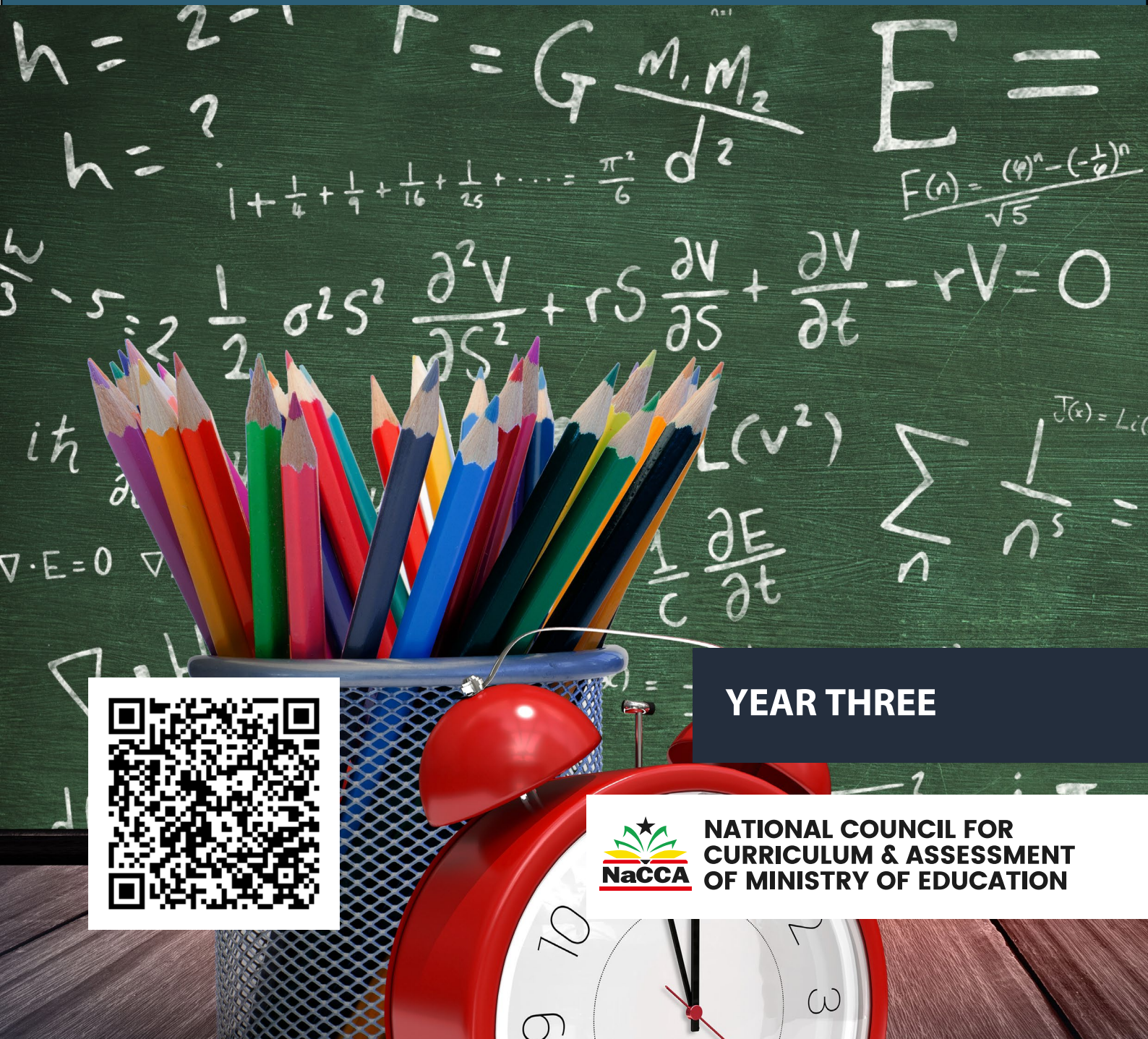
MINISTRY OF EDUCATION

Additional Mathematics

For Senior High Schools

TEACHER MANUAL



YEAR THREE

NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

MINISTRY OF EDUCATION



REPUBLIC OF GHANA

Additional Mathematics

For Senior High Schools

Teacher Manual

Year Three

NaCCA NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

Additional Mathematics Teacher Manual

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Introduction for the Year Three Teacher Manual

The National Council for Curriculum and Assessment (NaCCA) has developed the third and final set of curriculum materials to support the implementation of the new Senior High School curriculum.

This Year Three Teacher Manual for covers all aspects of the content, pedagogy, teaching and learning resources and assessment required to effectively support learners in their final year of Senior High School. All the necessary information for the Student Assessment Portal has been included in the Year Three Teacher Manual to simplify the development of weekly learning plans, meaning that teachers have a single reference document.

The Year Three Teacher Manual has two main objectives:

- To support learners to consolidate and deepen their understanding of the content, pedagogy, and assessment across all three years of the new Senior High School curriculum in preparation for the West Africa Senior Secondary Certificate Examination (WASSCE).
- To equip teachers with the tools and resources needed to deliver revision-focused, assessment-led lessons that build on the knowledge learners have acquired from Year One to Year Three.

In this third year, teachers will be supporting learners to prepare for the WASSCE. Each section of the Teacher Manual is therefore structured around the highest Depth of Knowledge (DoK) levels, allowing learners to build progressively on knowledge acquired from Year One to Year Three. For each week, learning tasks and pedagogy are built around the assessment, ensuring that classroom activities directly prepare learners for the demands of the examination. Two sets of sample examination papers are also provided; one administered by Week 12 as a first semester practice examination, and one by Week 20 as a mock examination, to gauge learners' competencies and identify areas requiring remediation.

To further support teachers in delivering Year Three, NaCCA, working with the Ghana Education Service, has developed a subject specific App for each subject.

The Year Three PLC sessions have been revised in response to findings from the 2025 annual evaluation survey. The survey confirmed that teachers have made strong progress in lesson planning, use of varied instructional methods, ICT integration and modes of assessment. To address the identified gap in explaining concepts clearly using examples familiar to learners, each PLC session now includes a dedicated Content Exploration segment. This ten-minute segment brings the subject group together to work through one challenging problem or concept from the week's Teacher Manual content, not to plan how to teach it, but to ensure that every teacher in the group fully understands it.

The PLC sessions in Year Three also place particular emphasis on connecting the week's content to equivalent topics taught in Years One and Two. Teachers are encouraged to note the recap weeks signposted throughout the Teacher Manual and to use the App to explore how content across all three years links together.

As in previous years, teachers are expected to integrate National Values, Gender Equality and Social Inclusion (GESI), Social and Emotional Learning (SEL), and 21st-century skills into their learning activities, assessment tasks, and rubrics and mark schemes. The subject-specific App will support teachers in doing this throughout the year.

Ghana Education Service acknowledges the dedication of the teachers who have implemented the new curriculum across Years One and Two and recognises that Year Three represents both the culmination of that effort and a critical moment for learners. The Year Three Teacher Manual and its accompanying resources have been developed to ensure that every learner, regardless of background, ability, or school context, is fully prepared to demonstrate what they know and can do in the WASSCE and to progress confidently to further studies, the world of work, and adult life.

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SECTION 1: LOGIC AND ITS APPLICATION

STRAND: MODELLING WITH ALGEBRA

Sub-Strand: Applications of Algebra

Learning Outcome: *Construct compound statements and truth tables using connectives*

Content Standard: *Appreciate the concept of logic and apply the concept to draw valid conclusion and deduction from arguments*

INTRODUCTION AND SECTION SUMMARY

Logic extends beyond abstract mathematics to form the foundation of rational thinking in countless practical contexts. This section examines the building blocks of logical reasoning, namely, simple statements, compound statements, truth tables, and argument validity as essential tools for navigating today's complex information landscape. Logical analysis serves as a practical framework for solving problems, making sound decisions, and communicating precisely. In our era of information overload, distinguishing valid arguments from flawed reasoning has become crucial for both personal and professional success. These logical concepts function across diverse fields: lawyers build cases through chains of reasoning; medical professionals use conditional reasoning (if-then structures) to move from symptoms to potential causes; and computer scientists employ Boolean logic in programming and AI development. In our daily lives, we apply these principles when weighing options and planning ahead. Understanding formal logical structures improves our thinking, helps us avoid fallacies, and strengthens our decision-making. By mastering these concepts, we develop transferable skills that enhance critical thinking and communication; essential abilities in our data-driven world where logical reasoning remains our most reliable tool for finding truth.

The weeks and the indicators covered by the section are:

- *Week 1*

1. Form statements including negation statements and comment on them
2. Draw implications from given statements and their converses

- *Week 2*

1. Identify simple statements and construct compound statements using the connectives.
2. Use conjunctions, disjunctions, implications and negations to construct truth tables of compound statement

Recap weeks

Year 1	**Week 2:** Establish the properties of operations on set, including commutative, associative, and distributive, sets algebra and apply them to solve problems.
Year 2	**Week 1:** Establish De Morgan's laws of set theory and use it to solve related problems

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires critical thinking and logical reasoning hence implements a multi-faceted approach to teaching logical reasoning through practical engagement and differentiated instruction. Learners will explore logical applications through individual analysis, debate construction, and team presentations. For individual work, learners will analyse real-world scenarios requiring logical reasoning, identify simple and compound statements, and construct truth tables to verify argument validity. The debate component engages learners in constructing logical arguments on contemporary issues, requiring them to identify premises, formulate implications, recognise fallacies, and build coherent chains of reasoning. Team presentations will challenge mixed-ability groups to apply logical reasoning to fields such as education, computer programming, legal reasoning or medical diagnosis. Each team member contributes according to their strengths with some focusing on concrete examples, others on symbolic representation, ensuring inclusive participation while maintaining rigor. Throughout these activities, assessment will track individual progress while emphasising collaborative problem-solving and the practical application of logical concepts to real-world situations.

ASSESSMENT SUMMARY

Table I contains the key assessment for each indicator in the section. The highest level of DOK for the indicator is denoted. The mode of assessment to be carried out and strategies that will aid in achieving the indicators are documented.

Indicators	DOK Level	Assessment Strategy and/or Mode
Form statements including negation statements and comment on them	3	Strategy: Peer/Self-assessment (in pairs) Mode: Classwork
Draw implications from given statements and their converses		
Identify simple statements and construct compound statements using the connectives.	3	Strategy: Debate Mode: Classwork
Use conjunctions, disjunctions, implications and negations to construct truth tables of compound statement		

WEEK 1

Learning Indicators

1. *Form statements including negation statements and comment on them*
2. *Draw implications from given statements and their converses*

FOCAL AREA 1: FORMING MATHEMATICAL STATEMENTS

A statement is a sentence that expresses a complete thought and can be clearly classified as either true or false, but not both at the same time.

Let us group the following expressions into statements and non-statements

a. Mathematics is easy.

b. All natural numbers are integers.

c. Add 16 to both sides of the equation.

d. Prime numbers are odd numbers.

e. π is a rational number.

Careful consideration will reveal that the expressions "a" and "c" are not statements because we cannot determine whether they are true or false. Mathematics being easy or not is heavily dependent on the individual you ask. Also, saying "Add 16 to both sides of the equation" is just given an instruction and the expression itself holds no truth value.

However, expressions "b", "d" and "e" are statements because we can determine whether they are true or false. The expressions "b" and "d" are true while "e" is false. There are established proofs to support the truth values of these statements.

Mathematical statements

A mathematical statement has a definite state of being either *True* or *False*. It cannot be both True and False. The statement may contain symbols or words.

Example 1

Determine whether the following statements are true or false.

a. $\sqrt{5}$ is rational

b. $\pi + 3 \geq 5$

Solution

a. False

b. True

Notation of statements

Mathematical statements are denoted by capital letters such as M, N, R, Q, etc.

T: $3 + 8 = 13$, simply means T is the statement $3 + 8 = 13$. T is obviously a false statement because $3 + 8 \neq 13$

Negation of a statement

Suppose that B is a statement. Then the negation of B, denoted by $\sim B$, is the statement claiming the opposite truth value to B. That is, $\sim B$ is false when B is true, and $\sim B$ is true when B is false.

