MINISTRY OF EDUCATION

MATHEMATICS ASSOCIATION OF GHANA

ADDITIONAL MATHEMATICS

for Senior High Schools

Year 3



Writers

Yaw Efa

Benedicta Ama Yekua Etuaful

Isaac Buabeng

Reviewer

Francis Tornyenor

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MAG



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FOREWORD

Ghana's new Senior High School Curriculum aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, knowledge, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, progress to further studies and enter the world of work. This is the first time that Ghana has developed a Senior High School Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

The Ministry of Education is proud to have overseen the production of these Learner Materials which can be used in class and for self-study and revision. These materials have been developed through a partnership between the Ghana Education Service, teacher unions (Ghana National Association of Teachers- GNAT, National Association of Graduate Teacher -NAGRAT and the Pre-Tertiary Teachers Association of Ghana-PRETAG) and National Subject Associations. These materials are informative and of high quality because they have been written by teachers for teachers with the expert backing of each subject association.

I believe that, if used appropriately, these materials will go a long way to transforming our Senior High Schools and developing Ghana so that we become a proud, prosperous and values-driven nation where our people are our greatest national asset.

Haruna Iddrisu MP
Minister for Education

SECTION

1

LOGIC AND ITS APPLICATION



MODELLING WITH ALGEBRA

INTRODUCTION

Logical reasoning is the foundation of sound mathematical thinking. It helps us form clear arguments and test whether conclusions follow from given statements. This section explores simple and compound statements and how we use truth tables to determine the validity of arguments. As we study logic, we will notice its strong link to Set theory studied in Year 1; especially in how union, intersection, and complement relate to logical connectors like "and", "or", and "not". In fact, De Morgan's Laws, covered in Year 2, which describe how set operations behave under complements, also apply directly to logic. Understanding these connections strengthens our reasoning skills and prepares us to handle more complex mathematical ideas with confidence and precision.

KEY IDEAS

- Compound statements – statements which are formed by combining two or more simple statements using logical connectives.
- Mathematical statements – statements which have a definite state of being either True or False. They cannot be both True and False. The statements may contain symbols or words.
- Truth table – a chart which uses logic to show all possible truth values of a statement or combination of statements. It helps us determine whether a compound statement is always true, sometimes true, or never true and whether an argument is valid.
- Validity of an argument – In logic, an argument is valid if, and only if, all the premises are true, then the conclusion must also be true.

FORMING MATHEMATICAL STATEMENTS

In our communication with others, we make statements all the time. Sometimes they are true, other times they are not. We make use of some information to determine whether the statements are true or not. In mathematics, we have "Mathematical Statements". These are sentences that have a definite state of being, either True or False. They cannot be both True and False. The statement may contain symbols or words. This implies that expressions whose truth value cannot be determined are not statements. If we cannot tell whether it is True or False, then it is not a statement.

Let us look at the sentence "3 is an odd number". Based on our knowledge of odd numbers, we can conclude that the sentence is true. Hence, it is a mathematical statement.

Example 1.1

Determine whether the following are True or False.

1. π is a rational number

2. If the product of two numbers is 0, then at least one of the numbers must be 0.
3. All parallelograms are rectangles
4. A number is divisible by 3 if the sum of its digits is divisible by 3.
5. 27 is a prime number
6. $\pi + 3 \geq 6$
7. The sum of the interior angles of a triangle is 240°.

Solution

1. False
2. True
3. False
4. True
5. False
6. True
7. False

Notation of Statements

Mathematical statements are denoted by capital letters such as A, B, D, K, etc.
D: $18 - 5 = 13$, simply means D is the statement $18 - 5 = 13$. D is obviously a true statement because $18 - 5 = 13$.

Example 1.2

Express the following as notations of statements using any variable of your choice and determine their truth value.

1. $\sqrt{7}$ is an irrational number.
2. $11 + \sqrt{4} \in E$ where, E is the set of even numbers.
3. $23 \in Q$ where, Q is the set of irrational numbers.
4. All integers are real numbers.

Solution

1. A: $\sqrt{7}$ is an irrational number. **(True)**
2. G: $11 + \sqrt{4} \in$ E where, E is the set of even numbers. **(False)**
3. K: $23 \in$ Q where, Q is the set of irrational numbers. **(False)**
4. Y: All integers are real numbers. **(True)**

Negation of a Statement

Assuming that Q is a statement, the negation of Q, denoted by $\sim Q$, is also a statement claiming the opposite truth value to Q. This means that if Q is true then $\sim Q$ is false. On the other hand, if Q is false then $\sim Q$ is true. Let us go through this activity on negation of statements.

Activity 1.1 Exploring negation of statements

1. Refer to the statements below
   a. A: $\sqrt{7}$ is an irrational number. **(True)**
   b. G: $11 + \sqrt{4} \in$ E where, E is the set of even numbers. **(False)**
   c. K: $23 \in$ Q where, Q is the set of irrational numbers. **(False)**
   d. Y: All integers are real numbers. **(True)**

2. To negate statements $1a - 1d$, change all positive statements to the negative and vice versa.
   a. ~A: $\sqrt{7}$ is **not** an irrational number.
   b. ~G: $11 + \sqrt{4} \notin$ E where, E is the set of even numbers.
   c. ~K: $23 \notin$ Q where, Q is the set of irrational numbers.
   d. ~Y: All integers are **not** real numbers.

3. Determine the truth value of the negated statements.
   a. ~A: $\sqrt{7}$ is not an irrational number. **(False)**
   b. ~G: $11 + \sqrt{4} \notin$ E where, E is the set of even numbers. **(True)**
   c. ~K: $23 \notin$ Q where, Q is the set of irrational numbers. **(True)**
   d. ~Y: All integers are not real numbers. **(False)**

4. Now, negate the already negated statements in Step 3 (double negation).
   a. ~(~A): $\sqrt{7}$ is an irrational number.
   b. ~(~G): $11 + \sqrt{4} \in$ E where, E is the set of even numbers.
   c. ~(~K): $23 \in$ Q where, Q is the set of irrational numbers.
   d. ~(~Y): All integers are real numbers.

5. Discuss your observations on the truth value of the original statement and the negated statement as well as the outcome of the double negation.

From the observation, you will realise that negating the original statement leads to a change in the truth value, where False becomes True and vice versa. Also, the double negation statements were the same as the original.

Double Negation

From ***Activity 1.1***, the truth value of G is the opposite of the truth value of ~G and the effect of taking double negation, that is ~(~G), was the same as that of G. Here we would say the statements ~(~G) and G are the same. However, in some cases, where statements Q and ~($\sim Q$) are not identical as statements, but have the same truth values for all possible choices of statements Q, we say that Q and ~(~Q) are logically equivalent, or that Q is logically equivalent to ~(~Q)and use the notation $\sim(\sim Q) \equiv Q$.

In everyday use, an example can be:

Q: All triangles have three sides

~Q: All triangles do not have three sides

~(~Q): It is not true that all triangles do not have three sides.

Here Q and ~(~Q) are not worded identically but hold the same truth value hence

$\sim(\sim Q) \equiv Q$.

Statements and Venn diagrams

So far, we have expressed statements using sentences and symbols. It is important to note that statements can be expressed as sets and subsequently represented on Venn diagrams. Recall set representation on Venn diagrams learnt in Year 1.

Example 1.3

Given the statement "M: Ignatius is a Computing student", identify the Universal set, Subsets and Complements that can be created from the statement and represent it on a Venn diagram.

Solution

Let U represent the Universal Set

Let C represent the Set of computing students

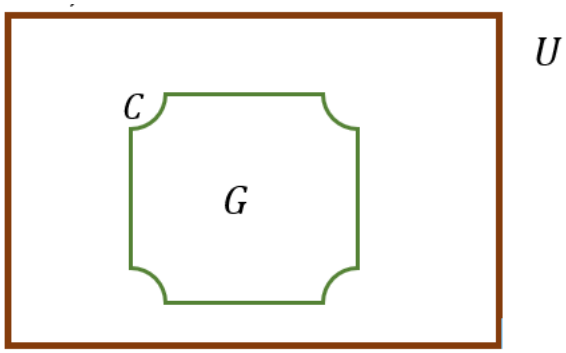
Let G represent Ignatius

U ={All students}

C ={Computing students}

G =Ignatius

Now, $G \in C$



Example 1.4

Given the statement "T: 18 is a multiple of 3, but 7 is not", identify the Universal set, Subsets and Complements that can be created from the statement and represent it on a Venn diagram.

Solution

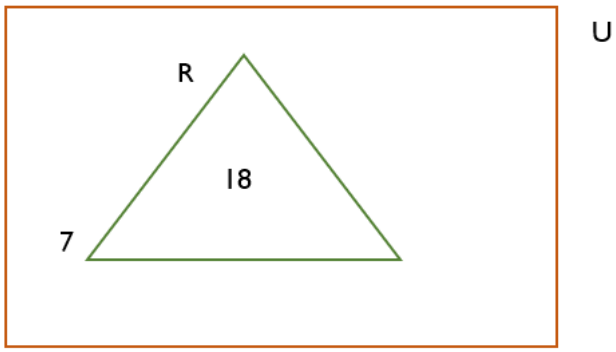
Let U represent the Universal Set

Let R represent the Set of multiples of 3

Let R′ represent non-multiples of 3

$U = \{\text{Real numbers}\}$

Now, $18 \in R$ and $7 \in R'$.



Compound Statements

We have been talking about statements for a while now, some of the statements we have come across have connectives (and, or, then, but) while others have no connectives. The statements made by combining two or more connectives are known as compound statements. Those without connectives are called simple statements.

Different connecting words yield different compound statements. For instance, the word AND is used for Conjunction, OR is used for Disjunction, IF-THEN for implication, etc. Let us go through a few examples of Compound Statements.

Example 1.5

1. 39 is an odd number and 8 is an even number
2. If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.
3. A set of real numbers is made up of rational and irrational numbers.
4. If a number is divisible by 5 then it ends with a 0 or a 5.

Note

A compound statement can have more than 1 logical connector.

DRAWING IMPLICATIONS FROM GIVEN STATEMENTS AND THEIR CONVERSES

In logical reasoning, drawing implications involves identifying what conclusions follow from certain statements. This includes working with implicative statements, exploring their converses, and applying the chain rule of implications to connect multiple ideas. Mastering these concepts helps in forming clear, valid arguments and understanding how truth flows through connected reasoning.

Implicative Statement

In English Language, we learn about conditional sentences and the various types. Conditional sentence Type 1 is the "If" statement. An example is "If Kwame goes to school, he will excel in his academics". In mathematics, that condition helps us generate what we call an Implicative Statement. It is a type of compound statement that establishes a conditional relationship between two simple statements using the "if-then" structure and is a fundamental concept in mathematical logic. Remember, the statements can be true or false. Let us go through some examples.

Example 1.6

1. If it is a prime number then, it is an odd number.
2. If three angles are supplementary then, their sum equals 180°.
3. If $\sqrt[3]{v} = 2$ then $v = 8$.
4. If a polygon is three-sided then, it is a triangle.

There is a symbolic way of representing implicative statements. That is two statements M and N expressed as "If M then N", can be written symbolically as $M \Longrightarrow N$.

M and N are referred to as the hypothesis and conclusion respectively.

Let us go through some examples to express statements symbolically

Example 1.7

Express the following statements symbolically.

1. If a polygon is a square, then it is a rectangle.
2. If a number is a multiple of 10, then it is also a multiple of 5.
3. If a matrix is invertible, then its determinant is not zero.
4. If a number is prime, then it has exactly two factors.

Solution

1. a polygon is a square $\Longrightarrow$ a polygon is a rectangle
2. Let $M_{10}(x)$: x a multiple of 10

   $M_5(x)$: x a multiple of 5

   Then symbolically, $M_{10}(x) \Longrightarrow M_5(x)$.
3. Let $Inv(B)$: matrix B is invertible

   det(B): determinant of matrix B

   Then symbolically, $Inv(B) \Longrightarrow (\det(B) \neq 0)$.
4. Let $P(x)$: x is prime

   $F(x)$: number of factors of x

   Then symbolically, $P(x) \Longrightarrow (F(x) = 2)$

Converse of an implication

Recall in implicative statements we have the "hypothesis" and "conclusion"? To create the converse of an implication, we switch the initial positioning of the "hypothesis" and "conclusion".

Let us look at the implicative statements in **Example 1.6**, and identify the hypothesis and conclusion of each statement.

Table 1.1: Identifying hypothesis and conclusion

Hypothesis	Conclusion
It is a prime number	It is an odd number.
Three angles are supplementary	Their sum equals 180°.
$\sqrt[3]{v} = 2$	$v = 8$.
A polygon is three-sided	It is a triangle.

To create converses of these statements we shall make the conclusion become the hypothesis and vice versa as seen in Table 1.2.

Table 1.2: Identifying hypothesis and conclusion

Hypothesis	Conclusion
It is an odd number.	It is a prime number
Their sum equals 180°.	Three angles are supplementary
$v = 8$.	$\sqrt[3]{v} = 2$
It is a triangle.	A polygon is three-sided

Therefore, the converses of the initial implications will read:

1. If it is an odd number, then it is a prime number.
2. If the sum of three angles equals 180°, then they are supplementary.
3. If $v = 8$, then $\sqrt[3]{v} = 2$.
4. If it is a triangle, then it is a polygon which is three-sided.

Note

You have to adjust the tenses and sentence structure where necessary as done for the second and fourth statements.

To show the repositioning of the hypothesis and conclusion symbolically, the converse may be represented using "$\Leftarrow$".

1. It is a prime number $\Leftarrow$ it is an odd number.
2. Three angles are supplementary $\Leftarrow$ their sum equals 180°.
3. $\sqrt[3]{v} = 2 \Leftarrow v = 8$.
4. A polygon is three-sided $\Leftarrow$ it is a triangle.

The chain rule of implication

A logical principle that provides a connection of multiple implicative statements to form a new valid statement is what we call "The chain rule of implication".

Given the two implicative statements Q and T:

Q: If a number is divisible by 9, then it is divisible by 3.

T: If a number is divisible by 3, then it is divisible by 1.

The chain-rule of implication will be; If a number is divisible by 9 then it is divisible by 1.

Let us go through more examples.

1 q qaA Q1

Example 1.8

Express the following as a chain of implications

1. If a figure is a square then it is a rectangle.

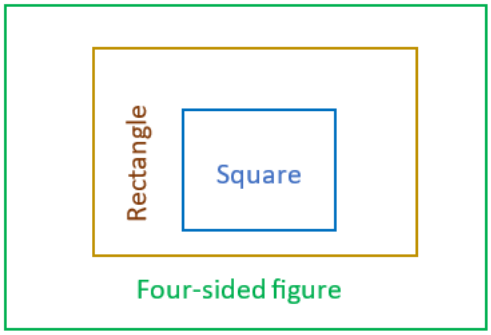
   If it is a rectangle then it has 4 sides.

2. If $y = 4$ then $y^2 = 16$.

   If $y^2 = 16$ then $y^2 - 16 = 0$.

Solution

1. If a figure is a square then it has four sides.



Venn diagram representation chain rule

2. If $y = 4$ then $y^2 - 16 = 0$.

IDENTIFY SIMPLE STATEMENTS AND CONSTRUCT COMPOUND STATEMENTS USING CONNECTIVES

In mathematics, we often work with ideas that can be stated as true or false. Before exploring how these ideas can be combined, it is important to first understand what simple statements are and how they relate to compound statements. This foundation will prepare us to see how logical connectives bring statements together to form meaningful expressions.

Identifying Simple Statements

A simple statement is a basic declarative statement with **no** logical connectives such as AND, OR, IF – THEN, NOT and IF AND ONLY IF. Such statements cannot be broken down into smaller logical sentences and express only a single, complete thought.

Examples of simple statements include:

1. The sun is a star.
2. 17 is a prime number
3. London is the capital city of Britain
4. Tamale is the capital town of Northern Region
5. $\sqrt{11}$ is an irrational number.

Forming Compound Statements Using Connectives

Earlier we mentioned connectives (AND, OR, IF-THEN, etc.) and their relevance in compound statements. To form a compound statement, we combine two simple statements using any of the logical connectives.

Let us explore using the statements R and V

R: 17 is a prime number

V: $\sqrt{9}$ is a rational number.

An example of a compound statement will be "17 is a prime number and $\sqrt{9}$ is a rational number".

Activity 1.2 Forming Compound Statements

Step 1: Provide 5 simple mathematical statements

Step 2: Using the statements created in Step 1, form 5 compound statements using any logical connective of your choice.

CONSTRUCTING TRUTH TABLES OF COMPOUND STATEMENTS

We have talked about the compound statements: Conjunction, Disjunction and Implication (Conditional). We will now examine their symbolic representations while also looking at Bi-conditional compound statements.

- **Conjunction statements (AND):** When two simple statements are combined with the logical connective "AND" we create a compound statement known as a conjunction statement. Given two simple statements E and R combined using the 'AND' connective, the compound statement can be written symbolically as $E \wedge R$.

Note

For a conjunction compound statement, both simple statements must be true for the compound statement to be true.

Example 1.9

1. $\sqrt{169}$ is 13 and $\sqrt[3]{64}$ is 4.

   Symbolically written as $\sqrt{169} = 13 \ \wedge \sqrt[3]{64} = 4$.
2. y is greater than 2 and less than 10 simultaneously

   Symbolically can be expressed as $(y > 2) \wedge (y < 10)$
3. A number is even and is a multiple of 4.

- **Disjunction statements (OR)**: When two simple statements are combined with the logical connective "OR" we create a compound statement known as a Disjunction

statement. The two simple statements represented as E and R can be combined using the OR connective and are written as $E \vee R$.

Note

In a disjunction statement, any of the simple statements must be true for the disjunction compound statement to be true.

Example 1.10

1. n is either less than 0 or greater than 15.

   Symbolically $(n < 0) \vee (n > 15)$

2. g is 6 or g is 18

   Symbolically, $(g = 6) \vee (g = 18)$

3. A number is prime or it is even.

- **Conditional Statement (IF-THEN)**: The connective used for a conditional statement is if - then. If Judith does well in the test, then she will be promoted to the next class. Here, the first statement, E, can be taken as the hypothesis, and the second statement, R, can be taken as the conclusion. We can express conditional statements of these two simple statements E, R as "If E then R".

Note

The conditional compound statement does not hold true if the hypothesis is true and the conclusion is false. But in all other situations, the conditional statement is true.

Example 1.11

1. If a and b are odd integers, then $a + b$ is even.

   Symbolically, $[(a \bmod 2 = 1) \wedge (b \bmod 2 = 1)] \Rightarrow [(a + b) \bmod 2 = 0]$

2. If n is a prime number greater than 2, then n is odd.

   Symbolic form: $[n > 2 \wedge \text{Prime}(n)] \Rightarrow [n \bmod 2 = 1]$

3. If the gradient of a line is 0, then the line is horizontal.

   Let m be the gradient of the line

   Symbolically, $(m = 0) \Rightarrow$ Horizontal line

- **Bi-conditional Statement (IF AND ONLY IF)**: The bi-conditional statement uses the connective 'If and only if,' which is represented by the symbol $\Leftrightarrow$.The two statements, E and R, are represented as a compound statement, $E \Leftrightarrow R$, and here, the first statement, E, is called the hypothesis (antecedent), and the second statement, R, is called the conclusion (consequent).

Note

The bi-conditional compound statement is true if both statements are either true or both

are false.

Example 1.12

1. A triangle is equilateral if and only if all its interior angles are 60°
   Symbolically, [Equilateral $(\triangle)$] $\Leftrightarrow$ [All angles $= 60^{\circ}$]
2. An integer is even if and only if it is divisible by 2.
   Symbolically, (n is even) $\Leftrightarrow$ ($n \bmod 2 = 0$)
3. Two figures are congruent if and only if they have the same size and shape.
   Let the figures be labelled A and B
   Symbolically, Congruent (A, B) $\Leftrightarrow$ (Same size $\wedge$ Same shape)

Truth Table of a Compound Statement

In simple statements, it is fairly easy to determine the truth value. We only need to critically analyse the statement and check if a counter example exists to determine whether the original statement is true or false. However, for a compound statement, you need to analyse the truth value of the individual simple statement then factor in the logical connective being used to determine the truth value. A more convenient way to establish the truth value of a compound statement is by making use of a truth table. We shall discuss the truth table representations for each on the logical connectives mentioned earlier.

Conjunction Truth Table

To create a conjunction truth table, we consider the situations where both simple statements are true, one of the statements is false and when both statements are false. Based on these, four different pairings (orderings) can be generated as shown in Table 1.4 for two simple statements D and K.

Table 1.4: Pairings for two statements D, K.

D	K
T	T
T	F
F	T
F	F

Table 1.5: Truth table for D $\wedge$ K

D	K	D $\wedge$ K
T	T	T
T	F	F
F	T	F
F	F	F

Observations from Table 1.5

1. For row 1 in Table 1.5, both statements (D, K) are true. Since "AND" requires both to be true, the result is true.
2. For row 2, statement D is true but statement K is false. However, "AND" needs both statements to be true resulting in a false result.
3. For row 3, statement D is false but statement K is true. However, "AND" needs both statements to be true resulting in a false result.
4. For row 4 in Table 1.5, both statements (D, K) are false. Since "AND" requires both to be true, the result is false.

In summary, the reasons for the compound statements being false is because one or both of the statements is false.

We can have a conjunction table for three or more statements. For instance, given three simple statements D, K, M, we will have;

Sample Space for (D, K, M)

D	K	M
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The conjunction truth table for the three statements

D	K	M	D ∧ K	D ∧ K ∧ M
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Remember that for a conjunction compound statement to be **True, both statements** must be **true**.

Disjunction Truth Table

This uses the connective 'or' to form a compound statement. Here, even if only one of the individual statements is true, the compound statement holds true.

Table 1.6: Truth table for $\mathbf{D \vee K}$

D	K	D ∨ K	Reasoning
T	T	T	Both statements are true. “OR” only needs at least one true statement thus the result is true.
T	F	T	Since at least one of the statements is true, the resulting statement is true
F	T	T	Since at least one of the statements is true, the resulting statement is true
F	F	F	Since both statements are false, the resulting statement is false.

Exploring a Disjunction table for three statements, we will have:

Sample Space for (D, K, M)

D	K	M
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The disjunction truth table for the three statements

D	K	M	D ∨ K	D ∨ K ∨ M
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

Remember that for a disjunction compound statement to be **True**, **one or both individual statements** must be **true**.

Implication truth table

This uses the If-then connective, which is represented as ⇒. Here, the statement **D** is referred to as the hypothesis, the statement **K** is referred to as the conclusion. The compound statement is true if the conclusion is true, irrespective of the hypothesis. Also, the compound statement is true if both the hypothesis and the conclusion are false. Table 1.7 shows the representation of the implicative connective, **D ⇒ K**.

Table 1.7: Truth table for **D ⇒ K**

D	**K**	**D ⇒ K**	**Reasoning**
T	T	T	If the hypothesis is true and the conclusion is true, then the implication is satisfied.
T	F	F	This is the only case where the implication is false because we claimed K must be true if D is true, but K turned out false.
F	T	T	If the hypothesis is false, the implication is considered true regardless of the conclusion; we didn't make any real promise to break.
F	F	T	Again, since the hypothesis is false, the implication doesn't make any false promise. So, the statement holds true.

Extended Activity

Create an implication truth table for three statements by yourself to show how well you have understood the concept.

Bi-conditional (If and only if) truth table

This uses the connective 'if and only if' and is represented as ⇔. Given the statements **D** and **K**, the first statement, **D**, is referred to as antecedent, and the second statement, **K**, is referred to as consequent. The validity of each of statement can be either true or false.

The bi-conditional compound statement is true if both first and second statements have the same truth values and is false otherwise. Displaying the if and only if connective, **D ⇔ K** in a table.

Table 1.8: Truth table for **D ⇔ K**

D	**K**	**D ⇔ K**
T	T	T
T	F	F
F	T	F
F	F	T

Let us go through some examples showing how we build truth tables using connectives

Example 1.13

If p and q are two simple statements such that p is true and q is false, construct the truth table for

a. $\sim p \wedge q$

b. $\sim(p \vee q)$

Solution

Truth table for $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim p \wedge q$
T	F	F	F

Truth table for $\sim(p \vee \sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	F	T	F

Valid and invalid arguments using the Truth table

Using truth tables, the validity of an argument can be checked by inspecting the premises and conclusion in the rows. If all the premises in the row are true and the conclusion is also true, then the argument will be valid. On the other hand, if all the premises are true and the conclusion is false, then the argument is invalid.

Note

A row in a truth table where all the premises are true is called a **critical row**.

Example 1.14

Refer to the statements below:

M: If a number is divisible by 4, then it is an even number.

N: 16 is divisible by 4

Q: 16 is an even number.

a. Represent the argument using symbols:

A: A number is divisible by 4

B: The number is even

b. Use a truth table to show whether this argument is valid or not.

Solution

a. "If a number is divisible by 4, then it is an even number" can be written as:

If A then B, A ⇒ B.

From the original statements:

1. A ⇒ B
2. A is true
3. So, B is true.

b. To test the validity of the argument let us create a truth table

Table 1.19: Truth Table for $A \Rightarrow B$

A	B	A ⇒ B
T	T	T
T	F	F
F	T	T
F	F	T

Let us rearrange the entries so the premises come before the conclusion

Table 1.20: Rearranged Truth Table for $A \Rightarrow B$

A	A ⇒ B	B
T	T	T
T	F	F
F	T	T
F	T	F

The premises are: "a number is divisible by 4", "If a number is divisible by 4, then it is an even number". The conclusion is "16 is an even number".

From Table 1.20, both premises are true and the conclusion is also true. Hence the argument is valid.

Example 1.15

Consider the following statements:

J: If it is raining, then Ama will carry an umbrella

K: Ama did not carry an umbrella

L: Therefore, it is not raining.

a. Represent the argument using symbols if:

G: It is raining

H: Ama will carry an umbrella

b. Use a truth table to test whether this argument is valid or not.

Solution

a. Now, $G \Rightarrow H \equiv J$

Then

Premise 1 is $G \Rightarrow H \equiv J$ (If it is raining, then Ama will carry an umbrella)

Premise 2 $\sim H \equiv K$ (Ama did not carry an umbrella)

Conclusion $\sim G \equiv L$ (It is not raining)

The argument is $G \Rightarrow H, \sim H, \sim G$.

b. Using the truth table, we have:

G	H	G ⇒ H	~ H	~ G
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	F	T

From the truth table, $G \Rightarrow H$ is True, $\sim H$ is False and $\sim G$ is False, in row 2.

In row 4, $G \Rightarrow H$ is True, $\sim H$ is False and $\sim G$ is True. Hence, we can conclude that the argument is not valid.

Recall that for an argument to be valid, all the premises in the row must be true as well as the conclusion.

Activity 1.3 Validity of Arguments

1. Choose a motion on the topic "Mathematics should be taught as a language rather than a science".
2. Prepare a half-page write-up in support of the motion you choose.
   Include examples and references to support your claim.
3. Take turns presenting your team's argument.
4. Identify the simple, compound statements and the logical connectives in your write-up.
5. Now, the opposing team should respond by stating the negation of the presented statements and evaluate whether the argument is logically valid or not.
6. After all statements have been debated, both teams should work together to construct at least three valid chain rule implications using any of the statements debated.

Example

a. If mathematics uses symbols, structure and rules of grammar, then mathematics functions like a language.
b. If mathematics functions like a language, then it should be taught using methods suited for teaching language.
c. If mathematics should be taught using methods suited for teaching language, then mathematics should be taught as a language rather than as a science.

7. Present your final chain rule implications to your classmates. Seek help from your teacher to confirm if the final chain rule implications are logically valid using truth tables or standard reasoning.

EXTENDED READING

- Logic in mathematics https://www.youtube.com/watch?v=A6xm9BBQyVg retrieved on 15th July, 2025.



- Truth table Tutorial-Discrete Mathematics logic https://www.youtube.com/watch?v=wRMC-ttjhwM retrieved on 15th July, 2025.



- Truth tables and Validity of Arguments https://www.youtube.com/watch?v=eWrzJP7r5cI retrieved on 15th July, 2025.



REVIEW QUESTIONS 1

1. A school policy states:

If a student fails to attend class regularly, then the student will not be allowed to sit the final exam.

If a student is not allowed to sit the final exam, then the student cannot pass the course.

a. Construct the symbolic form of this argument and apply the chain rule of implications to derive a new implication.

b. Using a truth table, determine whether the new implication is valid and always true. Justify your conclusion.

2. Consider the two compound statements:

a. Construct a truth table for both and .

b. Determine whether and are logically equivalent.

Justify your conclusions using observations from the truth table.

c. Explain how the response in (b) relates to De Morgan's Laws.

3. A learner makes the following argument:

If I study hard, I will pass the test.
I passed the test.
Therefore, I must have studied hard.

a. Express this argument in symbolic form.

b. Construct a truth table for this argument.

c. Determine whether the argument is valid or invalid using truth table analysis.

d. Explain the flaw in the learner's reasoning, if any.

4. Let P: "The student submitted the assignment."

Let Q: "The student passed the course."

Let R: "The student graduated."

$P \Rightarrow Q$

$Q \Rightarrow R$

You are told that:

$\sim R$: The student did not graduate

a. Using the chain rule of implication, represent a single implication involving P and R.

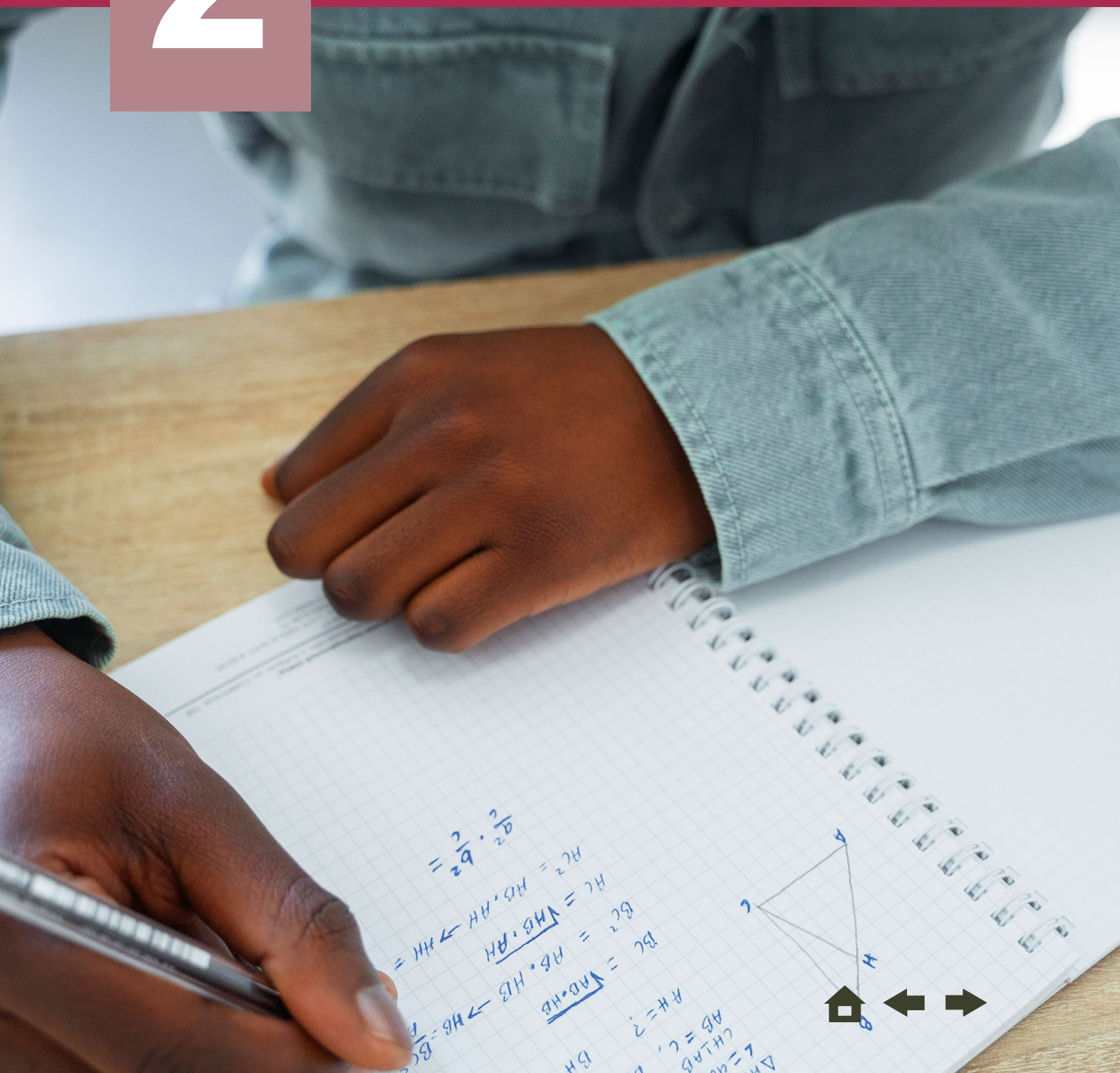
b. Construct a truth table for the resulting implication in (a) and $\sim R$.

c. Based on your truth table, determine if it is logically possible that the student submitted the assignment. Justify your answer using reasoning from the table.

SECTION

2

MATRICES



MODELLING WITH ALGEBRA

Applications of Algebra

INTRODUCTION

In this section on matrices and linear transformations, we will explore how matrices can serve as essential tools for solving linear equations and representing transformations. Matrices enable us to tackle real-world problems in various fields, such as engineering, economics, and computer graphics. By understanding matrix inverses, we can efficiently encode and decode messages, highlighting their significance in cryptography. We aim to equip you with the skills to apply matrices in practical scenarios and solve related problems. Ultimately, understanding these concepts will enhance your mathematical abilities and prepare you for advanced applications in mathematics.

KEY IDEAS

- **Composition of Transformations:** The composition of two transformations P and Q is denoted as P∘Q and is computed as the product of their matrices: PQ.
- **Finding the Image**: Given an object point and a transformation matrix, the image can be calculated by matrix multiplication.
- **Inverse Transformations:** The inverse of a transformation reverses the mapping, returning points to their original positions.
- **Linear transformation**: A linear transformation maps points (objects) in a plane to new points (images) using a matrix.
- **Transformations include**: Translation, rotation, reflection, and enlargement.

FINDING THE EQUATION OF A LINE AS AN IMAGE UNDER A LINEAR TRANSFORMATION

Concept of matrices in transformation

One critical area for the application of matrices is linear transformation, which represents the one-to-one mapping of a point called the **object** in a plane into a point called the **image** in the plane. The object is the original point, while the image is the point obtained. For example, if point $A(1, 2)$ is transformed to $A'(-2, 1)$, The object is taken as A and A' is the image. An object may undergo translation, rotation, reflection and enlargement. Details of these transformations: translation, rotation, reflection and enlargement will be discussed in the later part of this section.

Activity 2.1 Writing linear transformation in matrix form

1. Consider the linear transformation $x' = px + qy$ and $y' = mx + ny$.
2. Write a matrix for the linear transformation by writing the coefficient of x and y.
3. Discuss your result with your peers or teacher.

Note

- Given the linear transformation, $x' = px + qy$ and $y' = mx + ny$, we write the matrix as $\begin{pmatrix} p & q \\ m & n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
- x' and y' are the transformed coordinates
- x and y are the original coordinates
- p, q, m, and n are constants that define the transformation.

Example 2.1

Find the matrix of the linear transformation
$x' = 4x + 2y$ and $y' = 3x - 5y$

Solution

Write down the coefficient of x and y in both equations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the transformation matrix is $\begin{pmatrix} 4 & 2 \\ 3 & -5 \end{pmatrix}$

Example 2.2

Determine the matrix of the linear transformation
$x' = x - y$ and $y' = 10x + 3y$

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 10 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the transformation matrix is $\begin{pmatrix} 1 & -1 \\ 10 & 3 \end{pmatrix}$.

Example 2.3

Determine the matrix of the linear transformation
$x' = y$ and $y' = 2x$

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the transformation matrix is $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

Example 2.4

Determine the matrix for the linear transformation

$P: (x, y) \rightarrow \left(\frac{1}{2}x + 6y, 12x + 2\right)$

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 6 \\ 12 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the transformation matrix is $\begin{pmatrix} \frac{1}{2} & 6 \\ 12 & 2 \end{pmatrix}$

Equation of the image of a line under a given transformation

To find the equation of the image of a line under a translation, reflection, rotation, or enlargement, go through the following steps:

1. Find the coordinates of any two points on the line by choosing 2 numbers for x, and substituting into the equation to find the corresponding y values.
2. Find the coordinates of the images of the two points from Step 1.
3. Use the two image points from Step 2 to find the slope of the image line.
4. Substitute one of the image points and the slope from step 3 into the equation of a line formula to determine the equation of the image line.

Example 2.5

Find the equation of the image of the line $y = 3x + 4$, under the transformation $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Solution

Step 1: Find any two points that lie on line $y = 3x + 4$

If $x = 0$,
$y = 3(0) + 4$
$y = 4$

$A(0,4)$

If $x = 2$

$y = 3(2) + 4$

$y = 10$

$B(2,10)$

Hence, two points that lie on line $y = 3x + 4$ are $A(0, 4)$ and $B(2, 10)$.

Step 2: Add the points to the translation vector to obtain the image

$$A' = \begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$B' = \begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}2\\10\end{bmatrix} = \begin{bmatrix}7\\12\end{bmatrix}$$

The image of these points under the translation $\begin{bmatrix}5\\2\end{bmatrix}$ will be A′(5, 6) and $B'(7, 12)$.

Step 3: Find gradient using the image points; $m_1 = \frac{12-6}{7-5}$

$m_1 = 3$

Step 4: Substitute m_1 into $y - y_1 = m_1(x - x_1)$.

$y - 6 = 3(x - 5)$

$y - 6 = 3x - 15$

$y = 3x - 9$

Thus, the equation of the image line is $y = 3x - 9$.

Example 2.6

The line $y = 2x + 5$ is transformed by a translation vector $\begin{pmatrix}2\\3\end{pmatrix}$ find the image line.

Solution

Find any two points on the line using two arbitrary values of x.

$x = 0$, $y = 2(0) + 5$ $(0, 5)$

$x = 2$, $y = 2(2) + 5$ $(2, 9)$

Find the image points by adding the translation vector

$$\begin{pmatrix}0\\5\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2\\8\end{pmatrix}$$

$$\begin{pmatrix}2\\9\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}4\\12\end{pmatrix}$$

Find the gradient using the image points $\frac{12-2}{4-8} = -2.5$

Use one of the image points to form the equation of the image line

$y - 8 = -2.5(x - 2)$

$y = -2.5x + 13$

Example 2.7

The line $y = 4x + 1$ has been transformed by the linear equations $x' = 2x - 3y$ and $y' = 5x + 2y$. Find the equation of the resulting line.

Solution

Step 1: Write the equations as a matrix.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix is $\begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix}$

Step 2: Find two points that lie on the line $y = 4x + 1$.

If $x = 0$,

$y = 4(0) + 1$

$y = 1$

$A(0, 1)$

If $x = -3$

$y = 4(-3) + 1$

$y = -11$

$B(-3, -11)$

The two points are $\mathrm{A}(0, 1)$ and $\mathrm{B}(-3, -11)$

Step 3: Determine the images of the points through matrix multiplication

$$A' = \begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(0) - 3(1) \\ 5(0) + 2(1) \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$B' = \begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -11 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-3) - 3(-11) \\ 5(-3) + 2(-11) \end{pmatrix}$$

$$= \begin{pmatrix} 27 \\ -37 \end{pmatrix}$$

The image of these points under the $\begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix}$ will be $\mathrm{A}'(-3, 2)$ and $B'(27, -37)$.

Step 4: Find gradient using the image points;

$$m_1 = \frac{-37 - 2}{27 - (-3)}$$

$$m_1 = -\frac{13}{10}$$

Step 5: Substitute m_1 into $y - y_1 = m_1(x - x_1)$.

$$y - 2 = -\frac{13}{10}(x - (-3))$$

$$y - 2 = -\frac{13}{10}(x + 3)$$

$$10y - 20 = -13x - 39$$

The equation of the resulting line is $10y + 13x + 19 = 0$.

USING LINEAR TRANSFORMATION TO DETERMINE IMAGE AND OBJECT POINTS

We can use a linear transformation to find the image when we know the object.

To do this, assume we want to find the image of Q (x, y) under the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, go through the activity below.

Activity 2.2 How to find the image when given the object

1. Identify the given point, object or position vector, $\begin{bmatrix} x \\ y \end{bmatrix}$.
2. Identify the mapping represented by the 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
3. Multiply the position vector by the mapping to obtain the image of the point:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 2.8

Find the image of Q (-2, 3) under the matrix A= $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Solution

The object is given by Q (-2, 3)

Multiply the given matrix by the vector,

$$OQ' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(-2) + 3(3) \\ 2(-2) + 4(3) \end{bmatrix} = \begin{bmatrix} -2 + 9 \\ -4 + 12 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Therefore, $Q'(7, 8)$, is the image of Q (-2, 3) under the matrix A.

Note

Write the image in row form, just as the object was written.

Example 2.9

Find the image of the point P (4, -6) under the transformation associated with the matrix $\begin{bmatrix} 3 & 10 \\ -6 & 14 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 3 & 10 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 3(4) + 10(-6) \\ -6(4) + 14(-6) \end{bmatrix} = \begin{bmatrix} 12 - 60 \\ -24 - 84 \end{bmatrix} = \begin{bmatrix} -48 \\ -108 \end{bmatrix}$$

Thus, the image of P is $P'(-48, -108)$

Example 2.10

A point $S\ (a, b)$ was transformed by the matrix $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ to the image $S'(5, 9)$ Find S.

Solution

Using the same procedure as in the other examples,

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1(a) - 2(b) \\ 3(a) + 0(b) \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} a - 2b \\ 3a \end{bmatrix}$$

$5 = a - 2b$ ..... (1)

$9 = 3a$ .....(2)

$$\frac{9}{3} = \frac{3a}{3}$$

$a = 3$

$5 = 3 - 2b$

$5 - 3 = -2b$

$$\frac{2}{-2} = \frac{-2b}{-2}$$

$b = -1$

Hence $S\ (3, -1)$

Transformation of a given matrix

In pairs, or individually, write down the transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in linear transformation form.

The answer is $T(x, y) \rightarrow (ax + by,\ cx + dy)$.

To transform the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to a linear transformation, we multiply the matrix by the vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + by \end{bmatrix}$

Writing this in row form gives $T(x, y) \rightarrow (ax + by,\ cx + dy)$.

Example 2.11

Write down the following matrices in linear transformation form

a. $\begin{bmatrix} 2 & -4 \\ 9 & -1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{4} & 3 \\ 0 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 2\sqrt{2} & \sqrt{3} \\ \sqrt{3} & 5\sqrt{2} \end{bmatrix}$

Solution

a. $\begin{bmatrix} 2 & -4 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 2x - 4y \\ 9x - y \end{bmatrix}$

$T(x, y) \rightarrow (2x - 4y, 9x - y)$

b. $\begin{bmatrix} \frac{1}{4} & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} \frac{1}{4}x + 3y \\ 5y \end{bmatrix}$

$T(x, y) \rightarrow (\frac{1}{4}x + 3y, 5y)$

c. $\begin{bmatrix} 2\sqrt{2} & \sqrt{3} \\ \sqrt{3} & 5\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 2\sqrt{2}x + \sqrt{3}y \\ \sqrt{3}x + 5\sqrt{2}y \end{bmatrix}$

$T(x, y) \rightarrow (2\sqrt{2}x + \sqrt{3}y, \quad \sqrt{3}x + 5\sqrt{2}y)$

Geometric Transformation

A **geometric transformation** is a way to change a shape or figure in space. This change can affect the **position, size, shape**, or **orientation** of the figure.

There are four main types of geometric transformations:

1. **Translation**: Moving a shape from one place to another without changing its size or shape.
2. **Reflection**: Flipping a shape over a line, like looking in a mirror.
3. **Rotation**: Turning a shape around a point, like spinning a wheel.
4. **Dilation**: Changing the size of a shape while keeping its proportions the same, making it larger or smaller.

Example 2.12

Given that $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, describe the linear transformation T_A geometrically

Solution

Consider the starting points of (1, 0) and (0, 1).

The x coordinate has been unchanged as it remains (1, 0) as indicated by the first column.

The second column indicates that the point (0, 1) is now at (0, -1) which shows that it has been reflected in the x axis.

This means that the linear transformation T_A represented by the matrix A geometrically performs a reflection in the x −axis.

Example 2.13

Describe the linear transformation T_M geometrically of the matrix $M = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.

Solution

This matrix represents a dilation (scaling) with a scale factor of 5.

Geometric Interpretation: Every point moves away from the origin, making the shape larger while preserving its proportions.

Example 2.14

Describe the linear transformation T_M geometrically of the matrix $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution

This matrix represents a rotation by 180° about the origin. It negates both the x and y coordinates.

Geometric Interpretation: Every point moves to the opposite quadrant.

Example 2.15

Describe the linear transformation T_M geometrically of the matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution

Transformation: This matrix represents a reflection in the line $y = x$. It swaps the x and y coordinates.

Geometric Interpretation: Points are reflected in the line $y = x$.

FINDING THE COMPOSITION OF A LINEAR TRANSFORMATION.

Composition of linear transformation

Imagine you have two different linear transformations, represented by two different matrices, P and Q. If you first apply the transformation represented by matrix Q, and then immediately apply the transformation represented by matrix P, this combined transformation is called P ∘ Q (read as P composed with Q).

Mathematically, we can write this as $P \circ Q = PQ$, which means that the combined transformation is equal to the product of the two matrices, P and Q.

Similarly, Q ∘ P (P followed by Q) is given by the product of QP, i.e., Q ∘ P = QP

To calculate the composite transformation A ∘ B:

1. **Write the matrices A and B that represent the individual linear transformations:**
   a. Matrix A represents the first transformation.
   b. Matrix B represents the second transformation.
2. **Find the matrix product AB:**
   a. Multiply the matrices B and A in that order.
   b. The order is important because the second transformation (B) is applied first, followed by the first transformation (A).
3. **Write the linear transformation for AB:**
   a. The matrix AB represents the combined transformation, where the second transformation (B) is applied first, followed by the first transformation (A).

b. The combined transformation can be written as $AB(x, y)$, where (x, y) are the input coordinates.

4. **Observe the order of the matrices:**

a. When carrying out the matrix product, the matrix that represents the first transformation (A) is placed on the left, and the matrix that represents the second transformation (B) is placed on the right.

b. This order reflects the fact that the second transformation (B) is applied first, followed by the first transformation (A).

Example 2.16

If $M = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} 11 & 6 \\ 3 & -1 \end{bmatrix}$, evaluate:

i. $M \circ N$

ii. $N \circ M$

iii. Comment on your results in (i) and (ii).

Solution

i. $M \circ N = MN$

$$= \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 11 & 6 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2(11) + 3(3) & 2(6) + 3(-1) \\ 5(11) + 4(3) & 5(6) + 4(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 22 + 9 & 12 - 3 \\ 55 + 12 & 30 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 9 \\ 67 & 26 \end{bmatrix}$$

ii. $N \circ M = \begin{bmatrix} 11 & 6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 11(2) + 6(5) & 11(3) + 6(4) \\ 3(2) - 1(5) & 3(3) - 1(4) \end{bmatrix} = \begin{bmatrix} 22 + 30 & 33 + 24 \\ 6 - 5 & 9 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 57 \\ 1 & 5 \end{bmatrix}$$

iii. Since $M \circ N$ and $N \circ M$ are not equal, they are not commutative, confirming that Matrix multiplication is not commutative, thus $M \circ N \neq N \circ M$.

APPLYING LINEAR TRANSFORMATION IN FINDING TRANSLATION, REFLECTIONS, ROTATIONS AND ENLARGEMENT OF POINTS AND PLANE FIGURES

Activity 2.3 Finding the Translation of a Linear Transformation

1. Identify the points A and B plotted on the graph in Figure 2.1.
2. What are the coordinates of points A and B?
3. Starting at point A, perform the following activity

   Given the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

   i. Count 1 step to the right of A

      Remember that positive numbers on the x-axis are counted to the right, while negative numbers are counted to the left

   ii. Count 2 steps down from where you reached in 3(ai.)

      Positive numbers on the y −axis are counted upwards while negative numbers are counted downwards

4. Plot a point at the new destination and name it A' (..., ...)
5. Repeat the steps from 3 to 4 for point B
6. What did you obtain for point B'?

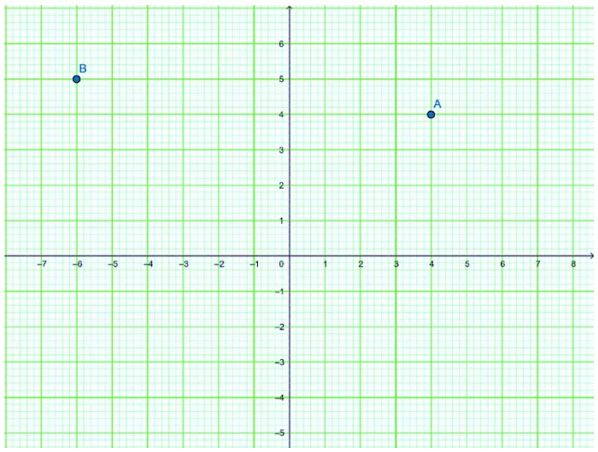


Figure 2.1: Translation of a vector

Points A and B are the objects, while A' and B' are the images obtained when they were translated by the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

The other way to obtain the images is given by

Image of object $=$ Object + Translation vector

Let us use this to find the images of A and B to confirm your results in the activity above

A is point (4, 4) and is translated by the vector $\begin{pmatrix}1\\-2\end{pmatrix}$,

Image $= \begin{pmatrix}4\\4\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$

Thus, $A'\ (5, 2)$

B is point (-6, 5) and is translated by the vector $\begin{pmatrix}1\\-2\end{pmatrix}$

Image $= \begin{pmatrix}-6\\5\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix}$

$= \begin{pmatrix}-5\\3\end{pmatrix}$

Hence, $B'(-5, 3)$

Example 2.17

A triangle ABC has the vertices $A(3, 4), B(-1, 5)$ and $C(0, 0)$.

Triangle ABC is translated by the vector $\begin{pmatrix}-2\\-3\end{pmatrix}$, find the coordinate of the image triangle A′B′C′.

Solution

Take each object and add the translated vector to it

For A, the image $A' = \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

For B, the image $B' = \begin{pmatrix}-1\\5\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$

For C, the image $C' = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix}$

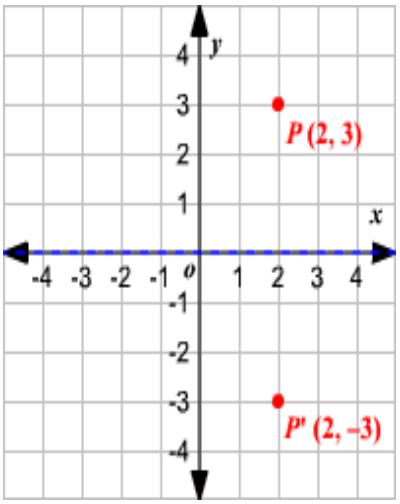
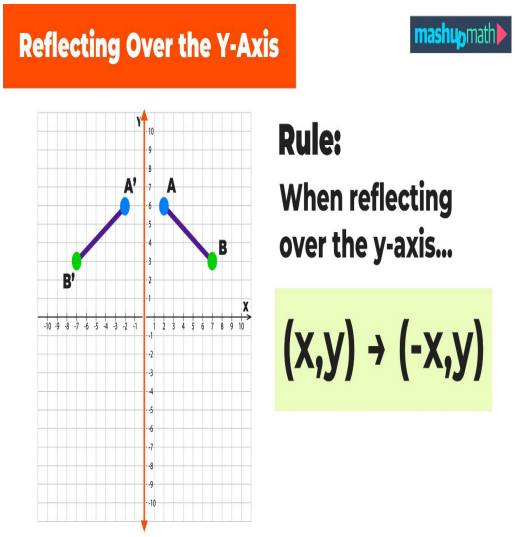
The coordinate of the vertices of image triangle $A'B'C'$ are $A'(1, 1), B'(-3,2)$ and $C'(-2, -3)$

Reflection and its properties

To reflect an object in a given line, for example the x-axis, use the x-axis as a mirror line and use one of the characteristics of a plane mirror, which is “the distance between the object and the mirror is the same as the distance between the mirror and the image”.

For example, in the diagram below we have a reflection in the x-axis.

The x-axis is used as a mirror. The perpendicular distance between the object $P(2, 3)$ and the mirror is 3 up, counting 3 down from the mirror line gives us $P'(2, -3)$

Reflection in the axis	Reflection in the axis
	
Given T (1,0) $\xrightarrow{x-axis}$ (1, 0) Also, T (0,1) $\xrightarrow{x-axis}$ (0, -1) If M is the matrix of the transformation, $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Given T (1,0) $\xrightarrow{y-axis}$ (-1, 0) Also, T (0,1) $\xrightarrow{y-axis}$ (0, 1) If M is the matrix of the transformation, $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

To find the image of an object, multiply the transformation matrix, M, by the object $\begin{pmatrix} x \\ y \end{pmatrix}$.
The table below summarises the various reflections and how to get the image.

Type of reflection	Point	Matrix to multiply by	Image
x-axis $(y = 0)$	(x, y)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
y-axis $(x = 0)$	(x, y)	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$y = x$	(x, y)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$y = -x$	(x, y)	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example 2.18

If $A\ (1, 2)$ is reflected in the x–axis to obtain A', find the coordinate of A'

Solution

For reflection in the x-axis,

$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

The object is $A(1, 2)$

Multiplying:

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1(1) + 0(2) \\ 0(1) - 1(2) \end{pmatrix} = \begin{pmatrix} 1 + 0 \\ 0 - 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Thus the image of $A\ (1, 2)$ is $A'(-1, 2)$.

Example 2.19

A triangle with vertices $E(2, 8)$, $F(4, 8)$ and $G\ (5, 8)$ is reflected in the line $y = x$ find the images $E'F'G'$

Solution

For reflection in the line $y = x$, the transformation matrix, $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The matrix for triangle EFG $= \begin{bmatrix} 2 & 4 & 5 \\ 8 & 8 & 8 \end{bmatrix}$

Multiplying the two matrices,

$$E'F'G' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ 8 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(8) & 0(4) + 1(8) & 0(5) + 1(8) \\ 1(2) + 0(8) & 1(4) + 0(8) & 1(5) + 0(8) \end{bmatrix} = \begin{bmatrix} 8 & 8 & 8 \\ 2 & 4 & 5 \end{bmatrix}$$

Thus the image triangle $E'F'G'$ has the coordinate $E'(8, 2), F'(8, 4), G'(8, 5)$.

Alternatively, we can do the calculation individually

For $E\ (2, 8)$

$$E' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0(2) + 1(8) \\ 1(2) + 0(8) \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

For $F\ (4, 8)$

$$F' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0(4) + 1(8) \\ 1(4) + 0(8) \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

For $G\ (5, 8)$

$$G' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0(5) + 1(8) \\ 1(5) + 0(8) \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Thus, the image triangle $E'F'G'$ has the coordinates $E'(8, 2), F'(8, 4), G'(8, 5)$.

Rotation and its properties

Rotation is the turning of an object or point about or around a fixed point (where the fixed point could be the origin). Rotation can be clockwise or anticlockwise through $90°$, $180°$, $270°$ and other angles, represented by θ.

Activity 2.4 Rotation through 90° clockwise

1. The graph shows the rotation of M through 90° degrees clockwise, about the origin.
2. Write down the coordinates of M and M'
3. What is your observation of the coordinates of M and M'?
4. Can you work out the transformation matrix for a rotation of 90° degrees clockwise, about the origin?

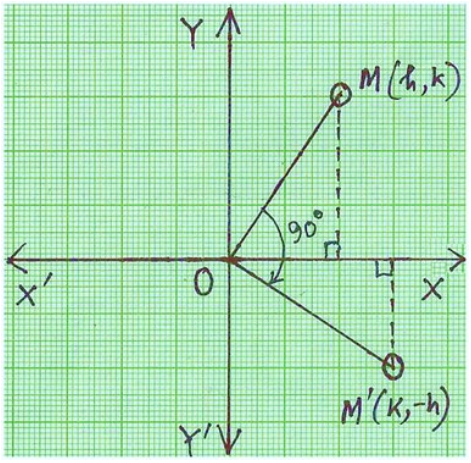


Figure 2.2: Rotation through 90° clockwise

Given T (1, 0) $\xrightarrow{90^{\circ}\ clockise}$ (0, 1)

Also, T (0,1) $\xrightarrow{90^{\circ}\ clockise}$ (-1, 0)

If M is the matrix of the transformation, $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Other forms of rotation are summarised in the table below:

Angle of rotation, about the origin	Point	Matrix to be multiplied	Image
90° Clockwise	(x, y)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
90° Anticlockwise	(x, y)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
180°	(x, y)	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
270° Clockwise	(x, y)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
270° Anticlockwise	(x, y)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
Any angle	(x, y)	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example 2.20

The vertices of a triangle are $A(1,2)$, $B(3,2)$ and $C(2,3)$. If triangle ABC is rotated through 90° anticlockwise, about the origin.

Find the vertices of the image $A'B'C'$.

Solution

Step 1: Determine the matrix to be multiplied under rotation through 90° anticlockwise.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Step 2: Multiply the objects by this matrix to obtain the corresponding image

For $A(1,2)$, we have :

$$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(1) - 1(2) \\ 1(1) + 0(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $B(3,2)$ we have:

$$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(3) - 1(2) \\ 1(3) + 0(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

For $C(2,3)$ we have:

$$C' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0(2) - 1(3) \\ 1(2) + 0(3) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Hence, triangle $A'B'C'$ has the vertices $A'(-2,1)$, $B'(-2,3)$ and $C'(-3,2)$.

Alternatively:

The vertices of the triangle ABC can be written in matrix form as $P = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

The matrix of the linear transformation, $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Now, find $\boldsymbol{MP}$

$$MP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0(1) - 1(2) & 0(3) - 1(2) & 0(2) - 1(3) \\ 1(1) + 0(2) & 1(3) + 0(2) & 1(2) + 0(3) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & -3 \\ 1 & 3 & 2 \end{bmatrix}$$

Hence, the vertices of the image are $A'(-2,1)$, $B'(-2,3)$ and $C'(-3,2)$

Example 2.21

Find the images of the rectangle $P(1,5)$, $Q(5,5)$, $R(5,-3)$ and $S(1,-3)$ if it is rotated through:

a. 180° about the origin.

b. 90° clockwise about the origin.

Solution

a. The matrix of the linear transformation, M, under rotation of 180°, about the origin, is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Multiply the objects by this matrix:

For $P(1,5)$, we have:

$$P' = \left[\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right]\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1(1) + 0(5) \\ 0(1) - 1(5) \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

For $Q(5,5)$, we have:

$$Q' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -1(5) + 0(2) \\ 0(5) - 1(5) \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

For $R(1,-3)$ we have:

$$R' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1(1) + 0(-3) \\ 0(1) - 1(-3) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

For $S(5,-3)$ we have:

$$S' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1(5) + 0(-3) \\ 0(5) - 1(-3) \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$P'Q'R'S'$, which is the image of rectangle $PQRS$, ar: $P'(-1,-5), Q'(-5,-5), R'(-1,3)$ and $S'(-5,3)$.

b. The matrix of the linear transformation, M, under rotation of 90° clockwise, $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Multiply the objects by this matrix:

For $P(1,5)$, we have:

$$P' = \left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right]\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0(1) + 1(5) \\ -1(1) + 0(5) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

For $Q(5,5)$ we have:

$$Q' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0(5) + 1(5) \\ -1(5) + 0(5) \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

For $R(1,-3)$ we have:

$$R' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0(1) + 1(-3) \\ -1(1) + 0(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

For $S(5,-3)$ we have:

$$S' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0(5) + 1(-3) \\ -1(5) + 0(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$P'Q'R'S'$, which is the image of rectangle $PQRS$, are $P'(5,-1), Q'(5,-5), R'(-3,-1)$ and $S'(-3,-5)$.

Example 2.22

The image of $R(8, 10)$ when rotated through 60° anticlockwise about the origin is $R'(a,\ b)$. Find the values of a and b.

Solution

The matrix of the linear transformation, M, under rotation of an angle about the origin is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The given angle is 60°. Substitute 60° into the matrix and multiply by the object given.

$$R' = \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix}\begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times 8 - \frac{\sqrt{3}}{2}\times 10 \\ \frac{\sqrt{3}}{2}\times 8 + \frac{1}{2}\times 10 \end{bmatrix} = \begin{bmatrix} 4 - 5\sqrt{3} \\ 5 + 4\sqrt{3} \end{bmatrix}$$

Thus, $R'(a,\ b) = R'\left(4 - 5\sqrt{3}\ ,\ 5 + 4\sqrt{3}\right)$ where, $a = 4 - 5\sqrt{3}$ and $b = 5 + 4\sqrt{3}$.

Example 2.23

Find the image of R $(-6, 1)$ when rotated through 135° anticlockwise about the origin.

The matrix of the linear transformation, M, under rotation of any angle about the origin is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The given angle is 135°. Substitute the 135° into the matrix and multiply by the object given.

$$R' = \begin{bmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{bmatrix}\begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -(1) \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1(-6) - 1(1) \\ 1(-6) - 1(1) \end{bmatrix} = \begin{bmatrix} 6 - 1 \\ -6 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$\therefore R'(5, -7)$

Dilation (Enlargement or Reduction) with centre of enlargement the origin

Dilation is where plane figures/shapes are transformed to make the figures bigger (enlargement) or smaller (reduction). The dilation matrix is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ where k is a dilation or scale factor and the centre of enlargement is the origin.

Activity 2. Finding the Dilation of a Linear Transformation

1. Observe the graphs in Figure 2.3 and Figure 2.4

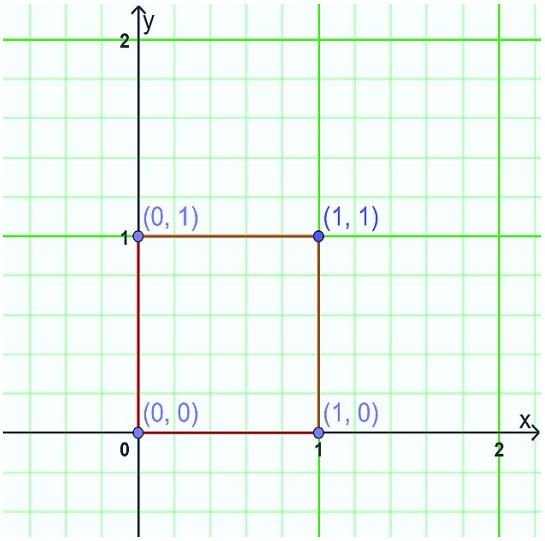


Figure 2.3

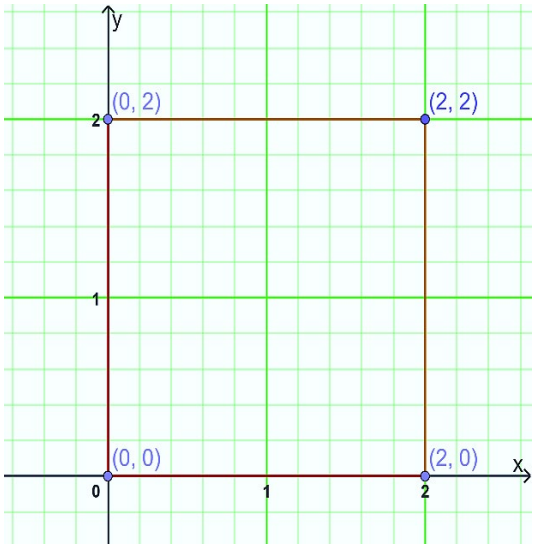


Figure 2.4

2. What is the relationship between Figure 2.3 and Figure 2.4?

Let the transformation be T, then:

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

Thus, the matrix of the transformation is $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$

If we assume a transformation under an enlargement whose scale factor is K,

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}K\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\K\end{bmatrix}$$

Thus, the matrix of the transformation is $\begin{bmatrix}K & 0\\0 & K\end{bmatrix}$

Properties of Enlargement and Reduction

1. There is a single invariant point, the centre of enlargement, O.
2. All other points P map onto their images P' so that $OP' = kOP$, where k is the scale factor of enlargement.
3. Under enlargement, any figure is mapped onto a similar figure.
4. If $|k|>1$, the image is larger than the original figure, but if $|k|<1$, it is smaller.
5. If $k < 0$, O lies between P and P' and the image figure is inverted.

6. A scale factor of -1 is equal to a rotation about the centre O of $180°$.
7. With a scale factor k, similar figures have lengths and areas in the ratio $k:1$ and $k^2:1$ respectively.

Example 2.24

Find the effect of matrix Q $= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ on a triangle whose vertices are $A\ (0,1), B(3,1)$ and $C(2,5)$.

Solution

Multiply the matrix Q by the vertices.

For vertex A, $A' = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5(0)+0(1) \\ 0(0)+5(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

For vertex B, $B' = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5(3)+0(1) \\ 0(3)+5(1) \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$

For vertex C, $C' = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5(2)+0(5) \\ 0(2)+5(5) \end{bmatrix} = \begin{bmatrix} 10 \\ 25 \end{bmatrix}$

Example 2.25

A matrix $M = \begin{bmatrix} x & y \\ w & z \end{bmatrix}$ transformed a triangle with vertices $A(1,2), B(2,6), C(3,3)$ into another triangle with vertices $A'(1,-2), B'(2,-6), C'(3,-3)$.

Find the components of the matrix M.

Solution

Let R and S be matrices for the triangle ABC and $A'B'C'$.

$R = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 3 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -6 & -3 \end{bmatrix}$

$MR = S$

$\begin{bmatrix} x & y \\ w & z \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -6 & -3 \end{bmatrix}$

$\begin{bmatrix} x+2y & 2x+6y & 3x+3y \\ w+2z & 2w+6z & 3w+3z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -6 & -3 \end{bmatrix}$

$x + 2y = 1 \quad \ldots\ldots\ldots(1)$

$2x + 6y = 2 \quad \ldots\ldots\ldots(2)$

$3x + 3y = 3.$ .....(3)

$w + 2z = -2$.....(4)

$2w + 6z = -6$.....(5)

$3w + 3z = -3$.....(6)

Solving equations (1) and (2) simultaneously,

$x = 1,\ y = 0,\ w = 0$ and $z = -1$

$\therefore M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Example 2.26

A company produces two local products: **kente cloth** (a traditional woven fabric) and a **smock**. The profit from selling one unit of kente cloth is GHC 800.00, and the profit from selling one unit of smock is GHC 400.00. The transformation matrix representing the change in production strategy is given by:

$T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

a. If the company initially produces 30 pieces of Kente cloth and 45 pieces of smock, calculate the new levels after the transformation

b. Discuss how this new strategy affect the company's total profit, considering the new production levels.

Solution

a. To calculate the new production levels after applying the transformation matrix T, we first represent the initial production levels as a vector:

Initial Production Vector $= \begin{pmatrix} 30 \\ 45 \end{pmatrix}$

Now, we apply the transformation matrix T:

$$T \cdot \begin{pmatrix} 30 \\ 45 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 45 \end{pmatrix}$$

Calculating the matrix multiplication:

$$T \cdot \begin{pmatrix} 30 \\ 45 \end{pmatrix} = \begin{pmatrix} 1(30) + 2(45) \\ 0(30) + 1(45) \end{pmatrix} = \begin{pmatrix} 30 + 90 \\ 0 + 45 \end{pmatrix} = \begin{pmatrix} 120 \\ 45 \end{pmatrix}$$

Thus, the new production levels are, Kente Cloth: 120 units and Smock: 45 units.

b. To discuss the impact on the company's total profit, we first calculate the total profit from the initial production levels:

For initial Profit:

Profit from Kente cloth $= 30 \times 800 =$ GH₵24 000.00

Profit from smock $= 45 \times 400 = \text{GH₵}18\,000.00$

Total Initial Profit $= 24\,000 + 18\,000 = \text{GH₵}42\,000.00$

For New Profit Calculation, calculate the total profit from the new production levels

Profit from Kente cloth $= 120 \times 800 = \text{GH₵}96\,000.00$

Profit from smock $= 45 \times 400 = \text{GH₵}18\,000.00$

Total new profit: $= 96\,000 + 18\,000 = \text{GH₵}114\,000.00$

The new strategy significantly increases the company's total profit from GH₵42 000.00 to GH₵114 000.00, indicating that the change in production strategy has been beneficial for the company. The production of Kente cloth has been prioritised, leading to a substantial increase in profit, which can be reinvested into the company to further enhance production or support local communities.

