1 = 0$.

Solution

$2\sin(x) - 1 = 0$

$2sinx = 1$

$\sin(x) = \frac{1}{2}$

$x = sin^{-1}\left(\frac{1}{2}\right)$

$x = 30°$

Example 4

Solve $sinx - 2sinxcosx = 0$.

Solution

$sinx - 2sinxcosx = 0$

$sinx(1 - 2cosx) = 0$

$sinx = 0$

$x = sin^{-1}(0)$

$x = 0°$

Or

$1 - 2cosx = 0$

$cosx = 1=$

$x = cos^{-1}\left(\frac{1}{2}\right)$

$x = 60°$

Quadratic Equations

Example 5

$2\,tan^2 x + 3tanx + 1 = 0$

Solution

$2\,tan^2 x + 3tanx + 1 = 02\,tan^2 x + 2tanx + tanx + 1 = 0$

$2tanx(tanx + 1) + 1(tanx + 1) = 0$

$(2tanx + 1)(tanx + 1) = 0$

$tanx = -\frac{1}{2}$ or $tanx = -1$

$x = tan^{-1}\left(-\frac{1}{2}\right)$ or $x = tan^{-1}(-1)$

$\therefore x = -26.6°\ or\ x = -45°$

Learning Tasks

1. Learners solve simple problems involving the sine and cosine rules in right-angled triangles.
2. Learners apply trigonometric identities to simplify expressions
3. Learners derive trigonometric identities and prove them using algebraic methods

PEDAGOGICAL EXEMPLARS

The aim of the lessons for the week is to assist all learners to derive the sine and cosine laws and solve trigonometric equations. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Experiential / Exploratory Learning**
   - **a.** Utilise visual aids such as diagrams, illustrations, and real-world examples to illustrate the relationships between sides and angles in triangles

     Provide printed notes and visual aids in multiple formats (large print, coloured backgrounds)
   - **b.** Break down concepts into smaller, manageable steps, providing clear explanations and ample opportunities for practice
   - **c.** Provide guided practice sessions where learners work through problems with support, gradually increasing independence
   - **d.** Conduct interactive demonstrations using technology or manipulatives to illustrate trigonometric identities and the sine and cosine rules

  e. Offer problem-solving challenges or puzzles that require application of trigonometric concepts in novel contexts
  f. Encourage exploratory learning experiences where students investigate more applications of the sine and cosine rules in real life independently or in small groups. Analyse how these rules apply to navigation, construction, and other fields. Incorporate stories and examples from diverse cultures and contexts

     Provide resources for independent study and research on advanced topics

2. **Collaborative learning**

   Using small heterogeneous group activities with clearly defined roles,

   a. learners in mixed-ability groups research how to derive the sine and cosine rules and share their findings in class
   b. learners in their mixed-ability groups solve questions on sine and cosine rules and present their solutions to the class
   c. learners work in mixed-ability groups to evaluate trigonometric equations
   d. learners recollect how to solve linear and quadratic equations and apply this knowledge to solve linear and quadratic trigonometric equations respectively

3. **Project-Based learning:** Task learners as individuals or in heterogenous groups to create projects where students must use the sine and cosine rules to design something or solve a complex problem. Allow for a variety of presentation formats, such as written reports, presentations, or digital projects.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Solve $2\,sin^2x - 2sinx - 1 = 0$
2. ABC is a triangle where $\angle A = 152°$ and $BC = 11\ cm$. Find the area of the that passes through all the vertices of the triangle (circumcircle) giving the answer to the nearest square centimetre

Assessment Level 3: Strategic Reasoning

1. A homeowner is trying to build a raised garden bed that has a triangular shape. His neighbour gave him two pieces of timber with lengths 20 feet and 8 feet and he puts them together to form two sides his triangular shape. He is planning to go to the store to buy the third piece of timber to finish the triangle. He wants to build the triangular garden so that third piece of timber will make an angle of 40 degrees with the piece that is 20 feet long. Will it be possible for him to make the garden? If so, how long should the third piece of lumber be?

2. A person rode a bicycle $7\sqrt{2}$ km east, and then he rode for another 21 *km*, 45° south of east. Determine the magnitude and direction of the displacement, rounding the direction to the nearest minute

Hint

- *The Recommended Mode of Assessment for Week 15* ***Individual Group Project****.*
- *Refer to Appendix E at the end of section 9 and the teacher assessment manual and toolkit for further guidance on how to go about the project.*

Section 9 Review

This section reviews all the lessons taught for weeks 14 and 15. These weeks discussed concepts of trigonometry which has built on the foundations of those which were introduced to learners in year 1. We explored trigonometric identities, the sine and cosine rules and trigonometric equations. It needs emphasising that all learners should be given the support needed to experience learning for themselve°s. The use of scientific calculators and / or any other relevant technological tools should be promoted during learning. Assessment strategies should be flexible to allow learners to use their preferred and personal methods to verify identities and decide on their preferred modes of showing evidence of learning.

APPENDIX E

Sample Individual Project Submission

Title: Trigonometric Analysis of a Triangular Roof Support System

Introduction

The project involves verifying trigonometric identities, deriving the sine and cosine rules, solving for unknown angles and sides, and calculating trigonometric values of multiple angles in a triangular roof support system. Measurements were taken from a triangular traffic sign support.

1. Measurements and Setup

i. Sides: $a = 3m$, $b = 4m$, $c = 5m$

ii. Angles: A, B, and C (to be calculated)

iii. Instrumentation: Meter rule, protractor, compass.

2. Verification of Trigonometric Identities

Angle Sum Property

$A + B + C = 180^\circ$

Measured angles

$A = 37^\circ, B = 53^\circ, C = 90^\circ$

Verification

$37^\circ + 53^\circ + 90^\circ = 180^\circ$

Pythagorean Identity

For $A = 37^\circ$

$sin^2(A) + cos^2(A) = 1$

Calculated values

$sin(37^\circ) = 0.6018, cos(37^\circ) = 0.7986$

$sin^2(37^\circ) + cos^2(37^\circ) = (0.6018)^2 + (0.7986)^2 = 1$

3. Derivation and Application of Sine and Cosine Rules

Sine Rule

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$

$$A = 37^\circ, B = 53^\circ, C = 90^\circ : \frac{3}{sin(37^\circ)} = \frac{4}{sin(53^\circ)} = \frac{5}{sin(90^\circ)}$$

Verification

$$\frac{3}{0.6018} = \frac{4}{0.7986} \quad = 5$$

Cosine Rule

$c^2 = a^2 + b^2 - 2ab \cdot cos(C)$

For $C = 90^\circ$:

$c^2 = 3^2 + 4^2 - 2(3)(4)cos(90^\circ)$

$c^2 = 9 + 16 - 0 = 25, \Rightarrow c = 5$

4. **Solving for Angles Using Algebraic Techniques**

Using the sine rule

$sin(A) = a \cdot \frac{sin(C)}{C}$

$sin(A) = \frac{3 \cdot 1}{5} = 0.6 \implies A = arcsin(0.6) = 37^\circ$

Similarly, for B

$sinB = \frac{4 \cdot 1}{5} = 0.8, \Rightarrow B = arcsin(0.8) = 53^\circ$

5. **Calculating Trigonometric Values for Multiple Angles**

Double Angle for $A = 37^\circ$:

$sin(2A) = 2sin(A)cos(A)$

$sin(2A) = 2(0.6018)(0.7986) = 0.9613$

Sum of Angles $\boldsymbol{A + B = 90^\circ}$:

$tan(A + B) = tan(90^\circ)\ (undefined)$

6. **Visual Representation**

   i. A well-labelled diagram showing sides $a = 3m, b = 4m, and\ c = 5\,m$ was provided with angles $A = 37^\circ, B = 53^\circ, C = 90^\circ$.

   ii. A Venn diagram illustrating trigonometric relationships was drawn.

Conclusion

The project successfully verified trigonometric identities, applied the sine and cosine rules, solved for unknown angles algebraically, and calculated multiple trigonometric values. The triangular roof support system is mathematically validated.

SECTION 10: DIFFERENTIATION

Strand: Calculus

Sub-Strand: Principles of calculus

Learning Outcome: *Determine the appropriate rule and use it to find the derivative of a function.*

Content Standard: *Determine the appropriate rule to use in finding the derivative of a function and relations.*

INTRODUCTION AND SECTION SUMMARY

In Week 17 of the year one teacher manual, learners were introduced to the concept of differentiation and differentiating functions by first principles or by using the power rule. In this section, learners will start with the power, product, quotient and chain rules to find derivatives of functions, which are crucial for solving real-world problems involving rates of change. Implicit differentiation allows learners to handle equations with interrelated variables and differentiating transcendental functions, such as exponential, logarithmic and trigonometric functions, which broadens their mathematical toolkit. These topics collectively build a solid foundation for advanced studies and diverse career applications

The weeks covered by the section are:

Week 16

1. *Identifying differentiation rules*
2. *Differentiating functions using differentiation rules*

Week 17: *Differentiating Implicit functions*

Week 18: *Differentiating transcendental functions*

SUMMARY OF PEDAGOGICAL EXEMPLARS

The focal areas in this section require that learners get the opportunity to practise differentiating functions ranging from polynomial functions through to transcendental functions and a combination of different functions. Therefore, pedagogical strategies which promote collaborative learning with accommodations for learners who might need more guidance are encouraged. Teachers are encouraged to allow learners to form small heterogeneous groups while they learn through tasks and present their findings to the larger group.

ASSESSMENT SUMMARY

Different forms of assessments should be carried out to ascertain learners' performance on the concepts that will be taught under this section. The assessments should cover a range of cognitive levels from recall to analysis and creativity. Thus, it should cover all levels of the DOK. Teachers are implored to administer these assessments and record the mandatory ones for onward submission into the Student Transcript Portal (STP). The following mandatory assessments would be conducted and recorded for each learner:

Week 18: *Mid-Semester Examination*

Note

For additional information on how to effectively administer these assessment modes, refer to the Appendices.

WEEK 16

Learning Indicators

1. *Identify the rules of differentiation*
2. *Apply the product and quotient rules to differentiate functions*

For more complex functions it is impractical to use differentiation from first principles to find derivatives. Simply put, there is just too much work to be done (finding limits and simplifying). Luckily, there are several formulas which can be used to find derivatives based on the nature of the function whose derivative is to be found. In this week's lessons, some differentiation rules will be introduced to learners.

FOCAL AREA 1: IDENTIFYING DIFFERENTIATION RULES

Derivative of a constant function

Consider a constant function, $f(x) = \alpha$, where α is a real number. It represents a horizontal line and like all horizontal lines, the gradient is zero. It has been discussed in previous lessons that the derivative of a function is the gradient function. We therefore expect that the gradient/derivative of any constant is zero and thus, if c is a real number and if $f(x) = c$, then $f'(x) = 0$ for all x.

We can use the definition of derivatives (differentiation from first principles) to prove this rule

If $f(x) = c$, then

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left(\frac{c-c}{h}\right) = \lim_{h\to 0}(0) = 0$$

Note that $f(x) = c$ implies that the functional value of any value in the domain of the function, f is equal to c and thus $f(x+h) = f(x) = c$

Example 1

If the cost of a book is fixed at *GH¢* 25.00 for one month, then in that period, the rate at which the price of the book changes is zero since there is no change at all.

Example 2

If $g(x) = -2$, then $g'(x) = 0$

Derivative of the Sum and difference

Given that $f(x) = x^3 + 2x$ and $g(x) = -3x + 5$,

Then $f'(x) = 3x^2 + 2$ and $g'(x) = -3$

$\therefore f'(x) + g'(x) = 3x^2 - 1$

Also, if we use $h(x)$ to represent the sum of the functions i.e.,

$h(x) = f(x) + g(x) = x^3 - x + 5$ then $h'(x) = 3x^2 - 1$

It can be inferred from the example above that if $f'(c)$ and $g'(c)$ exist, then so do $(f + g)'(c)$.

Moreover $(f + g)'(c) = f'(c) + g'(c)$

Also, if $f'(c)$ and $g'(c)$ exist, then so do $(f - g)'(c)$.

Moreover $(f - g)'(c) = f'(c) - g'(c)$

The difference rule can also be proven using the same functions thus

Let $k(x) = f(x) - g(x) = x^3 + 5x - 5$

$\Rightarrow k'(x) = 3x^2 + 5$

$f'(x) - g'(x) = 3x^2 + 5$

Similarly, if u and v are two functions of x, then

$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$ **and** $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Derivative of a linear combination of functions

Given that $a, b, c, \ldots$ are constants and $f(x), g(x), h(x), \ldots$ are functions, the sum of the products of the constants and functions is called the linear combination of the functions.

If $f'(c)$ exists, and if α is any constant, then $(\alpha \times f)'(c) = \alpha \times f'(c)$

Similarly, with the application of the sum rule,

If $f'(c)$ and $g'(c)$ exist, and if α and β are any constants, then $(\alpha \times f + \beta \times g)'(c)$ exists. Moreover $(\alpha \times f + \beta \times g)'(c) = \alpha \times f'(c) + \beta \times g'(c)$

Example 3

If $y_1 = 2x^2 - \sqrt{x}$ and $y_2 = -3x$, then

$\frac{d}{dx}(y_1) = 4x - \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}(y_2) = -3$

$$3 \times \frac{d}{dx}(y_1) - 2 \times \frac{d}{dx(y_2)} = 3(4x - \frac{1}{2\sqrt{x}}) - 2(-3)$$

$$= 12x + 6 - \frac{3}{2\sqrt{x}}$$

Let $g(x) = 3y_1 - 2y_2 = 3(2x^2 - \sqrt{x}) - 2(-3x) = 6x^2 + 6x - 3\sqrt{x}$

$\frac{d}{dx}(g(x)) = \frac{d}{dx}(6x^2 + 6x - 3\sqrt{x}) = 12x + 6 - \frac{3}{2\sqrt{x}}$

$3 \times \frac{d}{dx}(y_1) - 2 \times \frac{d}{dx}(y_2) = \frac{d}{dx}(3y_1 - 2y_2)$ and thus depicts the linear combination rule

Product Rule

This rule states that the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

If $f'(c)$ and $g'(c)$ exist, then $(f \times g)'(c)$ also exist and

$\mathbf{(f \times g)'(c) = f'(c) \times g(c) + f(c) \times g'(c)}$

Example 4

If $f(x) = -3 + 7x$ and $g(x) = 2x^3 - 4$, then

$\frac{d}{dx}(f(x)) = 7$ and $\frac{d}{dx}(g(x)) = 6x^2$

$\frac{d}{dx}(f(x) \times g(x)) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$

$= (2x^3 - 4)(7) + (-3 + 7x)(6x^2)$

$= 56x^3 - 18x^2 - 28$

We can confirm this derivative by finding the product of the original functions and differentiating the result.

$f(x) \times g(x) = 14x^4 - 6x^3 - 28x + 12$

$\frac{d}{dx}(f(x) \times g(x)) = 56x^3 - 18x^2 - 28$

Quotient Rule

If $f'(c)$ *and* $g'(c)$ exist and if $g(c) \neq 0$, $(f/g)'(c)$ exists.

Moreover, $\mathbf{\left(\frac{f}{g}\right)'(c) = \frac{g(c) \times f'(c) - f(c) \times g'(c)}{g^2(c)}}$

Example 5

Find the derivative of $y = \frac{t^2 - 1}{t^3 + 1}, t \neq -1$

Solution

$y = \frac{t^2 - 1}{t^3 + 1}, t \neq -1$ is a quotient function with numerator, $t^2 - 1$ and $t^3 + 1$ as the denominator

$y' = \frac{(t^3 + 1)(2t) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2}$

$$= \frac{2t^4 + 2t - (3t^4 - 3t^2)}{t^6 + 2t^3 + 1}$$

$$= \frac{2t + 3t^2 - t^4}{t^6 + 2t^3 + 1}$$

Reciprocal Rule

If $h'(c)$ exists and $h(c) \neq 0$, then $\left(\frac{1}{h}\right)'(c)$ exists too

Moreover, $\left(\frac{1}{\boldsymbol{h}}\right)'(\boldsymbol{c}) = -\frac{\boldsymbol{h}'(\boldsymbol{c})}{\boldsymbol{h}^2(\boldsymbol{c})}$

The reciprocal rule is actually derived from the quotient rule

$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{g^2(x)}$ with $f(x) = 1$ and $g(x) = h(c)$

Note that for $f(x) = 1 \Rightarrow f'(x) = 0$

Example 6

Find the derivative of $y = \frac{1}{-9x^2 + 25}$, $x \neq \sqrt{\frac{25}{9}}$

Solution

$y = \frac{1}{-9x^2 + 25}$, $x \neq \sqrt{\frac{25}{9}}$

Let $h(x) = -9x^2 + 25$

$\Rightarrow h^2(x) = (-9x^2 + 25)^2 = 81x^4 - 450x^2 + 625$ and

$h'(x) = -18x$

$$y'(x) = -\frac{h'(x)}{h^2(x)}$$

$$\Rightarrow y'(x) = -\frac{(-18x)}{81x^4 - 450x^2 + 625}$$

$$= \frac{18x}{81x^4 - 450x^2 + 625}$$

Chain Rule

Most functions are functions of other functions. This means that their functional values also depend on values which themselves depend on other values of one of more independent variables.

For example, in real life, the number of sales a food vendor will make could depend on how nice the food tastes and the taste of the food also depends on the ingredients used in preparing the food. Another example is the relationship between time, displacement and velocity: time is absolutely independent but the displacement of a body depends on the time interval and the velocity also depends on the displacement.

If $y = f[g(x)]$ then we can infer that f is a function of g which is also a function of x and thus,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \text{where } u = g(x)$$

Example 7

Given that $h(x) = \sqrt{4x^2 - 3x}$, find the derivative, $h'(x)$.

Solution

If we let $g(x) = 4x^2 - 3x$, then we can write $h(x)$ as $h(x) = \sqrt{g(x)}$

$g'(x) = 8x - 3$ and $h'(x) = \frac{1}{2}(g(x))^{-\frac{1}{2}} \times g'(x)$

$\Rightarrow h'(x) = \frac{1}{2}(4x^2 - 3x)^{-\frac{1}{2}} \times (8x - 3)$

$$h'(x) = \frac{4x - 3 \,\underline{\quad}\, 2}{\sqrt{4x^2 - 3x}}$$

Power Rule

If $y = [g(x)]^n$ then $\boldsymbol{y' = n[g(x)]^{n-1}g(x)'}$

Example 8

Identify which rule(s) of differentiation will be more appropriate for the given functions and why.

1. $f(x) = \frac{1}{x-3}$
2. $f(x) = 4x - 7x^2$
3. $f(x) = (2x + 5x^5)(x - 2)$
4. $f(x) = \frac{(2x + 5x^5)(x - 2)}{(x - 3)}$
5. $f(x) = (4x - 7x^2)(3x - 9)^5$
6. $f(x) = (3x - 4\sqrt{x})^7$
7. $f(x) = (3x - 9)^4$

Solution

1. $f(x) = \frac{1}{x-3}$ is a reciprocal function and thus will require the reciprocal rule.
2. $f(x) = 4x - 7x^2$ is a polynomial function in terms of only one variable: x and thus can be differentiated using the power rule combined with the sum/difference rule.