Double Negation

Since the truth value of B is the opposite of the truth value of $\sim B$, then the effect of taking double negation, that is $\sim(\sim B)$, will be same as that of B. In this situation, where B and $\sim(\sim B)$ are not identical as statements, but they do have the same truth values for all possible choices of statements B, we say that B and $\sim(\sim B)$ are logically equivalent, or that B is logically equivalent to $\sim(\sim B)$and use the notation $\sim(\sim B) \equiv B$.

Example 2

Let us take a simple example of the negation of a statement:

Q: Cape Coast is the capital of the Central Region in Ghana.

$\sim$Q: Cape Coast is **not** the capital of the Central Region in Ghana.

Denial of a statement refers to forming a new statement that asserts that the original statement is **not** the case. It usually transcends to the negation of a statement. The defining property of a negation is that it reverses the truth value of the original statement.

With normal intuition, denial and opposite statements may seem the same but are not so in mathematical logic. For instance, considering the statement "$m < n$", intuitively the opposite may be "$m > n$" because we may think of the opposite of less than$(<)$ as greater than$(>)$. This is erroneous as it does not consider cases where $m = n$. In order to avoid such mistakes, we make use of denial (negation) statements which cover all bases.

In our example, the denial statement for "$m < n$", will be "$m \geq n$" (i.e., **Statement 1**: m is less than n. **Statement 2**: m is not less than n).

Statements and Venn diagram

Statements can be expressed in forms of sets and represented on Venn diagrams.

Example 3

1. Identify the Universal set, Subsets and Complements from the given statements:

   P: Yayra is a General Science student.

   Q: 13 is a factor of 26, but 4 is not.

2. Represent each of the statements P and Q on a Venn diagram.

Solution

U = {All students}

S = {Science students}

y = Yayra

$y \in S$

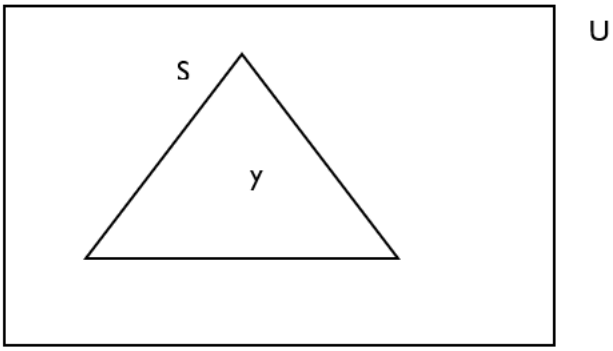


Figure 1.1: *Venn diagram representation of statement P*

i. U = {Integers}

G = {factors of 26}

13 ∈ factors of 26

G^{I} = {not factors of 26}

4 ∈ not factors of 26

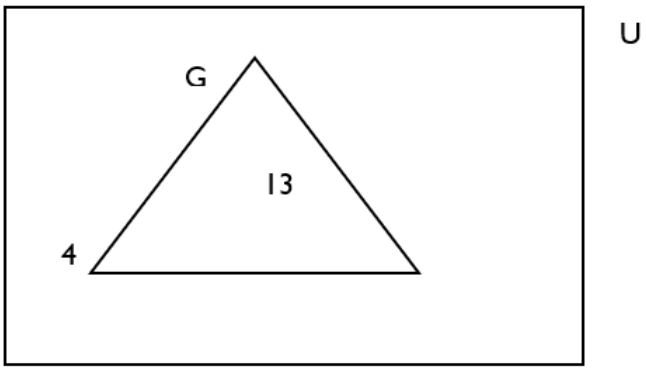


Figure 1.2: *Venn diagram representation of statement Q.*

Identifying simple statements

A simple statement is a basic declarative statement with **no** logical connectives such as AND, OR, IF – THEN, NOT and IF AND ONLY IF.

Simple statements cannot be broken down into smaller sentences. They express a single, complete thought.

For instance, "the sum of interior angles in a quadrilateral is 360^0" is a simple statement.

Example 4

1. Determine whether the following are statements or not.
   a. D: James always eats rice or yam for dinner.
   b. G: Zebras have black and white stripes.
   c. P: Mavis is a girl.
   d. Q: Leaders are born not made.
   e. R: All perfect squares are positive.

f. V: Visual art students appreciate graphic design works.

g. W: 25 is a prime number.

2. For those you claim to be statements in (1), show which ones are True or False.
3. Which of the statements identified in (1) is a simple statement?

Solution

1. All of them are statements **except** D.
   a. P: Mavis is a girl. (True)
   b. Q: Leaders are born not made. (False)
   c. V: Visual art students appreciate graphic design works. (True)
   d. W: 25 is a prime number. (False)
   e. G: Zebras have black and white stripes. (True)
   f. R: All perfect squares are positive. (True)
2. Statements P, V, W and R are simple statements because there are no logical connectives in the statements.

Compound Statements

A compound statement is formed by combining two or more simple statements using logical connectives. In mathematical logic, these are fundamental to constructing complex logical expressions.

Some of the connectives are Conjunction (And), Disjunction (Or), Implication (If -Then) and Negation (Not).

Example 5

a. 4 is an even number and 9 is an odd number

b. 3 is an odd number or 18 is an even number

c. If a real number is divisible by 6, then it is divisible by 3.

FOCAL AREA 2: DRAWING IMPLICATIONS FROM GIVEN STATEMENT AND THEIR CONVERSES

Implicative statement

An implicative statement is a type of compound statement that establishes a conditional relationship between two simple statements using the "if-then" structure. This is a fundamental concept in mathematical logic.

Suppose A and B are statements, an implicative statement connecting A and B can be written as "if A then B".

Symbolically, an implication can be written as $A \Longrightarrow B$.

A and B are referred to as the hypothesis and conclusion respectively.

Example 1

1. If a quadrilateral has four equal sides, then it is a rhombus.

   Symbolically, the statement can be expressed as:

   A quadrilateral has four equal sides $\Longrightarrow$ It is a rhombus

2. If two angles are complementary, then their sum equals 90°

   Symbolically; two angles are complementary $\Longrightarrow$ their sum equals 90°

3. If $y^3 = 64$ then $y = 4$.

   Symbolically; $y^3 = 64 \Longrightarrow y = 4$

Example 2

- A mathematics club is planning an event. The club treasurer makes this statement: *"If we charge at least GH₵15.00 per ticket, then we will have enough funds for both refreshments and prizes."*

  a. Express this statement in symbolic form using variables.

  b. Write the negation of the original statement in:

     i. symbolic form

     ii. words

Solution

a. Let p: charges at least GH₵15.00 per ticket

   Let v: have enough funds for both refreshment and prizes

   Symbolically, $p \Longrightarrow v$.

   i. $\sim(p \Longrightarrow v)$

   ii. We charge at least GH₵15 per ticket but we do not have enough funds for both refreshments and prizes.

Converse of an implication

For converse of an implication, there is a switch of hypothesis and conclusion statements.

Thus, the converse for "If a quadrilateral has four equal sides, then the quadrilateral is a rhombus" will be

"If a quadrilateral is a rhombus then the quadrilateral has four equal sides".

Symbolically, the converse will be:

A quadrilateral has four equal sides $\Longleftarrow$ a quadrilateral is a rhombus

Example 3

Write the converse of "if it is an equilateral triangle then the triangle has three equal sides" symbolically and explain.

Solution

Symbolically: It is an equilateral triangle $\Leftarrow$ a triangle has three equal sides.

This means that if a triangle has three equal sides, then it is an equilateral triangle.

The Chain-rule of implication

The chain rule of implication is a logical principle that allows us to connect multiple implications to form a new valid implication. For instance, given the statements:

M: If it rains then the compound gets wets

N: If the compound gets wet, then it becomes muddy.

The chain-rule of implications will result in the statement; If it rains then the compound becomes muddy.

Prompt: Assign debate topic and rubrics ahead of week 2

Learning Tasks

1. Bring out sentences for learners in pairs to determine if they are statements and share whether the statements are true or false.
2. Guide learners to discuss and construct negation of a statement in pairs and share with the larger class for comments.
3. Guide learners to discuss the difference between the opposite of a statement and the denial of a statement
4. Guide learners to discuss the representation of statements in Venn diagrams and the concept of implication.
5. Guide learners to discuss the chain rule of implications.

Pedagogical Exemplars

1. Review previous learning
2. Experiential learning – guide learners in pairs to create statements and determine whether the statements are mathematical or not.
3. Talk for learning – Provide the opportunity for learners to investigate and establish the real-world applications of mathematical statements through the use of Venn diagrams and algebraic analysis. Guide learners to work in pairs, where they can brainstorm, discuss, and explain their understanding of these concepts.

Key Assessment Level 3: Classwork

1. A school observes that:

   "If students use the school's Wi-Fi during lessons, then their devices must be registered with the school."

   a. Write the inverse and converse of this statement. **(2 marks)**

   b. Analyse whether the converse statement must be true if the original statement is true.

(2 marks)

Total (4 marks)

2. During a public health campaign, these statements were made:

   G: "If a person exercises regularly, then their risk of heart diseases decreases"

   K: "If a community improves access to fresh foods, then the obesity rates decline".

   a. For each statement identify the:

      i. hypothesis

      ii. conclusion

      iii. converse statement

   (6 marks)

   b. For statement G,

      i. analyse whether both the original and its converse statement are likely to be true.

      (2 marks)

      ii. if both the original and converse statements are true, how does it affect public health recommendations?

      (1 mark)

      iii. if only the original statement is true, how does it affect public health recommendations?

      (2 marks)

Total (11 marks)

PLC Session 1

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).

Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***classwork***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the classwork (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).

8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
    a. transfer your chats into your learning planner template.
    b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
    c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 2

Learning Indicators

1. *Identify simple statements and construct compound statements using the connectives*
2. *Use conjunctions, disjunctions, implications and negations to construct truth tables of compound statement*

FOCAL AREA 1: IDENTIFY SIMPLE STATEMENTS AND CONSTRUCTING COMPOUND STATEMENTS USING THE CONNECTIVES

Forming Compound statements using connectives

A compound statement is a statement composed of several individual statements, each of which is called a component statement. Each component statement can be true or false. Compound statements contain logical connectives such as AND, OR, IF – THEN, IF AND ONLY IF. Implicative statements studied in week 1 are examples of compound statements.

Example 1

Form compound statements using the logical connectives from the simple statements

V: They go to class regularly.

W: They passed the examinations.

Solution

1. They go to class regularly and they passed the examinations.
2. They go to class regularly or they passed the examinations.
3. If they go to class regularly then they passed the examinations.
4. They go to class regularly if and only if they passed the examinations.

Examples of compound statements

Identify the logical connectives in the following compound statements

1. Zebras have black and white stripes.
2. It is cold or it is sunny.
3. 13 is an odd and prime number.

Solution

1. And
2. Or

3. And

FOCAL AREA 2: CONSTRUCTING TRUTH TABLES OF COMPOUND STATEMENTS

Types of compound statements and their connectives

1. Conjunction statements (And): The compound statement of conjunction uses the connective 'AND' for connecting two simple statements. For this compound statement, both statements must be true for the compound statement to be true. The two simple statements B and E can be connected using the 'And' connective, and the compound statement can be written as B $\wedge$ E. For a conjunction compound statement, both simple statements should be true for the compound statement to be true.

Example 1

a. $2 + 6$ is 11 and $8 - 3$ is 4

Symbolically written as $2 + 6$ is $11 \wedge 8 - 3$ is 4

b. Ripe mangoes are orange in colour and ripe bananas are yellowish

2. Disjunction statements (Or): The connective used for two simple statements to form a compound statement, which is disjunction, is 'OR.' In a disjunction statement, any one of the statements must be true for the disjunction statement to be true. The two simple statements represented as B and E can be connected using OR connective and are written as B $\vee$ E. Here, any of the two statements should be true for the compound statement to be true.