Application of Matrices

One of the applications of the inverse Matrix is to provide a simple and effective procedure for encoding and decoding messages, known as cryptography.

To begin, assign the numbers 1–26 to the letters in the alphabet, as shown below:

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
	14	15	16	17	18	19	20	21	22	23	24	25	26

Assign the number 0 to a blank to provide for space between

Note

A more sophisticated code could include both uppercase and lowercase letters, as well as punctuation symbols.

Any matrix whose elements are positive integers and whose inverse exists can be used as an encoding matrix. For example, MATHEMATICS corresponds to:

M	A	T	H	E	M	A	T	I	C	S
13	1	20	8	5	13	1	20	9	3	19

Example 2.27

Encode the following message

a. GHANA IS BLESSED

b. EDUCATION IS THE KEY

Solution

a.

G	H	A	N	A		I	S		B	L	E	S	S	E	D
7	8	1	14	1	0	9	19	0	2	12	5	19	19	5	4

b.

E	D	U	C	A	T	I	O	N		I	S		T	H	E		K	E	Y
5	4	21	3	1	20	9	15	14	0	9	19	0	20	8	5	0	11	5	25

Given the matrix, A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to decode with the message 1 2 3 4 5 6 7 8
Follow the steps below to decode:

1. Find the inverse of the matrix A to obtain A^{-1}
2. Write the message [1 2 3 4 5 6 7 8] in 2×2 matrix because the matrix to be used for decoding is a 2×2 matrix say Q as $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$
3. Multiply the matrices AQ
4. Use the codes for the alphabet to decode the message

Example 2.28

$A = \begin{pmatrix} 1 & 2 \\ 1 & 3. \end{pmatrix}$ The following message was encoded with matrix A. Decode this message:
52 70 17 21 5 5 29 43 4 4 52 70 25 35 29 33 15 18 5 5.

Solution

1. Find the inverse of matrix A

$det|A| = 1(3) - 1(2)$

$det|A| = 1$

$A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

Write the message 52 70 17 21 5 5 29 43 4 4 52 70 25 35 29 33 15 18 5 5 in matrix form as

$B = \begin{pmatrix} 52 & 17 & 5 & 29 & 4 & 52 & 25 & 29 & 15 & 5 \\ 70 & 21 & 5 & 43 & 4 & 70 & 35 & 33 & 18 & 5 \end{pmatrix}$

Multiply A^{-1} by B= $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 52 & 17 & 5 & 29 & 4 & 52 & 25 & 29 & 15 & 5 \\ 70 & 21 & 5 & 43 & 4 & 70 & 35 & 33 & 18 & 5 \end{pmatrix}$

$$\begin{pmatrix} 156-140 & 51-42 & 15-10 & 87-86 & 12-8 & 156-140 & 75-70 & 87-66 & 45-36 & 15-10 \\ -52+70 & -17+21 & -5+5 & -29+43 & -4+4 & -52+70 & -25+35 & -29+33 & -15+18 & -5+5 \end{pmatrix}$$

$$\begin{pmatrix} 16 & 9 & 5 & 1 & 4 & 16 & 5 & 21 & 9 & 5 \\ 18 & 4 & 0 & 14 & 0 & 18 & 10 & 4 & 3 & 0 \end{pmatrix}$$

Write the matrix B in row form as it was given

16	18	9	4	5	0	1	14	4	0	16	18	5	10	21	4	9	3	5	0
P	R	I	D	E		A	N	D		P	R	E	J	U	D	I	C	E	

Remember we used this code:

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
	14	15	16	17	18	19	20	21	22	23	24	25	26

The message was PRIDE AND PREJUDICE.

Isometrics

Isometric transformations are the kind in which the images still have the same shape and size.

Reflected, rotated and translated shapes or objects still end up with the same look (shape and size).

Note

Under isometrics, any figure maps onto a congruent figure. Translation and rotation produce directly congruent figures; reflection is inversely congruent.

Inverse of a linear transformation

The inverse of a linear transformation moves a point back to its original position. This means it reverses the effect of the original transformation.

Here is a suggested step-by-step:

1. Write the transformation in matrix form.
2. Find the inverse of the matrix.
3. Express x and y in terms of their transformation x' and y' respectively.

Let us follows these steps to solve the following:

Example 2.29

Determine the inverse of the linear transformation $x' = 3x + 4y$ and $y' = -2x - 7y$.

Solution

Step 1: Writing the matrix gives:

$$\begin{bmatrix} 3 & 4 \\ -2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Let A $= \begin{bmatrix} 3 & 4 \\ -2 & -7 \end{bmatrix}$

Step 2: Find the inverse of A

Determinant, $|A| = 3(-7) - 4(-2) = -13$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} -7 & -4 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{13} & \frac{4}{13} \\ -\frac{2}{13} & -\frac{3}{13} \end{bmatrix}$$

Step 3: Express in terms of x' and y:

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{7}{13} & \frac{4}{13} \\ -\frac{2}{13} & -\frac{3}{13} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = \frac{7}{13}x' + \frac{4}{13}y'$$

$$y = -\frac{2}{13}x' - \frac{3}{13}y'$$

Example 2.30

Find the inverse of the linear transformation $x' = x + 12y$ and $y' = 3y$.

Solution

Step 1: Writing the matrix gives

$$\begin{bmatrix} 1 & 12 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Let A $= \begin{bmatrix} 1 & 12 \\ 0 & 3 \end{bmatrix}$

Step 2: Find the inverse of A

Determinant, $|A| = 1(3) - 0(12) = 3$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -12 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Step 3: Express in terms of x' and y' :

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' - 4y' \\ 0 + \frac{1}{3}y' \end{bmatrix}$$

$$x = x' - 4y'$$

$$y = \frac{1}{3}y'$$

Example 2.31

A transformation is defined by T: $(x, y) \rightarrow (3x + 9y, -x + 5y)$.

a. Write down the matrix T of the transformation.

b. Find the inverse of the matrix T.

c. If the point P is transformed by the matrix and the image is $(-10, 11)$, find the coordinates of P.

Solution

a. Matrix T= $\begin{pmatrix} 3 & 9 \\ -1 & 5 \end{pmatrix}$

b. $|T| = 3(5) - (-1)(9)$

$|T| = 15 + 9 = 24$

$$T^{-1} = \frac{1}{24}\begin{pmatrix} 5 & -9 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{24} & -\frac{9}{24} \\ \frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{24} & -\frac{9}{24} \\ \frac{1}{8} & \frac{3}{8} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{24}x' - \frac{9}{24}y' \\ \frac{1}{8}x' + \frac{3}{8}y' \end{pmatrix}$$

$$x = \frac{5}{24}x' - \frac{9}{24}y'$$

$$y = \frac{1}{8}x' + \frac{3}{8}y'$$

To find the point whose image under T is $(-10, 11)$, substitute the x and y values into the inverse of the linear transformation,

$$x = \frac{5}{24}(-10) - \frac{9}{24}(11)$$

$$y = \frac{1}{8}(-10) + \frac{3}{8}(11)$$

$$x = -\frac{149}{24}$$

$$y = \frac{23}{8}$$

Example 2.32

Determine the matrix A of the linear transformation defined by the mapping $(x, y) \rightarrow (x + 2y, 5x + 4y)$ as well as the inverse of A.
Hence, determine the coordinate of the point whose image under A is $(-6, -4)$.

Solution

Matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$

$det\ (A) = |A| = 1(4) - (5)(2)$

$|A| = 4 - 10$

$|A| = -6$

$$A^{-1} = -\frac{1}{6}\begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{5}{6} & -\frac{1}{6} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{5}{6} & -\frac{1}{6} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}x' + \frac{1}{3}y' \\ \frac{5}{6}x' - \frac{1}{6}y' \end{pmatrix}$$

$$x = -\frac{2}{3}x' + \frac{1}{3}y'$$

$$y = \frac{5}{6}x' - \frac{1}{6}y'$$

To find the point whose image under A is $(-6, -4)$, substitute the x and y values into the inverse of the linear transformation.

$$x = -\frac{2}{3}(-6) + \frac{1}{3}(-4)$$

$$y = \frac{5}{6}(-6) - \frac{1}{6}(-4)$$

$$x = \frac{8}{3}$$

$$y = -\frac{13}{3}$$

REVIEW QUESTIONS

1. The 2×2 matrix is given by M= $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$. The 2×2 matrix N satisfies $NM^2 = M$.

a. Find the element of N

b. Write N in linear transformation form.

2. Find the image of $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ under a rotation of 120° anticlockwise about the origin.

3. A transformation in the x-y plane is represented by a 2×2 matrix, T = $\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$.

A quadrilateral ABCD has vertices at the points with coordinates:
$A(20, 6), B(26, 9), C(50, 15)$ and $D(44, 12)$ respectively.

The vertices of quadrilateral ABCD are transformed by T to $A'B'C'D'$

a. Determine the position of the vertices $A'B'C'D'$

b. If $A'B'C'D'$ is also translated by the vector $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ to $A''B''C''D''$, find the vertices of $A''B''C''D''$.

4. The 2×2 matrix is given by B = $\begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix}$, where x and y are constants.

The point (3,1) is mapped by B onto the point with coordinates $(5, 13)$.

a. Determine the value of x and y.

b. Show that $B^2 = 5B + 2I$ where I is the identity element.

5. The matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, give the full geometrical description of the transformation.

6. Find the image of the line with the equation $2x + 3y = 10$ under the transformation represented by the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

7. The 2×2 matrix A represents a rotation by 90^0 anticlockwise about the origin O.
The 2×2 matrix B represents a reflection in the line $y = -x$.

a. Write down the matrices A and B.

b. The 2×2 matrix C represents a rotation by 90° anticlockwise about the origin O, followed by reflection about the straight line with equation $y = -x$.
Find the matrix C.

8. A 3×3 matrix A is defined in terms of the scalar constant k by

$$A = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{pmatrix}.$$

Given that $|A| = 8$, find the possible values of k.

9. Triangle ABC with vertices $A(1,3)\ B(-2,1)$ and $C(0,-2)$.

Find the image having translation vector $t = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

10. Write the following linear transformations in matrix form

a. $x' = x + 2y, y' = -x + y$

b. $x' = 0, y' = x$

11. A soft drink company that manufactures soft drinks are located in Goaso and Damango have its labour/hour and wage requirement for the processing of the drinks as below:

Labour/hour per machine **Hourly wage**

	Bottling	Assembling	Packaging
1 person	6	6	2
2 persons	10	9	3
3 persons	15	12	4

	Goaso	Damango
Bottling	12	6
Assembling	10	13
Packaging	11	16

Discuss the possible interpretation of the elements in the product LH, where L represent labour cost per machine and H represents the wage per hour.

12. The transformation matrix T_1 reflects triangle DEF in the x-axis.

The matrix T_2 enlarges the triangle by a scale factor of -3.

a. Find the matrices T_1 and T_2.

b. Given that the coordinates of the triangle are $D(2,1), E\ (6,1)$ and $F(4,5)$, find the coordinates of the images of D, E and F under the matrices $T_1\ T_2$.

SECTION

3

PARABOLAS



GEOMETRIC REASONING AND MEASUREMENT

Spatial Reasoning

INTRODUCTION

A parabola is a curve with applications in fields such as physics, engineering, and economics. Whether it is the path of a projectile, the shape of a satellite dish, or the beam from a flashlight, parabolas effectively model both natural and man-made phenomena. In this chapter, you will learn about the features of parabolas in detail. You will learn how to sketch and analyse parabolas derived from quadratic functions and geometric definitions. This section will equip you to:

1. Construct a parabola from a given quadratic equation of the form $y = ax^2 + bx + c$ and identify and explain its key features, such as the vertex, axis of symmetry, focus, directrix, and direction of opening.
2. Sketch a parabola, and from its symmetry and shape, derive the general geometric relationship that links a parabola's focus, directrix, and locus definition.
3. Sketch a parabola given the directrix and focus, using the formal geometric definition: a parabola is the set of all points equidistant from a fixed point (the focus) and a fixed line (the directrix). You will learn how to plot this step-by-step and derive its corresponding equation.
4. Derive the equations of the tangent and normal to a parabola at a given point on the curve.
5. Apply differentiation and geometric principles to interpret these lines and understand their relationship with the parabola.

KEY IDEAS

- **Definition of a Parabola:** A parabola is the set of all points in a plane that are equidistant from a fixed point (called the focus) and a fixed line (called the directrix).
- **Key Features of a Parabola**
  - Vertex: The highest or lowest point of the parabola.
  - Axis of Symmetry: A vertical or horizontal line that passes through the vertex and divides the parabola into two mirror-image halves.
  - Focus: A point inside the parabola that defines its shape.
  - Directrix: A fixed line outside the parabola used in its geometric definition.
  - Focal Length: The distance from the vertex to the focus (or to the directrix).

- **Forms of the Parabola**
  - Vertical (opens up or down):

    $(x-h)^2 = 4p(y-k)$

  - Horizontal (opens left or right):

    $(y-k)^2 = 4p(x-h)$

- **Sketching a Parabola:** A parabola can be sketched by
  - Identifying the vertex, axis of symmetry, and direction of opening.
  - Plotting key points (including the focus and directrix if available).
  - Drawing a smooth, symmetric curve.
- **Parabola from Focus and Directrix**
  - Given a focus and directrix, the parabola can be constructed by plotting all points equidistant from them.
  - The equation can then be derived using the distance formula.
- **Tangents and Normals**
  - The tangent to a parabola at a point touches it at exactly one point and has the same slope as the curve at that point.
  - The normal is perpendicular to the tangent and also passes through that point.
  - Both equations can be deduced using calculus or coordinate geometry methods.

DESCRIBING THE SHAPE OF THE GRAPH OF QUADRATIC FUNCTIONS

The general quadratic function is of the form $f(x) = y = ax^2 + bx + c$, where $a, b,$ and c are constants and $a \neq 0$. Also, x and y are variables. The basic form of the quadratic function is $y = x^2$.

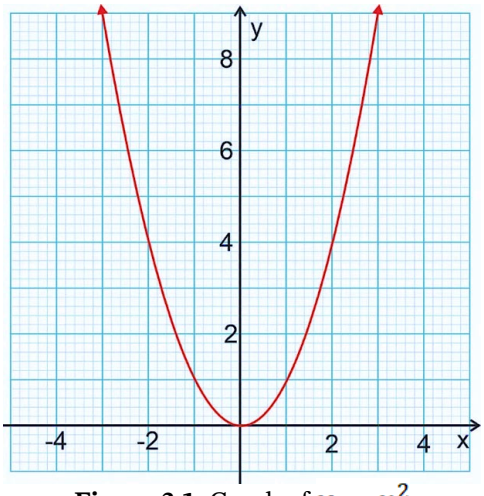


Figure 3.1: Graph of $y = x^2$

Another name for the quadratic function is parabola. A parabola is an example of a conic section. Other examples of conic sections are circles, ellipses, and hyperbolas. These shapes are called conics because they are generated from cones.

A parabola is generated when a plane intersects a cone parallel to the generating line (slant height) as illustrated below.

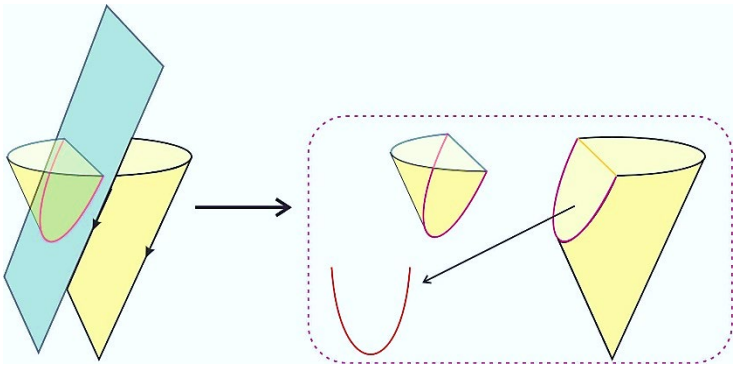


Figure 3.2: Parabola generating process

Activity 3.1 Discussing Real-life examples of Parabolas

1. Examine each image of real-life parabolas provided.
   a. Projectile Motion (Throwing or Launching Objects)

      The path of a ball when kicked or thrown follows a parabola.

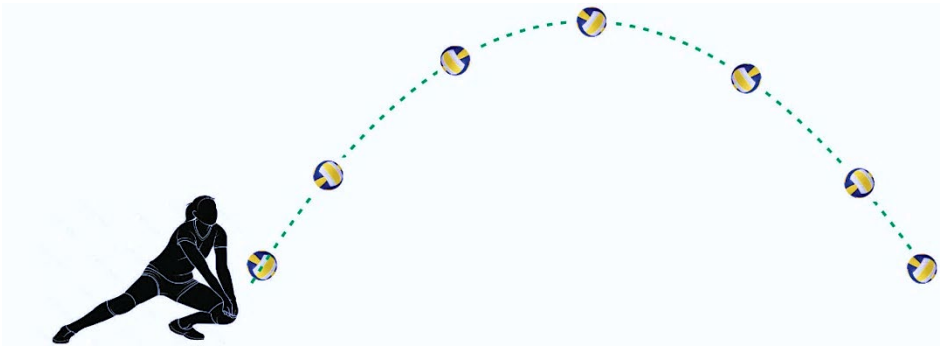


Figure 3.3: Motion of a projected ball

   b. Satellite Dishes

      Satellite dishes are shaped like **paraboloids** (3D parabolas) so they can focus signals to a single point — the receiver. This design ensures maximum signal strength.



Figure 3.4: Satellite dish

c. Car Headlights and Flashlights

The reflector inside is shaped like a parabola so that light rays from the bulb at the focus are reflected in parallel rays. This creates a strong, directed beam.

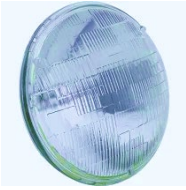


Figure 3.5: An automobile headlight

d. Bridges and Arches

Some suspension bridge cables and arch supports are shaped like parabolas



Figure 3.6: A bridge

e. Water Fountains

Water shooting from a fountain often follows a parabolic trajectory due to gravity.

The shape depends on the angle and force of the water jet.



Figure 3.7: Water fountain

2. Note the vertex position, direction of opening (upward or downward), and any visible symmetry.
3. Record observations in your notebook.
4. Share observations about the images with a peer or your teacher
5. Decide which image represents the clearest example of a parabola and explain why.
6. Discuss with a peer how the vertex and orientation of the parabola influence the object's function.
7. Select one image that interests you the most.

8. Sketch a possible quadratic function that represents it on graph paper or using a digital tool.
9. Label the vertex, axis of symmetry, and orientation.
10. Write a brief description (3–5 sentences) of your sketch, explaining how it relates to the real-life object.
11. Present your sketch and description to the class or group members.
12. Listen to others' presentations and ask at least one question or give one positive observation.
13. Reflect on how understanding parabolas can help in explaining or designing real-life objects.

Shape of a Quadratic Function

The graph of a parabolic function of the form $f(x) = y = ax^2 + bx + c$ is either of the diagrams below. These types of graphs are symmetrical, with the line of symmetry occurring at $x = -\frac{b}{2a}$

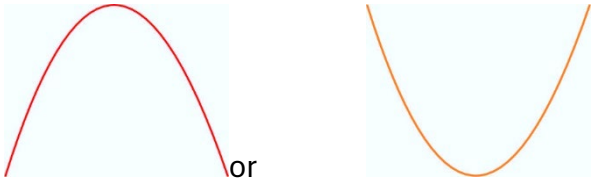


Figure 3.8: Graph of $y = -x^2$ and $y = x^2$

The orientation of the graph of is dependent on the value of a.

If the graph is written in the form $f(x) = a(x-h)^2 + k$ the minimum or maximum value is k and the line of symmetry is $x = h$ or $x - h = 0$.

Also, the vertex of the quadratic graph will be (h, k).

Maximum graph

If $a < 0$, the graph is shaped like the intersection symbol The graph will have a maximum turning point.

The coordinates of the turning point are $\left(-\frac{b}{2a},\ f(-\frac{b}{2a})\right)$ and the maximum value is $f\left(-\frac{b}{2a}\right)$

The line of symmetry is $x = -\frac{b}{2a}$

The graph is increasing when $x < -\frac{b}{2a}$

The graph is decreasing when $x > -\frac{b}{2a}$

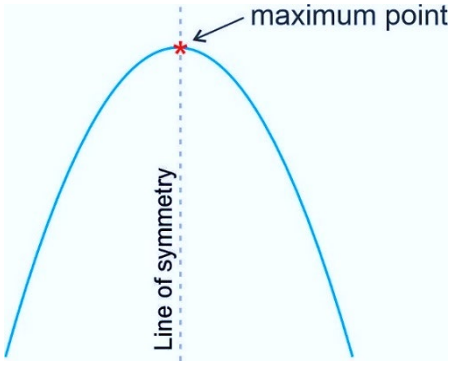


Figure 3.9: A maximum graph

The graph is increasing when $x < -\frac{b}{2a}$ or $\left(-\infty < x < -\frac{b}{2a}\right)$

The graph is decreasing when $x > -\frac{b}{2a}$ or $\left(-\frac{b}{2a} < x < \infty\right)$

The domain of the graph is $(-\infty < x < \infty) = x \epsilon R$

The range is $y > f\left(-\frac{b}{2a}\right)$ or $\left(-\infty < y < f\left(-\frac{b}{2a}\right)\right)$

Minimum graph

If $a > 0$, the graph is shaped like the Union symbol.

The graph will have a minimum turning point.

The coordinates of the turning point is

$\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$ and the minimum value is $f\left(-\frac{b}{2a}\right)$

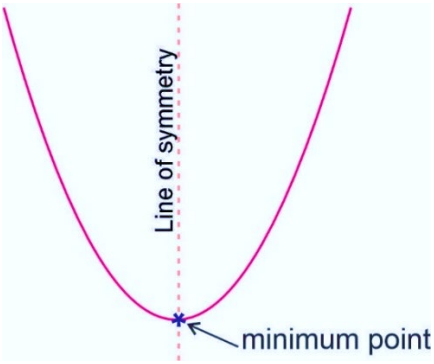


Figure 3.10: A minimum graph

The line of symmetry is $x = -\frac{b}{2a}$

The graph is increasing when $x > -\frac{b}{2a}$ or $\left(-\frac{b}{2a} < x < \infty\right)$

The graph is decreasing when $x < -\frac{b}{2a}$ or $\left(-\infty < x < -\frac{b}{2a}\right)$

The domain of the graph is $(-\infty < x < \infty) = x \epsilon R$

The range is $y > f\left(-\frac{b}{2a}\right)$ or $\left(f\left(-\frac{b}{2a}\right) < y < \infty\right)$

Example 3.1

Figure 3.11 shows the graph of a quadratic function.

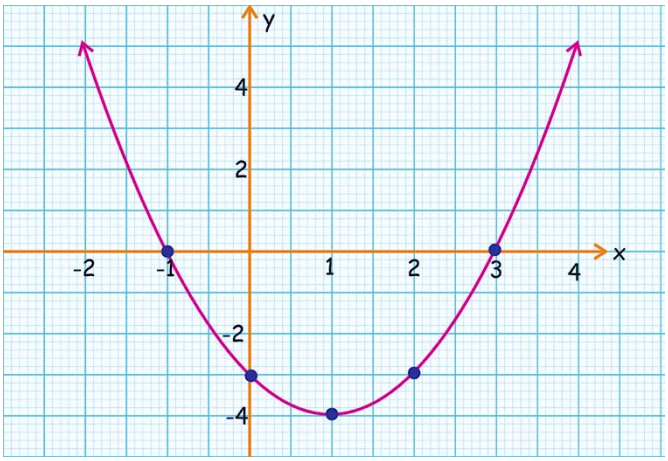


Figure 3.11: A quadratic function

a. Describe the shape of the quadratic function.

b. Write an equation to represent the function.

Solution

a. The graph opens upwards since it is a minimum graph. The equation of the line of symmetry is $x = 1$ or $x - 1 = 0$, the minimum point is (1, -4) and the minimum value is -4.

The graph is increasing when $1 < x < \infty$

The graph is decreasing when $-\infty < x < 1$

The domain of the graph is $-\infty < x < \infty$ or $x \in R$

The range is $-4 < y < \infty$

b. Using $y = a(x - h)^2 + k$ where (h, k) is the turning point and a is a parameter.

From the graph the turning point is $(1, -4)$. This means, $h = 1$ and $k = -4$

$y = a(x - 1)^2 - 4$

To find a, substitute any other ordered pair that satisfies the equation and solve for a.

The curve passes through the point (3, 0)

$0 = a(3 - 1)^2 - 4$

$0 = a(2)^2 - 4$

$4 = 4a$

$a = \frac{4}{4} = 1$

Therefore, $y = 1(x - 1)^2 - 4$

Now, expand and simplify

$y = (x^2 - 2x + 1) - 4$

$y = x^2 - 2x - 3$

Example 3.2

The path of a projectile is represented by the graph below, where H denotes the height (in metres) at time t seconds, and t is the time elapsed since the projectile was launched.

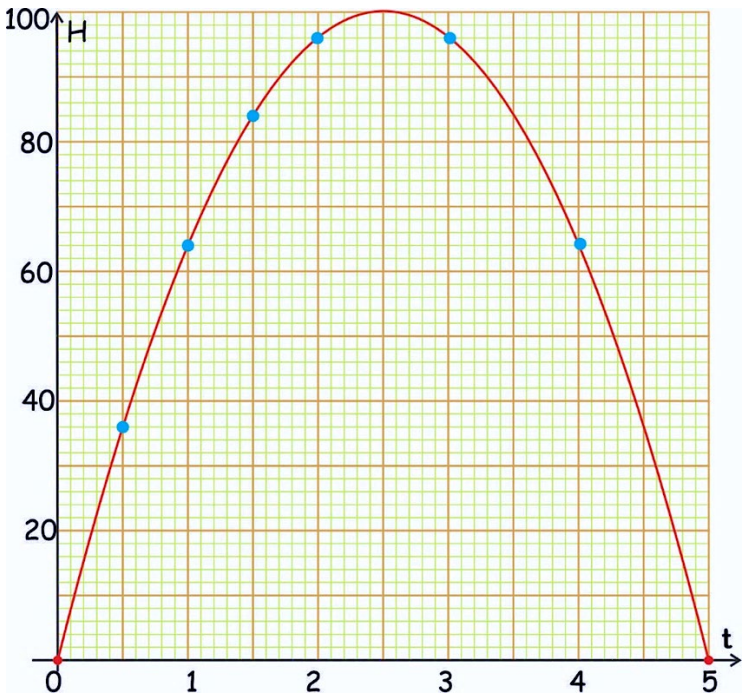


Figure 3.12: Path of a projectile

a. Write a quadratic equation to represent the path of the projectile.

b. Describe the shape of the H in F***igure 3.12***.

Solution

a. Using $H = a(t-h)^2 + k$ where (h, k) is the turning point and a is a parameter.

From the graph, the turning point is (2.5, 100). This means, $h = 2.5$ and $k = 100$

$H = a(t-2.5)^2 + 100$

To find a, substitute any other ordered pair that satisfies the equation, and solve for a.

The curve passes through the origin, (0, 0)

$0 = a(0-2.5)^2 + 100$

$0 = a\left(\frac{5}{2}\right)^2 + 100$

$-100 = \frac{25}{4}a$

$a = -100 \times \frac{4}{25} = -16$

Therefore, $H = -16(t-2.5)^2 + 100$

Now, expand and simplify

$H = -16(t^2 - 5t + 6.25) + 100$

$H = -16t^2 + 80t - 100 + 100$

$H = -16t^2 + 80t$

$H = 80t - 16t^2$

b. The graph opens downwards since it is a minimum graph.

The equation of the line of symmetry is $t = 2.5$ or $2t - 5 = 0$.

The maximum point is (2.5, 100), and the maximum value is 100.

The graph is increasing when $0 < t < 2.5s$

The graph is decreasing when $2.5s < t < 5.0s$

The domain of the graph is $0 < t < 5$

The range is $0 < H < 100$

EXPLORING THE GRAPHS OF QUADRATIC FUNCTIONS WITH DIFFERENT PARAMETERS

Before we start exploring the graphs of quadratic functions, let's simplify the general quadratic equation by completing the square.

The general form of a quadratic function is: n is: $f(x) = ax^2 + bx + c$.

By completing the square, we can write it in the form $f(x) = a(x + h)^2 + k$. The parameters a, h and k determines the shape of the graph of a quadratic function.

Let us now explore how the values of a, h and k affect the shape of the graph.

Graph of the quadratic function when $a = 1$, $h = 0$ and $k = 0$

Substituting $a = 1, h = 0$ and $k = 0$ into $f(x) = a(x + h)^2 + k$,

we get $f(x) = 1(x + 0)^2 + 0$

$f(x) = x^2$, which is the basic quadratic function

This means that the basic equation will have a vertex (0, 0) and a line of symmetry $x = 0$

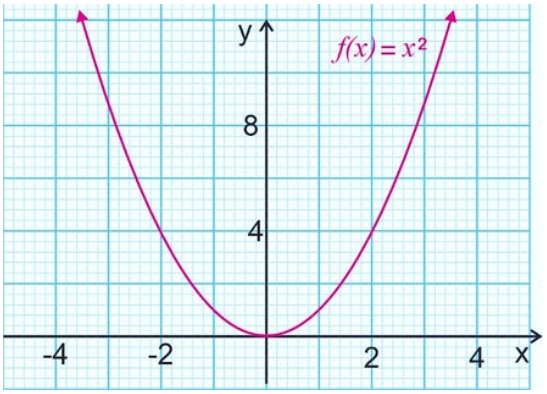


Figure 3.13: Graph of $f(x) = x^2$

The graph of $f(x) = x^2$ has the following properties:

1. It is U-shaped.

2. The turning point is located at the origin.

3. The turning point is a minimum point. $|a| > 1$

Activity 3.2 Graph of the quadratic equation when IaI>1, h = 0 and k = 0

Step 1: Substitute $a = 2, h = 0$ and $k = 0$ into $f(x) = a(x + h)^2 + k$.

Your answer should be $f(x) = 2(x + 0)^2 + 0$

$f(x) = 2x^2$

Step 2: Draw $f(x) = x^2$ and $g(x) = 2x^2$ on the same graph.

Your answer should be similar to

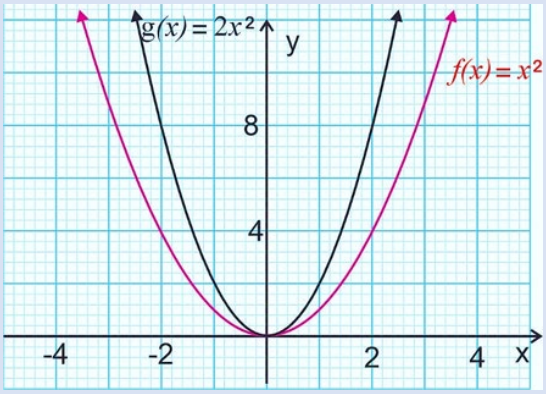


Figure 3.14: Graphs of $f(x) = x^2$ and $g(x) = 2x^2$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = 2x^2$ and state any observations you make.

Your observations should include the following.

1. They both have the same U-shape.
2. Their turning points are both located at the origin.
3. Their turning points are both minimum points.
4. The graph of $f(x) = x^2$ has been '*shrunk*' to obtain the graph of g$(x) = 2x^2$. This is known as a vertical stretch.

Step 4: From your observations in step 3, what conclusion can you make about the parameter a in a quadratic function?

Is your conclusion similar to the one below?

As the absolute value of "a" increases, the graph becomes narrower compared to the basic graph.

If your answer is no, recheck your graphs.

Activity 3.3 The graph of the quadratic equation when $-1 < a < 1$, $h = 0$ and $k = 0$

Step 1: Substitute $a = 0.5$, $h = 0$ and $k = 0$ into $f(x) = a(x + h)^2 + k$.

Your answer should be $f(x) = 0.5(x + 0)^2 + 0$

$f(x) = 0.5x^2$

Step 2: Draw $f(x) = x^2$ and $g(x) = 0.5x^2$ on the same graph.

Your answer should be similar to

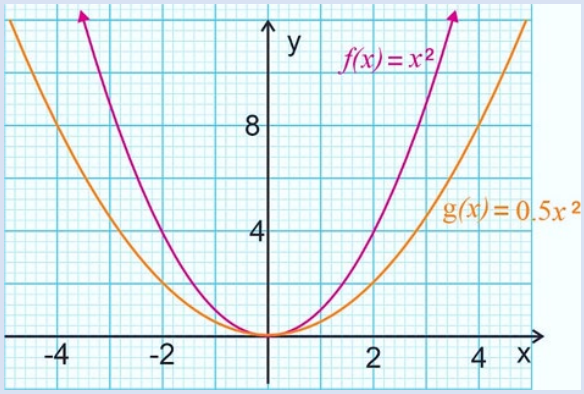


Figure 3.15: Graphs of $f(x) = x^2$ and $g(x) = 0.5x^2$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = 0.5x^2$ and state any observations you make.

Did you observe the following?

1. They are all U-shaped.
2. Their turning points are all located at the origin.
3. Their turning points are all minimum points.
4. The graph of $f(x)$ has been stretched vertically (dilated) to obtain graphs of g (x).
5. The graph of $f(x)$ is narrower compared to $g(x)$

If your answer is no, recheck your graphs.

Step 4: From your observations in step 3, what conclusion can you make about the parameter 'a' in a quadratic function.

Is your conclusion similar to the one below?

As the absolute value of 'a' is less than 1, the graph becomes wider as compared to the basic graph.

If your answer is no, recheck your graphs.

What happens when the values of 'a' is negative in all the situations that have been discussed?

Let us graph them and do comparisons.

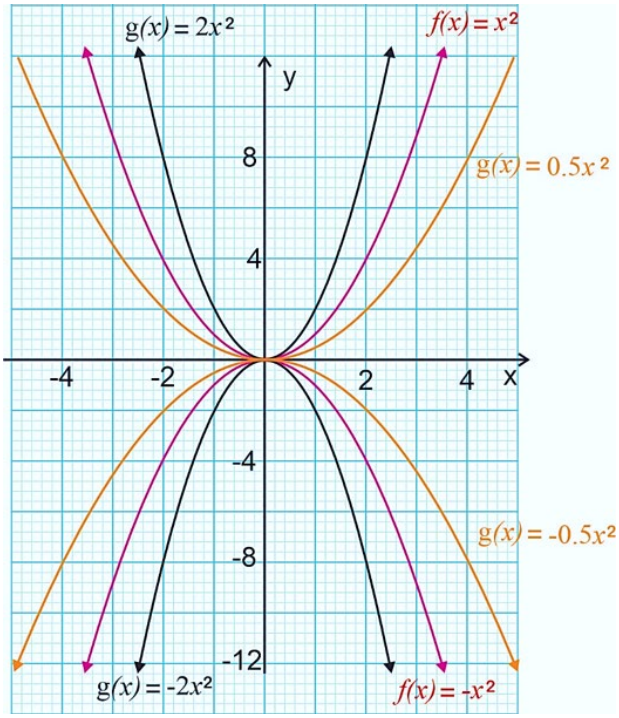


Figure 3.16: Graphs of $f(x)$, $g(x)$ and $h(x)$ and their reflections

It can be observed from Figure 3.16 that when the value of 'a' is negative, the resulting graph is a reflection of the original graph about the x – axis.

The following conclusions can be made from the graphs of Figure 3.16:

1. The greater the value of $|a|$, the narrower the curve.
2. The smaller the value of $|a|$, the wider the curve.
3. If a is negative, the resulting graph is a reflection about the x – axis.

Activity 3.4 Graph of the quadratic function when h is any number other than 0

a. When h is a positive number

Step 1: Substitute $h = 2$, $a = 1$ and $k = 0$ into $f(x) = a(x + h)^2 + k$

Was your answer similar to the one below?

$f(x) = (x + 2)^2$

Step 2: Draw $f(x) = x^2$ and $g(x) = (x + 2)^2$ on the same graph

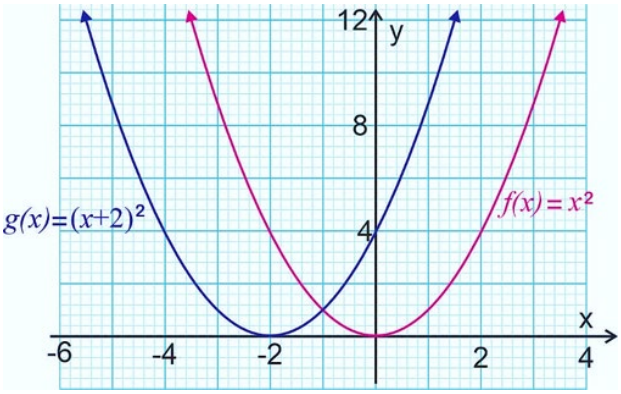


Figure 3.17: Graphs of $f(x) = x^2$ and $g(x) = (x + 2)^2$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = (x + 2)^2$ and state any observations you make.

Did you observe the following?

They are both U-shaped.

While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(-2, 0)$ indicating a horizontal shift of the graph of $f(x)$ two units to the left.

Their turning points are both minimum points.

The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrinkage or dilation (i.e., no stretching); they are merely translated.

If no, recheck your graphs

Step 4: From your observations in step 3, what conclusion can you make about the parameter ***h*** in a quadratic function.

Is your conclusion similar to the one below?
When h *is positive, the graph shifts* h *units to the left or translated by the vector* $\begin{pmatrix} -h \\ 0 \end{pmatrix}$.

If no, recheck your graphs

b. When ***h*** is a negative number

Step 1: Substitute $h = -1$, $a = 1$ and $k = 0$ into $f(x) = a(x + h)^2 + k$

Was your answer similar to the one below?
$f(x) = (x - 1)^2$

Step 2: Draw $f(x) = x^2$ and $g(x) = (x-1)^2$ on the same graph

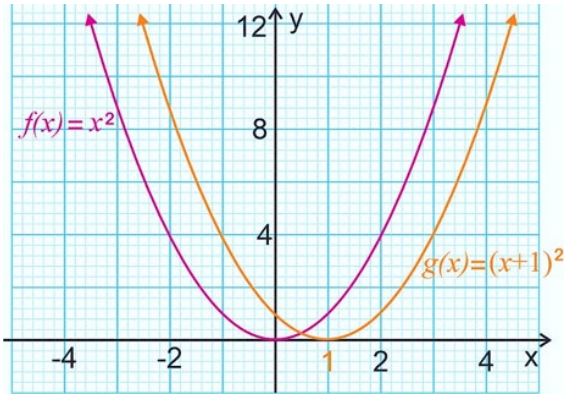


Figure 3.18: Graphs of $f(x) = x^2$ and $g(x) = (x+1)^2$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = (x-1)^2$ and state any observations you make.

Did you observe the following?

They are both U-shaped.

While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(1, 0)$ indicating a horizontal shift of the graph of $f(x)$ one unit to the right.

Their turning points are both minimum points.

The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrinkage or dilation (i.e., no stretching); they are merely translated.

If no, recheck your graphs

Step 4: From your observations in step 3, what conclusion can you make about the parameter h in a quadratic function?

Your conclusion should be similar to the one below:

When h *is negative, the graph shifts* h *units to the right or is translated by the vector*

$\begin{pmatrix} h \\ 0 \end{pmatrix}$.

General observation about the value of h

The following conclusions can be made about the effect of changing the value of h in a function defined as $f(x) = a(x+h)^2 + k$ from the graphs in Figure 3.17 through to Figure 3.18:

1. The value of "h" indicates the distance of horizontal shift of the resulting graph form the graph of $f(x) = x^2$.
2. The greater the value of "$|h|$", the farther the resulting curve is to the graph of $f(x) = x^2$.
3. The smaller the value of "$|h|$", the closer the resulting curve is to the graph of $f(x) = x^2$.
4. When $h < 0$, there is a shift to the right.
5. When $h > 0$, there is a shift to the left.

Activity 3.5 Shape of parabolic graphs when the value of "k" is not 0

When k is a positive number

Step 1: Substitute $k = 4$, $a = 1$ and $h = 0$ into $f(x) = a(x + h)^2 + k$

Your answer should be same as:
$f(x) = x^2 + 4$

Step 2: Draw $f(x) = x^2$ and $g(x) = x^2 + 4$ on the same graph

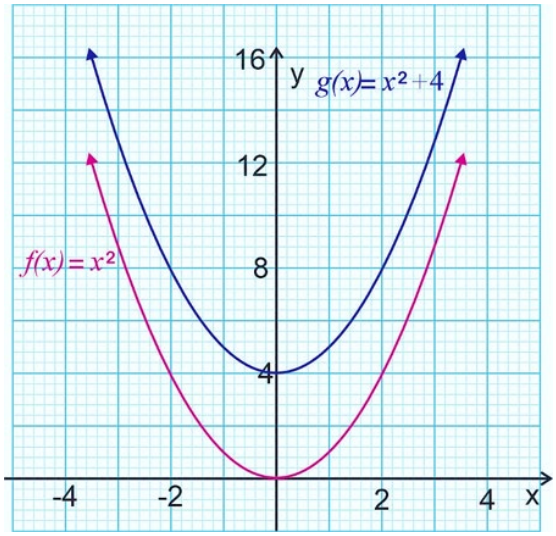


Figure 3.19: Graphs of $f(x) = x^2$ and $g(x) = x^2 + 4$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = x^2 + 4$ and state any observations you make.

Did you observe the following?

1. They are all U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(0, 4)$ indicating a vertical translation of the graph of $f(x)$ 4 units upward

3. Their turning points are both minimum.
4. The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrinkage or dilation; they are simply translations.

If no, recheck your graphs.

Step 4: From your observations in step 3, what conclusion can you make about the parameter k in a quadratic function?

Did you come to the conclusion below?

When k is positive, the graph shifts k units upwards or translated by the vector $\begin{pmatrix}0\\k\end{pmatrix}$.

If no, recheck your graphs.

When k is a negative number

Step 1: Substitute $k = -2$, $a = 1$ and $h = 0$ into $f(x) = a(x+h)^2 + k$

Your answer should be same as:
$f(x) = x^2 - 2$

Step 2: Draw $f(x) = x^2$ and $g(x) = x^2 - 2$ on the same graph

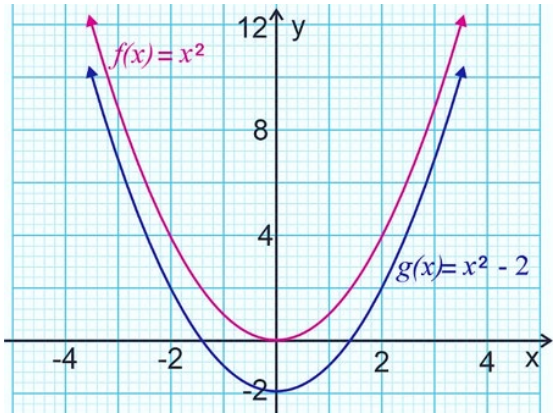


Figure 3.20: Graphs of $f(x) = x^2$ and $g(x) = x^2 - 2$

Step 3: Compare the graph of $f(x) = x^2$ to $g(x) = x^2 - 2$ and state any observations you make.

Did you observe the following?
They are all U-shaped.

While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(0, -2)$ indicating a vertical translation of the graph of $f(x)$, 2 units downwards.
Their turning points are both minimum.

The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrinkage or dilation; they are simply translations.

If no, recheck your graphs.

Step 4: From your observations in Step 3, what conclusion can you make about the parameter k in a quadratic function?

Did you come to the conclusion below?

When k is negative, the graph shifts k units downwards or is translated by the vector $\begin{pmatrix} 0 \\ -k \end{pmatrix}$.
If no, recheck your graphs.

General observation about the value of k

The following conclusions can be made about the effect of changing the value of "k" in a function defined as $f(x) = a(x + h)^2 + k$ from the graphs in Figure 3.19 and Figure 3.20:

1. The value of "$|k|$" indicates the distance of vertical shift of the resulting graph form the graph of $f(x) = x^2$.
2. The greater the value of "$|k|$", the farther the resulting curve is to the graph of $f(x) = x^2$.
3. The smaller the value of "$|k|$", the closer the resulting curve is to the graph of $f(x) = x^2$.
4. When $k < 0$, there is a downward shift.
5. When $k > 0$, there is an upward shift.

Example 3.3

Figure 3.21 shows a graph of a quadratic function.

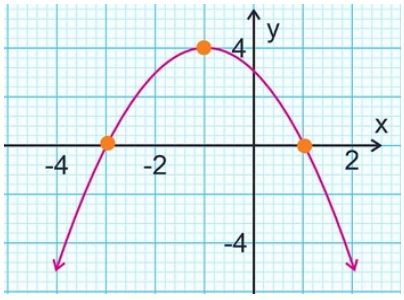


Figure 3.21: Graph of a quadratic function

a. Write the quadratic function in the form $y = a(x-h)^2 + k$ where a, h, and k are parameters.

b. Describe the shape of the graph when compared to $y = x^2$

Solution

a. The turning point is (-1, 4). This implies that $h = -1$ and $k = 4$

$y = a(x+1)^2 + 4$

Since (1, 0) satisfies the graph, substitute it into the equation and solve for a

$0 = a(1+1)^2 + 4$

$0 = 4a + 4$

$-4 = 4aa = -\frac{4}{4} = -1$

This means that $y = -(x+1)^2 + 4$.

b. The graph of $y = -(x+1)^2 + 4$ results from reflecting the graph of $y = x^2$ in the y-axis, followed by shifting it one unit to the left and 4 units upwards, or a translation by the vector $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

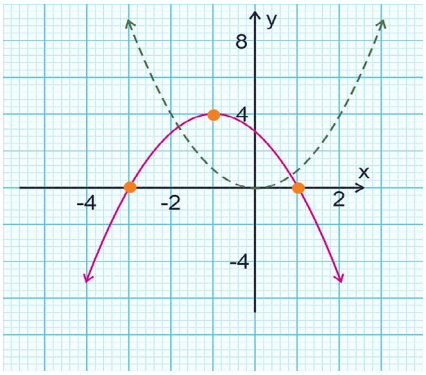


Figure 3.22: Graphs of $y = -(x+1)^2 + 4$ and $y = x^2$

Example 3.4

The diagram below shows a graph of a quadratic function.

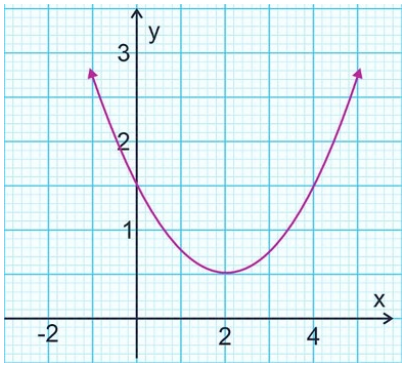


Figure 3.23: Graphs of quadratic function

a. Write the quadratic function in the form $y = a(x-h)^2 + k$ where a, h and k are parameters.

b. Describe the shape of the graph as compared to $y = x^2$

Solution

The turning point is (2, 0.5). This implies that $h = 2$ and $k = 0.5$

$y = a(x-2)^2 + 0.5$

Since (0, 1.5) satisfies the graph, substitute it into the equation and solve for a

$1.5 = a(0-2)^2 + 0.5$

$1.5 - 0.5 = 4a$

$1 = 4aa = \frac{1}{4} = 0.25$

This means that $y = \frac{1}{4}(x-2)^2 + 0.5$

a. The graph of $y = \frac{1}{4}(x-2)^2 + 0.5$ results from shifting the graph of $y = x^2$ two units to the right and 0.5 units upwards, or a translation by the vector $\begin{pmatrix} 2 \\ 0.5 \end{pmatrix}$.

Also, to obtain the graph of

$y = \frac{1}{4}(x-2)^2 + 0.5$, the graph of $y = x^2$ is widened.

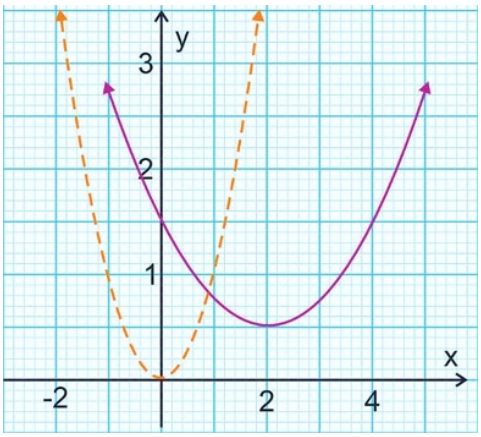


Figure 3.24: Graphs of $y = \frac{1}{4}(x-2)^2 + 0.5$ and $y = x^2$

SKETCHING QUADRATIC GRAPHS

Step 1: Express the function in the form $f(x) = a(x-h)^2 + k$.

When expressed in this form, (h, k) is the turning point or vertex. If a>0, the curve has a minimum and opens upwards. If a<0, the curve has a maximum and opens downwards.

Step 2: Sketch the function by comparing it to $f(x) = x^2$

Alternatively,

Step 1: Determine the shape. Either it is opening upwards, or opening downwards.

Step 2: Find the vertex or the turning point. This is given as $\left(-\frac{b}{2a},\ f(-\frac{b}{2a})\right)$

Step 3: Find the x-intercept(s), if any. For x-intercept put $f(x) = y = 0$.

Step 4: Find the y-intercept. To find the y-intercept, put $x = 0$

Step 5: Connect these points to form the curve.

Example 3.5

Sketch the graph $y = x^2 + 6x + 5$ and compare your graph to $y = x^2$.

Solution

Step 1: Write $y = x^2 + 6x + 5$ in the form $y = a(x-h)^2 + k$

$$y = x^2 + 6x + 5$$
$$y = x^2 + 3x + 3x + 9 - 9 + 5$$
$$y = x(x+3) + 3(x+3) - 4$$
$$y = (x+3)^2 - 4$$

Comparing to $y = a(x-h)^2 + k$, we have $a = 1, h = -3$ and $k = -4$

This means that the turning point is $(-3, -4)$. Since a is positive, the graph opens upwards (minimum graph).

This is a translation of the graph of $y = x^2$ by moving 3 units to the left and 4 units down, or the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$.

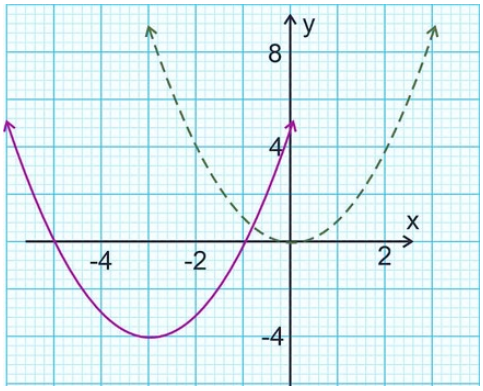


Figure 3.25: Graphs of $y = (x+3)^2 - 4$ and $y = x^2$

Example 3.6

Sketch the graph $y = 7 - 4x - 2x^2$ and compare your graph to $y = x^2$.

Solution

First, write $y = 7 - 4x - 2x^2$ in the form $y = a(x - h)^2 + k$

$y = -2x^2 - 4x + 7$ (factor -2 by dividing each term by -2)

$= -2(x^2 + 2x - 3.5)$ (complete the square)

$= -2(x^2 + x + x + 1 - 1 - 3.5)$

$= -2[(x + 1)^2 - 4.5]$

$= -2(x + 1)^2 + 9$

Comparing $y = -2(x + 1)^2 + 9$ to $y = a(x + h)^2 + k$ shows that $a = -2, h = -1$ and $k = 9$. This means that (-1, 9) is the turning point. Since a is negative, the graph is maximum. So, the graph will open downwards.

The graph of $y = -2(x + 1)^2 + 9$ is a reflection of $y = x^2$ in the x-axis, followed by a shift of 1 unit to the left and 9 units up, or a translation by the vector $v = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$.

Also, since |a|=2, the graph of $y = -2(x + 1)^2 + 9$ will be a vertical shrink of $y = x^2$.

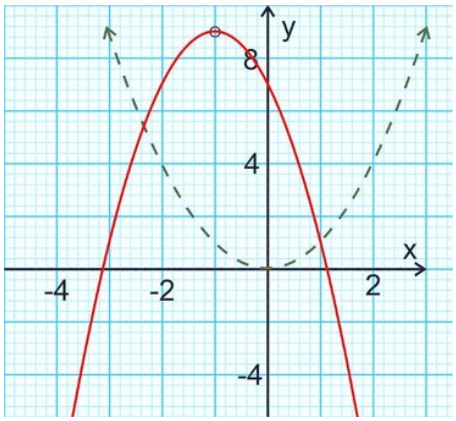


Figure 3.26: Graphs of $y = x^2$ and $y = -2x^2 - 4x + 7$

Example 3.7

Describe and sketch the graph of $y = -2x^2 + 4$.

Solution

$y = -2x^2 + 4$ is the same as $y = -2(x - 0)^2 + 4$

Compared to $y = a(x + h)^2 + k$ shows that $a = -2, h = 0$ and $k = 4$

The vertex is at (0, 4) and the line of symmetry is x=0

To obtain the graph of $y = -2x^2 + 4$, the graph of $y = x^2$ is shrunk (vertically stretched), reflected about the $x-$ axis and shifted upward by four units.

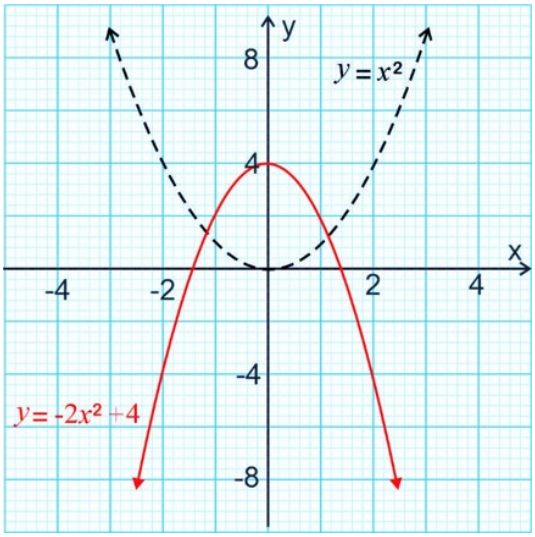


Figure 3.27: Graph of $y = -2x^2 + 4$ and $y = x^2$

DESCRIBING A GIVEN LOCUS AS A PARABOLA

Earlier, you learned that the graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

Definition of Parabola

A parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

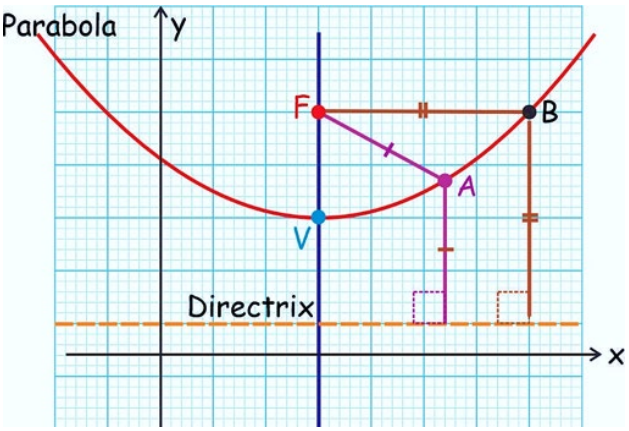


Figure 3.27: A parabola as a locus of points

In ***Figure 3.27***, the red curve illustrates a parabola. Points V, A, and B lie on the curve, where V represents the vertex or turning point of the parabola. The point labelled F is the focus, and the dotted horizontal line indicates the directrix.

By definition, a parabola is the set of all points that are equidistant from a fixed point (the focus) and a fixed line (the directrix). This means that, for any point chosen on the parabola, the distance to the focus is exactly equal to its perpendicular distance to the directrix.

So, $|AF|$ = distance between A and the directrix.

Likewise, |BF| = perpendicular distance between B and the directrix.

Also, $|FV|$ = distance between V and the directrix.

The Focus

The focus is a fixed point inside the parabola. Every point on the parabola is equidistant from the focus and a fixed line called the directrix. The focus lies on the axis of symmetry. It helps determine the shape and direction of the parabola. The vertex is always midway between the focus and the directrix.

Directrix of a Parabola

The directrix of a parabola is a fixed straight line used in the definition of the parabola.

Every point on the parabola is equidistant from a fixed point called the focus and the directrix. It lies opposite the focus, on the other side of the vertex. The vertex of the parabola is exactly halfway between the focus and the directrix. It helps determine the shape and orientation of the parabola.

Line of symmetry of a Parabola

Also called the axis of symmetry, it is a vertical or horizontal line that divides the parabola into two mirror-image halves and passes through the vertex and the focus.

Focal Length of a Parabola (p)

This is the distance between the vertex and the focus (or the vertex and the directrix, since they are equidistant). It tells us how "wide" or "narrow" the parabola is. The larger the focal length, the wider the parabola.

Vertex

This is the turning point or the point where the parabola changes direction. It is located halfway between the focus and the directrix and lies on the axis of symmetry.

Opening direction

A parabola can open in different directions depending on its equation

A parabola of the form $y = ax^2$	**A parabola of the form** $x = ay^2$
Opens upward if $a > 0$	Opens right if $a > 0$
Opens downward if $a < 0$	Opens left if $a < 0$

DESCRIBING THE FEATURES OF A PARABOLA

A parabola is a U-shaped curve that can open up, down, left, or right. It has specific key features that help describe its shape, position, and orientation. These key features are the focus, line of symmetry, focal length, vertex, and opening direction. We can also use these key features to sketch the parabola.

Let us use the activity below to generate the parabolic equation with vertex (0,0), focal length p, and a directrix which is parallel to the $x-axis$.

Activity 3.6 Generating the parabolic equation with vertex (0, 0), focal length p, and a directrix which is parallel to the x-axis.

Step 1: Make a copy of the graph below

The vertex is at the origin (0, 0). The focal length is p, the coordinates of the focus will be $F(0,p)$ and the equation of the directrix will intersect the y-axis at $-p$, R (x,y) is any point on the parabola. So, a perpendicular from R will intersect the directrix at A $(x,-p)$

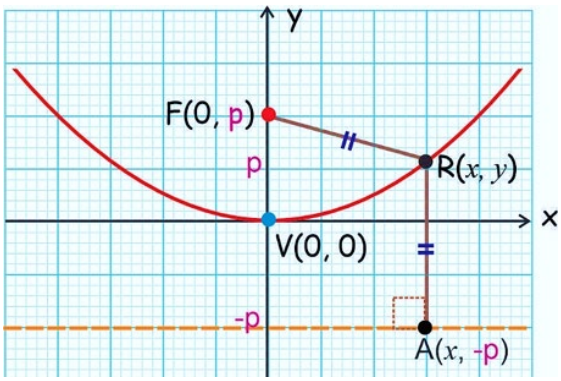


Figure 3.28: A parabola with vertex $(0,0)$ and focus $(0,p)$

Step 2: Find the distance between points R and A

Your answer should be same as:

$A(x,-p)$ and $R(x,y)$

So, $$|RA| = \sqrt{(y-(-p))^2 + (x-x)^2}$$

$$= \sqrt{(y+p)^2}$$

Step 3: Find the distance between points R and F

Your answer should be same as:

$F(0,p)$ and $R(x,y)$

So, $$|RF| = \sqrt{(y-p)^2 + (x-0)^2}$$

$$= \sqrt{(y-p)^2 + x^2}$$

Step 4: Equate |RF| to |RA| and simplify the resulting equation.

Your answer should be similar to:

$\sqrt{(y-p)^2+x^2} = \sqrt{(y+p)^2}$ Take square of both sides

$(y-p)^2+x^2 = (y+p)^2$ Let x^2 stand alone.

$x^2 = (y+p)^2 - (y-p)^2$ Expand RHS using difference of two squares

$x^2 = (y+p-y+p)(y+p+y-p)$ Simplify RHS

$x^2 = (2p)(2y)$

$x^2 = 4py$ or $4py = x^2$

This gives the basic equation of the parabola where the directrix is parallel to the x-axis.

Example 3.8

Describe the features of the parabola $y = x^2$

Solution

$y = x^2$ is the basic quadratic equation

$y = x^2$ ..... (1)

$4py = x^2$

making y the subject,

we have $y = \frac{1}{4p}x^2$ ..... (2)

Comparing equation 1 to equation 2, we have

$1 = \frac{1}{4p}$

$4p = 1$

$p = \frac{1}{4} = 0.25$

This means that the focal length of this parabola is 0.25,

The directrix is $y = -0.25$

The focus is (0, 0.25)

Figure 3.29 shows the graph of $y = x^2$

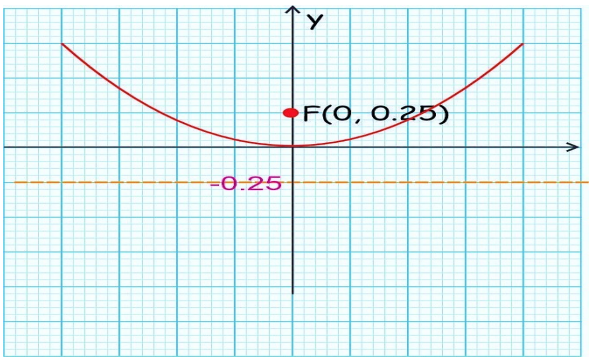


Figure 3.29: Features of parabola $y = x^2$

Example 3.9

Describe the features of the parabola $y = -x^2$

Solution

$y = -x^2$ ..... (1)

$4py = x^2$

making y the subject, we have

$y = -\frac{1}{4p}x^2$ ..... (2)

Comparing equation 1 to equation 2, we have $1 = -\frac{1}{4p}$

$4p = -1$

$p = -\frac{1}{4} = -0.25$

This means that the focal length of this parabola is 0.25,
The directrix is $y = 0.25$
The focus is $(0, -0.25)$

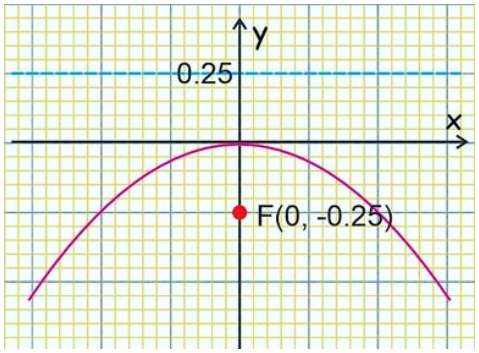


Figure 3.30: Features of parabola $y = -x^2$

Let us use the activity below to generate the parabolic equation with vertex (0, 0), focal length $\boldsymbol{p}$, and a directrix which is parallel to the y-axis.

Activity 3.7 Generating the parabolic equation with vertex (0, 0), focal length P, and a directrix which is parallel to the y-axis.

Step 1: Make a copy of the graph below

The vertex is at the origin (0,0).

The focal length is p, the coordinates of the focus will be F$(p, 0)$ and the equation of the

directrix will intersect the x-axis at

$-p, R(x, y)$ is any point on the parabola. So, a perpendicular

from R will intersect the directrix at A

$(-p, x)$

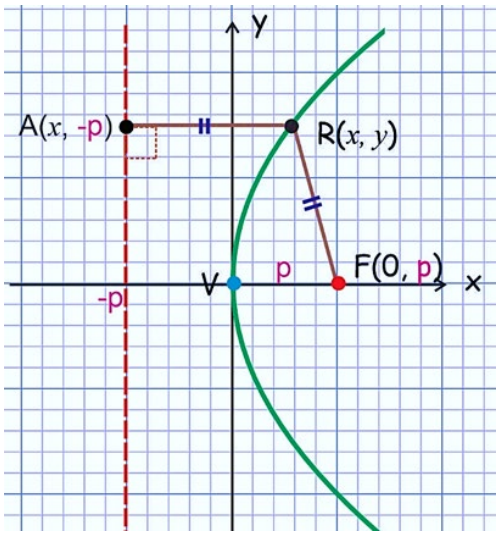


Figure 3.31: Graphs of parabola $x = y^2$

Step 2: Find the distance between points R and A.

Your answer should be the same as:

$A(-p, y)$ and $R(x, y)$

So, $|RA| = \sqrt{\left(x-(-p)\right)^2 + (y-y)^2}$

$= \sqrt{(x+p)^2}$

Step 3: Find the distance between points R and F

Is your result similar to the one below? If no, recalculate.

$F(p, 0)$ and $R(x, y)$

So, $|RF| = \sqrt{(x-p)^2 + (y-0)^2}$

$= \sqrt{(x-p)^2 + y^2}$

Step 4: Equate |RF| to |RA| and simplify the resulting equation.

Is your result similar to the one below? If no, recalculate.

$\sqrt{(x-p)^2 + y^2} = \sqrt{(x+p)^2}$	Take squares of both sides
$(x-p)^2 + y^2 = (x+p)^2$	Let x^2 stand alone.
$y^2 = (x+p)^2 - (x-p)^2$	Expand RHS using difference of two squares
$y^2 = (x+p-x+p)(x+p+x-p)$	Simplify RHS
$y^2 = (2p)(2x)$	
$y^2 = 4px$ or $4px = y^2$	

This gives the basic equation of the parabola where the directrix is parallel to the y-axis.

Example 3.10

Describe the features of the parabola $x = y^2$

Solution

$x = y^2$ ..... (1)

$4px = y^2$

making x the subject, we have $x = \frac{1}{4p}y^2$ ..... (2)

Comparing equation 1 to equation 2,

we have $1 = \frac{1}{4p}$

$4p = 1$

$p = \frac{1}{4} = 0.25$

This means that the focal length of this parabola is 0.25,

The directrix is $x = -0.25$

The focus is (0.25, 0)

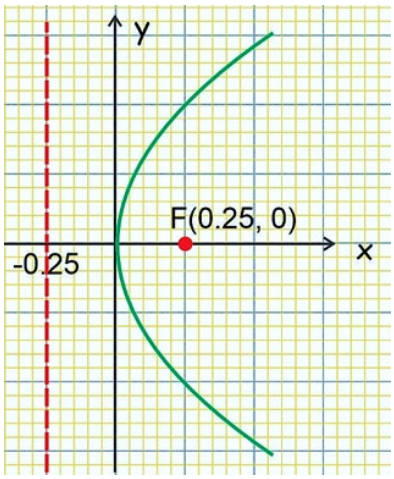


Figure 3.32: Features of parabola x= y^2

Example 3.11

Describe the features of the parabola $y = -x^2$

Solution

$x = -y^2$ ..... (1)

$4px = y^2$

making x the subject, we have $x = -\frac{1}{4p}y^2$ ..... (2)

Comparing equation 1 to equation 2, we have $1 = -\frac{1}{4p}$

$4p = -1$

$p = -\frac{1}{4} = -0.25$

This means that the focal length of this parabola is 0.25,

The directrix is $x = 0.25$

The focus is $(-0.25, 0)$

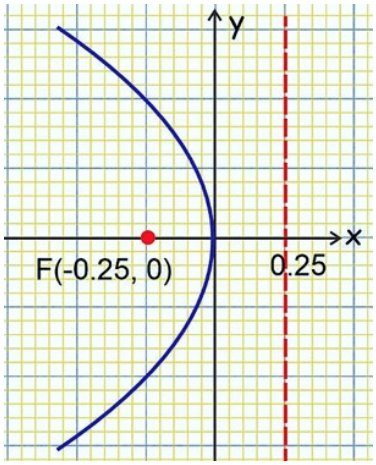


Figure 3.33: Features of parabola $x = -y^2$

Generally, for a parabolic equation of the form $y = a(x + h)^2 + k$:

The vertex is located at $V(-h, k)$

The Focus is located at $F\big(-h, (k + p)\big)$

The focal length is $p = \frac{1}{4a}$.