3. $f(x) = (2x + 5x^5)(x - 2)$ is the product of two polynomial functions: $2x + 5x^5$ and $x - 2$. The product rule can then be used to differentiate $f(x)$. The product can be expanded to obtain function which can be differentiated using the power and sum/difference rules.

4. $f(x) = \frac{(2x + 5x^5)(x - 2)}{(x - 3)}$ can be differentiated using the quotient rule followed by the product rule then the sum/difference rules.

   The numerator can be expanded so that the quotient rule followed by the sum/difference can be applied.

5. The expansion of $f(x) = (4x - 7x^2)(3x - 9)^5$ can be skipped by applying the product rule and later the chain rule on $(3x - 9)^5$ to complete the differentiation.

   Alternatively, the binomial theorem can be used to expand $(3x - 9)^5$ and the result multiplied to $4x - 7x^2$ to obtain a polynomial function which can be differentiated using the power rule.

6. The differentiation of $f(x) = (3x - 4\sqrt{x})^7$ requires an application of the chain rule coupled with the power rule.

7. An application of the chain rule followed by power rule can be used to differentiate $f(x) = (3x - 9)^4$

FOCAL AREA 2: DIFFERENTIATING FUNCTIONS USING DIFFERENTIATION RULES

We have identified under focal area 1, appropriate rules to differentiate the following functions. Let us now apply them to differentiate the following

1. $f(x) = \frac{1}{(x - 3)}$ (*Reciprocal Rule*)

   $$\Rightarrow f'(x) = \frac{-\frac{d}{dx}(x - 3)}{(x - 3)^2} = \frac{-1}{x^2 - 6x + 9}$$

2. $f(x) = 4x - 7x^2$ (*Power Rule*)

   $f'(x) = 4 - 14x$

3. $f(x) = (2x + 5x^5)(x - 2)$ (*Product Rule*)

   $$f'(x) = (x - 2) \times \frac{d}{dx}(2x + 5x^5) + (2x + 5x^5) \times \frac{d}{dx}(x - 2)$$

   $$= (x - 2)(2 + 25x^4) + (2x + 5x^5)(1)$$

   $$= 30x^5 - 50x^4 + 4x - 4$$

Alternatively, we could expand the function to obtain a simple polynomial function as such

$f(x) = (2x + 5x^5)(x - 2) = 5x^6 - 10x^5 + 2x^2 - 4x$

We now apply the power rule to obtain

$f'(x) = 30x^5 - 50x^4 + 4x - 4$

4. $f(x) = \dfrac{(2x + 5x^5)(x - 2)}{(x - 3)}$

Before we use the *Quotient Rule*, we would try to represent the various functions as such

Let $u = 2x + 5x^5$, $v = x - 2$ and $w = x - 3$

Also, let $h = uv = (2x + 5x^5)(x - 2)$ and $h' = v \times u' + u \times v'$ (*Product Rule*)

$\Rightarrow u' = x + 25x^4$, $v' = 1$, $w' = 1$ and

$h' = (x - 2)(x + 25x^4) + (2x + 5x^5)(1)$

$= 30x^5 - 50x^4 + x^2$

$\Rightarrow f(x) = \frac{h}{w}$ and $f'(x) = \dfrac{w \times h' + h \times w'}{w^2}$

$f'(x) = \dfrac{(x - 3)(30x^5 - 50x^4 + x^2) - ((2x + 5x^5)(x - 2))(1)}{(x - 3)^2}$

$= \dfrac{25x^6 - 130x^5 + 150x^4 + x^3 - 5x^2 + 4x}{x^2 - 6x + 9}$

5. $f(x) = (4x - 7x^2)(3x - 9)^5$

Let $h(x) = 4x - 7x^2$, $g(x) = 3x - 9$, $w(x) = (3x - 9)^5 = g^5$ and

$f(x) = h(x) \times w(x)$

$h'(x) = 4 - 14x$, $g'(x) = 3$ and

$w'(x) = 5g^4 \times g' = 5(3x - 9)^4(3)$

$= 15(3x - 9)^4$

$\Rightarrow f'(x) = w \times h' + h \times w'$

$= (3x - 9)^5(4 - 14x) + (4x - 7x^2)(15(3x - 9)^4)$

$= -11907x^6 + 158922x^5 - 838350x^4 + 2187000x^3 - 2854035x^2 + 1614006x - 236196$

Alternatively,

$(3x - 9)^5 = 243x^5 - 3645x^4 + 21870x^3 - 65610x^2 + 98415x - 59049$

$\Rightarrow f(x) = (4x - 7x^2)(243x^5 - 3645x^4 + 21870x^3 - 65610x^2 + 98415x - 59049)$

$f(x) = -1701x^7 + 26487x^6 - 167670x^5 + 546750x^4 - 951345x^3 + 807003x^2 - 236196x$

Now we use the *Power Rule*

$f'(x) = -11907x^6 + 158922x^5 - 838350x^4 + 2187000x^3 - 2854035x^2 + 1614006x - 236196$

6. $f(x) = (3x - 4\sqrt{x})^7$ *(Chain Rule)*

Let $g(x) = 3x - 4\sqrt{x} \Rightarrow g'(x) = 3 - \frac{2}{\sqrt{x}}$

$f(x) = (g(x))^7$

$f'(x) = 7g(x)^6 \times g'(x)$

$= 7 \times (3x - 4\sqrt{x})^6 \times \left(3 - \frac{2}{\sqrt{x}}\right)$

$= 15309x^6 - 132678x^{\frac{11}{2}} + 489888x^5 - 997920x^{\frac{9}{2}} + 1209600x^4 - 870912x^{\frac{7}{2}} + 344064x^3 - 57344x^{\frac{5}{2}}$

7. $f(x) = (3x - 9)^4$

Let $g(x) = 3x - 9 \Rightarrow g'(x) = 3$

$f'(x) = 4(3x - 9)^3 \times 3$

$= 324x^3 - 2916x^2 + 8748x - 8748$

Learning Tasks

1. differentiate polynomial functions using the power rule
2. write differentiation rules using functions of their own
3. identify which differentiation rules are applicable for differentiating given function
4. apply appropriate differentiation rules to obtain the derivatives of given functions

PEDAGOGICAL EXEMPLARS

1. **Inquiry-Based learning/Talk-for-learning:** Learners explore and derive differentiation rules through guided inquiry
   - **a.** Review previous lessons on finding derivatives using first principles or with the power rule
   - **b.** Have a direct teaching session on basic differentiation rules
   - **c.** Break the rules into smaller rules where possible

   d. Use interactive lectures to introduce new rules with opportunities for student questions and discussions

2. **Collaborative learning/Experiential learning:** Learners work in small heterogeneous groups and apply differentiation rules to find derivatives
   a. Encourage peer tutoring among learners
   b. Provide learners with the opportunity of guided practice sessions where they practise finding derivatives with support from peers or the facilitator
   c. Provide extra time for tasks and allow the use of assistive technology if necessary
   d. Encourage a supportive classroom environment where mistakes are seen as learning opportunities
   e. Encourage students to use graphing calculators to visualise functions and their derivatives

KEY ASSESSMENT

Assessment Level 2: Skills of conceptual understanding

Find the derivatives of the following functions

1. $y = -x^2 + 3$
2. $s = -2t^{-1} + 4t^{-2}$
3. $y = (x^2 + 1)(x^3 + 3)$
4. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
5. $y = 2x^3(x^2 - 3)^4$

Reminder

Learners' score on Individual Class Exercise should be ready for submission to the STP.

WEEK 17

Learning Indicator: *Find derivatives of functions and relations that are not functions (Implicit differentiation)*

Hitherto, we have discussed the concept of differentiation, the power rule and other rules for finding the derivatives of functions other than simple power functions. In the lessons for this week, we will practise finding the derivatives of more functions and also find the derivatives of functions which are not defined explicitly i.e., as an expression in terms of the independent variable only.

FOCAL AREA 1: FINDING THE DERIVATIVES OF FUNCTIONS

Example 1

Find the derivative of $f(x) = 5\sqrt{3x^2 + x}$

Solution

$f(x) = 5\left(3x^2 + x\right)^{\frac{1}{2}}$

$f'(x) = 5\left(\frac{1}{2}\right) \times \left(3x^2 + x\right)^{-\frac{1}{2}} \times \left(6x + 1\right)$

$= \dfrac{5(6x + 1)}{2\sqrt{3x^2 + x}}$

Example 2

Given that $f(x) = x^5 + \frac{1}{x}$ find $f'(x)$

Solution

$f(x) = x^5 + x^{-1}$

$f'(x) = 5x^4 - x^{-2}$

$= 5x^4 - \dfrac{1}{x^2}$

Example 3

If $y = 2\sqrt{x}$, provide an expression for $\dfrac{dy}{dx}$

Solution

$y = 2\left(x\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = 2 \times \frac{1}{2}(x)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

Example 4

Given that $y = (\sqrt{-2x + 1})\left(x^2 + 4\right)$, find the derivative of y with respect to x

Solution

$y = \left(-2x + 1\right)^{\frac{1}{2}} \times (x^2 + 4)$

$\frac{dy}{dx} = \frac{1}{2}(x^2 + 4)(-2)(-2x + 1)^{-\frac{1}{2}} + (2x)(-2x + 1)^{\frac{1}{2}}$

$= -\frac{x^2 + 4}{\sqrt{1 - 2x}} + 2x\sqrt{1 - 2x}$

FOCAL AREA 2: DIFFERENTIATING IMPLICIT FUNCTIONS

The function which can be easily written as $y = f(x)$ with the y variable on one side and the function of x on the other side, is called an explicit function. But in an implicit function, the x and the y variable cannot be written in the form $y = f(x)$ and an implicit function has more than one solution for the given function.

Further in an implicit function, the relationship between x and y is expressed as $g(x, y) = 0$, where y is an implicit function of x. More specifically the value of x defines the value of y such that on substitution of x, y in the expression on the left-hand side, equalizes it to zero. The expressions $y = x^2$, $y = ax + b$ and $y = \sqrt{x}$ are all examples of explicit functions. Expressions such as $ax^2 + bxy - y = 0$, $x^2 - y^2 = 0$, $ey + x - y + \log(y) = 0$, are examples of implicit functions.

Derivatives of implicit functions

We can go through the process to isolate the dependent variable in an equation of an implicit function to make it explicit so we can then use the rules of differentiation to find the required derivative.

For example, if $3x - y^2 = -4$, then

$y = \sqrt{3x + 4}$ and

$\frac{dy}{dx} = \frac{3}{2\sqrt{3x + 4}}$

It is however sometimes too difficult to convert an implicitly defined function to an explicit form. Therefore, we need to differentiate the original function implicitly.

The idea behind implicit differentiation is to treat y as a function of x (which is what we are trying to do anyway). To emphasise this, let us rewrite the relation $\sqrt{y} = x$, which is an implicit function by replacing y with $y(x)$

$\sqrt{y(x)} = x.$

Now we differentiate each side of this equation, and set their derivatives equal to each other. Since we do not know the formula for $y(x)$, we just leave its derivative as $y'(x)$

$\frac{1}{2}(y(x))^{-\frac{1}{2}} \times y'(x) = 1$

Finally, we solve for $y'(x)$ to get its formula

$y'(x) = 2\sqrt{y(x)}$

$y'(x) = 2\sqrt{y}$

Note that since y is in itself a function of x, the chain rule needs to be applied when differentiating terms which contains y

Example 5

Find the derivative of y with respect to x if $2x^3 + y^3 = 9xy$

Solution

$6x^2 + 3y^2\frac{dy}{dx} = 9\left(x\frac{dy}{dx} + y\right)$

$6x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y$

$6x^2 - 9y = \frac{dy}{dx}(9x - 3y^2)$

$\frac{dy}{dx} = \frac{6x^2 - 9y}{9x - 3y^2}$

$\frac{dy}{dx} = \frac{2x^2 - 3y}{3x - y^2}$

Example 6

$x^2y + 2 = 5y^4 - xy$

Solution

$x^2y + 2 = 5y^4 - xy$

$2xy + x^2\frac{dy}{dx} = 20y^3\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right)$

$$x^2\frac{dy}{dx} - 20y^3\frac{dy}{dx} + x\frac{dy}{dx} = -y - 2xy$$

$$(x^2 - 20y^3 + x)\,d\frac{y}{dx} = -y - 2xy$$

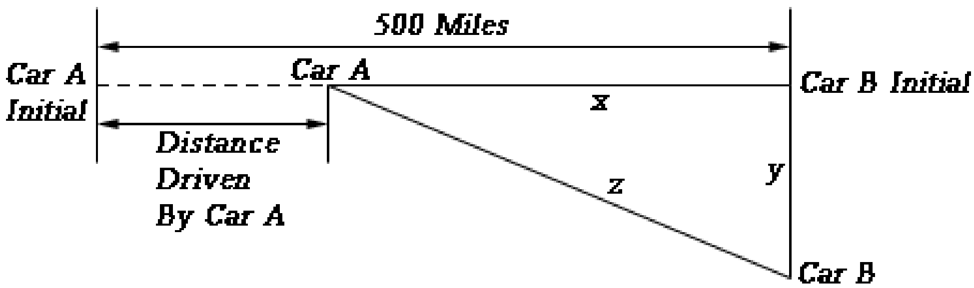
$$\frac{dy}{dx} = -\frac{y - 2xy}{x^2 - 20y^3 + x}$$

Everyday problems involving derivatives

Example 7

Two cars start out 500 miles apart. Car A is to the west of Car B and starts driving to the east (*i.e.*, towards Car B) at 35 mph, and at the same time, Car B starts driving south at 50 mph. After 3 hours of driving, at what rate is the distance between the two cars changing? Is it increasing or decreasing?

Solution



In this figure, y represents the distance driven by Car B, x represents the distance separating Car A from the initial position of Car B, and z represents the distance separating the two cars. After 3 hours of driving time, we have the following values of x and y

$x = 500 - 35(3) = 395$

$y = 50(3) = 150$

By Pythagoras theorem,

$z^2 = x^2 + y^2 = 395^2 + 150^2$

$= 178525$

$z = \sqrt{178525} = 422.5222$

Now to determine z', we apply implicit differentiation as all variables change with time.

Given $x' = -35$ *and* $y' = 50$

$z^2 = x^2 + y^2$

$\Longrightarrow 2zz' = 2xx' + 2yy'$

$z'(422.5222) = (395)(-35) + (150)(50)$

$z' = \frac{-6325}{422.5222} = -14.9696$

So, after three hours, the distance between them is decreasing at a rate of 14.9696 mph.

Example 8

Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution

The volume $V(t)$ and radius $r(t)$ are varying with time.

$V'(t) = 5,\ r'(t) = ?\ when\ r(t) = d\frac{=}{2} = 10cm$

Volume of a sphere is given by;

$Vt = \frac{4}{3}\pi[\ r(t)]^3$

By implicit differentiation;

$V' = 4\pi r^2 r'$

$5 = 4\pi(10^2)r'$

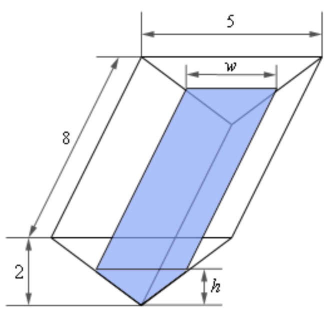
$\Longrightarrow r' = \frac{1}{80} cm/min$

Example 9

A trough of water is 8 metres in length, and its ends are in the shape of isosceles triangles whose width is 5 metres and height is 2 metres.

If water is pumped in at a constant rate of $6m^3/sec$, at what rate is the height of the water changing when the water has a height of 120 cm?

Solution



$V' = 6m^3/sec\ h' = ?\ and\ h = 1.2m$

$V = (Area\ of\ end)(depth)$

$= \left(\frac{1}{2} base \times height\right)(depth)$

$= \frac{1}{2}hw(8)$

$= 4hw$

$\frac{w}{5} = \frac{h}{2} \quad \Longrightarrow w = \frac{5}{2}h$

$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$

By implicit differentiation;

$V' = 20hh'$

$6 = 20(1.2)h'$

$\Longrightarrow h' = 0.25\ m/sec$

So, the height of the water is rising at a rate of 0.25 m/sec

$w = \frac{5}{2}h$

$\Longrightarrow w' = \frac{5}{2}h'$

$w' = \frac{5}{2}(0.25) = 0.625\ m/sec$

Therefore, the width is increasing at a rate of 0.625 m/sec.

Learning Tasks

Task learners to

1. provide examples of implicit functions, giving reasons.
2. convert implicit functions to corresponding explicit forms where possible.
3. find the derivative of an implicit function.

PEDAGOGICAL EXEMPLARS

1. **Collaborative learning/Talk-for-learning/Experiential learning:** Learners in heterogenous groups solve problems on implicit differentiation and share it across groups.
   a. Provide guidance and support and encourage learners to do same as they work in their groups.
   b. Assign practice problems that gradually increase in complexity. Encourage students to solve problems individually and then discuss their solutions in pairs or small groups.

      Encourage learners to share with other groups and the larger group, their solution to given tasks.

c. Use assistive technology or manipulatives for students who need additional help. Offer alternative ways to demonstrate understanding, such as oral explanations or drawings.

2. **Problem-Based learning:** Learners work as individuals or in small heterogeneous groups and apply implicit differentiation to solving real-life problems.

   a. Provide worksheets with problems of increasing difficulty. Walk around the classroom to give instant feedback and support.

   b. Encourage learners to explore and solve more problems on applications of implicit functions and differentiation.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Find $\frac{dy}{dx}$ given the equation $y^3 - y + 2x^3 - x = 8$

2. Find $\frac{dy}{dx}$ given that x and y are related by the equation $\sqrt{x^2 + y^2} - x^2 = 5$

Assessment Level 3: Strategic Reasoning

1. The curves $y = x^2$ and $x = y^3$ intersect at the points (1, 1) and (0, 0). Find the angle between the curves at each of these points

2. Two curves are said to be orthogonal if, at each point of intersection, the angle between their tangents is a right angle. Show that the curves are orthogonal.

a. The hyperbola $x^2 - y^2 = 5$ and the ellipse $4x^2 + 9y^2 = 72$.

b. The ellipse $3x^2 + 2y^2 = 5$ and $y^3 = x^2$.