Example 2

a. $3 + 7$ is 10 or $9 - 3$ is 4

Symbolically written as $3 + 7$ is $10 \vee 9 - 3$ is 4

b. Water is colourless or ripe bananas are yellowish

3. Conditional Statement (If-then): The connective used for a conditional statement is if - then. If Hanson does well in the test, then he will be promoted to the next class. Here, the first statement, B, can be taken as the hypothesis, and the second statement, E, can be taken as the conclusion. We can write condition statements of these two simple statements B, E as If B then E. The conditional compound statement does not hold true if the hypothesis is true and the conclusion is false. But in all other situations, the conditional statement is true.

Example 3

a. If Dzifa is bitten by a mosquito, then she will suffer from malaria.

Dzifa is bitten by a mosquito $\Longrightarrow$ she will suffer from malaria.

b. If you study for long hours then you will pass your examinations.

4. Bi-conditional Statement (If and only if)

The bi-conditional statement uses the connective 'If and only if,' which is represented by the symbol $\Longleftrightarrow$.The two statements, B and E, are represented as a compound statement, B $\Longleftrightarrow$ E, and here, the first statement, B, is called the hypothesis (antecedent), and the second statement, E, is

called the conclusion (consequent). Here, the bi-conditional compound statement is true if both statements are either true or both are false.

Example 4

a. Inter Milan will win the match against Barcelona if and only if Davide Frattesi scores the fourth goal.

b. A number belongs to the set of prime numbers if and only if it has exactly two factors,1 and the number itself.

Let P: A number belongs to the set of prime numbers

Let F: The number has exactly two factors,1 and the number itself

Symbolically, $P \iff F$

Truth table of a compound statement

The truth value of a compound statement depends on the truth value of the individual statements and also on the connective used to form the compound statement.

Conjunction Truth Table uses the connective 'and' to form the compound statement. Here, the compound statement is true only if both the individual statements are true. Even if one of the individual statements is false, then the compound statement is considered a false statement.

Given the statements B and E, each of the statement can be either true of false. Four different pairings (orderings) can be generated as shown in the table below;

Table 2.1: *Pairings for two statements B and E.*

B	E
T	T
T	F
F	T
F	F

Table 2.2: *Truth table for* $\boldsymbol{B \wedge E}$

B	E	$\mathbf{B \wedge E}$
T	T	T
T	F	F
F	T	F
F	F	F

For row 1 in Table 2.2, both statements are true. Since "AND" requires both to be true, the result is true. For row 2, statement B is true but statement E is false. However, "AND" needs both statements to be true resulting in a false result. In summary, the reasons for the compound statements in rows 3 and 4 being false is because one or both of the statements is false.

For three statements B, E and N, the sample space is TTT, TTF, TFT, FTT, FFF, FFT, FTF, TFF

Table 2.3: *Sample space for three statements B, E and N.*

B	E	N
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Table 2.4: *Truth table for* $B \wedge E \wedge N$

B	E	N	$B \wedge E \wedge N$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Example 5

If p and q are two simple statements such that p is true and q is false, construct the truth table for

a. $\sim p \wedge \sim q$

b. $\sim(p \vee \sim q)$

Solution

Table 2.5: *Truth table for* $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	F	F	T	F

Table 2.6: *Truth table for* $\sim(p \vee \sim q)$

p	q	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$
T	F	T	T	F

Example 6

Show that $\sim p \vee \sim q$ is equivalent to $\sim(p \wedge q)$

Solution

Table 2.7: *Truth table for* $\sim(p \wedge q)$ *and* $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	

Columns 6 and 7 have same entries showing that $\sim p \vee \sim q$ is equivalent to $\sim(p \wedge q)$.

Example 7

Show that the operation $\vee$ is distributive over $\wedge$.

Solution

Given three statements p, q and r, $\vee$ is distributive over $\wedge$ if $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Table 2.8: *Truth table showing* $\vee$ *is distributive over* $\wedge$

T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

Disjunction Truth Table uses the connective 'or' to form the compound statement. Here, even if one of the individual statements is true, then the compound statement also holds true.

Table 2.9: *Truth table for* $B \vee E$

B	E	$B \vee E$	Reasoning
T	T	T	Both statements are true. "OR" only needs at least one true statement thus the result is true.
T	F	T	Since at least one of the statements is true, the resulting statement is true
F	T	T	Since at least one of the statements is true, the resulting statement is true
F	F	F	Since both statements are false, the resulting statement is false.

Implication truth table

Uses If-then connective, which is represented as $\Rightarrow$. Here, the statement B is referred to as a hypothesis, the statement E is referred to as a conclusion. The compound statement is true if the conclusion is true, irrespective of the hypothesis. Also, the compound statement is true if both the hypothesis and the conclusion are false. Displaying the implicative connective, $B \Rightarrow E$ in a table,

Table 2.10: *Truth table for* $B \Rightarrow E$

B	E	$B \Rightarrow E$	Reasoning
T	T	T	If the hypothesis is true and the conclusion is true, then the implication is satisfied.
T	F	F	This is the only case where the implication is false because we claimed E must be true if B is true, but E turned out false.
F	T	T	If the hypothesis is false, the implication is considered true regardless of the conclusion; we didn't make any real promise to break.
F	F	T	Again, since the hypothesis is false, the implication doesn't make any false promise. So, the statement holds true.

Bi-conditional (If and only if) truth table

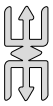
Uses the connective 'if and only if' and is represented as $\Leftrightarrow$. Given the statements B and E, the validity of each statement can be either true or false. Here, the first statement, B, is referred to as antecedent, and the second statement, E, is referred to as consequent. The bi-conditional compound statement is true if both first and second statements have the same truth values and is false otherwise. Displaying the if and only if connective, $B \Leftrightarrow E$ in a table:

Table 2.11: *Truth table for* $B \Leftrightarrow E$

B	E	$B \Leftrightarrow E$
T	T	T
T	F	F
F	T	F
F	F	T

Valid and invalid arguments using the truth table

Using truth tables, the validity of an argument can be checked by inspecting the premises and conclusion in the rows. If all the premises in the row are true and the conclusion is also true, then the argument will be valid. On the other hand, if all the premises are true and the conclusion is false, then the argument is invalid.



Note

A row in a truth table where all the premises are true is called a critical row.

Example 8

Consider the following statements.

p : the headmaster puts in a lot of correct learning measures

q : the standard of learning rises

S_1: If the standard of learning falls, then the headmaster fails to put up correct measures

S_2: The standard of learning is low

Conclusion: the headmaster fails to put up correct learning measures.

Determine whether the argument from the above reasoning is valid or not if p ⇒ q

Solution

Premises: $S_1 : = \sim q \Rightarrow \sim p$

$S_2 : = \sim q$

Conclusion: $\sim p$

Table 2.12: *Truth table for* $\sim q \Rightarrow \sim p$

T	T	F	F	T
T	F	F	T	F

F	T	T	F	T
F	F	T	T	T

From Table 2.12, the premises are columns 4 and 5 and the conclusion is column 3. Row 5 has all the premises true and the conclusion also true therefore the argument is valid. Remember no other row has all the premises true.

Example 9

Consider the following arguments

P_1 : If the government does not improve Agriculture, then he will invest in Gold

P_2 : If the government does not invest in Gold, then the economy will collapse.

Conclusion: If the government improves Agriculture, then the economy will collapse.

Determine if the argument is valid or not.

Solution

Premises:

A = Government does not improve Agriculture

G = Government invests in Gold

E = the Economy collapse

$P_1 : = \ A \Rightarrow \boldsymbol{G}$

$P_2 : = \sim G \Rightarrow E$

Conclusion: $\sim A \Rightarrow E$

There are three statements with two premises

Table 2.13: *Truth table to determine validity.*

T	T	T	F	F	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

From the table the premises are columns 6 and 7 and the conclusion is column 8. Checking the rows of the premises for all to be true rows 2, 3, 6, 7 and 8 are all true. The conclusion parts are all also true except row 7 which is false. Because of row 7 the whole argument is invalid

Learning Tasks

1. Learners in pairs discuss and construct simple statements and compound
2. Guide learners to discuss and construct truth tables of compound statements
3. Guide learners to determine the validity of an argument
4. Guide learners to identify and list simple statements relevant to the debate topic.
5. Guide learners to use logical connectives to construct compound statements that will strengthen their argument.
6. Assist learners in create a logical chain of reasoning using at least three implications leading to their final position
7. Guide learners to prepare counter-arguments using compound statements.
8. Create groups based on the stance individuals chose to argue on. Team structures their argument and present their teams position through a debate.

Pedagogical Exemplars

1. Individual work - Identify and list at least 10 simple statements relevant to the debate topic. They should use logical connectives to construct at least 5 compound statements that will strengthen their argument. Create a logical chain of reasoning using implications leading to their final position and prepare counter-arguments using compound statements.
2. Group work – individuals with same position team up and prepare their debate making use of the individual ideas.
3. Presentation – teams on opposing positions engage in the debate

Key Assessment Level 3: Debate

1. Prepare and present a debate on whether technology should replace traditional teaching methods in schools. Analyse arguments by identifying simple statements and constructing compound statements using logical connectives to strengthen your position. **(18 marks)**
2. Let M, N, and P be statement variables.

   Use a truth table to prove that $((M \Rightarrow N) \Rightarrow P) \equiv (P \vee M) \wedge (N \Longrightarrow P)$. **(10 marks)**
3. A mathematics club is planning an event. The club treasurer says:

   "If we charge at least GH₵15.00 per ticket, then we will have enough funds for both refreshments and prizes."

   Consider the **converse** of the treasurer's statement:

   "If we have enough funds for both refreshments and prizes, then we charged at least GH₵15.00 per ticket."

a. Justify whether or not the converse is always true. **(2 marks)**

b. Provide one numerical example that supports your answer . **2 marks**

c. Provide one counterexample (if possible) that disproves the converse. **2 marks**

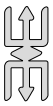
d. Use your findings to conclude whether the converse is logically valid. **2 marks**

Total 7 marks

PLC Session 2

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***debate***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the debate (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 1 Review

The first two weeks of the year three Teacher Manual focused on guiding teachers to assist learners in:

- Forming statements including negation statements and commenting on them
- Drawing implications from given statements and their converses
- Identifying simple statements and constructing compound statements using the connectives.
- Using conjunctions, disjunctions, implications and negations to construct truth tables of compound statement

Pedagogical strategies employed include talk for learning, collaborative discussions in pairs and groups as well as presentations. The mode of assessment was classwork with individual and pair work as well as debates as the assessment strategies.

MARKING SCHEMES FOR THE KEY ASSESSMENTS

WEEK 1

1. a. Inverse statement: If the students do not use the school's Wi-Fi during lessons, then their devices are not registered. **B1**

   Converse statement: If a student's device is registered with the school, then the student uses the school's Wi-Fi during lessons. **B1**

   a. The converse assumes that if a student's device is registered with the school, then the student uses the school's Wi-Fi during lessons. However, a student may have a registered device but can choose not to connect during lessons / Therefore, the converse is not necessarily true just because the original statement is true.