The equation of the axis of symmetry is $x = -h$

The equation of the directrix is $y = k - p$ as illustrated in Figure 3.34.

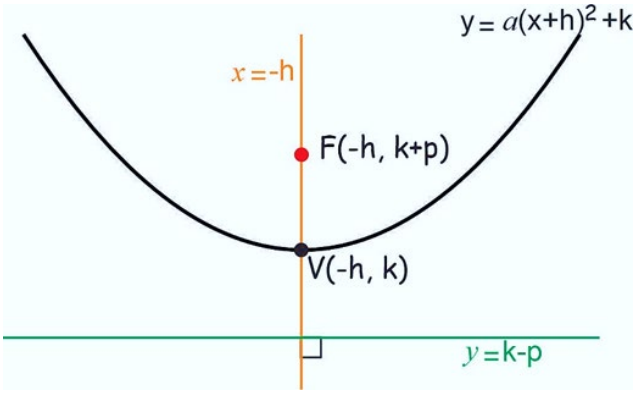


Figure 3.34: Features of a parabola with equation $y = a(x + h)^2 + k$

Generally, for a parabolic equation of the form $x = a(y + h)^2 + k$:

The vertex is located at $V(k, -h)$

The Focus is located at $F(k + p, -h)$

The focal length is $p = \frac{1}{4a}$.

The equation of the axis of symmetry is $y = -h$

The equation of the directrix is $x = k - p$ as illustrated in Figure 3.35

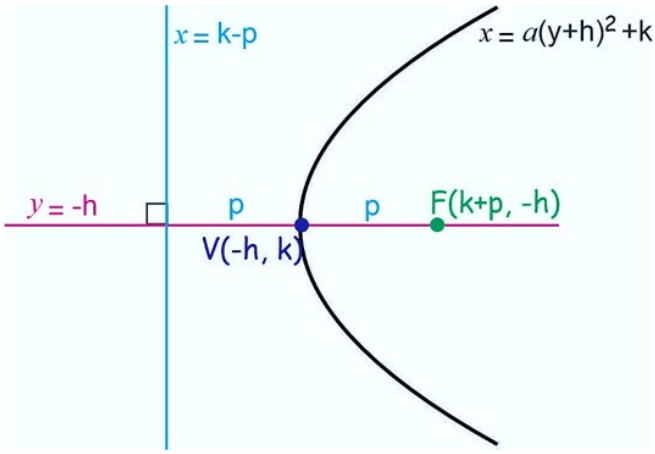


Figure 3.35: Features of a parabola with equation $x = a(y + h)^2 + k$

Example 3.12

Sketch the curve $x = 2y^2$.

Solution

Comparing $x = 2y^2$ and $x = a(y+k)^2 + h, a = 2, h = 0, k = 0$ and

$p = \frac{1}{4a}$

$p = \frac{1}{4(2)}$

$p = 0.125$

The vertex is located at $(0, 0)$

The focus is located at $F(0.125, 0)$

The directrix has equation: $x = -0.125$

The axis of symmetry has equation $y = 0$

The curve opens towards the right i.e., positive $x-$ axis

The curve is also obtained from a shrink of the curve of $x = y^2$

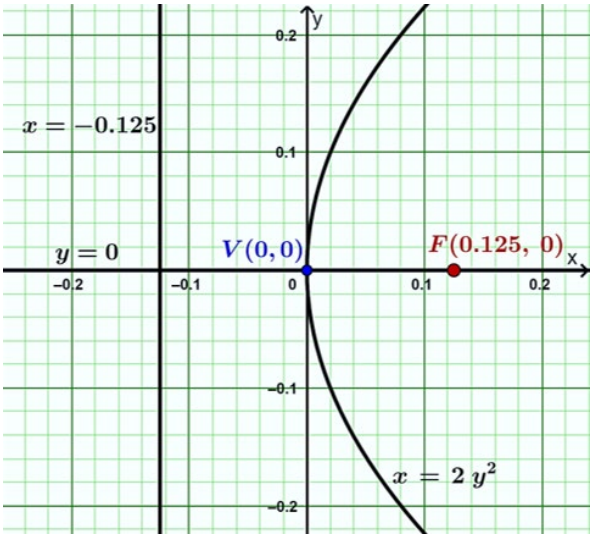


Figure 3.36: Graph of $x = 2y^2$

Example 3.13

Sketch the curve $x = -3y^2$.

Solution

Comparing $x = -3y^2$ and $x = A(y+B)^2 + C, A = -3, B = 0, C = 0$ and

$p = -\frac{1}{12}$

$p \approx -0.08$

The vertex is located at $V(0, 0)$

The focus is located at $F(-0.08, 0)$

The directrix has equation: $x = 0.08$

The axis of symmetry has equation $y = 0$

The curve opens towards the left i.e., negative $x-$ axis

The curve is also obtained from a shrink of the curve of $x = y^2$

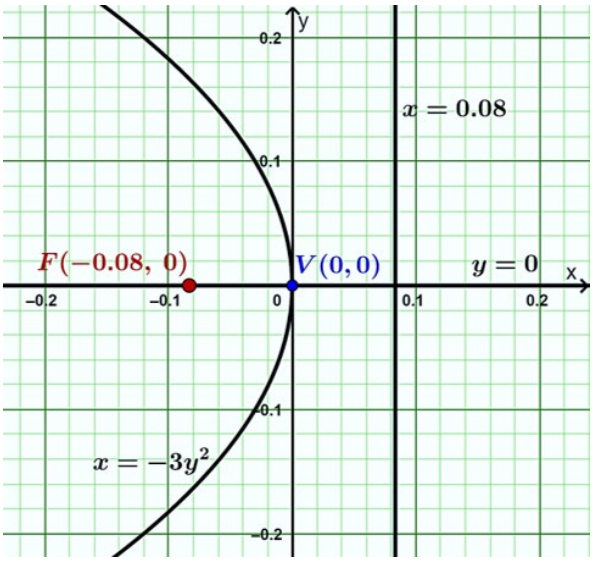


Figure 3.37: Graph of $x = -3y^2$

Example 3.14

Sketch the curve $y = 2x^2 - 12x + 22$.

Solution

$y = 2x^2 - 12x + 22$

$y = 2(x-3)^2 + 4$

Comparing $y = 2(x-3)^2 + 4$ and $y = a(x+h)^2 + k, a = 2, h = -3, k = 4$ and $p \approx 0.13$

The vertex is located at $V = (-h, k) = (3, 4)$

The focus is located at $F = (-h, k+p) = (3, 4.13)$

The directrix has equation: $y = k - p = 3.87$

The axis of symmetry has equation $x = 3$

The curve opens upward

The curve is also obtained from a shrink of the curve of $y = x^2$

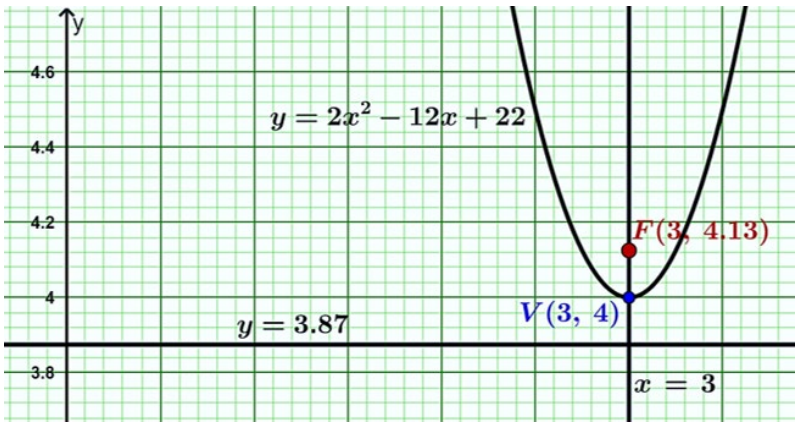


Figure 3.38: Graph of $y = 2x^2 - 12x + 22$

STANDARD EQUATION OF A PARABOLA

Earlier we derived the standard equation of a parabola with its vertex at the origin. In this lesson, we will build on that knowledge to derive the standard equation of a parabola whose vertex is located at a point other than the origin.

Deriving the standard equation of a parabola with a vertex not at the origin

1. If the vertex of the parabola is not at the origin, then it is translated.
2. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k).
3. This translation will cause x to be replaced with $(x - h)$ and y replaced with $(y - k)$ in the standard form of the equation.

Recall that:

A parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

So, for any chosen point (x, y) on the parabola, the distance between our chosen point on the parabola and the focus is equal to the perpendicular distance between our chosen point and the directrix.

Let us use the activity below to derive the equation.

We will first consider the situation where the directrix is parallel to the x − axis

Activity 3.8 Deriving the standard equation of a parabola with directrix parallel to the X - axis

Step 1: With the aid of a graph sheet, make a copy of the diagram below

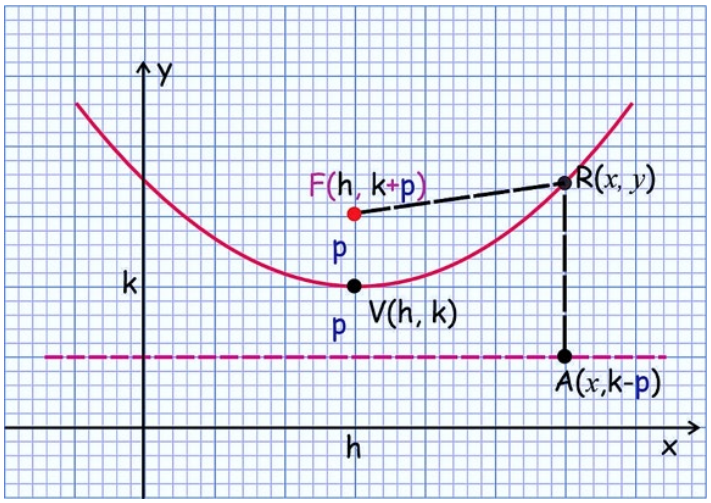


Figure 3.39: Features of a parabola with equation $y = a(x + h)^2 + k$

In Figure 3.39, R(x, y) is any point that satisfies the parabola or any point on the parabola,

(h, k) is the vertex or turning point, and P is the distance between the Focus and the vertex.

By definition, this means that the distance between the directrix and the vertex is also P, and the equation of the directrix is $y = k - p$.

Also, a perpendicular from point R(x, y) will intersect the directrix at point A

$(x, k - p)$ and the coordinates of the focus will be F$(h, k + p)$

Step 2: Calculate $|RF|$

Is your result similar to the one below? If no, recalculate.

$$|RF| = \sqrt{(x - h)^2 + [y - (k + p)]^2}$$

Step 3: Calculate $|RA|$

Is your result similar to the one below? If no, recalculate.

$$|RA| = \sqrt{(x - x)^2 + [y - (k - p)]^2}$$

Step 4: Equate |RF| to |RA| and simplify the resulting equation.

$\Rightarrow |RF| = |RA|$

$\Rightarrow \sqrt{(x-h)^2 + [y-(k+p)]^2} = \sqrt{(x-x)^2 + [y-(k-p)]^2}$

Remove square roots by squaring both sides of the equation.

$\Rightarrow (x-h)^2 + [y-k-p]^2 = (x-x)^2 + [y-k+p]^2$

Make $(x-h)^2$ stand alone

$(x-h)^2 = (0)^2 + [y-k+p]^2 - [y-k-p]^2$ Use the difference of two squares strategy to evaluate the RHS of the equation

$(x-h)^2 = [(y-k+p) + (y-k-p)][(y-k+p) - (y-k-p)]$

$(x-h)^2 = [2y-2k][2p]$

$(x-h)^2 = 2(y-k) \times 2p$

$(x-h)^2 = 4p(y-k)$

Therefore, the general equation of the parabola with vertex (h, k) and focal length p is given by; $(x-h)^2 = 4p(y-k)$

When the vertex is at the origin (0, 0), the equation becomes

$(x-0)^2 = 4p(y-0)$ which simplifies to

$x^2 = 4py$

Next, let us consider the situation where the directrix is parallel to the y-axis

Activity 3.9 Deriving the standard equation of a parabola with directrix parallel to the y - axis

Step 1: With the aid of a graph sheet, make a copy of the diagram below

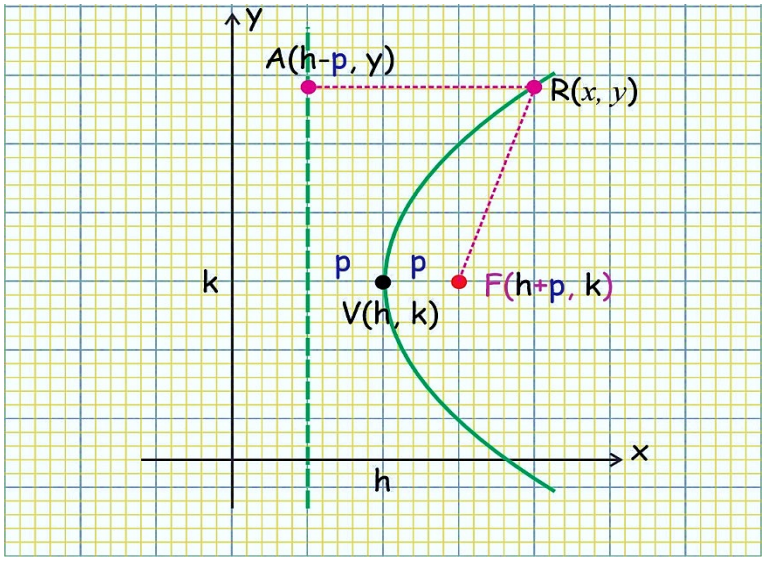


Figure 3.40: Features of a parabola with equation $x = a(y+h)^2 + k$

In Figure 3.40, R(x, y) is any point that satisfies the parabola or any point on the parabola? (h, k) is the vertex or turning point and P is the distance between the Focus and the vertex.

By definition, this means that the distance between the directrix and the vertex is also P, and the equation of the directrix is $x = h - p$.

Also, a perpendicular from point $R(x, y)$ will intersect the directrix at point A

$(h - p, y)$ and the coordinates of the focus will be F$(h + p, k)$

Step 2: Calculate |RF|

Is your result similar to the one below? If no, recalculate.

$$|RF| = \sqrt{(y-h)^2 + [x-(k+p)]^2}$$

Step 3: Calculate |RA|

Is your result similar to the one below? If no, recalculate.

$$|RA| = \sqrt{(y-y)^2 + [x-(k-p)]^2}$$

Step 4: Equate |RF| to |RA| and simplify the resulting equation.

$$\Rightarrow |RF| = |RA|$$

$$\Rightarrow \sqrt{(y-k)^2 + [x-(h+p)]^2} = \sqrt{(y-y)^2 + [x-(h-p)]^2}$$

Remove square roots by squaring both sides of the equation.

$$\Rightarrow (y-k)^2 + [x-h-p]^2 = (y-y)^2 + [x-h+p]^2$$

Make $(y-k)^2$ stand alone

$$(y-k)^2 = (0)^2 + [x-h+p]^2 - [x-h-p]^2$$ Use the difference of two squares strategy to evaluate the RHS of the equation

$$(y-k)^2 = [(x-h+p) + (x-h-p)][(x-h+p) - (x-h-p)]$$

$$(y-k)^2 = [2x-2h][2p]$$

$$(y-k)^2 = 2(x-h) \times 2p$$

$$(y-k)^2 = 4p(x-h)$$

Therefore, the general equation of the parabola with vertex (h, k) and focal length p is given by: $(y-k)^2 = 4p(x-h)$

Example 3.15

Find the equation of a parabola with a focus at $(1, 2)$ and vertex at $(-2, 2)$.

Solution

Vertex $= (-2, 2)$ Focus $= (1, 2)$

The y-coordinates of the vertex and the focus are the same.

This shows that the line of symmetry is $y = 2$

Also, the equation of the parabola is given by $(y - k)^2 = 4p(x - h)$

Substituting the vertex, we have

$$(y - 2)^2 = 4p(x - (-2))$$

$$\Rightarrow (y - 2)^2 = 4p(x + 2)$$

The focal length, p, is the distance between the focus and the vertex.

$$p = \sqrt{(2 - 2)^2 + (-2 - 1)^2}$$

$$= \sqrt{3^2}$$

$$= 3$$

The required equation is

$(y - 2)^2 = 4 \times 3(x + 2)(y - 2)^2 = 12(x + 2)$

Example 3.16

Find the equation of a parabola with a focus at $(4, -3)$ and vertex at $(4, 2)$.

Solution

Vertex $= (4, 2)$ Focus $= (4, -3)$

The x-coordinates of the vertex and the focus are the same.

This shows that the line of symmetry is $x = 4$

Also, the equation of the parabola is given by $(x - h)^2 = 4p(y - k)$

substituting the vertex, we have

$$(x - 4)^2 = 4p(y - 2)$$

$$\Rightarrow (x - 4)^2 = 4p(y - 2)$$

The focal length, p, is the distance between the focus and the vertex.

$$p = \sqrt{(-3 - 2)^2 + (4 - 4)^2}$$

$$= \sqrt{(-5)^2}$$

$$= \sqrt{-5 \times -5} = \sqrt{25} = 5$$

The required equation is

$(x - 4)^2 = 4 \times 5(y - 2)(x - 4)^2 = 20(y - 2)$

DEDUCING THE DIRECTRIX AND FOCUS FOR A PARABOLIC EQUATION AND VICE VERSA

A parabola can be described algebraically or geometrically.

Algebraically, it is written either as:

$y = a(x-h)^2 + k$ (opens up or down)

OR $x = a(y-k)^2 + h$ (opens right or left)

Geometrically, it is written as:

$(x-h)^2 = 4p(y-k)$ (opens up or down)

OR $(y-h)^2 = 4p(x-k)$ (opens left or right)

Understanding the relationship between its equation, focus, and directrix is essential to analysing and graphing parabolas.

How to Deduce the Focus and Directrix from the Equation

Step 1: Compare it to the standard equation

Step 2: Deduce the focus and the directrix from the standard equation.

- If it is of the form $(x-h)^2 = 4p(y-k)$, then the
  Focus: $(h, k+p)$, Directrix: $y = k-p$, and Focal length: |p|
- If it is of the form $(y-k)^2 = 4p(x-h)$, then the
  Focus: $(h+p, k)$, Directrix: $x = h-p$ and Focal length: |p|

Example 3.17

Find the focus and directrix of $y^2 = 12x$.

Solution

Compare the equation given to the standard equation of a parabola

$(y-k)^2 = 4p(x-h)$,

$y^2 = 12x$ (Write in standard form)

$(y-0)^2 = 12(x-0)$,

$\therefore 4p = 12$

$p = 3$

So, we have the vertex $(h, k) = (0, 0)$

Therefore, the focus of the parabola is $(0+3, 0) = (3, 0)$, and the directrix is $x = 0-3 = -3$.

Example 3.18

Find the directrix and focus of $(x+1)^2 = -8(y-4)$.

Solution

$(x+1)^2 = -8(y-4)$ is of the form $(x-h)^2 = 4p(y-k)$,

Comparing $(x+1)^2 = -8(y-4)$ to $(x-h)^2 = 4p(y-k)$,

we have the vertex of the parabola (h, k)= $(-1, 4)$

Also $4p = -8$, and $p = -2$

Since $p < 0$, the parabola opens down,

$\therefore$ the cordinates of the focus is $(h, k+p)$

$\therefore$ focus $= (-1, 4+(-2))$

$= (-1, 2)$

Directrix is $y = 4-(-2)$, so in this case it is $y = 6$

DESCRIBING THE ORIENTATION OF A PARABOLA WITH THE GIVEN EQUATION

At the beginning of this section, we explored the four orientations of a parabola. We learned how to find the equation of a parabola given its focus and directrix. Now we will apply everything we've learned to sketch parabolas. The orientation of a parabola is the direction in which the parabola opens on the coordinate plane.

Knowing the orientation will help sketch the parabola.

1. Vertical Parabola (opens up or down)

$y = a(x-h)^2 + k$ **or**
$(x-h)^2 = 4p(y-k)$

Opens up if $a > 0$ or $p > 0$

Opens down if $a < 0$ or $p < 0$

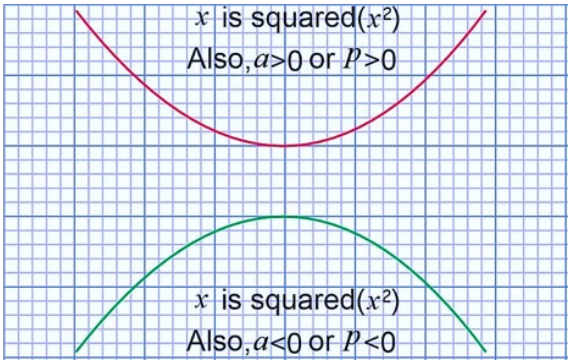


Figure 3.41: Graph showing orientation of parabola

$y = a(x-h)^2 + k$

2. Horizontal Parabola (opens left or right):

$x = a(y-k)^2 + h$ or

$(y-k)^2 = 4p(x-h)$

Opens right if $a > 0$ or $p > 0$

Opens left if $a < 0$ or $p < 0$

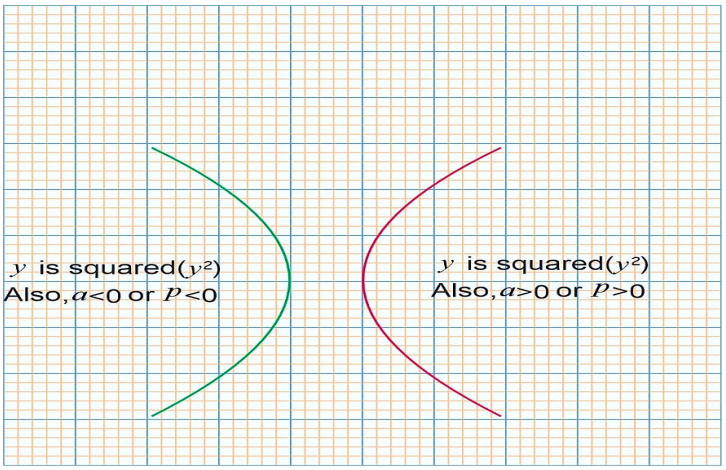


Figure 3.42: Graph showing orientation of parabola

$x = a(y-h)^2 + k$

How to Describe the Orientation:

Step 1: Identify which variable is squared.

If x is squared, it implies the parabola opens up or down

If y is squared, it implies the parabola opens left or right

Step 2: Check the sign of the coefficient of the squared term (a or p)

If a or p is positive, the parabola opens up (if vertical) or right (if horizontal)
If a or p is negative, the parabola opens down (if vertical) or left (if horizontal)

Example 3.19

Describe the orientation of the parabola with the equation $(y+1)^2 = -5(x-3)$.
Give the parabola's vertex and focus.

Solution

since y is squared, the parabola either opens to the right or the left.

It has a horizontal line of symmetry.
Comparing $(y+1)^2 = -5(x-3)$ with $(y-k)^2 = 4p(x-h)$,
we have $4p = -5$

$$p = -\frac{5}{4} = -1.25$$

Since P is negative, it means the parabola opens to the left.
The vertex is $(3, -1)$ and the focus is $(1.75, -1)$

Example 3.20

Describe the orientation of the parabola with the equation $(x-5)^2 = 4(y-2)$.
Give the parabola's vertex and focus.

Solution

Since x is squared, the parabola opens up or down. It has a vertical line of symmetry.

$(x-5)^2 = 4(y-2)$

The vertex is $(5, 2)$.

$4p = 4$

$p = 1$

Since P is positive, the parabola opens up.

The focus of the parabola is $(5, 3)$.

SKETCHING PARABOLAS GIVEN THE DIRECTRIX AND FOCUS

Step 1: Plot the focus and draw the directrix in the x-y plane or a coordinate grid.

Step 2: Find the vertex, which is halfway between the focus and the directrix.

Step 3: Determine the orientation. If the directrix is horizontal, the focus will open to the left or right. If the directrix is vertical, the parabola will open up or down.

Step 4: Sketch the parabola. The turning point is the vertex. Draw a smooth curve that passes through the vertex, curves around the focus, and moves away from the directrix.

Example 3.21

A parabola has a focus at $(-1, 2)$ and directrix $x = 1$.

a. Sketch the parabola

b. Write the equation of the parabola

Solution

a. Step 1: Plot the focus $(-1, 2)$ and draw the directrix $x = 1$

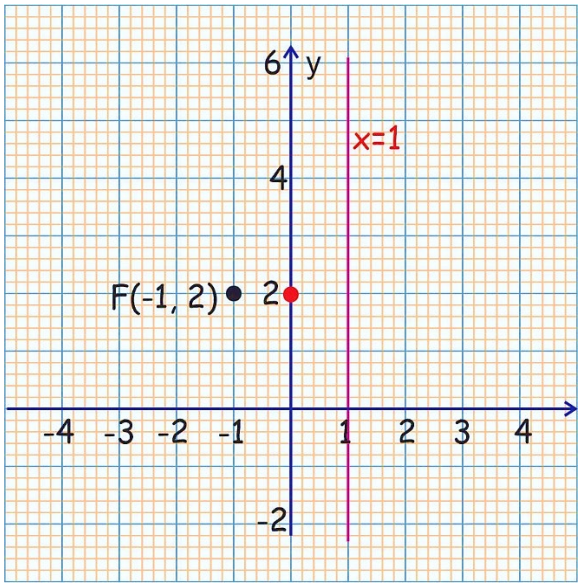


Figure 3.43: Graph $x = 1$ and point $F(-1, 2)$

Step 2: From Figure 3.43, the vertex will be located at (0, 2) as it is equidistant from the focus and the directrix.

Step 3: Since the parabola has to pass around the focus, it will open to the left.

Step 4: Draw a curve that opens to the left and passes through the vertex, (0, 2).

Note that figure is drawn to scale. Your sketch should not necessarily be drawn to scale.

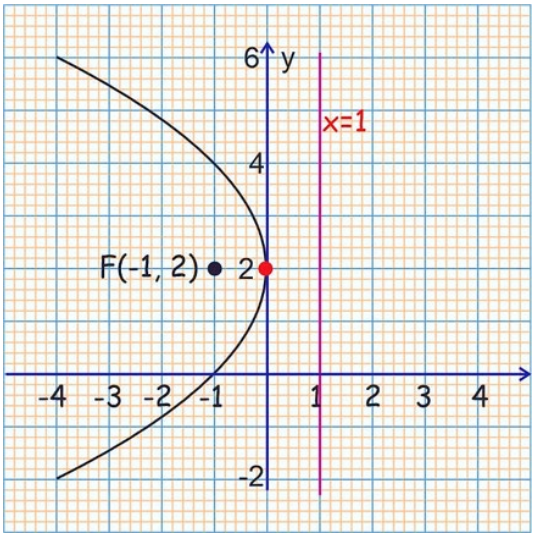


Figure 3.44: Parabola with focus at $(3, -3)$ and directrix $x = 4$

b. From Figure 3.44, the vertex is (0,2) and the focus is (-1, 2).

The focal length, p, is the distance between the vertex and the focus. From the graph, p=1. You can also calculate it using the distance formula

$$|p| = \sqrt{(2-2)^2 + (-1-0)^2} = \sqrt{0+1} = \sqrt{1} = 1$$

Since the parabola opens to the left, $p = -1$

Using $(y-k)^2 = 4p(x-h)$ by substituting (h, k)=(0, 2) and p=-1, we have:

$(y-2)^2 = 4(-1)(x-0)$

$(y-2)^2 = -4x$

Example 3.22

A parabola has a focus at $(3, 1)$ and directrix $y + 3 = 0$.

a. Make a sketch of the parabola

b. Write the equation of the parabola

Solution

a. Step 1: plot the focus (3, 1) and draw the directrix $y = -3$

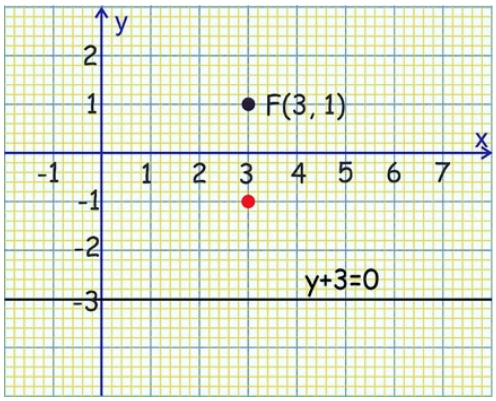


Figure 3.45: Graph $y = -3$ and point F $(3, 1)$

Step 2: From Figure 3.45, the vertex will be located at $(3, -1)$ as it is equidistant from the focus and the directrix.

Step 3: Since the parabola has to pass around the focus, it will open upwards.

Step 4: Draw a curve that opens upwards and passes through the vertex, $(3, -1)$.

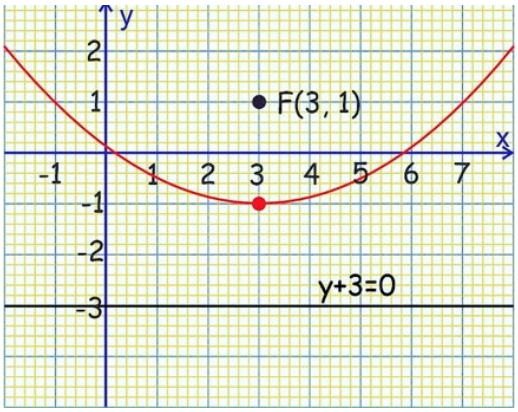


Figure 3.46: Parabola with focus at $(3, 4)$ and directrix $y = -3$

b. From the graph, the vertex is $(3, -1)$ and the focus is $(3, 1)$.

The focal length, p, is the distance between the vertex and the focus.

From the graph, |p|=2. You can also calculate it using the distance formula

$$|p| = \sqrt{\left(1-(-1)\right)^2 + (3-3)^2} = \sqrt{2^2} = 2$$

Since the parabola opens upwards, $p = 2$

Using $(x-h)^2 = 4p(y-k)$ by substituting $(h,k) = (3,-1)$ and $p = 2$, we have:

$(x-3)^2 = 4(2)\big(y-(-1)\big)$

$(x-3)^2 = 8(y+1)$

DESCRIBING THE EQUATIONS OF LINES AND DERIVATIVES OF FUNCTIONS

An equation of a straight line shows the relationship between x and y coordinates on a graph.

In slope-intercept form, it is given by: $y = mx + c$, where m is the gradient and c is the y-intercept.

The gradient is calculated using the formula, $m = \frac{\Delta y}{\Delta x}$ or $m = \frac{y_2 - y_1}{}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$
Description of a line equation

1. If $m > 0$, the graph strictly increases. The line slopes upwards from left to right.
2. when $m < 0$, it strictly decreases. The line slopes downwards from left to right
3. When $m = 0$, the line is horizontal or parallel to the x-axis. The equation of the line takes the form $y = k$, where k is a constant.
4. When $m = \infty$ (undefined), the line equation is vertical or parallel to the y-axis.
   The equation of the line takes the form $x = k$, where k is a constant

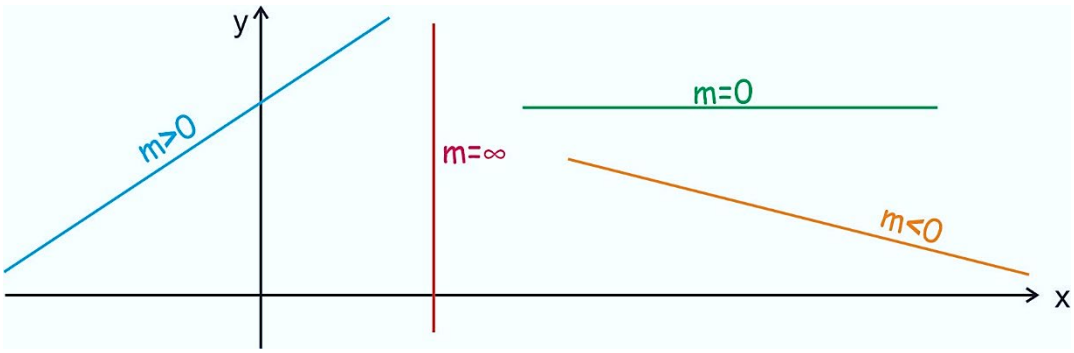


Figure 3.47: Graphs showing description of line equation.

The gradient of a function can also be obtained using the gradient function. The first derivative of a function gives the gradient function.

For parabolas (quadratic functions), the derivative will be a linear equation. The gradient of the parabola changes depending on the point of contact of the parabola and the line.

DETERMINING AND DESCRIBING THE EQUATIONS OF TANGENTS AND NORMAL TO A PARABOLA

Activity 3.10 Finding the equation of a tangent or normal to a parabola

Step 1: Locate the point at which the line intersects the parabola, (x_1, y_1) . If one of the coordinates of the points is provided, substitute it into the equation of the parabola and find the other coordinates. It is possible to have multiple points.

Step 2: Find the first derivative, $\left(\frac{dy}{dx}\right)$, of the parabola.

Step 3: Substitute (x_1, y_1) into the derivative function. This gives the gradient of the tangent.

$$\text{gradient } (m) = \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$$

$$\text{The gradient of the normal} = -\frac{1}{m}$$

Step 4: Use (x_1, y_1) and the gradient to find the tangent and/or normal equation.

The slope of a tangent is given by $\frac{dy}{dx}$.

The slope of the normal is given by $-\frac{dx}{dy} = -\frac{1}{\frac{dy}{dx}}$

Let us now generate an equation of a tangent to a parabola of the form $y^2 = 4px$

Differentiating $y^2 = 4px$ on both sides with respect to x will be

$$2y\frac{dy}{dx} = 4p$$

$$\frac{dy}{dx} = \frac{2p}{y}$$

Therefore, the slope (m) of the tangent to the parabola at the point (x_1, y_1) on it is $\frac{2p}{y_1}$.

Now, the equation of the tangent to the parabola at the point (x_1, y_1) on it is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow\ y - y_1 = \frac{2p}{y_1}(x - x_1)$$

$$\Longrightarrow yy_1 - y_1^2 = 2px - 2px_1$$

But $y_1^2 = 4px$

$$yy_1 - 4px_1 = 2px - 2px_1$$

$\therefore yy_1 = 2p(x + x_1)$ is the equation of the tangent at a point (x_1, y_1) on the parabola.

For a parabola with equation of the form $x^2 = 4py$,

Differentiating $x^2 = 4py$ on both sides with respect to x will be

$$2x = 4p\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{4p} = \frac{x}{2p}$$

Therefore, the slope (m) of the tangent to the parabola at the point (x_1, y_1) on it is $\frac{x_1}{2p}$.

Now, the equation of the tangent to the parabola at the point (x_1, y_1) on it is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow\ y - y_1 = \frac{x_1}{2p}(x - x_1)$$

$$\Longrightarrow xx_1 - x_1^2 = 2py - 2py_1$$

But $x_1^2 = 4py_1$

$$\Rightarrow xx_1 - 4py_1 = 2py - 2py_1$$

$\therefore 2py = xx_1 - 2p(2y_1 - y_1)$ is the equation of the tangent at a point (x_1, y_1) on the parabola.

Equation of a normal to a parabola of the form $y^2 = 4px$

Recall that the relationship between the gradients of two lines which a perpendicular, in this case, a tangent and a normal, given that their gradients are m_1 and m_2 is

$$m_1 \times m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

Since the gradient of the tangent to the parabola with equation $y^2 = 4px$ at (x_1, y_1) is $\frac{2p}{y_1}$, the slope (m) of the normal is $-\frac{y_1}{2a}$ since normal and tangent are perpendicular to each other.

$\therefore$ The equation of the normal at a point (x_1, y_1) on the parabola becomes

$$y - y_1 = -\frac{y_1}{2p}(x - x_1)$$

$$\Longrightarrow y_1x - 2py_1 - x_1y_1 + 2py = 0$$

Similarly, the slope (m) of the normal to the parabola with the equation $x^2 = 4py$ at (x_1, y_1) is $-\frac{2p}{x_1}$ Since the gradient of the tangent is $\frac{x_1}{2p}$.

$\therefore$ The equation of the normal at a point (x_1, y_1) on the parabola becomes

$$y - y_1 = -\frac{2p}{x_1}(x - x_1)$$

$$\Rightarrow x_1 y + 2px - x_1 y_1 - 2px_1 = 0$$

Example 3.23

A parabola $y^2 = 8x$ passes through the point $P(2, 4)$. Find:

a. Equation of the tangent to the parabola at the point P

b. Equation of the normal to the parabola at the point P

Solution

If $y^2 = 8x$, then comparing it to $y^2 = 4px,\ p = 2$

a. The equation of the tangent to the parabola is given as

$$yy_1 = 2p(x + x_1)$$

$$\Rightarrow y(4) = 2 \times 2(x + 2)$$

$$y = x + 2$$

Alternatively,

$$y^2 = 8x$$

$$\Rightarrow 2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{4}{y}$$

The gradient of the tangent line at P, $\left.\frac{dy}{dx}\right|_{y=4} = \frac{4}{4} = 1$

The equation of the tangent line that passes through P

$$y - 4 = 1(x - 2)$$

$$y = x + 2$$

b. The equation of the normal is given by

$$y_1 x - 2py_1 - x_1 y_1 + 2py = 0$$

$$\Rightarrow 4x - 2(2)4 - (2)(4) + 2(2)y = 0$$

$$y = -x + 6$$

Alternatively,

The gradient of the normal line at $P = -\frac{1}{1} = -1$

The equation of the tangent line that passes through P

$$y - 4 = -1(x - 2)$$

$$y = -x + 6$$

Example 3.24

The curve is defined by the equation $(y-3)^2 = -4x$

(a) Find the equation of the tangent to the curve at point $(-4,-1)$.

(b) Find the equation of the normal to the curve at point $(-4,7)$.

Solution

(a)The gradient of the tangent is $\frac{dy}{dx}$

$$\frac{d}{dx}((y-3)^2 = -4x)$$

$$2(y-3)\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = -\frac{4}{2(y-3)} = -\frac{2}{y-3}$$

$$= \frac{dy}{dx}\Big|_{(-4,-1)} = -\frac{2}{-1-3} = \frac{1}{2}$$

Equation of the tangent with gradient $\frac{1}{2}$ passing through $(-4,-1)$ is:

$$y-(-1) = \frac{1}{2}(x-(-4))$$

$$y+1 = \frac{1}{2}(x+4) y = \frac{1}{2}x + 1 \text{ or } 2y - x - 2 = 0$$

(b) At point $(-4,7)$, the gradient of the tangent is $= -\frac{2}{7-3} = -\frac{2}{4} = -\frac{1}{2}$

The gradient of the normal is $-1 \div -\frac{1}{2} = -1 \times -\frac{2}{1} = 2$

So, the equation of the normal at $(-4,7)$ is:

$$y-7 = 2(x-(-4))$$

$$y-7 = 2(x+4)$$

$$y = 2x + 8 + 7$$

$$y = 2x + 15$$

Example 3.25

A water jet follows the parabolic path $y = 14x - 2x^2$

Determine the point where the jet reaches its highest point.

Find the gradient of the tangent at $x = 2$.

A sensor is placed along the path, perpendicular to the direction of the water at $x = 2$. Find the equation of the sensor's line.



Figure 3.48: Water jet

Solution

a. For maximum height, put $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 14 - 4x$$

$$\Rightarrow 14 - 4x = 0$$

$$x = -\frac{14}{-4}$$

$$= 3.5$$

When $x = 3.5$, $y = 14 \times 3.5 - 2(3.5)^2$

$y = 24.5$ units.

The maximum point reached by the water jet is $(3.5, 24.5)$

b. The gradient of the tangent at $x = 2$ is $14 - 4(2) = 14 - 8 = 6$

c. When $x = 2, y = 14(2) - 2(2)^2 = 28 - 8 = 20$

The gradient of the path of the sensor is the same as the gradient of the normal to the curve $x = 2$

Therefore, the gradient of the path of the sensor $= -\frac{1}{6}$

The equation of the path of the sensor is;

$$y - 20 = -\frac{1}{6}(x - 2)$$

$$y = -\frac{1}{6}x + \frac{1}{3} + 20$$

$$y = -\frac{1}{6}x + 20\frac{1}{3}$$

Hence the equation of the sensor is $y = -\frac{1}{6}x + \frac{61}{3}$ or $6y + x - 122 = 0$

FINDING THE POINT OF INTERSECTION BETWEEN A LINE AND A PARABOLA

A straight line will intersect a parabola at a maximum of two points.

Note that a line that intersects a curve at two or more points is called a secant.

If the line intersects the parabola at one point, the line might be tangent to the parabola.

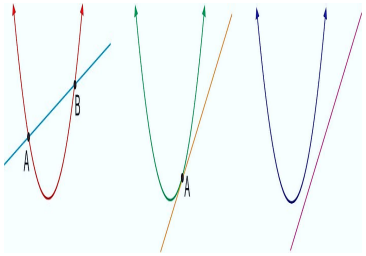


Figure 3.49: Line intersects parabola at two points, A and B

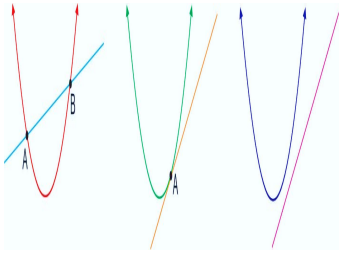


Figure 3.50: Line intersects parabola at one point, A.

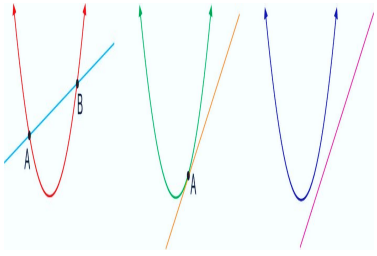


Figure 3.51: Line does not intersect parabola

How to find the point of intersection of a line and a parabola

Step 1: Set the equations equal. You might also make a variable in the linear equation the subject and substitute the result into the parabolic equation.

Step 2: Rearrange your result from step 1 to form a quadratic equation

Step 3: Solve the quadratic equation to find one of the variables.

[If the discriminant of the resulting quadratic is 0, you can be sure the line is tangent to the parabola. If the discriminant is less than 0, the line and the parabola do not intersect. In such cases, step 4 will not be necessary.]

Step 4: Using your result from step 3, find the corresponding x or y-values.

Example 3.26

The line $2y + 3x - 6 = 0$ intersects the parabola $y^2 = 6x$ at K and M respectively.

a. Find the coordinates of K and M.

b. Find the equation of the normal to the parabola at the point $\left(\frac{2}{3}, 2\right)$.

Solution

a. Step 1: Set the two equations equal

b. Let $2y + 3x - 6 = 0 \ldots\ldots (1)$ and

$$y^2 = 6x \ldots\ldots\ldots\ldots\ldots (2)$$

From (1), we have the line equation: $x = \frac{6-2y}{3}$

From (2), we have the parabolic equation: $x = \frac{1}{6}y^2$

Since the line and parabola intersect,

$$\frac{6-2y}{3} = \frac{y^2}{6}$$

Step 2: Rearrange $\frac{6-2y}{3} = \frac{y^2}{6}$ to form a quadratic equation.

multiply through by 6

$$6 \times \left(\frac{6-2y}{3}\right) = 6 \times \left(\frac{y^2}{6}\right)$$

$$2(6-2y) = y^2$$

$$12 - 4y - y^2 = 0 \quad \text{multiply through by } -1$$

$y^2 + 4y - 12 = 0$

Step 3: Solve the resulting quadratic equation

$y^2 + 6y - 2y - 12 = 0$

$y(y + 6) - 2(y + 6) = 0$

$(y - 2)(y + 6) = 0$

This means that either $(y - 2) = 0 \Rightarrow y = 2$

or $y + 6 = 0 \Rightarrow y = -6$

$y = 2$ or $y = -6$

Step 4: For each y-value, find its corresponding x-value

When $y = 2,\ x = \frac{1}{6}(2)^2 = \frac{2}{3}$ and when $y = -6, x = \frac{1}{6}(-6)^2 = \frac{1}{6} \times 36 = 6$

$\therefore K = \left(\frac{2}{3}, 2\right)$ and $M = (6, -6)$

c. The gradient function of the parabola is $\frac{d}{dx}(y^2 = 6x)$

$2y\frac{dy}{dx} = 6$ $\qquad m = \frac{dy}{dx} = \frac{6}{2y} = \frac{3}{y} = \frac{3}{2}$

The gradient of the normal is $= -1 \div \frac{3}{2} = -\frac{2}{3}$

The equation of the normal to the parabola at $\left(\frac{2}{3}, 2\right)$ is:

$y - 2 = -\frac{2}{3}\left(x - \frac{2}{3}\right)$

$y - 2 = -\frac{2}{3}x + \frac{4}{9}$

$9y - 18 = -6x + 4$

$9y + 6x - 22 = 0$

Alternatively,

Using the equation $2py - 2py_1 + xy_1 - x_1y_1 = 0$

We have $y^2 = 6x$ and $p = \frac{3}{2}$

The point is $\left(\frac{2}{3}, 2\right)$ therefore, the equation of the normal is

$2\left(\frac{3}{2}\right)y - 2\left(\frac{3}{2}\right) \cdot 2 + (2)x - 2 \cdot \left(\frac{2}{3}\right) = 0$

$9y + 6x - 22 = 0$

Example 3.27

The arch of a bridge, built in the shape of a parabola, is given by $y = 8x - x^2$. A cable is suspended straight along the line $y - 3x - 4 = 0$

Find the point(s) where the cable touches the bridge.

State whether the cable is a tangent or a secant to the bridge.

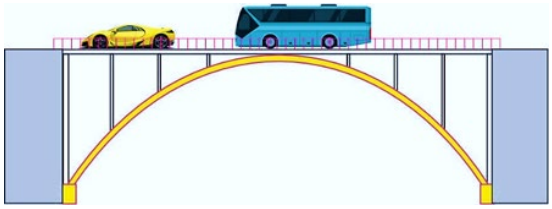


Figure 3.52: A bridge

Solution

a. We will follow the steps to answer the questions

Step 1: Set the two equations equal

Parabolic equation: $y = 8x - x^2$.

Line equation: $y = 3x + 4$

Setting the equations equal, we have: $8x - x^2 = 3x + 4$

Step 2: Rearrange $8x - x^2 = 3x + 4$ to form a quadratic equation.

$8x - x^2 - 3x - 4 = 0$ $\quad -x^2 + 5x - 4 = 0$ (multiply through by -1)

$x^2 - 5x + 4 = 0$

Step 3: Solve $x^2 - 5x + 4 = 0$ for the values of x

$x^2 - 5x + 4 = 0$

$x^2 - x - 4x + 4 = 0 x(x - 1) - 4(x - 1) = 0\ (x - 1)(x - 4) = 0$

So, $x - 1 = 0$ and/or $x - 4 = 0$

This implies that $x = 1$ and/or $x = 4$

Step 4: For each x-value, find its corresponding y-value.

If $x = 1$, $y = 3(1) + 4 = 7$

If $x = 4$, $y = 3(4) + 4 = 16$

So, the cable touches the bridge at (1, 7) and (4, 16).

b. The cable is secant to the arch of the bridge since it intersects the bridge at two points.

Example 3.28

A parabola has a focus at $(0, -3)$ and directrix $x - 2 = 0$.

a. Write the equation of the parabola

b. Find the point at which the parabola intersects the line $2y + x + 1 = 0$ and state the relationship between the parabola and the line.

c. Find the equation of the tangent to the parabola at $y = -5$

Solution

A sketch of the parabola shows that the vertex is $(1, -3)$ and the focal length is $|p| = 1$

Since the parabola opens to the left, it means $p = -1$.

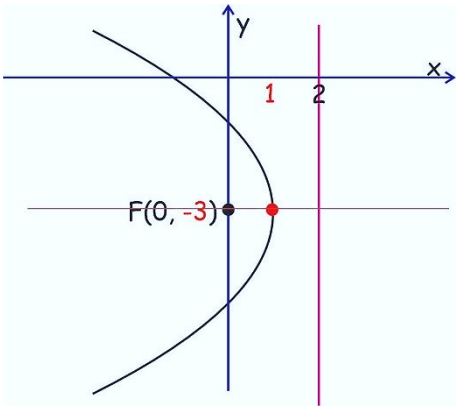


Figure 3.53: Parabola with focus $(0, -3)$ and directrix $x = 2$

a. Therefore, the equation of the parabola is given by:

$$(y + 3)^2 = 4(-1)(x - 1)$$

$$(y + 3)^2 = -4(x - 1)$$

b. The equation of the parabola is: $(y + 3)^2 = -4(x - 1)$

The line equation is: $2y + x + 1 = 0$

From the line equation, we have, $x = -2y - 1$

Substituting $x = 2y - 1$ into the parabolic equation, we have:

$(y + 3)^2 = -4((-2y - 1\ ) - 1)$

Simplify and solve the resulting quadratic equation.

$$(y + 3)^2 = -4(-2y - 2)$$

$$y^2 + 6y + 9 = 8y + 8$$

$$y^2 - 2y + 1 = 0 \qquad y^2 - y - y + 1 = 0$$

$$y(y - 1) - 1(y - 1) = 0 \qquad (y - 1)(y - 1) = 0$$

$$y - 1 = 0$$

$$\Rightarrow y = 1$$

If $y = 1, x = -2(1) - 1 = -3$

So, the point of intersection of the parabola and the line is $(-3, 1)$. Since the line intersects the parabola at one point, the line is tangent to the parabola at $(-3, 1)$

c. We will first find the value of x for which $y = -5$.

To do this, we will substitute $y = -5$ into the parabola and solve for x

$(-5+3)^2 = -4(x-1)$ $4 = -4(x-1)$ divide each side by -4

$$\frac{4}{-4} = -\frac{4}{-4}(x-1)$$

$$-1 = x - 1$$

$$-1 + 1 = x$$

$$0 = x$$

This means that the tangent intersects the parabola at $(0, -5)$

Next, we find the gradient:

$$\frac{d}{dx}[(y+3)^2 = -4(x-1)]$$

$$2(y+3)\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = -\frac{4}{2(y+3)}$$

At $(0, -5)$, the gradient of the tangent is $= \frac{dy}{dx} = -\frac{4}{2(-5+3)} = -\frac{4}{-4} = 1$

Therefore, the equation of the tangent is:

$y - (-5) = 1(x - 0)$

$y + 5 = x$

$y - x + 5 = 0$

REVIEW QUESTIONS 3

1. Outline 5 real-life examples of parabolas.
2. In each of the following graphs,

   a. Describe the shape. Your description should include a statement about the domain, range, line of symmetry, orientation, and maximum or minimum value

   b. Write the quadratic function in the form $y = a(x + h)^2 + k$ where a, h, and k are parameters.

   c. Describe the shape of the graph as compared to $y = x^2$

(i)

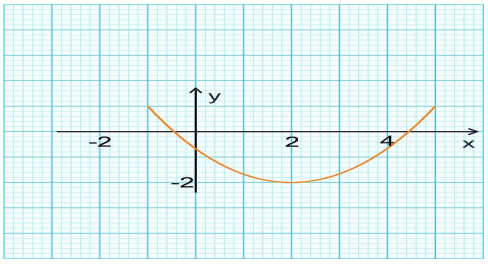


Figure 3.54: Parabola 1

(ii)

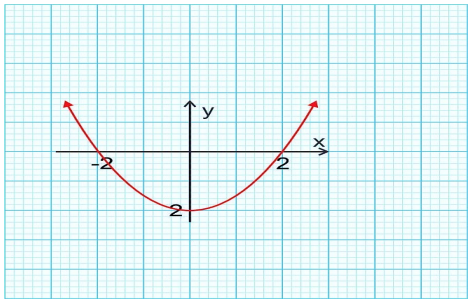


Figure 3.55: Parabola 2

(iii)

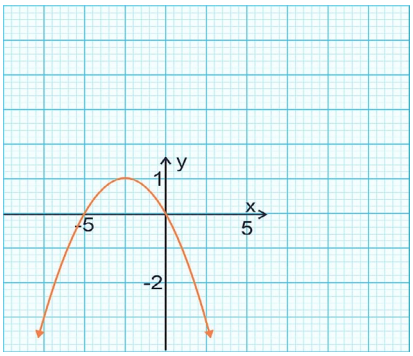


Figure 3.56: Parabola 3

(iv)

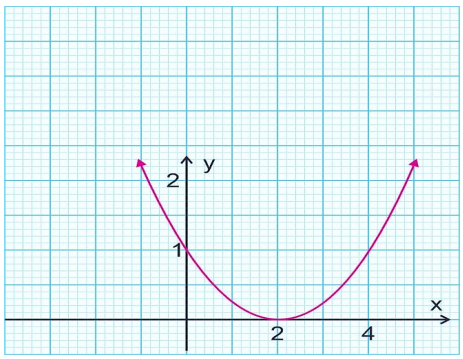


Figure 3.57: Parabola 4

(v)

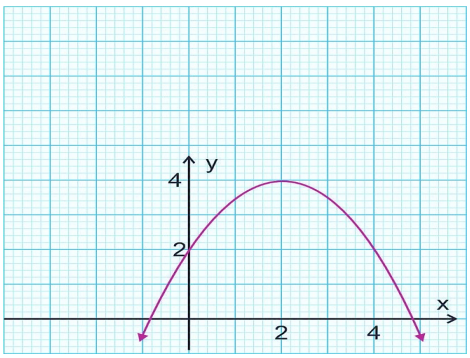


Figure 3.58: Parabola 5

(vi)

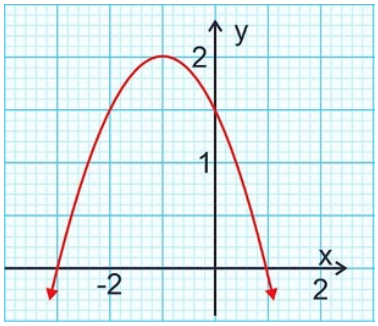


Figure 3.59: Parabola 6

3. The parabola is defined by: $y(y - 2) = 4x - 3$.

a. Sketch the parabola. Your sketch should include the coordinates of the focus, the coordinates of the vertex, the line of symmetry, and the points at which the parabola intersects with the x-axis and the y-axis.

b. Give a brief description of the orientation of the parabola.

4. The graph below shows a cross-section of a drainage in the form of a parabola.



Figure 3.60: Parabola representing cross-section of a drainage

a. Write the equation of the parabola in the form $y = a(x - h)^2 + k$ where a, h, and k are parameters.

b. Describe the shape of the graph as compared to $y = x^2$

c. State the focus, the equation of the directrix, and the focal length.

d. Find the points of intersection of the parabola and the line 7y+4x-4=0

5. Describe and sketch the graph of $y = -x^2 + 3$.

6. Sketch the graph $y = 3 - 5x - 2x^2$ and compare your graph to $y = x^2$.

7. Sketch the graph $y = x^2 - 7x + 6$ and compare your graph to $y = x^2$.

8. The diagram below shows a graph of a parabola. Point V is the vertex.

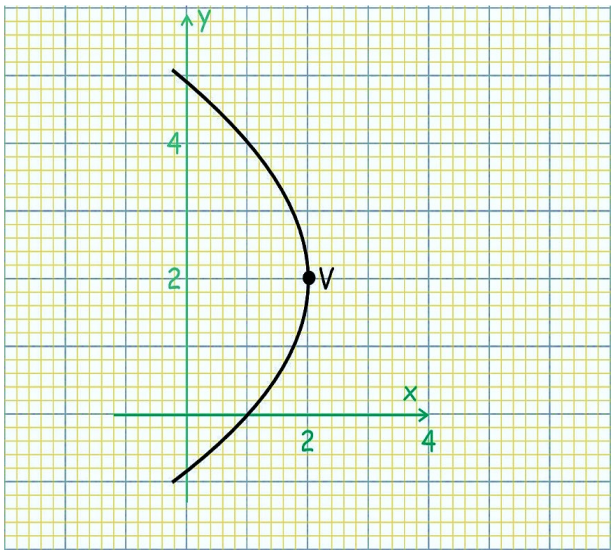


Figure 3.61: Graph of a parabola

a. Write the equation of the parabola.

b. Describe the shape of the graph as compared to $x = y^2$

9. Figure 3.62 shows a graph of a quadratic function.

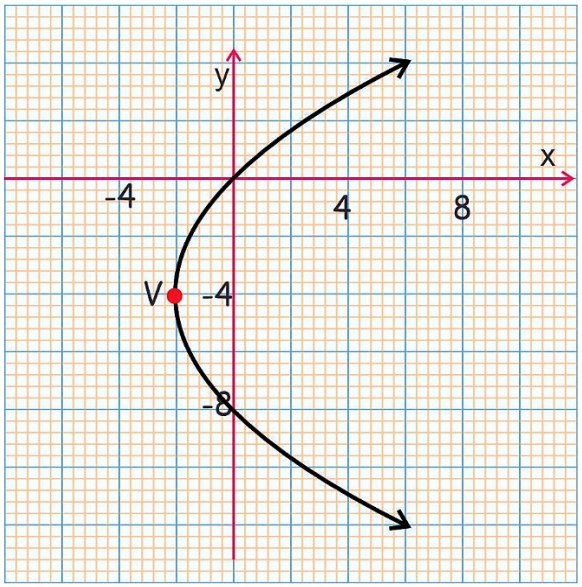


Figure 3.62: Graph of a parabola

a. Write the quadratic function in the form $x = a(y - h)^2 + k$ where a, h, and k are parameters.

b. Describe the shape of the graph when compared to $x = y^2$

10. The graph below represents the path of a volleyed ball, where ***H*** is the height of the ball in meters, and ***t*** is the time in seconds after it was hit.

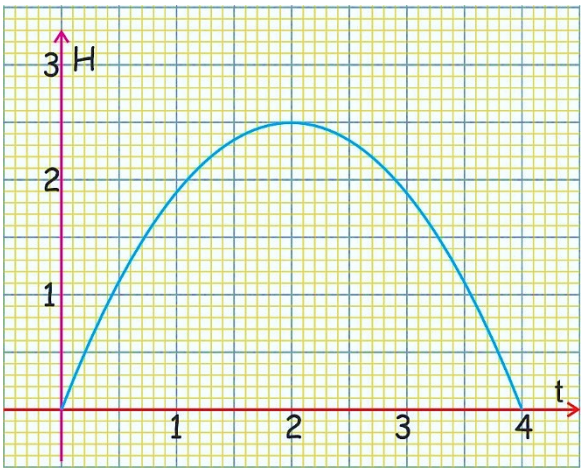


Figure 3.63: Graph of a parabola representing the path of a volleyball

a. Write a quadratic equation to represent the path of the projectile.

b. Describe the shape of the H in Figure 3.63

11. Sketch and describe the features of the parabola $x + 12y^2 = 0$.

12. A parabola has a focus at $(-4,-2)$ and vertex at $(-2,-2)$.

a. Find the equation of the parabola

b. Sketch the parabola. Your sketch should include the directrix, the coordinate of the x-axis, focus and vertex.

13. A parabola has a focus at $(-1,3)$ and vertex at $(2,3)$.

a. Find the equation of the parabola.

b. Sketch the parabola. Your sketch should include the directrix, the coordinate of the x-axis, focus and vertex.

14. Find the vertex, focus and directrix of the following parabolas:

a. $y^2 = -7x$

b. $(x-2)^2 = -8(y-5)$

c. $(y-1)^2 = -8(x+4)$

15. Describe the orientation of the parabola with the following equations equation

a. $y^2 = -8x$

b. $(x-2)^2 = -8(y-5)$

c. $(y-1)^2 = -8(x+4)$.

16. A parabola has a focus at $(-3,3)$ and directrix $x = 0$.

a. Sketch the parabola. Your sketch should include the point of intersection with the x −axis, the coordinates of the vertex.

b. Write the equation of the parabola

17. A parabola has a focus at $(2,-1)$ and directrix $y + 4 = 0$.

a. Sketch the parabola. Your sketch should include the points of intersection with the coordinates, the coordinates of the vertex.

b. Write the equation of the parabola

18. A Parabola $y^2 = 3x$ passes through the point $P(3,3)$.

a. Find the equation of the tangent to the parabola at P

b. Find the equation of the normal to the parabola at the point P.

c. The parabola intersects $2y - 3x + 3 = 0$ at points P and Q. Find

i. the coordinates of Q

ii. PQ, leave your answer in surds

iii. the equation of the tangent to the parabola at point Q.

19. The curve is defined by the equation: $(y-4)^2 = -8x$

a. Find the equation of the tangent to the curve at the point where the parabola intersects the x −axis.

b. Find the equation of the normal to the curve at $y = 8$.

c. Find the point at which the parabola intersects $y - 3x - 4 = 0$

20. A parabola has a vertex at $(-1, -1)$ and directrix $2y - 1 = 0$.

a. Write the equation of the parabola

b. Find the point at which the parabola intersects the line $3y + 2x + 5 = 0$ and state the relationship between the parabola and the line.

c. Find the equation of the tangent to the parabola at its y-intercept.

d. Find the equation of the normal to the parabola at $x = -4$

SECTION

4

TRIGONOMETRY



GEOMETRIC REASONING AND MEASUREMENT

INTRODUCTION

In this section on trigonometric functions, we will discover how ideas from compound angles can be used to rewrite expressions in harmonic form. This skill makes it easier to find the highest and lowest points of a vibrating system, such as the motion of a string on a guitar or the swing of a pendulum. We will also explore the shapes of sine, cosine, and tangent graphs, noticing their repeating patterns and how they rise and fall. By sketching these graphs by hand and experimenting with digital tools like GeoGebra, we can see how mathematics describes waves, rhythms, and cycles that appear all around us. Linking these graphs to the unit circle will give us a clear picture of why trigonometric functions repeat and how their values change as we move around the circle. In the end, you will gain not only the ability to analyse trig graphs and find their maximum and minimum values but also an appreciation of how these ideas explain real-life phenomena such as sound waves, tides, and even satellite motion.

KEY IDEAS

- The maximum, minimum, and periodic behaviour of trigonometric graphs come directly from their definitions on the unit circle.
- Trigonometric expressions of the form $a\cos\theta + b\sin\theta$ can be rewritten in harmonic form using compound angle identities.
- The amplitude in harmonic form shows the maximum value, and the minimum is its negative (with vertical shifts considered).
- The graphs of $y = \sin x$ and $y = \cos x$ oscillate between maximum and minimum values and are periodic.
- The graph of $y = \tan x$ repeats every 180° and has vertical asymptotes where $\cos x = 0$.

REVISION ON PROPERTIES OF A UNIT CIRCLE

In Year 1 Section 7, we discovered about quadrantal angles and their special properties. We established that they are angles that terminate on the x and y axes. The angles are $0, 90°, 180°, 270°$ and $360°$.

In Year 2 Section 9, we learnt that a unit circle is a circle with a radius of 1 unit. If the circle is centred at the origin $(0, 0)$ it has the equation $x^2 + y^2 = 1$.

Regardless of the scale you choose, a unit circle will always intersect the x and y axes at the points $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ as represented in Figure 4.1.

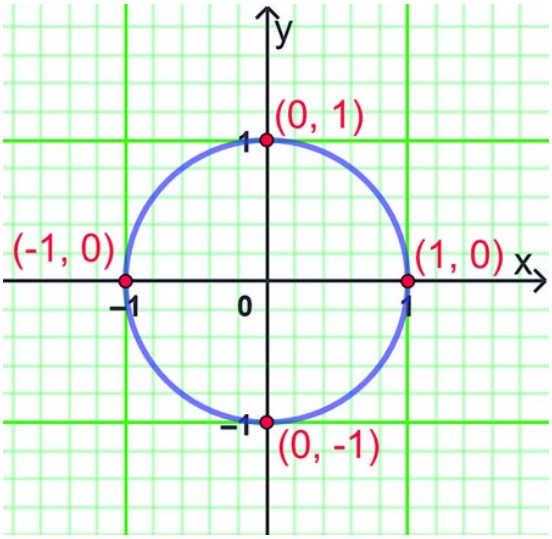


Figure 4.1: Representation of a Unit Circle

Now, let us explore the relationship between the coordinates of a point on a unit circle and the cosine/sine value of the angle between the line that joins the point to the origin and the x or y axis using a diagram.

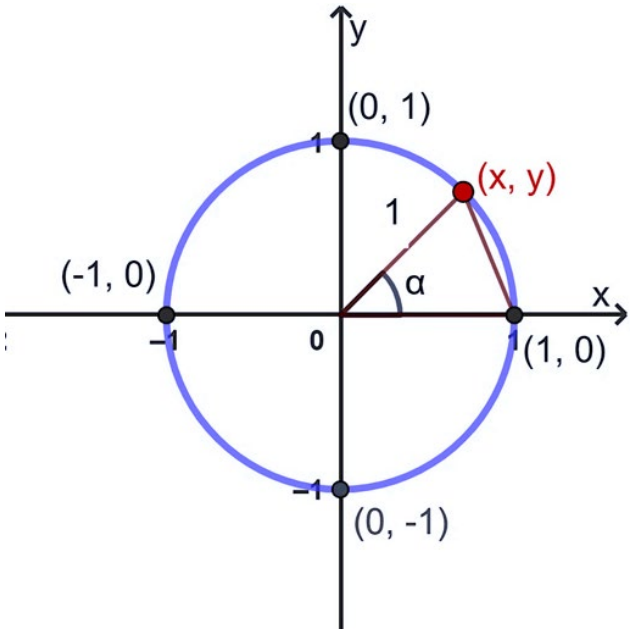


Figure 4.2: Unit circle with a point

From Figure 4.2, $\sin(\alpha) = y$ and $\cos(\alpha) = x$. This means that for any point on the unit circle, the $x -$ coordinate is the value of the cosine value of the angle between the line that joins the point to the origin and the $x - axis$. The $y -$ coordinate of a point on the unit circle is the value of the sine value of the angle between line that joins the point to the origin and the $x - axis$.

Combining what we know about unit circles and quadrantal angles we may have a representation as seen in Figure 4.3.

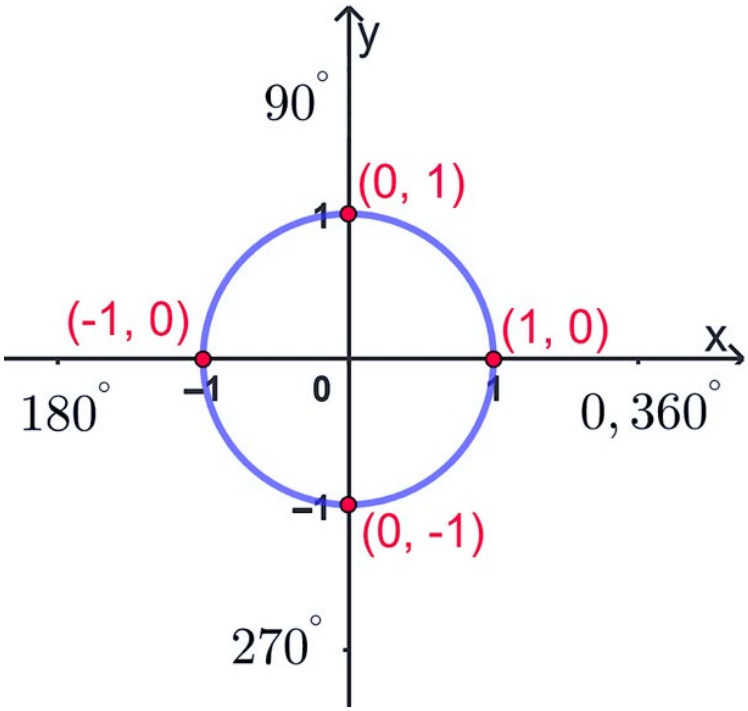


Figure 4.3: Unit Circle with Quadrantal angles

Table 4.1: Functional values of quadrantal angles for sine and cosine

	Quadrantal Angles				
	0°	90°	180°	270°	360°
Point	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$	$(1,0)$
Sine	0	1	0	-1	0
Cosine	1	0	-1	0	1

Table 4.1 contains details of the sine and cosine values of the quadrantal angles in relation to the positioning of the points on the unit circle that intersect with the x and y axes.