Hint

The graphs show the curves with their points of intersection

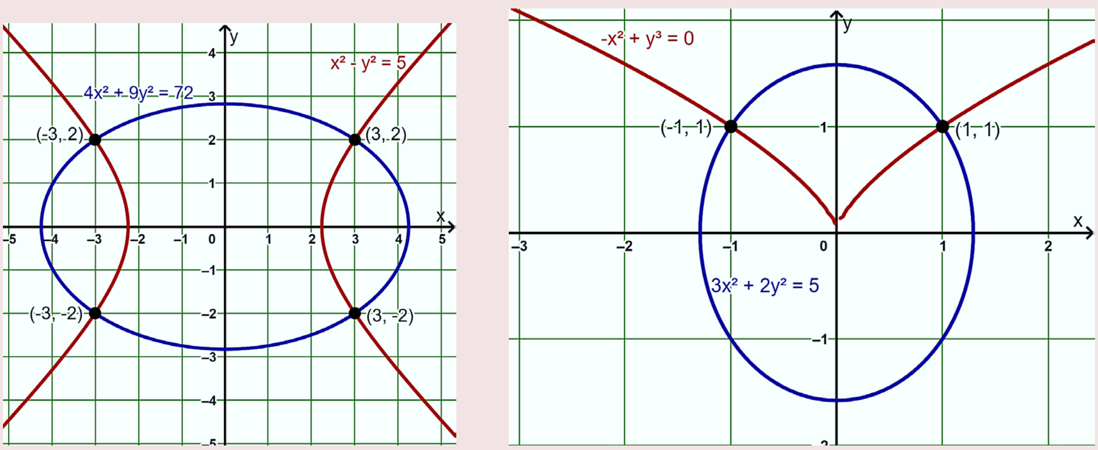


Figure 10.1

Assessment Level 4: Extended critical thinking and reasoning

A major audiotape manufacturer is willing to make x thousand ten-packs of metal alloy audiocassette tapes available in the marketplace each week when the wholesale price is \$p per ten-pack. It is known that the relationship between x and p is governed by the supply equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of tapes changing when the price per ten-pack is \$11, the quantity supplied is 4000 ten-packs, and the wholesale price per ten-pack is increasing at the rate of \$.10 per ten-pack each week

WEEK 18

Learning Indicator: *Identify and apply techniques of differentiation to solve problems involving transcendental functions*

FOCAL AREA 1: DIFFERENTIATING TRANSCENDENTAL FUNCTIONS

Transcendental functions are those that are not algebraic. They include trigonometric, exponential and logarithmic functions.

Derivative of Exponential and Logarithm functions

If $g(x) = e^{f(x)}$ **then** $\frac{d}{dx}(g(x)) = f'(x) \times e^{f(x)}$

and

If $f(x) = \log_a(g(x))$ **then** $\frac{d}{dx}f(x) = \frac{1}{\ln(a)} \times \frac{1}{g(x)} \times g'(x)$

Example 1

Find the derivative of the following;

i. $g(x) = e^x$

ii. $g(x) = e^{3x}$

iii. $g(x) = xe^x$

iv. $g(x) = e^{x^2-6x}$

v. $g(x) = \ln(x)$

vi. $g(x) = \ln(x+2)$

vii. $g(x) = \ln(x+4x^2)$

viii. $g(x) = \ln(\ln(x+4x^2)) + e^{3x}$

ix. $g(x) = x\ln(x+4x^2)$

Solution

i. $g(x) = e^x$

$g'(x) = e^x$

ii. $g(x) = e^{3x}$

$g'(x) = 3e^{3x}$

iii. $g(x) = xe^x$

$g'(x) = \frac{d}{dx}(x) \times e^x + \frac{d}{dx}(e^x) \times x$

$= e^x + xe^x$

iv. $g(x) = e^{x^2-6x}$

We can make $g(x)$ look simpler as such

Let $y = x^2 - 6x \Rightarrow g(x) = e^y$

Also, $y' = 2x - 6$

$g'(x) = y' \times e^y$

$= (2x - 6)e^{x^2-6x}$

v. $g(x) = \ln(x) = \log_e(x)$

$g'(x) = \frac{1}{\ln(e)} \times \frac{1}{x}$

$= \frac{1}{x}$

vi. $g(x) = \ln(x + 2)$

$g'(x) = \frac{1}{x+2} \times 1$

$= \frac{1}{x+2}$

vii. $g(x) = \ln(x + 4x^2)$

$g'(x) = \frac{1}{x+4x^2} \times (1 + 8x)$

$= \frac{1+8x}{x+4x^2}$

viii. $g(x) = \ln(\ln(x + 4x^2)) + e^{3x}$

Let $u = x + 4x^2$, $v = \ln(x + 4x^2) = \ln(u)$

$\Rightarrow g(x) = \ln(v) + e^{3x}$

$u' = 1 + 8x,$

$v' = \frac{1}{u} \times u' = \frac{1}{x+4x^2}(1 + 8x) = \frac{1+8x}{x+4x^2}$

$g'(x) = \frac{1}{v} \times v' + 3(e^{3x})$

$= \frac{1}{\ln(x+4x^2)} \times \frac{1+8x}{x+4x^2} + 3e^{3x}$

$\frac{8x+1}{(4x^2+x)\ln(4x^2+x)} + 3e^{3x}$

ix. $g(x) = x\ln(x + 4x^2)$

$h = \ln(x + 4x^2) \Rightarrow g(x) = x \times h$

Also, $h' = \dfrac{1 + 8x}{x + 4x^2}$

$\Rightarrow g'^{(x)} = x \times h' + h \times \frac{d}{dx}(x)$

$= xh' + h$

$= x\left(\dfrac{1 + 8x}{x + 4x^2}\right) + h$

$= \dfrac{1 + 8x}{1 + 4x} + \ln(x + 4x^2)$

Example 2

Find the derivative of the following:

i. $y = \sqrt[3]{\ln x}$

ii. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Solution

i. $y = (\ln(x))^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{3}(\ln(x))^{-\frac{2}{3}} \times \dfrac{1}{x}$

$= \dfrac{1}{3x} \times \dfrac{1}{\sqrt[3]{(\ln(x))^2}}$

$\mathbf{= \dfrac{1}{3x\sqrt[3]{(\ln(x))^2}}}$

ii. $\dfrac{dy}{dx} = \dfrac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$= \dfrac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$

$= \dfrac{(e^x - e^{-x} - e^x - e^{-x})(e^x - e^{-x} + e^x + e^{-x})}{(e^x - e^{-x})^2}$

$= \dfrac{(-2e^{-x})(2e^x)}{(e^x - e^{-x})^2}$

$= -\dfrac{4}{(e^x - e^{-x})^2}$

Derivatives of trigonometric functions

If $\boldsymbol{y = \sin(x), \dfrac{dy}{dx} = \cos(x)}$

Also, if $\boldsymbol{y = \cos(x)\ \dfrac{dy}{dx} = -\sin(x)}$

Example 3

Find the derivative of $g(x) = 3\cos^2(x)$

Solution

$g(x) = 3\cos^2(x)$

$g(x) = 3\cos(x) \times \cos(x)$

Let $u(x) = u = \cos(x)$, $g(x) = 3u^2$

$\Rightarrow u'(x) = -\sin(x)$, $g'(x) = 3 \times 2u \times u'$

$g'(x) = -6\cos(x)\sin(x)$

Example 4

Find an expression for $\frac{dy}{dx}$ if $y = \sin(2x + 7)$

Solution

$y = \sin(2x + 7)$

Let $u(x) = 2x + 7$

$\Rightarrow \frac{du}{dx} = 2$

$y = \sin(u)$

$\frac{dy}{dx} = \frac{du}{dx} \times \cos(u)$

$= 2\cos(2x + 7)$

Example 5

Using implicit differentiation, differentiate the following

$e^{2y}x + 2\sin(\sin(y)) = 5x + 2 - 3xy$

Solution

$e^{2y}x + 2\sin(\sin(y)) = 5x + 2 - 3xy$

$\Rightarrow x(e^{2y})\left(2\frac{dy}{dx}\right) + e^{2y} + 2\cos(\sin(y))\left(\cos(y)\frac{dy}{dx}\right) = 5 - 3\left(y + x\frac{dy}{dx}\right)$

$\frac{dy}{dx}(2xe^{2y} + 2\cos(\sin(y))\cos(y)) = 5 - 3y - 3x\frac{dy}{dx}$

$\frac{dy}{dx}(2xe^{2y} + 2\cos(\sin(y))\cos(y) + 3x) = 5 - 3y$

$\frac{dy}{dx} = \frac{5 - 3y}{2xe^{2y} + 2\cos(\sin(y))\cos(y) + 3x}$

Learning Tasks

Task learners to

1. Recall differentiation rules.
2. Provide examples of transcendental functions.
3. Differentiate transcendental functions.

PEDAGOGICAL EXEMPLARS

Collaborative learning/Experiential learning

1. Learners work in small heterogeneous groups and apply differentiation rules to find derivatives of transcendental functions.
2. Learners work through examples of differentiating basic exponential functions (e.g., $y = e^x$), logarithmic functions (e.g., $\log_b(x)$ and $\ln(x)$), and trigonometric functions (e.g., $\sin(x)$, $\cos(x)$) with support from peers and the teacher.
3. Learners work on challenging differentiation problems involving combinations of transcendental functions.
4. Provide problems that require multi-step solutions and critical thinking.
5. Learners complete worksheets with a mix of guided problems and simple, step-by-step differentiation exercises.
6. Include fill-in-the-blank sections to reinforce steps in the differentiation process.
7. Provide one-on-one support, use simple language and offer extra time for practice when necessary.
8. Investigate real-life scenarios where differentiation of transcendental functions is applicable, such as growth models, decay processes, and wave functions.
9. Encourage peer tutoring among learners.
10. Provide learners with the opportunity of guided practice sessions where they practise finding derivatives with support from peers or the teacher.
11. Allow students to choose projects that interest them, ensuring relevance to diverse backgrounds.
12. Provide extra time for tasks and allow the use of assistive technology if necessary.
13. Encourage a supportive classroom environment where mistakes are seen as learning opportunities.
14. Encourage students to use graphing calculators to visualise functions and their derivatives.

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

Determine the domain and the derivative of the following functions

1. $f(x) = \cos(\ln(x))$

2. $g(x) = x^{\sin(x)}$

3. $h(x) = (\cos(x))^{x^2+1}$

Assessment Level 3: Strategic Reasoning

Let $f(x) = xe^{-1}$ for all real values of x

1. On what intervals does $f(x)$ increase?

2. On what intervals does $f(x)$ decrease?

Hint

- *The Recommended Mode of Assessment for Week 18 is* ***Mid-semester Examination.***
- *Refer to Appendix F at the end of section 10 for further information on how to go about the mid-semester examination.*

Section 10 Review

In this section, learners were introduced to rules for differentiating functions. These rules are necessary because they tend to be more practical than differentiation by first principles. These rules were used to find the derivatives of implicit functions, transcendental and combinations of different groups of functions.

Pedagogical approaches that provide learners with the opportunity to practise applying differentiation rules particularly experiential and problem-based learning in collaborative environments were encouraged. Accommodations should be made to support learners who may have special educational needs to promote learning while learners should be challenged with more challenging tasks

APPENDIX F

Table of Test Specification

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
13	1. Application of the laws of logarithms 2. Graphing logarithmic functions	Multiple Choice	2	1	1		4
		Short answer		1			1
		Application					
14	1. Trigonometric identities	Multiple Choice	1	1	1		3
		Short answer		1			1
		Application		1			1
15	1. Deriving and applying the sine and cosine Rule 2. Solving Trigonometric Equations	Multiple Choice	1	1	1		3
		Short answer		1			1
		Application		1	1		2
16	1. Identifying differentiation rules 2. Differentiating functions using differentiation rules	Multiple Choice	1	1			2
		Short answer		1			1
		Application		1			1
17	1. Differentiating Implicit functions	Multiple Choice	1	1	1		3
		Short answer		1			1
		Application					
	Total		6	13	5	0	24

Structure of Mid-Semester Examination Questions

a) *Cover contents from weeks 13-17*

b) *Take into consideration DoK levels*

- **i.** *Section A- Multiple Choice (15 questions)*
- **ii.** *Section B- (5 fill-in questions and short answers, all to be answered)*
- **iii.** *Section C- Real-life Application (4 questions, 1 to be answered).*

c) *Time: 1 hour 30 minutes.*

d) *Total Score: 100 marks*

SECTION 11: INTEGRATION

Strand: Calculus

Sub-Strand: Principles of calculus

Learning Outcome: *Estimate the area under a curve using the trapezoid rule.*

Content Standard: Demonstrate a conceptual understanding of the connection between integration and limits and integration as a reverse process of differentiation.

INTRODUCTION AND SECTION SUMMARY

In this section, learners will explore the foundational concepts of calculus, focusing on partitioning areas under curves, the connection between limits and integrals, the Fundamental Theorem of Calculus and the process of finding integrals. The lessons in this section start with how partitioning an area under a curve into simple geometric shapes approximates the area, leading to an understanding of limits as this partitioning becomes more refined. This understanding is crucial for grasping definite integrals, which calculate areas precisely. The Fundamental Theorem of Calculus, which connects differentiation and integration, making the process of finding areas under curves more straightforward through antiderivatives will then be discussed. In the third year, these concepts will be built upon to discover how integrals are used in real-world applications, such as calculating areas and volumes, ensuring that learners not only understand the theoretical concepts but also see their practical value.

The weeks covered by the section are

Week 19

1. *Partitioning of an area within an interval*
2. *Area under a Curve*

Week 20

1. *Connection between limits and integrals*
2. *Fundamental Theorem of Calculus*
3. *Finding indefinite integrals*

SUMMARY OF PEDAGOGICAL EXEMPLARS

Effective learning of the focal areas in the lessons in this section depends on strong prerequisite understanding of intervals, the concept of limits and the basis for differentiating functions. Talk-for-learning approaches are therefore suggested to provide the opportunity for learners to review these essentials concepts. It is also suggested that exploratory, collaborative, experiential and inquiry pedagogical approaches be adopted to promote learning. The use of computer assisted technologies is also promoted as it helps learners to visualise concepts and perform manipulations/simulations to improve learning. It must be noted that pedagogical exemplars provided are guides and should be adapted to cater for inclusive learning environments.

ASSESSMENT SUMMARY

Formative assessments such individuals task sheets, classroom quizzes, peer evaluations, charts on graphs, use of GeoGebra and more should be used to assess learners. Group presentations will also be used as an assessment strategy to provide the teacher with an opportunity to assess learners' ability to explain their understanding of learning and provide learners with the opportunity to develop national values as they work together. Assessment items which range from Level 1 to Level 4 the DOK will be utilised. Records of performances of learners should be kept for continuous assessment records.

WEEK 19

Learning Indicators

1. *Distinguish between partitioning an interval for a given step size and the number of subintervals*
2. *Partition intervals for a given step size or number of sub-intervals and find area under curves through graphics*
3. *Compare and judge the effect of reduction in step size and increase in the number of subintervals in an interval for a given function on the area under a curve*

In year one, learners determined the area bounded by linear functions. The areas discussed were in the shape of trapezia. The areas obtained were accurate since the required areas were perfect trapezia. For some curves and by extension some required areas like in Figure 11.1, the choice of using a trapezium leaves some areas not catered for. That area is the error region. We can get a more accurate approximation of the area by using smaller trapezia and summing the areas of the trapezia. In this week's lessons, we will discuss partitioning the area under curves.

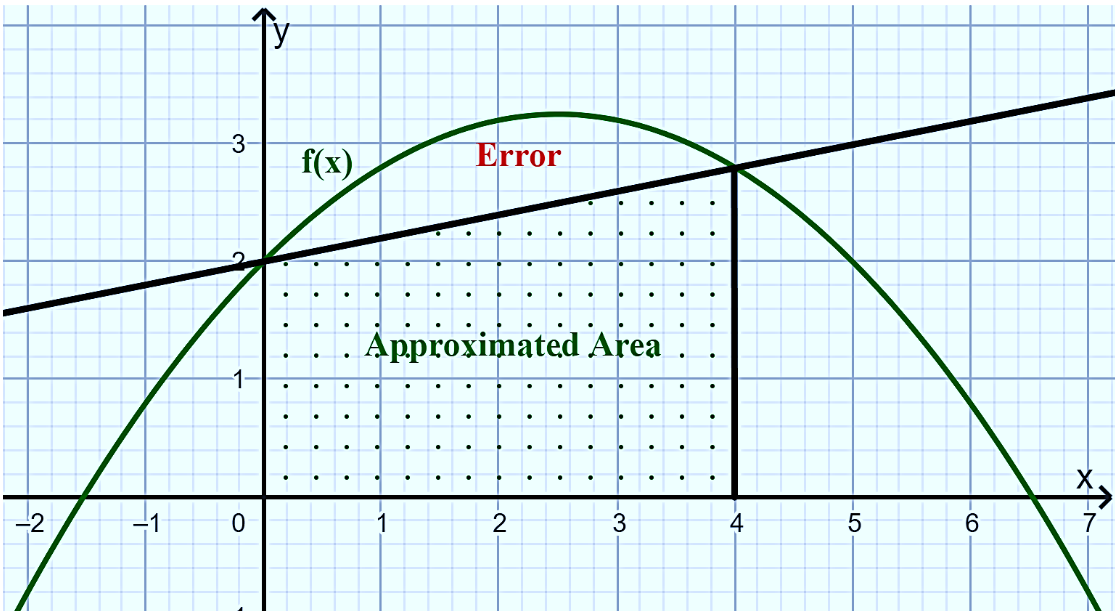


Figure 11.1

FOCAL AREA 1: PARTITIONING AN AREA WITHIN AN INTERVAL

Before we zoom into the trapezoidal rule, learners should be guided to determine step sizes of given intervals as depicted in the following examples.

Example 1

How many subintervals will be from [0, 10] for a step size of 2?

Solution

The step size 2 as in this case provides the difference in each interval

The first interval will be from 0 to $0 + 2 = 2$ i.e., $[0, 2)$ or $(0, 2]$

The next interval will be from 2 to $2 + 2 = 4$ i.e., $[2, 4)$ or $(2, 4]$

The next interval will be from 4 to $4 + 2 = 6$ i.e., $[4, 6)$ or $(4, 6]$

The next interval will be from 6 to $6 + 2 = 8$ i.e., $[6, 8)$ or $(6, 8]$

The last interval will be from 8 to $8 + 2 = 10$ i.e.,$[8, 10)$ or $(8, 10]$

We end the addition of the step size here since we have obtained the upper limit of the original interval i.e., 10

We have 5 intervals

Alternatively,

$$\textit{Number of subintervals} = \frac{\textit{Upper limit} - \textit{Lower limit}}{\textit{Step size}} = \frac{10-0}{2} = 5$$

Example 2

Construct n subinterval on the interval $[a, b]$

Solution

Step 1: Compute the step $h = \frac{b-a}{n}$

Step 2: Get the subintervals as follows

$[a, a+h), [a+h, a+2h), \ldots\ldots\ldots, [a+(n-1)h, a+nh]$

NB:

1. The length of each subinterval is the step size.
2. The relationship between the step size and the number of subintervals allows for partitioning once we know the step size or the number of subintervals.