   B1 for acknowledging that a device could be registered but not used in lessons

   B1 for stating that the converse statement need not be true

   i. Hypothesis for statement G is "a person exercises regularly". **B1**

      Hypothesis for statement K is "a community improves access to fresh foods". **B1**

   ii. Conclusion for statement G is "their risk of heart diseases decreases". **B1**

      Conclusion for statement K is "the obesity rates decline". **B1**

   iii. Converse for statement G is "If a person's risk of heart diseases decreases, then they exercise regularly". **B1**

      Converse for statement K is "If obesity rates decline then the community has improved access to fresh foods". **B1**

   iv. The original statement is true because there is existing research to support that regular exercises decreases risk of heart diseases. **B1**

      The converse statement is unlikely to be true because heart disease risk can decrease due to other factors such as medication, changes in diet, etc. **B1**

   v. If both the original and converse were true, then regular exercises will be the ultimate recommendation for persons with risk of heart diseases. Public health recommendations will focus strictly on exercising programmes. **B1**

   vi. If only the original statement is true, then regular exercises will be one of the recommendations for persons with risk of heart diseases and public health officers will recommend other risk-reducing strategies in addition to regular exercises. **B2**

WEEK 2

Table 2.14: *Rubrics for Key Assessment Question 1 in week 2*

CRITERIA	EXCELLENT (3)	PROFICIENT (2)	GOOD (1)
Identification of Simple Statements	*Clearly identifies and utilises at least 8 –10 relevant simple statements. Each statement is precise, accurately formulated, and has clear truth value.*	*Identifies and utilises 5–7 relevant simple statements. Most statements are correctly formulated with generally clear truth values.*	*Identifies fewer than 5 simple statements or inaccurate simple statements*

Construction of Compound Statements	*Constructs at least 5 compound statements using a variety of logical connectives (AND, OR, IF-THEN, NOT). All connectives are used appropriately with precise logical relationships.*	*Constructs 3-4 compound statements using at least 2 different logical connectives. At least 2 of the connectives are used appropriately with generally correct logical relationships.*	*Constructs 2 or fewer compound statements and uses connectives incorrectly, creating logically flawed statements*
Logical Chain of Reasoning	*Presents a logical chain with based on at least 3 implicative statements with clear premises leading to sound conclusions. Implications follow logically with no gaps in reasoning. Avoids all logical fallacies.*	*Presents logical chain based on at least 2 implicative statements. Most implications follow logically. May contain minor logical fallacies that don't significantly impact the overall argument.*	*Presents weak or flawed logical chains with significant gaps in reasoning. Contains major logical fallacies that undermine the argument.*
Evidence and Support	*Provides strong, relevant evidence for all key statements with at least 4 credible citations.*	*Provides adequate evidence for most statements with at least 3 credible citations.*	*Provides minimal evidence at least 2 credible citations.*
Rebuttal Quality	*Precisely identifies logical weaknesses in opposing arguments. Rebuttals directly address opponents' premises and conclusions using logical analysis.*	*Generally, identifies some logical weaknesses in opposing arguments. Rebuttals address major opposing points with some logical analysis.*	*Struggles to identify logical weaknesses in opposing arguments. Rebuttals are weak or focus on irrelevant aspects rather than logical structure.*
Debate Delivery and Communication	*Articulates logical statements with exceptional clarity. Uses precise language that enhances understanding of logical relationships. Maintains excellent time management and engagement.*	*Articulates logical statements with general clarity. Uses mostly correct terminology. Maintains adequate time management and engagement.*	*Articulates logical statements unclearly. Uses imprecise or incorrect terminology. Poor time management or engagement.*

2.

Table 2.15: *Truth table for Key Assessment Question 2 in week 2*

M	N	P	$\mathbf{M \Rightarrow N}$	$\mathbf{(M \Rightarrow N) \Rightarrow P}$	$\mathbf{P \vee M}$	$\mathbf{N \Rightarrow P}$	$\mathbf{(P \vee M) \wedge (N \Rightarrow P)}$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T

F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	F

B8 (0.5 mark for each logically correct entry in the table)

Then justifies that the entries of the tables for $(\mathbf{M} \Rightarrow \mathbf{N}) \Rightarrow \mathbf{P}$ are sequentially the same as the entries of $(\mathbf{P} \vee \mathbf{M}) \wedge (\mathbf{N} \Rightarrow \mathbf{P})$, thus $((M \Rightarrow N) \Rightarrow P) \equiv (P \vee M) \wedge (N \Longrightarrow P)$.

B2 for logically accurate concluding statement

3. a. The converse is not always true.

B2 for not always true WITH a correct justification

b. Provides a valid example; e.g., charging GH¢15.00 and having enough funds. **B2**

c. Provides a counter example: e.g., charging GH¢12.00 and having enough funds through sponsorships **or** charging GH¢ 15.00 but still having fewer funds. **B2**

d. Provides a conclusion where the converse is not logically valid. **B2**

(E.g., Having enough funds does not necessarily imply charging at least GH¢15.00 per ticket. There can be possible alternative sources of fund like donations, sponsorships, etc.).

SECTION 2: MATRICES

STRAND: MODELLING WITH ALGEBRA

Sub-Strand: Applications of Algebra

Learning Outcome: *Apply linear transformations in finding images of points and object points, finding reflection and rotation of points and plane figures.*

Content Standard: *Demonstrate the ability to use and apply knowledge of matrices in linear transformation and apply a linear transformation to solve problems in context.*

INTRODUCTION AND SECTION SUMMARY

Matrix transformations are a powerful and elegant way to understand and visualise how shapes and objects move or change in space. They turn complex geometric operations such as rotations, reflections and enlargements into simple matrix calculations, revealing the deep connection between algebra and geometry. With just a few numbers arranged in rows and columns, matrix transformations allow us to manipulate entire coordinate systems, explore symmetry, and solve real-world problems in computer graphics, engineering, robotics, and more. This section introduces learners to the concept of matrices in representing transformations and how linear transformations affect points and lines on the plane. Learners will be able to determine the image of a point or line under a given transformation, apply matrix methods to identify or describe transformations, understand the properties of various geometric transformations and find the inverse of a linear transformation. The overall outcome for this learning area is to enable learners to confidently apply matrix representations in solving transformation problems, interpret the geometric effects of transformations, and make connections between algebraic and spatial reasoning. The key concepts to be studied include the use of matrices to represent transformations, image and object relationships, transformation properties, isometric mappings, and the computation and interpretation of inverse transformations.

The weeks and the indicators covered by the section are:

- *Week 3*

1. Find the equation of the image of a line under a linear transformation
2. Use linear transformation to determine image and object point.

- *Week 4*

1. Find the composition of linear transformations
2. Apply linear transformation in finding translations, reflections, rotations and enlargement of points and plane figures

Recap weeks

Year 1	Week 12 Find the determinant of a 2 x 2 matrix Add and subtract matrices Multiply a matrix by a scalar and a matrix by a matrix (2 x 2 matrices) Week 13 Recall the formula for finding gradient of a line and apply it to find the equation of a straight line in various forms. Use standard algebraic manipulations to find the equation of parallel and perpendicular lines, including the equation of the perpendicular bisector of a line.
Year 2	**Week 9**: Inverse of 2 x 2 matrices **Week 10** Using matrices to solve systems of linear equations Using matrices to model and solve real-life problems

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section focuses on active learning through group discussions and simulations to explore learners' ability to use and apply knowledge of matrices in linear transformation. Learners will work collaboratively to find the equation of the image of a line under a linear transformation and determine image and object points. These tasks promote critical thinking, communication, and inclusion, with attention to crosscutting issues in group formation. For composition of linear transformations and applications involving translations, reflections, rotations, and enlargements, learners will engage with simulations (digital or physical) to develop deeper conceptual understanding through visualisation. Teaching approaches should emphasise experiential and project-based learning. All learners must be supported to participate fully, with differentiated tasks to cater for varying abilities. Advanced learners should be challenged with real-life applications and computer-based exploration. This inclusive strategy ensures equitable learning opportunities while encouraging collaboration, creativity, and self-directed learning in the study of linear transformations.

ASSESSMENT SUMMARY

The table comprises the key assessment for each indicator in the section. The highest level of DOK for the indicator is denoted. The mode of assessment to be carried out and strategies that will support the achievement of the indicators are outlined.

Indicators	DOK Level	Assessment Strategy and/ or Mode
Find the equation of the image of a line under a linear transformation Use linear transformation to determine image and object point.	3	Strategy: Group discussion Mode: Classwork
Find the composition of linear transformations Apply linear transformation in finding translations, reflections, rotations and enlargement of points and plane figures	3	Strategy: Simulation Mode: Performance assessment

WEEK 3

Learning Indicators

1. *Find the equation of the image of a line under a linear transformation*
2. *Use linear transformation to determine image and object point*

FOCAL AREA 1: FINDING THE EQUATION OF A LINE AS AN IMAGE UNDER A LINEAR TRANSFORMATION

Concept of matrices in transformation

Matrices can be transformed through translation, rotation, reflection or enlargement.

The image of a point D under transformation is generally denoted by D′, expressed as $D \rightarrow \mathrm{D}'$. The transformation T can also be represented by using the functional notation $T(D) = \mathrm{D}'$.

Given the linear transformation, $x' = ax + by$ and $y' = cx + dy$, the matrix of the linear transformation is expressed as: $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & \mathrm{d} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 1

Find the matrix of the linear transformation, $x' = 5x + 2y$ and $y' = 3x - 4y$.

Solution

$x' = 5x + 2y$

$y' = 3x - 4y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix of the linear transformation is $\begin{bmatrix} 5 & 2 \\ 3 & -4 \end{bmatrix}$.

Example 2

Determine the matrix of the linear transformation, $u' = -3u + v$ and $v' = 6u + 9v$.

Solution

$u' = -3u + v$

$v' = 6u + 9v$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Thus, the matrix of the linear transformation is $\begin{bmatrix} -3 & 1 \\ 6 & 9 \end{bmatrix}$.

Equation of the image of a line under a given transformation

To find the equation of the image of a line under a translation, reflection, rotation, or enlargement:

1. Find the coordinates of any two points on the line.
2. Find the coordinates of the images of the two points from step 1.
3. Use the two image points from step 2 to find the slope of the image line.
4. Substitute one of the image points and the slope from step 3 into the equation of a line formula to determine the equation of the image line.

Example 3

Find the equation of the image of the line $y = 5x + 2$, under the transformation $\begin{bmatrix}2\\3\end{bmatrix}$.

Solution

Two points that lie on line $y = 5x + 2$ are A(0, 2) and B($^{-}1$, $^{-}3$)

The image of these points under the translation $\begin{bmatrix}2\\3\end{bmatrix}$ will be $A'(2, 5)$ and $B'(1, 0)$. (As the translation is $\begin{bmatrix}2\\3\end{bmatrix}$, we add 2 to our x coordinate and add 3 to our y coordinate to find the images of the points under the translation.)

Find gradient using the image points; $m_1 = \frac{5-0}{2-1} = 5$

Using $y - y_1 = m_1(x - x_1)$,

$y - 5 = 5(x - 2)$

$y - 5 = 5x - 10$

$y = 5x - 5$ or $y = 5(x - 1)$ is the equation of the image of the line.

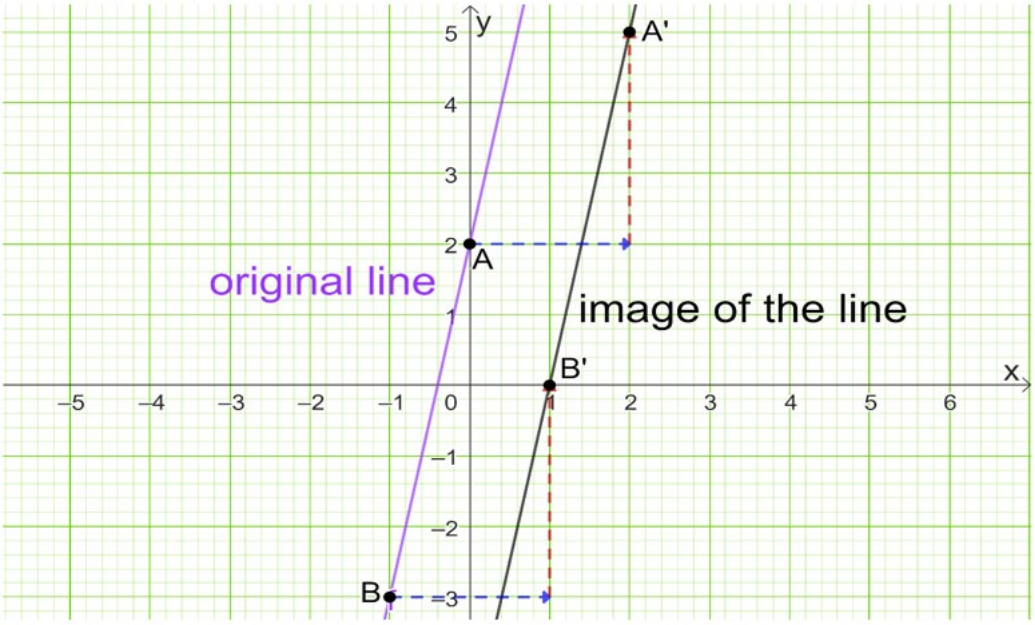


Figure 3.1: *Graphical Representation of* $y = 5x + 2$ *and its image* $y = 5x - 5$.