SKETCHING BASIC TRIGONOMETRIC GRAPHS

Graphs of basic trigonometric functions

In Year 2 Section 9 while learning about verification of identities, we explored how to use technological tools such as GeoGebra and Desmos to graph given trigonometric functions and determine their equivalence. Now, we are going to explore how to graph the basic trigonometric functions (sine, cosine and tangent) and identify their distinctive properties.

Activity 4.1 Graphs of basic trigonometric functions

1. Refer to Table 4.1, use the quadrantal angles as x and represent the values of $\sin x$.

x	0°	90°	180°	270°	360°
$\sin x$	0	1	0	-1	0

2. Using the same quadrantal angles as in Step 1, represent the values of $\cos x$.

x	0°	90°	180°	270°	360°
$\cos x$	1	0	-1	0	1

3. Using a scale of $2\text{cm}: 90^\circ$ on the x –axis and $2\text{cm}: 0.5$ units on the y –axis, represent $y = \sin x$ and $y = \cos x$ on the same graph sheet. Remember to draw the graphs using smooth curves.

   Your resulting graph should be similar to the one in Figure 4.4 below.

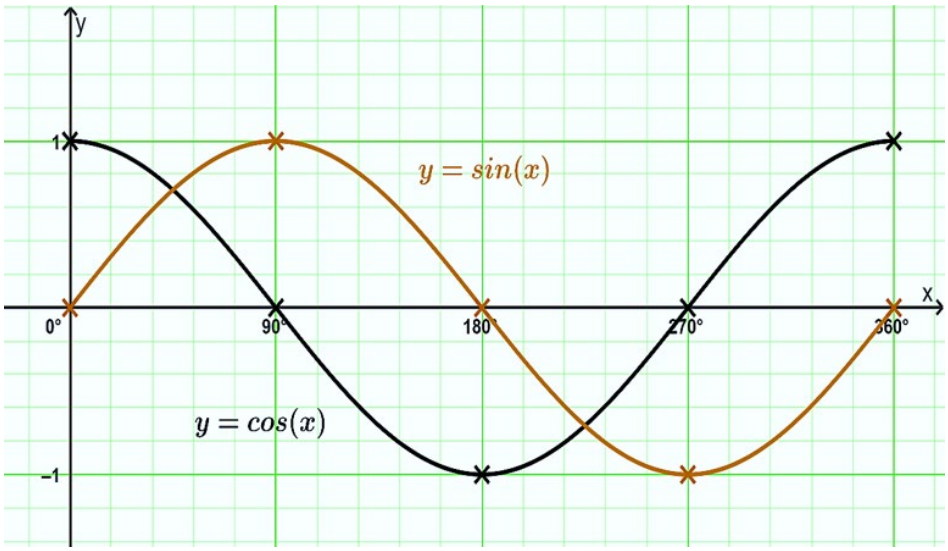


Figure 4.4: Graphical illustration of $sin\, x$ and $cos\, x$ where $0^\circ \leq x \leq 360^\circ$.

4. Record your observations from the graph in your notebook and share with a classmate.
5. Recall the $\tan x = \frac{\sin x}{\cos x}$, now, create a table of values for $\tan x$ using $30^\circ, 45^\circ, 60^\circ$ and the quadrantal angles as x.

x	0°	30°	45°	60°	90°	120°	150°	180°	240°	270°	330°	360°
tan x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	unde-fined	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0	$\sqrt{3}$	unde-fined	$-\frac{\sqrt{3}}{3}$	0

6. Represent $y = \tan x$ on a different graph sheet/book.

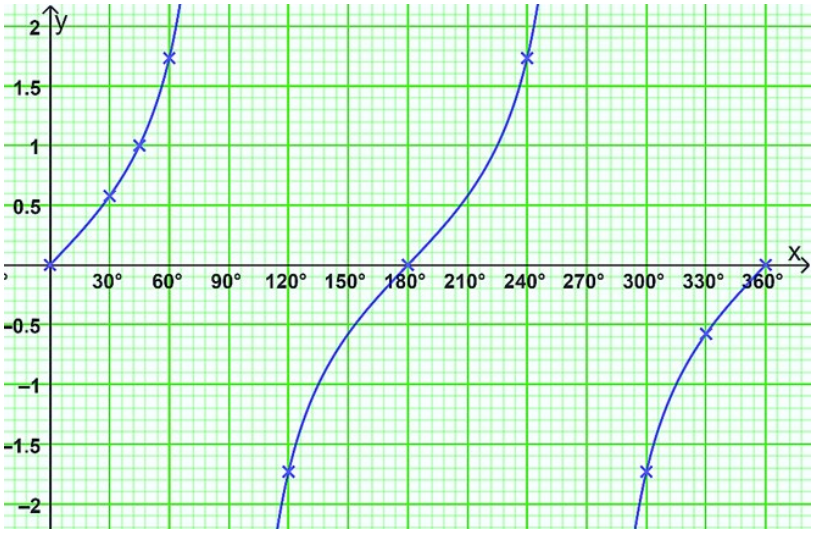


Figure 4.5: Graphical illustration of $tan\ x$, where $0^\circ \leq x \leq 360^\circ$.

Generalisations

1. For the **sine** graph you should arrive at the conclusion that:
   a. The range of the graph is $[-1, 1]$.
      The graph has a maximum value of 1 and minimum value of -1.
   b. The graph increases before decreasing when starting from the origin.
   c. The graph cuts the $x-$ axis at $0, 180°$ and $360°$ i.e., values of $x = 180n°$ for integer values of n.
   d. The graph is symmetric about the origin, confirming that it is an odd function.
   e. The $x-$ axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.
2. For the **cosine** graph you should arrive at the conclusion that:
   a. The range of the graph is $[-1, 1]$.
      The graph has a maximum value of 1 and minimum value of -1.
   b. The graph decreases before increasing when starting from the origin.
   c. The graph cuts the $x-$ axis at $90°$ and $270°$ i.e., values of $x = 90° + 180n°$ for integer values of n.

d. The graph is symmetric about the $y-$ axis, confirming that it is an even function.

e. The $x-$ axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.

3. For the graph of the **tangent** function, you should arrive at the conclusion that:

a. The range of the graph is $(-\infty, \infty)$.

b. The graph cuts the x- axis at $0, 180°$ and $360°$ i.e., values of $x = 180n°$ for integer values of n.

c. There are asymptotes at $x = 90°$ and $x = 270°$ i.e., values of $x = 90° + 180n°$ for integer values of n. (Remember that an asymptote is a line that a graph approaches but never meets.)

d. The graph has no maximum or minimum values.

e. The graph is symmetric about the origin, confirming that it is an odd function.

f. The x- axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.

Transformation of graphs of trigonometric functions

In Section 2, while learning about Matrices, we discovered various ways in which they can be transformed. From explorations and observations, we realised that transformations sometimes lead to a change in the size of the shape of a plane figure or even the direction. In the same way, transformations of trigonometric graphs bring about changes in the graphs. The original graph might express a shift up or down, a shift to the left or right, a stretch/compression vertically or horizontally. Sometimes, the graph can even experience a reflection across the x −axis.

To allow for transformation, basic trigonometric functions can be stated in a more general way.

The sine function can be expressed as $f(x) = A\sin(Bx - C) + D$ and the cosine function can be expressed as $f(x) = A\cos(Bx - C) + D$. Let us delve deeper into the various forms of transformation that may occur.

Transformation of the form A sin x or A cos x

Let us go through this activity to explore transformation of trigonometric functions.

Activity 4.2 Transformation of the form A sin x

1. Using the quadrantal angles as x values, create a table of values for $\sin x$, $3\sin x$, $0.5\sin x$, and $-2.5\sin x$.

   Your table of values should be similar to this:

x	0°	90°	180°	270°	360°
$\sin x$	0	1	0	-1	0
$3 \sin x$	0	3	0	-3	0
$0.5 \sin x$	0	0.5	0	-0.5	0
$-2.5 \sin x$	0	-2.5	0	2.5	0

2. On the same graph sheet, draw the curves of points for $y = \sin x$, $y = 3 \sin x$, $y = 0.5 \sin x$ and $y = -2.5 \sin x$.

   Your graphical representation should be similar to Figure 4.6.

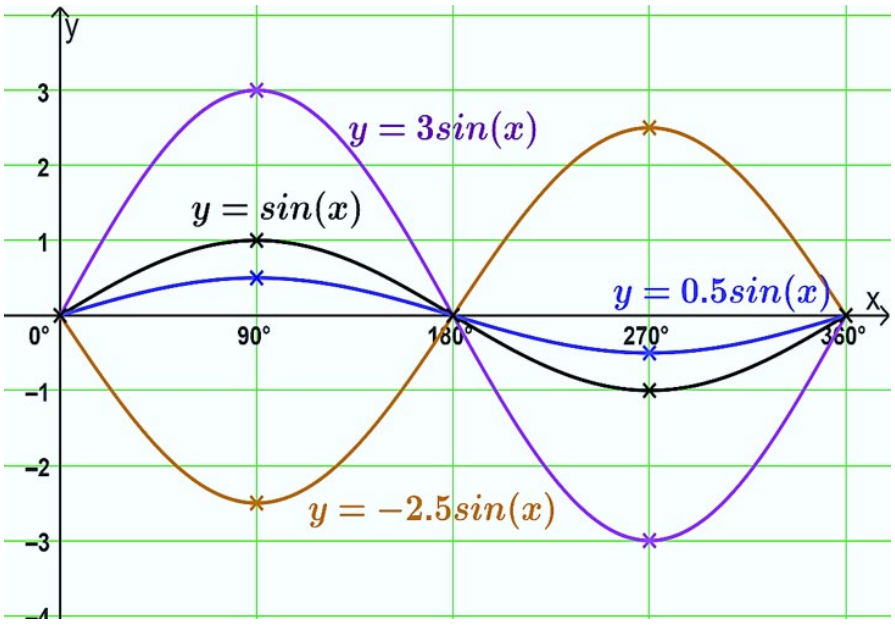


Figure 4.6: Graphical representation of $y = \sin x, y = 3 \sin x, y = 0.5 \sin x$ and $y = -2.5 \sin x$

3. The functions you have can be represented generally as A $\sin x$ where A assumes different values.
4. Post your graphical illustration on the walls of your class, move around and observe how your classmates did theirs.
5. Now, record your observation on the different functions as the value of A changes and share with your classmates.

Observations

When:

1. $A > 1$, the graph of the function stretches vertically.
2. $0 < A < 1$, the graph of the function compresses or shrinks vertically.
3. $A < 0$, the graph of the function reflects across the $x - axis$.

For these graphs, the amplitude, $|A|$ is the largest value the functions attain. $|A|$ is the distance between the midline and the highest or lowest point on the graph.

1. the graph of $y = A\sin(x)$ stretches or shrinks vertically as the value of $|A|$ changes.
2. when the value of $|A|$ gets larger, the graph is stretched more and as the value of $|A|$ gets smaller, it is compressed.
3. when $A > 0$, the graph increases before decreasing.
4. when $A < 0$, the graph decreases before increasing.
5. the $x -$ intercepts remain the same.
6. The width (domain of the function for a full revolution) does not change.
7. The midlines for all three graphs remain at $y = 0$.

Activity 4.3 Transformation of the form A cos x

1. Using the quadrantal angles as x values, create a table of values for $\cos x$, $3\cos x$, $0.5\cos x$, and $-2.5\cos x$.

   Your table of values should be similar to this:

x	0°	90°	180°	270°	360°
cos x	1	0	−1	0	1
3 cos x	3	0	−3	0	3
0.5 cos x	0.5	0	−0.5	0	0.5
-2.5 cos x	−2.5	0	2.5	0	−2.5

2. On the same graph sheet, draw the curves of points for $y = \cos x$, $y = 3\cos x$, $y = 0.5\cos x$ and $y = -2.5\cos x$.

Your graphical representation should be similar to Figure 4.7.

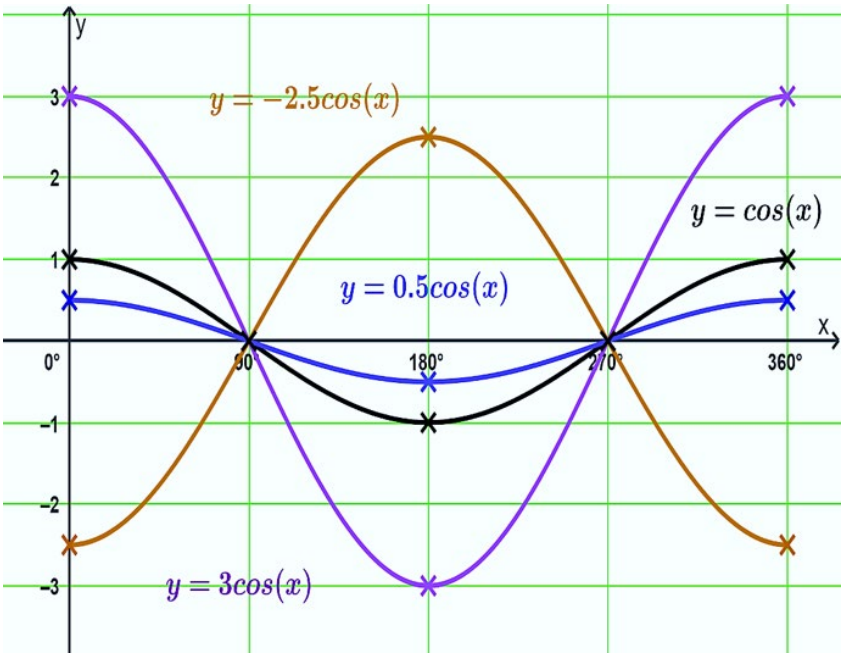


Figure 4.7: Graphical representation of $y = \cos x$, $y = 3\cos x$, $y = 0.5\cos x$ and $y = -2.5\cos x$

3. The functions you have can be represented generally as A $\cos x$ where A assumes different values.
4. Post your graphical illustration on the walls of your class, move around and observe how your classmates did theirs.
5. Now, record your observation on the different functions as the value of A changes and share with your classmates.

Example 4.1

Plot the graphs of $y = 2\sin x$ and $y = 4\sin x$ on the same axes and:

a. Explain how the amplitude affects the graph without changing the period.

b. Justify, using both algebraic reasoning and a diagram, why the maximum and minimum values change as they do.

Solution

x	0	90°	180°	270°	360°
$2\sin(x)$	0	2	0	−2	0
$4\sin(x)$	0	4	0	−4	0

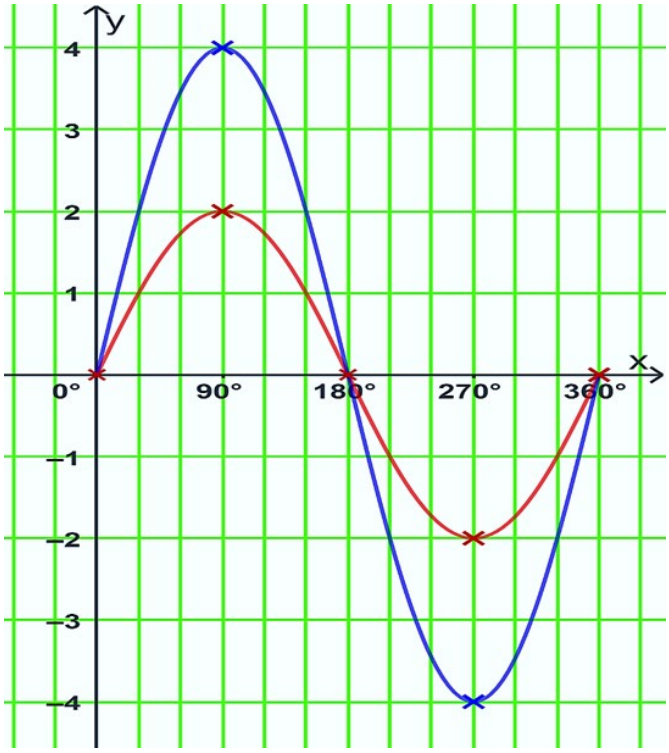


Figure 4.8: Graphical representation of $y = 2\sin x$ and $y = 4\sin x$

a. Compare $y = 2\sin x$ and $y = 4\sin x$

For $y = A\sin x$, amplitude $= |A|$, period $= 2\pi = 360°$.

So $2\sin x$ has amplitude 2; $4\sin x$ has amplitude 4; both have period 2π $(or\ 360°)$.

b. Max/min & range (algebraic justification):

Since $-1 \leq \sin x \leq 1$, multiplying by A gives $-|A| \leq A\sin x \leq |A|$.

Thus:

$y = 2\sin x$: range $[-2, 2]$; maxima $= 2$ at $x = \frac{\pi}{2} + 2k\pi$; minima $= -2$ at $x = \frac{3}{2}\pi + 2k\pi$.

$y = 4\sin x$: range $[-4, 4]$; same x-locations for peaks/troughs, but doubled height.

Geometrically there is a vertical stretch by factor 2 from the first to the second as seen in the graphical illustration.

Example 4.2

A student claims that the graph of $y = -3\cos x$ is the same as $y = 3\cos(x + 180^{\circ})$

a. Determine whether the claim is correct.

b. Support your answer with a clear explanation, a sketch, and an example showing the correspondence between the two graphs.

Solution

a. Using the identity $\cos(x+\pi) = -\cos x$, where $\pi \equiv 180^{\circ}$.

Then $3\cos(x+\pi) = 3(-\cos x) = -3\cos x$, so the claim is correct.

b. Point correspondence check:

When $x = 0$:

LHS	RHS
$-3\cos 0 = -3$	$3\cos\pi = 3(-1) = -3$

When $x = \frac{\pi}{2}$

LHS	RHS
$-3\cos\left(\frac{\pi}{2}\right) = 0$	$3\cos\left(\frac{3\pi}{2}\right) = 0$

Using table of values

x	0	90°	180°	270°	360°
$-3\cos x$	-3	0	3	0	-3
$3\cos(x+180^{\circ})$	-3	0	3	0	-3

The whole graph is a phase shift by π (half-period) of $3\cos x$, which exactly flips the sign.

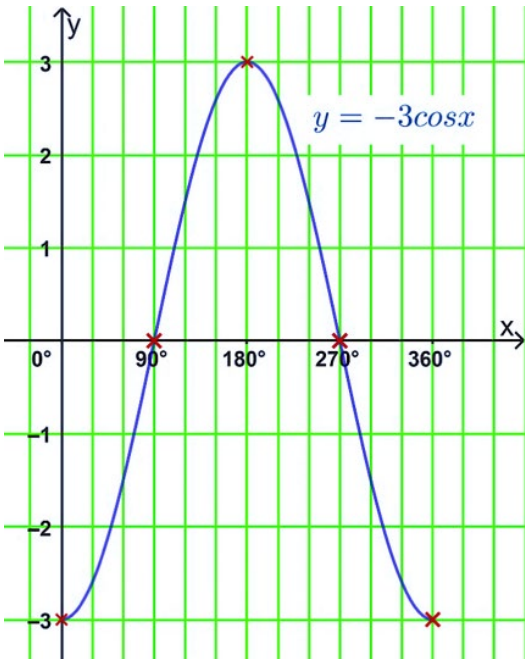


Figure 4.9: Graphical Illustration of

$y = -3\cos x$

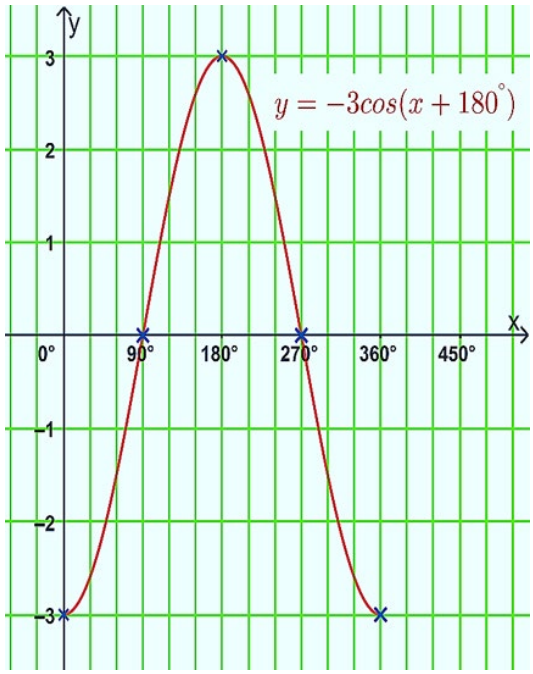


Figure 4.10: Graphical Illustration of

$y = 3\cos(x+180^{\circ})$

Transformation of the form *sin Bx or cos(Bx), A sin (Bx - C) or A cos(Bx - C)*

Recall in Year 1 Section 7 we learnt about **radian** measure. We explored how to convert radians to degrees and vice versa. Let us go through this example quickly as a refresher.

Example 4.3

Convert the following to degrees.

1. 1 radian
2. π radians
3. 4 radians
4. 5 radians

Solution

1. $1 \times \frac{180^\circ}{\pi} = 57.30^\circ$
2. $\pi \times \frac{180^\circ}{\pi} = 180^\circ$
3. $4 \times \frac{180^\circ}{\pi} = 229.18^\circ$
4. $5 \times \frac{180^\circ}{\pi} = 286.48^\circ$

Example 4.4

Express the following in radians.

1. 180°
2. 75°
3. 32°

Solution

1. $180^\circ \times \frac{\pi}{180^\circ} = \pi$
2. $75^\circ \times \frac{\pi}{180^\circ} = \frac{5}{12}\pi$
3. $32^\circ \times \frac{\pi}{180^\circ} = \frac{8}{45}\pi$

Now that we remember how to change radians into degrees and vice versa, we shall make use of the two interchangeably while dealing with transformation of trigonometric functions.

The graphs of the functions $\sin(Bx)$ or $\cos(Bx)$ have amplitude $|A|$ and period $\frac{2\pi}{B}$ or $\frac{360°}{B}$. The appropriate interval of one complete period is $\left[0, \frac{2\pi}{B}\right]$ or $[0, 360°]$.

Note

- $\sin(Bx) = A\sin(Bx - C)$ where $A = 1, C = 0$
- $\cos(Bx) = A\cos(Bx - C)$ where $A = 1, C = 0$

Activity 4.4 Transformation of the form A sin(Bx - C) or A cos(Bx - C)

1. Exploration on Amplitude (A): Watch the video https://youtu.be/Wvpa6He9qf0 and pay close attention to the changes happening to the functions.



   a. What do you observe about the maximum and minimum values of the graph when you change the value of A?
   b. If you set $A = 1$, then increase it to $A = 3$, what changes and what stays the same?
   c. Watch this video https://youtu.be/q7HS9tnPrQU. What happens to the graph when A becomes negative?



   d. Describe the effect of a negative amplitude.
   e. Based on your observations, how would you define the amplitude of the function?

2. Exploration on Frequency (B) and Period. Watch this video https://youtu.be/lQbxO7YDhiI and answer the questions that follow.



a. What happens to the distance between repeating patterns (periods) as you adjust the values for B?
b. Compare the graph of $\sin(x)$ and $\sin(2x)$ and identify the difference on how often each repeat.
c. How would you calculate the period of the graph using B?
d. What do you notice about the graphs when $B = 0$?

3. Exploration for Phase Shift

a. Draw the graphs of $y = \sin x$, $y = \sin(x - 2)$ and $y = \sin(x + 2)$ using the interval, $0 \leq x \leq 360^{\circ}$.
b. What happens to the position of the starting point of the sine wave when C changes?
c. What is the horizontal shift from $C = 0$ to when $C = 2$?
d. What is the horizontal shift from $C = 0$ to when $C = -2$?
e. What is the horizontal shift when $C = 0.5$?
f. Set $B = 1$ and make changes to C and explore how far the graph shifts with each change.
g. Explore same using the cosine function.

4. Discuss your general observations about the transformation of the form $A \sin(Bx - C)$ and $A \cos(Bx - C)$ with your classmates.

Note

Without graphing software, you would have to build a table of values for each function with different values for A, B and C and graph them to observe the changes in Amplitude, Period and Phase Shift.

Table 4.2: Conclusions for transformations of the form $A\ sin(Bx - C)$ or $A\ cos(Bx - C)$

$\mathbf{A\ sin(Bx - C)}$ or $\mathbf{A\ cos(Bx - C)}$
Amplitude $= \mathbf{|A|}$
It has a period of $\frac{2\pi}{B}$ or $\frac{360^{\circ}}{B}$
The interval of one cycle is $\frac{C}{B} \leq x \leq \frac{C}{B} + \frac{360^{\circ}}{B}$

Note

"$Bx - C$" in the equation represents a shift/translation to the right while "$Bx + C$" represents a shift/translation to the left.

Example 4.5

Describe the transformations performed on the graph of $y = 3\sin(x)$ to obtain the graph of one cycle of $y = 4\sin(2x - 60°)$.

Refer to Figure 4.11 for the graphs.

The broken lines indicate a second period thus the first period ends at 180° for $y = 4\sin(2x - 60°)$.

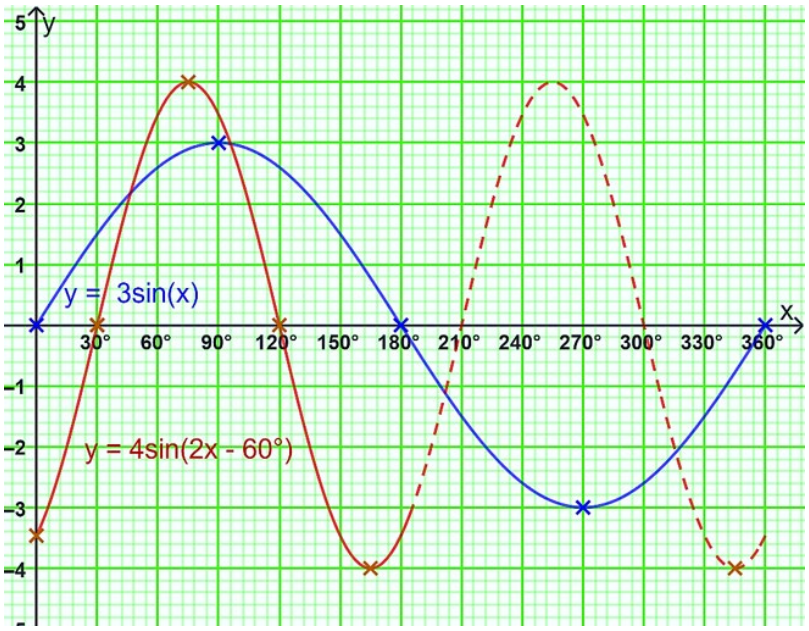


Figure 4.11: Graphical Illustration of $y = 4\sin(2x - 60°)$ and $y = 3\sin(x)$.

Solution

The graph of $y = 4\sin(2x - 60^\circ)$ has a period of 180°, i.e., from 30° to 210°which is half the period of $y = 3\sin(x)$. This implies a horizontal stretch of the graph of $y = 3\sin(x)$ by a scale factor of $\frac{1}{2}$.

The graph of $y = 4\sin(2x - 60^\circ)$ starts from $x = 30^\circ$ while the graph of $y = 3\sin(x)$ starts from the origin. This depicts a horizontal shift to the right by 30° confirmed numerically by $\frac{C}{B} = \frac{60^\circ}{2} = 30^\circ$.

The maximum and minimum values of the graph of $y = 4\sin(2x - 60^\circ)$ are 4 and -4 respectively, whereas those of $y = 3\sin x$ are 3 and -3. This implies a vertical stretch by a scale factor of $\frac{4}{3}$.

The midline remains unchanged at $y = 0$, showing that there is no vertical translation.

Example 4.6

The up-and-down motion of a child on a rope swing can be modelled by $y = 3\sin\left(2x - \frac{\pi}{6}\right)$.

a. Without graphing, describe when the child will be at the highest point, lowest point, and midline.

b. Explain how amplitude and frequency affect the swing's motion.

c. Why would a parent need to know the maximum height of the swing?

Solution

a. For $y = 3\sin\left(2x - \frac{\pi}{6}\right)$, $A = 3, B = 2, C = \frac{\pi}{6}$.

Period $(T) = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase shift $\left(\frac{C}{B}\right) = \left(\frac{\pi}{6}\right) \div 2 = \frac{\pi}{12}$ to the right.

Since there is no vertical shift, the midline is $y = 0$.

Since $-1 \leq \sin x \leq 1$, multiplying by A gives $-|A| \leq A\sin x \leq |A|$.

For $y = 3\sin\left(2x - \frac{\pi}{6}\right)$: range $[-3, 3]$;

Let $\theta = 2x - \frac{\pi}{6}$ then $y = 3\sin\theta$

For the midline, $\theta = k\pi, (k \in \mathbb{Z})$

$$2x - \frac{\pi}{6} = k\pi$$

$$x = \frac{k\pi}{2} + \frac{\pi}{12}.$$

So, the child is at midline when $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \ldots$

For maximum height, the swing reaches its highest point when $\sin\theta = 1$

$$\theta = \frac{\pi}{2} + 2k\pi$$

$$2x - \frac{\pi}{6} = \frac{\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{3} + k\pi$$

At such x values, $y = 3$

For minimum height, the swing reaches its lowest point when $\sin\theta = -1$

$$\theta = \frac{3\pi}{2} + 2k\pi$$

$$2x - \frac{\pi}{6} = \frac{3\pi}{2} + 2k\pi$$

$$x = \frac{5\pi}{6} + k\pi$$

At such x values, $y = -3$

b. For the amplitude, the child swings 3 units above and below the midline. Thus, the larger the amplitude, the higher the swing.

For the frequency, since the period is π, the swing completes one full cycle every π units of time. A larger B will make the swing faster.

c. A parent needs to know the maximum height to decide if the swing is too high for the child's age. It helps ensure the swing does not collide with the frame or nearby objects and aids in taking all necessary precautions.

Example 4.7

Draw the graph of $y = 2\cos\left(3x - \frac{\pi}{2}\right)$ and describe the transformations performed on the graph of $y = \cos x$ to obtain $y = 2\cos\left(3x - \frac{\pi}{2}\right)$.

Solution

The broken lines indicate a second period.

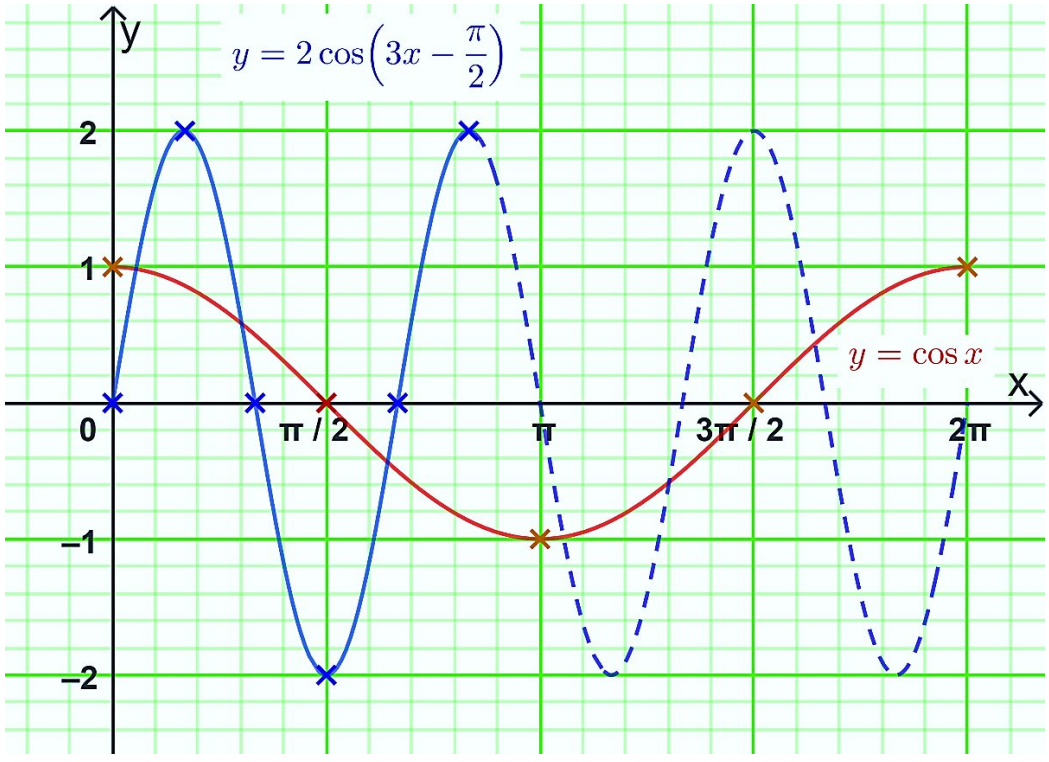


Figure 4.12: Graphical Illustration of $y = 2\cos\left(3x - \frac{\pi}{2}\right)$ and $y = \cos(x)$.

Amplitude $= 2$

Period $= \frac{2\pi}{B} = \frac{2\pi}{3}$

Step $= \frac{2\pi}{3} \times \frac{1}{2} = \frac{\pi}{3}$.

Phase shift (Left endpoint): $\frac{C}{B} = \frac{\pi}{2} \div 3 = \frac{\pi}{6}$

Right endpoint: $\frac{C}{B}$ + period $= \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

The interval is $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

Points are $\left(\frac{\pi}{6}, 2\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{\pi}{2}, -2\right), \left(\frac{2\pi}{3}, 0\right)$ and $\left(\frac{5\pi}{6}, 2\right)$.

The graph of $y = 2\cos\left(3x - \frac{\pi}{2}\right)$ has period $\frac{2\pi}{3}$, i.e., one cycle runs from $x = \frac{\pi}{6}$, to $x = \frac{5\pi}{6}$ which is one third of the period of $y = \cos x$. This corresponds to a horizontal shrink by a factor of 3. The graph is shifted right by $\frac{\pi}{6}$. The maximum and minimum values are 2 and -2, so the graph is vertically stretched by a factor of 2. The midline remains $y = 0$, so there is no vertical translation.

Transformation of the form *A sin (Bx - C) ± D or A cos (Bx - C) ± D*

Activity 4. 5 Daylight hours modelling

1. Working in pairs, go through this scenario:

   In Accra, Ghana, the number of daylight hours varies slightly over the course of a year due to the tilt of the Earth's axis, but it remains within a narrow range, approximately 11.9 to 12.4 hours per day. The daylight hours over 12 months can be modelled by the function $\mathrm{M}(t) = \mathrm{A}\cos(\mathrm{B}t - \mathrm{C}) + \mathrm{D}$ and are presented in the table.

Month	Jan.(0)	Feb.(1)	Mar.(2)	Apr. (3)	May(4)
Daylight hours (t)	11.9	12	12.1	12.3	12.4

Jun.(5)	Jul.(6)	Aug. (7)	Sep.(8)	Oct. (9)	Nov.(10)	Dec. (11)
12.3	12.2	12.1	12.0	11.9	11.8	11.9

2. Using an appropriate scale, plot the data presented on your graph sheet.
3. Join the points to form a curve.
4. From your graph, estimate the:

   a. Average daylight (D)

b. Amplitude of the curve
c. The period $\left(\frac{2\pi}{B}\right)$
d. The phase shift $\left(\frac{C}{B}\right)$

5. Substitute the values obtained to build your model.
6. According to your model, in which months will Accra experience the shortest and longest daylight hours?
7. How many hours of daylight are there in August? Use your model to estimate.
8. If you lived in a country near the poles, how would the amplitude and midline of the graph change?
9. Why is a cosine graph a better model than a straight line for this situation?

Example 4.8

Draw the graph of $y = -3\sin\left(2x + \frac{\pi}{3}\right) + 1$ and describe the transformations performed on the graph of $y = \sin(x)$ to obtain one cycle of the given graph.

Solution

Comparing $y = -3\sin\left(2x + \frac{\pi}{3}\right) + 1$ to $y = A\sin(Bx - C) + D$, $A = -3, B = 2, C = -\frac{\pi}{3}$, $D = 1$.

Hence, the Amplitude $|A| = 3$ but reflects in the midline because $A < 0$.

Period $= \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase shift (left endpoint) $= \frac{C}{B} = -\frac{\pi}{3} \div 2 = -\frac{\pi}{6}$.

Right endpoint: $\frac{C}{B} + period = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$.

Points in one cycle are $\left(\frac{\pi}{6}, 1\right), \left(\frac{\pi}{12}, -2\right), \left(\frac{\pi}{3}, 1\right), \left(\frac{7\pi}{12}, 4\right), \left(\frac{5\pi}{6}, 1\right)$.

There is an upward vertical shift of 1 because $D = 1$. Hence the midline $y = 1$.

The broken lines in Figure: 4.13 indicate a second period.

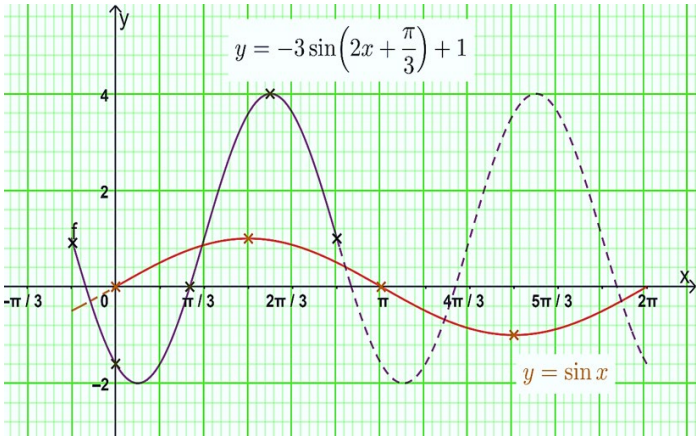


Figure: 4.13: Graphical Illustration of $y = -3\sin\left(2x + \frac{\pi}{3}\right) + 1$ and $y = \sin(x)$ $y = \sin(x)$.

The graph of $y = -3\sin\left(2x + \frac{\pi}{3}\right) + 1$ has period π (one cycle from $x = -\frac{\pi}{6}$ to $\frac{5\pi}{6}$), which is half the period of $y = \sin x$; this corresponds to a horizontal shrink by factor 2.

The "$\frac{\pi}{3}$" inside causes a shift left by $\frac{\pi}{6}$ (since $\frac{C}{B} = -\frac{\pi}{6}$).

The amplitude is 3 but A is negative, so the graph is reflected in its midline and vertically stretched by a factor of 3.

Finally, the whole graph is shifted up by 1 unit so the midline is $y = 1$.

Because A is negative, it reverses the usual rising/falling pattern of $y = \sin x$.

Example 4.9

Compare the graphs of $y = \cos x$ and $y = -\frac{1}{2}\cos(4x - \pi) - 3$.

Explain how the negative sign, the coefficient of x and the constants inside and outside the function affect the graph of $y = \cos x$.

Solution

For $y = \cos x$, Amplitude $= 1$, Period $= 2\pi$, Midline $y = 0$, Range $= [-1, 1]$, Starts at $(0, 1)$, symmetric about the y-axis.

Compared to $y = \cos x$ the graph of $y = -\frac{1}{2}\cos(4x - \pi) - 3$ is reflected in the x-axis because of the negative sign of A. The coefficient of the cosine being $-\frac{1}{2}$, implies an amplitude $|A|$ of $\frac{1}{2}$ thus vertically shrinking the graph by $\frac{1}{2}$.

For the coefficient of x the "4" inside $\cos(4x - \pi)$, the period becomes $\frac{2\pi}{4} = \frac{\pi}{2}$. This means the cosine wave completes a cycle four times faster, a horizontal compression by factor 4.

The constant inside $(-\pi)$, causes a phase shift $\left(\frac{C}{B}\right) = \frac{\pi}{4}$ to the right.

The constant outside (-3), translates the entire graph down by 3 units hence the midline moves from $y = 0$ to $y = -3$. The range also shifts from $[-1, 1]$ to $[-3.5, -2.5]$.

MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRIC GRAPHS

In Section 3, you learnt about Parabolas and how to determine the minimum and maximum points of such functions. We are going to apply that knowledge to determine minimum and maximum values of trigonometric graphs.

When the graphs are sketched, we can use inspection to determine the maxima/minima values. Another way to find the maxima and minima of trigonometric expressions of the form $A\sin(B\theta \pm C) + D$ or $A\cos(B\theta \pm C) + D$ without sketching is to multiply A by 1 and -1 and add the D.

Mathematically:

Minimum or maximum value $= A + D$ or

Minimum or maximum value $= -A + D$

Note

1. D could be negative or positive.
2. The smaller values of "$A + D$" and "$-A + D$" is the minimum value of the expression.
3. The larger values of "$A + D$" and "$-A + D$" is the maximum value of the expression.

Example 4.10

If $y = 5\sin(\theta)$, determine the maximum and minimum value of y.

Solution

Rewriting the function in the form $A\sin(B\theta \pm C) + D$, $A = 5$ and $D = 0$

Using $\mathrm{A}(-1) + \mathrm{D}$, substitute the values of A and D

Now, $\mathrm{A}(-1) + \mathrm{D} = 5(-1) + 0 = -5$

Using $A(1) + D$,

$A(1) + D = 5(1) + 0 = 5$

Since $-5 < 5$, -5 is the minimum value while 5 is the maximum value of y

Example 4.11

Given the function $f(\phi) = 3\cos(\phi)$, determine the maximum and minimum value of $f(\phi)$.

Solution

Rewriting the function in the form $A\cos(B\theta \pm C) + D$, $A = 3$ and $D = 0$

Using $\mathrm{A}(-1) + \mathrm{D}$, substitute the values of A and D

Now, $\mathrm{A}(-1) + \mathrm{D} = 3(-1) + 0 = -3$

Using $A(1) + D$,

$A(1) + D = 3(1) + 0 = 3$

Since $-3 < 3$, -3 is the minimum value while 3 is the maximum value of $f(\phi)$.

Example 4.12

Determine the minima and maxima values of:

a. $h(x) = -3\cos\left(\frac{1}{4}x\right) + 2$
b. $f(\gamma) = 3\sin(2\gamma) + 4$

Solution

a. From the function, $A = -3$ and $D = 2$

For A + D,
$A + D = -3 + 2 = -1$

For $-A + D$,
$-A + D = 3 + 2 = 5$

For $h(x)$, the maximum value is 5 and the minimum value is -1.

b. From the function, $A = 3$ and $D = 4$

For A + D,
$A + D = 3 + 4 = 7$

For $-A + D$,
$-A + D = -3 + 4 = 1$

For $f(\gamma)$, the maximum value is 7 and the minimum value is 1.

REVIEWING TRIGONOMETRIC IDENTITIES

In year 2 section 9, you learnt about compound angle identities, multiple and double angle identities. You engaged in activities where you researched the algebraic proof of the identities and solved examples to know when to apply these identities. The compound angle identities are listed below to serve as a reminder.

1. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
2. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
3. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
4. $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
5. $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$
6. $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$

The multiple angle identities are:

1. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

2. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$
3. $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

We are going to make use of these identities in deriving harmonic functions.

DERIVING HARMONIC IDENTITIES

Expression of *a cos (0) + b sin(0) as R cos(0 - a)*

We have $a\cos(\theta) + b\sin(\theta)$ as a trigonometric expression made up of two trigonometric functions. It can however, be expressed as $R\cos(\theta - \alpha)$ which comprises of only one trigonometric identity called a Harmonic Identity. There are several of such harmonic identities but we will go through an activity to generate the first one while you carry out personal research to prove the others.

Activity 4.6 Deriving Harmonic Identity *R cos(0 - a)*

1. Graph the function $y = 3\cos(\theta) + 4\sin(\theta)$ and analyse it.
2. Do you think the function can be rewritten as a single sine or cosine function?
3. Express $3\cos(\theta) + 4\sin(\theta)$ generally as $a\cos(\theta) + b\sin(\theta)$
4. Expand $R\cos(\theta - \alpha)$ using the appropriate compound angle identity.

   You should get $R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\alpha\sin\theta$.

   When rearranged, should be $R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$

5. Compare the rearranged function to $a\cos(\theta) + b\sin(\theta)$

   $R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$ and $a\cos(\theta) + b\sin(\theta)$

6. Equate and match the coefficients of $\cos\theta$ and $\sin\theta$ to get;

   $a = R\cos\alpha, b = R\sin\alpha$

   This means that $a\cos(\theta) + b\sin(\theta) = R\cos(\theta - \alpha)$

7. From Step 5, square each of the terms on your LHS and RHS and add the two equations.

   $a^2 = R^2\cos^2\alpha, b^2 = R^2\sin^2\alpha$

   $a^2 + b^2 = R^2(\cos^2\alpha + \sin^2\alpha)$

8. Recall from Pythagorean identities that $\cos^2\alpha + \sin^2\alpha = 1$. Thus $a^2 + b^2 = R^2$.
9. Take the square root of both sides of the equation;

   $\sqrt{a^2 + b^2} = R$

10. Does the final equation look familiar?
11. Yes, it looks exactly like Pythagoras' theorem.

12. From Step 5, taking a ratio of the two equations results in $\frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$
13. This can be simplified further to obtain $\tan\alpha = \frac{b}{a}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.
14. Discuss your observations from this activity with a classmate and your teacher.

Trigonometric Expression	Harmonic identities
$a\cos(\theta) + b\sin(\theta) + c$	$R\cos(\theta - \alpha) + c$
$a\cos(\theta) - b\sin(\theta) + c$	$R\cos(\theta + \alpha) + c$
$a\cos(\theta) + b\sin(\theta) + c$	$R\sin(\theta + \alpha) + c$
$a\cos(\theta) - b\sin(\theta) + c$	$R\sin(\theta - \alpha) + c$

Example 4.13

Express $3\cos(\theta) - 2\sin(\theta)$ in the form $R\cos(\theta + \alpha)$.

Solution

Comparing $3\cos(\theta) - 2\sin(\theta)$ to $a\cos(\theta) - b\sin(\theta) + c$, $a = 3$, $b = 2$ and $c = 0$.

Recall that $R = \sqrt{a^2 + b^2}$

$R = \sqrt{3^2 + 2^2}$

$R = \sqrt{13}$ units

Also, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

Thus $\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.6901^\circ$

Now, $R\cos(\theta + \alpha) = \sqrt{13}\cos(\alpha + 33.6901^\circ)$

$\therefore\ 3\cos(\theta) - 2\sin(\theta) = \sqrt{13}\cos(\alpha + 33.6901^\circ)$.

Example 4.14

The height of a point on the surface of the sea due to a wave is given by $y = 7\cos(x) + 24\sin(x)$.

Calculate the maximum height of the wave above sea level.

Solution

Given functions in the form $A\sin(B\theta \pm C) + D$ or $A\cos(B\theta \pm C) + D$, the maximum value is $A + D$.



Figure: 4.14: Wave

Our function $y = 7\cos(x) + 24\sin(x)$ can be expressed as $A\sin(B\theta \pm C) + D$ or $A\cos(B\theta \pm C) + D$ when rewritten as a harmonic identity.

Step 1: Express the given function as a harmonic identity

Comparing $7\cos(x) + 24\sin(x)$ to $a\cos(\theta) + b\sin(\theta) + c$, $a = 7$, $b = 24$ and $c = 0$.

Making use of a harmonic identity $R\sin(\theta + \alpha) + c$

Now, $7\cos(x) + 24\sin(x) = R\sin(\theta + \alpha) + c$ but $c = 0$

Step 2: Compare the harmonic identity chosen to $A\sin(B\theta \pm C) + D$ or $A\cos(B\theta \pm C) + D$ to determine the maximum value.

Comparing $R\sin(\theta + \alpha) + c$ to $A\sin(B\theta \pm C) + D$, $A = R, B = 1, C = \alpha, D = c$

But maximum value is $A + D \equiv R + c$

Since $c = 0$, the maximum value is $R + 0$ hence the maximum height will be;

$$R = \sqrt{a^2 + b^2}$$

$$R = \sqrt{7^2 + 24^2}$$

$$= \sqrt{576 + 49}$$

$$= 25$$

∴ The maximum height of the wave above sea level is 25 units.

Note

In using any of the harmonic identities;

- maximum value is $R + c$
- minimum value is $-R + c$

Example 4.15

The displacement of a pendulum bob from its rest position is modelled by:
$y = 5\sin(x) - 12\cos(x)$.

Find the greatest displacement of the pendulum from the equilibrium position.

Solution

Step 1: Rearrange the function

$5\sin(x) - 12\cos(x) = -12\cos(x) + 5\sin(x)$

Now, $y = -12\cos(x) + 5\sin(x)$

Step 2: Express the given function as a harmonic function (identity)

Now, $-12\cos(x) + 5\sin(x) = a\cos(\theta) + b\sin(\theta) + c$,

where $a = -12, b = 5, c = 0$.

Also, $a\cos(\theta) + b\sin(\theta) + c = R\cos(\theta - \alpha) + c$

Step 3: Determine the maximum value

Since $c = 0$, maximum value will be R.

$R = \sqrt{a^2 + b^2}$

$R = \sqrt{(-12)^2 + 5^2}$

$= \sqrt{169}$

$= 13$

∴ the greatest displacement of the pendulum from the equilibrium position is 13 units.

Example 4.16

The vertical displacement of a vibrating spring is given by $y = 9\cos(\theta) - 40\sin(\theta) - 3$. Calculate the maximum and minimum displacement of the spring.

Solution

Step 1: Express the given function as a harmonic function (identity)

$9\cos(\theta) - 40\sin(\theta) - 3 = a\cos(\theta) - b\sin(\theta) + c$,

where $a = 9, b = 40, c = -3$.

Also, $a\cos(\theta) - b\sin(\theta) + c = R\sin(\theta - \alpha) + c$

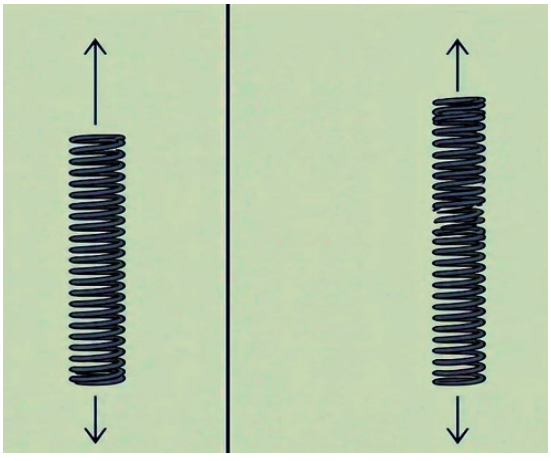


Figure: 4.15: Displaced Spring

Step 2: Determine the maximum value

Since $c = -3$, maximum value will be $R + c$.

$R + c = \sqrt{a^2 + b^2} + c$

$R + c = \sqrt{9^2 + 40^2} + (-3)$

$= \sqrt{1681} - 3$

$= 41 - 3$

$= 38$

Step 3: Determine the minimum value

Since $c = -3$, and $R = 41$, the minimum value will be $-R + c$

$-R + c = -41 - 3$

$= -44$

$\therefore$ the maximum and minimum displacement of the spring is 38 units and -41 units respectively.

REVIEW QUESTIONS 4

1. Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{3}\right) + 1$ for $0 \leq x \leq 2\pi$. From your graph:

   a. Identify its amplitude, period, phase shift and vertical shift.

   b. Explain how these transformations can be interpreted geometrically on the unit circle.

2. The function $y = -\cos\left(\frac{1}{2}x\right)$ models alternating current in physics.

   Sketch the graph of the function for $0 \leq x \leq 4\pi$ and using your graph:

   a. Identify its amplitude and period.

   b. Discuss the meaning of the negative sign and the coefficient of x in terms of rotations on the unit circle.

3. Compare the graphs of $y = \sin(x)$ and $y = \sin(x) + 3$ for $-360^\circ \leq x \leq 360^\circ$.

4. The height of the tide at a port can be modelled by $H(t) = 5\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) + 7$ where t is the time in hours after midnight.

   Using the unit circle and transformations, explain how to determine the tide is at its highest and lowest.

5. Show that $\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

   Hence state the maximum and minimum values of $\cos x + \sin x$.

6. Express $3\cos x - 4\sin x$ in the form $R\cos(x + \alpha)$ where $R > 0$ and α (to 2 decimal places) is acute.

   Determine the maximum value of the expression.

7. The vertical displacement of a vibrating string is modelled by $g = 4\cos\beta - 7\sin\beta$. Write the function in harmonic form and determine the least displacement of the string to the nearest whole number.

8. Find the maximum and minimum values of the function $y = -4\sin x + 2$.

9. A trigonometric function is defined by $t = 3\sin v + 4$.

   a. Determine the maximum and minimum values of the function.

   b. Compare the values obtained in (a) with those of $y = 3\sin x$ and explain the effect of the constant 4.

10. Write down the identities for:

    a. $\sin(A + B)$

    b. $\cos(A - B)$

    c. $\tan(A - B)$

SECTION

5 INTEGRATION



PRINCIPLES OF CALCULUS

INTRODUCTION

This topic builds upon the concepts you learned in your second year and is a fundamental aspect of calculus. Integration is a branch of calculus that supports us in finding the approximate values of functions. It is used to find the area under curves, marginal cost, etc. It is also a prerequisite for engineering science, physics, etc.

In this section, we will look at the integration of functions, including logarithms and natural logarithms. We shall also look at how to approximate the area under a curve using the trapezium rule.

KEY IDEAS

- **Antiderivatives**: If f′(x) represents the derivative of a function $f(x)$, then $f(x)$ is known as an antiderivative of $f'(x)$.

  This is expressed as $\int f'(x)\ dx = f(x) + C$.
- **Finding Antiderivatives**: For $f(x) = x^n$
  - Differentiate: $\frac{d}{dx}(x^n) = nx^{n-1}$
  - Integrate: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (where $n \neq -1$).
- **Methods of Substitution**: When an integral can be expressed in the form $\int f[g(x)]g'(x)\,dx$, use $u = g(x)$ to simplify the integral.
- **Transcendental Functions**: Functions like e^x and $ln(x)$ that cannot be expressed as algebraic equations.

  Integration examples include $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ and $\int In|x|\,dx = xIn|x| + C$.

IDENTIFYING AND APPLYING THE INTEGRATION RULES TO EVALUATE INTEGRALS

Understanding Antiderivatives

If $f'(x)$ represents the derivative of a function $f(x)$, then $f(x)$ is known as an antiderivative of $f'(x)$. This relationship can be mathematically expressed as: $\int f'(x)\,dx = f(x) + c$, where c is the constant of integration. This constant is crucial because there are infinitely many antiderivatives corresponding to any given derivative, each differing by a constant.

Finding Antiderivatives

For the function $f(x) = x^n$: Recall the following steps:

1. Differentiation:

   Multiply the function by the power n and subtract 1 from the power to obtain: nx^{n-1}
2. Integration (Reverse of Differentiation):
   a. Add 1 to the power n to get $n + 1$: x^{n+1}
   b. Divide the function by the new power $\frac{x^{n+1}}{n+1} + c$

Activity 5.1 Understanding Antiderivatives through Integration

Concept Overview

When we integrate the derivative of a function $f'(x)$, we obtain the original function $f(x)$ plus a constant: $\int f'(x)\,dx = f(x) + c$

For example, if $f(x) = x^3$

$$\frac{dy}{dy} = 3x^2$$

$$f'(x) = 3x^2$$

How will change or reverse the result, $\frac{dy}{dy} = 3x^2$ back to $f(x) = x^3$?

The procedure of reversing $f'(x) = 3x^2$ to $f(x) = x^3$ is using Integration denoted by the symbol $\int$

Integrating Power Functions

1. **Given the function**: $f(x) = x^n$
2. **Integration Steps**:

   Step 1: Add 1 to the exponent n to obtain $n + 1$

   Step 2: Divide by the new exponent:

   $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

   Where c is the constant of integration.

Indefinite Integral is when we have to integrate a function which has no limits. For example, $\int x^4 dx$. On the other hand, $\int_1^2 x^4 dx$, is a definite integral, because it has a limit [1, 2].

Other rules of integration are as follows

Rule	Function	Procedure
Power rule	$f(x) = ax^n$	$\int f(x)dx = \int ax^n dx = \frac{ax^{n+1}}{n+1} + c, n \neq -1$
Sum rule	$f(x) + h(x)$	$\int [f(x) + h(x)]\,dx = \int f(x)dx + \int h(x)\,dx$
Difference rule	$f(x) - h(x)$	$\int [f(x) - h(x)]\,dx = \int f(x)dx - \int h(x)\,dx$
Multiplication rule	$kf(x)$	$\int kf(x)dx = k\int f(x)dx$

Example 5.1

Identify the various rules that will be applicable in evaluating the following

1. $\int 5x^3 dx$
2. $\int 6x dx$
3. $\int (2x - 8x^2)dx$
4. $\int (3x^3 + 4x^2)dx$
5. $\int (\frac{x^4}{x^2} + 4)dx$

Solution

1. The power rule and the multiplication rule
2. The power rule
3. The difference rule, the multiplication rule
4. The sum rule and the multiplication rule
5. The sum rule

Definite Integral

Definite integrals are integrals that have upper and lower limits and can be evaluated as a numerical value.

The Second Fundamental Theorem of Calculus is a key concept in calculus that connects two important ideas: **definite integrals** and **antiderivatives**

Definite Integral: The definite integral $\int_a^b f(x)dx$ represents the **area under the curve** of the function $f(x)$ from $x = a$ to $x = b$. It helps us find the total accumulation of a quantity (like area, distance, etc.) over an interval.

Antiderivative: An antiderivative of a function $f(x)$ is another function $F(x)$ such that when you take the derivative of $F(x)$, you get back $f(x)$. In other words, $F'(x) = f(x)$. For example, if $f(x) = 2x$, then an antiderivative $F(x)$ could be x^2, because the derivative of x^2 is $2x$.

The Second Fundamental Theorem of Calculus states: $\int_a^b f(x)dx = F(b) - F(a)$

This means that to find the area under the curve $f(x)$ from a to b, we:

1. Find an Antiderivative $F(x)$ of $f(x)$.
2. Evaluate $F(x)$ at the endpoints b and a.
3. Subtract the value at a from the value at b: $F(b) - F(a)$.

The Second Fundamental Theorem of Calculus simplifies the process of finding the area under a curve by allowing us to use antiderivatives instead of relying solely on approximations. It shows the powerful relationship between differentiation and integration, making calculus a more manageable and effective tool for solving real-world problems.

Example 5.2

Evaluate the following:

a. $\int_{-1}^{2}(x^2 + 2x)dx$
b. $\int_{0}^{3}(6 - 3x + 2x^2)dx$
c. $\int_{-2}^{4} 5dx$
d. $\int_{0}^{\pi}(sinx)dx$

Solution

a. $\int_{-1}^{2}(x^2 + 2x)dx$

Find an Antiderivative: $F(x)$ as $\left[\frac{x^3}{3} + x^2\right]_{-1}^{2}$

Evaluate: $F(x)$ at the endpoints 2 and -1 as

$F(2) = \left[\frac{2^3}{3} + (2)^2\right]$

$F(-1) = \left[\frac{(-1)^3}{3} + (-1)^2\right]$

Subtract the value at a from the value at b: $F(2) - F(-1)$.

$$F(2) - F(1) = \left[\frac{8}{3} + 4\right] - \left[\frac{-1}{3} + 1\right]$$

$$= \left[\frac{20}{3}\right] - \left[\frac{2}{3}\right]$$

$\int_{-1}^{2}(x^2 + 2x)dx = 6$

b. $\int_0^3 (6 - 3x + 2x^2)dx$

Find an Antiderivative $F(x)$: as $\left[6x - \frac{3x^2}{2} + \frac{2x^3}{3}\right]_0^3$

Evaluate $F(x)$ at the endpoints 3 and 0 as;

$$F(3) = \left[6(3) - \frac{3(3)^2}{2} + \frac{2(3)^3}{3}\right] F(0) = \left[6(0) - \frac{3(0)^2}{2} + \frac{2(0)^3}{3}\right]$$

Subtract the value at a from the value at b: $F(3) - F(0)$.

$$F(3) - F(0) = \left[18 - \frac{27}{2} + 18\right] - [0]$$

$$= \left[\frac{45}{2}\right] - [0]$$

$$\int_0^3 (6 - 3x + 2x^2)dx = \frac{45}{2}$$

c. $\int_{-2}^4 5dx = [5x]_{-2}^4$

$$= [5(4)] - [5(-2)]$$

$$= [20] - [-10]$$

$$= 30$$

Hence $\int_{-2}^4 5dx = 30$

d. $\int_0^\pi (\sin x)dx = [-\cos x]_0^\pi$

$$= [-\cos(\pi)] - [-\cos(0)]$$

$$= [-(-1)] - [-1]$$

$$= [1 + 1]$$

$$= [2]$$

$$\int_0^\pi (\sin x)dx = 2$$

IDENTIFYING AND APPLYING APPROPRIATE TECHNIQUES FOR INTEGRATING A FUNCTION

Methods of Substitution

If an integral can be written in the form $\int f[g(x)]g'(x)\,dx$, then we can set $u = g(x)$ and integrate $\int f(u)\,du$. In other words, when an integral is not in a standard algebraic or trigonometric form, it can often be simplified using substitution.

Example 5.3

Let us go through this process step-by-step using the integral $\int(\sqrt{2x+4})dx$ as an example.

Step 1: Choose a Substitution

Let $u = 2x + 4$

Step 2: Differentiate the Substitution.

Next, we differentiate u with respect to x: $\frac{du}{dx} = 2$

Step 3: Solve for dx or make dx the subject.

Now, we rearrange this equation to express dx in terms of du: $dx = \frac{1}{2}du$

Step 4: Substitute into the Integral.

Now we substitute u and dx into the original integral. The integral becomes $\int(\sqrt{2x+4})dx = \int \sqrt{u} \times \frac{1}{2}du$

Step 5: Simplify the Integral

This can be rewritten as: $\frac{1}{2}\int(\sqrt{u})du$

Example 5.4

In each of the following, rewrite the integral in the form $\int f[g(x)]g'(x)\,dx$. Identify $g(x)$ and $g'(x)$.

1. $\int \cos(2x)\,dx$
2. $\int x^2\sqrt{3x^3+4}\,dx$
3. $\int 4t^3(-6t^4+10)dt$
4. $\int \frac{5x^3}{\sqrt{3+10x^4}}\,dx$

Solution

1. Let $u = g(x) = 2x$

$$\frac{du}{dx} = g'(x) = 2$$

$$du = 2dx$$

$$dx = \frac{du}{2}$$

$$\int \cos(2x)\,dx = \int \cos(u) \times \frac{du}{2}$$

$$= \frac{1}{2}\int \cos(u)\,du$$

Now, $\int \cos(2x)\,dx = \frac{1}{2}\int \cos u\,du$

2. $\int x^2\sqrt{3x^3+4}\,dx$

Let $u = g(x) = 3x^3 + 4$

$\frac{du}{dx} = g'(x) = 9x^2$

$du = 9x^2 dx$

$dx = \frac{du}{9x^2}$

$\int x^2(\sqrt{u}) \times \frac{du}{9x^2}$

$\frac{1}{9}\int (\sqrt{u})du$

Now, $\int x^2\sqrt{3x^3+4}\,dx = \frac{1}{9}\int(\sqrt{u})du$

3. $\int 4t^3(-6t^4+10)dt$

Let $u = g(t) = -6t^4 + 10$

$\frac{du}{dt} = g'(t) = -24t^3$

$du = -24t^3 dt$

$dt = \frac{du}{-24t^3}$

$\int 4t^3(-6t^4+10)dt = \int 4t^3(u) \times \frac{du}{-24t^3}$

$= -\frac{1}{6}\int (u)du$

Now, $\int 4t^3(-6t^4+10)dt = -\frac{1}{6}\int(u)du$

4. $\int \frac{5x^3}{\sqrt{3+10x^4}}\,dx$

Let $u = g(x) = 3 + 10x^4$

$\frac{du}{dx} = g'(x) = 40x^3$

$du = 40x^3 dx$

$dx = \frac{du}{40x^3}$

$\int \frac{5x^3}{\sqrt{3+10x^4}}\,dx = \int \frac{5x^3}{\sqrt{u}} \times \frac{du}{40x^3}$

$= \frac{1}{8}\int \frac{1}{\sqrt{u}}du$

Now, $\int \frac{5x^3}{\sqrt{3+10x^4}}\,dx = \frac{1}{8}\int \frac{1}{\sqrt{u}}du$

Activity 5.2

Working in small groups, evaluate the following indefinite integral using the substitution method.

$$\int \cos(4x + 3)\, dx$$

Step 1: Choose a Substitution

Let: $u = 4x + 3$

Step 2: Differentiate the Substitution

Differentiate u with respect to x: $\frac{du}{dx} = 4$

Step 3: Solve for dx or make dx the subject

Express dx in terms of du: $dx = \frac{1}{4}du$

Step 4: Substitute into the Integral
Substitute u and dx into the original integral. The integral becomes:

$$\int \cos(4x + 3)\, dx = \int \cos u \times \frac{1}{4} du$$

Step 5: Simplify the Integral

This can be rewritten as: $\frac{1}{4}\int \cos u\, du$

Step 6: Integrate

$$\frac{1}{4}\int \cos u\, du = \frac{1}{4}[\sin u] + C = \frac{1}{4}\sin u + C$$

Step 7: Substitute Back

Finally, we substitute back $u = 4x + 3$ to obtain $\frac{1}{4}\sin(4x + 3) + C$.

Thus, the integral $\int \cos(4x + 3)\, dx$ evaluates to $\frac{1}{4}\sin(4x + 3) + C$.

Example 5.5

Evaluate $\int 2xe^{x^2} dx$

Solution

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{1}{2x}du$$

$$\int 2xe^{x^2}dx = \int 2xe^u \times \frac{1}{2x}du$$

$$\int 2xe^{x^2}dx = \int e^u du$$

$$= e^u + c$$

Thus $\int 2xe^{x^2}dx = e^{x^2} + c$

Example 5.6

Simplify $\int x^3 \cos x^4\, dx$

Solution

Let $u = x^4$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{1}{4x^3}du$$

$$\int x^3 cosx^4 dx = \int x^3 \cos u \times \frac{1}{4x^3}du$$

$$\int x^3 cosx^4 dx = \frac{1}{4}\int \cos u\, du$$

$$= \frac{1}{4}(\sin u) + c$$

$$= \frac{1}{4}\sin\ (x^4) + c$$

Example 5.7

Evaluate $\int x^2(x^3 + 5)^9 dx$

Solution

Let $u = x^3 + 5$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{1}{3x^2}du$$

$$\int x^2(x^3+5)^9 dx = \int x^2(u)^9 \times \frac{1}{3x^2} du$$

$$\int x^2(x^3+5)^9 dx = \frac{1}{3}\int u^9 du$$

$$= \frac{1}{3}\left[\frac{u^{10}}{10}\right] + C$$

$$= \frac{u^{10}}{30} + C$$

$$= \frac{(x^3+5)^{10}}{30} + C$$

The procedure for evaluating a definite integral follows the same steps as we have witnessed above. However, we must change the original given limits to conform with $g(x) = u$.

Example 5.8

Evaluate $\int_0^1 x\sqrt{x^2+2}\, dx$

Step 1: Choose a Substitution

Let's define a new variable u based on the expression inside the square root:

Let $u = x^2 + 2$

Step 2: Change the Limits of Integration

Next, we need to change the limits of integration according to our substitution:

When $x = 1$,

$u = 1^2 + 2$

$u = 3$

At $x = 0$

$u = 0^2 + 2$

$u = 2$

Step 3: Find du

Now, we need to find the differential du:

$u = x^2 + 2$

$\frac{du}{dx} = 2x$

$du = 2xdx$

$dx = \frac{du}{2x}$

Step 4: Substitute into the Integral

Now, we substitute u and dx into the integral:

$\int_2^3 x\sqrt{u} \times \frac{du}{2x}$

$\frac{1}{2}\int_2^3 \sqrt{u}\, du$

Step 5: Rewrite the Integral

Rewrite the integral as $\frac{1}{2}\int_2^3 u^{\frac{1}{2}} du$

Step 6: Integrate

Now, we can integrate:

$\frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^3$

$\frac{1}{3}\left[u^{\frac{3}{2}}\right]_2^3$

Step 7: Evaluate the Limits

Now we evaluate the limits:

$\frac{1}{3}\left(\left[3^{\frac{3}{2}}\right] - \left[2^{\frac{3}{2}}\right]\right)$

$\frac{1}{3}(3\sqrt{3} - 2\sqrt{2})$

Working in pairs, or small groups, solve the following example.