Example 3

You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road in miles. [1 *mile* = 5280 *ft*]

Time / sec	0	10	20	30	40	50	60	70	80	90	100	110	120
Velocity / *ft*/*sec*	0	44	15	35	30	44	35	15	22	35	44	30	35

Solution

Our assumption is that the velocity of the car remains constant for the 10-seconds windows / intervals. For example, from the table, from the 1st second to the 10th second, the car is travelling at 44 *ft*/*sec* and the car is travels at 15 feet per second from the 61st second to the 70th second.

These two bits of information can be used to obtain the distances / displacements travelled in each interval

Recall that $displacement = velocity \times time\ duration$

The time will remain constant i.e., 10 *seconds* as the intervals are of equal time intervals of 10 *seconds*

Time / sec	0	10	20	30	40	50	60	70	80	90	100	110	120
Velocity / *ft*/*sec*	0	44	15	35	30	44	35	15	22	35	44	30	35
Distance ($Velocity \times 10$)	0	440	150	350	300	440	350	150	220	350	440	300	350

The total distance covered from the journey is the sum of all the distances in the intervals

$440 + 150 + 350 + \ldots + 440 + 300 + 350 = 3{,}840\ ft$

$3{,}840\ ft = \frac{3{,}840}{5280}\ mi = 0.72\ mi$

$\therefore$ the length of the road is 0.72 miles

FOCAL AREA 2: AREA UNDER A CURVE

The area in Figure 1 can be approximated by simply finding the area of the trapezium bounding the required area. The trapezium has lengths of size 2 units and 2.8 units and spans along the $x - axis$, from 0 to 4 and thus has a width of 4 units. The approximated area under the curve of $f(x)$ between $x = 0$ and $x = 4$ is therefore $\frac{1}{2}(4)(2.8 + 2) = 9.6$ *squared units*

A comparison of figure 19.1 and figure 19.2 shows that the error region in figure 19.1 is bigger than in figure 19.2. Effectively, the area which will be obtained in figure 19.2 will be closer to the expected area.

$A_1 = \frac{1}{2}(1)(2 + 2.8) = 2.4$ *squared units*

$A_2 = \frac{1}{2}(1)(2.8 + 3.2) = 3$ *squared units*

A_2 is equal in area to A_4 hence $A_4 = 3$ *squared units*

$A_3 = (1)(3.2) = 3.2$ *squared units*

The approximated area is $A_1 + A_2 + A_3 + A_4 = 2.4 + 2(3) + 3.2 = 11.6$ *squared units*. The exact area under the curve between $x = 0$ and $x = 4$ is 11.7333 *squared units*

The difference between the illustrations in Figure 11.1 and Figure 11.2 is that the interval i.e., [0, 4] has been partitioned into four smaller trapezia with a step size of 1 *unit*. The higher the number of trapezia, the more accurate the approximate area under the curve

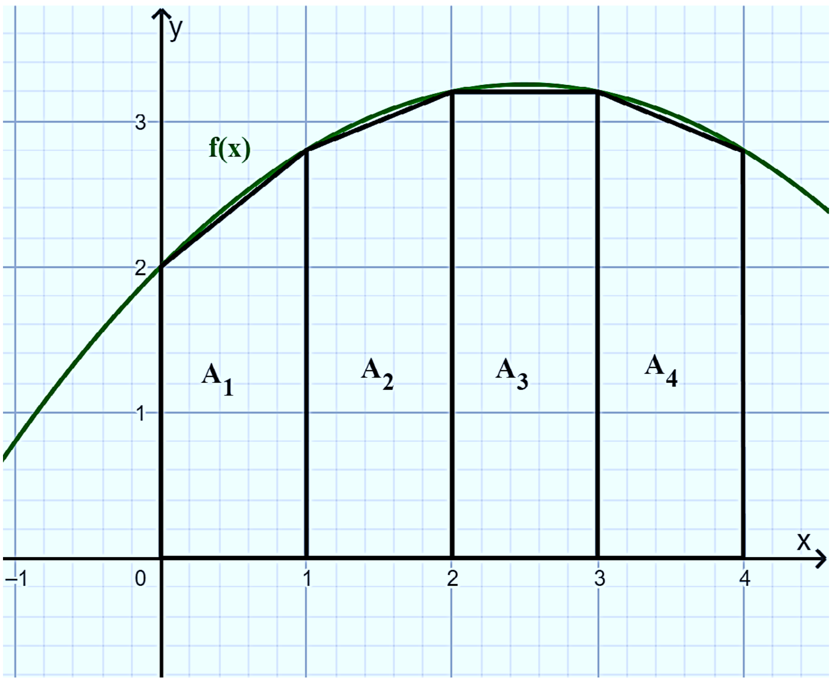


Figure 11.2

Example 4

Investigate the area under the curve, $y = x$ for $x \in [0,8]$ when the number of intervals, N is 5 and 10. Try to approximate the area of the rectangles constructed under subintervals.

$N = 5$, *Step size*, $h = \frac{8-0}{5} = 1.6$

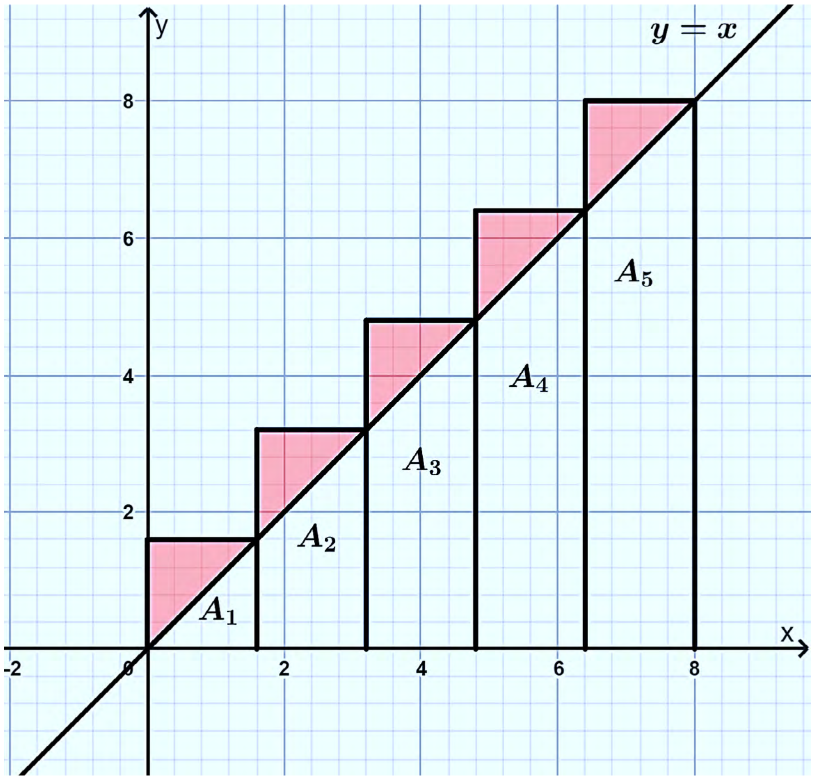


Figure 11.3

We would find and sum the areas of the individual rectangles but before we do, we need the heights of the rectangles. Theses can be found by finding the functional values of the x values of the right sides of the rectangles. We will represent the values as x and the corresponding heights by $y(x)$.

Note that the width of the rectangles are equal and it equals the step size, h

$x = 0 + h = 0 + 1.6 = 1.6$

$A_1 = 1.6 \times y(1.6) = 1.6 \times 1.6 = 2.56$

$x = 1.6 + h = 1.6 + 1.6 = 3.2$

$A_2 = 1.6 \times y(3.2) = 1.6 \times 3.2 = 5.12$

$x = 3.2 + h = 3.2 + 1.6 = 4.8$

$A_3 = 1.6 \times y(4.8) = 1.6 \times 4.8 = 7.68$

$x = 4.8 + h = 4.8 + 1.6 = 6.4$

$A_4 = 1.6 \times y(6.4) = 1.6 \times 6.4 = 10.25$

$x = 8$

$A_5 = 1.6 \times y(8) = 1.6 \times 8 = 12.8$

$Approximate\ Area = A_1 + A_2 + A_3 + A_4 + A_5$

$= 2.56 + 5.12 + 7.68 + 10.25 + 12.8$

$= 38.41$ *squared units*

$N = 10,\ h = 0.8$

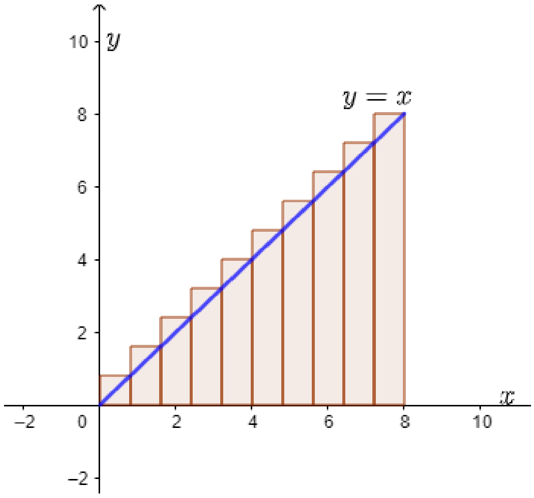


Figure 11.4

$x = 0 + h = 0 + 0.8 = 0.8$

$A_1 = 0.8 \times y(0.8) = 0.8 \times 0.8 = 0.64$

$x = 0.8 + h = 0.8 + 0.8 = 1.6$

$A_2 = 0.8 \times y(1.6) = 0.8 \times 1.6 = 1.28$

$x = 1.6 + h = 1.6 + 0.8 = 2.4$

$A_3 = 0.8 \times y(2.4) = 0.8 \times 2.4 = 1.92$

$x = 2.4 + h = 2.4 + 0.8 = 3.2$

$A_4 = 0.8 \times y(3.2) = 0.8 \times 3.2 = 2.56$

$x = 3.2 + h = 3.2 + 0.8 = 4$

$A_5 = 0.8 \times y(4) = 0.8 \times 4 = 3.2$

$x = 4 + h = 4 + 0.8 = 4.8$

$A_6 = 0.8 \times y(4.8) = 0.8 \times 4.8 = 3.84$

$x = 4.8 + h = 4.8 + 0.8 = 5.6$

$A_7 = 0.8 \times y(5.6) = 0.8 \times 5.6 = 4.48$

$x = 5.6 + h = 5.6 + 0.8 = 6.4$

$A_8 = 0.8 \times y(6.4) = 0.8 \times 6.4 = 5.12$

$x = 6.4 + h = 6.4 + 0.8 = 7.2$

$A_9 = 0.8 \times y(7.2) = 0.8 \times 7.2 = 5.76$

$x = 7.2 + h = 7.2 + 0.8 = 8$

$A_{10} = 0.8 \times y(8) = 0.8 \times 8 = 6.4$

$Approximate\ Area = A_1 + A_2 + \ldots + A_9 + A_{10}$

$= 0.64 + 1.28 + 1.92 + \ldots + 5.12 + 5.76 + 6.4$

$= 35.2\ squared\ units$

The exact area under the line, $y = x$ for x values between 0 and 8 is in the shape of a triangle and thus can be calculated thus

$$Exact\ area = \frac{1}{2} base \times height$$
$$= \frac{1}{2} \times 8 \times 8 = 32$$

The approximation of the area using ten rectangles (obtained as 35.2 *squared units*) is closer to the exact area (32 squared units) than the approximation obtained using five rectangles (38.41 squared units). This suggests that the higher the number rectangles (shape) used for the approximation, the better the approximation

Learning Tasks

Task learners to

1. use a number line to partition given intervals
2. calculate the step size for a given interval and specified number of partitions
3. find the sum of the area of rectangles constructed for different numbers of subintervals or step sizes on a close interval of a function
4. compare the effect of different numbers of sub-intervals on the accuracy of the area estimation under a curve

PEDAGOGICAL EXEMPLARS

1. **Talk-for-learning:** Lead a whole class discussion on the concept of partitioning intervals and estimating areas under curves
   a. Review basic concepts of intervals and partitioning
   b. Discuss the importance of finding the area under a curve
   c. Use visual and concrete examples
   d. Discuss the application of partitioning in calculus, such as Riemann sums

2. **Experiential Learning:** Learners in small heterogenous or as individuals partition intervals and estimate areas under curves
   - **a.** Provide worksheets with pre-drawn number lines
   - **b.** Task learners to partition intervals using different step sizes
   - **c.** Encourage the use of graph paper to visualise sub-intervals
   - **d.** Introduce software tools or graphing calculators for complex calculations
   - **e.** Assign real-life application problems, such as using partitioning to estimate distances or areas in practical scenarios
   - **f.** Tailor assignments to meet the needs of each learner, allowing for varying levels of difficulty and complexity
   - **g.** Provide opportunities for peer teaching and mentoring
   - **h.** Provide opportunity for learners to use manipulatives, such as rulers or strips of paper, to physically partition intervals

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Figure 11.5 shows the graph of $y = f(x)$. Use a step size of 2 *units* and suitable rectangles or trapezia to estimate the area under the curve between $x = 1$ and $x = 9$

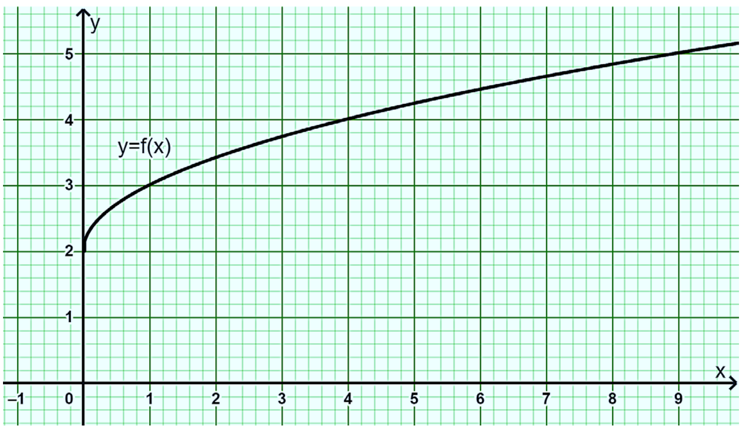


Figure 11.5

2. The velocity function of a projectile fired straight into the air is $v(t) = (160 - 9.8t)$ m/s. Use 3, 6 and 9 subintervals to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m

Assessment Level 3: Strategic Reasoning

The table below gives the velocity of a vintage sports car accelerating from 0 to 229 *km/h* in 36 *sec* with its corresponding graph

Time / h	0	10	20	30	40	50	60	70	80	90	100
Velocity / *km/h*	0	64	100	132	154	174	187	201	212	220	229

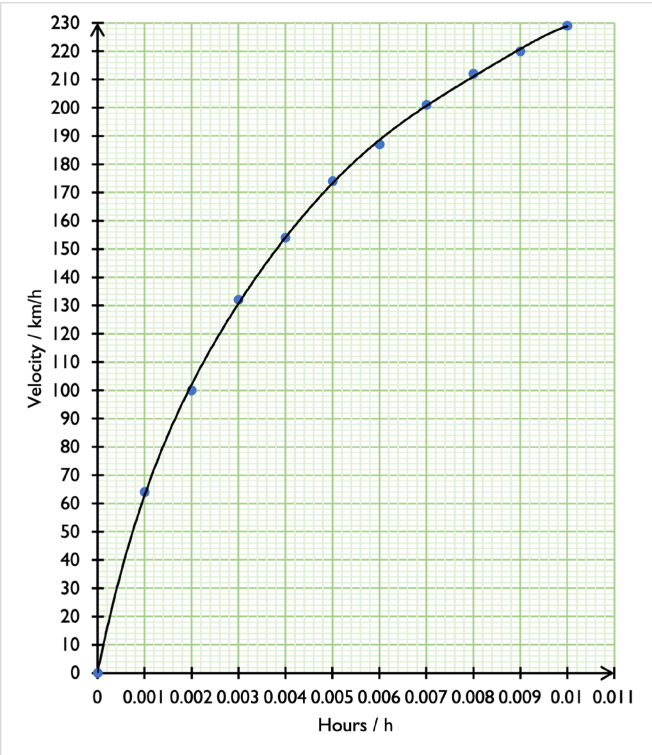


Figure 11.6

1. Use rectangles or trapezia to estimate how far the car travelled during the 36 sec it took to reach 229 km/h
2. Approximately how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

Reminder

Remind learners of their portfolio assessment. Find out about their progress and offer assistance where required.

WEEK 20

Learning Indicators

1. *Identify and write a definite integral notation and its connection to the limits of the partial sum of areas*
2. *Identify integration as a reverse process of differentiation*

FOCAL AREA 1: CONNECTION BETWEEN LIMITS AND INTEGRALS

As we realised in the previous week's lessons, the more the number of partitions used for approximating the area under a curve within a given interval, the better the approximation. In fact, the approximated area will get "closer and closer" to the exact value.

We could write the step-by-step computations that we used in the previous week's lesson as an iterative function as follows

Provided that there are n number of rectangles formed between $x = a$ and $x = b$ to approximate the area under the curve $f(x)$, we would need the height and the width to find the areas of rectangles used to approximate the required area.

The width of the rectangles are equal and is h. For the heights, same as the functional values of the x values of either side of the rectangles. If we choose to use the right side of the rectangles, then the x coordinates could be obtained as follows

$x_1 = a + h$

$x_2 = a + 2h$

$x_3 = a + 3h$

$\vdots$

$x_n = b = a + nh$

The heights of the rectangles are also gotten from

$f(x_1) = f(a + h)$

$f(x_2) = f(a + 2h)$

$f(x_3) = f(a + 3h)$

$\vdots$

$f(x_n) = f(b) = f(a + nh)$

The areas are thus

$A_1 = h \times f(a + h)$

$A_2 = h \times f(a + 2h)$

$A_2 = h \times f(a + 3h)$

$\vdots$

$A_n = h \times f(a + nh)$

The required area is the sum of all the n areas i.e., $A_1 + A_2 + A_3 + \ldots + A_n$. The sum is called the Riemann sum and it can be written as $R(n)$.

$$\Rightarrow R(n) = h \times \sum_{i=1}^{n}(f(a + ih))$$

We have noticed that the higher the number of divisions/intervals used, the better the approximation. If the number of partial areas was infinity, we would have the exact area. Unfortunately, infinity is not a real number and we can only assume to get closer to it. The limit of the function that provides the partial is just right to obtain such an approximate area.

$$= h \times \sum_{i=1}^{n} f(x_i)$$

$$Exact\ Area = \lim_{i\to\infty}(R(n))$$

The sum of the areas of all infinitely many rectangles or trapezia (as the case might be) gives the actual area under the curve.

We will now introduce an operator to represent the summation of the infinitely many areas called the integral sign and denoted by $\int$. When we use this symbol just as it is, we refer to an indefinite sum of areas and therefore say that the resulting area is an indefinite integral.

The area under the line $y = x$ written as a function of x i.e., $f(x) = x$ over its entire domain, $(-\infty, \infty)$ can be denoted by $\int f(x)dx$ or $\int ydx$

If we were interested in finding the area under the curve of a function, $f(x)$ between the boundaries $x = a$ and $x = b$ then we would be referring to some definite integral and we denote that by $\int_a^b f(x)dx$ where a is the lower limit while b is the upper limit.