Figure 3.1: *shows the graphical representation of a line AB and its image* $A'B'$ *with emphasis on how the line was transformed.*

Example 4

The line $y = 2x - 5$ is transformed by a translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ find the image line.

Solution

Find any two points on the line using two arbitrary values of x.

$x = 0$, $\quad y = 2(0) - 5 \quad (0, -5)$

$x = 2$, $\quad y = 2(2) - 5 \quad (2, -1)$

Find the image points by adding the translation vector

$\begin{pmatrix} 0 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Find the gradient using the image points $\frac{2+2}{0+2} = 2$

Use one of the image points to form the equation of the image line

$y - 2 = 2(x - 0)$

$y = 2x + 2$

Example 5

The line $y + 5x = 3$ undergoes a transformation when rotated $270°$ clockwise about the origin. Find the equation of the image line.

Solution

Find two points on the line using two arbitrary values of x.

When $x = 0$, $y + 5(0) = 3$ $(0, 3)$

When $x = 1$, $y + 5(1) = 3$ $(1, -2)$

$x \qquad -y$

Find the image of $(0, 3)$ and $(1, -2)$ under the rotation $270°$ clockwise $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

$\begin{pmatrix} 0 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Use the two image points to find the gradient $\frac{1-0}{2+3} = \frac{1}{5}$

Substitute the gradient and (2, 1) into $y = mx + c$

$1 = \frac{1}{5}(2) + c$

$C = 1 - \frac{2}{5} \quad C = \frac{3}{5}$

$y = \frac{1}{5}x + \frac{3}{5}$

Example 6

The line $2y + x = 3$ is reflected on the line $y + x = 0$ find the image line.

Solution

Find the coordinates of any two points on the line $2y + x = 3$.

When $x = 0 \rightarrow 2y + 0 = 3, y = \frac{3}{2}$, $\left(0, \frac{3}{2}\right)$

When $x = -1 \rightarrow 2y - 1 = 3, y = 2, (-1, 2)$

Perform the reflection on the line $y = -x$

$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ -x \end{pmatrix}$

$\begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Gradient of image points $\frac{1-0}{-2+\frac{3}{2}} = -2$

Using one of the points $(-2, 1)$

$y - 1 = -2(x + 2)$

$y = -2x - 3$

FOCAL AREA 2: USING LINEAR TRANSFORMATION TO DETERMINE IMAGE AND OBJECT POINTS

Image of a point under linear transformation

To find the image of a point under a given linear transformation;

1. Identify the given point or the position vector, $\begin{bmatrix} x \\ y \end{bmatrix}$.
2. Identify the mapping represented by the 2×2, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
3. Multiply the position vector on the left by the mapping to obtain the image of the point:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax & by \\ cx & dy \end{bmatrix}.$$

Example 1

Find the image of the points $D(2, -2), E(6, -1)$ and $F(1, 4)$ under the mapping represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution

Using the matrix $\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$

Let image of D = D'

$$D' = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix}$$

$$= \begin{bmatrix}(2\times 2) + (1\times(-2))\\(0\times 2) + (2\times(-2))\end{bmatrix}$$

$$= \begin{bmatrix}2\\-4\end{bmatrix}$$

Let image of E = E'

$$E' = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}6\\-1\end{bmatrix}$$

$$= \begin{bmatrix}(2\times 6) + (1\times(-1))\\(0\times 6) + (2\times(-1))\end{bmatrix}$$

$$= \begin{bmatrix}11\\-2\end{bmatrix}$$

Let image of F = F'

$$F' = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}$$

$$= \begin{bmatrix}(2\times 1) + (1\times 4)\\(0\times 1) + (2\times 4)\end{bmatrix}$$

$$= \begin{bmatrix}6\\8\end{bmatrix}$$

Therefore $D'(2,-4)$, $E'(11,-2)$ and $F'(6,8)$ are the images of points D, E and F respectively.

Alternatively, we can do these in one calculation:

	D	**E**	**F**		**D′**	**E′**	**F′**

$$\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}2 & 6 & 1\\-2 & -1 & 4\end{bmatrix} = \begin{bmatrix}2 & 11 & 6\\-4 & -2 & 8\end{bmatrix}$$

Aside finding image of a point under linear transformations, the equation of the image of a straight line can also be determined under a linear transformation. Let us go through the following examples to understand how it works.

Example 2

The line $y = 2x - 3$ is transformed by the linear equation

$x' = x - 2y$ and $y' = 3x + y$ find the equation of the image line.

Solution

Substitute y into the transformation

$x' = x - 2(2x - 3)$ $x' = 6 - 3x$

$y' = 3x + 2x - 3\ y' = 5x - 3$

Use two arbitrary values of x to find the image point When $x = 0$

$x' = 6 - 3(0) = 6$

$y' = 5(0) - 3 = -3\ (6, -3)$

when $x = 1$,

$x' = 6 - 3(1) = 3$

$y' = 5(1) - 3 = 2\ (3, 2)$

Find the gradient of the image points $(6, -3)$ and $(3, 2)$

Gradient $= \frac{2+3}{3-6} = \frac{5}{-3}$

Use one of the image points (3, 2) to form the equation

$2 = -\frac{5}{3}(3) + c$ C = 7

$y = -\frac{5}{x}x + 7$ is the equation of the image line.

Example 3

Find the equation of the image line of the straight line $5x - 2y = 3$ under the linear transformation $T(x, y) \rightarrow (x - 5y, -3x + 4y)$.

Solution

We can use another approach

Matrix $= \begin{bmatrix} 1 & -5 \\ -3 & 4 \end{bmatrix}$

Find two points on the line using two arbitrary values of x

When $x = 1, 5(1) - 2y = 3, y = 1$ point $(1, 1)$.

When $x = 3, 5(3) - 2y = 3, y = 6$ point $(3, 6)$.

Find the image points by multiplying matrix and the points

$$\begin{bmatrix} 1 & -5 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 5 \\ -3 + 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -5 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 - 30 \\ -9 + 24 \end{pmatrix} = \begin{pmatrix} -27 \\ 15 \end{pmatrix}$$

Thus, the points are $(-4, 1)$ and $(-27, 15)$.

Find the gradient of the image points $(-4, 1)$ and $(-27, 15)$

Gradient $= \frac{15-1}{-27+4} = -\frac{14}{23}$

Form the equation using $(-4, 1)$

$$y - 1 = -\frac{14}{23}(x + 4)$$

$23y - 23 = -14x - 56$

$23y + 14x = -33$

Transformation of given a matrix

Given the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the linear transformation T, represented by the matrix, is:
$T(x, y) \rightarrow (ax + by,\ cx + dy)$.

Example 4

Write down the linear transformation defined by the matrix $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution

Let the matrix be $Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x + y \\ x + 0y \end{bmatrix} = \begin{bmatrix} -2x + y \\ x \end{bmatrix}$$

The linear transformation, T, defined by Q, is $T(x, y) \rightarrow (-2x + y, x)$.

Geometric transformation

A geometric transformation is a rule or operation that changes the position, size, shape or orientation of a geometric figure in a plane or space. Such a point or shape can be transformed by a translation, reflection, rotation or dilation.

Example 5

Let $Y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Describe the linear transformation T_Y geometrically.

Solution

Consider the points (1, 0) and (0, 1). They have transformed to (-1, 0) and (0, 1). So the x coordinate has been reflected in the y axis, whilst the y coordinate is unchanged because reflecting this in the y axis has no effect.

Therefore, horizontal directions are "flipped" so that (x, y) is sent to $(-x, y)$, and T_Y is reflection in the y-axis.

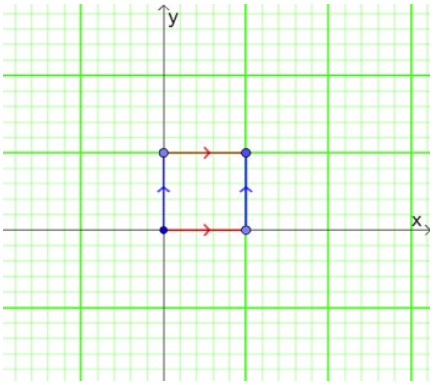


Figure 3.2: *Parallelogram of height 1unit*

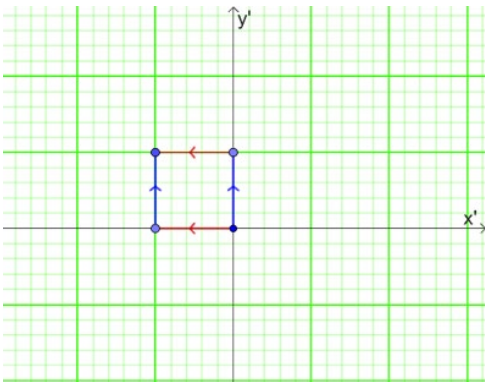


Figure 3.3: *Parallelogram transformed by matrix Y*

Learning Tasks

1. Guide learners to apply knowledge in finding gradients and substitutions to determine the equation of the image of a line under a linear transformation through group discussions.
2. Guide learners to apply multiplication of matrices to find the image of a point under linear transformations through group discussions.

Pedagogical Exemplars

1. Review previous learning
2. Experiential learning – encourage learners in groups to determine the equation of a line under a linear transformation.
3. Talk for learning – encourage learners in groups to discuss how apply multiplication of matrices to find the image of a point.

Key Assessment Level 3: Group Project

1. Given a linear transformation matrix $V = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and a line defined by the equation $y = 3x + 2$, find the equation of the image of the line under the transformation.

(10 marks)

2. a. A line passes through the points A(1, 2) and B(3, 6) and is mapped under the linear transformation $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the image of the line under the transformation M.

(8 marks)

b. Given the matrix $Q = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, determine the image of L(2, 1) under the transformation of Q.

(2 marks)

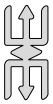
HINT

Assign a group project to learners and give them a reasonable time to complete it. Use the SS App to generate the project and the rubric. Learners' projects should be marked immediately and scores entered into the SAP.

PLC Session 3

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous session, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand.
   b. identify three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q).
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



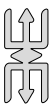
Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Use your app to plan and administer the mode of assessment for the Student Assessment Portal (SAP).
   a. The key assessment for the week is ***group project***. Chat with your app to plan this assessment by reviewing or developing the assessment items/tasks and marking schemes/rubrics for the project (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or topic/idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through the problem individually for 3–4 minutes
   c. the group then works through it together the problem, step by step
   d. highlight the most common errors made at each step (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*

- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 4

Learning Indicators

1. *Find the composition of linear transformations*
2. *Apply linear transformation in finding translations, reflections, rotations and enlargement of points and plane figures*

FOCAL AREA 1: FINDING THE COMPOSITION OF LINEAR TRANSFORMATION

Composition of linear transformation

Suppose A and B are the matrices of two linear transformations, then combined transformation

$A \circ B$ (computed as B followed by A) results in the product AB. Note that $B \circ A$ (computed as A followed by B) results in the product BA.

Matrix multiplication is **not** commutative thus $A \circ B \neq B \circ A$.

To calculate $A \circ B$;

1. Write the matrix A and B of the linear transformation.
2. Find the matrix product AB.
3. Write the linear transformation for AB. That is $AB(x, y)$.
4. Notice that the matrix that represents the first transformation, A, is placed on the right, and the second transformation, B, is placed on the left when carrying out the matrix product.