Example 5.9

Evaluate $\int_{-1}^{0} x^3(4x^4 + 2)^3\, dx$

Solution

Let $u = 4x^4 + 2$

$\frac{du}{dx} = 16x^3$

$dx = \frac{du}{16x^3}$

When $x = -1$,

$u = 4(-1)^4 + 2$

$u = 6$

When $x = 0$,

$u = 4(0)^4 + 2$

$u = 2$

$$\int_{-1}^{0} x^3(4x^4 + 2)^3\,dx = \int_{6}^{2} x^3(u)^3 \frac{du}{16x^3}$$

$$\int_{-1}^{0} x^3(4x^4 + 2)^3\,dx = \frac{1}{16}\int_{6}^{2} u^3 du$$

$$= \frac{1}{16}\left[\frac{u^4}{4}\right]_6^2$$

$$= \frac{1}{16}\left(\left[\frac{(2)^4}{4}\right] - \left[\frac{(6)^4}{4}\right]\right)$$

$$= \frac{1}{16}\left(\left[\frac{16}{4}\right] - \left[\frac{1296}{4}\right]\right)$$

$$= \frac{1}{16}([4] - [324])$$

$$= -20$$

Therefore, $\int_{-1}^{0} x^3(4x^4 + 2)^3 = -20$.

Indefinite integrals

What is an Indefinite Integral?

- An indefinite integral is an integral that does not have specific limits (i.e., no starting and ending points).
- It is written as $\int f(x)\,dx$ and represents a family of functions that are all antiderivatives of $f(x)$.
- Since there are many antiderivatives for a function (they can differ by a constant), we include an arbitrary constant C to represent all possible antiderivatives.

Example 5.10

Evaluate the following

a. $\int (x^6 + 2)dx$

b. $\int \sqrt{(x)}dx$

c. $\int (2x^2 + 3x + 4)dx$

d. $\int (e^{2x} + 5x)dx$

Solution

a. $\int (x^6 + 2)dx = \frac{x^{6+1}}{6+1} + \frac{2x^{0+1}}{0+1} + c$

$= \frac{x^7}{7} + 2x + c$

d. $\int \sqrt{(x)}dx = \int (x)^{\frac{1}{2}} dx$

$= \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{2}{3}x^{\frac{3}{2}} + C$

e. $\int (2x^2 + 3x + 4)dx = \frac{2x^{2+1}}{2+1} + \frac{3x^{1+1}}{1+1} + \frac{4x^{1+0}}{1+0} + C$

$= \frac{2x^3}{3} + \frac{3x^2}{2} + 4x + C$

f. $\int (e^{2x} + 5x)dx = \frac{1}{2}e^{2x} + \frac{5x^2}{2} + C$

Integration can be applied to solve real-life problems involving marginal cost, velocity, distance and acceleration. Some terms and symbols are represented in the table below.

Quantity	Definition	Notation
Displacement	Distance from a fixed origin	s, y, x
Velocity	Rate of change of displacement with time	$v = \frac{ds}{dt}$
Acceleration	Rate of change of velocity with time	$a = \frac{dv}{dt}$

Example 5.11

A particle Q moves in a straight line so that at time t, seconds, its acceleration is $(12t + 3)ms^{-2}$. Q passes through the point O with velocity $5ms^{-1}$.
Find the:

a. velocity of Q in terms of t

b. distance Q from O when $t = 3s$

Solution

a. The velocity of Q in terms of t, acceleration $a = \frac{dv}{dt}$

$$\int dv = \int a dt$$

$$v = \int a dt$$

$$v = \int (12t + 3) dt$$

$$v = 6t^2 + 3t + C$$

But at $t = 0$, the velocity was $5ms^{-1}$

$$5 = 6(0)^2 + 3(0) + C$$

$$C = 5$$

Hence, the velocity of Q in terms of t is $v = 6t^2 + 3t + 5$

b. $v = \frac{ds}{dt}$

$$ds = \int v dt$$

$$s = \int v dt$$

$$s = \int (6t^2 + 3t + 5) dt$$

$$s = 2t^3 + \frac{3}{2}t^2 + 5t + C$$

At the point $O, s = 0, t = 0$

$$0 = 2(0)^3 + \frac{3}{2}(0)^2 + 5(0) + C$$

$$C = 0$$

$$s = 2t^3 + \frac{3}{2}t^2 + 5t$$

At $t = 3s$,

$$s = 2(3)^3 + \frac{3}{2}(3)^2 + 5(3)$$

$$s = 54 + \frac{27}{2} + 15$$

$$82.5m$$

IDENTIFYING AND APPLYING APPROPRIATE TECHNIQUES OF INTEGRATION TO INTEGRATE TRANSCENDENTAL FUNCTIONS

Activity 5.3 Exploring Transcendental Functions

This activity is to enable you to research, understand, and present information about transcendental functions, their properties, and applications.

Working in small groups:

1. Research the properties, graphs, and applications of exponential functions, including growth and decay models.
2. Investigate the properties, graphs, and real-world applications of logarithmic functions, including their role in solving exponential equations.
3. Explore the properties and applications of trigonometric functions, including their use in modelling periodic phenomena.
4. Research inverse trigonometric functions, their properties, and applications in geometry and calculus.
5. Investigate hyperbolic functions, their definitions, properties, and applications in engineering and physics.
6. Make use of reliable online resources such as educational websites, academic articles, and videos (e.g., Khan Academy (khanacademy.org), Wolfram Alpha (wolframalpha.com), Math is Fun (mathsisfun.com) or university resources).



7. Take notes on key points.

Transcendental functions are special types of functions that cannot be expressed as solutions to simple algebraic equations (like polynomials). In other words, you cannot write them as a finite combination of addition, subtraction, multiplication, division, and taking roots.

Example of Algebraic Functions

Example of Algebraic Functions

For example, consider the equation:

$y^2 - 2x = 0.$

You can solve this equation for y to get:

$y = \mp\sqrt{2x}$

This means y is an algebraic function because we can express it using a finite number of operations.

Example of Transcendental Functions

Now, let's look at a transcendental function like:

$y = e^x$.

This function represents the exponential function, which grows very rapidly. There is no polynomial equation that can describe it simply. Other examples of transcendental functions include trigonometric functions (like sine and cosine) and natural logarithms (like ln(x)). Here we will concentrate on the integration of exponential functions as well as natural logarithms.

Figure 5.1 represents an algebraic function ($y = x^2$) and transcendental functions ($y = e^x$).

What is your observation?

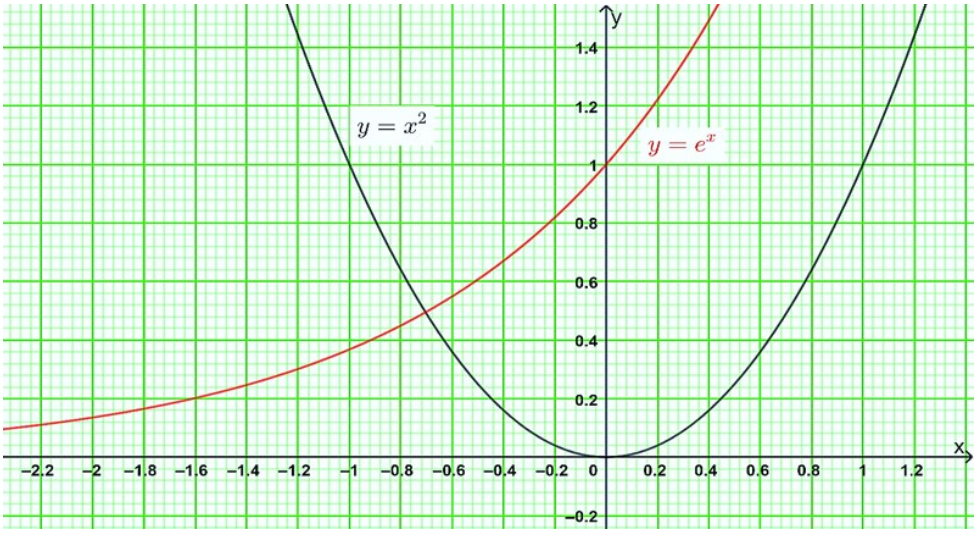


Figure 5.1: Graphical Illustration of $y = x^2$ and $y = e^x$

Example 5.12

Evaluate

Solution

Applying $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

We have: $\int e^{3x}dx = \frac{1}{3}e^{3x} + C$

Alternatively, we can consider:

Let $u = 3x$

$\frac{du}{dx} = 3$

$du = 3dx$

$dx = \frac{du}{3}$

$\int e^u \frac{du}{3} = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$

Example 5.13

Evaluate $\int e^{2-7x}dx$

Solution

Let $u = 2 - 7x$

$\frac{du}{dx} = -7$

$du = -7dx$

$dx = -\frac{du}{7}$

$\int -e^u \frac{du}{7} = -\frac{1}{7}\int e^u du = -\frac{1}{7}e^u + C = -\frac{1}{7}e^{2-7x} + C$

Example 5.14

Evaluate $\int e^{2x}\sqrt{1+e^{2x}}dx$

Solution

Let $u = 1 + e^{2x}$

$\frac{du}{dx} = 2e^{2x}$

$du = 2e^{2x}dx$

$dx = \frac{du}{2e^{2x}}$

$\int e^{2x}(\sqrt{u}) \times \frac{du}{2e^{2x}} = \frac{1}{2}\int (\sqrt{u})du = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(1+e^{2x})^{\frac{3}{2}} + C$

Example 5.15

Evaluate $\int 4x^2 e^{2x^3}dx$

Solution

Let $u = 2x^3$

$$\frac{du}{dx} = 6x^2$$

$$du = 6x^2 dx$$

$$dx = \frac{du}{6x^2}$$

$$\int 4x^2(e^u) \times \frac{du}{6x^2} = \frac{2}{3}\int (e^u)du = \frac{2}{3}[e^u] + C = \frac{2}{3}\left(e^{2x^3}\right) + C = \frac{2e^{2x^3}}{3} + C$$

Example 5.16

Evaluate $\int e^x(3e^x - 2)^2\, dx$

Solution

Let $u = 3e^x - 2$

$$\frac{du}{dx} = 3e^x$$

$$dx = \frac{du}{3e^x}$$

$$\int e^x(u)^2 \times \frac{du}{3e^x}$$

$$\frac{1}{3}\int (u)^2 du = \frac{1}{3}\left[\frac{u^3}{3}\right] + C = \frac{u^3}{9} + C = \frac{(3e^x - 2)^3}{9} + C$$

Example 5.17

Evaluate $\int_1^2 e^{1-2x} dx$

Solution

Let $u = 1 - 2x$

$$\frac{du}{dx} = -2$$

$$dx = -\frac{du}{2}$$

When $x = 1$

$u = 1 - 2(1)$

$u = -1$

When $x = 2$

$u = 1 - 2(2)$

$u = -3$

$$\int_1^2 e^{1-2x}dx = \int_{-1}^{-3} e^u \times -\frac{du}{2} = -\frac{1}{2}\int_{-1}^{-3} e^u du = -\frac{1}{2}[e^u]_{-1}^{-3} = -\frac{1}{2}([e^{-3} - e^{-1}])$$

$$= -\frac{1}{2}(0.367879 - 0.0497871) = 0.1590$$

Example 5.18

$$\int_0^1 x\left(e^{4x^2+1} - 8\right)dx$$

Solution

Let $u = 4x^2 + 1$

$$\frac{du}{dx} = 8x$$

$$dx = \frac{du}{8x}$$

When $x = 0$

$u = 4(0)^2 + 1$

$u = 1$

When $x = 1$

$u = 4(1)^2 + 1$

$u = 5$

$$\int_0^1 x\left(e^{4x^2+1} - 8\right)dx = \int_1^5 x(e^u - 8)\frac{du}{8x} = \frac{1}{8}\int_1^5 (e^u - 8)du = \frac{1}{8}[e^u - 8u]_1^5$$

$$= \frac{1}{8}([e^5 - 8(5)] - [e^1 - 8(1)]) = \frac{1}{8}([e^5 - 40] - [e^1 - 8])$$

$$= \frac{1}{8}([148.413 - 40] - [2.71828 - 8]) = \frac{1}{8}([108.413] - [-5.28172])$$

$$= \frac{1}{8}(113.69472) = 14.21184$$

Thus, $\int_0^1 x\left(e^{4x^2+1} - 8\right)dx = 14.211\,8$ (approximated to four decimal places).

Integrating natural logarithmic functions

The integral of a natural logarithmic function in the form

Integrand	Brief explanation
$\int \frac{1}{x}dx = \ln\|x\| + C$	The absolute value is important because the logarithm is only defined for positive numbers. By using $\|x\|$, we can handle both positive and negative values of x.

$\int \frac{u'}{u} dx = \ln\|u\| + C$	This integral shows that if you have a function u and its derivative u', you can integrate the fraction $\frac{u'}{u}$ to get the natural logarithm of u. u can be any differentiable function.
$\int \ln\|x\| dx = x\ln\|x\| - x + C$	The integral of the natural logarithm function $\ln\|x\|$ results in a more complex expression involving x multiplied by $\ln\|x\|$, minus x, plus a constant C. **NB:** This integral is often solved using a technique called integration by parts, which is based on the formula: $\int u\, dv = uv - \int v\, du$.

Example 5.19

Evaluate the following

a. $\int \frac{2}{x} dx$

b. $\int \frac{1}{4x-1} dx$

c. $\int \frac{7}{3-x} dx$

d. $\int \frac{x^2-4}{x} dx$

e. $\int \frac{3x^2+1}{x^3+x} dx$

f. $\int \frac{\sec^2 x}{\tan x} dx$

g. $\int \frac{x}{x^2+2} dx$

Solution

a. $\int \frac{2}{x} dx = 2\int \frac{1}{x} dx = 2\ln|x| + C$

b. $\int \frac{1}{4x-1} dx$

Let $u = 4x - 1$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

$$dx = \frac{1}{4} du$$

$$\int \frac{1}{4x-1} dx = \int \frac{1}{u} \times \frac{1}{4} du = \frac{1}{4}\int \frac{1}{u} du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|4x-1| + C$$

Thus $\int \frac{1}{4x-1}dx = \frac{1}{4}\ln|4x-1| + C$

c. $\int \frac{7}{3-x}dx$

Let $u = 3 - x$

$\frac{du}{dx} = -1$

$du = -dx$

$dx = -du$

$\int \frac{7}{3-x}dx = \int \frac{7}{u} \times -du = -7\int \frac{1}{u}du = -7\ln|u| + C = -7\ln|3-x| + C$

d. $\int \left(\frac{x^2-4}{x}\right)dx$

Divide each term by x

$\int \left(\frac{x^2}{x} - \frac{4}{x}\right)dx = \int \left(x - \frac{4}{x}\right)dx = \int xdx - \int \frac{4}{x}dx = \frac{x^2}{2} - \ln|x| + C$

e. $\int \frac{3x^2+1}{x^3+x}dx$

Let $u = x^3 + x$

$du = (3x^2 + 1)dx$

$dx = \frac{du}{3x^2+1}$

$\int \frac{3x^2+1}{x^3+x}dx = \int \frac{3x^2+1}{u} \times \frac{du}{3x^2+1} = \int \frac{\cancel{3x^2+1}}{u} \times \frac{du}{\cancel{3x^2+1}} = \int \frac{1}{u}du = \ln|u| + C$

$= \ln|x^3 + x| + C$

Thus $\int \frac{3x^2+1}{x^3+x}dx = \ln|x^3 + x| + C.$

f. $\int \frac{\sec^2 x}{\tan x}dx$

Let $u = tanx$

$\frac{du}{dx} = sec^2x$

$dx = \frac{du}{sec^2x}$

$\int \frac{\sec^2 x}{\tan x}dx = \int \frac{sec^2x}{u} \times \frac{du}{sec^2x} = \int \frac{1}{u}du = \ln|u| + C = \ln|\tan x| + C$

Thus $\int \frac{3x^2+1}{x^3+x}dx = \ln|\tan x| + C.$

g. $\int \frac{x}{x^2+2}dx$

Let $u = x^2 + 2$

$\frac{du}{dx} = 2x$

$$dx = \frac{du}{2x}$$

$$\int \frac{x}{x^2+2}\,dx = \int \frac{x}{2x\,(u)}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

Thus $\int \frac{x}{x^2+2}\,dx = \frac{1}{2}\ln|u| + C$

SOLVING PROBLEMS INVOLVING INTEGRALS WITH NO EXACT ANTIDERIVATIVE

Activity 5.4 Researching the Trapezium Rule

In small groups, or individually, go through the following activity to enable you to research and understand the trapezium rule, its applications, and how it is used in numerical integration.

1. Using reliable online resources such as educational websites, academic journals, and videos (e.g., Khan Academy, YouTube, or university websites):
   a. Research the origins of the trapezium rule and its historical significance in mathematics.
   b. Find and explain the derivation of the trapezium rule formula.
   c. Investigate real-world applications of the trapezium rule in fields such as physics, engineering, or economics.
   d. Find examples of problems that use the trapezium rule and solve them, demonstrating the method step-by-step.
2. Share your findings with the other groups or your teacher.

Some functions can be challenging to integrate because they lack simple solutions for their integrals. For instance, integrals like $\int_a^b \frac{1}{\sqrt{x+1}}\,dx$ and $\int_a^b x\sin x\,dx$ do not have exact formulas. In these cases, numerical methods can help us find approximate solutions. One effective method is the **Trapezium Rule**, which will be our focus. The **Trapezium Rule** estimates the area under a curve (which represents an integral) by dividing it into smaller trapezoids. Here's a step-by-step breakdown of how it works. For example, given the function $f(x) = (x+1)^2$.

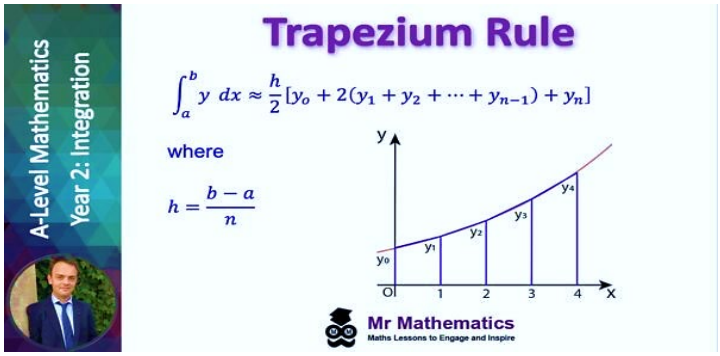


Figure 5.2: Graphical Illustration of $f(x) = (x+1)^2$.

1. How many subintervals can you count in the Figure 5.2?

   There are 4 subintervals

2. Identify the limits.

   The limits are $[0, 4]$?

Recall that the area of a trapezium is $A = \frac{1}{2}h(a + b)$

In this instance, we determine h, as $h = \frac{b-a}{n}$

Where a = starting value of partition,

b = ending value of partition,

n = number of partitions.

From the Figure 5.2, $h = \frac{4-0}{4}$

$h = 1$

Find the area of each trapezoid:

First trapezoid $= \frac{1}{2}h(y_0 + y_1)$

Second trapezoid $= \frac{1}{2}h(y_1 + y_2)$

Third trapezoid $= \frac{1}{2}h(y_2 + y_3)$

Fourth trapezoid $= \frac{1}{2}h(y_3 + y_4)$

Adding all to get the total area

$$= \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \frac{1}{2}h(y_3 + y_4)$$

Factorising $\frac{1}{2}h$ gives:

$$= \frac{1}{2}h[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 + y_4]$$

$$= \frac{1}{2}h[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{1}{2}h[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

Example 5.20

Based on Figure 5.3 and given the function, $\int_{0.5}^{3} \frac{2}{x}\, dx$:

a. Find the number of partitions, n.
b. Find h, the height.
c. Use the trapezium rule to approximate the value of the integral.

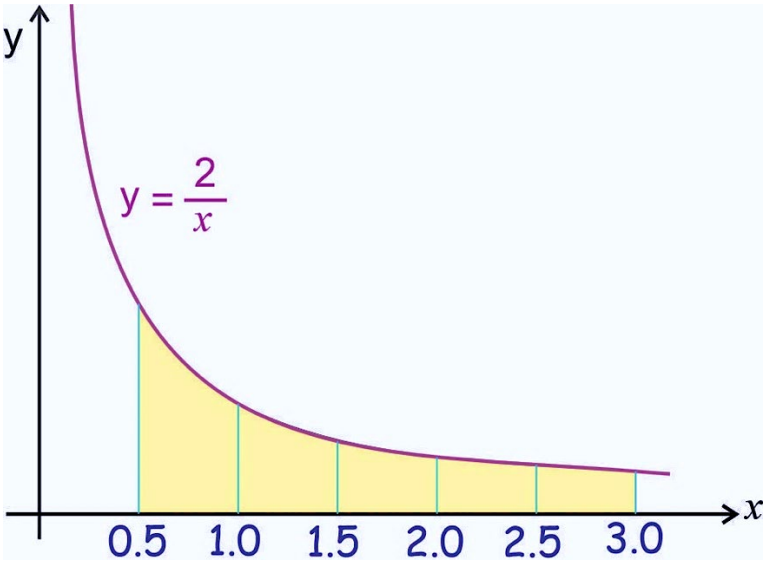


Figure 5.3: Graphical Illustration of $y = \frac{2}{x}$

Solution

a. From the diagram, the number of partitions is 5

b. $h = \frac{3-0.5}{5}$

$h = 0.5$

c. We need to find the y values for $x = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$.

x	0.5	1.0	1.5	2.0	2.5	3.0
$y = \frac{2}{x}$	4.0	2.0	1.3333	1.00	0.8	0.6667

Using, $\frac{1}{2}h[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$

$$\int_{0.5}^{3} \frac{2}{x}\,dx = \frac{1}{2}(0.5)[(4.0 + 0.6667) + 2(2 + 1.3333 + 1 + 0.8)]$$

$$= \frac{1}{2}(0.5)[4.6667 + 10.2666] = \frac{1}{2}(0.5)(14.9333) = 3.3733$$

Thus, $\int_{0.5}^{3} \frac{2}{x}\,dx = 3.3733$.

Example 5.21

Estimate $\int_0^7 \frac{1}{x+1}\,dx$ using the trapezoidal rule with 7 intervals.

Solution

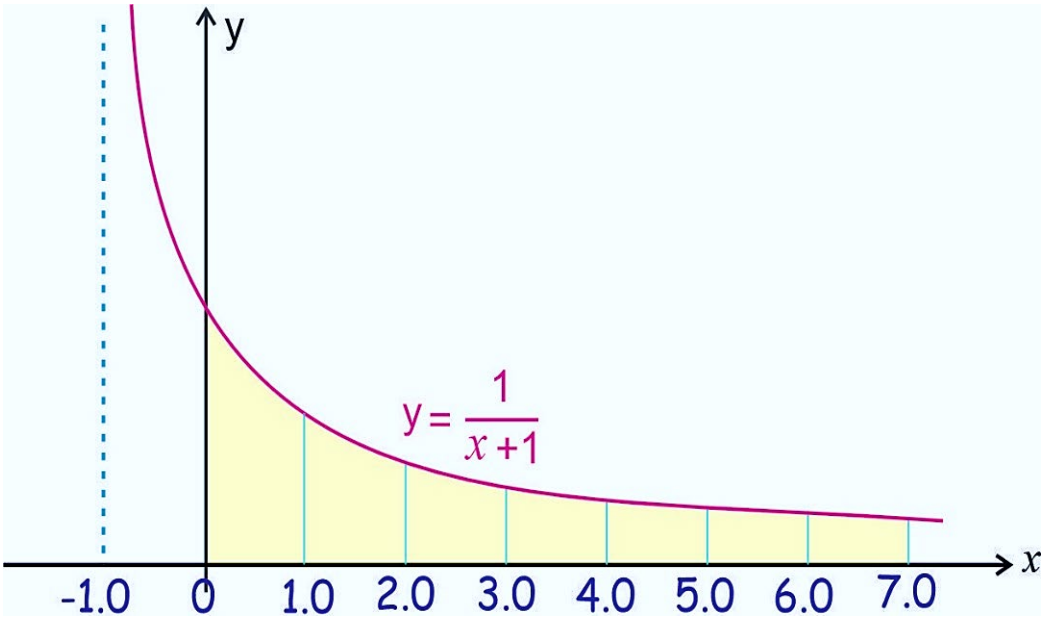


Figure 5.4: Graphical Illustration of $y = \frac{1}{x+1}$

$$h = \frac{7-0}{7}$$
$$\therefore\ h = 1$$

x	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0
$x+1$	1	2	3	4	5	6	7	8
$y = \frac{1}{x+1}$	1.0	0.5	0.3333	0.25	0.2	0.1667	0.1429	0.125

Using, $\frac{1}{2}h[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$

$$\int_0^7 \frac{1}{x+1}\,dx = \frac{1}{2}(1)[(1.0 + 0.125) + 2(0.5 + 0.3333 + 0.25 + 0.2 + 0.1667 + 0.1429)]$$

$$= \tfrac{1}{2}(1.0)[(1.125) + (3.1858)] = \tfrac{1}{2}(1.0)[(4.3108)] = 2.1554$$

Thus, $\int_0^7 \frac{1}{x+1}\,dx = 2.1554$.

Example 5.22

Using the trapezium rule, estimate $\int_{-3}^{6} \frac{4x}{x^2+2}\,dx$ with 9 intervals.

Solution

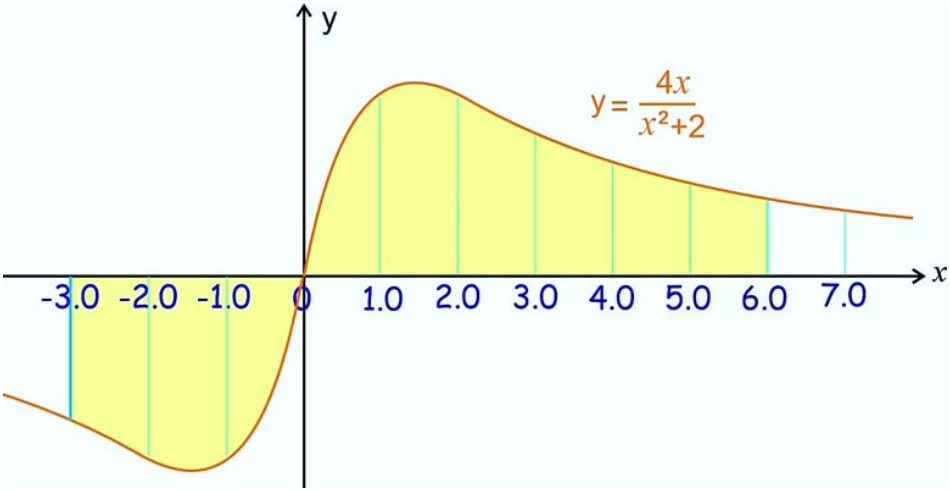


Figure 5.5: Graphical Illustration of $y = \frac{4x}{x^2+2}$

$$h = \frac{6-(-3)}{9}$$

$$\therefore\ h = 1$$

x	-3	-2	-1	0	1	2	3	4	5	6
$4x$	-12	-8	-4	0	4	8	12	16	20	24
x^2	9	4	1	0	1	4	9	16	25	36
x^2+2	11	6	3	2	3	5	11	18	27	38
$y = \frac{4x}{x^2+1}$	-1.090	-1.333	-1.333	0	-1.333	-1.333	1.0909	0.8889	0.7407	0.6316

Using, $\frac{1}{2}h[(y_0+y_4)+2(y_1+y_2+y_3)]$

$$\int_{-3}^{6}\frac{4x}{x^2+2}\,dx = \frac{1}{2}(1)[(-1.0909+0.6316)+2(-1.3333-1.3333+0+1.3333+1.3333 + 1.0909+0.8889+0.7407)]$$

$$= \tfrac{1}{2}(1.0)[-0.4593+2.7205] = \tfrac{1}{2}(1.0)(2.2612) = 1.1306$$

Thus $\int_{-3}^{6}\frac{4x}{x^2+2}\,dx = 1.1306.$

REVIEW QUESTIONS 5

1. Use the trapezium rule with 5 equally spaced ordinates to estimate the value of $\int_2^{18} \ln\left|\frac{2}{\sqrt{4+\sqrt{x}}}\right| dx$.
2. Using the trapezium rule with 4 equally spaced strips, find an estimate for $\int_0^{\frac{\pi}{3}} cos^2 x dx$
3. Find the estimate for the integral $\int_1^2 e^{\frac{1}{10}x^2} dx$.
4. Evaluate the following:
   a. $\int_0^2 e^{1-\frac{x}{2}} dx$
   b. $\int_{-1}^0 \frac{dx}{x-2}$
5. If $\int_a^{-1} \frac{dx}{5+x} = In\ 2$, find the value of a
6. a) Find the price demand equation for a particular brand of toothpaste at a supermarket when the demand is 50 tubes per week at GHC 23.50 per tube given that the marginal price demand function $P'(x) = -0.015e^{-0.01x}$
   b) If the supermarket chain sells 500 tubes per week, what price should it sell?
7. By using the substitution $u = 4 + 3x^2$ or otherwise, find $\int \frac{2x}{(4+3x^2)^2} dx$
8. Show that $\int_0^{\frac{1}{3}} xe^{3x} dx = \frac{1}{9}$
9. Given $\frac{3x-5}{x-1} = A + \frac{B}{x-1}$
   a. Determine the value of each of the constants, A and B.
   b. Find, $\int \frac{3x-5}{x-1} dx$.
10. Given that $y = \frac{3x}{2+x-x^2}$
    a. Copy and complete the table

x	0	0.25	0.5	0.75	1.00
y	0				1.5

    b. Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for $\int_0^1 \frac{3x}{2+x-x^2} dx$
    c. Use a suitable method to find the exact value of $\int_0^1 \frac{3x}{2+x-x^2} dx$
    d. Comment on your results in b) and c)
11. An object is moving along the x-axis with values of the velocity v in m/s at various times t given in the table below.

    Given that the distance travelled may be found by calculating the area under a velocity/time graph, use the trapezium rule to estimate the distance travelled by the particle in the first 5 seconds.

t	0	1	2	3	4	5
v	1.5	1.3	1.4	2.0	2.4	2.7

SECTION

6 APPLICATIONS OF INTEGRATION



CALCULUS

Application of Calculus

INTRODUCTION

In this section on applications of integration, we will explore how integration allows us to measure the space enclosed by curves and estimate the volume of solids formed by revolving those regions. The disc method provides a systematic way to compute volumes by slicing a solid into circular cross-sections and summing their contributions. This technique is valuable in fields such as engineering, architecture, and manufacturing, where precise volume calculations are essential. By mastering these ideas, you will gain the skills to apply definite integrals to real-world problems, deepen your understanding of three-dimensional geometry, and build a strong foundation for more advanced topics in calculus and applied mathematics.

KEY IDEAS

- **Definite integrals** – identifying these as areas to create an understanding of how integration measures the exact area between curves over a specified interval.
- **Setting up bounds** – identifying intersection points of curves to establish correct limits of integration.
- **Solids of revolution** – recognising that revolving a plane region around an axis generates a three-dimensional solid.
- **Disc method** – slicing the solid perpendicular to the axis of revolution and expressing each cross-section's area as to determine the volume of the solid.

FINDING THE AREA UNDER A CONSTANT FUNCTION, CURVES AND DISTINGUISHING BETWEEN TOTALS, NET AND SIGNED AREA

In mathematics, area is not just about counting space; it tells a deeper story when functions rise above or fall below the x - axis. Whether a curve soars or dips, the regions it creates carry meaning: sometimes adding up, other times cancelling out. We will explore how these areas differ, and sort them into totals, nets, or signed values based on their position above or below the x-axis. In Years 1 and 2 we learnt how to represent graphs of linear and non-linear functions. We will now go through an activity to establish a few theories.

Activity 6.1 Exploring Area Above and Below the x - axis

In pairs, or in small groups, work through the following activity.

Step 1: On the same coordinate grid, carefully sketch the graphs of the following functions from $x = -4$ to $x = 4$:

a. $f(x) = 3$
b. $g(x) = -2$
c. $h(x) = x$
d. $k(x) = x^2 - 4$

Your sketch should look like this.

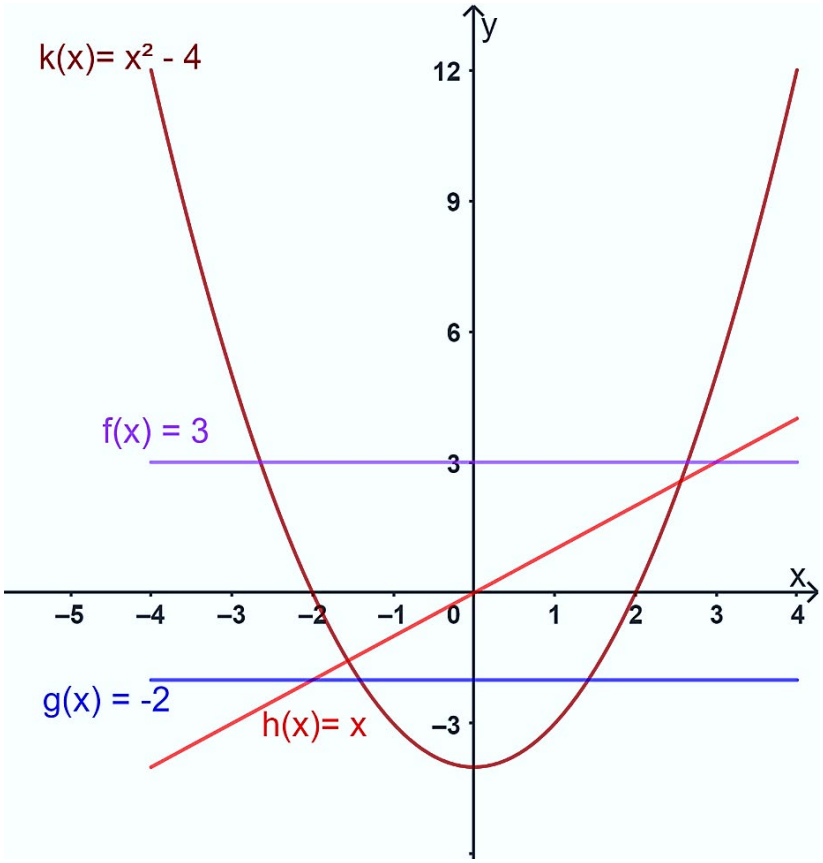


Figure 6.1: Graphical representations of functions in Step 1.

Step 2: For each graph, shade the region between the line/curve and the x −axis within the interval given. Use two different colours:

a. One colour for parts of the graph above the x −axis
b. Another colour for parts below the x −axis

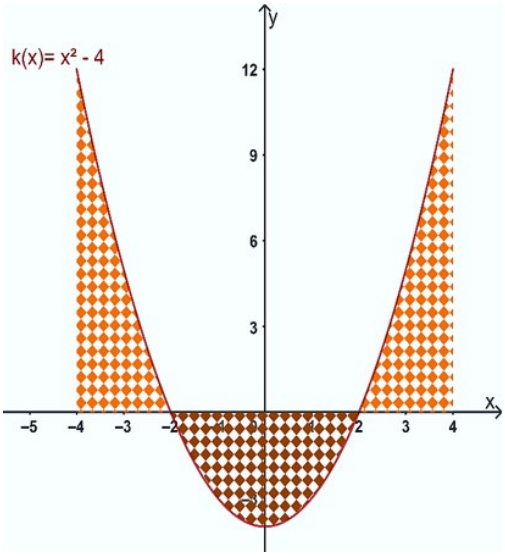


Figure 6.2: Graphical representation of function $k(x) = x^2 - 4$

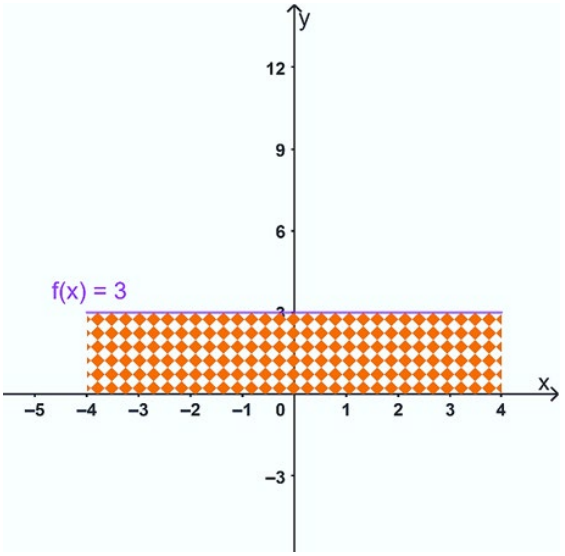


Figure 6.3: Graphical representation of function $f(x) = 3$

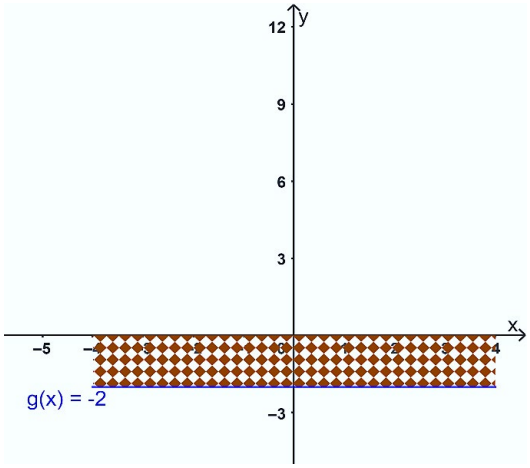


Figure 6.4: Graphical representation of function $g(x) = -2$

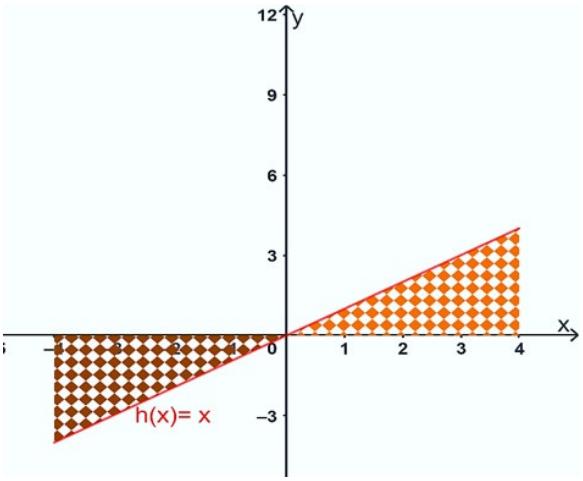


Figure 6.5: Graphical representation of function h $(x) = x$

Step 3: For each function, write the following in your notebook:

a. Which parts of the graph lie above the x − axis?

b. Which parts lie below the x − axis?

Observations

Function	Part above x- axis (+ area)	Part below x- axis (- area)
$f(x) = 3$	$[-4, 4]$	None
$g(x) = -2$	None	$[-4, 4]$
$h(x) = x$	$[0, 4]$	$[-4, 0]$
$k(x) = x^2 - 4$	$[-4, -2]$ and $[2, 4]$	$[-2, 2]$

From ***Activity 6.1***, you have come across different kinds of functions. There are the constant functions, $(f(x) = 3, g(x) = -2)$, a linear function $(h(x) = x)$ and a quadratic function $(k(x) = x^2 - 4)$.

To be able to determine the area covered by these functions and the x-axis, you need to find the definite integral of the functions based on the interval they cover.

It is important to note that there are three different kinds of area to be determined, the total area, net area and signed area.

Total Area

In determining the total area, we are only interested in the physical area between the function and the $x-$ axis. We are not concerned if a portion is below or above the $x-$ axis. We treat all the areas as positive and add them up. Mathematically, we say a total area refers to the sum of the absolute values of all areas between the function and the $x-$ axis. It is always positive or zero and represents the actual physical area irrespective of the position relative to the $x-$axis.

It is denoted by $\int |f(x)|dx$.

Given an interval $[a, b]$, total area $= \int_a^b |f(x)|dx$ or

1. Divide the interval into subintervals where $f(x)$ is entirely above or below the $x-$ axis.
2. Add the absolute values of all areas.

Note

If $f(x)$ is a trigonometric function, analyse how $f(x)$ behaves in specific quadrants and apply the signs needed to make them positive when calculating the total area.

Net Area

When it comes to the net area, we particularly observe whether a portion of the area is above or below the $x-$ axis. The portion above is treated as positive while the one below is treated as a negative area. We do not find the sum of the absolute values.

Mathematically, **net area** is the algebraic sum of areas between the function and the $x-$ axis. It can be positive, negative or zero and represents the balance between positive and negative contributions.

Given an interval $[a, b]$, net area $= \int_a^b f(x)dx$.

The shaded region below the x −axis will have a negative area and will affect the final result.

Signed area

Signed area is equivalent to net area but emphasises that the result carries a sign indicating position relative to the x −axis.

Example 6.1

Determine the total, net and signed area of the functions over the interval $x = -4$ to $x = 4$.

a. $f(x) = 3$

b. $g(x) = -2$

c. $h(x) = x$

d. $k(x) = x^2 - 4$

Solution

a. Total area $= \int_{-4}^{4} |3|\, dx$

$= |[3x]_{-4}^{4}|$

$= |3(4) - (3 \times (-4))|$

$= 24$ units squared

Net Area $= \int_{-4}^{4} 3\, dx$

$= [3x]_{-4}^{4}$

$= 3(4) - (3 \times (-4))$

$= 24$ units squared

Signed area $= \int_{-4}^{4} 3\, dx$

$= [3x]_{-4}^{4}$

$= 3(4) - (3 \times (-4))$

$= +24$ units squared

b. Total area $= \int_{-4}^{4} |-2|\, dx$

$= |[-2x]_{-4}^{4}|$

$= |-2(4) - (-2 \times (-4)|$

$= 16$ units squared

$$\text{Net Area} = \int_{-4}^{4} -2\,dx$$
$$= [-2x]_{-4}^{4}$$
$$= -2(4) - (-2 \times (-4))$$
$$= -16 \text{ units squared}$$

$$\text{Signed area} = \int_{-4}^{4} -2\,dx$$
$$= [-2x]_{-4}^{4}$$
$$= -2(4) - (-2 \times (-4))$$
$$= -16 \text{ units squared}$$

c. $$\text{Total area} = \int_{-4}^{0} |x|\,dx + \int_{0}^{4} |x|\,dx$$
$$= \left|\left[\frac{x^2}{2}\right]_{-4}^{0}\right| + \left|\left[\frac{x^2}{2}\right]_{0}\right|$$
$$= \left|\left(\frac{0^2}{2} - \frac{(-4)^2}{2}\right)\right| + \left|\left(\frac{4^2}{2} - \frac{0^2}{2}\right)\right|$$
$$= |-8| + |8|$$
$$= 16$$

$$\text{Net Area} = \int_{-4}^{0} x\,dx + \int_{0}^{4} x\,dx$$
$$= \left[\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4}$$
$$= \left(\frac{0^2}{2} - \frac{(-4)^2}{2}\right) + \left(\frac{4^2}{2} - \frac{0^2}{2}\right)$$
$$= -8 + 8$$
$$= 0$$

$$\text{Signed area} = \int_{-4}^{0} x\,dx + \int_{0}^{4} x\,dx$$
$$= \left[\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4}$$
$$= \left(\frac{0^2}{2} - \frac{(-4)^2}{2}\right) + \left(\frac{4^2}{2} - \frac{0^2}{2}\right)$$
$$= -8 + 8$$
$$= 0$$

d. Total Area $= \int_{-4}^{-2}|x^2-4|\,dx + \int_{-2}^{2}|x^2-4|\,dx + \int_{2}^{4}|x^2-4|\,dx$

$$= \left|\left[\frac{x^3}{3}-4x\right]_{-4}^{-2}\right| + \left|\left[\frac{x^3}{3}-4x\right]_{-2}^{2}\right| + \left|\left[\frac{x^3}{3}-4x\right]_{2}^{4}\right|$$

$$= \left|\left[\left(\frac{(-2)^3}{3}-4(-2)\right)-\left(\frac{(-4)^3}{3}-4(-4)\right)\right]\right|$$

$$+ \left|\left[\left(\frac{2^3}{3}-4(2)\right)-\left(\frac{(-2)^3}{3}-4(-2)\right)\right]\right| + \left|\left[\left(\frac{4^3}{3}-4(4)\right)-\left(\frac{2^3}{3}-4(2)\right)\right]\right|$$

$$= \left|\frac{32}{3}\right| + \left|-\frac{32}{3}\right| + \left|\frac{32}{3}\right|$$

$$= 32 \text{ units squared}$$

Net Area $= \int_{-4}^{-2}(x^2-4)\,dx + \int_{-2}^{2}(x^2-4)\,dx + \int_{2}^{4}(x^2-4)\,dx$

$$= \left[\frac{x^3}{3}-4x\right]_{-4}^{-2} + \left[\frac{x^3}{3}-4x\right]_{-2}^{2} + \left[\frac{x^3}{3}-4x\right]_{2}^{4}$$

$$= \left[\left(\frac{(-2)^3}{3}-4(-2)\right)-\left(\frac{(-4)^3}{3}-4(-4)\right)\right] + \left[\left(\frac{2^3}{3}-4(2)\right)-\left(\frac{(-2)^3}{3}-4(-2)\right)\right]$$

$$+ \left[\left(\frac{4^3}{3}-4(4)\right)-\left(\frac{2^3}{3}-4(2)\right)\right]$$

$$= \frac{32}{3}-\frac{32}{3}+\frac{32}{3}$$

$$= \frac{32}{3} \approx 10.\dot{6} \text{ units squared}$$

Signed Area $= \int_{-4}^{-2}(x^2-4)\,dx + \int_{-2}^{2}(x^2-4)\,dx + \int_{2}^{4}(x^2-4)\,dx$

$$= \left(\frac{x^3}{3}-4x\right)\Big|_{-4}^{-2} + \left(\frac{x^3}{3}-4x\right)\Big|_{-2}^{2} + \left(\frac{x^3}{3}-4x\right)\Big|_{2}^{4}$$

$$= \left[\left(\frac{(-2)^3}{3}-4(-2)\right)-\left(\frac{(-4)^3}{3}-4(-4)\right)\right] + \left[\left(\frac{2^3}{3}-4(2)\right)-\left(\frac{(-2)^3}{3}-4(-2)\right)\right]$$

$$+ \left[\left(\frac{4^3}{3}-4(4)\right)-\left(\frac{2^3}{3}-4(2)\right)\right]$$

$$= \frac{32}{3}-\frac{32}{3}+\frac{32}{3}$$

$$= +\frac{32}{3} \approx 10.\dot{6} \text{ units squared}$$

Ask yourself the following questions:

Is it possible to have functions with the same net area but different total area?

If such situations exist, what might be the possible explanations?

To check your answers work through the following activity individually, or in pairs.

Activity 6.2 Functions with equal net areas but different total areas

Step 1: On separate grids, plot the following functions over the given intervals:

a. $f(x) = x$ for $-2 \leq x \leq 2$
b. $g(x) = 2x$ for $-2 \leq x \leq 2$

Step 2: For each function shade the part of the graph

a. above the x –axis in green
b. below the x –axis in red.

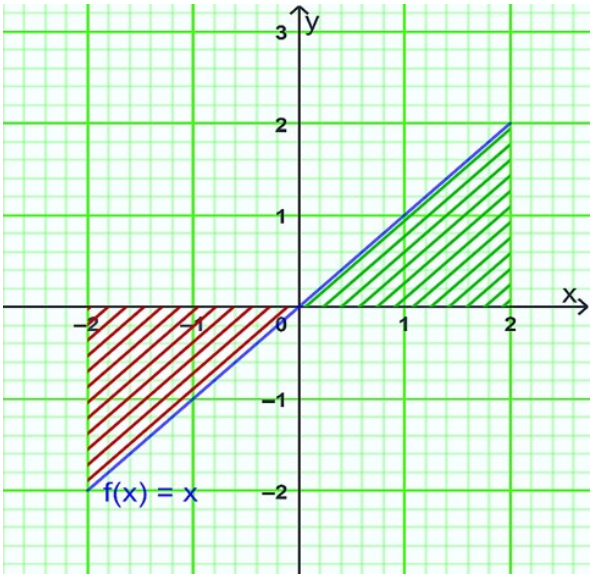


Figure 6.6: Graphical representation of function $f(x) = x$

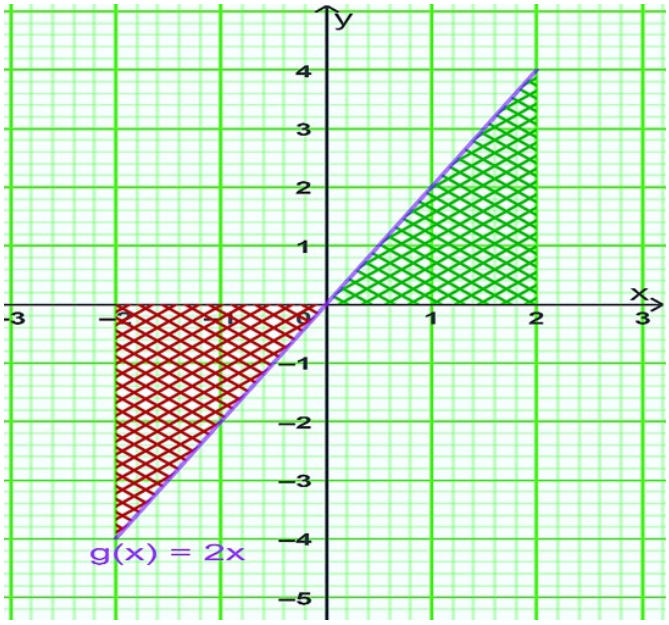


Figure 6.7: Graphical representation of function $g(x) = 2x$

Step 3: Calculate the area

a. above the x −axis.
b. below the x −axis (take the magnitude only).

Step 4: For each function calculate the total and net area

Step 5: Answer the following questions in your notebook:

a. Are the net areas the same or different? Why?
b. Are the total areas the same or different? Why?
c. Which measure tells you "Overall effect" and which tells you "Total space covered"?

Step 6: Give a real-world situation where net area would be more useful.

Step 7: Give a real-world situation where total area would be more useful.

Area under constant function

From Activity 6.1, we observed $f(x) = 3$ and $g(x) = -2$ as constant functions because, they had same values as outputs regardless of the input in the domain. Such functions are usually represented by $f(x) = k$ where k is a constant and x is the independent variable.

Note: If $f(x) = k$ where k is a constant,

1. $\frac{d}{dx}(k) = 0$.
2. $\int k\,dx = kx + C$.

To determine the area under a constant function $f(x) = k$ from $x = a$ to $x = b$, the area will be;

$$\int_a^b f(x)\,dx = \int_a^b k\,dx$$

$$= k\int_a^b 1\,dx$$

$$= k(b - a)$$

Note:

1. If $k > 0$ the area is above the $x - axis$
2. If $k < 0$ the area is below the $x - axis$ just as seen in Figure 6.3 and Figure 6.4 above.

Example 6.2

Determine the total and net areas between the following functions and the x −axis.

a. $f(x) = 4$ using interval $[-3, 1]$
b. $h(x) = 6$ using interval $[2, 5]$
c. $g(x) = -4$ using interval $[-3, 1]$
d. $r(x) = 5$ using interval $[0, 3]$
e. $k(x) = -7$ using interval $[-2, 3]$

Solution

a. Total area $= \int_{-3}^{1} |4|\, dx$

$= |4x|_{-3}^{1}|$

$= |4(1) - 4(-3)|$

$= 16$ units squared

Net Area$= \int_{-3}^{1} 4\, dx$

$= 4x|_{-3}^{1}$

$= 4(1) - 4(-3)$

$= +16$ units squared

b. Total area $= \int_{2}^{5} |6|\, dx$

$= |6x|_{2}^{5}|$

$= |6(5) - 6(2)|$

$= 18$ units squared

Net Area$= \int_{2}^{5} 6\, dx$

$= 6x|_{2}^{5}$

$= 6(5) - 6(2)$

$= 18$ units squared

c. Total area $= \int_{-3}^{1} |-4|\, dx$

$= |-4x|_{-3}^{1}|$

$= |-4(1) - (-4(-3))|$

$= 16$ units squared

Net Area$= \int_{-3}^{1} -4\, dx$

$= -4x|_{-3}^{1}$

$= -4(1) - \big(-4(-3)\big)$

$= -16$ units squared

d. Total area $= \int_{0}^{3} |5|\, dx$

$= |5x|_{0}^{3}|$

$= |5(3) - 5(0)|$

$= 15$ units squared

Net Area$= \int_{0}^{3} 5\, dx$

$= 5x|_{0}^{3}$

$$= 5(3) - 5(0)$$

$$= 15 \text{ units squared}$$

e. Total area $= \int_{-2}^{3} |-7|\, dx$

$$= |-7x|_{-2}^{3}|$$

$$= |-7(3) - (-7)(-2)|$$

$$= 35 \text{ units squared}$$

Net Area $= \int_{-2}^{3} -7\, dx$

$$= -7x|_{-2}^{3}$$

$$= -7(3) - (-7)(-2)$$

$$= -35 \text{ units squared}$$

Note

If you are required to find the area under a constant function and no clear indicator is given on the kind of area, calculate the **total area**.

Area under curves

If you need to find the area under a curve between two x -values a and b, the tool you use is the **definite integral**. You set up the integral of the function from a to b and evaluate it to get a number. Remember that the area can be Net area (regions below the x-axis count as negative) or Total area (every region is treated as positive, no matter where it sits).

Example 6.3

A drone's vertical speed (in m/s) is given by $v(t) = 2t - t^2$ for $0 \leq t \leq 3$. Determine the total area and net area under the curve created by the function of the vertical speed.

Solution

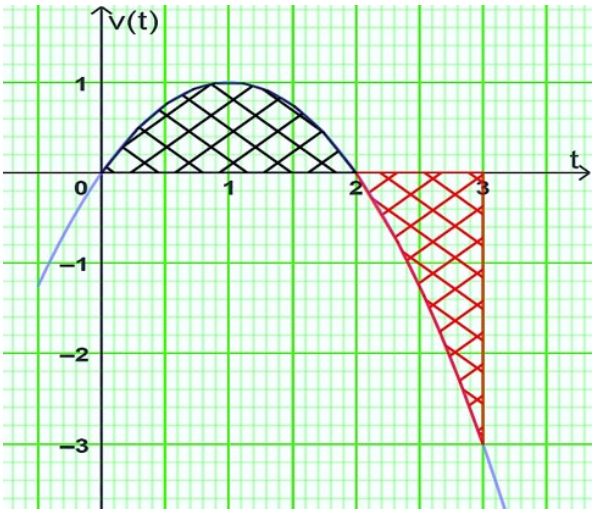


Figure 6.8: Graphical representation of function $v(t) = 2t - t^2$ for $0 \leq t \leq 3$.

Total Area

$$\int_0^3 |2t - t^2|\ dt = \int_0^2 |2t - t^2|\ dt + \int_2^3 |2t - t^2|\ dt$$

$$= \left|\left[t^2 - \frac{t^3}{3}\right]_0^2\right| + \left|\left[t^2 - \frac{t^3}{3}\right]_2^3\right|$$

$$= \left|\left(2^2 - \frac{2^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right)\right| + \left|\left(3^2 - \frac{3^3}{3}\right) - \left(2^2 - \frac{2^3}{3}\right)\right|$$

$$= \left|\frac{4}{3}\right| + \left|-\frac{4}{3}\right|$$

$$= \frac{8}{3}$$

$$= 2\frac{2}{3} \text{ units squared}$$

The total area under the curve created by the function of the vertical speed is $2\frac{2}{3}$ units squared.

Net Area

$$\int_0^3 2t - t^2\ dt = \int_0^2 2t - t^2\ dt + \int_2^3 2t - t^2\ dt$$

$$= \left[t^2 - \frac{t^3}{3}\right]_0^2 + \left[t^2 - \frac{t^3}{3}\right]_2^3$$

$$= \left[\left(2^2 - \frac{2^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right)\right] + \left[\left(3^2 - \frac{3^3}{3}\right) - \left(2^2 - \frac{2^3}{3}\right)\right]$$

$$= \frac{4}{3} - \frac{4}{3}$$

$$= 0 \text{ units squared}$$

The net area under the curve created by the function of the vertical speed is 0 units squared.

Example 6.4

The Department of Marine and Fisheries Sciences in University of Ghana is studying wave-energy patterns during a powerful coastal storm. The instantaneous wave power (in kilowatts per metre of wave front) can be modelled over a 12-hour observation window by the function $P(t) = 20 + 12\sin\left(\frac{\pi t}{6}\right) + 6e^{-\frac{(t-12)^2}{40}}$, where t represents time in hours after midnight $(0 \le t \le 12)$. As a marine engineering student evaluating the safety of an offshore wind-farm platform, carry out the following analyses:

a. Estimate the total wave-energy impact (in kilowatt-hours per metre) during the 12-hour storm by evaluating; $\int_0^{12} P(t)\,dt$. Use six equal sub-intervals and an appropriate numerical method to obtain your result.

b. Determine the 2-hour period during which the wave power was, on average, the greatest. Justify your answer by sketching the graph of $P(t)$.

c. The platform is designed to withstand $280\ kWh\ m^{-1}$ of wave energy during any 24-hour day. Based on your total-energy calculation, decide whether the recorded 12-hour storm would exceed the daily safety threshold if such conditions persisted for a full day.

Identify all time intervals within $0 \le t \le 12$ when the wave power exceeded $30kW\ m^{-1}$, signalling a high-risk condition for equipment and personnel.

Solution

a. $$\int_0^{12} |P(t)|\,dt = \int_0^{12} \left|20 + 12\sin\left(\frac{\pi t}{6}\right) + 6e^{-\frac{(t-12)^2}{40}}\right| dt$$

$$= \int_0^{12} |20|\,dt + \int_0^{12} \left|12\sin\left(\frac{\pi t}{6}\right)\right| dt + \int_0^{12} \left|6e^{-\frac{(t-12)^2}{40}}\right| dt$$

$$\int_0^{12} |20|\,dt = [20t]_0^{12} = 240$$

$$\int_0^{12} \left|12\sin\left(\frac{\pi t}{6}\right)\right| dt = 12\int_0^{12} |\sin u|\,\frac{6}{\pi}\,du, u = \frac{\pi t}{6}.$$

$$\frac{du}{dt} = \frac{\pi}{6}, dt = \frac{6}{\pi}du.$$

When $t = 0, u = 0$ and when $t = 12, u = 2\pi$

$$\int_0^{12} \left|12\sin\left(\frac{\pi t}{6}\right)\right| dt = \frac{72}{\pi}\int_0^{2\pi} |\sin(u)|\,du$$

$$\frac{72}{\pi}\int_0^{2\pi} |\sin(u)|\,du = \frac{72}{\pi}[|-\cos(u)|]_0^{2\pi}$$

$$= \frac{72}{\pi}|-\cos(2\pi) + \cos(0)|$$

$$= \frac{72}{\pi}$$

Using the trapezium rule and $h = \frac{12-0}{6} = 2$,

$$\int_0^{12} \left| 6e^{-\frac{(t-12)^2}{40}} \right| dt$$

$$= \frac{2}{2}\left[\left(6e^{-\frac{(0-12)^2}{40}}\right) + \left(6e^{-\frac{(12-12)^2}{40}}\right) + 2\left(6e^{-\frac{(2-12)^2}{40}} + 6e^{-\frac{(4-12)^2}{40}} + 6e^{-\frac{(6-12)^2}{40}} + 6e^{-\frac{(8-12)^2}{40}} + 6e^{-\frac{(10-12)^2}{40}}\right)\right]$$

$$= 33.3524$$

$$\int_0^{12} \left| 20 + 12\sin\left(\frac{\pi t}{6}\right) + 6e^{-\frac{(t-12)^2}{40}} \right| dt = 240 + \frac{72}{\pi} + 33.3524$$

$$= 296.271 \text{KWh m}^{-1}$$

a. The 2-hour period with highest intensity was $2-4$ hours after midnight.

b. The platform tolerates $280\text{kWh } m^{-1}$ in a 24-hour day. If the 12-hour storm conditions persist for 24 hours at the same average rate, total energy will be $2 \times 296.271 = 592.542\text{kWh } m^{-1}$ which is well above 280. Hence the recorded storm, if sustained, would exceed the daily threshold so the current design is not adequate.

c. From the graph, the values exceed $30\text{kW } m^{-1}$ around $1.75-4.45$ hours where $1.75 \leq t \leq 4.45$.

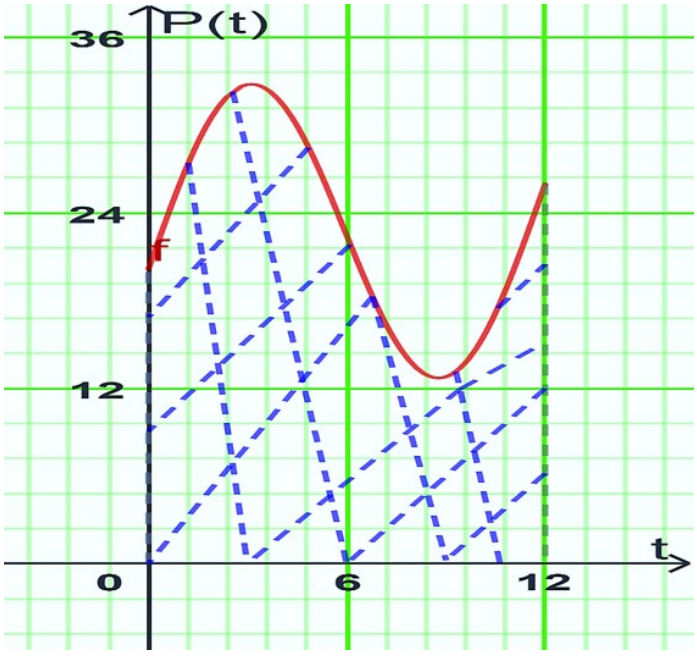


Figure 6.9: Graphical representation of function $P(t) = 20 + 12\sin\left(\frac{\pi t}{6}\right) + 6e^{-\frac{(t-12)^2}{40}}$

Example 6.5

Find the exact area of the region bounded by the curve $f(x) = -x^3 - 2x^2 + x + 1$ and the x-axis between $x = -2$ and $x = 0.8$.

Solution

To find the area (A), integrate the given function such that $A = \int_{-2}^{0.8} f|(x)|\,dx$.

From the graph in Figure 6.10, the area between $x = -2$ and $x = -0.55$ is below the x-axis while the area between $x = -0.55$ and $x = 0.8$ is above the x-axis.

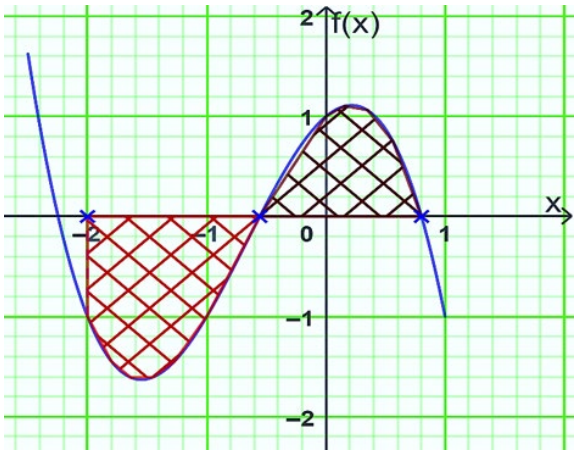


Figure 6.10: Graphical representation of function $f(x) = -x^3 - 2x^2 + x + 1$

Hence, $\int_{-2}^{0.8}|f(x)|\,dx = \int_{-2}^{-0.55}|f(x)|\,dx + \int_{-0.55}^{0.8}|f(x)|\,dx$

$$= \int_{-2}^{-0.55} |-x^3 - 2x^2 + x + 1|\,dx + \int_{-0.55}^{0.8} |-x^3 - 2x^2 + x + 1|\,dx$$

$$= \left[\left|-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + x\right|\right]_{-2}^{-0.55} + \left[\left|-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + x\right|\right]_{-0.55}^{0.8}$$

$$= \left|\left(-\frac{(-0.55)^4}{4} - \frac{2(-0.55)^3}{3} + \frac{(-0.55)^2}{2} + (-0.55)\right)\right.$$
$$\left. - \left(-\frac{(-2)^4}{4} - \frac{2(-2)^3}{3} + \frac{(-2)^2}{2} + (-2)\right)\right|$$
$$+ \left|\left(-\frac{(0.8)^4}{4} - \frac{2(0.8)^3}{3} + \frac{(0.8)^2}{2} + 0.8\right)\right.$$
$$\left. - \left(-\frac{(-0.55)^4}{4} - \frac{2(-0.55)^3}{3} + \frac{(-0.55)^2}{2} + (-0.55)\right)\right|$$

$$= |-1.6440| + |0.9873|$$

$$= 2.6313 \text{ units squared}$$

Therefore, the exact area bounded by the curve is 2.6313 units squared.

Note

When a question/activity requires you to determine the area under the curve without stating whether it is total area or net area, calculate the **total area**.

FINDING THE AREA BETWEEN TWO CURVES

The area between two regions

So far we have explored areas between curves and the x –axis. Now, we will take a look at determining the area between two curves.

If $y = f(x)$ lies above $y = g(x)$ as illustrated in Figure 6.12 from $x = a$ to $x = b$, the area of the region between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is;

$$A = \sum_{i=1}^{n}[f(c_i) - g(c_i)]h, \text{ where } h = b - a.$$

$$A = \int_a^b [f(x) - g(x)]dx.$$

Activity 6.3: Calculating areas of specific regions using appropriate techniques

Let us estimate the area of grass between a curved hedge and a cement walkway edge in your school using three simple steps:

- Manual estimate
- Simple calculation
- Exact method.

Materials Needed

- Measuring tape/string/metre ruler
- Chalk or sticks for marking
- Graph paper and



Figure 6.11: Curved hedge and cement walkway

Note

Red arrow – curved hedge edge

Purple arrow- cement walkway edge

Step 1:

1. Choose a straight **base line** about $6 - 8\ m$ along the hedge.
2. Mark every 1 m along this line (these are your x-values).
3. At each mark, measure the perpendicular distance to;
    a. the hedge edge (upper boundary)
    b. the walkway edge (lower boundary).
    c. Record these as two sets of (x, y) pairs.

Step 2: Choose one of these methods for manual area estimation

1. Counting Squares:
    a. Plot both sets of points on graph paper and shade the region.
    b. Count full squares inside and estimate halves on the edges.
    c. Multiply by the real area of one square to get a rough total.
2. Strip Averaging:
    a. For each $1m$ interval, find the average width (hedge distance – walkway distance) of the strip.
    b. Multiply by $1m$ to get each strip's area.
    c. Add all strip areas.

Step 3: Apply learnt techniques and compare results

1. Use the same measurements to calculate the area formally with the trapezium rule. If the points look like a line or a U-shape, create an equation:
    a. Line: use 2 points to find $y = mx + c$
    b. Quadratic: Use 3 points to solve for $y = ax^2 + bx + c$
2. Use $\int_{x_0}^{x_n}[y_{hedge} - y_{walk}]dx$ for an exact answer.
3. Check how close the **manual**, **trapezium**, and (if done) **integral** results are.
4. Think about why small differences appear and which method is easiest for your setting

Observations

Going through the activity, these are some realisations you might come to:

1. **Comparison of Methods**
   a. Estimating the area by counting squares or averaging strip widths gives a quick result but may differ slightly from the value obtained using the trapezium rule or definite integral.
   b. The trapezium rule usually provides a closer approximation because it uses the measured widths more systematically.
   c. When an equation of the curve is known, integration produces the most exact value.
2. **Effect of Measurement**
   a. Small measuring errors, such as uneven ground or a tape that is not straight, can affect all calculations.
   b. Using shorter intervals (more measuring points) improves the accuracy of both manual and trapezium estimates.
3. **Nature of the Curves**
   a. The hedge and walkway edges are not perfectly smooth.
   b. If the boundaries follow a regular pattern, it is easier to guess a simple linear or quadratic equation.
4. **Practical Importance**

   The same techniques can be applied to real tasks such as estimating the size of a lawn for cutting grass, planning a flower bed, or costing materials for a walkway.
5. **Mathematical Insight**

   The activity shows that integration and area approximation methods are not only classroom ideas but also tools for solving everyday problems in the school environment.