For a function $f(x)$ and with a step size of h (resulting in n subintervals) being used to find the area between $x = a$ and $x = b$,

$$Area = Definite\ Integral = \int_a^b f(x)dx$$

$$\approx h \times \sum_{i=1}^{n}(f(a + ih))$$

Or

$$\int_a^b f(x)dx \approx [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \text{ where } \Delta x = \frac{b-a}{n},$$

Example 1

Using $f(x) = x^2$ on the interval $0 \leq x \leq 3$, find the definite integral $\int_0^3 f(x)dx$

On the interval [0,3], get three subintervals with their corresponding rectangles and estimate the integral by summing the areas. Repeat the process 10, 15 and 20 subintervals

Solution

$f(x) = x^2, 0 \leq x \leq 3$

$a = 0, b = 3$

For $n = 3$, $h = \frac{3-0}{3} = 1$

i	$x_i = a + ih$	$f(x_i) = (x_i)^2$
1	$0 + 1(1) = 1$	$1^2 = 1$
2	$0 + 2(1) = 2$	$2^2 = 4$
3	$0 + 3(1) = 3$	$3^2 = 9$
		$\sum_{i=1}^{3}(f(x_i)) = 1 + 4 + 9 = 14$

$\int_0^3 f(x)dx \approx h \times \sum_{i=1}^{3}(f(a + ih))$

$\approx 1 \times \sum_{i=1}^{3}(f(x_i))$

$\approx 1(14)$

≈ 14 *squared units*

For $n = 10$, $h = \frac{3-0}{10} = 0.3$

i	$x_i = a + ih$	$f(x_i) = (x_i)^2$
1	0.3	0.09
2	0.6	0.36
3	0.9	0.81
4	1.2	1.44
5	1.5	2.25
6	1.8	3.24
7	2.1	4.41
8	2.4	5.76
9	2.7	7.29
10	3	9
		$\sum_{i=1}^{3}(f(x_i)) = 34.65$

$\int_0^3 f(x)dx \approx 0.3 \times \sum_{i=1}^{3}(f(x_i))$

$\approx 0.3(34.65)$

≈ 10.395 *squared units*

For $n = 15$, $h = \frac{3-0}{15} = 0.2$

i	$x_i = a + ih$	$f(x_i) = (x_i)^2$
1	0.2	0.04
2	0.4	0.16
3	0.6	0.36
4	0.8	0.64
5	1	1
6	1.2	1.44
7	1.4	1.96
8	1.6	2.56
9	1.8	3.24
10	2	4
11	2.2	4.84
12	2.4	5.76
13	2.6	6.76
14	2.8	7.84
15	3	9
		$\sum_{i=1}^{3}(f(x_i)) = 49.6$

$\int_0^3 f(x)dx \approx 0.2 \times \sum_{i=1}^{3}(f(x_i))$

$\approx 0.2(49.6)$

≈ 9.92 *squared units*

For $n = 20$, $h = \frac{3-0}{20} = 0.15$

i	$x_i = a + ih$	$f(x_i) = (x_i)^2$
1	0.15	0.0225
2	0.3	0.09
3	0.45	0.2025
4	0.6	0.36
5	0.75	0.5625

i	$x_i = a + ih$	$f(x_i) = (x_i)^2$
6	0.9	0.81
7	1.05	1.1025
8	1.2	1.44
9	1.35	1.8225
10	1.5	2.25
11	1.65	2.7225
12	1.8	3.24
13	1.95	3.8025
14	2.1	4.41
15	2.25	5.0625
16	2.4	5.76
17	2.55	6.5025
18	2.7	7.29
19	2.85	8.1225
20	3	9
		$\sum_{i=1}^{3}(f(x_i)) = 64.575$

$\int_0^3 f(x)dx \approx 0.15 \times \sum_{i=1}^{3}(f(x_i))$

$\approx 0.15(64.575)$

≈ 9.68625 *squared units*

Until this point, we relied on the understanding that an integral is the limit of Riemann's sum. It is however not practical to find integrals using this approach to obtain approximations with minimal errors. In this week's lessons, learners will consider the fundamental theorem of calculus and its implications for finding integrals.

FOCAL AREA 2: FUNDAMENTAL THEOREM OF CALCULUS

This theorem establishes the relationship between differentiation and integration and thus provides an alternative for finding integrals. It is in two parts.

Part one

Suppose that $f(x)$ is a continuous function over the interval $[a, b]$. To $f(x)$ we associate an area function $F(x)$ that is defined by $F(x) = \int_a^b f(x)dx$ $(a < x < b)$. $f(x)$ and $F(x)$ are such that $F'(x) = f(x)$ over $[a, b]$

This part of the theorem suggests that the inverse of a derivative function is an integral function and vice versa.

Consider the function, $f(x) = x^2$, its derivative, $f'(x)$ can be obtained, using the power rule by *reducing* the exponent by 1 and *multiplying* the resulting expression by the original exponent. The reverse would be done to obtain the integral of $f(x)$ i.e., the exponent will be *increased* by one and the new exponent used to *divide* the resulting expression.

The integral of $f(x)$ represented by $F(x) = \int f(x)dx = \frac{1}{3}x^3$.

In fact, this part of the fundamental theorem of calculus can be proven thus

If we let the area under the curve of $f(x)$ between $x = a$ and $x = b$ be $F(x) = \int_a^b f(x)dx$, we could find the rate of change of the area with respect to change in x as such

$$\Delta F = A(x + \Delta x) - A(x)$$

$$\frac{\Delta F}{\Delta x} = \frac{A(x + \Delta x) - A(x)}{\Delta x}$$

As the change in the values of x, Δx becomes smaller, i.e. $\Delta x \to 0$, we obtain the derivative of $F(x)$,

$$\frac{dF}{dx} = F'(x) = \lim_{\Delta \to 0}\left(\frac{AF}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{A(x + \Delta x) - A(x)}{\Delta x}\right)$$

Note that there is an x value between x and $x + \Delta x$ which we would represent with x_m and a corresponding $f(x_m)$ which represents the height of the rectangle we are using to approximate the area with such that the area we seek is simply $\Delta x \times f(x_m)$

$$F'(x) = \lim_{\Delta x \to 0}\left(\frac{\Delta x \times f(x_m)}{\Delta x}\right)$$ and that leaves us with

$$F'(x) = \lim_{\Delta x \to 0}(f(x_m))$$ essentially equivalent to

$F'(x) = f(x)$ since any value of x between x and $x + \Delta x$ as $\Delta x \to 0$ equals x;

Part two

If $f(x)$ is a continuous function on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

It has been established that $\int_a^b f(x)dx$ represents some area. Which can also be found using Riemann's sum. It can be recalled that the n subintervals have the boundaries located at $x_0, x_1, x_2, x_3, \ldots, x_n$ or simply, x_i, $i = 0, 1, 2, \ldots, n$. Also, $x_0 = a$ while $x_n = b$

$\Rightarrow$ *Required Area* $= F(b) - F(a) = \sum_{i=0}^{n-1}\left(F(x_{i+1}) - F(x_i)\right)$

However, the width of the rectangles is $x_{i+1} - x_i$ and the heights obtained from $f(x_i) = F'(x_i)$

We now have $F(b) - F(a) = \sum_{i=0}^{n-1}\left((x_{i+1} - x_i) \times f(x_i)\right)$

As the number of subintervals, n is increased, the value of $x_{i+1} - x_i$ becomes smaller (approaches 0), we can then introduce the integral operator and refer to $f(x_i)$ as $f(x)$

$$F(b) - F(a) = \int_a^b f(x)dx$$

In year 3, we would use the combination of the two parts of the fundamental theorem of calculus to find indefinite and definite integrals and find the area under curves in specified intervals among other applications of integration.

FOCAL AREA 3: FINDING INDEFINITE INTEGRALS

Consider the table below,

Function, $f(x)$	Derivative, $f'(x)$
$x^3 + 4$	$3x^2$
$x^3 - 7$	$3x^2$
$-2x^4 + 1.5$	$-8x^3$
$-2x^4 + 5.7$	$-8x^3$
$-x^2 + x + 3$	$-2x + 1$
$-x^2 + x - 8$	$-2x + 1$

It can be observed that $x^3 + 4$ and $x^3 - 7$ have the same derivative i.e., $3x^2$ just like $-2x^4 + 1.5$ and $-2x^4 + 5.7$ have $-8x^3$ and $-x^2 + x + 3$ and $-x^2 + x - 8$ have $-2x + 1$

This means that the only difference any two expressions of a pair is the addition (or subtraction) of a constant. In the case of the first two, x^3 remains while 4 is added to the first and 7 subtracted from the second. For the last pair, $-x^2 + x$ remains

To obtain a specific antiderivative, $f(x)$ we will need to cater for the constant.

The antiderivative of a derivative function, $f'(x)$ for our purposes, denoted by $\int f'(x)dx = F(x) + C$, where C is the constant of integration. For a polynomial function of the form x^n, the corresponding antiderivative is basically $\frac{1}{n+1}x^{n+1}$.

Example 2

Simplify $\int(x^2 - 2x + 5)dx$

Solution

Let $F(x) = \int(x^2 - 2x + 5)dx$ where $F(x)$ represents the antiderivative of $f(x) = x^2 - 2x + 5$

We will now have to perform the reverse of differentiation (using the power rule) on each term of $f(x)$

$$F(x) = \frac{1}{2+1}x^{2+1} - \frac{2}{1+1}x^{1+1} + 5x^{0+1} + C$$

$$= \frac{1}{3}x^3 - x^2 + 5x + C$$

Example 3

Find the antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = 1$

Solution

The general antiderivative of $f(x) = 3x^2$ is given by $F(x) = \frac{3}{3}x^{2+1} + C$

$\Rightarrow F(x) = x^3 + C$

The specific antiderivative we seek meets the condition $F(1) = 1$

$\Rightarrow F(1) = 1^3 + C$

$1 = 1 + C$

$C = 0$

The required antiderivative is $F(x) = x^3$

Example 4

Find the most general antiderivative

a) $\int\left(3t^2 + \frac{t}{2}\right) dt$

b) $\int\left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

c) $\int\left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

Solution

a) $\int\left(3t^2 + \frac{t}{2}\right) dt$

$$F(t) = t^3 + \frac{1}{4}t^2 + C$$

b) $\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx$

$F(x)=\frac{1}{5}x+\frac{1}{x^2}+x^2+C$

c) $\int\left(\frac{\sqrt{x}}{2}+\frac{2}{\sqrt{x}}\right)dx$

$F(x)=\frac{\sqrt{x^3}}{3}+4\sqrt{x}+C$

Learning Tasks

Task learners to

1. calculate the area under a straight line using a few rectangles and share their observations about the limit process as the rectangles become narrower
2. find the approximate area under a curve for a given function using Riemann sums with different numbers of rectangles
3. compute estimates for definite integrals for functions, using Riemann sums and areas of rectangles under the corresponding curves
4. illustrate the process of differentiating polynomial functions using the power rule
5. explain the fundamental theorem of calculus
6. use the reverse of differentiation to find the indefinite integrals

PEDAGOGICAL EXEMPLARS

1. **Talk-for-learning:** Lead a whole class discussion on
   - **a.** the connection between limits of the sum of areas of partitions and definite integrals
     - i. Review partitioning of intervals
     - ii. Discuss the Riemann sum by dividing the area under a curve into small rectangles
     - iii. Discuss how the sum of these areas approaches the integral as the number of rectangles increases
     - iv. Incorporate examples from diverse cultures or local contexts that relate to the students' lives, making the content more relatable and inclusive
   - **b.** the relationship between differentiation and integration
     - i. Begin with a recap of differentiation, followed by a discussion on integration as its reverse process

ii. Discuss how the sum of these areas approaches the integral as the number of rectangles increases

iii. Incorporate examples from diverse cultures or local contexts that relate to the students' lives, making the content more relatable and inclusive

2. **Experiential / Collaborative Learning**: Learners in small heterogenous or as individuals find integrals

   a. Task learners to write the sum of areas of rectangles used to estimate the area under a curve as a formula which involves the summation operator

   b. Task learners to find the limit as the number of rectangles are increased to infinity of the sum (Riemann sum) they have obtained

   c. Introduce the concept of definite integrals as the limit of Riemann sums

   d. Use visual aids, like graphs and animations, to show how the area under a curve can be approximated by summing the areas of rectangles under the curve

   e. Challenge students to derive the area under more complex functions

   f. Introduce software tools or graphing calculators for complex calculations

   g. Pair students with diverse abilities so they can support each other, fostering an inclusive environment

   h. Tailor assignments to meet the needs of each learner, allowing for varying levels of difficulty and complexity

   i. Break down the tasks into smaller, manageable parts, offering additional support where necessary

   j. Provide supplementary materials, like video tutorials or interactive simulations, to cater to diverse learning needs. Use graphing software or physical graphing tools to show how a curve's slope changes. Then reverse the process by "filling in" the area under the curve to demonstrate integration

   k. Work through basic problems with the class, such as integrating simple polynomials

   l. Use matching games where students pair derivatives with their corresponding integrals, helping them visualise the reverse process

   m. Use colour coding and diagrams to differentiate between the processes of differentiation and integration

   n. Provide worksheets with basic integrals and ask students to solve them. Monitor progress and give immediate feedback

   o. Give students word problems that require them to set up and solve integrals

p. Present problems that require the application of integration in real-world contexts, like calculating the work done by a variable force

q. Include multiple representations of problems, such as word problems, graphs, and algebraic expressions

3. **Inquiry Learning:** Learners in small heterogenous or as individuals explore finding integrals of other functions apart from polynomial functions and research on some applications of antiderivatives

   a. Introduce more complex integrals, such as trigonometric or exponential functions. Encourage students to explore the integration process independently

   b. Offer access to advanced resources, such as online calculus simulators or university-level problem sets, for further exploration

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

1. Find the following definite integrals using the indicated number of subintervals

   a. $\int_1^4 (x^3 + 2x)dx$; 8 subintervals

   b. $\int_1^8 (2\sqrt{x} + 7x^2)dx$; 4 subintervals

   c. $\int_0^{\frac{\pi}{2}} \cos(x)\, dx$; 4 subintervals

2. What mathematical expression can be used to represent the area under the graph of $\sin(x)$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ if the number of rectangles used to approximate the area is increased indefinitely?

3. Find the indefinite integral $\int (x^2 + x + x^{-3})dx$

4. Evaluate the integral $\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

Assessment Level 3: Strategic Reasoning

Ejisuman rural bank opened two branches on 1st September, ($t = 0$). Branch A is located at Fumesua and branch B is located Ejisu. The net rate at which money was deposited into the two branches in the first 180 business days is given by the graphs of $f(t)$ and $g(t)$, respectively. Which branch has a larger amount on deposit at the end of 180 business days? Justify your answer.

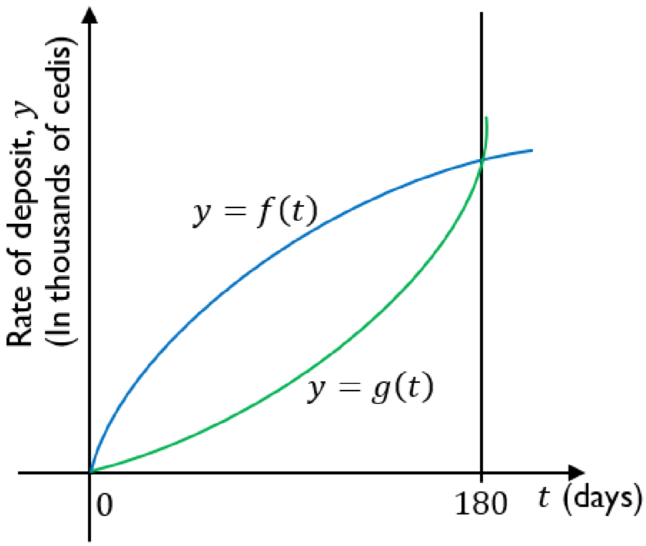


Figure 11.7

REMINDER

Learners Individual Project should be ready for submission.

Section 11 Review

This section built on learners' understanding on differentiation, partitioning intervals, finding areas of rectangles and limits to introduce them to the basis for and basics of integration. Pedagogical approaches such as whole class discussions and demonstrations, group and individual activities and inquiry learning were suggested. Considerations for creating inclusive learning environments were also suggested in the pedagogical exemplars. Furthermore, assessments tasks of varying Depths of Knowledge levels that test learners' understanding of concepts were also suggested.

SECTION 12: APPLICATIONS OF DIFFERENTIATION

Strand: Calculus

Sub-Strand: Applications of calculus

Learning Outcome: *Investigate the turning point of a function.*

Content Standard: *Find* maximum and minimum values and points of a function, sketch the functions, and solve some real-life problems.

INTRODUCTION AND SECTION SUMMARY

Understanding the nature of gradients, turning points, sketching polynomial functions and applying differentiation to real-life problems equips senior high school learners with essential calculus skills. These concepts enable learners to analyse and interpret function behaviour, identify critical points and visualise polynomial graphs. Applying differentiation in practical scenarios, such as optimising costs and analysing motion, bridges theoretical maths with real-world problems, enhancing critical thinking and problem-solving abilities. This holistic approach ensures learners grasp both mathematical theories and their practical significance, fostering deeper engagement with the subject. In week 16 to 18 of the current manual, learners learnt to use differentiation rules to find the derivatives of functions. In this section, learners will build on that learning to sketch polynomial functions and solve some real-life problems.

The weeks covered by the section are:

Week 21

1. *Determining the nature of gradients*
2. *Investigating turning points*
3. *Sketching polynomial functions*

Week 22: *Applying differentiation to solve real-life problems*

SUMMARY OF PEDAGOGICAL EXEMPLARS

Problem-based learning in small heterogeneous groups is promoted for use in the lessons in this section. This is to provide learners with an environment that challenges learners with sufficient support through peer tutoring from other learners and emotional support to apply understanding of differentiation to solving real-life problems

ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests and homework assignments, can be used to evaluate learners understanding of concepts and their ability to solve problems. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for STP.

WEEK 21

Learning Indicator: *Use the second derivative of a function to classify the maximum, minimum and saddle point of that function and perform curve sketching*

FOCAL AREA 1: DETERMINING THE NATURE OF GRADIENTS

In year one, learners discovered the relevance of finding the derivative of a function. It was discussed that the derivative of a function at a point represents the gradient of the tangent line that touches the curve of the function at that point. Because the gradient of a curve (other than straight lines) at different points on the curve differ, the derivatives are functions in themselves. We can therefore consider the nature of the gradient at different points on the curve.

We can use our knowledge of the nature/behaviour of constant and linear functions as discussed in year one to do the classification.