Example 1

Given the matrices $B = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ find the matrix of the transformation

i. B followed by M.

ii. M followed by B

Solution

i. $M \circ B = B$ followed by M

$$M \circ B = MB$$

$$M \circ B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \times 4) + (3 \times 1) & (2 \times 1) + (3 \times 0) \\ (1 \times 4) + (5 \times 1) & (1 \times 1) + (5 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 3 & 2 + 0 \\ 4 + 5 & 1 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 \\ 9 & 1 \end{bmatrix}$$

ii. $B \circ M = M$ followed by B

$$B \circ M = BM$$

$$B \circ M = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \times 2) + (1 \times 1) & (4 \times 3) + (1 \times 5) \\ (1 \times 2) + (0 \times 1) & (1 \times 3) + (0 \times 5) \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 1 & 12 + 5 \\ 2 + 0 & 3 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 17 \\ 2 & 3 \end{bmatrix}$$

Example 2

Let S: $(x, y) \rightarrow (-x, 2y)$ and $T(x, y) \rightarrow (-y, x)$ be two linear transformations in the x–y plane. Find a single matrix representing the composite transformation TS.

Solution

S: $(x, y) \rightarrow (-x, 2y)$

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$T(x, y) \rightarrow (-y, x)$

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$TS = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (0 \times (-1)) + (-1 \times 0) & (0 \times 0) + (-1 \times 2) \\ (1 \times (-1)) + (0 \times 0) & (1 \times 0) + (0 \times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

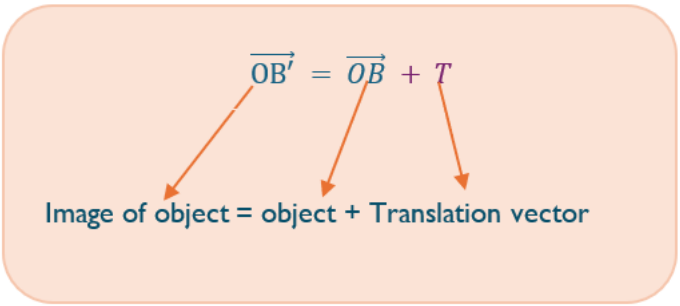
FOCAL AREA 2: APPLYING LINEAR TRANSFORMATION IN FINDING TRANSLATION, REFLECTIONS, ROTATIONS AND ENLARGEMENT OF POINTS AND PLANES FIGURES

Translation and its properties

In translation, every point of the plane moves the same distance in the same direction; for all points $B(x, y)$, $\overrightarrow{OB'} = \overrightarrow{OB} + T$, where T is the translation vector.

Given $\overrightarrow{OB} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $T = \begin{pmatrix} a \\ b \end{pmatrix}$ then $\overrightarrow{OB'} = \overrightarrow{OB} + T = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$.

In summary for translation of a point of an object, Image of object = object + Translation vector.



Properties of Translation

1. All points B map unto their images B' so that $B - B' = T$.
2. The vector gives the magnitude and direction of the translation.
3. There is no invariant point.
4. It requires a translation vector, T.

Example 1

Given the coordinates of a point V(5, 7) and the translation vector as $\begin{pmatrix}-2\\1\end{pmatrix}$, determine the coordinates of the image of V'.

Solution

Let V' have coordinates (x, y)

Using $\overrightarrow{OV'} = \overrightarrow{OV} + T$,

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5 + (-2)\\7 + 1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\8\end{pmatrix}$$

Thus, the image of V is $V'(3, 8)$.

Example 2

In a translation in the plane, the image of M$(-5, 3)$ is (2, 4).

Find the image of N$(-3, 0)$ under the translation.

Solution

If image of M, i.e., M' has coordinates (2, 4), then

$$\overrightarrow{OM'} = \overrightarrow{OM} + T$$

$$\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-5\\3\end{pmatrix} + T$$

$$\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}-5\\3\end{pmatrix} = T$$

$$\begin{pmatrix} 2- & (-5) \\ 4-3 & \end{pmatrix} = T$$

$$T = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

Thus, the translation vector is $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$.

For the image of N,

$$\overrightarrow{ON'} = \overrightarrow{ON} + T$$

$$\overrightarrow{ON'} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

$$\overrightarrow{ON'} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The image of N is N'(4, 1).

Example 3

Determine the translation vector that transforms ABCD to A′B′C′D′.

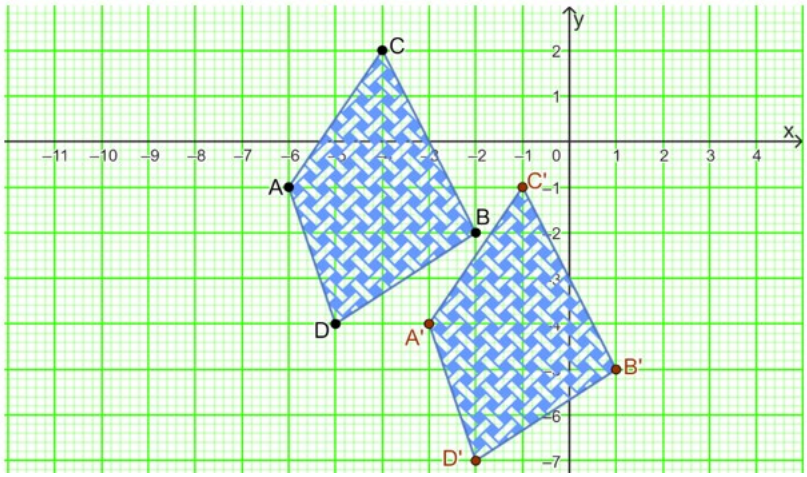


Figure 4.1: *Graphical representation of ABCD and its image*

Solution

ABCD was transformed by the translation vector $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ because each point moves 3 units to the right and 3 units downwards.

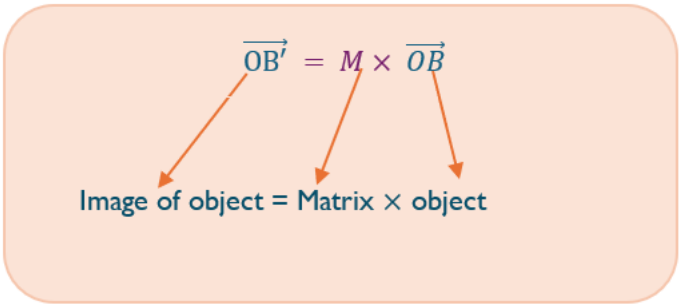
Reflection and its properties

Reflection as a form of transformation is characterised by the flipping of a point or object to create a mirror over a line of symmetry. The lines of symmetry can be the x-axis, y-axis, $y = x$ and $y = -x$ or any given line. Matrix multiplication is used to reflect points and plane figures.

Table 4.1 comprises the respective matrices to be used in the matrix multiplication to create images of a point over a given line of symmetry.

Table 4.1: *Reflecting a point in a given line of symmetry by a matrix.*

Line of symmetry (reflection)	Point	Matrix to multiply by	Image
x-axis ($y = 0$)	(x, y)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
y-axis ($x = 0$)	(x, y)	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$y = x$	(x, y)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$y = -x$	(x, y)	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



Example 4

Find the image of the point Y(3, -1) under a reflection in the:

1. x-axis
2. y-axis
3. $y = x$
4. $y = -x$

Solution

1. Using the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for reflection in the x-axis

   Image = matrix × object

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1(3) + 0(-1) \\ 0(3) + (-1)(-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

   The point Y(3, -1) when reflected in the x-axis becomes Y'(3, 1).

2. Using the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ for reflection in the y-axis

   Image = matrix × object

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (-1)(3) + (0)(-1) \\ (0)(3) + (1)(-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

The point Y$(3, -1)$ when reflected in the y-axis becomes Y'$(-3, -1)$.

3. Using the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for reflection in the line $y = x$.

Image = matrix $\times$ object

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(3) + 1(-1) \\ 1(3) + 0(-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The point Y$(3, -1)$ when reflected on the line $y = x$ becomes Y'$(-1, 3)$.

4. Using the matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ for reflection in the line $y = -x$.

Image = matrix $\times$ object

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(3) + (-1)(-1) \\ (-1)(3) + 0(-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The point Y$(3, -1)$ when reflected on the line $y = -x$ becomes Y'$(1, -3)$.

Note

When reflecting plane figures, apply the reflection to each vertex of the shape.

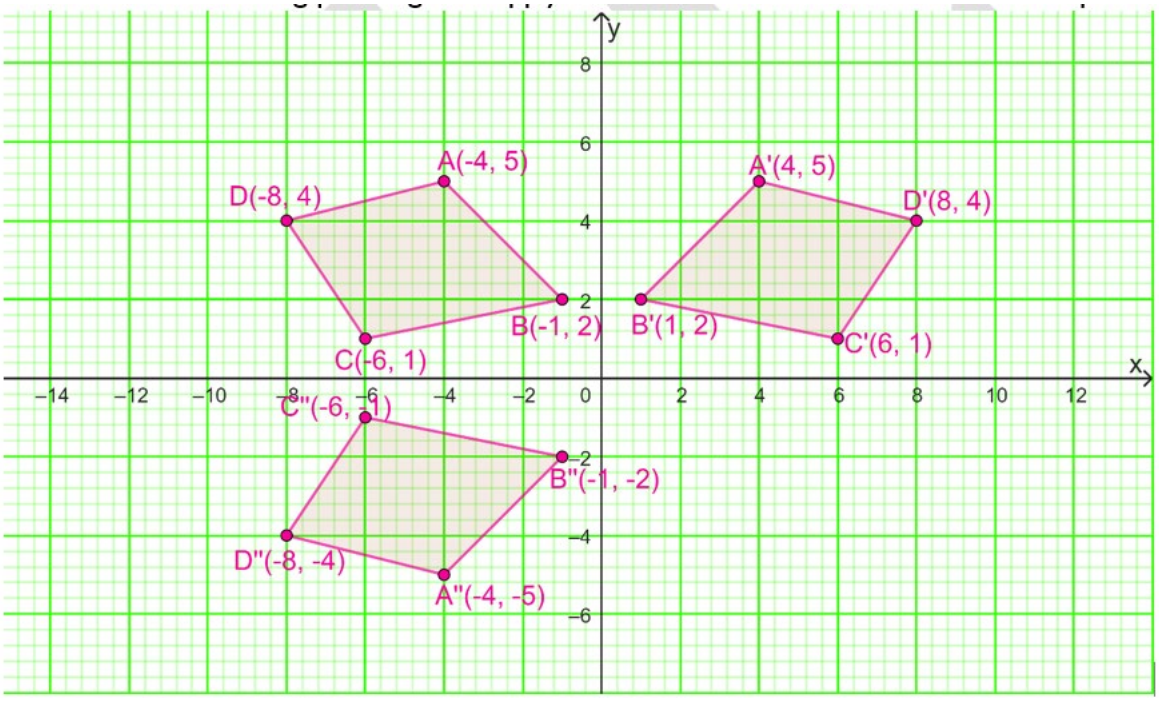


Figure 4.2: *Reflection of ABCD in the x-axis and y-axis*

In Figure 4.2, the shape ABCD is reflected in the y-axis to create $A'B'C'D'$ and reflected in the x-axis to create $A''B''C''D''$.

Rotation and its properties

Rotation is the turning of an object or point about or around a fixed point (where the fixed point could be the origin). Rotation can be clockwise or anticlockwise through 90^0, 180^0, $270^0$and other angles, represented by θ.

Table 4.2 comprises of the respective matrices to be used in matrix multiplication to create images of a point rotated about the origin through given angles.