Example 6.6

Find the area of the region between $y = 2x^2 - 3x + 3$ and $y = -3x^2 + 4x$ from $x = 0$ to $x = 1$·

Solution

$$\int_0^1 [(2x^2 - 3x + 3) - (-3x^2 + 4x)]dx$$

$$= \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 3x + \frac{3x^3}{3} - \frac{4x^2}{2}\right]_0^1$$

$$= \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 3x + x^3 - 2x^2\right]_0^1$$

$$= \left[\frac{5}{3}x^3 - \frac{7}{2}x^2 + 3x\right]_0^1$$

$$= \frac{5}{3}(1)^3 - \frac{7}{2}(1)^2 + 3(1)$$

$$= \frac{7}{6}$$

$= 1\frac{1}{6}$ units squared

Note that the two functions can be subtracted first to obtain $\int_0^1 \left[\frac{5}{3}x^3 - \frac{7}{2}x^2 + 3x\right] dx$. Then integrated to result in $1\frac{1}{6}$ units squared.

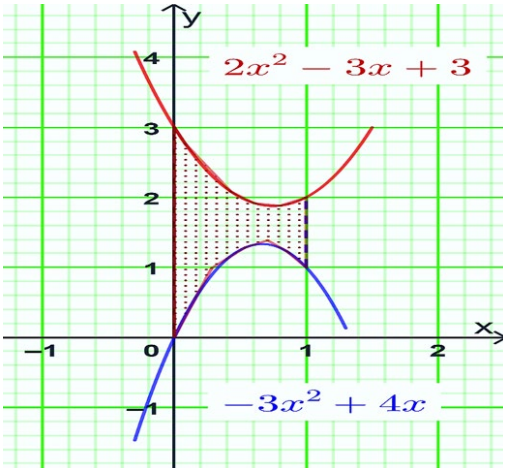


Figure 6.12: Graphical representation of area between $y = 2x^2 - 3x + 3$ and $y = -3x^2 + 4x$

Example 6.7

Determine the area of the region enclosed between $y = x^3 - 2x^2 + 4$ and $y = -2x^2 + 3x + 2$ from $x = -1$ to $x = 2$.

Solution

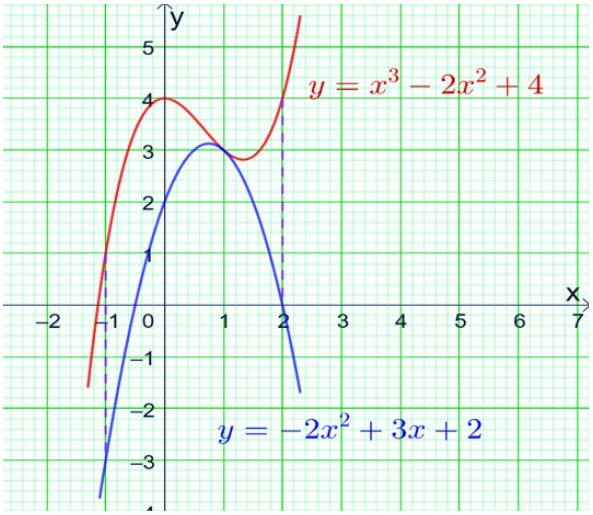


Figure 6.13: Graphical representation of area between $y = x^3 - 2x^2 + 4$ and $y = -2x^2 + 3x + 2$

From Figure 6.13 the curve of $y = x^3 - 2x^2 + 4$ lies on top for the given interval $[-1, 2]$.

Area integral: $A = \int_{-1}^{2}[(x^3 - 2x^2 + 4) - (-2x^2 + 3x + 2)]\,dx$

Simplify: $A = \int_{-1}^{2}(x^3 - 3x + 2)\,dx$

Integrate: $A = \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right]_{-1}^{2}$

$$= \left(\frac{1}{4}(2)^4 - \frac{3}{2}(2)^2 + 2(2)\right) - \left(\frac{1}{4}(-1)^4 - \frac{3}{2}(-1)^2 + 2(-1)\right)$$

$$= 2 - \left(-\frac{13}{4}\right)$$

$$= \frac{21}{4}$$

Example 6.8

Determine the area of the region bounded by $y = 5 - x^2$ and $y = 2x + 1$ from $x = -2$ to $x = 1$.

Solution

Top curve on $[-2, 1]$ is $y = 5 - x^2$.

Area integral: $A = \int_{-2}^{1}[(5 - x^2) - (2x + 1)]\,dx$

Simplify: $A = \int_{-2}^{1}(-x^2 - 2x + 4)\,dx$

Integrate: $A = \left[-\frac{1}{3}x^3 - x^2 + 4x\right]_{-2}^{1}$

$$= \left(-\frac{1}{3}(1)^3 - (1)^2 + 4(1)\right) - \left(-\frac{1}{3}(-2)^3 - (-2)^2 + 4(-2)\right)$$

$$= \frac{8}{3} - \left(-\frac{28}{3}\right)$$

$$= 12$$

The area enclosed by two graphs

Sometimes, two curves may intersect at some point(s), creating a region enclosed (bounded) by such graphs. To determine the area enclosed by the curves let us go through this activity.

Activity 6.4: Exploring areas between enclosed curves

Step 1: Draw the graph of the first function $y = -x^2 + 4$ and shade the area between the function and the x-axis.

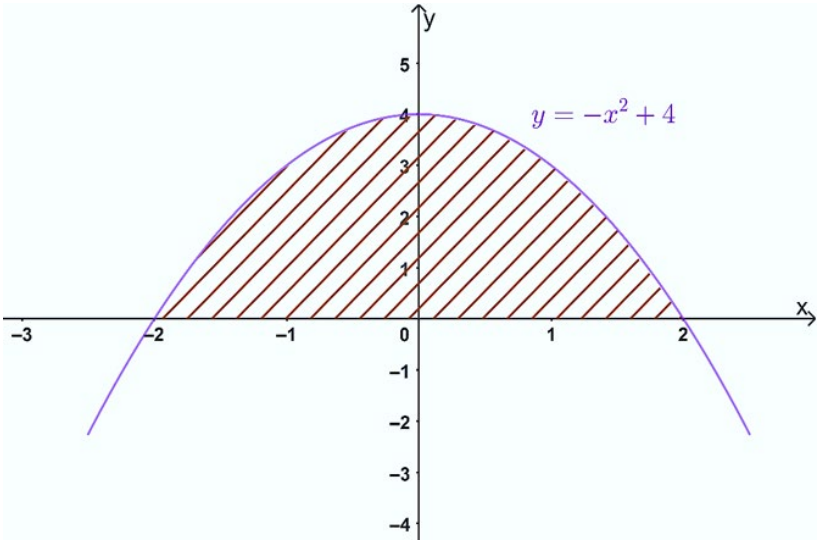


Figure 6.14: Graphical representation of area between $y = -x^2 + 4$ and the x-axis.

Step 2: On the same axes, draw the graph of the second function $y = x^2 + 2$ and shade the area between the function and the x-axis with a different style.

Step 3: Find and label all points where the two parabolas intersect.

Step 4: Identify the different regions enclosed (bounded) between the curves.

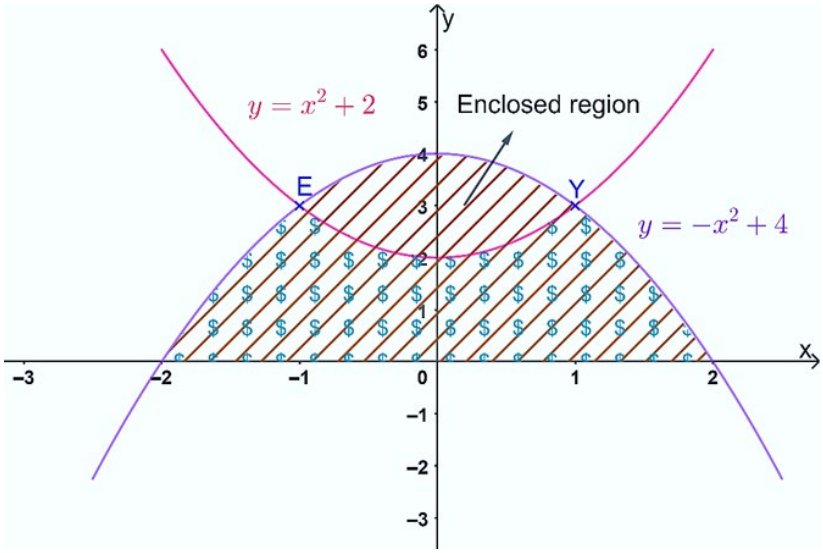


Figure 6.15: Graphical representation of area between $y = -x^2 + 4$ and $y = x^2 + 2$

Step 5: For each region, note which curve is above and which one is below across that interval of x.

Step 6: Noting the **Area** under each of the curves, explain how you think we can determine the area of the enclosed region.

Step 7: Express the area under curve using definite integrals and use your explanation to express the area of the enclosed region as a definite integral

(Show it to your teacher. You do not need to solve the integral at this point).

Generalisation

From Activity 6.4, it can be observed that the area enclosed by two curves can be expressed as; $A = \int_a^b |f(x) - g(x)|dx$ where $f(x)$ lies above $g(x)$ in the given interval, just as illustrated in Figures 6.15 and 6.16.

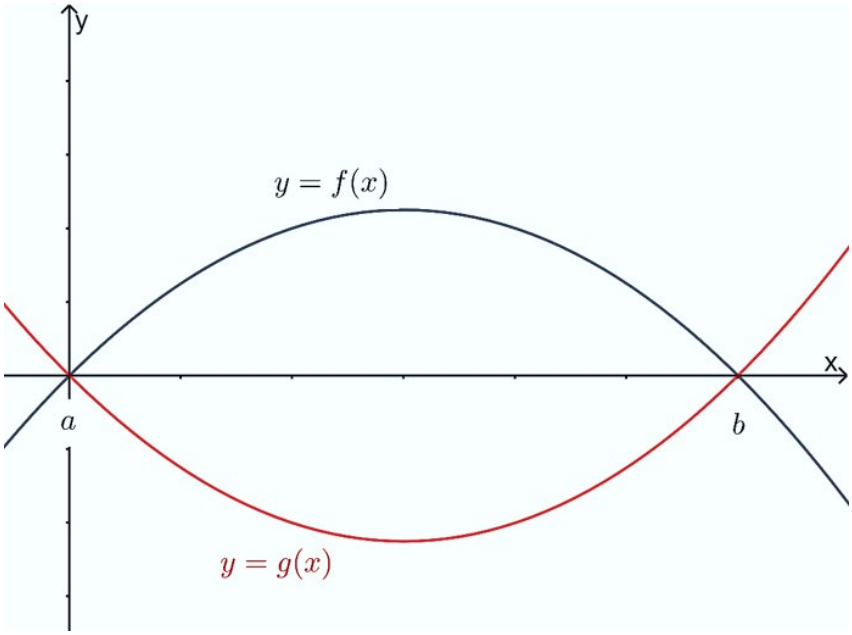


Figure 6.16: Graphical representation of area between $y = f(x)$ and $y = g(x)$.

Example 6.9

Draw the graphs of the curves $y = \frac{x^2}{4} + 1$ and $y = \sqrt{x+2}$, where $-2 \leq x \leq 3$ on the same axes. From your graph:

i. Determine their points of intersection.

ii. Calculate the area enclosed by the two curves.

Solution

i. From the graphical illustrations, the curves intersect at $(-0.72, 1.13)$ and $(2, 2)$ within the interval $[-2, 3]$.

ii. Enclosed Area $= \int_{-0.72}^{2} \left[\left(\sqrt{x+2}\right) - \left(\frac{x^2}{4}+1\right)\right] dx$

$$= \left[\frac{2(x+2)^{\frac{3}{2}}}{3} - \left(\frac{x^3}{12}+x\right)\right]_{-0.72}^{2}$$

$$= \left[\left(\frac{2(2+2)^{\frac{3}{2}}}{3}\right) - \left(\frac{2^3}{12}+2\right)\right] - \left[\left(\frac{2(-0.72+2)^{\frac{3}{2}}}{3}\right) - \left(\frac{(-0.72)^3}{12}+(-0.72)\right)\right]$$

$$= 0.9501$$

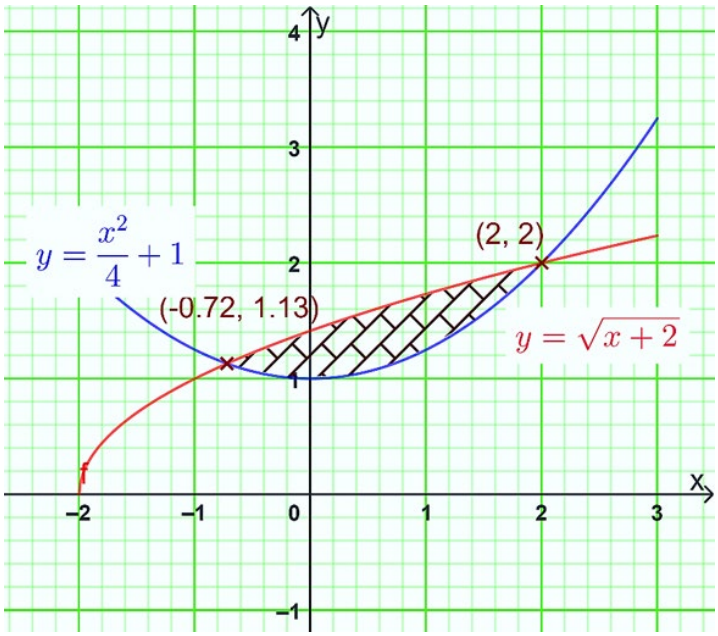


Figure 6.17: Graphical illustration of area between $y = \frac{x^2}{4} + 1$ and $y = \sqrt{x+2}$.

USING THE TECHNIQUES OF INTEGRATION TO SOLVE REAL-LIFE PROBLEMS

We have familiarised ourselves with calculating areas using integration. Now we will explore its usefulness in real-life using these examples.

Example 6.10

A hot-air balloon is rising vertically, and its upward velocity is modelled by $v(t) = 20(1 - e^{-0.4t}) m/s$, where t is the time in seconds after take-off. Find the total height of the balloon after the first 8 seconds of ascent.

[Clue: The total height is the area under the velocity-time graph].

Solution

$v(t) = 20(1 - e^{-0.4t})\ m/s$,

The height of the balloon can be determined by integrating the function $v(t)$ from 0 to 8 seconds.

Hence height of balloon $= \int_0^8 v(t)dt$

$$\int_0^8 v(t)dt = \int_0^8 20(1 - e^{-0.4t})\ dt$$

$$\int_0^8 v(t)dt = \int_0^8 20(1 - e^{-0.4t})\ dt$$

$$= 20\left(\int_0^8 1\ dt - \int_0^8 e^{-0.4t}\ dt\right)$$

Solve $\int_0^8 e^{-0.4t}\ dt$ using $\int_a^b e^{kt}dt = \left[\frac{1}{k}(e^{kt})\right]_a^b$ where $k = -0.4, a = 0, b = 8.$

$$= 20\left[t - \left(-\frac{1}{0.4}e^{-0.4t}\right)\right]_0^8$$

$$= 20\left[t + \frac{1}{0.4}e^{-0.4t}\right]_0^8$$

$$= 20\left[\left(8 + 2.5e^{-0.4(8)}\right) - \left(0 + 2.5e^{-0.4(0)}\right)\right]$$

$$= 20(5.5 + 2.5e^{-3.2})$$

$$= 110 + 50e^{-3.2}$$

$$= 112.04m$$

Therefore, the balloon is about $112.04m$ high after 8 seconds.

Example 6.11

Water is being pumped into and drained out of a swimming pool at different times of the day. The net rate of change of water in the pool is given by $r(t) = 150(t - 3)$ litres per minute, where $0 \leq t \leq 6$.

1. Determine the time at which the water being pumped into the pool and the water being drained is equal.
2. If the pool initially contains $1\,000$ litres of water at $t = 0$, find the total amount of water in the pool at $t = 6$ minutes.

Solution

1. Water level decreases if $r(t) < 0$ so,

$150(t-3) < 0$

$150(t-3) = 0$

$150t - 450 = 0$

$150t = 450$

$t = 3$

At $t = 3\ minutes, r(3) = 0$, the water level is neither increasing nor decreasing.

2. Amount of water at time t is $A(t) = A(0) + \int_0^t r(t)\,dt$

For $t = 6$

$$A(6) = 1000 + \int_0^6 150(t-3)\,dt$$

$$= 1000 + 150\left[\frac{t^2}{2} - 3t\right]_0^6$$

$$= 1000 + 150\left(\frac{6^2}{2} - 3(6)\right)$$

$$= 1000 \text{ litres}$$

The pool still has 1000 litres of water after 6 seconds.

FINDING THE VOLUME OF A SOLID FORMED AFTER ROTATION ABOUT THE HORIZONTAL OR VERTICAL AXIS

When two-dimensional figures are rotated about an axis. A solid is formed. For example, a rectangle on the x-axis as shown in Figure 6.18 and Figure 6.19.

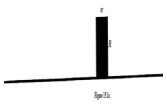


Figure 6.18: Representation of a rectangle

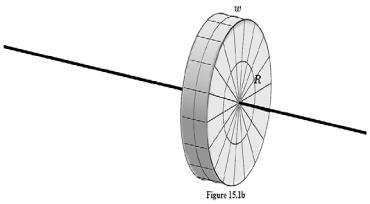


Figure 6.19: solid of evolution of a rectangle

The width of the rectangle is given by w. Its height is represented by R. When the rectangle is rotated about the axis, it creates a cylinder whose volume, V can be found as the product of the area of the circle formed from the rotation (πR^2) and the thickness of the cylinder (same as the width of the rectangle, w) i.e., $V = \pi R^2 w$.

Method of discs

A variety of interesting solids can be obtained from revolving the curve represented by $f(x)$ between a given interval of x, say $[a, b]$ about the x-axis or an interval of y about the y-axis.

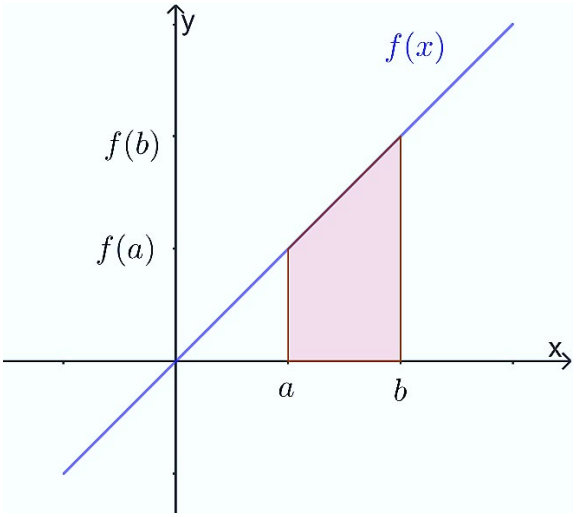


Figure 6.20: Graphical representation of $f(x)$ at interval $[a, b]$

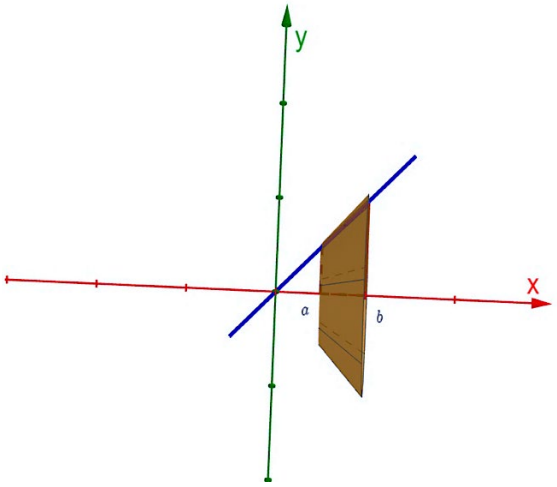


Figure 6.21: Resulting solid of evolution of Figure 6.20

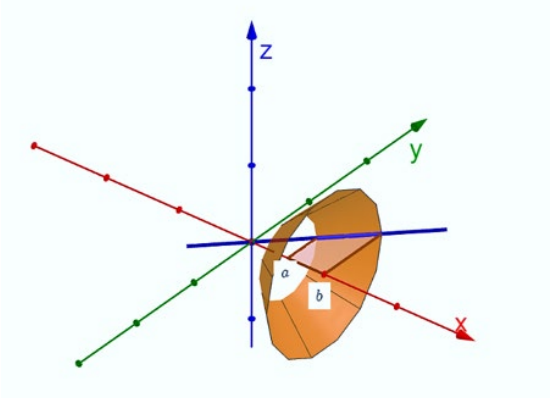


Figure 6.22: 3-dimensional view of Figure 6.21

Since the curve may not be horizontal (for revolutions about the x-axis) or vertical (for revolutions about the y-axis), the heights of the curve at the extremities of the intervals i.e., $f(a)$ and $f(b)$ may be unequal i.e., $f(a) \neq f(b)$ and thus, R cannot be used to represent the height. This means we cannot simply generalise that the volume of the solid obtained is $V = \pi R^2 w$, where, $w = b - a$.

A way to take care of this problem is to divide the interval, $[a, b]$ into n smaller intervals of width Δx or Δy so small that the heights of the curve at the ends of those intervals (x_i , for $i = 0, 1, 2, \ldots, n$ or y_i, for $i = 0, 1, 2, \ldots, n$) are the significantly the same. This would mean that many rectangles of height $f(x_i)$, for $i = 0, 1, 2, \ldots, n$ and width, Δx or Δy are created. The revolutions of these rectangles would produce n circles (c_i, for $i = 0, 1, 2, \ldots, n$) of radii, $f(x_i)$ and thereby n cylinders (discs) of thickness, Δx or Δy.

The volume of the resulting solid is thus the summation of all the volumes of the n cylinders (discs) obtained i.e.,

$$V = \sum_{i=1}^{n} [A(c_i)]\,\Delta x$$

This process is referred to as the **method of discs**.

Computing the definite integral over the interval $[a, b]$ removes the need for repeated calculations.

In practical terms, the method of discs can be described as follows:

Let $f(x) \geq 0$ be a continuous function in the interval $[a, b]$, consider the region bounded by the curve $y = f(x)$ and the x-axis, for $a \leq x \leq b$. When this region is revolved about the x- axis, it generates a solid (Figure 6.23 and Figure 6.24). The volume of this solid can be obtained by slicing it perpendicular to the $x - axis$ noting that each cross section is a circular disk of radius $r = f(x)$. The volume of the solid is therefore given by:

$$V = \sum_{i=1}^{n} [A(c_i)]\,\Delta x = \int_a^b \pi [f(x)]^2 dx$$

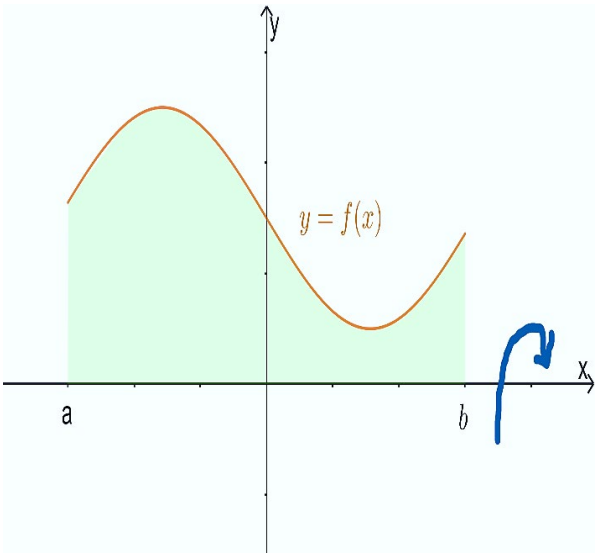


Figure 6.23: Graphical representation of $y = f(x)$ rotated about the x-axis

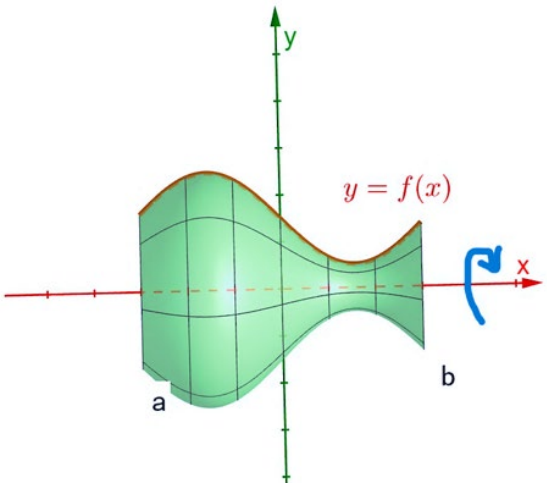


Figure 6.24: Graph of the resulting solid of revolution of $y = f(x)$

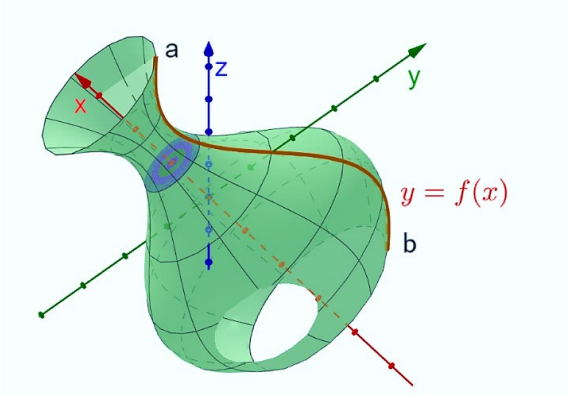


Figure 6.25: 3-dimensional view of Figure 6.24

In a 3-dimensional plane, this is what the rotated curve looks like and the purple circle within the green solid shade shows the cross section is a circular disk

Example 6.12

Revolve the region under the curve $y = \sqrt{x}$ on the interval $[1, 5]$ about the $x - axis$ and find the volume of the resulting solid of revolution.

Solution

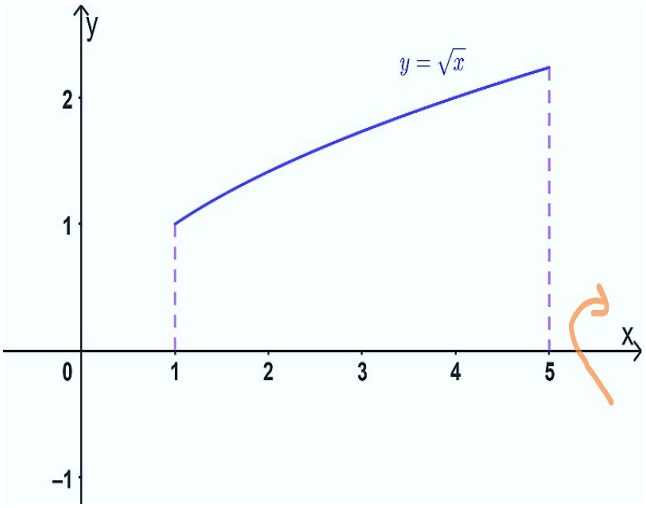


Figure 6.26: Graph of $y = \sqrt{x}$ when rotated about the x-axis

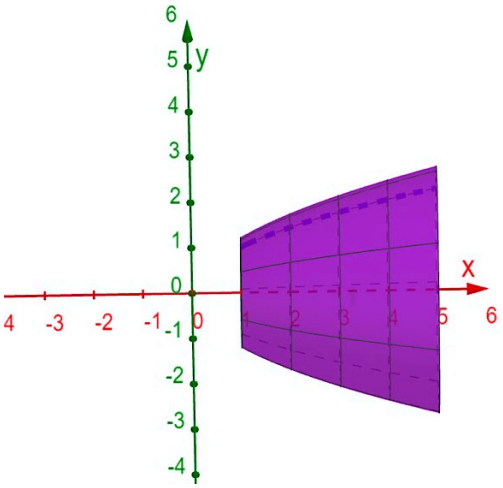


Figure 6.27: The resulting solid of revolution of $y = \sqrt{x}$

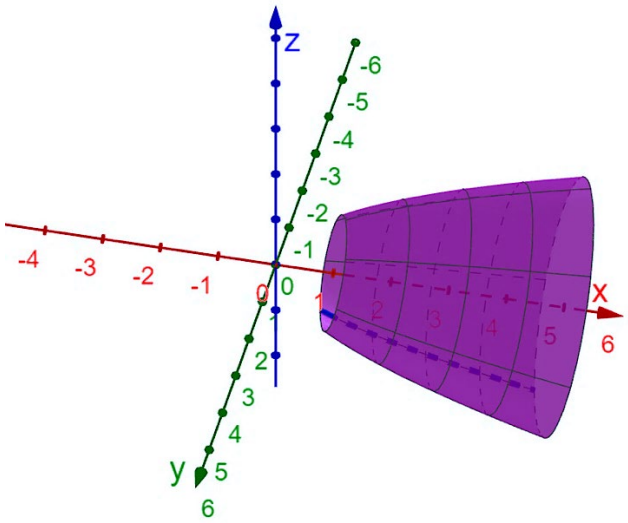


Figure 6.28: 3-dimensional view of Figure 6.27

$$The\ circular\ area\ of\ a\ disc = \pi y^2$$

$$= \pi\left(\sqrt{x}\right)^2\ squared\ units$$

$$Volume\ of\ solid = \int_1^5 \pi(\sqrt{x})^2\,dx$$

$$= \pi \int_1^5 (\sqrt{x})^2\,dx$$

$$= \pi \int_1^5 x\,dx$$

$$= \pi \left[\frac{x^2}{2}\right]_1^5$$

$$= \pi \left[\frac{5^2}{2} - \frac{1^2}{2}\right]$$

$$= 12\pi \text{ cubic units}$$

Revolving about the y-axis

It is important to note that when the shape revolves around the y-axis, then the integration is done with respect to y(i.e., dy) and **not** with respect to x.

Example 6.13

Determine the volume of the solid resulting from revolving the region bounded by the curve $y = 2 - x^2$ from $y = 1$ to $y = 3$.

Solution

$y = 2 - x^2, y = 1, y = 3$

By solving for x, we have;

$x = \sqrt{2 - y}$

$$\text{Volume} = \int_1^3 \pi(\sqrt{2-y})^2\,dy$$

$$= \int_1^3 \pi(2 - y)\,dy$$

$$= \pi \left[2y - \frac{y^2}{2}\right]_1^3$$

$$= \pi \left[(8 - 4.5) - \left(2 - \frac{1}{2}\right)\right]$$

$$= 2\pi \text{ cubic units}$$

Example 6. 14

Determine the volume of the solid resulting from revolving the portion of the curve $y = 1 - \frac{x^2}{3}$ from $x = 0$ to $x = 2$ about the $y - axis$.

Solution

$y = 1 - \frac{x^2}{3}$

$\Rightarrow x = \sqrt{3 - 3y}$

When $x = 0, y = 1$

When $x = 2, y = -\frac{1}{3}$

Note that area and volume is a scalar quantity and hence it would not matter which limit is used as the lower or upper limit.

$$\int_{1}^{-\frac{1}{3}} \pi\left(\sqrt{3-3y}\right)^2 dy = \int_{-\frac{1}{3}}^{1} \pi\left(\sqrt{3-3y}\right)^2 dy$$

$$\int_{-\frac{1}{3}}^{1} \pi\left(\sqrt{3-3y}\right)^2 dy = \pi \int_{-\frac{1}{3}}^{1} (3-3y)\,dy$$

$$= \pi\left[3y - \frac{3y^2}{2}\right]_{-\frac{1}{3}}^{1}$$

$$= \pi\left[\left(3(1) - \frac{3(1)^2}{2}\right) - \left(3(0) - \frac{3(0)^2}{2}\right)\right]$$

$$= 1.5\pi \text{ cubic units}$$

REVIEW QUESTIONS 6

1. A car accelerates with velocity $v(t) = 20\ln(t+1)$ for $0 \leq t \leq 5$. Find the total distance travelled.
2. Determine the area enclosed between $y = \sqrt{x+1}$ and $y = \frac{x}{2} + 1$ for $0 \leq x \leq 3$.
3. A runner's velocity is given by $v(t) = 10 - 2t$ for $0 \leq t \leq 8$.

   Calculate the signed displacement and the total distance travelled.
4. Find the volume of the solid formed when the region under the curve

   $y = \sqrt{4-x^2}, 0 \leq x \leq 2$ is rotated about the x-axis.
5. The marginal revenue function for producing and selling x units of a product is $R'(x) = \frac{200}{x+2}$. If no revenue is earned at $x = 0$, find the revenue function.
6. Find the total area between $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \pi$.
7. A water pipe delivers water at a rate of $R(t) = 100\,e^{-\frac{t}{3}}$ litres per hour for $0 \leq t \leq 6$.

   Calculate the total volume of water supplied.
8. Evaluate the signed area $\int_0^{2\pi} \cos(2x)\ dx$ and explain why the signed and total areas differ.
9. Estimate the area bounded by the curves $y = e^{-x}$ and $y = 1 - x$ on $[0,1]$ using the trapezium rule with 4 intervals.
10. The region bounded by $y = \sin x$, $y = 0$, $0 \leq x \leq \pi$ is rotated about the y-axis. Find the volume generated.
11. A drone's vertical speed is $v(t) = \sin\left(\frac{\pi t}{6}\right)$ for $0 \leq t \leq 6$.

    Use the trapezium rule with 6 intervals to estimate its total vertical distance.
12. Calculate the area under the constant function $f(x) = -5$ from $x = -2$ to $x = 3$. State clearly the signed area and the total area.
13. The function $f(x) = \sin x - \frac{1}{2}$ is defined for $0 \leq x \leq 2\pi$.

    Find the signed area under the curve and compare it with the total area.
14. A car moves such that its acceleration is $a(t) = 4t - 6$.

    If its initial velocity is $10\ m/s$, find the velocity function and displacement at $t = 5$.

15. Find the volume of the solid obtained by rotating the region under $y = e^{-x}$, $0 \leq x \leq 1$ about the x-axis.

16. Find the area between $y = \ln(x + 1)$ and $y = \frac{x}{2}$ for $0 \leq x \leq 2$.

17. The function $f(x) = \frac{1}{x}$ is considered on $-2 \leq x \leq -1$.

Find both the net area and the total area between the curve and the x-axis.

18. Calculate the volume of the solid formed when the area under $y = \ln(x + 1)$, $0 \leq x \leq 2$ is rotated about the x-axis.

19. The region between $y = \frac{1}{x+1}, x = 0, x = 2$ and the x-axis is rotated about the x-axis. Before rotation, calculate the area between the curve and the axis.

20. Find the total displacement of a boat moving with velocity $v(t) = t^2 - 4t + 3$ for $0 \leq t \leq 5$.

Interpret your result in context.

SECTION

7 STATISTICS



HANDLING DATA

Organisation, Presentation, and Interpreting Data

INTRODUCTION

In statistics, analysing relationships between variables is crucial for making informed decisions and solving real-world problems. Regression and correlation are essential concepts that help us understand how variables are related and how one can be used to predict another. While correlation measures the strength and direction of a relationship, regression goes further by providing a mathematical model for predictions. By collecting data through suitable research methods and tools, we can apply statistical techniques to fit a linear function and identify the line of best fit. This line allows us to make accurate forecasts and solve contextual problems based on the data. Effective analysis, interpretation, and presentation of findings not only improve understanding but also support evidence-based decision-making in fields such as business, health, education, and science. By the end of this section, you should be able to:

1. Describe the concept of regression and distinguish between correlation and regression, and describe situations that call for the use of regression.
2. Fit a linear function and use the line of best fit to solve problems in the context of the data.
3. Undertake research to gather data using appropriate tools.
4. Perform analysis on a data set using appropriate techniques and tools and interpret findings.

KEY IDEAS

- **Correlation**: A measure of the strength and direction of a relationship between two variables.
- **Regression**: This provides a mathematical model to predict one variable based on another.
- **Line of Best Fit**: A straight line that best represents the data in a scatter plot, found through regression analysis.
- **Data Collection**: Use appropriate research tools such as questionnaires, interviews, observations, or online databases to gather accurate information.
- **Data Analysis**: Apply suitable statistical techniques (e.g., scatter plots, regression equations) to examine the data.
- **Interpretation of Results**: Draw conclusions from the analysis to answer questions or solve problems in the given context.

DESCRIBING THE CONCEPT OF REGRESSION

In statistics, we often want to know how two or more numeric variables are related. For example, a teacher may want to know if there is a relationship between the grades of students on a mid-term exams and on their end-of-term exams. If there is a relationship, what is the relationship, and how strong is it? In another example, a business owner may want to know how advertising expenditure affects sales, or a farmer may want to know how rainfall influences crop yield.

The type of data described in the examples is bivariate data (two variables). In reality, statisticians use multivariate data, meaning many variables. In this chapter, we will study the simplest form of regression, linear regression, with one independent variable (x) and one dependent variable (y). This involves data that fit a line in two dimensions. We will also revise correlation since it is related to regression. Before we learn the process of fitting a linear function to a set of data, we will first look at the meaning of variable as used in statistics and the difference between correlation and regression. Although correlation and regression are related, they serve different purposes.

Variable

In statistics, a variable is a characteristic or measurement that can be determined for a population. Variables may describe values like weight in kg, height in metres, favourite food, satisfaction level, age, and gender.

Example 7.1

Given the table below, identify the variables involved.

Table 7.1: Expenditure versus number of cars sold

Expenditure on advertisement (GHc)	Number of cars sold
25,000	25
30,000	40
35,000	43
40,000	48

Solution

Variable 1: Expenditure on advertisement

Variable 2: Number of cars sold

Independent and dependent variables

Independent Variable

This is the variable you manipulate to see how it affects another variable. It is also referred to as the Predictor or Explanatory variable. Example: Hours studied (predicts exam score), Price of an item (determines the quantity sold), Temperature (determines the amount of water consumed).

Usually, the independent variable can be determined when used in a context. Ask yourself: "What is being changed, used, or controlled to see its effect?" This is the independent variable.

Characteristics of the Independent variable

1. It is also known as an input, predictor, or explanatory variable.
2. It is the cause or factor being tested.
3. It does not depend on other variables in the analysis.
4. In a graph, it is usually displayed on the x-axis.

Dependent Variable

This is the variable that changes as a result of changes in the independent variable. It is also referred to as the Outcome or Response variable. Example: Exam scores (outcome of hours of study), Expenditure (determined by the income), Amount of water consumed (determined by the temperature).

Usually, the dependent variable can be determined when used in a context. Ask yourself: "What changes because of what I did or something that was done?" This is the dependent variable.

Characteristics of the dependent variable

1. It is also known as an output, response, or measured variable.
2. It is what the researcher is trying to explain, predict, or observe.
3. It is NOT directly controlled. It shows a response to some other change.
4. In a graph, it is usually displayed on the y-axis.

Example 7.2

State the independent and dependent variables in the following research scenarios

1. Capacity and sizes of bottles
2. Speed and Distance Travelled
3. Effect of study time and test scores
4. Effect of temperature and sales of ice cream
5. Area of plane shapes and their dimensions.

Solution

1. Capacity and sizes of bottles

   Independent Variable = size of the bottles

   Dependent Variable = Capacity of the bottles

   Reason: The capacity is dependent on the size of the bottles

2. Speed and Distance Travelled

   Independent variable = speed

   Dependent variable = distance travelled.

   Reason: The distance travelled is dependent on the speed of the car.

3. Effect of study time and test scores

   Independent variable = Study time

   Dependent variable = Test scores

   Reason: Test scores are usually dependent on the amount of time you study.

4. Effect of temperature and sales of ice cream

   Independent variable = temperature

   Dependent variable = Sales of ice cream

   Reason: Sales of ice cream are usually dependent on the outdoor temperature.

5. Area of plane shapes and their dimensions.

   Independent variable = dimensions of the plane shape

   Dependent variable = Area of the plane shape

   Reason: Without the dimensions, you cannot find the area.

Correlation

Corelation is a measure of the nature and strength of the relationship between two or more variables. It describes how well two variables go together. The relation between two variables may be positive, negative or non-existent (no correlation).

- A positive correlation between two variables means that an increase in the value of one variable is likely to increase the value of the other. Likewise, a decrease in one of the variables will cause a reduction in the other. It is a relationship that moves in tandem (in the same direction). It shows a direct relation between two variables. An example of a positive correlation is the relationship between temperature and water consumption.
- A negative correlation describes an inverse relationship. As one variable increases, the other decreases. It describes the relationship between two variables that change in opposite directions. An example is the relationship between age and agility. As your age increases, your agility decreases. Another example is speed and travel time. The higher the speed, the shorter the travel time. Other examples include preparation and mistake, supply and price, exercise and body weight, inflation and purchasing power.

- Zero or no correlation. This occurs when there is no relationship between the two variables. It also means changes in one variable do not predict changes in the other. An example is weight and intelligence. Your weight has nothing to do with your intelligence, and an increase in your weight has nothing to do with your intelligence. Other examples are blood type and prosperity, shoe size and favourite colour, weight and income, and race and intelligence.

Concept of Regression

Regression is a statistical method used to model the relationship between two variables and provide an equation that can be used to make predictions.

In simple linear regression, we have: $y = mx + c$

Where:

- y = dependent variable (predicted value)
- x = independent variable (predictor)
- c = intercept (value of y when $x = 0$)
- m = slope (rate of change of y with respect to x)

Typically, you choose a value to substitute for the independent variable and then solve for the dependent variable.

Table 7.2: Distinguishing Between Correlation and Regression

Feature	Correlation	Regression
Purpose	Measures strength and direction of relationship	Predicts the value of one variable from another.
Nature of Result	Single value (correlation coefficient, r) between -1 and 1	An equation of a line ($y = mx + c$)
Direction	No distinction between dependent and independent variables	One variable is dependent, the other is independent
Interpretation	Describes association	Describes relationship and enables prediction

Situations that Call for the Use of Regression

Regression is useful when:

1. Predicting Future Values – e.g., predicting sales from advertising expenditure.
2. Forecasting Trends – e.g., estimating future population growth.
3. Estimating Missing Data – e.g., filling in gaps in a dataset.
4. Analysing Cause-and-Effect – e.g., examining how temperature affects crop growth.
5. Optimisation – e.g., determining the level of resources needed to maximise profit.

FITTING A LINEAR FUNCTION AND USING THE LINE OF BEST FIT TO SOLVE PROBLEMS IN THE CONTEXT OF THE DATA

In year 2, we learned how to plot data points on a scatter plot which enabled us to visually observe the relationship between two variables. We have noticed that many of these scatter plots show a clear, linear trend, where the points seem to cluster around a straight line. This straight line is called the **line of best fit**, and it is a powerful tool in statistics that allows us to model the relationship between variables mathematically.

Now we will look at the process of fitting a linear function to a set of data. We will learn how to find the equation of the line of best fit and, most importantly, how to use this line to make predictions and solve problems in real-world contexts.

The Concept of the Line of Best Fit

Data rarely fits a straight line exactly. Usually, we must be satisfied with rough predictions. A straight line that represents the relationship between a dependent variable (Y) and one or more independent variables (X) in a linear regression model is called a line of best fit. It is a single straight line that best represents a trend in the data.

The line of best fit is usually used to predict data values within the given data. This is known as **interpolation**. When used to predict data values outside a given data, it is called **extrapolation**. Pairs of data values (points) relatively far from the line of best fit are called outliers. Extrapolation is a lot less reliable that interpolation.

Fitting a Linear Function to Data

While we can visually draw a line of best fit by hand, a more accurate method is to use a statistical technique called **linear regression**. Here, we will focus on the **least squares method**, which finds the line that minimises the sum of the squared vertical distances (residuals) from each point to the line.

Method of Least Squares

Least-Squares Criteria for Best Fit

The process of fitting the best-fit line is called linear regression. We assume that the data are scattered about a straight line. To find that line, we minimise the sum of the squared errors (SSE), or make it as small as possible. Any other line you might choose would have a higher SSE than the best-fit line. This best-fit line is called the least-squares regression line.

To understand the SSE, consider Figures 7.1 and 7.2, which show the same data with different lines fitted by hand.

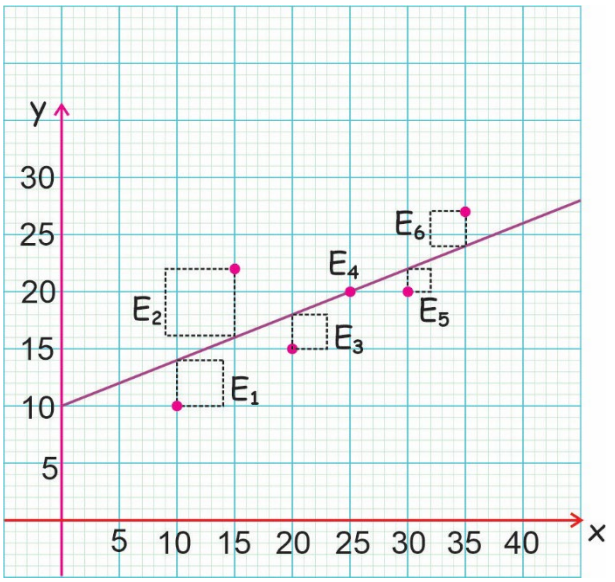


Figure 7.1: Error squares of a fitted line

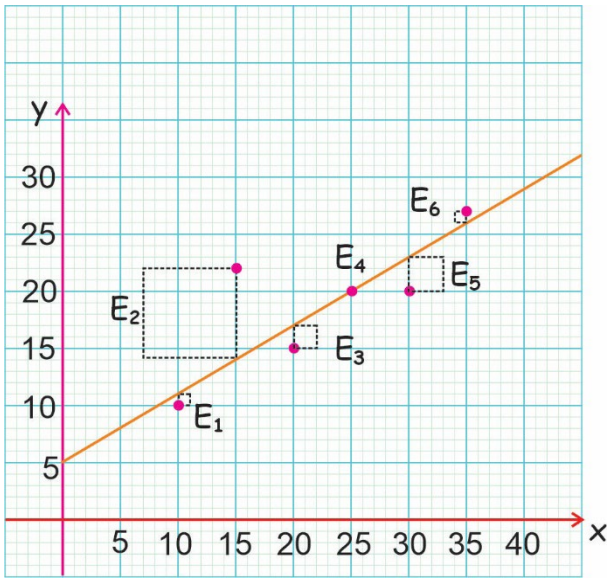


Figure 7.2 : Error squares of a fitted line

Which of the lines of best fit gives the least SSE?

If the line of best fit in Figure 7.1 is used for predicting the values of y the results would be 14, 16, 18, 20, 22, and 24 as can be read from the graph. Comparing this to the actual y-values, this respectively gives errors of:

$$E_1 = 10 - 14 = -4,$$
$$E_2 = 22 - 16 = 6,$$
$$E_3 = 15 - 18 = -3,$$
$$E_4 = 20 - 20 = 0,$$
$$E_5 = 20 - 22 = -2$$
$$E_6 = 27 - 24 = 3$$

If the line of best fit in Figure 7.2 is used for predicting the values of y the results would be 11, 14, 17, 20, 23, and 26 as can be read from the graph. Comparing this to the actual data values, this respectively gives errors of:

$$E_1 = 10 - 11 = -1,$$
$$E_2 = 22 - 14 = 8,$$
$$E_3 = 15 - 17 = -2,$$
$$E_4 = 20 - 20 = 0,$$
$$E_5 = 20 - 23 = -3$$
$$E_6 = 27 - 26 = 1$$

To calculate the SSE, we will first draw a vertical line from our supposed line of best fit to each of the plotted data points, find the square of each distance and sum the result.

For Figure 7.1	For Figure 7.2
$E_1 = -4 \implies (E_1)^2 = (-4)^2 = 16$ $E_2 = 6 \implies (E_2)^2 = 6^2 = 36$ $E_3 = -3 \implies (E_3)^2 = (3)^2 = 9$ $E_4 = 0 \implies (E_4)^2 = 0^2 = 0$ $E_5 = -2 \implies (E_5)^2 = (-2)^2 = 4$ $E_6 = 3 \implies (E_6)^2 = 3^2 = 9$ SSE$= 16 + 36 + 9 + 0 + 4 + 9 = 74$	$E_1 = -1 \implies (E_1)^2 = 1$ $E_2 = 8 \implies (E_2)^2 = 64$ $E_3 = -2 \implies (E_3)^2 = 4$ $E_4 = 0 \implies (E_4)^2 = 0$ $E_5 = -3 \implies (E_5)^2 = 9$ $E_6 = 1 \implies (E_6)^2 = 1$ SSE$= 16 + 64 + 4 + 0 + 9 + 1 = 79$

This means that the line of best fit in Figure 7.1 gives the least SSE.

The least squares method aims to obtain the line for which the total area of these squares is the smallest.

In statistics, it is known (though we will not prove) that the best-fitting line for a set of data always passes through the point $(\bar{x}, \bar{y})$. Here, $\bar{x}$ is the mean of all the x-values, and $\bar{y}$ is the mean of all the y-values.

Given a dataset with n points: (x_i, y_i) for $i = 1, 2, 3, \ldots, n$, a line with equation $y = mx + c$ can be found to model the data and predict other data points. Note that:

$$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

The value of c can be found by substituting $(\bar{x}, \bar{y})$ into the general linear equation.

Example 7.3

The table shows values of two related variables.

Table 7.3: Value of two related variables, x and y

x	1	2	3	4	5	6	7	9	8	5
y	9	10	6	4	7	1	2	1	1	4

Use the least squares method to determine the equation of the line of best fit.

Solution

Let the equation of the line of best fit be $y = mx + c$

Mean of x values, $\bar{x} = \frac{1+2+3+4+5+6+7+9+8+5}{10} = 5.0$

Mean of y values, $\bar{y} = \frac{9+10+6+4+7+1+2+1+1+4}{10} = \frac{45}{10} = 4.5$

$(\bar{x}, \bar{y}) = (5.0, 4.5)$

$m = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

$(x_i - \bar{x}) = (x_i - 5)$ and $(y_i - \bar{y}) = (y_i - 4.5)$

Table 7.4: Least Squares Regression Computation

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	9	-4	4.5	-18	16
2	10	-3	5.5	-16.5	9
3	6	-2	1.5	-3	4
4	4	-1	-0.5	0.5	1
5	7	0	2.5	0	0
6	1	1	-3.5	-3.5	1
7	2	2	-2.5	-5	4
9	1	4	-3.5	-14	1
8	1	3	-3.5	-10.5	1
5	4	0	-0.5	0	0
				$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = -70$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 37$

$m = \frac{-70}{37} = -1.89189 \approx -1.89$

Since $(\bar{x}, \bar{y}) = (5.0, 4.5)$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$\therefore 4.5 = (-1.89 \times 5) + c$

$c = 13.959 \approx 14.0$

Therefore, the equation of the best-fit line is $y = -1.89x + 14.0$

Figure 7.3 shows the graph of $y = -1.89x + 14.0$.

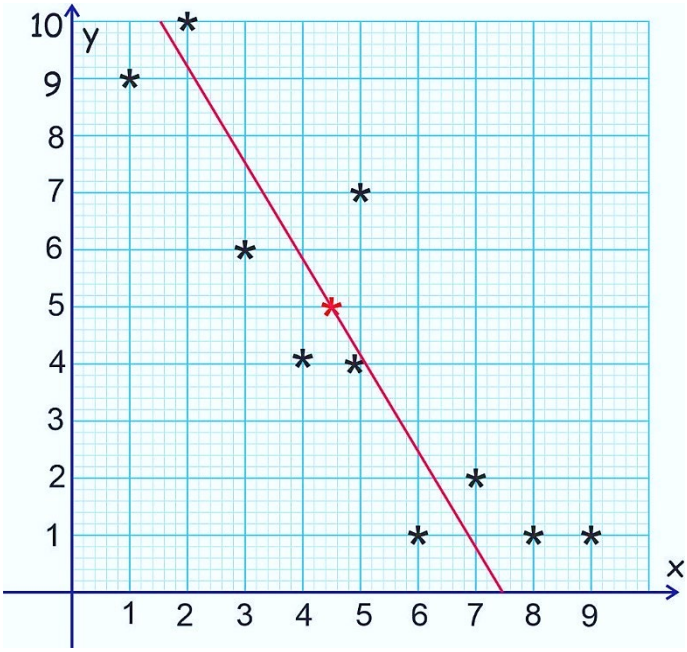


Figure 7.3: Scatter plot for the data showing the line of best fit

Activity 7.1: Analysing Data Using the Line of Best Fit

Work through this activity in pairs.

The diagram shows a scatter plot of the weight and height of a sample of 10 students.

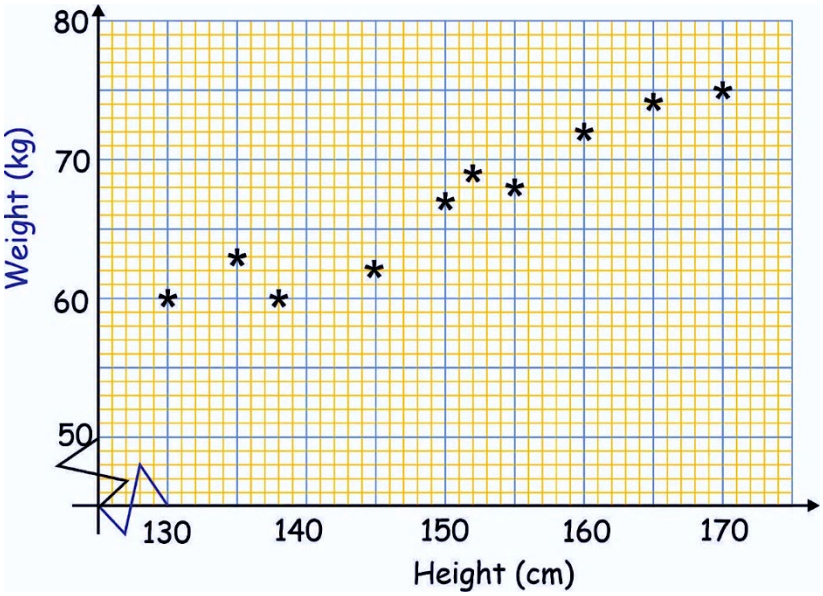


Figure 7.4: Scatter plot of Height versus Weight

Step 1: Observation

a. Examine the scatter plot provided and describe what it suggests about the relationship between height and weight.
b. Indicate whether the relationship is positive, negative or shows no correlation.
c. Copy and complete the table below

Table 7.5: Partially completed table of Height and Weight.

Height (x)	130	135	138	145	150	152	155	160	165	170
Weight (y)	60	63	60	62	67	69	68	72	74	75

Step 2: Fitting the model

a. Using the least squares method, calculate the equation of the line of best fit for the given data.
b. Express the equation in the form: $y = mx + c$ where m is the gradient and c is the y-intercept.
c. Interpret the gradient of the line of best fit.

Step 3: Drawing the Line of Best Fit

a. Make a copy of Figure 7.4 in your graph book. Using the equation you obtained, draw the line of best fit on the diagram.
b. Ensure the line passes through the point $(\bar{x}, \bar{y})$

Step 4: Prediction

Using your linear model,

a. Estimate the height of a student who weighs 70 kg. Show all working clearly.
b. Estimate the weight of a student whose height is $165cm$

Solution

Step 1: Observation

a. The scatter plot shows a positive correlation between height and weight. This means that taller people in this data tend to weigh more.
b. Find below the completed table.

Table 7.6: Height and Weight

Height (x)	130	135	138	145	150	152	155	160	165	170
Weight (y)	60	63	60	62	67	69	68	72	74	75

Step 2: Fitting the model

a. Let the equation of the line of best fit be $y = mx + c$

Mean of x values, $\bar{x} = \frac{130+135+138+145+150+152+155+160+165+170}{10} = \frac{1500}{10} = 150$

Mean of y values, $\bar{y} = \frac{60+63+60+62+67+69+68+72+74+75}{10} = \frac{670}{10} = 67$

$(\bar{x}, \bar{y}) = (150, 67)$

$m = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

$(x_i - \bar{x}) = (x_i - 150)$ and $(y_i - \bar{y}) = (y_i - 67)$

Table 7.7: Least Squares Regression Computation.

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
130	60	-20	-7	140	400
135	63	-15	-4	60	225
138	60	-12	-7	84	144
145	62	-5	-5	25	25
150	67	0	0	0	0
153	69	3	2	6	9
155	68	5	1	5	25
160	72	10	5	50	100
165	74	15	7	105	225
170	75	20	8	160	400
				$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 635$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 1553$

$m = \frac{635}{1553} = 0.408886027 \approx 0.41$

Since $(\bar{x}, \bar{y}) = (150, 67)$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$, we have:

$67 = (0.41 \times 150) + c$

$c = 67 - 61.33 = 5.667 \approx 5.67$

Therefore, the equation of the best-fit line is $y = 0.41x + 5.67$

c. The gradient for the line of best fit is approximately 0.41. This means that for every 1 cm increase in height, the weight increases by about 0.41 kg.

Step 3: Drawing the Line of Best Fit

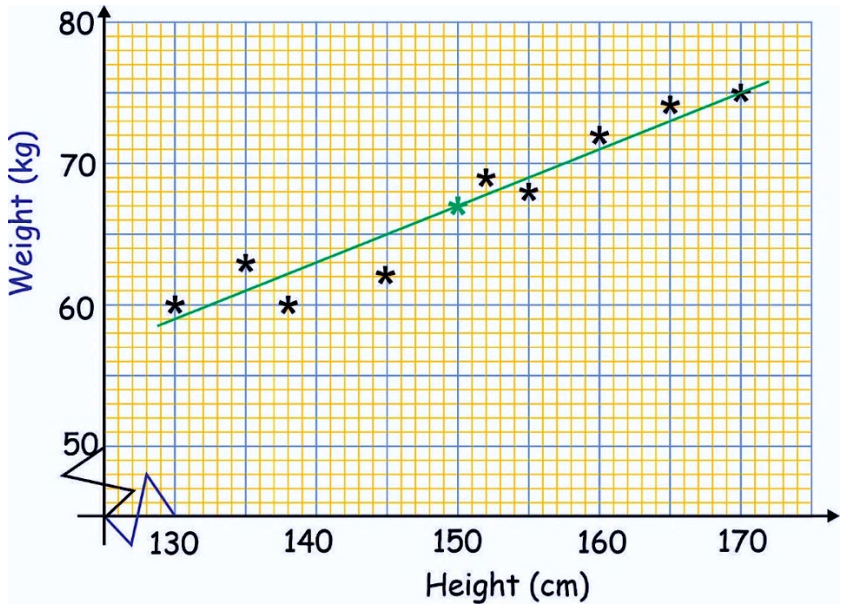


Figure 7.5: Scatter plot and regression line of Height versus Weight

Step 4: Prediction

Using your linear model,

a. $y = 0.41x + 5.67$

When y=70, we have $70 = 0.41x + 5.67$

$0.41x = 70 - 5.67$

$0.41x = 64.33x = \frac{64.33}{0.41} = 156.902 \approx 157$

This means that a person who weighs 70kg is expected to be approximately 157cm tall.

b. Estimate the weight of a student whose height is 165cm

From the model, when $x = 165$, $y = 0.41 \times 165 + 5.67 = 73.32$kg

This means that a person who is 165cm tall is expected to weigh approximately 73.32kg.

Activity 7.2: Investigating the Effect of Study Time on Test Performance

Working in pairs, or small groups carry out the following activity.

A group of students conducted an experiment to determine how the number of hours spent studying affects performance in a mathematics test. Eight students recorded the number of hours they studied during the week before the test and their corresponding scores out of 20 marks. The data below shows the number of hours studied (x) and the test scores (y):

Table 7.8: Hours studied and Test Scores

Hours Studied (x)	2	5	8	3	7	6	1	4	0
Test Score (y)	6	11	16	9	14	14	4	12	4

Tasks:

1. Construct a scatter plot using the data above with the x-axis as hours studied and the y-axis as test scores.
2. Describe the trend shown by the scatter plot. What does it suggest about the relationship between hours of study and test scores?
3. Fit a linear model to the data using the method of least squares. Express the equation in the form: $y = mx + c$. Using the equation you obtained, draw the line of best fit on the scatter plot.
4. Prediction and Reliability: If the margin of error for predicting test scores is ± 1.5 marks, use your model to estimate the score of a student who studied for 5.5 hours. Would the model be reliable for this prediction?

Solution

1. The scatterplot is shown in Figure 7.6

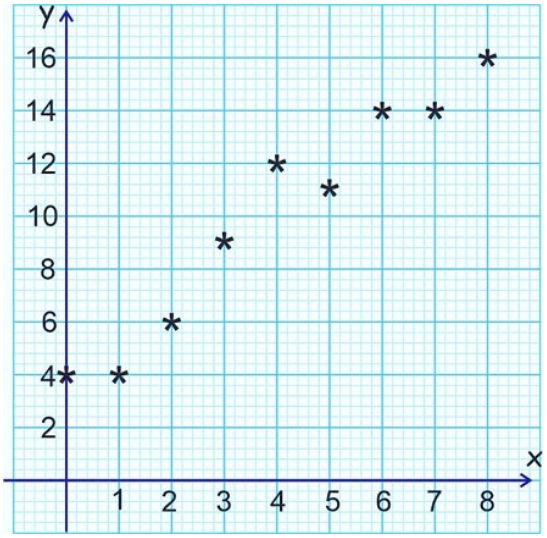


Figure 7.6: Scatter plot for data in Table 7.8

2. The scatter plot shows a general upward trend: as the number of hours studied increases, test scores also increase. This suggests a **positive linear relationship** between hours of study and test performance.

3. Let the linear model of best fit be $y = mx + c$

Table 7.9: Least Squares Regression Computation.

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
2	6	-2	-4	8	4
5	11	1	1	1	1
8	16	4	6	24	16
3	9	-1	-1	1	1
7	14	3	4	12	9
6	14	2	4	8	4
1	4	-3	-6	18	9
4	12	0	2	0	0
0	4	-4	-6	24	16
$\sum x = 36$	$\sum y = 90$			$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 96$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 60$

$$Mean\ of\ x\ values, \bar{x} = \frac{\sum x}{9} = \frac{36}{9} = 4$$

$$Mean\ of\ y\ values, \bar{y} = \frac{\sum y}{9} = \frac{90}{9} = 10$$

$$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$m = \frac{96}{60} = \frac{8}{5} = 1\frac{3}{5} = 1.6$$

Since $(\bar{x}, \bar{y})$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$$\therefore 10 = \left(\frac{8}{5} \times 4\right) + c$$

$$c = 10 - \frac{32}{5} = \frac{18}{5} = 3\frac{3}{5} = 3.6$$

Therefore, from the least-squares method, the equation of the line of best fit is:

$y = \frac{8}{5}x + \frac{18}{5}$ or $y = 1.6x + 3.6$ or $5y - 8x - 18 = 0$.

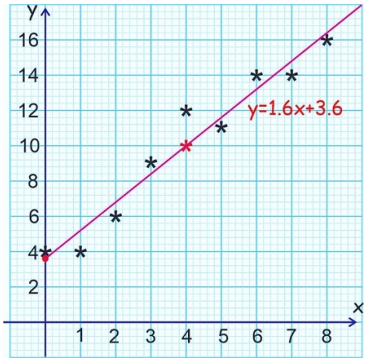


Figure 7.7: Scatter plot and graph of $y = 1.6x + 3.6$

4. **Prediction and Reliability**

   Using the model, the estimated test score of a student who studied for 6 hours is: $y = 1.6 \times 6 + 3.6 = 13.2$.

   However, from the data obtained from actual readings, the score of the student who studied for 6 hours is 14, which gives an error of $14 - 13.2 = 0.8$

   So, the error committed in using the model is 0.8. which falls between the interval -1.5 and 1.5. Therefore, the model is appropriate for the prediction.

UNDERTAKE RESEARCH TO GATHER AND ANALYSE DATA USING APPROPRIATE TOOLS

Research to Gather Data Using Appropriate Tools

Research is a process of collecting information to answer a specific question, solve a problem, or understand a situation. In statistics, research begins with identifying the type of data needed and the most effective tools to collect it.

Undertaking research helps us discover concepts we may not have grasped in the classroom. It may also offer us the opportunity to expand our understanding of concepts. Here we will apply the knowledge we have gained previously to analyse the data we will collect. We will also have the opportunity to present our findings to our classmates.

Steps in Data Collection:

1. Identify the purpose of the research
   Clearly define the question or problem to be addressed.
2. Determine the type of data required
   The data could either be primary or secondary.
3. Select appropriate data collection tools
   Depending on the research topic, a questionnaire, a face-to-face Interview, an observation checklist, or take measurements may be appropriate.
4. Collect the data.
   Ensure all data is recorded accurately. If dealing with measurements, maintain consistency.
5. Ensure ethical considerations.
   For instance, if we need students' grades from a class or a year group, obtain consent from the authorities or participants and keep the collected information confidential.

Reasons to define the research question clearly

1. Provides Focus and Direction

   A clear research question acts as a compass, guiding us towards a specific objective. Instead of a broad topic like "studying crops," a well-defined question such as "How does the amount of fertiliser affect cocoa yield?" gives a precise goal. This focus ensures

that every step of the research contributes to answering that specific inquiry.

2. Guide Data Collection

   The research question determines the specific data to be collected and the tools required for the task. In the cocoa yield example, a well-defined question indicates that data should be gathered on both the amount of fertiliser used and the final cocoa yield. This precision helps prevent collecting irrelevant information.

3. Saves Time and Resources

   By knowing exactly what we are looking for means that we can avoid the time-consuming process of gathering unnecessary data. A clear question helps to create a lean and efficient research plan, ensuring all efforts are directed only at what's essential.

4. Determines Analysis Methods

   The nature of the research question directly determines the statistical tools to be used. A question about the relationship between two variables, like fertiliser and yield, will probably need regression or correlation analysis. A question about a group's average performance, on the other hand, might simply require calculating the mean.

5. Simplifies Interpretation

   When the research question is clear from the start, the findings can be presented as a direct and concise answer. This makes interpreting and discussing the results easier, allowing us to clearly explain what has been discovered and how it relates to the original problem.

Activity 7.3: Investigating the Relationship Between Study Time and Test Scores

Working in pairs, or individually, carry out the following activity.

Objective:

To design and carry out a small research project that collects bivariate data using appropriate data collection tools.

Task Instructions

1. Define the Research Question:

   Example: "Is there a relationship between the number of hours students' study in a week and their last mathematics test scores?"

2. Identify the Variables:

   For this particular research question,

   Independent variable (x) = Hours of study per week

   Dependent variable (y) = Mathematics test scores.

3. Decide the Type of Data:

   The data to be collected will be primary data because it will be collected directly from students.

4. Select an Appropriate Data Collection Tool:

   You can use:

   - Questionnaire: Design a simple questionnaire that will solicit the study hours and test scores in mathematics for a particular term or semester.
   - Interview: Have a face-to-face interaction with learners to collect the data.
   - Class record: You may ask learners about their study hours and obtain their scores in mathematics from class records.

5. Prepare the Data Collection Tool:

   - For a questionnaire, write clear, concise questions (e.g., "How many hours did you study mathematics last week?").
   - For an interview, prepare a short list of guiding questions.

6. Collect the Data:

   - Survey at least 10–15 students.
   - Record responses neatly in a table with two columns: Hours Studied and Test Score.

7. Organise the Data:

   - Present the collected data in a table.
   - Make sure both variables are correctly labelled.

8. Class Discussion

   Share with the class the:

   - The research question.
   - The type of data collected (primary or secondary).
   - The tool used and the reason for choosing it.
   - The raw data table.