A quick recap

1. Constant functions of the form $f(x) = c$, $c \in \mathbb{R}$, being representations of horizontal lines, the gradient is zero

2. For functions of the form $y(x) = mx + c$,

a. They represent straight lines that slope upward from left to right if the value of m is positive i.e., $m > 0$

b. they represent straight lines that slope downward from left to right if the value of m is negative i.e., $m < 0$

We now combine all these facts and state that given a function $y = f(x)$, if

1. $\frac{dy}{dx}|_{x=c} > 0$ implying that the gradient of $y = f(x)$ is positive at $x = c$, then the tangent(s) touching the curve when $x = c$ slopes upwards

2. $\frac{dy}{dx}|_{x=b} < 0$ i.e., the gradient of $y = f(x)$ at $x = b$ is negative, then the tangent(s) touching the curve when $x = c$ slopes downwards.

3. $\frac{dy}{dx}|_{x=d} = 0$ then the gradient of $y = f(x)$ is zero, the point $(d, f(d))$ is a turning point or a saddle point and the tangent at that point is a horizontal line

Example 1

Find the gradient and classify the behaviour of the curve at the point. State whether the gradient is positive or negative or a turning point.

1. $f(x) = \frac{1}{x-3}$ at $x = 4$

2. $f(x) = 4x - 7x^2$ at $x = -3$

3. $f(x) = (2x + 5x^5)(x - 2)$ at $x = 1$

4. $f(x) = \dfrac{(2x + 5x^5)(x - 2)}{(x - 3)}$ at $x = -2$

5. $f(x) = (4x - 7x^2)(3x - 9)^{10}$ at $x = 0$

Solution

1. $f(x) = \dfrac{1}{x - 3}$ at $x = 4$

$f'(x) = \dfrac{-1}{(x - 3)^2}$

$f'(4) = -\dfrac{1}{(4 - 3)^2} = -1$

The gradient is negative when $x = 4$ thus the tangent slopes downwards from left to right and the curve is decreasing around the point where $x = 4$

2. $f(x) = 4x - 7x^2$ at $x = -3$

Let $y = 4x - 7x^2$

$\Rightarrow \dfrac{dy}{dx} = 4 - 14x$

$\Rightarrow \dfrac{dy}{dx}\Big|_{x=-3} = 4 - 14(-3)$

$= 46$

We obtain a positive gradient and thus the tangent line passing though the point of $f(x)$ where $x = -3$ slopes upwards from the left to right and the curve is increasing around the point where $x = -3$

3. $f(x) = (2x + 5x^5)(x - 2)$ at $x = 1$

Let $u(x) = 2x + 5x^5$ and $v(x) = x - 2$, then $f(x) = u(x) \times v(x)$

$u'(x) = 2 + 25x^4$ and $v'(x) = 1$

$f'(x) = v(x) \times u'(x) + u(x) \times v'(x)$

$= (x - 2)(2 + 25x^4) + (2x + 5x^5)(1)$

$= 30x^5 - 50x^4 + 4x - 4$

$f'(1) = 30(1)^5 - 50(1)^4 + 4(1) - 4$

$= -20$

Since the gradient is negative, the tangent line where $x = 1$ slopes downwards from left to right and the curve is decreasing around the point where $x = 1$

4. $f(x) = \frac{(2x + 5x^5)(x - 2)}{(x - 3)}$ at $x = -2$

Let $h(x) = (2x + 5x^5)(x - 2) = 5x^6 - 10x^5 + 2x^2 - 4x$ and $g(x) = x - 3$ then $f(x) = \frac{h(x)}{g(x)}$

$h'(x) = 30x^5 - 50x^4 + 4x - 4$ and $g'(x) = 1$

$$f'(x) = \frac{(g(x) \times h'(x)) - (h(x) \times g'(x))}{(g(x))^2}$$

$$= \frac{(x - 3)(30x^5 - 50x^4 + 4x - 4) - (5x^6 - 10x^5 + 2x^2 - 4x)(1)}{(x - 3)^2}$$

$$= \frac{30x^6 - 140x^5 + 150x^4 + 4x^2 - 16x + 12 - 5x^6 + 10x^5 - 2x^2 + 4x}{x^2 - 6x + 9}$$

$$= \frac{25x^6 - 130x^5 + 150x^4 + 2x^2 - 12x + 12}{x^2 - 6x + 9}$$

$$f'(-2) = \frac{25(-2)^6 - 130(-2)^5 + 150(-2)^4 + 2(-2)^2 - 12(-2) + 12}{(-2)^2 - 6(-2) + 9}$$

$$= 328.16$$

The gradient is positive and thus the tangent line slopes upwards from left to right and the curve is increasing around the point where $x = -2$

5. $f(x) = (4x - 7x^2)(3x - 9)^{10}$ at $x = 0$

Let $h(x) = 4x - 7x^2$ and $g(x) = (3x - 9)^{10}$ then $f(x) = h(x) \times g(x)$

$h'(x) = 4 - 14x$, $g'(x) = 30(3x - 9)^9$ and $f'(x) = g(x) \times h'(x) + h(x) \times g'(x)$

$f'(x) = (4 - 14x)(3x - 9)^{10} + (4x - 7x^2)(30(3x - 9)^9)$

$f'(0) = (4 - 14(0))(3(0) - 9)^{10} + (4(0) - 7(0)^2)(30(3(0) - 9)^9)$

$= 13{,}947{,}137{,}604$

The gradient is positive and therefore, the tangent slopes upward from left to right and the curve is increasing around the point where $x = 0$

Example 2

Given that the gradient of the curve $f(x) = cx^2 + dx + 4$ at $x = 2$ is 5 and the curve turns at $x = 1$, find the roots and the turning point of $f(x)$.

Solution

$f(x) = cx^2 + dx + 4$

The gradient of $f(x)$ is given by $f'(x) = 2cx + d$

The gradient at $x = 2$ which is $f'(2) = 4c + d$

But $f'(2) = 5$ hence $4c + d = 5$

If the curve turns at $x = 1$ then $f'(1) = 0$

But $f'(1) = 2c + d$ and hence $2c + d = 0$

We now solve the system $\left.\begin{array}{l} 4c + d = 5 \\ 2c + d = 0 \end{array}\right\}$ to obtain $c = 2.5$ and $d = -5$

$\Rightarrow f(x) = 2.5x^2 - 5x + 4$

At the roots of $f(x)$, $f(x) = 0$

$2.5x^2 - 5x + 4 = 0$

The discriminant of $f(x)$, found as $(-5)^2 - 4(2.5)(4) = -15$ is negative and thus suggest that $f(x)$ has imaginary / complex roots and thus no zeroes in the set of real numbers

The complex roots however, are $x = -\frac{\sqrt{15}}{15}\boldsymbol{i} + 1$ and $x = \frac{\sqrt{15}}{15}\boldsymbol{i} + 1$

We know the turning point can be located where $x = 1$. When $x = 1$,

$y = 2.5(1)^2 - 5(1) + 4 = 1.5$ and hence the turning point is located at $(1, 1.5)$

FOCAL AREA 2: INVESTIGATING TURNING POINTS

Theorem: Let $f(x)$ be a differentiable function on an open interval and suppose that $f'(c) = 0$ at a point c inside this interval

1. If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c
2. If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c
3. If f' does not change sign at c for $x < c$ or $x > c$. Even though $f(c)=0$), f has neither a local minimum nor maximum at c.

Example 3

Find the turning point(s) and determine which are maximum or minimum

1. $y = x^2 - x - 6$
2. $y = x^2 - 3x - 4$
3. $y = x^3 - 9x^2 - 21x - 4$

Solution

1. $y = x^2 - x - 6$

   $\frac{dy}{dx} = 2x - 1$

   $\Rightarrow f'(x) = 2x - 1$

   At the turning point,

   When $\frac{dy}{dx} = 0$

   $2x - 1 = 0$

$x = 0.5$

At $x = 0.5$, $f(0.5) = -6.25$ hence $(0.5, -6.25)$ is the turning point.

We will now find take an x value less than $x = 0.5$ say $x = 0$

$f'(0) = 2(0) - 1$

$= -1$

$\Rightarrow f'(0) < 0$ therefore, the gradient is decreasing heading 'into' the turning point.

Taking a value of x greater than 0.5 i.e. $x = 1$

$f'(1) = 2(1) - 1$

$= 1$

$\Rightarrow f'(0) > 0$, therefore, the gradient is increasing heading 'out' of the turning point.

Since $f'(0) < 0$ as $0 < 0.5$ and $f'(1) > 0$, where $1 > 0$, it can be concluded that $(0.5, -6.25)$ is a minimum

2. $y = x^2 - 3x - 4$

$\Rightarrow f'(x) = 2x - 3$

At the turning point,

When $f'(x) = 0$

$2x - 3 = 0$

$x = 1.5$

At $x = 1.5$, $f(1.5) = -6.25$ hence $(1.5, -6.25)$ is the turning point

The curve is a parabola since its equation is quadratic and thus from the lessons in week 8 of the year one manual, we can conclude that the turning point is minimum since the coefficient of the leading term, 1 in this case, is greater than zero

3. $y = f(x) = x^3 - x^2 - 9x + 9$

Step 1: Find the derivative of y

$\frac{dy}{dx} = 3x^2 - 2x - 9$

$\Rightarrow f'(x) = 3x^2 - 2x - 9$

Step 2: Find x coordinates of the turning points by equating the derivative function to zero

$\frac{dy}{dx} = 3x^2 - 2x - 9 = 0$

$3x^2 - 2x - 9 = 0$

Solving the above, we have $x = -1.43$ or 2.10

Step 3: Find and investigate the turning points

At $x = -1.43$, $f(-1.43) = 16.90$ is a critical value; hence $(-1.43, 16.9)$ is a turning/critical point.

At $x = 2.10$, $f(2.10) = -5.05$ is a critical value; hence $(2.10, -5.05)$ is a turning point.

We will now investigate the turning points

For $(-1.43, 16.9)$,

Let us take a value of x less than -1.43 and find the corresponding gradient of $f(x)$. We will take -2

$f'(-2) = 3(-2)^2 - 2(-2) - 9$

$= 7$

$\Rightarrow f'(-2) > 0$, as the gradient is positive, the graph is increasing leading into the turning point.

We now take a value of x greater than -1.43 and find the corresponding gradient of $f(x)$. We will take 0

$f'(0) = 3(0)^2 - 2(0) - 9$

$= -9$

$\Rightarrow f'(0) < 0$, as the gradient is negative, the graph is decreasing leading out of the turning point.

Since $f'(-2) > 0$ as $-2 < -1.43$ and $f'(0) < 0$, where $0 > -1.43$, it can be concluded that $(-1,7)$ is a local maximum in the interval where $x \in (-2, 0)$

For $(2.10, -5.05)$,

Let us take a value of x less than 2.10 and find the corresponding gradient of $f(x)$. We will take 2

$f'(2) = 3(2)^2 - 2(2) - 9$

$= -1$

$\Rightarrow f'(2) < 0$, as the gradient is negative, the graph is decreasing leading into the turning point.

We now take a value of x greater than 2.10 and find the corresponding gradient of $f(x)$. We will take 3

$f'(3) = 3(3)^2 - 2(3) - 9$

$= 12$

$\Rightarrow f'(3) > 0$, as the gradient is positive, the graph is increasing leading out of the turning point.

Since $f'(2) < 0$ as $2 < 2.10$ and $f'(3) > 0$, where $3 > 2.10$, it can be concluded that $(7, -249)$ is a local minimum in the interval where $x \in (2, 3)$

FOCAL AREA 3: SKETCHING POLYNOMIAL FUNCTIONS

To sketch the graph of a polynomial,

Step 1: Find the roots of the function; find the values of x for which $f(x) = 0$.

Step 2: Find the turning points.

Step 3: Investigate the turning points, whether it is maximum or minimum.

Step 4: Sketch.

Example 4

Sketch the graph of $y = x^2 - x - 6$

Solution

The turning point have already been found as $(0.5, -6.25)$ and it has been confirmed to be a minimum turning point

For the $x -intercepts$, $y = 0$

$x^2 - x - 6 = 0$

$\Rightarrow x = -2$ or $x = 3$

We can find the $y - intercept$ by finding the corresponding y coordinate for $x = 0$. Note that this y value is the same as the constant term in the equation. For this equation, the y value is -6 hence the $y - intercept$ is $(0, -6)$

We now plot our relevant points and draw a curve to pass through them

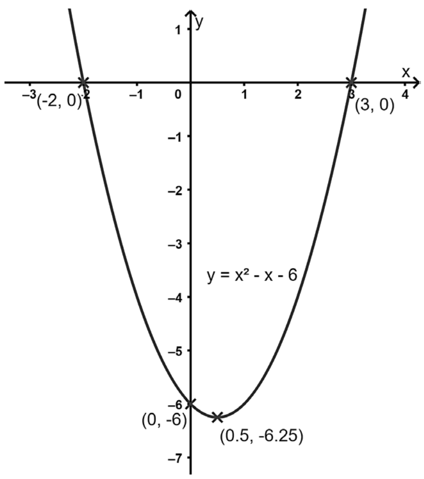


Figure 12.1

Example 5

Sketch the graph of $y = x^2 - 3x - 4$

Solution

The turning point have already been found as (1.5, −6.25) and it has been confirmed to be a minimum turning point

For the $x - intercepts$, $y = 0$

$x^2 - 3x - 4 = 0$

$\Rightarrow x = -1$ or $x = 4$

We can find the $y - intercept$ by finding the corresponding y coordinate for $x = 0$. Note that this y value is the same as the constant term in the equation. For this equation, the y value is -4 hence the $y - intercept$ is $(0, -4)$

We now plot our relevant points and draw a curve to pass through them

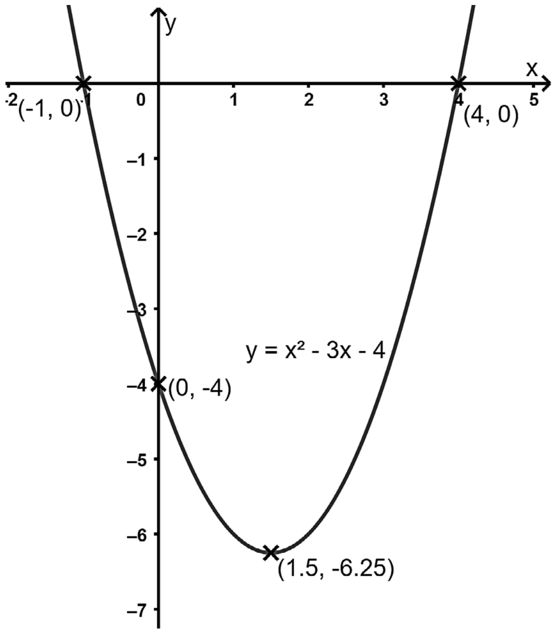


Figure 12.2

Example 6

Sketch the graph of $y = x^3 - x^2 - 9x + 9$

Solution

It has already been confirmed that $y = x^3 - x^2 - 9x + 9$ has a local maximum and minimum turning points at (−1.43, 16.9) and (2.10, −5.05) respectively.

For the $x - intercepts$, $y = 0$

$x^3 - x^2 - 9x + 9 = 0$

$\Rightarrow x = -3, x = 1$ or $x = 3$

The $y - intercept$ is the same as the constant term in the equation. For this equation, the y value is 9 hence the $y - intercept$ is (0, 9)

We now indicate our relevant points and draw a curve to pass through them

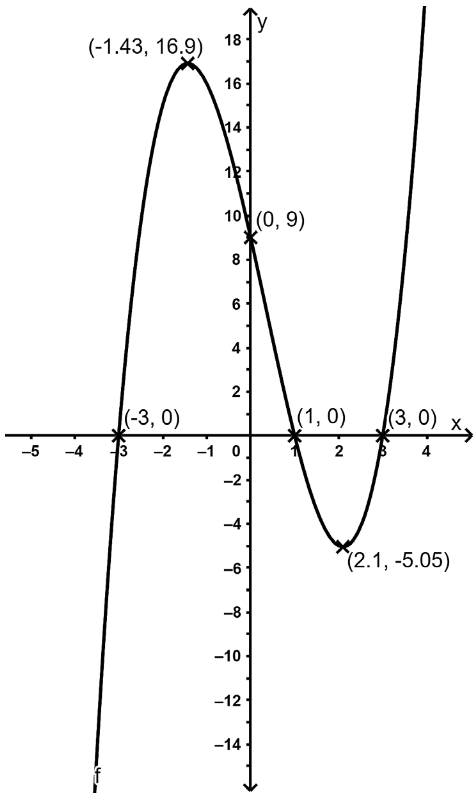


Figure 12.3

Learning Tasks

Task learners to

1. find and classify gradients of a curve at a point
2. discuss how to find maximum and minimum values and points

PEDAGOGICAL EXEMPLARS

1. **Experiential / Exploratory Learning:** Learners interact with objects and computer software to investigate the nature of gradients
   - **a.** Use a physical ramp or incline to demonstrate positive, negative, and zero gradients
   - **b.** Use interactive gradient sliders on dynamic computer software such as GeoGebra to illustrate the nature of gradients
   - **c.** Incorporate multimedia resources to cater to different learning styles
   - **d.** Task learners to analyse and sketch polynomial functions of higher degrees, considering all critical points and behaviour at infinity

   e. Provide guided practice sessions where learners work through problems with support, gradually increasing independence

   Provide resources for independent study and research on advanced topics

   f. Provide text-to-speech software, graphing tools, and other aids to support diverse learning needs

2. **Collaborative learning:** Learners work in small heterogeneous groups to sketch polynomial graphs with clear step-by-step instructions

   a. provide extra time for practice

   b. Use heterogeneous grouping to promote peer learning

   c. Rotate groups regularly to ensure diverse interactions and equal opportunities

KEY ASSESSMENTS

Assessment Level 2: Skills and Conceptual Understanding

1. For the following derivatives, determine on what intervals they are increasing or decreasing, stating the local minimum and maximum values if they exist

   a. $f'(x) = x(x - 1)$

   b. $f'(x) = (x - 1)^2(x + 2)^2$

   c. $f'(x) = \frac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, x \neq -1, 3$

2. Sketch the graphs of the following functions

   a. $y = x^3 - 3x + 3$

   b. $y = (x - 2)^3 + 1$

   c. $y = x^4 - 2x^2$

Assessment Level 3: Strategic Reasoning

1. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer

WEEK 22

Learning Indicator: *Find the maximum and minimum values and points of a function, sketch the functions and solve some real-life problems*

FOCAL AREA 1: APPLYING DIFFERENTIATION TO SOLVE REAL-LIFE PROBLEMS

Example 1

Suppose that a ball is thrown straight up into the air and its height after t seconds is: $4 + 48t - 16t^2$ feet.

Determine how long it will take for the ball to reach its maximum height and determine the maximum height.