Table 4.2: *Anticlockwise rotations about the origin (0, 0).*

Angle of rotation	Point	Matrix to multiply by	Image
90°	(x, y)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
180°	(x, y)	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
270°	(x, y)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
Any angle	(x, y)	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Notes

- *For clockwise rotations, replace θ with $-\theta$ in $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.*
- *In determining the image of plane figures, rotate each point of the shape to get the image.*

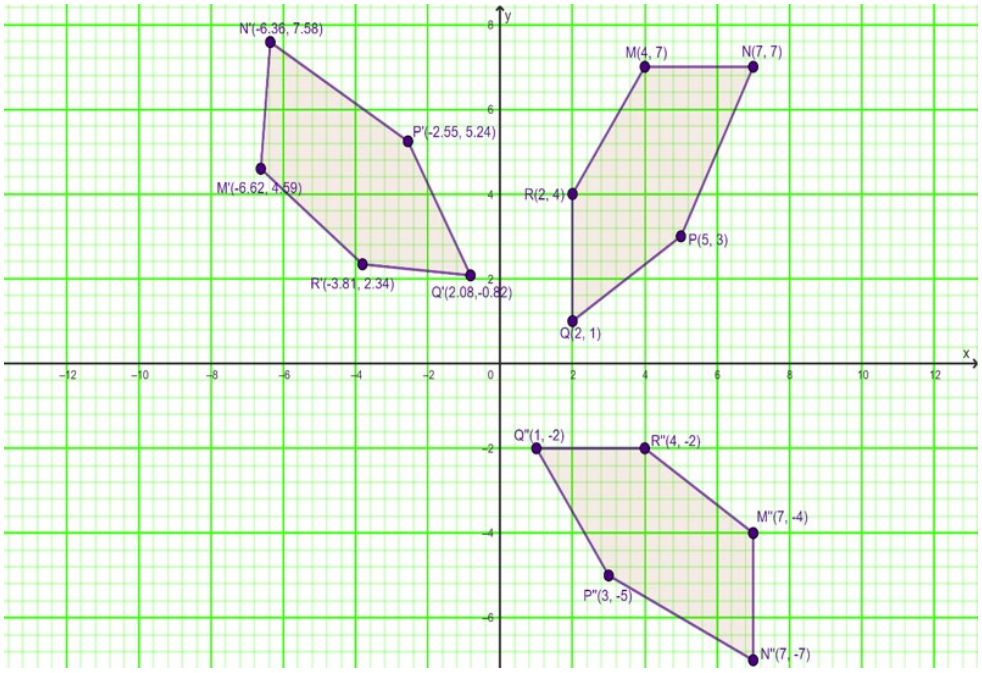


Figure 4.3: *Rotation of shape MNPQR about the origin*

In Figure 4.3, figure MNPQR is rotated about the origin through $85°$ anticlockwise to obtain $M'N'P'Q'R'$ and through $270°$ anticlockwise to obtain $M''N''P''Q''R''$.

Example 5

Find the image of the point M(8, 3) rotated anticlockwise about the origin through;

1. $90°$
2. $180°$
3. $270°$
4. $75°$

Solution

1. Using the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for rotation through $90°$ about the origin,

$$Image = matrix \times object$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(8) + (-1)(3) \\ 1(8) + 0(3) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

The point M(8, 3) when rotated through $90°$ anticlockwise about the origin becomes $M'(-3, 8)$.

2. Using the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ for rotation through $180°$ about the origin, $\text{Image} = \text{matrix} \times \text{object}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1(8) + 0(3) \\ 0(8) + (-1)(3) \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \end{bmatrix}$$

The point M(8, 3) when rotated through $180°$ anticlockwise about the origin becomes $M'(-8, -3)$.

3. Using the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for rotation through $270°$ about the origin

$$Image = matrix \times object$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(8) + 1(3) \\ (-1)(8) + 0(3) \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$$

The point M(8, 3) when rotated through $270°$ anticlockwise about the origin becomes $M'(3, -8)$.

4. Using the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for rotation through $75°$ about the origin,

$$\text{Image} = \text{matrix} \times \text{object}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 75° & -\sin 75° \\ \sin 75° & \cos 75° \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (8)\cos 75° - (3)\sin 75° \\ (8)\sin 75° + (3)\cos 75° \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.07 - 2.90 \\ 7.73 + 0.78 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.83 \\ 8.51 \end{bmatrix}$$

The point M(8, 3) when rotated through $75°$ anticlockwise about the origin becomes $(-0.83, 8.51)$.

Note

Assist learners to discover the relationship between:

i. *90^0 clockwise and 270^0 anticlockwise rotations about the origin.*

ii. *270^0 clockwise and 90^0 anticlockwise rotations about the origin.*

Properties of Rotation

1. There is a single invariant point, O, of the rotation.
2. All other points P map onto their images P' so that $PO = P'O, \angle POP' = \theta$
3. The angle θ gives the magnitude of the rotation, the sign convention being positive for anticlockwise and negative for clockwise.
4. The angle between any line and its image, for example, PQ and $P'Q'$, is equal to θ.

Dilation (Enlargement or Reduction)

A way of transforming plane figures/shapes where the figures are made bigger (enlargement) or smaller (reduction). The dilation matrix is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ where k is a dilation or scale factor and the centre of enlargement is the origin.

Properties of Enlargement and Reduction

1. There is a single invariant point, the centre of enlargement, *O*.
2. All other points *P* map onto their images *P′* so that $OP' = kOP$, where k is the scale factor of enlargement.
3. Under enlargement, any figure is mapped onto a similar figure.
4. If $|k|>1$, the image is larger than the original figure, but if $|k| <1$, it is smaller.
5. If $k < 0$, O lies between P and P' and the image figure is inverted.
6. A scale factor of -1 is equal to a rotation about the centre O of $180°$.
7. With a scale factor k, similar figures have lengths and areas in the ratio $k:1$ and $k^2:1$ respectively.

Note

To enlarge or reduce a given plane figure, apply the scale factor to each of the points of the shape to find the image as shown in Figure 4.4.

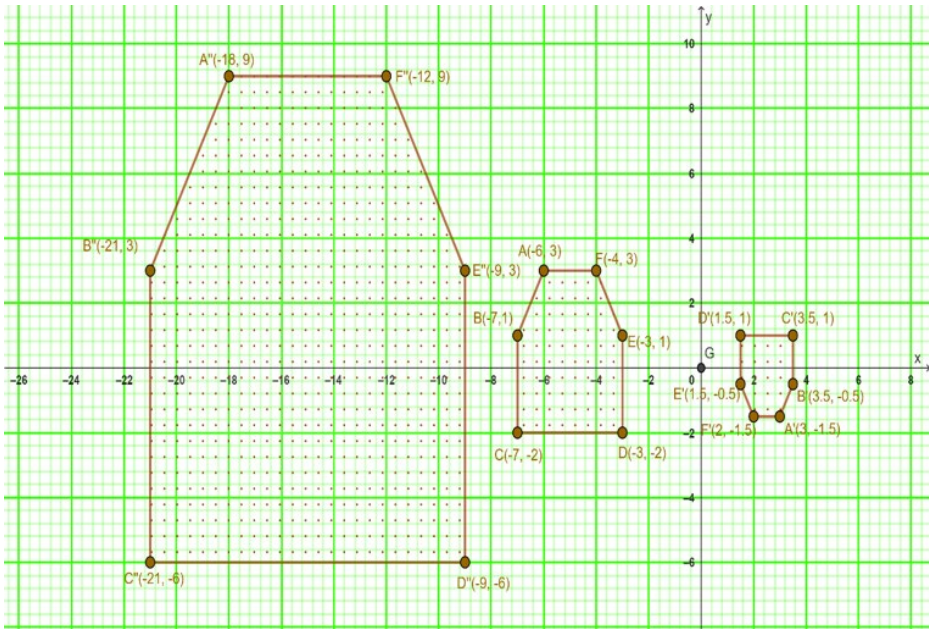


Figure 4.4: *Enlargement and reduction of shape ABCDEF.*

In Figure 4.4, figure ABCDE is enlarged by scale factor 3 about the origin to obtain $A'B'C'D'E'$and by scale factor -0.5 about the origin to obtain $A''B''C''D''E''$.

Isometrics

Isometric transformations are the kind in which the images still have the same shape and size.

Reflected, rotated and translated shapes or objects still end up with the same look (shape and size).

Note

Under isometrics, any figure maps onto a congruent figure. Translation and rotation produce directly congruent figures; reflection is inversely congruent.

HINT

Guide learners to determine whether some shapes are isometric or not after undertaking a transformation.

Inverse of a linear transformation

The inverse of a linear transformation moves a point back to its original position. That is, it reverses the effect of the original transformation.

Given a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with a determinant $k = ad - bc$, and a linear transformation $x' = ax + by$ and $y' = cx + dy$, the inverse of the linear transformation becomes:

$x = \frac{1}{k}(dx' - by')$ and $y = \frac{1}{k}(-cx' + ay')$ or, in matrix form, $\frac{1}{k}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Note

The inverse of a linear transformation exists only if the transformation is invertible.

Example 6

Determine the inverse of the linear transformation $x' = 4x + 2y$ and $y' = -3x + 5y$.

Solution

Let $a = 4, b = 2, c = -3$ and $d = 5$,

$k = 4(5) - 2(-3)$

$k = 26$

Using $x = \frac{1}{k}(dx' - by')$ and $y = \frac{1}{k}(-cx' + ay')$.

$x = \frac{1}{26}(5x' - 2y')$

$x = \frac{5}{26}x' - \frac{1}{13}y'$ while

$y = \frac{1}{26}(-(-3)x' + 4y')$

$y = \frac{3}{26}x' + \frac{2}{13}y'$.

Learning Tasks

Guide learners to determine the composition of linear transformations in convenient group discussions. Visit https://www.geogebra.org/m/m3BMum5s?utm to explore how the composition of linear maps corresponds to the matrix multiplication.

Guide learners to translate, reflect, rotate and dilate points and plane figures through simulations.

A video from https://www.youtube.com/watch?v=sc9M1Nhx3LM can assist with using GeoGebra to transform shapes.

1. Guide learners to identify and discuss the properties of each of the transformation processes after going through the simulation.
2. Assist learners to use matrix multiplication to confirm the obtained images in the simulations.

Pedagogical Exemplars

Demonstrations - Use technological software (e.g., GeoGebra) to demonstrate transformations of shapes. A simulation worksheet at https://www.geogebra.org/m/yq2nnewu?utm may serve as a guide to creating a similar one.

Experiential learning -Assist learners to combine different transformations, adjusting parameters to see and understand the resulting effects. Visit https://www.geogebra.org/m/PRxxqcFx?utm for a sample.

Talk for learning – encourage learners to share their observations from the simulations and verification with matrix multiplications.

Key Assessment Level 3: Simulation

1. The matrix M of the linear transformation $M: (x, y) \rightarrow (ax + by, cx + dy)$ is given by $\begin{bmatrix} -3 & 4 \\ 2 & 7 \end{bmatrix}$.

   a. Determine the values of the constants a, b, c and d.

   2 marks

   b. Find the matrix of the inverse of the transformation M.

   4 marks

   c. Determine the image of $(1, -3)$ under a linear transformation of the inverse matrix of M.

   4 marks

   Total 10 marks

2. a. Write down the matrices P and Q representing the following transformations respectively:

   i. a reflection in the line $y = -x$.

   1 mark

   ii. the transformation $(x, y) \rightarrow \left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)$

   1 mark

   a. Find the image of the point $(3, -1)$ under the transformation QP

   5 marks

   b. Find the matrix Q^3 and identify which transformation it represents.

   3 marks

   Total 10 marks

PLC Session 4

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).

Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***simulation***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the simulation (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).

8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 2 Review

Weeks 3 and 4 of the year three Teacher Manual focused on guiding teachers to assist learners in:

- Finding the equation of the image of a line under a linear transformation.
- Using linear transformation to determine image and object point.
- Finding the composition of linear transformations.
- Applying linear transformation in finding translations, reflections, rotations and enlargement of points and plane figures.