PERFORM ANALYSIS ON A DATA SET USING APPROPRIATE TECHNIQUES AND TOOLS AND INTERPRET FINDINGS

Data analysis is the process of examining, cleaning, transforming, and modelling data to discover useful information, draw conclusions, and support decision-making.

Steps in Data Analysis:

1. Organise the data
   - o Arrange data in tables or spreadsheets for easier handling.
2. Select appropriate analysis techniques
   - o Descriptive statistics: Mean, median, mode, range, percentages, frequency tables.
   - o Graphical methods: scatter plots.
   - o Inferential statistics: Correlation, regression.

3. Use appropriate tools
   - Manual calculations with calculators.
   - Software tools such as Microsoft Excel, Google Sheets, SPSS, or Python.
4. Analyse the data
   - Apply formulas to find averages and other statistical measures.
   - Draw graphs to visualise patterns or relationships.
5. Interpret the findings
   - Explain what the results mean in relation to the original research question.
6. Draw conclusions and make recommendations
   - Summarise the key insights from the analysis.
   - Suggest possible actions based on findings.

Here is an activity specifically using regression for the indicator

Activity 7.4: Predicting Crop Yield Using Regression Analysis

Working individually, or in pairs, undertake the following activity.

Objective:

Apply regression techniques to analyse data, find the equation of the line of best fit, and use it for predictions.

Scenario

A group of agricultural researchers collected data on the amount of fertiliser applied to a plot of land and the corresponding crop yield. Use regression analysis to determine how fertiliser quantity affects crop yield.

Table 7.10: Fertiliser and Crop Yield

Fertiliser (cups) (x)	Crop Yield (bags) (y)
1	16
2	21
3	23
4	26
5	31
6	34
7	40
8	41

Instructions

Step 1 – Plot the Data

Draw a scatter plot with fertiliser quantity on the x-axis and crop yield on the y-axis.

Step 2 – Perform Regression Analysis

a. Using the method of least squares, calculate:

   i. The gradient (m)

   ii. The y-intercept (c)

b. Write the regression equation in the form: $y = mx + c$

Step 3 – Interpretation

a. Explain what the gradient means in the context of crop yield and fertiliser use.

b. Explain what the y-intercept means in the context of crop yield

c. State the relationship (positive/negative correlation) between fertiliser and yield.

Step 4 – Prediction

a. Use your regression equation to predict the yield when 7 cups of fertiliser are applied. State whether your prediction seems reasonable based on the plotted data.

b. Use your regression equation to predict the yield when 18 cups of fertiliser are applied. State whether your prediction seems reasonable based on the plotted data.

Solution

Step 1 – Plot the Data

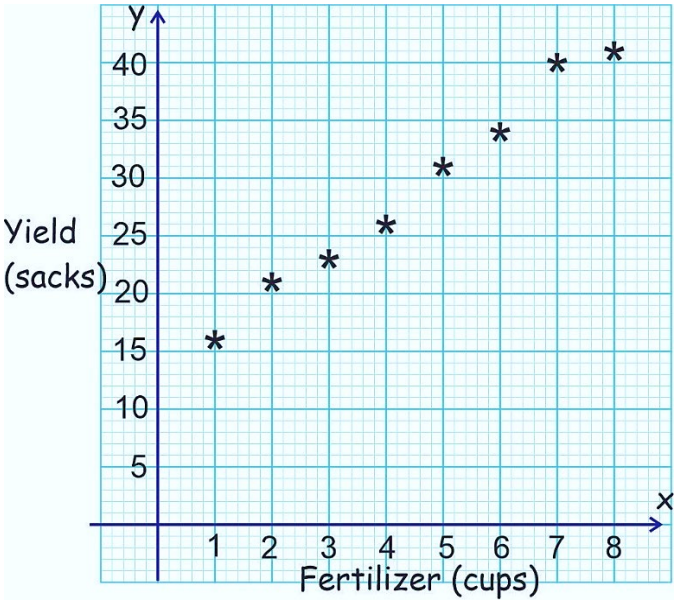


Figure 7.8: Scatter plot of Fertiliser versus Yield

Step 2 – Perform Regression Analysis

a. Let the equation of the line of best fit be $y = mx + c$

Mean of x values, $\bar{x} = \frac{1+2+3+4+5+6+7+8}{8} = 4.5$

Mean of y values, $\bar{y} = \frac{16+21+23+26+31+34+40+41}{8} = \frac{232}{8} = 29$

$(\bar{x}, \bar{y}) = (4.5, 29)$

$m = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

$(x_i - \bar{x}) = (x_i - 4.5)$ and $(y_i - \bar{y}) = (y_i - 29)$

Table 7.11: Least Squares Regression Computation.

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	16	-3.5	-13	45.5	12.25
2	21	-2.5	-8	20	6.25
3	23	-1.5	-6	9	2.25
4	26	-0.5	-3	1.5	0.25
5	31	0.5	2	1	0.25
6	34	1.5	5	7.5	2.25
7	40	2.5	11	27.5	6.25
8	41	3.5	12	42	12.25
				$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 154$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 42$

i. $m = \frac{1540}{42} = \frac{11}{3} = 3\frac{2}{3}$

ii. Since $(\bar{x}, \bar{y}) = (4.5, 29)$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$$\therefore 29 = \left(\frac{11}{3} \times 4.5\right) + c$$

$$c = 12.5$$

b) Therefore, the regression line is $y = \frac{11}{3}x + \frac{25}{2}$ or $6y - 22x - 75 = 0$

Figure 7.9 shows the graph of $y = \frac{11}{3}x + \frac{25}{2}$

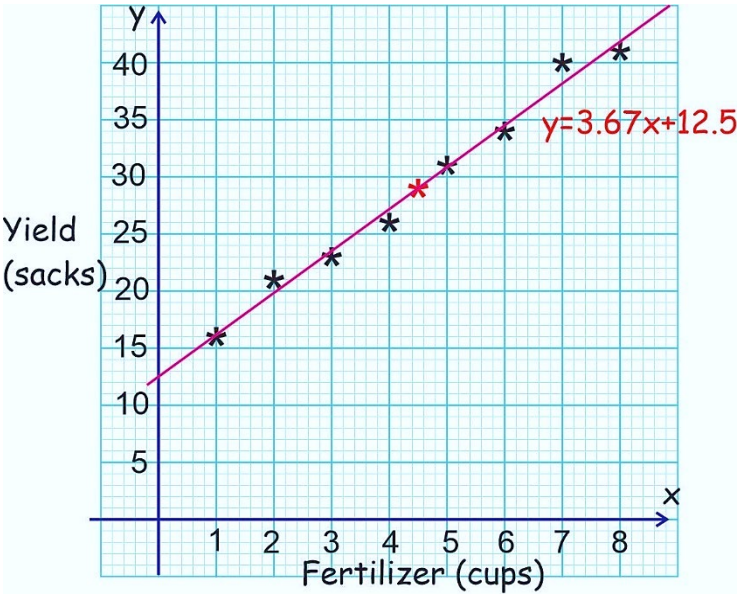


Figure 7.9: Scatter plot and graph of $y = 3.6x + 12.5$

Step 3 – Interpretation

a. The gradient of the regression line is $3\frac{2}{3} = 3.67$

   This means that, on average, each additional cup of fertiliser is associated with a yield increase of about 3.67 bags. In other words, increasing fertiliser by 1 cup raises yield by approximately 3.67 bags.

b. The y-intercept of the regression line is 12.50

   This gives the model's predicted yield when no fertiliser is applied (i.e., when $x = 0$). This means that the model predicts about 12.5 bags when no cups of fertiliser are applied. (In practice, this is an extrapolated value and should be interpreted cautiously — it may reflect a baseline yield from other factors (soil, rainfall), but the experiment did not include $x = 0$, so the intercept is mainly a model parameter rather than a direct measured value.)

c. Correlation: There is a clear positive correlation between cups of fertiliser applied and crop yield. As fertiliser increases, yield tends to increase.

Step 4 – Prediction

a. Using the regression equation ($y = 3.67x + 12.50$), the predicted crop yield for 7 cups of fertiliser is: $y = 3.67(7) + 12.50 = 25.69 + 12.50 = 38.19$

   The predicted yield is 38.19 bags.

   This prediction seems reasonable, as it is very close to the actual observed yield of 40 bags for 7 cups of fertiliser, as seen in the scatter plot.

b. Using the regression line, when 18 cups of fertiliser are applied, the yield will be:

$y = 3.6667(18) + 12.5 \approx 78.5$ sacks. The prediction is NOT reasonable because of the following:

i. The original data cover fertiliser amounts from 1 to 8 cups. Predicting at x=18 is far outside that range, so the linear trend may not continue — yields could level off, fall, or behave nonlinearly beyond the observed range.

ii. Crop yield typically cannot increase linearly without bound. Too much fertiliser may damage plants or give diminishing returns.

Activity 7.5: Relationship Between Price and Mileage of Used Cars

Woking in pairs, or individually, carry out the following activity.

Objective

Apply regression techniques to analyse the price and mileage of used cars, find the equation of the line of best fit, and use it for predictions.

Background

In the automobile market, the value of a used car is influenced by several factors such as brand, age, fuel efficiency, and mileage. One common belief is that as the mileage of a car increases, its price decreases.

Investigate whether this relationship holds using the secondary data below.

Table 7.12: Mileage and Price

Mileage (in 1,000km)	Price (in 1,000.00)
80	30
70	40
40	50
30	60
20	70
90	20
50	46
20	80
40	60
10	84

Procedure

Step 1 – Plot the Data

Draw a scatter plot with:

- Mileage on the x-axis
- Price on the y-axis

Step 2 – Perform Regression Analysis

Mean Mileage $= \frac{(8+7+3+4+2+9+5+2+4+1)}{10} = 450 \div 10 = 45$

Mean Price

$= (30 + 40 + 50 + 60 + 70 + 20 + 46 + 80 + 60 + 84) \div 10 = 540 \div 10 = 54$

$$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$(x_i - \bar{x}) = (x_i - 45)$ and $(y_i - \bar{y}) = (y_i - 54)$

Table 7.13: Least Squares Regression Computation.

Mileage (in 1,000km $)x$	**Price (in GH₵ 1,000)** y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
80	30	35	−24	−840	1225
70	40	25	−14	−350	625
40	50	−5	−4	20	25
30	60	−15	6	−90	225
20	70	−25	16	−400	625
90	20	45	−34	−1530	2025
50	46	5	−8	−40	25
20	80	−25	26	−650	625
40	60	−5	6	−30	25
10	84	−35	30	−1050	1225
				$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = -4960$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 6650$

(i) $m = -\frac{4960}{6650} = -0.745865 \approx -0.746$

Gradient of the line is -0.746. This means that the price decreases by about Ghc 0.75 or 75 pesewas for each 1km driven.

ii) Since $(\bar{x}, \bar{y}) = (45, 54)$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$\therefore 54 = (-0.746 \times 45) + c$

$c = 87.564$

The y-intercept is GHC87,564

b) Therefore, the regression line is $y = -0.746x + 87.564$

Figure 7.11 shows the graph of $y = -0.746x + 87.564$

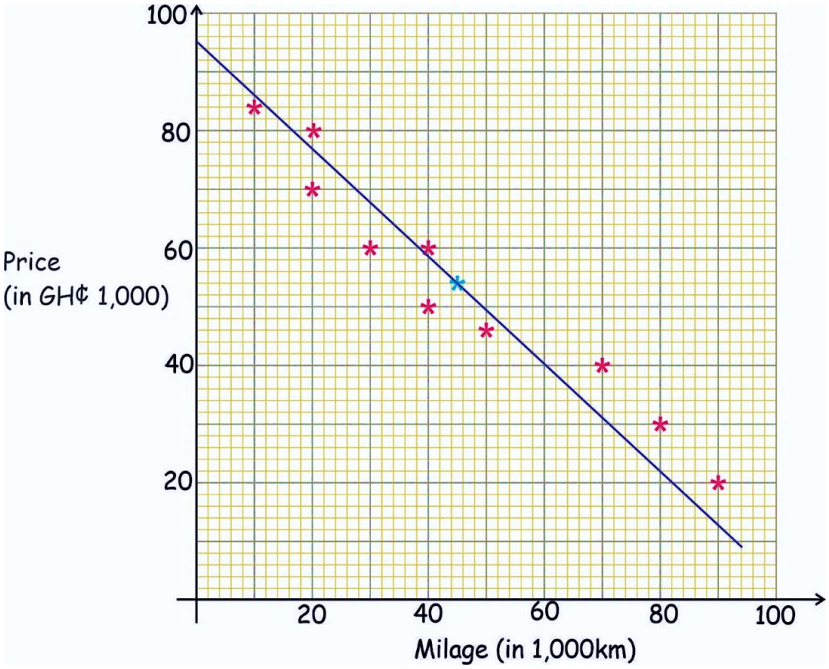


Figure 7.11: Scatter plot and graph of $y = -0.746x + 87.564$

Step 3 – Interpret Your Findings

- Gradient: –0.746 means that for every extra 1 km on the odometer, the price drops by about 75 pesewas.
- Intercept: When mileage is 0, the estimated price is GH₵87,564 (theoretical, since cars are never truly at zero mileage).
- The relationship is negative — the higher mileage the lower the price.

Step 4 – Correlation Coefficient

Table 7.14: Correlation Coefficient Computation.

Mileage (in 1,000km)	**Price (in GHC1,000)**	R_m	R_p	$d = R_m - R_p$	d^2
80	30	2	9	$2 - 9 = -7$	$(-7)^2 = 49$
70	40	3	8	$3 - 8 = -5$	$(-5)^2 = 25$
40	50	5.5	6	$5.5 - 6 = -0.5$	$(-0.5)^2 = 0.25$
30	60	7	4.5	$7 - 4.5 = 2.5$	$(2.5)^2 = 6.25$
20	70	8.5	3	$8.5 - 3 = 5.5$	$5.5^2 = 30.25$
90	20	1	10	$1 - 10 = -9$	$(-9)^2 = 81$

50	46	4	7	$4-7 \quad =-3$	$(-3)^2=9$
20	80	8.5	2	$8.5-2 \quad =6.5$	$6.5^2=42.25$
40	60	5.5	4.5	$5.5-4.5=1$	$1^2=1\times1=1$
10	84	10	1	$10-1=9$	$9^2=81$
					$\sum d^2=325$

$$r_x = 1-\frac{6\sum d^2}{n(n^2-1)}$$

$$r_x = 1-\frac{6\times 325}{10(10^2-1)}$$

$$r_x = 1-\frac{1956}{10(99)}$$

$$r_x = 1-1.976$$

$$r_x = -0.975$$

A correlation coefficient of -0.975 shows a strong negative correlation between mileage and price, confirming that as the mileage of the car increases, the price decreases.

Step 5 – Prediction

When the mileage is 60 000km, using the model, the price

$= -0.746(60\,000) + 87564 = \text{GH₵}42{,}804$

The prediction is reasonable since it is within the range of the observed values.

REVIEW QUESTIONS 7

The first four questions are multiple choice. Choose the correct option.

1. Which of the following is an example of primary data?

A. Data from last year's school records

B. Information from an online encyclopaedia

C. Measurements you take during a science experiment

D. Statistics from a published government report

2. Which data collection tool is most suitable for finding out the opinions of many students in a short time?

A. Interview

B. Observation

C. Questionnaire

D. Weighing scale

3. In data collection, ethical consideration means:

A. Asking only your friends to respond

B. Getting consent and protecting participants' privacy

C. Making sure data supports your opinion

D. Only collecting data you like

4. A thermometer used to record daily temperatures is an example of:

A. Interview

B. Measuring instrument

C. Observation checklist

D. Questionnaire

5. State a difference between primary data and secondary data.

6. List three tools you can use to collect primary data.

7. Why is it important to define the purpose of your research before collecting data?

8. You have been assigned to find out how many hours students in your class spend on the internet each week.

a. State whether you would use primary or secondary data. Give a reason.

b. Name two appropriate tools you could use for data collection.

c. Explain one advantage of using each tool.

9. A science club wants to know the effect of different amounts of sunlight on bean plant growth.

   a. Suggest a suitable data collection tool.

   b. Describe how you would use this tool in the experiment.

10. Explain the difference between correlation and regression.

11. Give two examples each of positive correlation and negative correlation.

12. A company finds that monthly sales (y) can be predicted from advertising cost (x) using the equation $y = 1000 + 15x$.

   a. Interpret the values 1000 and 15.

   b. Predict sales when GH₵200.00 is spent on advertising.

13. Mention three real-life situations where regression analysis would be useful.

14. Define the term "line of best fit" and explain its importance.

15. Given the data below, find the equation of the line of best fit and use it to predict y when $x = 8$.

Table 7.15: Values of two related variables, x and y.

x	2	3	4	6	8
y	5	7	9	13	17

16. State three real-life situations where the line of best fit can be applied.

17. The data below gives information about the price and age of 10 used cars.

Table 7.16: Age and Price of cars.

Age (years)	Price (in GH₵1 000.00)
10	20
9	40
6	50
5	60
4	70
11	20
7	46
4	80
6	60
4	100

Tasks:

a. Construct a scatter plot using the data in Table 7.16.

b. Describe the trend shown by the scatter plot. What does it suggest about the relationship between age and the price of cars?

c. Fit a linear model to the data using the method of least squares. Express the equation in the form: $y = mx + c$. Using the equation you obtained, draw the line of best fit on the scatter plot.

d. Prediction and Reliability: If the margin of error for predicting used car prices is ± 1400, use your model to estimate the price of a 7-year-old car.

18. The table shows values of two related variables.

Table 7.17: Values of two related variables, x and y.

x	11	12	13	14	15	16	17	18	19	15
y	90	100	60	40	70	10	20	10	10	40

Tasks:

a. Construct a scatter plot for the data.

b. Describe the trend shown by the scatter plot.
What does it suggest about the relationship between x and y values?

c. Fit a linear model to the data using the method of least squares. Express the equation in the form: $y = mx + c$. Using the equation you obtained, draw the line of best fit on the scatter plot. Interpret the gradient of the line of best fit and the y-intercept in the context of your data.

d. Calculate r to measure the strength and direction of the relationship. Interpret whether the relationship is weak, moderate, or strong.

e. If the margin of error is ± 3, use your model to estimate the value of y when $x = 14$. Will the model be reliable for this prediction?

19. The diagram shows a scatter plot of the drop height and bounce height of a tennis ball.

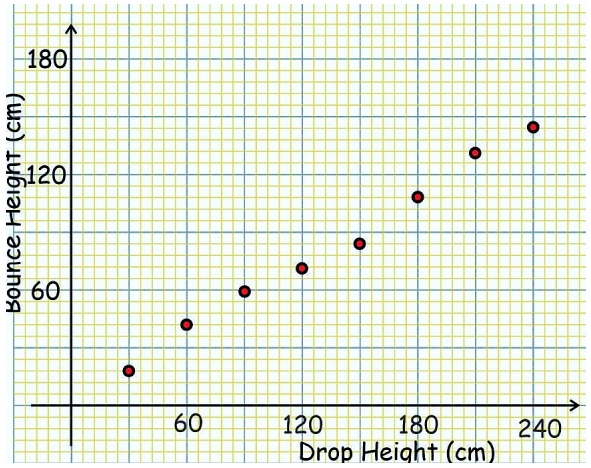


Figure 7.12: Scatter plot of Drop Height versus Bounce Height

Step 1: Observation

a. Examine the scatter plot provided and describe what it suggests about the relationship between drop height and bounce height.

b. Indicate whether the relationship is positive, negative, or shows no correlation.

c. Copy and complete the table below

Table 7.18: Drop height and Bounce height.

Drop height (x)	30	60	90	120	150	180	210	240
Bounce height (y)	18			72	84		132	144

Step 2: Fitting the model

a. Using the least squares method, calculate the equation of the line of best fit for the given data.

b. Express the equation in the form: $y = mx + c$ where m is the gradient and c is the intercept on the y axis.

c. Interpret the gradient of the line of best fit.

Step 3: Drawing the Line of Best Fit

a. Make a copy of the scatter plot in your graph book. Using the equation you obtained, draw the line of best fit on the diagram.

b. Ensure the line passes through the point $(\bar{x}, \bar{y})$

Step 4: Prediction

Using your linear model,

a. Estimate the bounce height when the drop height is $198cm$.

b. Estimate the drop height if the bounce height is $90cm$.

20. A group of students experimented to determine how the number of hours spent studying affects performance in a Physics test.

Ten students recorded the number of hours they studied during the week before the test and their corresponding scores out of 30 marks. The data below shows the number of hours studied (x) and the test scores (y):

Table 7.19: Hours Studied and Test Scores

Hours Studied (x)	**2**	**5**	**8**	**3**	**7**	**6**	**1**	**4**	**9**	**5**
Test *Score* (y)	17	21	27	20	24	25	15	22	27	22

Tasks:

a. Construct a scatter plot using the data above with the x-axis as hours studied and the y-axis as test scores.

b. Describe the trend shown by the scatter plot. What does it suggest about the relationship between hours of study and test scores?

c. Fit a linear model to the data using the method of least squares. Express the equation in the form: $y = mx + c$.

Using the equation you obtained, draw the line of best fit on the scatter plot. Interpret the gradient of the line of best fit and the y intercept in the context of your data.

d. Calculate r to measure the strength and direction of the relationship. Interpret whether the relationship is weak, moderate, or strong.

e. If the margin of error for predicting test scores is ± 2 marks, use your model to estimate the score of a student who studied for 6 hours. Would the model be reliable for this prediction?

21. Screen Time and Sleep

An online health article reported data collected from teens aged 15–18 on average daily screen time and hours of sleep.

Table 7.20: Screen time and sleep time

Screen time(hrs.)	**3**	**6**	**8**	**5**	**1**	**2**	**1**	**6**	**4**	**7**
Sleep time (hrs.)	7.5	6.5	5.2	6.7	8.2	8	8.5	6.1	7	5.7

Tasks:

a. Construct a scatter plot using the data.

b. Describe the trend shown by the scatter plot.
What does it suggest about the relationship between screen time and sleep time?

c. Fit a linear model to the data using the method of least squares. Express the equation in the form: $y = mx + c$.
Using the equation you obtained, draw the line of best fit on the scatter plot. Interpret the gradient of the line of best fit and the y-intercept in the context of your data.

d. Calculate r to measure the strength and direction of the relationship. Interpret whether the relationship is weak, moderate, or strong.

e. If the margin of error for predicting sleep time is ± 0.5 hours, use your model to estimate the sleep time of a student whose screen time is 3 hours. Would the model be reliable for this prediction? Justify your answer.

22. Working in small groups carry out the following project.

Project: Finding the relationship between two variables using a scatter plot

Project Goal: To create and interpret scatter plots to analyse the strength and nature of the relationship between two quantitative variables.

Question

Select a real-world topic involving two variables, gather or find data for these related variables, create a scatter plot, and then analyse the correlation (or absence of one) between them. Present your findings and discuss the implications of your observations.

The materials you will need include:

- Graph paper or graphing software (e.g., GeoGebra, Microsoft Excel, Google Sheets)
- Rulers
- Calculators (optional)
- Access to the internet for data collection (optional)
- Presentation tools (e.g., poster board, Google Slides, PowerPoint)

Project Steps:

Phase 1: Brainstorming and Data Selection

a. Introduction to Scatter Plots: In your groups, review what scatter plots are, their purpose, and the concept of correlation (positive, negative, no correlation).

b. Brainstorming Variables: Brainstorm pairs of quantitative variables that might have a relationship and are easily assessable.
You may choose one of the examples listed below:

- Hours studied and test scores in a particular subject.
- Daily temperature and water sales.
- Height of a student and his/her shoe size
- Population of regional capitals in Ghana and the number of JHS.
- Distance from school and travel time
- Mid-term and end-of-term scores.
- Weight of a student and his/her height.
- Income and expenditure.

c. Topic Selection: Choose ONE pair of variables you are interested in investigating.

- Define the Research Question

  Example: “Is there a relationship between the number of hours students’ study in a week and their last mathematics test scores?”

- Clearly define your independent (x-axis) and dependent (y-axis) variables.

d. Select an Appropriate Data Collection Tool:

You can use:

- Questionnaire. Design a simple questionnaire that will solicit their study hours and test scores in mathematics for a particular term or semester.
- Interview. Have face-to-face interactions with learners to collect the data.
- Class record. You may ask permission from authorities to obtain records of students.

e. Prepare the Data Collection Tool:

- For a questionnaire, write clear, concise questions
- For an interview, prepare a short list of guiding questions.

f. Data Collection Strategy:
   - Collect at least 20 data points for your chosen variables.
   - Option A: Self-Collected Data: Collect your own data
   - Option B: Found Data: You can research data online from reliable sources (e.g., government statistics websites, sports statistics, scientific studies, reputable news organisations).
     You must cite your sources.
   - Option C: Teacher-Provided Data: You may ask a teacher or a parent to provide you with data.

Phase 2: Creating the Scatter Plot

a. Organise your collected data in a table with two columns (one for each variable).

b. Decide whether to create the scatter plot by hand on graph paper or using technology.
   - By Hand: Provide the scale used for each axis, label the axes, and give the graph a clear title.
   - Using Technology: Use an Excel Spreadsheet or any appropriate software but still ensure everything is correctly labelled.

Phase 3: Analysis and Interpretation

a. Analyse your generated scatter plot.

b. Use your scatter plot to answer the following questions
   - What is your independent variable (x-axis)? What are its units?
   - What is your dependent variable (y-axis)? What are its units?
   - Describe the overall pattern you observe in the scatter plot.
   - Does there appear to be a correlation between your two variables?
     If so, what type (positive, negative, or no correlation)?
   - Calculate the spearman's rank correlation coefficient between the variables and interpret your answer.
   - What conclusions can you draw about the relationship between your two variables based on your scatter plot?
   - Obtain the least square regression line for the data. Interpret the gradient and the y-intercept in the context of your data.
   - Calculate the sum of squared error.
   - Does correlation imply causation?
     Explain this concept in the context of your data.

Phase 4: Presentation

a. Prepare a presentation (e.g., poster, slide show, written report) that includes:
   - Project Title
   - Names of group members/learners
   - A brief introduction of the chosen variables and the research question.
   - A displayed data table.
   - Your scatter plot (large and clear).

- Answers to all analysis questions from Phase 3.
- A concluding summary of your findings.
- State the sources you used for the data and sources of your data, research materials (if any).
- Provide a brief statement acknowledging any individual or institution other than your group members and your teacher who contributed to the successful completion of your project. (Optional)
  Example: We would like to express our sincere gratitude to [Name of Individual or institution], whose insights and support were invaluable to the successful completion of this project.

b. Presentation: You will present your project to your class and answer any questions your peers ask.

Assessment and Scoring:

You will be assessed on:

- **Topic Selection and Relevance**: It should be clear, realistic and relevant
- **Data Collection/Sourcing:** Appropriateness of data, sufficient number of points, proper citation (if applicable).
- **Data Representation**: Is the scatter plot accurate, with clear labels and accurate point plotting
- **Data Analysis and Interpretation:** Thoroughness and accuracy of answers to analysis questions and understanding of correlation.
- **Conclusions and Implications:** Ensure your findings are well explained with well-considered implications and/or solutions
- **Presentation:** Clarity, organisation, engagement, ability to explain your findings.

SECTION

8

PROBABILITY



HANDLING DATA

Making Prediction

INTRODUCTION

Probability and counting are fundamental concepts that play a crucial role in solving real-world problems. Understanding these concepts will equip us with the tools to analyse situations involving uncertainty, make predictions, and make informed decisions based on data. We will explore several key topics, including conditional probability, equally likely events, permutations and combinations and the binomial distribution. Understanding these concepts is not just theoretical; they have practical applications in everyday life, including:

1. School Leadership Selection: When choosing a student council, schools can use combinations to determine how many different groups can be formed from a larger pool of candidates.
2. Business Planning: Companies can apply binomial probability to assess risks and make informed decisions about investments or product launches.
3. Industrial Processes: In manufacturing, understanding permutations can help in optimising the arrangement of products on assembly lines.

KEY IDEAS

- **Binomial probability** deals with experiments that have two possible outcomes, such as success and failure.
- **Combinations** refer to the selection of objects where the order does not matter.
- **Conditional probability** is the probability of an event occurring given that another event has already occurred.
- **Events** are considered equally likely if they have the same chance of occurring.
- **Permutations** refer to the different ways to arrange a set of objects where the order matters. For instance, arranging books on a shelf involves permutations.

EXPLAINING THE CONCEPTS THAT GIVE RISE TO CONDITIONAL PROBABILITIES

Activity 8.1 Probability Concepts Activity

This activity is designed to help us engage with key probability concepts. We will work individually or in small groups to apply what we have learned about experiments, the multiplication rule, mutually exclusive events, the law of total probability, and dependent events.

Materials Needed

- Coins (at least 2 per group)
- Dice (1 per group)
- A bag containing coloured balls (e.g., 2 red, 3 blue)
- Paper and pens/pencils for calculations

Part 1: Experiments and Outcomes

Task: Conduct a simple experiment and observe the outcomes.

1. Coin Flip Experiment
    - Flip a coin 3 times. Record the outcomes (heads or tails).
    - Calculate the experimental probability of getting all heads.
    - Discuss how this relates to the theoretical probability of $\frac{1}{8}$.
2. Dice Roll Experiment
    - Roll a die. Record the outcomes.
    - Calculate the experimental probability of rolling a 3.
    - Compare your results with the theoretical probability of $\frac{1}{6}$.

Part 2: Multiplication Rule of Probability

Task: Solve problems using the multiplication rule.

3. Dependent Events
    - Using the bag of coloured balls, randomly draw one ball without replacement, record its colour, then draw a second ball.
    - Calculate the probability of drawing a red ball first and then a blue ball second.
    - Use the formula: $P(A \cap B) = P(A) \cdot P(B \mid A)$
4. Independent Events
    - Flip a coin and roll a die at the same time. Record the outcomes.

- Calculate the probability of getting heads on the coin and a 4 on the die.
- Use the formula: $P(A \cap B) = P(A) \times P(B)$

Part 3: Mutually Exclusive Events

Task: Identify mutually exclusive events (two events that cannot occur at the same time).

Group Discussion

- Think of examples of mutually exclusive events in daily life (e.g., weather conditions: sunny vs. rainy).
- Create a list of at least 3 pairs of mutually exclusive events and explain why they cannot occur simultaneously.

Part 4: Law of Total Probability

Task: Apply the law of total probability.

1. Fruit Basket Scenario

- Imagine you have a basket with 3 apples, 2 bananas, and 5 oranges. Calculate the probability of picking a banana using the law of total probability.
- Use the formula:
  $P(B) = P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A2) + P(B \mid A_3) \cdot P(A_3)$
- Discuss how partitioning the sample space helps in calculating probabilities.

Part 5: Dependent Events (Sampling Without Replacement)

Task: Explore dependent events through selection.

Group Leader and Secretary Selection

- From a group of 5 learners, randomly select a group leader and then a secretary.
- Discuss how the selection of the secretary is dependent on who was chosen as the leader.
- Calculate the probability of selecting a specific learner as the leader and a different specific learner as the secretary.

How conditional probabilities arise

In probability theory, we often encounter situations involving multiple events, particularly when we need to determine the likelihood of an event occurring based on the occurrence of another event. This is where the concept of conditional probability comes into play. Conditional probability measures the probability of an event happening given that we know another event has already occurred.

For example, if we want to find the probability of drawing a red card from a deck of cards after knowing that the first card drawn was a heart, we recognise that the first event

influences the outcome of the second. Understanding this relationship helps us make more informed predictions and decisions based on prior knowledge.

To illustrate, consider a scenario where a student takes a mathematics test and a science test. If we know that the student scored well on the mathematics test, we might want to determine the probability that they will also score well on the science test. This situation exemplifies conditional probability because the outcome of the mathematics test can provide insight into the performance on the science test. By analysing how these events are related, we gain a deeper understanding of the underlying patterns and dependencies in various situations, which is crucial for making accurate assessments in real-life scenarios.

Example 8.1

A box contains 12 identical balls, 5 of which are black and the rest are green. Two balls are drawn at random from the bag without replacement. What is the probability of getting one black ball and one green ball?

Solution

To find the probability of drawing one black ball and one green ball from a box containing 12 identical balls (5 black and 7 green), we can follow these steps:

Step 1: Determine the Total Number of Balls

Total Balls: 12

Black Balls: 5

Green Balls: $12 - 5 = 7$

Step 2: Calculate the Total Ways to Draw Two Balls

The total number of ways to choose 2 balls from 12 is given by the combination formula:

Total Ways = ${}^{12}C_2 = 66$

Step 3: Calculate the Ways to Draw One Black and One Green Ball

To find the number of ways to draw one black ball and one green ball, we can choose:

1. One Black Ball: The number of ways to choose 1 black ball from 5 = ${}^{5}C_1 = 5$
2. One Green Ball: The number of ways to choose 1 green ball from 7 = ${}^{7}C_1 = 7$

Step 4: Calculate the Total Ways to Draw One Black and One Green

The total ways to draw one black ball and one green ball is the product of the two combinations:

Ways to choose 1 black and 1 green = $5 \times 7 = 35$

Step 5: Calculate the Probability

Now, the probability of drawing one black ball and one green ball is given by the ratio of the favourable outcomes to the total outcomes:

$$\text{P(1 black and 1 green)} = \frac{\text{Ways to choose 1 black and 1 green}}{\text{Total Ways}} = \frac{35}{66}$$

Example 8.2

A bag contains 8 counters: 2 pink, 1 green, and 5 yellow. A counter is drawn at random from the bag and not replaced. A second counter is then drawn.

Find the probability that both counters drawn are yellow.

Solution

To find the probability that both counters drawn are yellow, we can follow these steps:

1. Total Ways to Choose 2 Counters from 8: Total Ways = ${}^{8}C_{2} = 28$
2. Ways to Choose 2 Yellow Counters:

   Since there are 5 yellow counters, the number of ways to choose 2 yellow counters is ${}^{5}C_{2} = 10$

3. Probability of Both Counters Being Yellow

$$\text{P(Both Yellow)} = \frac{\text{Ways to choose 2 yellow}}{\text{Total Ways}} = \frac{{}^{5}C_{2}}{{}^{8}C_{2}}$$

$$\text{P(Both Yellow)} = \frac{10}{28} = \frac{5}{14}$$

Example 8.3

Two marbles are drawn randomly from a bag containing 5 red, 7 white, and 8 blue marbles, one after the other, without replacement.

Find the probability that:

a) both are red

b) one is white and the other is blue

Solution

Red marbles = 5 White marbles = 7 Blue marbles = 8

Total number of marbles = 5 + 7 + 8 = 20

Total ways of selecting 2 marbles out of 20 $= {}^{20}C_{2} = 190$

a) Ways of selecting two red marbles from 5 marbles is ${}^{5}C_{2} = 10$

$$\text{P (both red balls)} = \frac{{}^{5}C_{2}}{{}^{20}C_{2}} = \frac{10}{190} = \frac{1}{19}$$

b) Ways of selecting 1 white marble from 7 is ${}^{7}C_{1} = 7$

Ways of selecting 1blue marble from 8 marbles is ${}^{8}C_{1} = 8$

$$\text{P (1 white and 1 blue marbles)} = \frac{{}^{7}C_{1} \times {}^{8}C_{1}}{{}^{20}C_{2}} = \frac{7 \times 8}{190} = \frac{28}{95}$$

DESCRIBING THE PROBABILITY OF TWO EQUALLY LIKELY EVENTS OCCURRING FROM GIVEN EXPERIMENTS

Conditional probability $P(B|A)$ is the probability of event B happening given the information that A has already taken place.

$P(B|A) = \frac{P(B \cap A)}{P(A)}, \ P(A) > 0.$

What will happen to the value of $P(B|A)$ if the two events are mutually exclusive?

In this special case, $P(B|A) = 0$.

Example 8.4

A fair die is rolled once.

Let A be the event that the outcome is 2.

Let B be the event that the outcome is even.

Find $P(A|B)$

Solution

$S = \{1,2,3,4,5,6\} A = \{2\}, \quad B = \{2,4,6\}, \quad (A \cap B) = \{2\} P(A) = \frac{1}{6},$

$P(B) = \frac{1}{2}, P(A \cap B) = \frac{1}{6}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A|B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

Example 8.5

A box has 3 red marbles and 2 black ones. 2 marbles are randomly selected one after the other without replacement.

What is the probability that the second marble is red, given that the first one is red?

Solution

Let R_2 be the event that the second marble is red and R_1 the event that the first one is red.

Total marbles, $3 + 2 = 5$

$P(R_2) = \frac{3}{5}$

$P((R_1 \cap R_2)) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

$P(R_1|R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)}$

$$P(R_1|R_2) = \frac{\frac{3}{10}}{\frac{3}{5}} = \frac{1}{2}$$

Alternatively:

Selecting 2 from 5, $5_{C_2} = 10$

Selecting 2 red marbles from 3 red marbles, $3_{C_2} = 3$

Selecting 0 marbles from 2 black marbles, $2_{C_0} = 1$

P (selecting two red) $= \frac{3_{C_2} \times 2_{C_0}}{5_{C_2}} = \frac{3 \times 1}{10} = \frac{3}{10}$

$$P(R_2) = \frac{3}{5}$$

$$P(R_1|R_2) = \frac{\frac{3}{10}}{\frac{3}{5}} = \frac{1}{2}$$

Example 8.6

In a survey among high school learners, 60% use Facebook, 40% use WhatsApp and 20% use both.

A person is chosen at random and it is given that they already use WhatsApp, find the probability that they also use Facebook.

Solution

Let F and W represent those who use Facebook and WhatsApp

$$P(F) = \frac{60}{100} P(F) = 0.6$$

$$P(W) = \frac{40}{100} P(F) = 0.4$$

$$P(F \cap W) = \frac{20}{100}$$

$$P(F \cap W) = 0.2$$

$$P(F|W) = \frac{0.2}{0.4}$$

$$P(F|W) = 0.5$$

MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING CONDITIONAL PROBABILITIES

Conditional probabilities can be applied to solve real-life problems. We will now work through the following examples to understand how conditional probabilities can be applied to real life.

We know that:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \quad P(A) > 0.$$

Example 8.7

In a school with 50 students, 11 females study French while 12 females study Arabic. In addition, 7 males study French and 20 males study Arabic.

a) Organise the information in tabular form.

b) If a student is selected at random, what is the probability that they are

: i) studying Arabic, given that they are female.
ii) male.

Solution

a)

	French (Fr)	Arabic (A)	Total
Female (Fe)	11	12	23
Male (M)	7	20	27
Total	18	22	50

b) i) $P(A|Fe) = \frac{12}{23}$

ii) $P(M) = \frac{27}{50}$

Example 8.8

The probability that a student is absent on a Friday is 0.06. The probability that today is Friday is 0.2.

Find the probability that a student is absent given that today is Friday.

Solution

$$P(absent|Friday) = \frac{P(\text{absent} \cap \text{Friday})}{P(Friday)}$$

$$P(absent|Friday) = \frac{0.06}{0.2}$$

$$P(absent|Friday) = 0.30$$

Example 8.9

In a community, 50% of families own a car. Of those families, 90% have a garage. Among families that do not own a car, only 20% have a garage.

If a family is selected at random and found to have a garage, what is the probability that they own a car?

Solution

Let:

C: Event that a family owns a car

C′: Event that a family does not own a car

G: Event that a family has a garage.

Now, $P(C|G) = \frac{P(C \cap G)}{P(G)}$

Also, $P(C) = 0.50, P(C') = 1 - 0.50 = 0.50$

$P(G|C) = 0.90\ P(G|C') = 0.20$

We determine the probability that a family owns a car and has a garage

$$P(C \cap G) = P(C) \times P(G|C)$$

$$= 0.50 \times 0.90$$

$= 0.45$For the probability that a family does not own a car and has a garage, $P(C' \cap G)P(C' \cap G) = P(C') \times P(G|C')$

$$= 0.50 \times 0.20$$

$= 0.10$The probability that a family owns a garage, P(G)

$$P(G) = P(C \cap G) + P(C' \cap G)$$

$$= 0.45 + 0.10$$

$$= 0.55$$

Now, we calculate the probability that a family owns a car given that the family has a garage.

$$P(G|C) = \frac{P(C \cap G)}{P(G)}$$

$$= \frac{0.45}{0.55}$$

$= 0.8182$ Thus, the probability that a family owns a car given that it has a garage is 0.83 to 2 d. p.

Example 8.10

In a class of 70 students, 25 use tablets for their studies. Among the tablet users, 12 are on MTN. Of the students who do not use tablets, 7 are on the MTN network.

If a student is selected at random and is found to be on MTN, what is the probability that this student uses a tablet for his/her studies?

Solution

	Use Tablet	Does not use a tablet	Total
Uses MTN	12	7	19
Does not use MTN	13	38	51
Total	**25**	**45**	**70**

Let T = students who use tablets for their studies.
and M = students on MTN.

This means that:

$$P(T) = \frac{25}{70} = \frac{5}{14}$$

$$P(M) = \frac{19}{70}$$

Probability of selecting a student who is on MTN, given that the student uses a tablet

$$P(M|T) = \frac{12}{25}$$

$$P(T|M) = \frac{P(M|T) \times P(T)}{P(M)}$$

$$P(T|M) = \frac{\left(\frac{12}{25}\right) \times \left(\frac{5}{14}\right)}{\frac{19}{70}}$$

$$P(T|M) = \frac{12}{19}$$

Therefore, the probability that a student uses a tablet given that the student is on MTN is $\frac{12}{19}$

MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING BINOMIAL PROBABILITIES

In many real-life situations, we encounter events that can have only two possible outcomes. These outcomes can be categorised as success or failure. The study of these scenarios falls under the umbrella of binomial probability.

Examples of Binomial Events

1. Winning an Election: A candidate can either win (success) or lose (failure) an election.
2. Gender of a new-born child: When a baby is born, it can be either male or female. one gender can be described as “success” and the other, “failure”.

3. Conducting a Malaria Test: A malaria test can yield a positive result (success) or a negative result (failure).

The probability of success (event of interest) is usually denoted by the letter p and the probability of failure by the letter q.

Recall, $P(A) + P(A') = 1$

$$P(A) = 1 - P(A')$$

This means that $p + q = 1$ and

$$q = 1 - p$$

To find the probability of obtaining exactly x successes in n trials, we use the **binomial probability formula**: $\mathrm{P}(X = x) = {}^{n}C_{x}p^{x}q^{n-x}$

Understanding binomial probability is crucial for analysing events with two possible outcomes. It helps us make informed predictions and decisions in various fields, such as healthcare, finance, and social sciences.

Example 8.11

A farmer observed that the probability of a coconut seed germinating is 0.40. She plants 12 coconut seeds. What is the probability that exactly 5 of them will germinate?

Solution

Let P(a seed germinating) $= p$

Let P(a seed not germinating) $= q$

$p = 0.4$

$q = 1 - 0.4 = 0.6$

N = total number seeds planted= 12

x = number of germinated seeds (success)=5

$\mathrm{P}(X = x) = {}^{n}C_{x}p^{x}q^{n-x}$

$P(X = 5) = {}^{12}C_{5}(0.4)^{5}(0.6)^{12-5}$

$P(X = 5) = {}^{12}C_{5}(0.4)^{5}(0.6)^{7}$

$P(X = 5) = 0.227$

There is about a 22.7% chance that exactly 5 out of the 12 coconut seeds will germinate.

Example 8.12

A factory produces bulbs, and the probability that a bulb is defective is 0.35.

A quality inspector randomly checks 15 bulbs. Find the probability that:

a) None of the bulbs is defective.

b) Exactly 7 bulbs are defective.

c) At least 10 bulbs are defective.

d) More than 4 bulbs are defective.

Solution

Let P(a bulb is defective)= $p = 0.35$

Let P(a bulb is not defective) = $q = 1 - 0.35 = 0.65$

n = total number of bulbs inspected=15

x = the number of defective bulbs among those inspected.

a)$x = 0$ (none of the inspected bulbs were defective)

$P(X = x) = {}^{n}C_{x}p^{x}q^{n-x}$

$P(X = 0) = {}^{15}C_{0}(0.35)^{0}(0.65)^{15-0}$

$P(X = 0) = {}^{15}C_{0}(0.35)^{0}(0.65)^{15}$

$P(X = 0) = 0.00152069$

b) Exactly 7 bulbs are defective.

Let $x = 7$

$P(X = x) = {}^{n}C_{x}p^{x}q^{n-x}$

$P(X = 7) = {}^{15}C_{7}(0.35)^{7}(0.65)^{15-7}$

$P(X = 7) = {}^{15}C_{7}(0.35)^{7}(0.65)^{8}$

$P(X = 7) = 0.0525$

c) At least 10 bulbs are defective

P(At least 10 bulbs are defective) = $P(x \geq 10)$

$P(x \geq 10) = P(x = 10) + P(x = 11) + P(x = 12) + P(x = 13) + P(x = 14) + P(x = 15)$

We need to calculate the probability of each and sum them.

$P(X = 10) = {}^{15}C_{10}(0.35)^{10}(0.65)^{15-10}$ $= 0.00961175$
$P(X = 11) = {}^{15}C_{11}(0.35)^{11}(0.65)^{15-11}$ $= 0.00235253$
$P(X = 12) = {}^{15}C_{12}(0.35)^{12}(0.65)^{15-12}$ $= 0.00042225$

$P(X = 13) = {}^{15}C_{13}(0.35)^{13}(0.65)^{15-13}$ $= 0.00005247$
$P(X = 14) = {}^{15}C_{14}(0.35)^{14}(0.65)^{15-14}$ $= 0.000004036$
$P(X = 15) = {}^{15}C_{15}(0.35)^{15}(0.65)^{15-15}$ $= 0.000000148$

Finding the sum:

$0.00961175 + 0.00235253 + 0.00042225 + 0.00005247 + 0.000004036 + 0.000000148$

$P(x \geq 10) = 0.00124$

d) More than 4 bulbs are defective

P(More than 4 bulbs are defective) $= P(x > 4)$

$= P(x = 5) + P(x = 6) + \ldots + P(x = 15)$

This is the same as finding $1 - P(x \leq 4)$

$P(X = 0) = {}^{15}C_{0}(0.35)^{0}(0.65)^{15-0}$ $= 0.001562$
$P(X = 1) = {}^{15}C_{1}(0.35)^{1}(0.65)^{15-1}$ $= 0.01262$
$P(X = 2) = {}^{15}C_{2}(0.35)^{2}(0.65)^{15-2}$ $= 0.04755$
$P(X = 3) = {}^{15}C_{3}(0.35)^{3}(0.65)^{15-3}$ $= 0.11096$
$P(X = 4) = {}^{15}C_{4}(0.35)^{4}(0.65)^{15-4}$ $= 0.17923$

$0.001562 + 0.01262 + 0.04755 + 0.11096 + 0.17923 = 0.35192$

P(More than 4 bulbs are defective) $= 1 - 0.35192$

P(More than 4 bulbs are defective) $= 0.64808 \approx 0.65$ (2 decimal places)

Example 8.13

A fair coin is tossed 9 times. Find the probability of obtaining exactly 4 heads

Solution

Probability of success, tossing to get a head, $p = \frac{1}{2}$

$q = 1 - \frac{1}{2}$

$q = \frac{1}{2}$

a) exactly 4 heads

$x = 4$

$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}$

$P(X = 4) = {}^{9}C_{4} \left(\frac{1}{2}\right)^{4} \left(\frac{1}{2}\right)^{9-4}$

$P(X = 4) = {}^{9}C_{4} \left(\frac{1}{2}\right)^{4} \left(\frac{1}{2}\right)^{5}$

$P(X = 4) = \frac{63}{256}$

SOLVING REAL-LIFE PROBLEMS IN BINOMIAL PROBABILITY

Example 8.14

The probability that a customer orders a particular model of a new car in silver is 0.2. In the next 10 random orders, find the probability that

a) Exactly 2 cars are ordered in silver.

b) At most 3 cars are ordered in silver.

Solution

a) Exactly 2 cars are ordered in silver.

Let $x = 2$

$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}$

$P(X = 2) = {}^{10}C_{2} (0.2)^{2} (0.8)^{10-2}$

$P(X = 2) = {}^{10}C_{2} (0.2)^{2} (0.8)^{8}$

$P(X = 2) = 0.3020$

b) At most 3 cars are ordered in silver.

We can find $x = 0, 1, 2, 3$ and sum the results.

$P(X = 0) = {}^{10}C_0(0.2)^0(0.8)^{10-0}$ $= 0.107374$
$P(X = 1) = {}^{10}C_1(0.2)^1(0.8)^{10-1}$ $= 0.268435$
$P(X = 2) = {}^{10}C_2(0.2)^2(0.8)^{10-2}$ $= 0.301990$
$P(X = 3) = {}^{10}C_3(0.2)^3(0.8)^{10-3}$ $= 0.201327$

$P(x \leq 3) = 0.107374 + 0.268435 + 0.301990 + 0.201327$

$= 0.879126$

Thus, the probability that at most 3 cars are ordered in silver is 0.87912.

Example 8.15

The proportions of people with blood groups A, B, AB, and O in the village are in the ratio. $20: 15: 10: 35$, respectively.

Determine the probability that a random sample of 8 people from the village contains:

(a) Exactly 5 with blood group O

(b) at most 2 with blood group A

Solution

Total ratio $= 20 + 15 + 10 + 35 = 80$

Let $A =$ event that a randomly selected individual has blood group A

Let $B =$ event that a randomly selected individual has blood group B

Let $AB =$ event that a randomly selected individual has blood group AB

Let $O =$ event that a randomly selected individual has blood group O

This means that;

$P(A) = \frac{20}{80} = \frac{1}{4}$ $P(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4}$ $P(B) = \frac{15}{80} = \frac{3}{16}$ $P(\bar{B}) = 1 - \frac{3}{16} = \frac{13}{16}$

$P(AB) = \frac{10}{80} = \frac{1}{8}$ $P(\overline{AB}) = 1 - \frac{1}{8} = \frac{7}{8}$ $P(O) = \frac{35}{80} = \frac{7}{16}$ $P(\bar{O}) = 1 - \frac{7}{16} = \frac{9}{16}$

Let Sample $(n) = 8$ people

Let $x =$ number of people out of the sample of 8 with a particular blood group

(a) P(Exactly 5 with blood group O) $= {}^8C_5\left(\frac{7}{16}\right)^5\left(\frac{9}{16}\right)^3$

$= 56 \times 0.0160284 \times 0.17797852$

$= 0.15975$

(b) P(at most 2 with blood group A) $= P(x = 0) + P(x = 1) + P(x = 2)$

$P(x = 0) + P(x = 1) + P(x = 2) = {}^8C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^8 + {}^8C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^7 + {}^8C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6$

$= \frac{6561}{65536} + \frac{8 \times 2187}{65536} + \frac{28 \times 729}{65536}$

$= \frac{6561}{65536} + \frac{17496}{65536} + \frac{20412}{65536}$

$= \frac{44469}{65536}$

$= 0.6785$

MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING COMBINATIONS AND PERMUTATIONS

Fundamental Counting Principle

It states that:

If one event can occur in m ways and a second event can occur in n ways, then the two events together can occur in $m \times n$ ways.

This can be extended to more events:

If event 1 can occur in a ways, event 2 in b ways, and event 3 in c ways, then all three events together can occur in $a \times b \times c$ ways

Example 8.16

A student has 5 shirts and 3 shorts. How many ways can the student dress?



Solution

Number of ways to choose a shirt = 5

Number of ways to choose shorts = 3

Total number of outfits = $5 \times 3 = 15$

So, the student can dress in 15 different ways.

Example 8.17

A license plate is made up of 2 letters followed by 4 digits. How many number plates are possible?



Solution

There are 26 letters of the English alphabet. i.e., A, B, C, D, …., Z

Number of ways to choose 2 letters = 26×26

The decimal number system has 10 digits. i.e., 0, 1, 2, 3, …., 9.

Number of ways to choose 4 digits = $10 \times 10 \times 10 \times 10 = 10\,000$

Total = $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6\,760\,000$

So, there are 6 760 000 possible license plates.

Example 8.18

A password consists of 3 letters (A–Z) followed by 2 digits (0–9). How many passwords can be formed?



Solution

Letters: $26 \times 26 \times 26$

Digits: 10×10

Total = $17\,576 \times 100 = 1\,757\,600$

So, 1 757 600 different passwords can be formed.

Example 8.19

If a sports club decides to elect a chairperson and a secretary from 5 members, assuming the same person cannot hold both positions, find the number of ways this can be done.

Solution

Number of people to choose a chairperson from = 5

Number of people to choose a secretary from = 4

Number of ways to choose a chairperson then a secretary $5 \times 4 = 20$.

Therefore, there are 20 ways in which this can be done.

MODELLING AND SOLVING REAL-LIFE PROBLEMS USING COMBINATIONS

A combination is a selection of items from a set where the order of selection does not matter.

The combination formula is:

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

Where:

- n = total number of items
- r = number of items chosen

Situations Where Order of Arrangement is NOT Important (Combination)

1. Choosing a Committee
   Example: Selecting 2 prefects from a class of 30.
   The 2 students will be chosen regardless of the order.
2. Lottery Games
   Example: Selecting 4 winning numbers from 98.
3. Food Choices
   Example: Choosing 2 drinks out of 5 options in a shop.
   Sobolo and Asaana are the same as Asaana and Sobolo.
4. Subject Selection
   Example: If Naadu selects 4 elective subjects out of a total of 10 subjects in the Arts department, the order in which she selects the subjects is not important.
5. Team Selection
   Example: Choosing 11 players from 15 to form a football team.
   The lineup order doesn't matter at this stage (positions not yet assigned).
6. Clothing Choices
   Example: Picking 4 shirts from 10 in a wardrobe.
   Order of picking the shirts does not matter.
7. Test Question Selection
   Example: Kofi chooses 5 questions out of 8 in an exam.
   The order of chosen questions is irrelevant.

8. Meal Menu
   Example: Selecting a set of dishes for a party.
   Whether fufu is chosen before or after konkonte does not matter.
9. Selecting Students for a Prize
   Example: Picking any 2 students to receive awards.
   The order of calling their names is not important.
10. Choosing Books to Borrow
    Example: Borrowing 3 books from a library.
    It does not matter in what order the books are selected.

How to Answer a Combination Question

Step 1: Identify if it is a combination

Ask yourself: Does order matter?

If NO, it's a combination.

Look for words like: choose, select, pick, form, group, committee.

Step 2: Recall the formula

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

Where n = total number of items and r = number of items being chosen

Step 3: Substitute values

Put in the given values for n and r.

Step 4: Simplify factorials

Cancel out terms where possible to make calculations simpler.

Step 5: State the final answer

Write your result clearly, with units if needed (e.g., "ways", "groups", "teams").

Example 8.20

A committee of 3 people is to be selected from 7 teachers.

How many different committees can be formed?

Solution

1. Check order: Committees don't depend on order → Combination.
2. Formula: $^{n}C_{r} = \frac{n!}{r!(n-r)!}$
3. Substitute: we have n=7 and r=3

$$^{7}C_{3} = \frac{7!}{3!\,(7-3)!}$$

4. Simplify:

$$^{7}C_{3} = \frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!} = 35$$

5. Answer: There are 35 different committees.

Example 8.21

A DJ is making a 5-track playlist from 7 afrobeat, 4 highlife, and 3 reggae tracks, but track order does not matter for selection.

In how many ways can the DJ select tracks if the playlist contains:

(i) exactly 3 afrobeat and 2 highlife?

(ii) at least one reggae?

(iii) at most 2 highlife?

Solution

This is a combination question since the track order does not matter.

(i) selecting 3 afrobeats from 7 can be done in $^{7}C_{3}$ ways =

Selecting 2 highlife tracks from 4 can be done in $^{4}C_{2}$ ways =

exactly 3 afrobeat and 2 highlife = (exactly 3 afrobeat) ×(2 highlife)

$= {}^{7}C_{3} \times {}^{4}C_{2} = 35 \times 6 = 210$ ways

So, the number of ways the DJ can select 3 afrobeats and 2 highlights from the list of available tracks is 210.

(ii) The are a total of 3 reggae and $(7 + 4) = 11$ non-reggae tracks the DJ can choose from.

Choosing at least one reggae track

= [Choosing 1 reggae and 4 non-reggae tracks or
Choosing 2 reggae track and 3 non-reggae tracks or
Choosing 3 reggae tracks and 2 non-reggae tracks]

Number of ways the DJ can choose 1 reggae and 4 non-reggae tracks

$= {}^{3}C_{1} \times {}^{11}C_{4} = 3 \times 330 = 990$

Number of ways the DJ can choose 2 reggae and 3 non-reggae tracks

$= {}^{3}C_{2} \times {}^{11}C_{3} = 3 \times 165 = 495$

Number of ways the DJ can choose 3 reggae and 2 non-reggae tracks

$= {}^{3}C_{3} \times {}^{11}C_{2} = 1 \times 55 = 55$

At least one reggae track $= 990 + 495 + 55 = 1\,540$ ways.

(iii) There are a total of 4 highlife and $(3 + 7) = 10$ non-highlife tracks the DJ can choose from.

Choosing at most 2 highlife tracks

= [Choosing 0 or no highlife and 5 non-highlife tracks or
Choosing 1 highlife track and 4 non-highlife tracks or
Choosing 2 highlife tracks and 3 non-highlife tracks]

Number of ways the DJ can choose 0 highlife and 5 non-highlife tracks

$= {}^{4}C_0 \times {}^{10}C_5 = 1 \times 252 = 252$

Number of ways the DJ can choose 1 highlife track and 4 non-highlife tracks

$= {}^{4}C_1 \times {}^{10}C_4 = 4 \times 210 = 840$

Number of ways the DJ can choose 2 highlife tracks and 3 non-highlife tracks

$= {}^{4}C_2 \times {}^{10}C_3 = 6 \times 120 = 720$

At most two highlife tracks $= 252 + 840 + 720 = 1\,812$ ways.

Example 8.22

If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, find the value of ${}^{r}C_2$.

Solution

We know ${}^{n}C_r = \frac{n!}{r!(n-r)!}$

So, ${}^{n}C_r = 84$ is the same as $\frac{n!}{r!(n-r)!} = 84$ .....(1)

Likewise, ${}^{n}C_{r-1} = 36$ is same as $\frac{n!}{(r-1)!(n-r+1)!} = 36$ ..... (2)

And ${}^{n}C_{r+1} = 126$ is same as $\frac{n!}{(r+1)!(n-r-1)!} = 126$ .....(3)

Divide equation (1) by equation (2)

$$\frac{n!}{r!\,(n-r)!} \div \frac{n!}{(r-1)!\,(n-r+1)!} = 84 \div 36$$

Change ÷ to × by reciprocating the second fractions on each side of the equation

$$\frac{\cancel{n!}}{r!\,(n-r)!} \times \frac{(r-1)!\,(n-r+1)!}{\cancel{n!}} = 84 \times \frac{1}{36}$$

Recall that $r! = r(r-1)!$

Also, $(n-r+1)! = (n-r+1)(n-r)!$

Cancel out $n!$, and substitute the expansions of $r!$ and $(n-r+1)!$

$$\frac{\cancel{n!}}{r(r-1)!\,(n-r)!} \times \frac{(r-1)!\,(n-r+1)(n-r)!}{\cancel{n!}} = 7 \times \frac{1}{3}$$

Cancel out the common factors

$$\frac{n-r+1}{r} = \frac{7}{3}$$

Cross multiplying, we have $3n - 3r + 3 = 7r$

$3n - 10r = -3$ .....(4)

Divide equation (1) by equation (3)

$$\frac{n!}{r!\,(n-r)!} \div \frac{n!}{(r+1)!\,(n-r-1)!} = 84 \div 126$$

Change ÷ to × by reciprocating the second fraction on each side of the equation

$$\frac{\cancel{n!}}{r!\,(n-r)!} \times \frac{(r+1)!\,(n-r-1)!}{\cancel{n!}} = 84 \times \frac{1}{126}$$

Recall that $(r+1)! = (r+1)r!$

Also, $(n-r)! = (n-r)(n-r-1)!$

Cancel out $n!$, and substitute the expansions of $(r+1)!$ and $(n-r)!$

$$\frac{1}{r!\,(n-r)(n-r-1)!} \times \frac{(r+1)r!\,(n-r-1)!}{1} = 2 \times \frac{1}{3}$$

Cancel out the common factors

$$\frac{r+1}{n-r} = \frac{2}{3}$$

Cross-multiplying, we have $2n - 2r = 3r + 3$

$2n - 5r = 3$ .....(5)

Multiply (5) by 2 and subtract (4) from the result

$(4n - 10r) - (3n - 10r) = 6 - -3$

$n = 9$

Substitute n into (5)

$2(9) - 5r = 3$

$-5r = 3 - 18$

$-5r = -15$

$r = 3$

Therefore, ${}^{r}C_2 = {}^{3}C_2 = 3$

Permutations

A permutation is an arrangement of items where the order of the arrangement is important.

The general formula is:

$$^{n}C_{r} = \frac{n!}{(n-r)!}$$

Where:

n = total number of objects

r = number of objects arranged

Situations Where Order of Arrangement is Important (Permutations)

1. Password and Security Codes
   Example: A 3-digit PIN, such as "457", is **not** the same as "574".
   The order of the digits matters.
2. Seating Arrangements
   Arranging students in seats: (Kofi, Naadu, Afi, Oko) sitting in order is different from (Naadu, Afi, Oko, Kofi).
3. Award Positions
   Selecting 1st, 2nd, and 3rd place winners in a race.
   Who comes first, second, or third is important.
4. Job Assignments
   Assigning roles like Manager, Secretary, and Treasurer.
   The same three people in different roles create different outcomes.
5. Telephone, vehicle, index, account, and other identification numbers
   A change in the position of just one digit will render it wrong.
6. Arranging Books on a Shelf
   The order of books (Math, Science, English) is different from (Science, Math, English).
7. Forming words with letters
   Example: Cork and rock, listen and silent, state and taste, secure and rescue.
   These pairs of letters use the same letters but have very different meanings.
8. Dance or Performance Line-Up
   The order of dancers or speakers affects the presentation.
9. Sports Team Formation with Positions
   In sports, the position of players is important.
10. Queue/Line Order
    Whether it is a waakye queue or a queue at a bank or a trotro station, the order is important.

How to Answer a Permutation Question

Step 1: Identify if it's a permutation

Ask yourself: Does order matter?

If YES, it's a permutation.

Step 2: Recall the formula

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Where n = total number of items and r = number of items being chosen.

Step 3: Substitute values and simplify the factorials

Put in the given numbers for n and r.

Step 4: State the final answer

Write your result clearly, with units as necessary.

Example 8.23

A computer password is created by selecting 5 distinct letters from the English alphabet (26 letters).

How many possible passwords can be formed?

Solution

Step 1: The order is important since it forms a password.

Step 2: We will use the formula $^{n}P_{r} = \frac{n!}{(n-r)!}$

Where n = total number of items $= 26$ and

r = number of items being chosen $= 5$

Step 3: Substitute values and simplify the factorials

$$^{26}P_{5} = \frac{26!}{(26-5)!} = \frac{26!}{21!} = \frac{26 \times 25 \times 24 \times 23 \times 22 \times 21!}{21!}$$

$$= 26 \times 25 \times 24 \times 23 \times 22$$

$$= 7\,893\,600$$

Step 4: Answer: $7\,893\,600$ possible passwords.

Example 8.24

(a) In how many different ways can the letters of the word CRATES be arranged?

(b) A group of 10 people is to elect a president, a secretary, and a treasurer. In how many ways can these three officers be chosen if no person can hold more than one position?

(c) In how many ways can the letters of the word STATISTICS be arranged if all the vowels must be together?

Solution

(a) CRATES has 6 letters. None is repeated. So, the number of arrangements is

$${}^{6}P_{6} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \qquad = 6! = 720 \text{ ways}$$

(b) $n = 10, r = 3$

$${}^{10}P_{3} = \frac{10!}{(10-3)!} = 720 \text{ ways}$$

(c) The word is STATISTICS has 10 letters

Letter counts: $S = 3$ counts, $T = 3$ counts, $A = C = 1$ count, $I = 2$ counts.

Number of vowels = 3 counts (I, I, A)

Since all vowels must be together, treat them as a single letter.

This will give us 7 consonants and 1 vowel = 8 items to arrange.

But some of the consonants repeat. S has 3 counts, and T has 3 counts.

So, the number of arrangements of the 8 items is $\frac{8!}{3!\times 3!}$

Number of arrangements of the vowels (I, I, A)$= \frac{3!}{2!} = 3$

So, total arrangements $= 3 \times \frac{8!}{3!\times 3!} = 3 \times 1120$

$= 3\,360$ ways

Example 8.25

In a certain country, car licence plates consist of 5 characters. The first two characters are letters, chosen from one of the fixed prefixes: GE, AS, WR, VR, or UE. The last three characters are digits chosen from {0,1,3,4,5,6,7,8,9}. No letter or digit is repeated within a plate.

How many different licence plates can be formed?

Solution

Since the first two characters have been chosen already, we will focus on the last three characters. We must choose and arrange 3 distinct digits from 10. Order matters (each digit goes into a specific position), so we use permutations:

$${}^{10}P_{3} = 10 \times 9 \times 8 = 720$$

To find the total number of plates, multiply the number of prefixes (5) by the number of digit arrangements.

This gives: $5 \times 720 = 3\,600$ ways

Example 8.26

In a certain town, all telephone numbers have eight digits, the first three always being 058 or 139 or 246 or 732.

How many distinct telephone numbers can be formed if all eight digits must be different?

Solution

Each telephone number is 8 digits long and must begin with one of the four fixed three-digit prefixes: 058 or 139 or 246 or 732. In each prefix, the three starting digits are already fixed and are different from each other.

After choosing a prefix (4 choices), the remaining 5 digits (4^{th} to 8^{th} positions) must be chosen from the 10 digits $0,1,\ldots,9$, but must be different from each other and from the three digits already used in the prefix.

That leaves $10 - 3 = 7$ available digits, and we must arrange 5 distinct digits in order:

This arrangement can be done in ${}^{7}P_{5} = 2\,520$

To find the total number of telephone numbers that have all eight digits distinct, multiply the number of arrangements by the 4 possible prefixes:

$4 \times 2\,520 = 10\,080$ ways

Therefore, 10 080 telephone numbers have all eight digits distinct.

Permutations vs. Combinations

Aspect	Permutation	Combination
Definition	An arrangement of objects where order matters.	A selection of objects where order does not matter.
Key Question	"Does the order/position make a difference?"	"Does the order/position NOT make a difference?"
Formula	${}^{n}P_{r} = \frac{n!}{(n-r)!}$	${}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$
Order	Important	Not important
Real-Life Applications	Passwords & security codes, seating plans, job assignments, race results, sports line-up with positions	Committee selection, food/menu choices, subject selection, team formation, lottery, exam question selection
Key Words to Watch	Arrange, Line-up, Order, Position, Rank, Schedule	Select, Choose, Pick, Form, Group, Combination

Mixed Questions: Permutations or Combinations

Example 8.27

Read each situation carefully. State whether the problem is a permutation (order matters) or a combination (order does not matter).

Justify your answer.

1. A school is selecting 3 prefects (without specific roles) from a group of 12 students.
2. A football coach is arranging 11 players into field positions.
3. A customer chooses 2 drinks from 5 available in a shop.
4. The manager wants to assign Chairperson, Secretary, and Treasurer roles to 3 people from a group of 10 workers.
5. A lottery game requires selecting 6 numbers from 49.
6. A student forms a password using 3 letters from the alphabet without repetition.
7. A teacher selects 4 questions from a test bank of 10 questions to form a mini test.

Solution

1. Combination – order doesn't matter in team selection.
2. Permutation – field positions make order important.
3. Combination – choosing drinks doesn't depend on order.
4. Permutation – roles assigned make order important.
5. Combination – lottery numbers are the same regardless of order.
6. Permutation – passwords depend on order.
7. Combination – selection of questions doesn't depend on order.

Example 8.28

A school choir has 10 students: 6 girls and 4 boys. The music director wants to form a 4-member singing group to perform a special piece. The group must include at least 1 boy.