Solution

Let h be the height of the ball. Such that $h = f(t)$, t in seconds

$f(t) = 4 + 48t - 16t^2 = 48 - 32t$

At turning point, $f'(t) = 0$

$48 - 32t = 0$

$t = \frac{3}{2} = 1.5$ *seconds*

Maximum height of the ball

$f(1.5) = 4 + 48(1.5) - 16(1.5)^2$

$= 40$ *feet*

The ball reaches its maximum height of 40 feet in 1.5 seconds

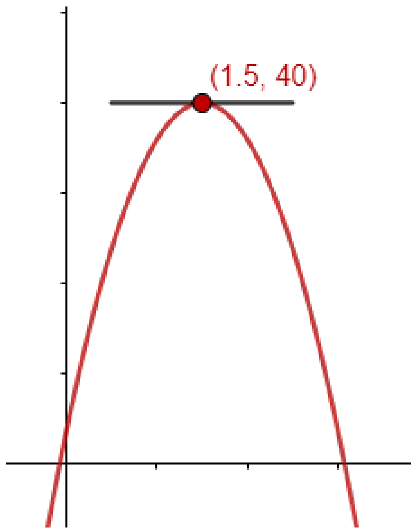


Figure 12.4

Example 2

A person wants to plant a rectangular garden along one side of a house, with a picket fence on the other three sides of the garden. Find the dimensions of the largest garden that can be enclosed using 40 feet of fencing.

Solution

Let w *and* x denote the dimensions of the rectangular garden, the area, $A = wx$

The perimeter of the rectangular garden fencing 3 sides

$2x + w = 40$

$w = 40 - 2x$

$\therefore A = (40 - 2x)x$

$= 40x - 2x^2$

At turning point, $A' = 0$

$40 - 4x = 0$

$x = 10$ feet

$w = 40 - 2(10) = 20$ feet

The maximum area, $A = 40(10) - 2(10)^2 = 200$ *square feet* with dimensions 10 by 20 feet.

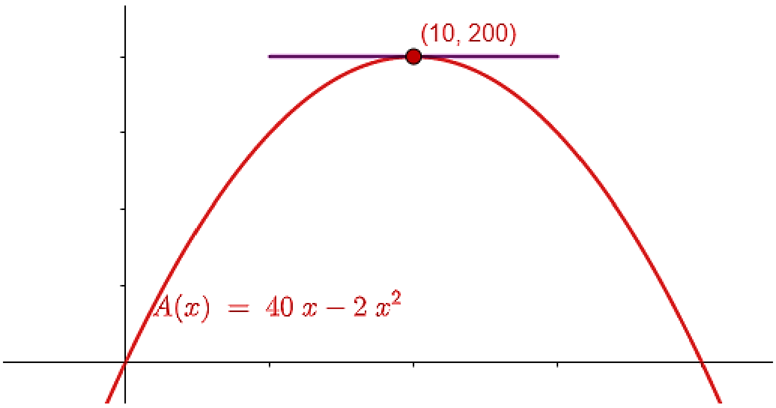


Figure 12.5

Example 3

The manager of a department store wants to build a 600-square-foot rectangular enclosure on the store's parking plot in order to display some equipment. Three sides on the enclosure will be built of redwood fencing at a cost of *GH¢* 14.00 per running foot. The fourth side will be built of cement blocks at a cost of *GH¢* 28.00 per running foot.

Find the dimensions of the enclosure that will minimise the total of the building materials.

Solution

Let x *and* y be the length of the side built out of cement blocks and the adjacent side, respectively.

Cost of redwood= $(x + 2y) \bullet 14 = 14x + 28y$

Cost of cement blocks $= 28x$

If C denotes the total cost of the materials, then

$C = (14x + 28y) + 28x$

$C = 42x + 28y$

Area, $A = xy = 600$

$y = \frac{600}{x}$

$C = 42x + 28\left(\frac{600}{x}\right)$

$C = 42x + \frac{16,800}{x}$

$\frac{dC}{dx} = 42 - \frac{16800}{x^2}$

When $\frac{dC}{dx} = 0$ we have the turning point and the minimum value for x.

$42 - \frac{16800}{x^2} = 0$

$42x^2 = 16800$

$x = 20$

When $x = 20$, $y = 30$. Therefore, the minimum costs are incurred when the dimensions are 20 by 30 feet.

We can confirm this by sketching the curve

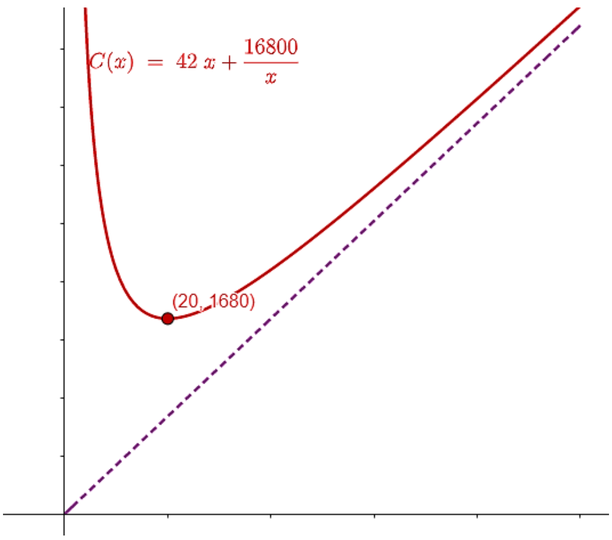


Figure 12.6

The minimum total cost of *GH¢* 1,680.00 occurs at $x = 20$, $y = 30$.

Example 4

Ghana's regulations on posting parcels state that packages must have a length plus circumference of no more than 84 inches. Find the dimensions of the cylindrical package of the greatest volume that is mailable by parcel post.

Solution

Let l and r denote the length of the package and radius of the circular end, respectively.

The volume,

$V = \pi r^2 l$The circumference of the circular end $= 2\pi r$

$length + girth = 84$

$l + 2\pi r = 84$

$l = 84 - 2\pi r$

$\Longrightarrow V = \pi r^2(84 - 2\pi r)$

$= 84\pi r^2 - 2\pi^2 r^3$

At maximum, $V' = 0$

$\therefore V' = 168\pi r - 6\pi r^2 = 0$

$r = \frac{28}{\pi}$

The maximum volume,

$V = 84\pi\left(\frac{28}{\pi}\right)^2 - 2\pi^2\left(\frac{28}{\pi}\right)^3$

$= \frac{28^3}{\pi}$

$l = 84 - 2\pi\left(\frac{28}{\pi}\right) = 28$

$circumference = 2\pi\left(\frac{28}{\pi}\right) = 56$

$\therefore l = 28\ inches,\ r = \frac{28}{\pi}\ inches$

$circumference = \mathbf{56}\ inches$

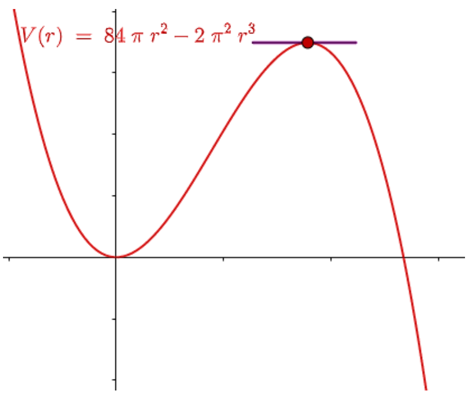


Figure 12.7

Example 5

Suppose that, from Takoradi to Ho, a VIP bus carries 8000 passengers per month, each paying *GH¢* 50.00. The carrier wants to increase the fare. However, the market research department estimates that for each *GH¢* 1.00 increase in fare, the company will lose 100 passengers. Determine the price that maximises the company's revenue.

Solution

Let x and n be the price per ticket and number of passengers, respectively.

Revenue, $R = nx$

Number of passengers

=original number of passengers-(number of passengers lost due to fare increase)

$n = 8000 - (x - 50) \bullet 100$

$= 13{,}000 - 100x$

$\therefore R = (13{,}000 - 100x)x = 13{,}000x - 100x^2$

At maximum, $R' = 0$

$13{,}000 - 200x = 0$

$x = 65$

The maximum revenue occurs when the price per ticket is *GH¢* 65.00.

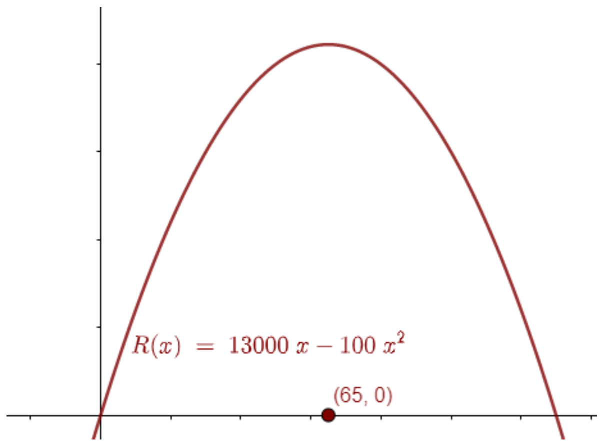


Figure 12.8

Learning Tasks

1. Task learners to sketch functions and describe the behaviour around specific points
2. Task learners to translate real-life problems into functions
3. Task learners to find the derivative of functions and interpret the results

PEDAGOGICAL EXEMPLAR

Collaborative/Problem-based group learning: Learners work in small heterogeneous groups discuss and apply differentiation to solve life problems

1. Use a story or scenario relevant to learners' lives, such as calculating the speed of a moving car
2. Introduce real-life scenarios, such as optimising the area of a garden or maximising profit in a business
3. Offer choice in problem selection to cater to diverse interests
4. Guide learners through setting up equations, differentiating and solving
5. Encourage learners to explain their reasoning and solutions in writing
6. Allow extra time for task completion for learners who might need it and offer frequent check-ins
7. Assign projects where learners select a real-life problem of interest, formulate it mathematically and apply differentiation techniques to solve it
8. Allow multiple means of expression

KEY ASSESSMENTS

Assessment Level 2: Skills and Concepts

Find the turning point(s) of the following curves and determine whether they are minimum or maximum

1. $2x^2 - 5x - 3 = 2y$
2. $4x^3 + 4x^2 - 69x - 135 = 4y$

Assessment Level 3: Strategic Reasoning

After an injection, the concentration of the drug in a muscle varies according to a function of time, $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Determine the time when the maximum concentration of the drug occurs.

Assessment Level 3: Strategic Reasoning

1. Suppose a large computer file is sent over the internet. If the probability that it reaches its destination without any errors is x, then the probability that an error is made is $1 - x$. The field of Information Theory studies such situations. An important quantity is entropy (a measure of unpredictability), defined by $H = -x\ln(x) - (1 - x)\ln(1 - x)$ *for* $0 < x < 1$.

   Find the value of x that maximises this quantity. Explain why this value makes sense to the probability that maximises entropy.

2. A tetramer is a protein with four subunits. In the study of tetramer binding, the equation:

$$Y = \frac{x + 3x^2 + 3x^3 + 10x^4}{1 + 4x + x^2 + 4x^3 + 10x^4}$$

expresses a typical relationship between saturation Y and ligand concentration x($x \geq 0$). Ordinarily, the variable Y/x is plotted as a function of x.

Explain why Y/x has an absolute maximum value. Find this value to three decimal places.

3. Freddy Phones Company limited markets its product in Kumasi and Accra and can charge different amounts in each city. Let x be the number of units to be sold in Kumasi and y the number of units to be sold in Accra. Due to the law of demand, Freddy phones must set the price at $(97 - ())$ Ghanaian cedis in Kumasi and $\left(83 - \left(\frac{y}{20}\right)\right)$ Ghanaian cedis in Accra in order to sell all the units. The cost of producing these units is $20{,}000 + 3(x + y)$. Find the values of x and y that maximise the profit.

4. Find the path to minimise pigeon flight energy if a pigeon, released 1 mile from the shore, needs to reach a point on the shore 2 miles from the closest shore point, and the pigeon needs 4/3 more energy to fly over water.

Section 12 Review

In this Section, learners applied their knowledge about gradient of functions, graphs of polynomial functions and differentiation rules to investigate the nature of gradients, turning points of curves and to sketch polynomial functions. Learners were further assisted to translate real-life problems into mathematical equations and functions, apply differentiation (where required) and interpret the results in the right context.

Pedagogical strategies that provide opportunities for learners to apply differentiation to solving real-life problems are promoted. Learners should be provided with a lot of problems of varying difficulty and social contexts.

SECTION 13: PROBABILITY

Strand: Handling Data

Sub-Strand: Making predictions with data

Learning Outcome: *Solve problems using the axioms and the laws of probability.*

Content Standard: *Apply the addition and multiplication laws and axioms of probability to solve real life problems.*

INTRODUCTION AND SECTION SUMMARY

Most activities in life depend on our ability to make good and sound judgement of the choices available to us on the premise that most events are entirely unpredictable and therefore have to be left to chance. Understanding probability is essential in senior high school mathematics, as it quantifies uncertainty and aids in making predictions. In this section, we will discuss the addition and multiplication laws of probability. The axioms of probability which underpin these laws will also be discussed. Mastering these concepts equips learners to solve complex probability problems and apply them in real-world contexts. The contents of the lessons in this section will be developed on those learnt in year one.

This section consists of

Week 23

1. *Applying the addition and multiplication laws and axioms of probability*
2. *Investigating the Axioms of Probability*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in translating real-life situations that depend on chance into mathematics expressions, apply the laws of probability to solve them and interpret the results in context to the problems. Therefore, collaborative learning, experiential learning and initiate talk for learning, problem-based learning should dominate the lessons on these concepts. Using practical examples and interactive activities will help learners understand and appreciate these concepts and make the learning of probability enjoyable. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings.

However, make considerations and accommodations for the different groups. Then, extend activities for some learners to use formulas and computer applications to solve problems.

ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests and homework assignments, can be used to evaluate learners understanding of concepts and their ability to solve problems. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for STP.

Week 23: *Portfolio Assessment submission.*

WEEK 23

Learning Indicators

1. *Use De Morgan's law in set theory to establish the addition and multiplication laws of probability and apply them to solve real life problems*
2. *Investigate the axioms of probability: For any event A, there is $0 \le P(A) \le 1$; that is, the probability of an event A is a number between 0 and 1 inclusive*
3. *Create and solve problems using the axioms and the laws of probability*

Introduction

Most real-life activities occur either together or separately. We have already learnt that probability is the likelihood of events occurring. The addition and multiplication rules are useful when finding the likelihood of an event happening out of several events or multiple events occurring in sequence.

THEME OR FOCAL AREA 1: APPLYING THE ADDITION AND MULTIPLICATION LAWS AND AXIOMS OF PROBABILITY

The addition and multiplication laws are the fundamental laws of probability to help to calculate the probabilities for multiple events.

Addition laws of probability

Remember that if events are mutually exclusive, they cannot happen at the same time.

The addition laws of probability states that the probability of two events which are **no**t mutually exclusive (this means that they can happen at the same time) is the sum of the events minus the events happening simultaneously.

An example of events that are not mutually exclusive is throwing a die and obtaining an even number and a prime number, picking a club and an ace from a deck of cards, driving and listening to music on the radio.

Mathematically, the addition rule of probability is expressed as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Representing the addition law of probability in a Venn Diagram

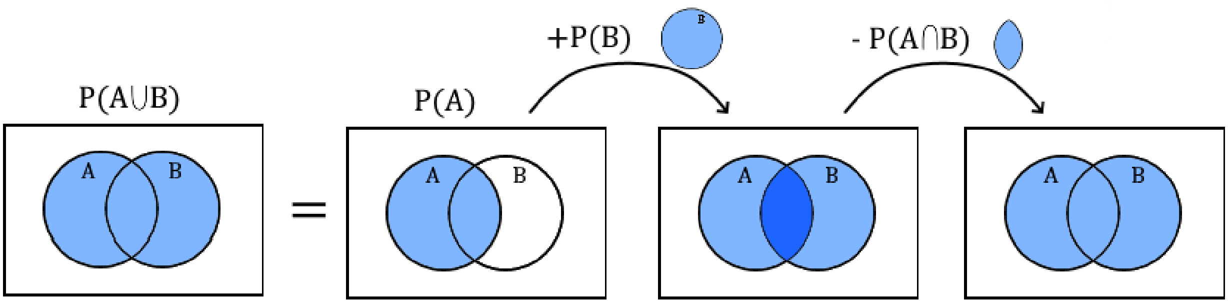


Figure 13.1

In words this is

Probability of event A **or** B occurring = Probability of event A occurs + Probability of event B occurs − Probability that both events A and B occur.

In some instances, there will be events that are mutually exclusive events (cannot happen at the same time). In this instance $\boldsymbol{P(A \cap B) = 0}$**,** then the addition law of probability becomes

$\boldsymbol{P(A \cup B) = P(A) + P(B)}$

An example of a mutually exclusive event is the outcome of rolling a die, tossing a coin, being a boy and a girl at the same time etc.

The addition law of probability is also known as the ***OR*** rule.

Example 1

Given a deck of cards find the probability of drawing

a) either a Jack or a Queen

b) a Jack and a Club.

Solution

a) $P(J) = \frac{4}{52} = \frac{1}{13}$ and

$P(Q) = \frac{4}{52} = \frac{1}{13}$

$\therefore P(J \cup Q) = P(J) + P(Q)$ (These events are mutually exclusive)

$= \frac{1}{13} + \frac{1}{13}$

$= \frac{2}{13}$

b) $P(J \cup C) = P(J) + P(C) - P(J \cap C)$ (These events are not mutually exclusive)

$= \frac{4}{13} + \frac{13}{52} - \frac{1}{52}$

$= \frac{16}{52}$

Example 2

Two events R and Q are such that $(R \cup Q) = \frac{23}{30}$, $P(R) = 0.6$ and $P(R \cap Q) = 0.5$. Find $P(Q)$.

Solution

$P(R \cup Q) = P(R) + P(Q) - P(R \cap Q)$

$\frac{23}{30} = 0.6 + P(Q) - 0.5$

$P(Q) = \frac{2}{3}$

Multiplication laws of probability

The multiplication law of probability is also known as the joint probability formula. It is also known as the ***AND*** rule. For two *independent* events the multiplication law of probability states that

$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

However, when the two events are *dependent* then the probability of events A and B occurring is given by

$$\boldsymbol{P(A \cap B) = P(A) \times P(B|A)}$$

This means the probability A multiplied by the probability B has occurred when A has occurred.

Example 3

Given a deck of cards find the probability of drawing both a 5 and a heart.

Solution

$P(5) = \frac{4}{52}$

$P(heart) = \frac{13}{52}$

$\therefore P(5 \cap heart) = \frac{4}{52} \times \frac{13}{52}$

$= \frac{1}{52}$

This means that there exists only one card out the 52 that can be a 5 and a heart at the same time.

FOCAL AREA 2: INVESTIGATING THE AXIOMS OF PROBABILITY

Axioms are statements that are accepters without proof. Axioms of probability are the basic principles essential for understanding the values in probability. The axioms help us to make sense of the calculations.