The order in which the singers stand on stage matters (positions are important).

a. In how many different ways can the music director select and arrange 4 students to form the singing group under the given condition?

b. Explain why your solution involves both combination and permutation, and how the model reflects the conditions of the problem.

Solution

a. Ways of selecting 4 members with at least 1 boy

$$({}^{6}C_3 \times {}^{4}C_1) + ({}^{6}C_2 \times {}^{4}C_2) + ({}^{6}C_1 \times {}^{4}C_3) + ({}^{6}C_0 \times {}^{4}C_4)$$

$$(20 \times 4) + (15 \times 6) + (6 \times 4) + (1 \times 1)$$

$$80 + 90 + 24 + 1 = 195 \text{ ways}$$

After selecting the 4 students, the teacher needs to arrange them. The number of ways of arranging 4 students $= 4! = 24$ ways

Therefore, the number of ways the teacher can select and arrange 4 students

$= 24 \times 195 = 4\,680$ ways

b. First, we choose which students form the group. When choosing who will be in the group, we use combinations to calculate the choices since the order is not important. Once the 4 students are chosen, the order in which they stand on stage matters. For each chosen set of 4, there are 4! ways to arrange them; this is the permutation part.

DETERMINING AREAS IN REAL-LIFE WHERE PERMUTATIONS AND COMBINATIONS CAN BE APPLIED TO IMPROVE THE SYSTEMS

Applications of Permutations and Combinations

Permutations and combinations are mathematical tools used to count, arrange, and select items under different conditions. Beyond the classroom, these concepts have applications in business, commerce, industry, and many aspects of daily life. Understanding these applications helps us appreciate mathematics as a tool for solving real-world problems. The list below gives a few examples of their applications.

Applications in Business and Commerce

1. Marketing Strategies – Companies can use permutations to calculate the number of ways products can be arranged on shelves to attract customers.
2. Menu Planning – Restaurants can use combinations to determine the different possible sets of meals that can be served to customers.
3. Staff Scheduling – Businesses can use combinations to select workers for shifts or teams efficiently.

Example 8.29

A supermarket wants to display 5 different brands of fruit juice on a shelf to attract customers.

a) In how many different ways can the 5 brands be arranged in a row?

b) If the manager insists that two particular brands must always be placed next to each other, how many arrangements are possible?

c) If the manager decides that one brand must always be placed at the centre of the shelf, how many possible arrangements are there?

Solution

(a) Arranging 5 brands in a row

Number of arrangements $= {}^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

So, the five brands can be arranged in 120 ways,

(b) Two particular brands are always together.

Treat the two brands as one item. So, instead of 5 brands, we now have 4 items.

These 4 items can be arranged in ${}^4P_4 = 4!$ Ways.

However, within the block, the 2 brands can be arranged in 2! Ways.

So, total arrangements $= 4! \times 2! = 24 \times 2 - 48$ ways.

Therefore, if two brands are placed next to each other, the total number of arrangements is 48 ways.

(c) One brand fixed in the centre

If one brand is fixed at the centre, the remaining 4 brands can be arranged in any order without any restrictions.

Number of arrangements ${}^4P_4 = 4! = 24$

So, if one brand is fixed at the centre, the total number of arrangements = 24 ways.

Applications in Industry

1. Password and Security Codes – Permutations are used to calculate the possible arrangements of letters and digits for creating strong passwords and PIN codes.
2. Colour Schemes in Fashion and Manufacturing – Factories apply permutations in arranging colour patterns for clothing, textiles, or packaging designs.
3. Product Design – Combinations are used in deciding features to include in different versions of a product (e.g., car models with different accessories).

Example 8.30

A company requires its employees to create a 4-digit security code using the digits 0–9. Each digit can be used only once in a code.

a) How many different security codes can be created?

b) If the first digit cannot be 0, how many valid codes can be formed?

c) If the code must begin with an odd number and end with an even number, how many possible codes are there?

Solution

(a) Total number of security codes
We are choosing and arranging 4 digits out of 10 digits (0 – 9).
This is a permutation: ${}^{10}P_4 = 5\,040$

5 040 different security codes can be created.

(b) The first digit cannot be 0

Choices for the first digit = 9 (since it can be any digit except 0).
After choosing the first digit, we have 9 digits left for the second, 8 digits for the third, and 7 digits for the fourth.

Number of codes $= 9 \times 9 \times 8 \times 7 = 4\,536$

Thus, 4 536 valid codes can be formed.

(c) Code begins with an odd number and ends with an even number

Odd digits available = {1, 3, 5, 7, 9}. This gives 5 choices for the first digit.

Even digits available = {0, 2, 4, 6, 8}. This also gives 5 choices for the last digit.

After choosing the first and last digits, 8 digits remain for the second, and 7 digits remain for the third.

Number of codes $= 5 \times 8 \times 7 \times 5 = 1\,400$

Hence, there will be 1 400 possible codes.

Applications in Science and Technology

1. Cryptography and Network Security – Permutations and combinations help in designing secure systems for communication and online banking.
2. Computer Architecture – Combinations are used in memory allocation and coding.
3. Molecular Biology – Scientists use permutations and combinations to study arrangements of DNA sequences and protein structures.
4. Pattern Analysis – Applications are found in language processing, artificial intelligence, and image recognition.

Example 8.31

In molecular biology, DNA sequences are made up of 4 nitrogen bases: Adenine (A), Thymine (T), Cytosine (C), and Guanine (G).

a) How many different sequences can be formed if a DNA chain has 4 bases arranged in a specific order (repetition allowed)?

b) If no base is repeated, how many possible sequences can be formed using the 4 bases?

Proteins are built from 20 amino acids.

c) In how many ways can a protein segment of 3 amino acids be arranged if order matters and no amino acid is repeated?

Solution

a) DNA sequence of 4 bases, repetition allowed

Each of the 4 positions can be filled with any of the 4 bases.

Total arrangements = $4^4 = 4 \times 4 \times 4 \times 4 = 16 \times 16 = 256$

The number of sequences that can be formed is 256

b) DNA sequence of 4 bases, no repetition

We are arranging 4 distinct items (A, T, C, G).

Total arrangements = $4! = 24$

24 sequences can be formed using the 4 bases.

c) Protein segment of 3 amino acids, no repetition

We are arranging 3 amino acids out of 20.

This is a permutation:

$$^{20}P_3 = \frac{20!}{(20-3)!} = \frac{20 \times 19 \times 18 \times 17!}{17!}$$

$$= 20 \times 19 \times 18 = 6\,840$$

So, given the restrictions, a protein segment of 3 amino acids can be arranged in 6 840 possible ways.

This shows how permutations are applied in molecular biology to model DNA and protein arrangements.

Applications in Games and Sports

1. Team Selection – Coaches use combinations to select players from a squad for a match.
2. Tournament Arrangements – Permutations are applied in fixing fixtures and arranging match orders.
3. Card Games and Lotteries – Both permutations and combinations are used in calculating the probability of outcomes.

Example 8.32

A football coach has 15 players available and wants to select a team of 11 players for a match.

a) In how many different ways can the coach select the team of 11 players?

b) If the coach must include 2 particular players (the captain and the goalkeeper), how many different teams can be formed from the remaining players?

c) If the coach wants to select a captain in addition to the 11 players, how many possible selections are there?

Solution

(a) Selecting 11 players out of 15
This is a combination:

The number of ways = ${}^{15}C_{11} = 1\,365$

(b) If 2 particular players are always included

Then we only need to choose 9 more players from the remaining 13.

Number of ways ${}^{13}C_9 = 715$ different teams

(c) Selecting a captain in addition to the 11 players

First select 11 players from 15: ${}^{15}C_{11}$=1 365.

Then select a captain from these 11 players: ${}^{11}C_1 = 11$.

Total selections = $1\,365 \times 11 = 15\,015$

There are $15\,015$ more possible selections.\

Activity 8.2 Application of Permutations and Combinations

Investigate how permutations and combinations are applied in:

1. Business and commerce (e.g., menu planning, marketing strategies, scheduling)
2. Industry (e.g., product designs, passwords, colour schemes in factories)
3. Games and sports (e.g., team selection, card games, tournaments)
4. Technology and science (e.g., cryptography, communication networks, computer memory, molecular biology)

REVIEW QUESTIONS 8

1. In a class of 40 students, 18 study Physics, 12 study Chemistry, and 8 study both Physics and Chemistry.

   If a student is selected at random:

   a) Find the probability that the student studies Physics given that they study Chemistry.

   b) Find the probability that the student studies Chemistry given that they study Physics.

2. A card is drawn at random from a standard pack of 52 playing cards.

   a) Find the probability that it is a King given that it is a face card.

   b) Find the probability that it is red given that it is a heart.

3. In a school, 60% of the students play football, and 30% of those who play football also play basketball. If 25% of all students do not play either game:

   a) What is the probability that a student plays basketball given that they play football?

   b) What is the probability that a student plays football given that they play basketball?

4. The probability that it rains on a particular day is 0.4. If it rains, the probability that students bring umbrellas is 0.8. If it does not rain, the probability that students bring umbrellas is 0.1.

   a) Find the probability that it rains given that a student has brought an umbrella.

   b) Interpret your result.

5. A box contains 3 red balls and 5 blue balls. Two balls are drawn at random without replacement.

   a) Find the probability that the second ball is red given that the first ball was red.

   b) Find the probability that the second ball is blue given that the first ball was red.

6. A fair coin is tossed 6 times.

   a) Find the probability of getting exactly 3 heads.

   b) Find the probability of getting at least 4 heads.

7. A box contains red and white balls. The probability of picking a red ball in a single draw is 0.3. If a ball is drawn and replaced 5 times,

   a) Find the probability of obtaining exactly 2 red balls.

   b) Find the probability of obtaining more than 3 red balls.

8. A factory produces bulbs, and each bulb has a probability of 0.1 of being defective. If a random sample of 12 bulbs is chosen,

   a) Find the probability that exactly 2 are defective.

b) Find the probability that at most 3 are defective.

9. In a certain city, the probability of rain on any given day is 0.4. Consider a week (7 days).

a) Find the probability that it rains on exactly 4 days in the week.

b) Find the probability that it rains on at least 5 days in the week.

10. The probability that a student passes a mathematics test is 0.7. If 10 students take the test,

a) Find the probability that exactly 8 students pass.

b) Find the probability that fewer than 6 students pass.

11. A basketball player has a free-throw success rate of 0.75. If she takes 8 free throws:

a) Find the probability that she scores all 8.

b) Find the probability that she scores at least once.

12. A restaurant offers 3 types of rice dishes, 2 types of stew, and 4 types of drinks. How many possible meal combinations can a customer order?

13. A student has 5 pairs of shoes, 3 trousers, and 4 shirts. In how many ways can the student dress up?

14. A computer system generates codes using 2 letters followed by 1 digit. How many codes can be formed?

15. (a) A school prefect is to be chosen from 8 candidates.

In how many ways can a prefect and 2 assistants be chosen?

(b) A company wants to form a committee of 4 workers from 12 staff members.

How many committees can be formed?

(c) A basketball coach must select 5 players out of 12.

In how many ways can the team be formed?

(d) A shop has 7 varieties of fruits.

In how many ways can a customer select 3 fruits for a fruit salad?

(e) In how many ways can 3 science subjects be selected from the 5 core science subjects?

16. An organisation needs a 4-member panel selected from 6 teachers, 5 parents, and 3 alumni. In how many ways can the panel be chosen if:

(i) it must include exactly 2 teachers and 2 parents?

(ii) it must include at least one alumnus?

(iii) it must include no parents?

17. A designer must select 4 items to feature in a set from 5 shirts, 4 trousers, and 3 jackets. In how many ways can the designer choose the set if:

(i) it must include 1 jacket, 2 shirts, and 1 trouser?

(ii) it must include at least two shirts?

(iii) the set has no jacket?

18.

(a) If ${}^{n}C_{r-1} = 10$, ${}^{n}C_{r} = 45$ and ${}^{n}C_{r+1} = 120$, find ${}^{r}C_{2}$

(b) If ${}^{n}C_{r-1} = 15$, ${}^{n}C_{r} = 35$ and ${}^{n}C_{r+1} = 70$, find ${}^{r}C_{3}$

(c) If ${}^{n}C_{r} = 210$, ${}^{n}C_{r+1} = 252$ and ${}^{n}C_{r+2} = 210$, find ${}^{r}C_{2}$

(d) If ${}^{n}C_{r-2} = 21$, ${}^{n}C_{r-1} = 56$ and ${}^{n}C_{r} = 126$, find ${}^{r}C_{4}$

19. In a small village, there are 17 families, of which 11 families have at most 2 children. In a rural development program, 7 families are to be chosen for assistance, of which at least 3 families must have at most 2 children. In how many ways can the choice be made?

20. In an examination, there are three multiple-choice questions, and each question has 4 choices. Find the number of ways a student can fail to get all answers correct.

21. In how many ways can 4 books be arranged on a shelf among 7 different books?

22. How many 3-digit numbers can be formed using the digits 1, 2, 3, 4, 5 if repetition is not allowed?

23. In how many ways can the letters of the word "MATH" be arranged?

24. In how many different ways can a school assign 5 prefects to 5 distinct offices?

25. A company has 8 candidates for 3 different job positions (Manager, Secretary, Treasurer).

In how many ways can the positions be filled?

26. A security code is created using 4 different digits chosen from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. How many different codes can be formed if digits cannot be repeated?

27. In how many ways can the letters of the word MARKET be arranged?

28. From a committee of 12 members, a chairperson, a vice-chairperson, and a financial secretary are to be chosen.

In how many ways can these positions be filled if no one can hold more than one office?

29. In the arrangement of the letters of the word SUCCESS, in how many ways can the letters be arranged if the two C's are always together?

30. A car registration number consists of 2 letters followed by 3 digits. The first two letters must be chosen from the prefixes: BA, KO, NI, SE, or TU. The digits are chosen from 0–9, but no digit is repeated. How many possible car registration numbers can be formed?

3. A certain mobile network assigns 7-digit phone numbers. The first two digits are fixed as $05, 07$, or 09, while the remaining five digits are distinct and chosen from $0 - 9$.

31. How many such numbers can be assigned?

32. A school is organising a cultural performance, and 12 students have volunteered to participate; 7 girls and 5 boys. The teacher wants to form a 5-member dance group such that the group must contain at least 2 girls, and the order in which students perform on stage matters.

(a) In how many different ways can the teacher select and arrange 5 students to form the dance group under the given condition?

(b) Explain how your model accounts for both selection and order, and why a combination or permutation (or both) is appropriate in this context.

33. For each question: State whether it is a Permutation or a Combination and solve the problem.

a) A school is selecting 3 prefects from a class of 15 students.
How many different groups of prefects can be chosen?

b) A password is formed using 4 different letters from the 26 letters of the English alphabet. How many possible passwords can be created?

c) A football coach wants to assign the roles of Captain, Assistant Captain, and Goalkeeper from a team of 18 players.
In how many ways can this be done?

d) From 12 books, a student chooses 5 books to borrow from the library.
In how many ways can this selection be made?

e) In a race with 10 athletes, how many different ways can the Gold, Silver and Bronze medals be awarded?

f) A restaurant offers 6 types of drinks. A customer wants to choose 2 drinks.
How many different drink choices are possible?

g) A factory wants to arrange 5 different products in a row for display on a shelf.
How many possible arrangements are there?

h) In a quiz competition, 8 questions are given. A student must answer 3 questions.
How many ways can the questions be chosen?

ANSWERS TO REVIEW QUESTIONS – SECTION ONE

Let W: Learner fails to attend class regularly

Let X: Learner is not allowed to sit the final exam

Let Y: Learner cannot pass the course

Symbolic representation

$W \Rightarrow X$

$X \Rightarrow Y$ thus

Chain rule implication $W \Rightarrow Y$

Using the truth table

W	X	Y	$W \Rightarrow X$	$X \Rightarrow Y$	$W \Rightarrow Y$	$(W \Rightarrow X) \wedge (X \Rightarrow Y)$	$(W \Rightarrow X) \wedge (X \Rightarrow Y) \Rightarrow (W \Rightarrow Y)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Entries from the final column provide evidence to support that the new implication is valid and always true.

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

From the truth table $\sim (P \vee Q)$ and $(\sim P \wedge \sim Q)$ have identical truth values in all cases. Hence, they are logically equivalent.

From De Morgan's Law if P and Q are sets, $(P \cup Q)' \equiv P' \cap Q'$ which is similar to $\sim (P \vee Q) \equiv (\sim P \wedge \sim Q)$. Hence the logical equivalence of M and N reflect De Morgan's law.

Let Q: I study hard

R: I pass the test

Argument:

Premise 1: Q $\Rightarrow$ R

Premise 2: R

Conclusion: $\therefore$ Q

Q	**R**	**Q $\Rightarrow$ R**
T	T	T
T	F	F
F	T	T
F	F	T

Line 3 suggests the argument is invalid.

If the learner passes the test but did not study hard gives a flaw in the reasoning.

a. $P \Longrightarrow R$ i.e., If the student submitted the assignment, then the student graduated.

b.

P	***R***	**P $\Rightarrow$ R**	**$\sim$ R**
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	T

It is **NOT** logically possible that the student submitted the assignment because If P were true, then $P \Rightarrow R$ would force R to be true. This however contradicts the given $\sim R$ thus P must be false.

ANSWERS TO REVIEW QUESTIONS – SECTION TWO

a. element of N $= \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$ (It must be the inverse of N to satisfy $NM^2 = M$.)

$x' = 4x - 3y, y' = -x + y$

$\begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$

a. $A' = (2,4), B' = (-1,10), C' = (5,10), D' = (8,4)$

b. $A'' = (-3,6), B'' = (-6,12), C'' = (0,12), D'' = (3,6)$

a. $x = 1$ and $y = 4$

b. $B^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

$5B + 2I = 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ as required

90° clockwise rotation about the origin

$y = -11x + 70.$

a. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b. $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$k = -2$ or $k = -8$

$A'(5, 2), B'(2, 0)$ and $C'(4, -3)$.

a. $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The product LH can be interpreted as the **total labour cost** associated with operating a specific machine for a given period. It could support the company to understand and assess the cost-effectiveness of their operations as well as aid in resource allocation and inform strategic business decisions.

a. $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $T_2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

b. $(-6,3)(-18,3)(-12,15)$

ANSWERS TO REVIEW QUESTIONS – SECTION THREE

1. Real-life examples of parabolas:

Some **roller coaster** designs have hills shaped like parabolas for smooth acceleration and deceleration.

Design of some suspension bridges

Some projectiles take the shape of parabolas

Satellite dishes are shaped like a parabola

Water jets that shoot into the air and fall back down follow a parabolic path.

2. Table 3.3: Summary to answers, Question 2

(i)	(a) Domain: $x \in \mathbb{R}$ Range: $y \geq -2$ line of symmetry: $x = 2$ Orientation: opens upwards Minimum Value: -2 Minimum point: $(2, -2)$	(b) $y = \frac{1}{3}(x-2)^2 - 2$ (c) it is wider, shifted 2 units to the right and 2 units down.
(ii)	(a) Domain: $x \in \mathbb{R}$ Range: $y \geq -2$ line of symmetry: $x = 0$ Orientation: opens upwards Minimum Value: -2 Minimum point: $(0, -2)$	(b) $y = \frac{1}{2}x^2 - 2$ (c) It is wider and shifts 2 units down.
(iii)	(a) Domain: $x \in \mathbb{R}$ Range: $y \leq 1$ line of symmetry: $x = -2.5$ Orientation: opens downwards Maximum Value: 1 Maximum point: $(-2.5, 1)$	(b) $y = -\frac{4}{25}(x+2.5)^2 + 1$ (c) Reflected in the x-axis, it is wider, shifted 2.5 units to the left and 1 unit up.
(iv)	(a) Domain: $x \in \mathbb{R}$ Range: $y \geq 0$ line of symmetry: $x = 2$ Orientation: opens upwards Minimum Value: 0 Minimum point: $(2, 0)$	(b) $y = \frac{1}{4}(x-2)^2$ (c) It is wider since the coefficient of x^2 is less than 1. It is also shifted 2 units to the right.

(v)	(a) Domain: $x \in \mathbb{R}$ Range: $y \leq 4$ line of symmetry: $x = -2$ Orientation: opens downwards Maximum Value: 4 Maximum point: $(2, 4)$	(b) $y = -\frac{1}{2}(x-2)^2 + 4$ (c) It is wider since the coefficient of x^2 is less than 1. Also, it is reflected in the x-axis, shifted 2 units to the right and 4 units up.
(vi)	(a) Domain: $x \in \mathbb{R}$ Range: $y \leq 4$ line of symmetry: $x = -1$ Orientation: opens downwards Maximum Value: 2 Maximum point: $(-1, 2)$	(b) $y = -\frac{1}{2}(x+1)^2 + 2$ (c) It is wider since the coefficient of x^2 is less than 1. Also, it is reflected in the x-axis, shifted 1 unit to the left and 2 units up.

3. (a) Vertex= (0.5, 1), Focus =(1.5, 1), Line of symmetry y=1, x-intercept (0.75, 0) direction: opens right. The graph does not touch the y-axis

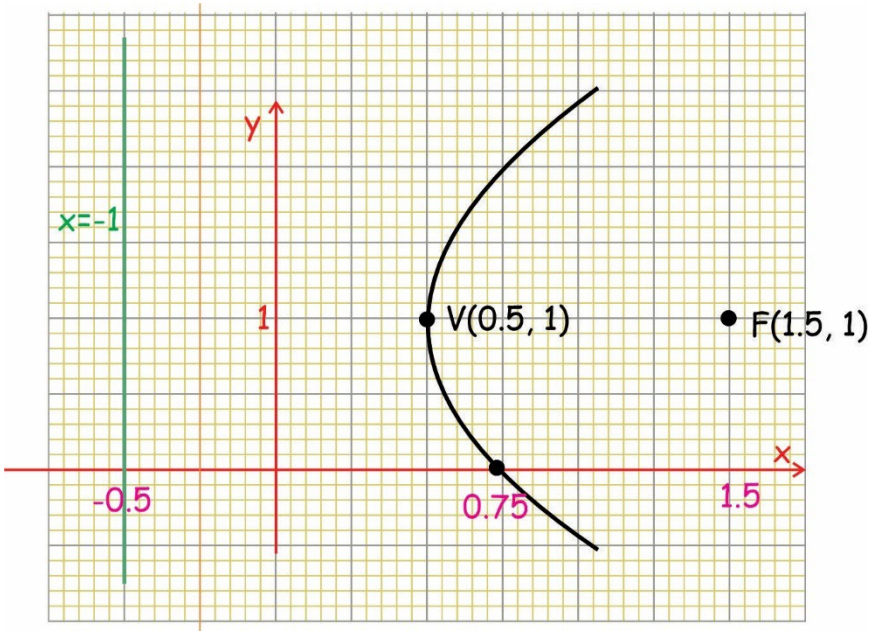


Figure 3.64: Graph of $x = \frac{1}{4}(y-1)^2 + \frac{1}{2}$

(b) Equation of the parabola $x = \frac{1}{4}(y-1)^2 + \frac{1}{2}$. The coefficient of the squared term is positive, so it opens to the right.

4. (a) $y = \frac{4}{49}(x-8)^2 - 4$

(b) $y = \frac{4}{49}(x-8)^2 - 4$ is wider compared to $y = x^2$ as the coefficient of the squared term is less than 1. They are both open upwards. $y = \frac{4}{49}(x-8)^2 - 4$ is a result of shifting $y = x^2$ 8 units to the right and 4 units down.

(c) Focus=(8, -15/16) Equation of the directrix: 16y+113=0 Focal length=49/16 units.

(d) (1,0) and (8, -4). The line is a secant.

5. The vertex is at (0, 3)

The parabola opens downward

It is symmetrical about the y-axis

It crosses the x-axis at about −1.73 and 1.73

it crosses the y-axis at 3

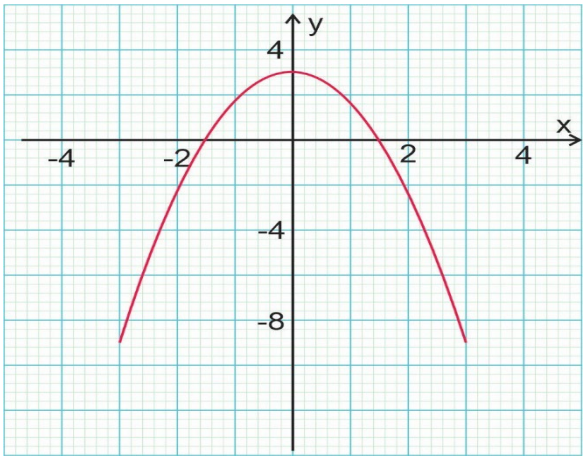


Figure 3.65: Graph of $y = -x^2 + 3$

6. vertex at (−1.25, 6.125).

y-intercept at (0, 3).

x-intercepts at (−3, 0) and (0.5, 0).

Draw a downward-opening curve passing through these points.

Draw the line of symmetry: a dashed vertical line through x=−1.25

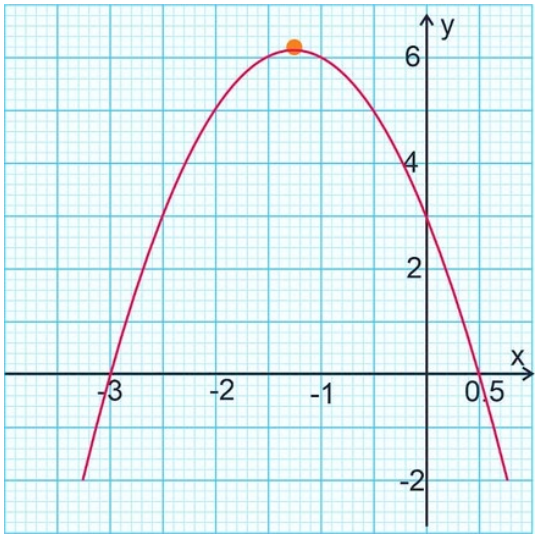


Figure 3.66: Graph of $y = -2x^2 - 5x + 3$

It shows a reflection in the y-axis. It opens downward, is narrower, and is shifted both horizontally to the right and vertically up as compared to $y = x^2$.

7. Table 3.4: comparing features of $y = x^2 - 7x + 6$ to $y = x^2$

Feature	$y = x^2$	$y = x^2 - 7x + 6$
Shape	U-shaped parabola	U-shaped parabola
Vertex	(0, 0)	(3.5, -6.25)
Opens	Upwards	Upwards
x-intercepts	(0, 0)	(1, 0) and (6, 0)
y-intercepts	(0,0)	(0, 6)
Shift	None	Shifted right and down

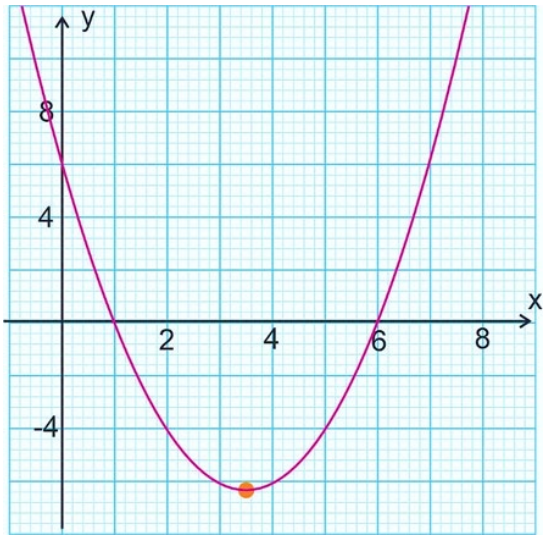


Figure 3.67: Graph of $y = x^2 - 7x + 6$

8. a. $(y-2)^2 = -4(x-2)$

b. Table 3.5: comparing features of $(y-2)^2 = -4(x-2)$ to $x = y^2$

Feature	$x = y^2$	$(y-2)^2 = -4(x-2)$
Orientation	Opens rightward	Opens leftward (because of negative sign)
Vertex	(0, 0)	(2, 2)
Line of Symmetry	Horizontal: y = 0	Horizontal: y = 2
Direction of Opening	To the right along x-axis	To the left along x-axis
Stretch/Width	Standard width	Standard width
Translation	None	Translated 2 units right and 2 units up

9. a. $x = \frac{1}{8}(y+4)^2 - 2$

b. Table 3.6: comparing features of $x = \frac{1}{8}(y+4)^2 - 2$ to $x = y^2$

Feature	$x = y^2$	$x = \frac{1}{8}(y+4)^2 - 2$
Vertex	(0,0)	(−2,−4)
Direction of opening	Rightwards	Rightwards (same direction)
Stretch/Width	Standard width	Wider (because coefficient of the squared term is <0)
Translation	None	Shifted 2 units left and 4 units down

10. a. $H = -0.625(t-2)^2 + 2.5$

b. Table 3.7: Features of $H = -0.625(t-2)^2 + 2.5$

Feature	Description
Vertex	(2, 2.5) — the maximum height of the ball
Axis of symmetry	$t = 2$ — the ball is symmetrical around 2 seconds
Direction	Opens downward because a =−0.625<0
Maximum height	2.5 meters at $t = 2$ seconds
Domain (time)	Typically, t∈[0,4] — from when the ball is hit to when it lands (approx.)
Range (height)	H∈[0,2.5] meters — the ball rises and falls back to the ground

11. Sketch guide:

Vertex (0, 0)

Opens to the left. Examples of points the graph passes through are

(-12, 1), (-12, -1), and (-3, 0.5)

Table 3.8: Features of $x = -12y^2$

Feature	Description
Vertex	(0,0)
Direction	Opens to the left (since a=−12<0)
Axis of symmetry	The horizontal line y=0
Focus	$\left(-\frac{1}{48}, 0\right)$
Directrix	$x = \frac{1}{48}$

Description:

This parabola is narrow and opens leftward. Compared to the standard $y = x^2$ which opens upward, this one:

Opens in the horizontal direction

Has its arms getting wider in the vertical direction (y-axis), but the movement is along the x-axis

Has a maximum point to the right of the vertex and minimum x-values as it opens left

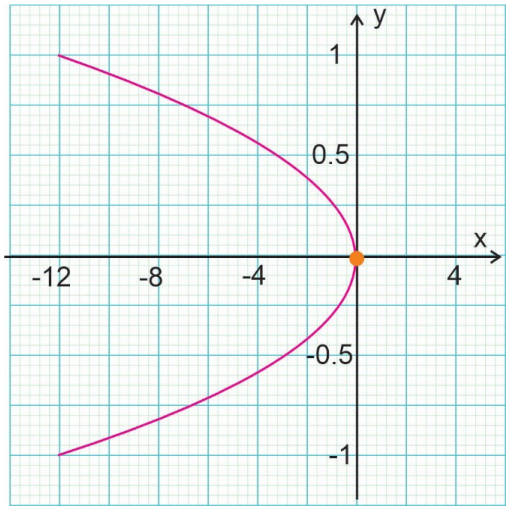


Figure 3.68: Graph of $x = -12y^2$

12.

a. $(y+2)^2 = -8(x+2)$

b. Sketch Description

Your sketch should include:

Vertex: $(-2, -2)$

Focus: $(-4, -2)$

Directrix: x=0

Axis of symmetry: Horizontal line $y = -2$

Orientation: Opens to the left

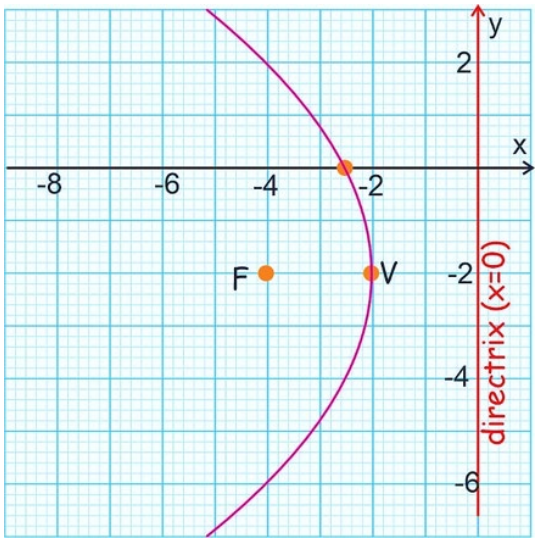


Figure 3.69: Graph of $(y+2)^2 = -8(x+2)$

13.a. $(y-3)^2 = -12(x-2)$

sketch guide:

Vertex: (2, 3)

Focus: (-1, 3)

Directrix: x=5

Axis of symmetry: y = 3

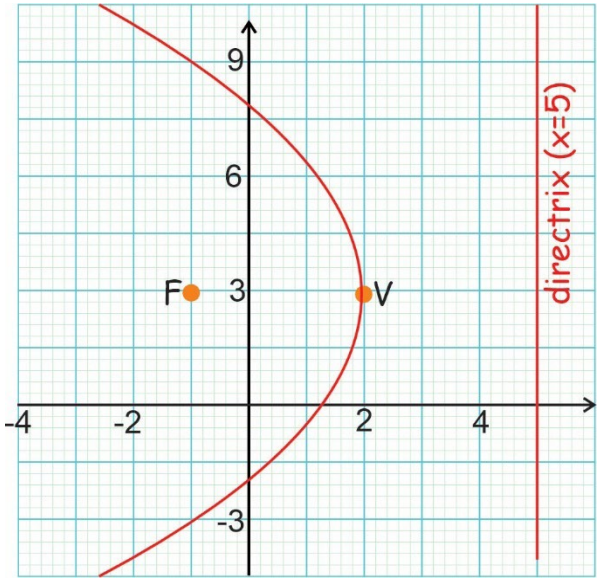


Figure 3.70: Graph of $(y-3)^2 = -12(x-2)$

Table 3.9: Answer summary, Question 14.

	Equation	Vertex	Focus	Directrix
(a)	$y^2 = -7x$	(0, 0)	$\left(-\frac{7}{4}, 0\right)$	$x = \frac{7}{4}$
(b)	$(x-2)^2 = -8(y-5)$	(2, 5)	(2, 3)	y=7
(c)	$(y-1)^2 = -8(x+4)$	(-4, 1)	(-6, 1)	x=−2

14. Table 3.10: Answer summary, Question 15.

	Equation of parabola	p Value	Vertex	Axis of Symmetry	Orientation
(a)	$y^2 = -8x$	p=−2	(0,0)	y=0	Opens leftward
(b)	$(x-2)^2 = -8(y-5)$	p=−2	(2,5)	x=2	Opens downward
(c)	$(y-1)^2 = -8(x+4)$	p=−2	(−4,1)	y=1	Opens leftward

15. a. Sketch guide

Focus: (-3,3)

Directrix: x=0

Vertex: (-1.5, 3)

Direction: Since the parabola opens leftward (focus is left of vertex), it's a horizontal parabola

x-intercept: (-3, 0)

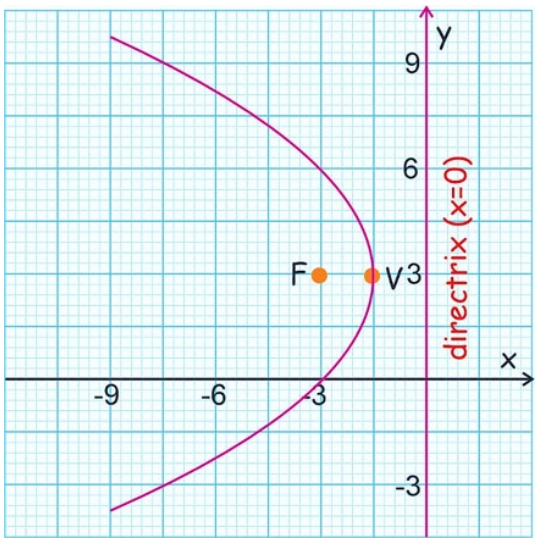


Figure 3.71: Graph of $(y-3)^2 = -6(x+1.5)$

b. $(y-3)^2 = -6(x+1.5)$

16. Sketch guide:

Vertex: (2, –2.5)

Focus: (2, –1)

Focal length $p = 1.5$

Directrix: $y = -4$

Orientation: Opens upward

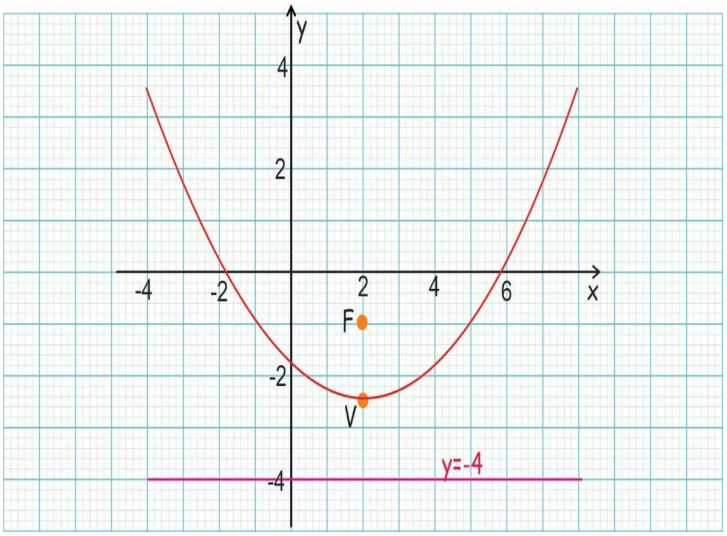


Figure 3.72: Graph of $(x-2)^2 = 6(y+2.5)$

b. $(x-2)^2 = 6(y+2.5)$

17. a. Tangent at P is $x - 2y + 3 = 0$

b. Normal at P is $2x + y - 9 = 0$

c. i. coordinates of Q is (1/3, -1)

ii. PQ $= \frac{4}{3}\sqrt{13}$ units

iii. Tangent at Q is $3x + 2y + 1 = 0$

18. a. The point of intersection is $(-2, 0)$. The equation of the tangent is $y = x + 2$

b. When $y = 8, x = -2$. The equation of the normal is $y = x + 10$

c. (0, 4) and $\left(-\frac{8}{9}, \frac{4}{3}\right)$

19. a. $(x+1)^2 = -6(y+1)$

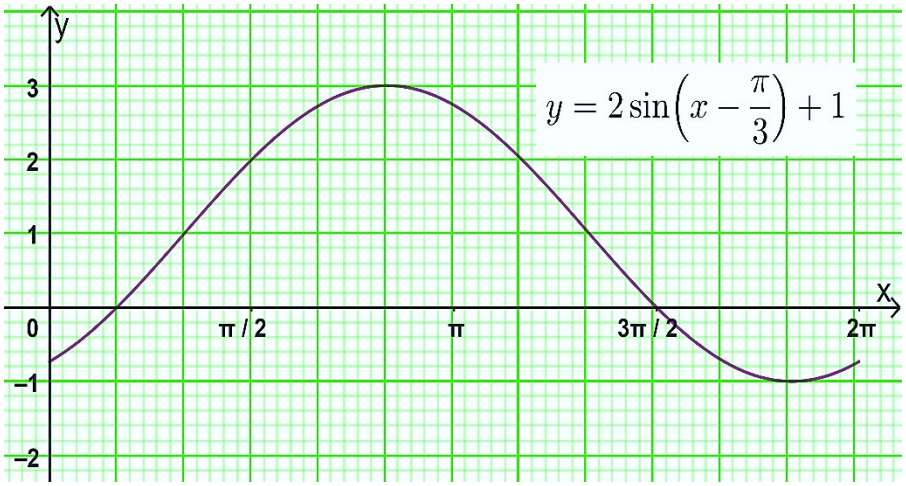
b. intersects the parabola at points (3, -11/3) and (-1, -1)

Since the parabola and line intersect at two distinct points, the line cuts the parabola. The line is a secant.

c. At the y-intercept, the tangent equation is: $y = -\frac{1}{3}x - \frac{7}{6}$

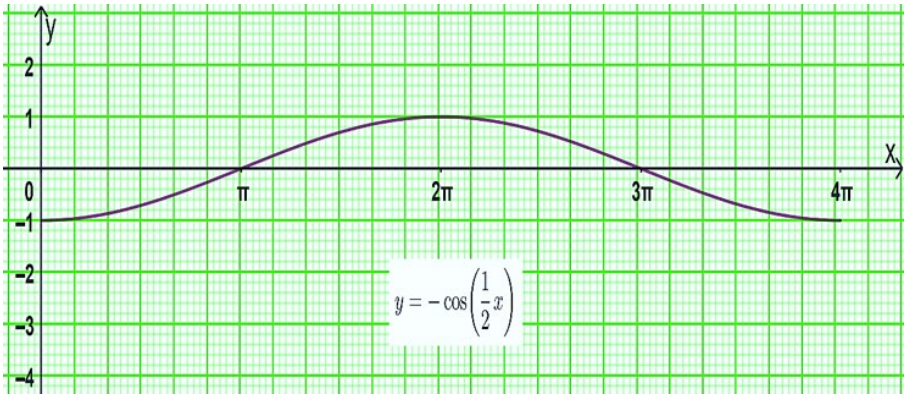
d. The equation of the normal to the parabola at $x = -4$ is: $y = -x - \frac{13}{2}$

ANSWERS TO REVIEW QUESTIONS – SECTION FOUR



Amplitude $|A| = 2$, period $= 2\pi$, phase shift $= \frac{\pi}{3}$, vertical shift $= 1$

There is a maximum value of 3 starting at $\frac{5\pi}{6}$ and repeats every 2π. Also, there is a minimum value of -1 starting at $\frac{11\pi}{6}$ repeating every 2π.



Amplitude $|A| = 1$, Period $= 4\pi$.

There is a maximum value of 1 starting at $x = 2\pi$. Also, there is a minimum value of -1 at $x = 0$ and $x = 4\pi$.

For $y = \sin(x)$, Amplitude is 1, period is 2π, range is $[-1, 1]$.

For $y = \sin(x) + 3$, Amplitude is 1, period is 2π, range is $[2, 4]$ with a vertical shift of 3.

On the unit circle, the sine function is largest at the top $(y = 1)$ and smallest at the bottom $(y = -1)$. In the tide model, the angle $\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$ traces the unit circle as time passes. The amplitude 5 stretches the sine values and the vertical shift 7 moves them up. Therefore, the tide is highest when $\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) = 1$ and lowest when

$\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) = -1$, after applying these transformations.

$\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

Maximum value is $\sqrt{2}$ while the minimum value is $-\sqrt{2}$

$3\cos x - 4\sin x = 5\cos(x + 53.13^{\circ})$

The maximum value is 5.

$4\cos\beta - 7\sin\beta = 8\cos(\beta + 60)$

The least displacement is -8 to the nearest whole number.

Maximum value is 6 while the minimum value is -2.

a. Maximum value is 7 while the minimum value is 1

For $y = 3\sin x$, the maximum value is 3 and minimum is -3.

Compared with $y = 3\sin x + 4$, the maximum and minimum values of $y = 3\sin x + 4$ increase by 4.

Also, the midline shifts from $y = 0$ for $y = 3\sin x$ to $y = 4$ for $y = 3\sin x + 4$.

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

ANSWERS TO REVIEW QUESTIONS – SECTION FIVE

-4.5344

0.6703

1.2696

a. $2e^{-2}$ b. $\ln\left(\frac{2}{3}\right)$

$a = -3$

a) $P(x) = 1.5e^{-0.01x} + 22.59025$

GHC 22.60

$-\frac{1}{3(4+3x^2)} + C$

$\int_0^{\frac{1}{3}} xe^{3x}dx = \frac{1}{9}$

a. $A = 3, B = -2$

b. $3x - 2\ln|x-1| + C$

a.

x	0	0.25	0.5	0.75	1.00
y	0	0.3429	0.6667	1.0286	1.5

b. 0.6976

c. 0.693147

d. The estimated value using the trapezium rule was approximately 0.6976, while the exact value of the integral is $\ln(2) \approx 0.693147$, indicating a small difference that demonstrates the effectiveness of the trapezium rule for this integral. This close approximation suggests that numerical integration methods are useful when exact solutions are challenging to obtain, and increasing the number of intervals could further enhance accuracy.

Distance$\approx$ 9.2 metres

ANSWERS TO REVIEW QUESTIONS – SECTION SIX

$120\ln(6) - 100 = 115.011$ units

$\frac{7}{12} = 0.5833$ units^2

Signed displacement $= 16$; Total distance $= 34$ units

Volume $= \frac{16\pi}{3}$ units^3

$R(x) = \int \left(\frac{200}{x+2}\right) dx = 200\ln(x+2) + C;$

$At\ x = 0, R = 0$

$\Rightarrow C = -200\ln 2$

$Area = 2\sqrt{2}$ units^2

Volume $= 259.4\ L$

Signed Area $= 0$; Total area $= 4$ units^2

Explanation: The signed integral counts areas above the x-axis as positive and areas below as negative for $\cos(2x)$ these cancel exactly over 0 to 2π. The total area uses absolute value, so all "lobes" contribute positively; hence a positive value (4) that measures the actual geometric area between the curve and the axis.

Area $= 0.1354$ units^2

Volume $= 2\pi^2$ units^3

Distance $= 3.7321$ units

Signed area $= -25$; Total area $= 25$ units^2

Signed area $= -\pi$; Total area $= 4.5113$ units^2

Velocity function $(v(t)) = 2t^2 - 6t + 10$; Displacement $= \frac{175}{3} = 58.3333$ units

Volume $= 1.36$ units^3

Area $= 3\ln(3) - 3 \approx 0.2958$ units^2

Net area $= -\ln 2$; Total area $= \ln 2$ units^2

Volume $= \pi[3(ln3)^2 - 6ln3 + 4] \approx 3.2332$ units^3

Area $= \ln(3) = 1.0986$ units^2

Displacement $= \frac{20}{3} = 6.667$ units.

Interpretation: The net displacement over the 5-unit time interval is $\frac{20}{3}$ units in the positive direction. That means the boat's final position is $\frac{20}{3}$ units ahead of its starting position.

ANSWERS TO REVIEW QUESTIONS -SECTION SEVEN

C – Measurements you take during a science experiment.

C – Questionnaire.

B – Getting consent and protecting participants' privacy.

B – Measuring instrument.

Primary data is collected directly from the original source by the researcher while secondary data is obtained from already existing records or publications.

Questionnaire, Interview, Observation checklist, Measuring instrument (e.g., ruler, thermometer), etc.

Importance of defining research purpose before collecting data, for example: Helps determine the type of data to collect.

Ensures selection of appropriate data collection tools and methods.

a. Primary data – because the information will be collected directly from students.

b. Questionnaire, Interview.

c. Advantages

Questionnaire – can reach many students quickly.

Interview – allows for more detailed responses.

a. Measuring instrument (e.g., ruler) and observation checklist.

b. Example description: "Record plant heights every two days using a ruler and note the heights on an observation sheet to track growth."

Correlation measures the strength and direction of the relationship between two variables. It shows how closely the variables move together, but does not provide a predictive formula. The correlation coefficient (r) ranges from -1 to $+$. However, regression describes the relationship with an equation, allowing us to predict one variable based on the other. Regression produces the line of best fit with a slope and an intercept.

Examples of Positive Correlation (both variables increase together):

Height and weight of children (taller children often weigh more).

Number of hours studied and test scores.

Negative Correlation (one variable increases while the other decreases):

Mileage of a car and its selling price.

Number of workers absent and total production output.

Given: $y = 1000 + 15x$

Interpretation:

1000: The base sales amount (GH₵) when no money is spent on advertising.

15: The increase in sales (GH₵) for every extra GH₵1.00 spent on advertising.

Prediction: If GH₵200.00 is spent: Predicted sales: GH₵4,000.00

For example:

Predicting crop yield based on rainfall amount.

Estimating house prices based on size.

Forecasting electricity consumption based on external temperature.

The line of best fit is a straight line drawn through a scatter plot that best represents the relationship between the two variables.

Importance:

It summarises the trend in the data.

It allows predictions for values not directly observed.

It reduces the effect of random variations in the data by averaging them out.

Equation of the line of best fit is $y = 2x + 1$. When $x = 8, y = 17$.

For example:

Economics & Business. Predicting sales based on advertising spending. A company may plot past data (advertising budget vs. sales revenue) and use the line of best fit to forecast future sales if they increase their spending on advertisement.

Healthcare & Medicine. Analysing the relationship between drug dosage and patient recovery rate. Doctors can use regression to determine the optimal drug dosage by studying how different doses affect recovery times in clinical trials.

Education & Psychology. Studying the correlation between study time and exam scores.

Educators can predict a student's expected test performance based on the number of hours they study, helping in academic planning.

Finance: Predicting stock prices based on market trends.

Engineering: Estimating material strength under different stress conditions.

Sports Science: Analysing training hours vs. athlete performance improvements.

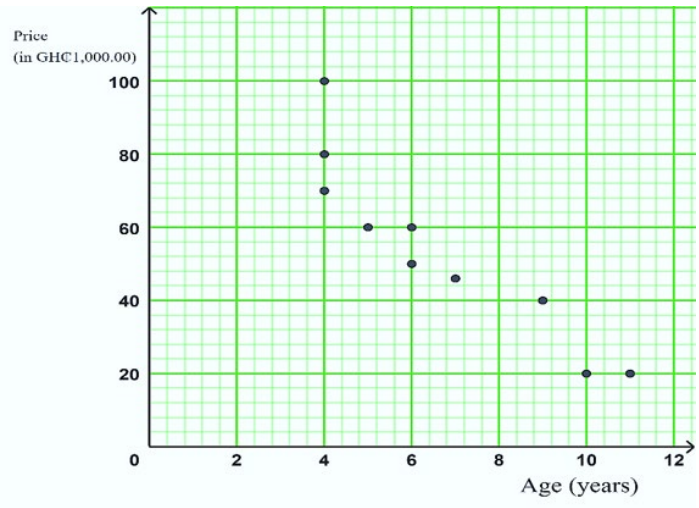


Figure 7.13: Scatter plot of Age of car versus Price

The scatter plot shows a negative correlation such that as the age of the car increases, its price generally decreases. This indicates that older cars are typically less expensive.

Regression equation; $y \approx -8.97x + 113.80$

Thus, Price decreases by about GH₵8 970.00 for each additional year of age.

A brand-new car (age 0) would be predicted to cost about GH₵113 800.00.

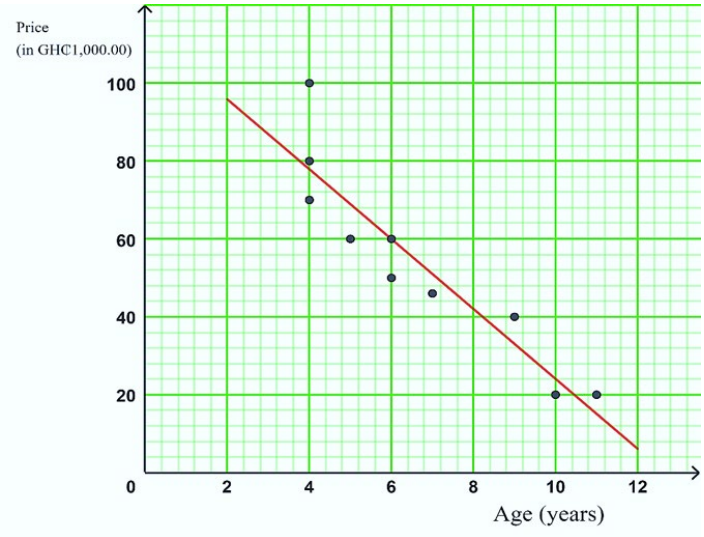


Figure 7.14: Scatter plot of Age of car versus Price with a linear model

Prediction for a 7-year-old car is $\approx$ GH₵51 010.00. Since 7 years falls within the observed data range, the prediction appears reliable.

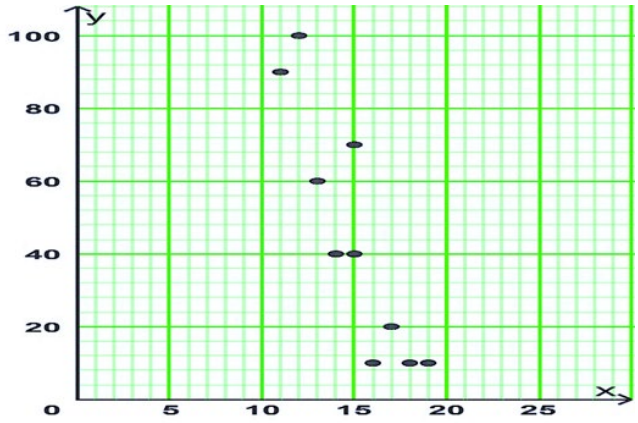


Figure 7.15: Scatter plot of x versus y

As x increases y decreases significantly.

Fit the Regression Model

Gradient $= -11.67$

y – intercept $= 220.05$

Regression equation: $y \approx -11.67x + 220.05$

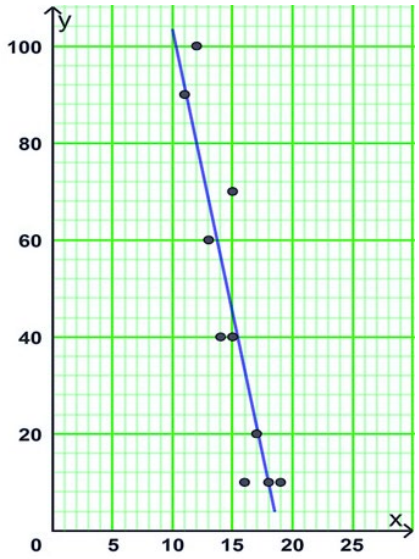


Figure 7.16: Best fit line of x versus y

Gradient (-11.67): For every 1 unit increase in x, y decreases by about 11.67.

Intercept (220.05): When $x = 0$, y is predicted to be about 220.05.

The Correlation Coefficient $r \approx -0.90$. This means that there is a strong negative relationship between x and y.

When $x = 14, y \approx 56.67$. Since $x = 14$ is within the observed data range (11–19), the prediction is reliable.

Step 1: Observation

As drop height increases, bounce height also increases.

The relationship is positive and appears to be close to linear.

Completed table

Table 7.21: Drop height and Bounce height

Drop height (x)	30	60	90	120	150	180	210	240
Bounce height (y)	18	42	60	72	84	108	132	144

Step 2: Fitting the model

The equation of the line of best fit is: $y = 0.537x + 10$

$y = 0.537x + 10$, gradient is 0.537

Interpretation of Gradient:

The gradient, $m = 0.537$, means that for every $1cm$ increase in drop height, the bounce height increases by about $0.54cm$.

Step 3: Drawing the Line of Best Fit

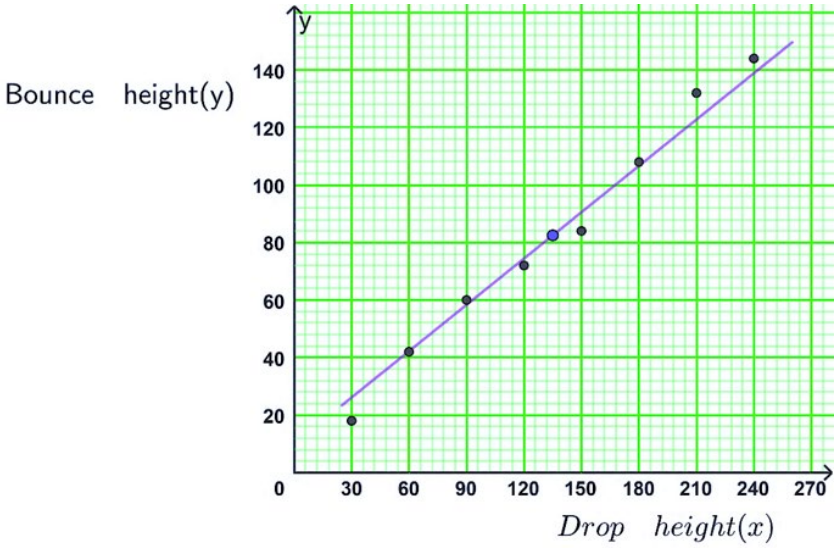


Figure 7.17: Line of best fit of Drop Height versus Bounce Height

The point $(\bar{x}, \bar{y})$ corresponds to $(135, 82.5)$ plotted in blue ink in Figure 7.16.

Step 4: Prediction
Using the linear model,

Bounce height $\approx 116.4cm$ at $198cm$ drop.

Drop height $\approx 149.2cm$ if bounce is $90cm$.

Scatter plot

Scatter plot representing hours studied and physics test scores

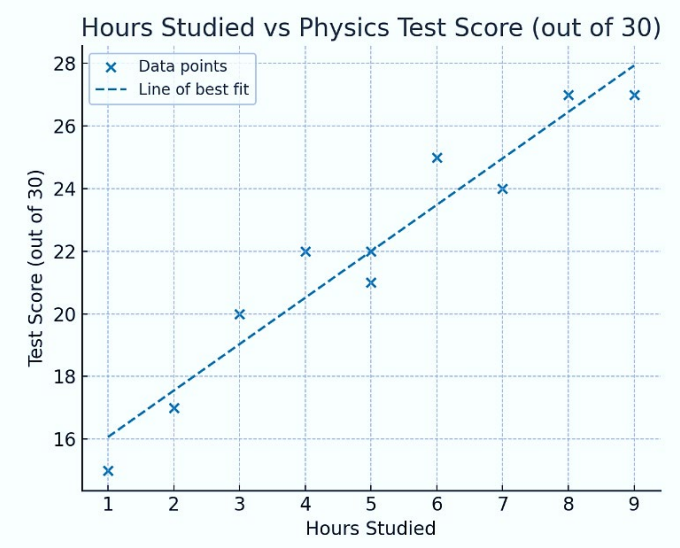


Figure 7.18: Scatter plot and graph of $y = 1.4833x + 14.5833$

Clear **positive** trend: more study hours → higher test scores.

The linear model is $y = 1.4833x + 14.5833$

Interpretation:

Gradient (1.4833): For every extra hour of study there is an increase of about 1.48 marks (on average).

Intercept (14.58): predicted score if a student studied 0 hours (14.6). Treat this as a model baseline, not a recommendation.

The correlation coefficient is $r \approx 0.964$. There is very strong positive relationship.

Since 6 hours lies well inside the observed range (1–9), this is a reliable interpolation.

Screen time and sleep

Scatter plot of Screen time and sleep time

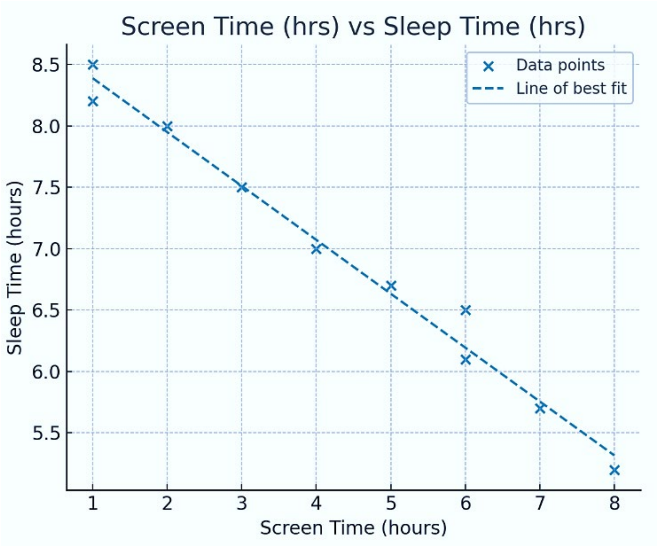


Figure 7.19: Scatter plot and graph of $y \approx -0.4389x + 8.8271$

There is a clear negative trend: more screen time → fewer hours of sleep.

The linear model is $y \approx -0.4389x + 8.8271$

Interpretation:

Gradient (-0.4389): Each extra hour of screen time is associated with 0.44 fewer hours of sleep.

Intercept (8.83): predicted sleep when screen time is 0 hours.

Correlation Coefficient

$r \approx -0.992$ shows a very strong negative relationship.

Prediction and Reliability:

3 hours is within the observed range (1–8), this is a reliable interpolation.

22. Project Rubric

Table 7.22: Rubric for project.

Criteria	Excellent (4 marks)	Very good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 marks)
1. Topic Selection & Relevance	The topic is clear, realistic, and relevant; it allows easy data collection	The topic is appropriate but slightly broad or lacks clarity	The topic is chosen, but not clearly related to real-world variables	The topic is unclear or not suitable for bivariate data
2. Data Collection	Data is accurate, complete, and from credible sources	Data is mostly complete and reliable	Some data is missing, or source reliability is unclear	The data is insufficient or unrelated
3. Data Representation	Charts/graphs are clear, accurate, and appropriate for the data	Visuals are mostly appropriate but may lack clarity or labelling	Visuals are present but unclear, mislabelled, or poorly chosen	Charts/graphs are missing or unrelated to the data
4. Data Analysis	Analysis clearly identifies patterns, trends, and relationships	Some patterns or trends are identified, but with limited explanation	Basic analysis with minimal understanding shown	No meaningful analysis provided
5. Conclusion & Implications	Findings are well explained with thoughtful implications or solutions	Findings are clear, but implications are basic or general	Basic summary given with little insight or relevance	The conclusion is missing or unrelated to the findings
6. Presentation & Organisation	The presentation is well-organised and clear. All required components are present. Student articulates findings clearly.	The presentation is well-organised, but it is missing a component or lacks clarity in certain areas. Students can articulate findings.	The presentation is well-organised, but it is missing some components or lacks clarity in certain areas. Students may struggle to articulate findings.	Presentation is disorganised, unclear, or missing several key components. Student struggles significantly to articulate findings.

Grading Scale:

19 – 24 marks: Excellent

13 – 18 marks: Very Good

7 – 12 marks: Satisfactory

Below 7 marks: Needs Improvement

ANSWERS TO REVIEW QUESTIONS – SECTION EIGHT

1. (a) $\frac{2}{3} = 0.6667$ (b) $\frac{4}{9} = 0.4444$.

2. (a) $\frac{1}{3} = 0.3333$ (b) 1

3. (a) 0.30 (b) $\frac{6}{11} = 0.5455$

4. (a) 0.8421 (b) Interpretation: If a student brings an umbrella, there is about an 84.2% chance that it is raining — the umbrella is a strong indicator of rain.

5. (a) $\frac{2}{7} = 0.2857$ (b) $\frac{5}{7} = 0.7143$

6. (a) 0.3125 (b) 0.34375

7. (a) 0.3087 (b) 0.03087

8. (a) 0.23013 (b) 0.97436

9. (a) 0.193536 (b) 0.096256

10. (a) 0.23347 (b) $0.1502684 \approx 0.1503$

11. (a) 0.1001129 (b) 0.9999847

12. 12. 24

13. **60**

14. The system can generate 6 760 codes (with repetition allowed) or 6 500 codes (without repetition).

15. (a) 168 ways (b) 495 ways (c) 792 ways (d) 35 ways (e) 10 ways

16. (a) 150 (b) 671 (c) 126

17. 17. (a) 120 (b) 285 (c) 126

18. (a $n = 10, r = 2$, answer 1

(b)$n = 29, r = 9$, answer 84

(c) $n = 10, r = 4$ answer 6

(d) There is no integer solution as $r = \frac{44}{5}$

19. 19 107

20. 63

21. 840

22.60

23. 24

24. 120

25. 336

26. 5040

27. 720

28. 1320

29. 120

30.. 3600

31. 90720

32.(a) 90720 ways

(b) First select the students using combinations, since the order of selection doesn't matter. After selecting the 5 students, we need to arrange them in a line on the stage. The order of performance matters, so we use permutations to find the number of ways to arrange the chosen group.

33.(Type + Solution)

Combination: 455

Permutation: 358 800

Permutation: 4 896

Combination: 792

Permutation: 720

Combination: 15

Permutation: 120

Combination: 56

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GLOSSARY

Amplitude	Half the vertical distance between the peak and trough of a wave; measures how far the graph rises/falls from its midline.
Antiderivative	A function whose derivative is the given function.
Axis of Revolution	Fixed line around which a region is rotated to form a solid.
Axis of Symmetry	Line through the vertex dividing a parabola into two mirror halves.
Binomial Probability	Probability distribution for experiments with two outcomes (success/failure).
Combination	Selection of objects without regard to order.
Compound Statements	Mathematical statements formed by combining simple ones with logical connectives.
Conditional Probability	Probability of one event given another has occurred.
Connectives	Words/symbols joining statements to form compound statements.
Correlation	Statistical measure of strength/direction of relationship between variables.
Cross-Section	2D slice obtained by cutting a solid perpendicular to an axis.
Data Analysis	Examining data to identify patterns or trends.
Decision-Making	Using analysis findings to support choices or predictions.
Definite Integral	Calculates accumulated area/volume between bounds; yields a number.
Dependent Events	Events where one outcome affects the other.
Determinant	Scalar value of a square matrix; indicates invertibility.
Dilation	Transformation enlarging/reducing a shape while keeping proportions.
Directrix	Fixed line used in parabola definition.
Disc Method	Technique for finding volumes of solids of revolution using discs.
Event	Subset of sample space (e.g., rolling an even number).
Experiment	Activity producing an outcome.
Factorial	Product of first n positive integers (n!).
Focal Length	Distance from vertex to focus/directrix of parabola.
Focus	Fixed point used in parabola definition.
Forecasting	Using data/statistics to predict future outcomes.
Frequency	Number of cycles per unit interval.
Geometric Transformation	Change in position, size, shape, or orientation of a figure.
Image	Result of a transformation applied to a point/figure.
Indefinite Integral	Integral without limits; represents a family of functions.
Independent Events	Events where one outcome does not affect the other.

Integration	Process of finding the integral of a function.
Interpretation	Explaining meaning of data findings.
Intersection Point	Coordinates where two curves meet.
Interval	Range of values over which a region is measured.
Line of Best Fit	Straight line representing data on a scatter plot.
Linear Transformation	Mapping between vector spaces preserving addition and scalar multiplication.
Logarithmic Function	Function of form log_b(x); inverse of exponential.
Logical	Reasoning based on valid principles of thought.
Mathematical Statements	Declarative sentences that are true or false.
Matrix	Rectangular array of numbers/expressions.
Maximum or Minimum Value	Vertex gives max/min depending on parabola orientation.
Mutually Exclusive Events	Events that cannot occur simultaneously.
Negative Correlation	Increase in one variable linked to decrease in another.
Normal to a Parabola	Line perpendicular to tangent at parabola point.
Orientation	Direction parabola opens (up, down, left, right).
Outcome	Possible result of an experiment.
Parabola	Set of points equidistant from focus and directrix.
Period	Horizontal distance for one full cycle.
Permutation	Arrangement of objects in specific order.
Phase Shift	Horizontal displacement of a periodic function.
Planar Region	2D area drawn on a flat surface.
Positive Correlation	Increase in one variable linked to increase in another.
Power Rule	Rule for integrating functions of form x^n.
Prediction	Estimating future/unknown value using data.
Probability of an Event	Measure of likelihood of an event.
Radius Function	Distance from axis of revolution to curve.
Reflection	Transformation flipping a shape over a line.
Region	Planar area enclosed by curves/axes.
Regression	Statistical method predicting one variable from another.
Rotation	Transformation turning a shape around a point.
Sample Space	Set of all possible outcomes.
Scale Factor	Number describing enlargement/reduction in dilation.
Statistical Model	Mathematical representation of a real-world situation.
Substitution	Technique simplifying integrals by changing variables.

Tangent to a Parabola	Line touching parabola at one point with same slope.
Transcendental Function	Function not expressible as finite algebraic operations.
Transformation	Operation changing graph position/size/orientation.
Transformation Matrix	Matrix representing a linear transformation.
Translation	Transformation moving a shape a constant distance.
Vector	Quantity with magnitude and direction.
Vertex	Turning point of parabola.
Volume of Revolution	3D space formed when a region is revolved around an axis.

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This book is intended to be used for the Year Three Additional Mathematics Senior High School (SHS) Curriculum. It contains information and activities to support teachers to deliver the curriculum in the classroom as well as additional exercises to support learners' self-study and revision. Learners can use the review questions to assess their understanding and explore concepts and additional content in their own time using the extended reading list provided.

All materials can be accessed electronically from the Ministry of Education's Curriculum Microsite.

[Scan the QR Code to visit the website]

Ghana Education Service (GES)

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