The axioms of probability

Given that A is a probabilistic event

1. The probability(P) of the sample space(S) is one, that is $P(S) = 1$, where $A = S$, the sample space
2. $P(\varnothing) = 0$, $A = \varnothing$, A is a null set
3. $0 \le P(A) \le 1$, where A is a proper of subsets of S. That is the probability of an event must be between 0 and 1, where 0 represents an event that will never happen, and 1 represents an event that will definitely happen.
4. The complement rule is $P(A) = 1 - P(A')$

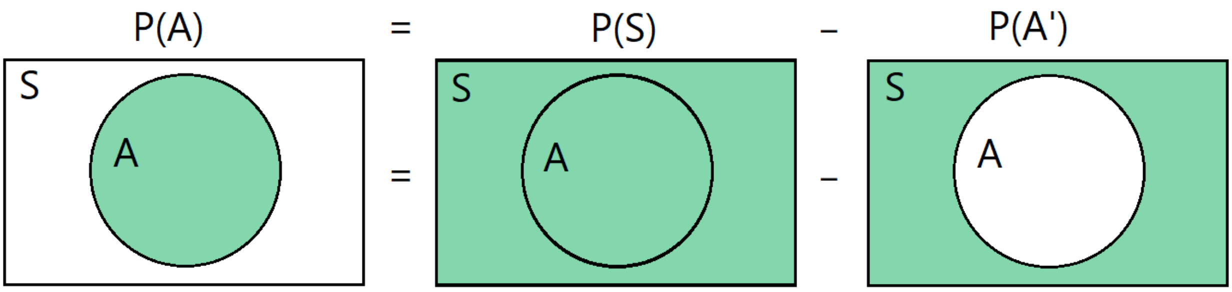


Figure 13.2

Example 4

The probabilities of Ben and Alice solving a mathematics problem correctly are 0.7 and 0.8 respectively. Find the probability that;

a) only Alice solves it correctly

b) both Ben and Alice fail to solve it

c) only one of them solves it correctly

d) What is the probability of at least one of them solving the problem correctly?

Solution

Let P(Alice solving correctly)$= P(A)$ and

P(Ben solving correctly) $= P(B)$

$P(A) = 0.8, P(A') = 0.2$

$P(B) = 0.7, P(B') = 0.3$

a) $P(\text{only Alice solving}) = P(A) \times P(B')$
$= 0.8 \times 0.3$
$= 0.24$

b) $P(\text{both failed to solve}) = P(B') \times P(A')$
$= 0.3 \times 0.2$
$= 0.06$

c) $P(\text{only one solved it}) = P(A) \bullet P(B') + P(A') \bullet P(B)$
$= 0.8 \times 0.3 + 0.2 \times 0.7$
$= 0.24 + 0.14$
$= 0.38$

d) $P(\text{at least one of them solving it correctly}) = 1 - P(\text{none solving})$
$= 1 - P(\text{None solving})$
$= 1 - P(A') \bullet P(B')$
$= 1 - 0.06$
$= 0.94$

Example 5

The probabilities that two friends, Kojo and Yayra will be in school are $\frac{2}{3}$ and $\frac{3}{5}$ respectively. Find the probability that on a given day;

a) Kojo is absent from school but Yayra is present.

b) both are absent from school

c) at least one of them will be in school

Solution

$P(\text{Kojo Present}) = \frac{2}{3}, P(\text{Kojo Absent}) = \frac{1}{3}$

$P(\text{Yayra Present}) = \frac{3}{5}, P(\text{Yayra Absent}) = \frac{2}{5}$

a) $P(\text{Kojo Absent, Yayra Present}) = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$

b) $P(\text{both absent}) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

c) $P(\text{at least one of them will be in school}) = 1 - P(\text{none in school}) = 1 - \frac{2}{15} = \frac{13}{15}$

Learning Tasks

Task learners to

1. solve probability problems by applying De Morgan's laws of probability
2. investigate axioms of probability
3. solve problems using the axioms of probability

PEDAGOGICAL EXEMPLAR

Problem-based / Exploratory Learning

1. Learners in small heterogenous groups solve probabilistic problems applying De Morgan's law
2. Present real-world problems that require the use of probability axioms
3. Task learners to work in groups to solve these problems
4. Conduct experiments like flipping coins or drawing marbles from a bag to collect data
5. Encourage exploration of conditional probability, Bayes' theorem and other advanced topics
6. Task learners to guide their peers through sections of the lessons
7. Tailor assignments to meet the needs of each learner, allowing for varying levels of difficulty and complexity

KEY ASSESSMENTS

Assessment Level 2: Skills and conceptual understanding

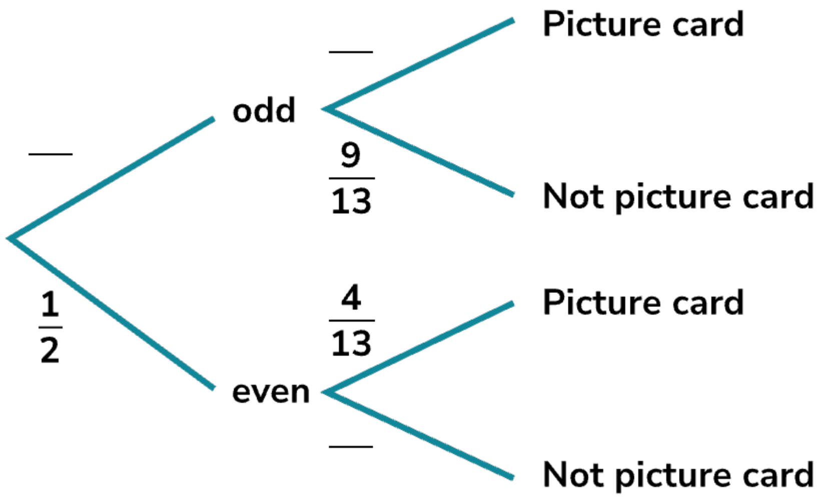
There are 18 girls and 12 boys in a class. $\frac{2}{9}$ of the girls and $\frac{1}{4}$ of the boys walk to school.

One of the students who walks to school is chosen at random.

Find the probability that the student is a boy

Assessment Level 3: Strategic Reasoning

1. Awini runs a game at a fair. To play the game, you must roll a dice and pick a card from a deck of cards. To win the game you must roll an odd number and pick a picture card (including ace cards) i.e., J, Q K and Ace. The game can be represented by the tree diagram below



Awini charges players *GH¢* 1.00 to play and gives *GH¢* 3.00 to any winners. If 260 people play the game, how much profit would Awini expect to make?

2. The blood groups of 200 people is distributed as follows: 50 have type **A** blood, 65 have **B** blood type, 70 have **O** blood type and 15 have type **AB** blood.

If a person from this group is selected at random, what is the probability that this person has O blood type?

REMINDER

Learners should submit their portfolios for marking. Ensure to mark them promptly and submit the scores.

SECTION 13 REVIEW

This section reviews all the lessons taught in week 23. The lessons were developed on the basis concepts taught in weeks 22 and 24 of the year one manual and focused on the addition and multiplication laws and axioms of probability. It needs emphasising that all learners be given the necessary support to experience learning for themselves and be challenged to apply their learning to everyday situations. The use of scientific calculators and / or any other relevant technological tools should be promoted during learning. Assessment strategies should be flexible to allow learners to use their preferred and personal methods and modes of showing evidence of learning.

SECTION 14: COMBINATIONS AND PERMUTATIONS

Strand: Handling Data

Sub-Strand: Making predictions with data

Learning Outcome: *Solve real life problems using combinations and permutations.*

Content Standard: *Apply the concepts of permutations and combinations to solve real life problems.*

INTRODUCTION AND SECTION SUMMARY

In weeks 22 and 23 of the year one manual, learners were introduced to fundamental counting rules, permutations and combinations. Learners looked at the definitions and applied same to simplifying expressions as well as finding the number of arrangements and sample spaces of experiments. In this section, we will delve more into the applications of permutations and combinations to solving problems of arrangements and probability

This section consists of:

Week 24

1. *Fundamental counting rules*
2. *Solving problems involving permutations*
3. *Solving problems involving combinations*

SUMMARY OF PEDAGOGICAL EXEMPLARS

For the lessons in this section, learners will be required to apply their understanding of permutations and combinations to solve more application problems. Therefore, collaborative learning, experiential learning and initiate talk for learning, problem-based learning should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. Then, extend activities for some learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Different forms of assessments should be carried out to ascertain learners' performance on the concepts that will be taught under this section. Teachers are entreated to administer these assessments and record them for onward submission into the Student Transcript Portal (STP). The following assessment would be conducted and recorded for each learner:

Week 24: *End of Semester Examination*

Note

For additional information on how to effectively administer these assessment modes, refer to the Appendices.

WEEK 24

Learning Indicators

1. *Describe the number of ways objects can be arranged using fundamental counting principles*
2. *Solve real life problems involving permutations*
3. *Model and solve real life problems involving combinations*

Introduction

When computing the probability of an event, it is important to know the number of outcomes in the sample space and the number of outcomes in the event. Under describing the number of ways objects can be arranged we will consider three techniques that is the fundamental counting principles, permutations and combinations.

FOCAL AREA 1: FUNDAMENTAL COUNTING RULES

The fundamental counting principle is used when you want to find how many possibilities can exist when combining choices, objects or results. The fundamental counting principle is used when you want to know how many outcomes there are in all the events together.

For example, a learner wants to buy a meal from a restaurant. In the restaurant there are four rice dishes served with either fish or chicken. How many possible meals are there to buy?

	Plain rice	Jollof rice	Saffron rice	Fried rice
Fish	Plain rice and Fish	Jollof rice and Fish	Saffron rice and Fish	Fried rice and Fish
Chicken	Plain rice and Chicken	Jollof rice and Chicken	Saffron rice and Chicken	Fried rice and Chicken

Listing all the possible ways the meal can be served shows that there are 8 ways.

That is $4 \times 2 = 8$

From the above example the fundamental counting principle can be generalised as:

For independent events A and B, if there are n outcomes in event A and m outcomes in event B, then there are $n \times m$ outcomes for the events together.

Example 1

If a sports club decides to elect a chairman and a secretary from 5 members, assuming the same person cannot hold both positions, find the number of ways this can be done?

Solution

Thus, number of ways of choosing a chairman = 5

And the number of ways for choosing a secretary = 4

$= 5 \times 4$

$= 20$

Example 2

A restaurant offers 2 choices for starters, 4 for main course, and 4 for dessert. How many meal variations are possible if a diner chooses 1 of each?

Solution

There are $2 \times 4 \times 4 = 32$ different meal variations.

FOCAL AREA 2: SOLVING PROBLEMS INVOLVING PERMUTATIONS

Permutations relates to the act of arranging all the members of a set into some specific sequence or order. The order in which the items are arranged is important. For example, when the teacher wants to arrange the names of learners in an alphabetical order. That is *ab* is different from *ba* and *abc*, *acb*, *bac* etc, are all different arrangements.

The formula for finding the number of ways to choose and arrange *r* objects from *n* objects is ${}^{n}P_{r} = \frac{(n!)}{(n-r)}$

Example 3

How many numbers can be formed from 2,3, 4 and 5, if each number must be divisible by 5?

Solution

Think through the problem as follows

For 2,3, 4,5 ;

i. Single digit is divisible by 5 = $1P_1 = 1$ (5 only)

ii. Two digits ending 5 = $3P_1 = 3$ (25, 35, 45)

iii. Three digits ending 5 = $3P_2 = 6$ (235, 245, 325, 425, 345, 435)

iv. Four digits ending 5 = $3P_3 = 6$ (2345, 3245, 2435, 4235, 3425, 4325)

$\therefore$ Total number of arrangements $= 1 + 3 + 6 + 6$

$= 16$

Example 4

The Department of Vehicle and Licensing Authority designs the registration of cars with 2 letters from E, R, G, T and 3 digits from 1,4,5,6 on a given day.

How many cars can be registered on that day under this arrangement?

Solution

Firstly, how many 2-letter arrangements can be forms from the 4 letters, E, R, G, T?

$$\text{This is } {}^4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$$

Similarly, arranging 3 digits from the digits 1,4,5,6 is $4P_3$

$$ {}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$$

FOCAL AREA 3: SOLVING PROBLEMS INVOLVING COMBINATIONS

Combinations are a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter. The order in which the items are arranged is not important. The formula for finding the number of ways to choose r objects from n objects is ${}^nC_r = \frac{(n!)}{r!(n-r)}$

Example 5

In a certain examination, a candidate is to answer 6 out of 8 questions.

How many choices has the candidate if;

a. the first 2 questions must be answered

b. at least three of the first four questions must be answered?

Solution

a. If the first two questions are selected, then there are only six questions left from which four need to be selected. That translates to ${}^6C_4$ ways.

$${}^6C_4 = \frac{6!}{4!2!} = 15 \text{ ways}$$

b. A candidate can select three out of the first four questions translating to ${}^4C_3$ and then select the other three questions from the remaining four questions i.e., ${}^4C_3$ again. This option for selecting the 6 questions can be written as ${}^4C_3 \times {}^4C_3$

The other option is to select all four questions out of the first four questions translating to ${}^4C_4$ and then select the other two questions from the remaining four questions i.e., ${}^4C_2$. This option can be written as ${}^4C_4 \times {}^4C_2$At least 3 out of the first 4 questions gives

The number of ways of selecting at least three of the first four questions can be represented by $({}^4C_3 \times {}^4C_3) + ({}^4C_4 \times {}^4C_2)$

$i.e.(4 \times 4) + (1 \times 6) = 22$

Example 6

A committee of 5 teachers is to be formed from 5 males and 4 females. Find the number of ways of forming the committee if

a. there must be only 2 females

b. one particular man must be included

Solution

a. The committee must comprise of 3 men and 2 women

Thus, giving us ${}^5C_3 \times {}^4C_2 = 60\ ways$

b. If we select that particular man, there are 8 persons remaining, and we need any 4 to make up the 5.

$So\ 1 \times {}^8C_4 = \frac{8 \bullet 7 \bullet 6 \bullet 5}{4 \bullet 3 \bullet 2} = 70$ ways

Learning Tasks

1. Learners state theoretical probabilities of some events
2. Learners distinguish between events which are mutually exclusive or otherwise
3. Learners use counting rules to determine sample spaces of events and solve probability problems

PEDAGOGICAL EXEMPLAR

Experiential / Exploratory / Problem-Based Learning

1. Learners apply counting rules, combinations and permutations to solve real-life problems
2. Use clear and simple language to explain each axiom

3. Provide printed notes and visual aids in multiple formats (large print, coloured backgrounds)
4. Break down concepts into smaller, manageable steps, providing clear explanations and ample opportunities for practice
5. Engage students with interactive activities like drawing cards or rolling dice to illustrate probability concepts
6. Task learners to calculate probabilities based on these activities
7. Use simple, real-life examples to demonstrate the axioms
8. Work on problems together as a class before allowing students to try independently
9. Encourage exploration of more complex scenarios, such as overlapping events

KEY ASSESSMENTS

Assessment Level 2: Skills of Conceptual Understanding

1. A restaurant offers the following options

   Starter - Soup or salad

   Main dish – Chicken, fish or vegetarian

   Dessert – Ice cream or cake

   How many possible different combinations of starter, main and dessert are there?

Assessment Level 4: Extending Thinking and Reasoning

1. A soccer team is made up of 1 goalkeeper and 10 players (consisting of defenders, midfielders and attackers). Kotoko football club registered 3 goalkeepers, 7 defenders, 10 midfielders and 5 attackers. Find how many possible ways there are to compose a first team with the following conditions
   a. The formation 3-5-2: 3 defenders, 5 midfielders and 2 attackers,
   b. 3 or 4 defenders, a minimum of 2 and maximum of 4 midfielders with the rest of the team being attackers

   Note that any team setup requires 1 goalkeeper

2. A family of mother, father, older sister, brother and younger sister will be randomly assigned seats A, B, C, D and E, in a row in a bus. Seat A is next to the window.
   a. In how many different ways can the family be assigned seats?
   b. In how many different ways can the family be assigned seats, if the older sister is assigned the seat next to the window?
   c. What is the probability that the older sister will randomly be assigned the seat next to the window?

d. What is the probability that the older sister will not be randomly assigned the seat next to the window?

Hint

- *The Recommended Mode of Assessment for Week 24 is* ***End of Semester Examination.***
- *Refer to Appendix G at the end of Section 6 for further information on how to go about the group presentation.*

Section 14 Review

This section reviews all the lessons taught in week 24. This week discussed concepts of permutations and combinations and their applications which has built on the foundations introduced to learners in year 1. The use of scientific calculators and / or any other relevant technological tools should be promoted during learning.

APPENDIX G

Table of Test Specification

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
13	1. Application of the laws of logarithms 2. Graphing logarithmic functions	Multiple Choice	1	1	1		3
		Fill-in/short answer					
		Real-life Application					
14	Trigonometric identities	Multiple Choice	1	1	1		3
		Short answer					
		Application					
15	1. Deriving and applying the sine and cosine Rule 2. Solving Trigonometric Equations	Multiple Choice	1	1	1		3
		Fill-in/short answer		1			1
		Real-life Application			1		1
16	1. Identifying differentiation rules 2. Differentiating functions using differentiation rules	Multiple Choice	1	1			2
		Fill-in/short answer					
		Real-life Application		1			1
17	Differentiating Implicit functions	Multiple Choice	1	1	1		3
		Fill-in/short answer		1			1
		Real-life Application					

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
18	Differentiating transcendental functions	Multiple Choice		1	1		2
		Fill-in/short answer					
		Real-life Application					
19	1. Partitioning of an area within an interval 2. Area under a Curve	Multiple Choice		1	1		2
		Fill-in/short answer					
		Real-life Application			1		1
20	Indefinite and definite integrals	Multiple Choice	1	1	1		3
		Fill-in/short answer					
		Real-life Application					
21	1. Determining the nature of slopes 2. Investigating turning points 3. Sketching polynomial functions	Multiple Choice			1		1
		Fill-in/short answer			1		1
		Real-life Application					
22	Applying differentiation to solve real-life problems	Multiple Choice			1		1
		Fill-in/short answer					
		Real-life Application				1	1

Week	Focal Area(s)	Nature of Questions	DoK Levels				Total
			1	2	3	4	
23	1. Applying the addition and multiplication laws and axioms of probability 2. Investigating the Axioms of Probability	Multiple Choice	1	1	2		4
		Fill-in/short answer			1		1
		Real-life Application					
24	1. Fundamental counting rules 2. Solving problems involving permutation 3. Solving problems involving combinations	Multiple Choice	1	2			3
		Fill-in/short answer			1		1
		Real-life Application					
	Total		8	14	16	1	39

Structure Of End-of-Semester Examination Questions

a) *Cover content from weeks 13 – 24*

b) *Take into consideration DoK levels*

- **i.** *Section A- Multiple Choice (30 questions, all to be answered)*
- **ii.** *Section B- (5 fill-in questions and short answers, all to be answered)*
- **iii.** *Section C- Real-life Application (4 questions, 2 to be selected).*

c) *Time: 2 hours.*

d) *Total Score: 100 marks.*

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