MARKING SCHEMES FOR THE KEY ASSESSMENTS

WEEK 3

1. Find two points that lie on line $y = 3x + 2$

B2 (1 mark for each correct point)

Example: A(0, 2) and B(1, 5)

Use the transformation matrix to obtain images of the two points. Example:

$OA' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **M1**

$A'(4, 2)$ **A1**

$OB' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **M1**

$B'(11, 5)$ **A1**

Determine gradient of image line

$m_1 = \frac{5-2}{11-4}$ **M1**

$m_1 = \frac{3}{7}$ **A1**

Substitute one point into point-slope or slope-intercept form of equation of line

Example: $y - 5 = \frac{3}{7}(x - 11)$ M1

Obtaining equation of image line as $7y - 3x - 2 = 0, or\ equivalent$ **A1**

a. Use the transformation matrix to obtain images of the two points. Example:

$OA' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-2 \\ 1+0 \end{bmatrix}$ **M1**

$A'(-2, 1)$ **A1**

$OB' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0-6 \\ 3+0 \end{bmatrix}$ **M1**

$B'(-6, 3)$ **A1**

Determine gradient of image line

$m_1 = \frac{3-1}{-6-(-2)}$ **M1**

$m_1 = -\frac{1}{2}$ **A1**

Substitute one point into point-slope or slope-intercept form of equation of line

Example: $y - 3 = -\frac{1}{2}(x + 6)$ M1

Obtaining equation of image line as $y = -\frac{1}{2}x$ **A1**

b. Use the transformation matrix to obtain image of L(2, 1).

$OL' = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -2+2 \end{bmatrix}$ **M1**

$L'(3, 0)$ **A1**

WEEK 4

1. a. $a = -3, b = 4, c = 2$ and $d = 7$

B2 (0.5 for each)

a. Determinant of $M = -3(7) - 4(2) = -29$ **M1, A1**

Inverse of M; $M^{-1} = -\frac{1}{29}\begin{bmatrix} 7 & -4 \\ -2 & -3 \end{bmatrix}$ $M^{-1} = \begin{bmatrix} -\frac{7}{29} & \frac{4}{29} \\ \frac{2}{29} & \frac{3}{29} \end{bmatrix}$ **for each in the matrix)**

b. Using $x' = ax + by$, $y' = cx + dy$ and

Now, $a = -\frac{7}{29}, b = \frac{4}{29}, c = \frac{2}{29}$ and $d = \frac{3}{29}$

$x' = -\frac{7}{29}(1) + \frac{4}{29}(-3)$ **M1**

$= -\frac{19}{29}$

$y' = \frac{2}{29}(1) + \frac{3}{29}(-3)$ **M1**

$= -\frac{7}{29}$

Now, the image of $(1, -3)$ under M^{-1} is $\left(-\frac{19}{29}, -\frac{7}{29}\right)$. **A2 (1 for each)**

2. a. i. Matrix $P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **B1**

i. Matrix $Q = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **B1**

c. Matrix multiplication of Q and P: $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **M1**

Simplifying to get $QP = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$ **A2 ($\frac{1}{2}$ for each correct entry)**

Image of $(3, -1) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **M1**

$= \begin{bmatrix} \frac{3\sqrt{3}+1}{2} \\ \frac{\sqrt{3}-3}{2} \end{bmatrix}$

Stating image as a point $\left(\frac{3\sqrt{3}+1}{2}, \frac{\sqrt{3}-3}{2}\right)$ **A1**

d. Computing Q^2 with all correct entries $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **A1**

Computing Q^3 with all correct entries $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **A1**

Stating that Q^3 is a $180°$ rotation **A1**

An **alternative route** is for students to realise that Vector Q is a 60^0 rotation anticlockwise, so doing this 3 times makes a $180°$ rotation. This would be awarded all 3 marks.

SECTION 3: PARABOLAS

STRAND: GEOMETRIC REASONING AND MEASUREMENT

Sub-Strand: Spatial Reasoning

Learning Outcomes

1. **Week 5 & 6:** *Construct a parabola of a given quadratic equation ($y = ax^2 + bx + c$) and explain its key features.*
2. **Week 7:** *Sketch a parabola and use it to deduce the relation $y^2 = 4ax$.*
3. **Week 8:** *Sketch a parabola given the directrix and focus.*
4. **Week 9:** *Deduce the equation of the tangent and normal to a parabola.*

Content Standard: *Demonstrate an understanding of parabola and its properties.*

INTRODUCTION AND SECTION SUMMARY

Parabolas form an important part of the study of quadratic functions in Senior High School Mathematics. At the Year Three level, learners are expected to deepen their understanding of parabolas not just as curves on a graph, but as mathematical models that describe real-life situations such as the path of a thrown object or the shape of satellite dishes. Teaching parabolas begins with exploring the general form of a quadratic function, $f(x) = ax^2 + bx + c$, and guiding learners to recognise key features such as the vertex, axis of symmetry, direction of opening, and the roots or intercepts.

By using visual aids, graphing tools, and contextual problems, teachers can help learners see how changes in the parameters in the parabolic equation affect the shape and position of the parabola. Linking the algebraic expressions to their graphical representations allows learners to develop both analytical and visual understanding. Ultimately, this topic prepares learners for more advanced mathematical concepts and practical problem-solving.

The weeks and the indicators covered by the section are:

- *Week 5*

1. *Describe the shape of the graphs of quadratic functions.*
2. *Explore the graphs of quadratic functions with different parameters using technological tools such as GeoGebra, Geometer's sketchpads, etc., other available materials, or by hand.*

- *Week 6*

1. *Explore and describe the features of a parabola (focus, directrix, axis of symmetry and vertex) using a technological tool and relate it to real life.*
2. *Describe a given locus as a parabola.*

- *Week 7*

1. *Apply the formula of distance between two points to find the general equation of a parabola.*
2. *Deduce the directrix and focus from a parabolic equation and vice-versa.*

- *Week 8*

1. *Describe the appropriate orientation of the parabola to the given equation.*
2. *Use directrix and focus to sketch a parabola.*

- *Week 9*

1. *Describe the equation of lines and derivatives of functions.*
2. *Determine the equations of tangents and normal to a parabola.*
3. *Find the point of intersection for a line and a parabola.*

Recap weeks

Year 1	Week 8: Recognise and construct linear and parabolic functions by hand and with the aid of technology, where appropriate show the intercepts and investigate the turning points. Week 10 1. Use the method of completing the square to transform any quadratic equation in, say x into an equation of the form $(x-p)^2 = q$ that has the same solutions and explain how the quadratic formula is derived from this form. 2. Draw the graph of a polynomial function with degree up to 3 by hand and by using technology (e.g., GeoGebra, Demos, PhET Simulations, and Geometer's Sketch Pad) where appropriate. Week 13 1. Use standard algebraic manipulations to find the equation of parallel and perpendicular lines, including the equation of the perpendicular bisector of a line. 2. Deduce the shortest distance between a point and a line and use the knowledge of intercepts and right-angled triangles to find the perpendicular distance from an external point to a line. Week 18: Use knowledge of differentiation to determine the equation of tangents and normal to curves at a given point.
Year 2	Week 7: Deduce relations of various loci under given conditions. Week 22 1. Find the maximum and minimum values and points of a function, sketch the functions and solve some real-life problems. 2. Use the second derivative of a function to classify the maximum, minimum and saddle point of that function and perform curve sketching.

SUMMARY OF PEDAGOGICAL EXEMPLARS

Teaching parabolas to final-year high school students requires engaging strategies that enhance exam readiness and cater to diverse abilities. Methods like talk-for-learning facilitate understanding through discussions and peer explanations. Hands-on activities, such as paper folding and sketching, help connect algebraic equations with their graphical forms. Project-based learning allows students to model real-life scenarios involving parabolas, while inquiry-based learning explores how changing coefficients affect their shape. Collaborative learning encourages peer interaction through group tasks and projects. To support all learners, teachers can use visual aids and structured steps for those struggling, while offering advanced tasks for quicker learners. In resource-limited classrooms, simple materials like string, graph paper, and chalkboard illustrations can effectively demonstrate parabolic curves. These pedagogies foster an inclusive and active learning environment that builds understanding and exam confidence.

ASSESSMENT SUMMARY

Assessing learners' understanding of parabolas can be achieved effectively through a combination of varied, inclusive methods. Classwork provides immediate feedback on individual comprehension of concepts such as vertex, axis of symmetry, and graphing. It allows the teacher to identify and support learners who need reinforcement. Group discussions encourage peer learning, allowing learners of differing abilities to collaborate, explain reasoning, and clarify misconceptions.

A gallery walk is suggested to enable learners to display and review each other's work—such as sketches or applications of parabolas—promoting active engagement and visual learning. This method can be effective even in resource-limited classrooms using chart paper or reused cardboard. The mid-semester exam provides a formal assessment of conceptual understanding and procedural fluency. To support varied learning needs, include a mix of multiple-choice, short-answer, and open-ended questions.

Gamification, such as quiz competitions or matching games, enhances motivation and can be adapted with minimal materials. It supports kinaesthetic and auditory learners especially well.

A group project on modelling real-life parabolic paths is suggested to promote deeper learning and teamwork. Allowing flexible presentation formats ensures all learners can contribute according to their strengths. This blend of methods fosters inclusion and adaptability in diverse classroom settings.

Examples of assessment tasks for the suggested assessment modes and strategies are provided as a guide. The table provides a summary of the assessment used against the indicator.

Indicators	DOK Level	Assessment Strategy and/or Mode
Week 5		
Describe the shape of the graphs of quadratic functions	1	Mode: Classwork Strategy: Group discussion Note: Gallery walk for hand drawn graphs and projections of graphs from technological tools Prompt: Prepare for mid-semester examination in week 6
Explore the graphs of quadratic functions with different parameters using technological tools such as GeoGebra, Geometer's sketchpads, etc., other available materials, or by hand	4	
Week 6		

Explore and describe the features of a parabola (focus, directrix, axis of symmetry and vertex) using a technological tool and relate it to real life.	2	Mode: Mid-semester examination Strategy: 20 objectives and 2 theory questions from 4 options Prompt: Assign group project of parabola to be submitted in week 9
Describe a given locus as a parabola	3	
Week 7		
Apply the formula of distance between two points to find the general equation of a parabola	2	Mode: Classwork Strategy: Individual or in pairs
Deduce the directrix and focus from a parabolic equation and vice-versa	4	
Week 8		
Describe the appropriate orientation of the parabola to the given the given equation	3	Mode: Classwork Strategy: Gamification / Homework Note: Gamification for LI 7 and Homework for LI 8
Use directrix and focus to sketch a parabola	3	
Week 9		
Describe the equation of lines and derivatives of functions	2	Mode: Group project Strategy: Presentation on group project of parabola assigned in week 6
Determine the equations of tangents and normal to a parabola	4	
Find the point of intersection for a line and a parabola	3	

WEEK 5

Learning Indicators

1. *Describe the shape of the graphs of quadratic functions*
2. *Explore the graphs of quadratic functions with different parameters using technological tools such as GeoGebra, Geometer's sketchpads, etc., other available materials, or by hand*

FOCAL AREA 1: DESCRIBING THE SHAPE OF THE GRAPH OF QUADRATIC FUNCTIONS

In Week 8 of the Year One manual, learners learned to recognise the shape of linear and quadratic (parabolic) functions. Linear graphs were described as straight lines while the graphs of functions with degree two and hence classified as quadratic were described as parabolic. In this week we will describe the shape of quadratic functions and explore the graphs when the parameters in the parabolic function are changed.

In our environment, there are several objects whose shape are parabolic. The path taken by a ball kicked upwards, the arc formed from bridges and the shape of a satellite dish, rainbow or banana are all models of parabolas around us.

Shape of quadratic functions

The graph of a quadratic function in terms of x i.e., $f(x) = ax^2 + bx + c$ is either shaped like the intersection symbol, $\cap$ thus opening downwards or the union symbol, $\cup$ opening upward.

Note

For a parabolic function, $f(x) = ax^2 + bx + c$ with vertex located at (u, v):

1. If $a > 0$,
   a. The graph of $f(x)$ is "$\cup$" shaped
   b. It has a minimum point, same as the turning point
   c. It decreases in the interval: $x = (-\infty, u)$ while increasing in the interval: $x = (u, \infty)$
   d. Its range is the interval: $y = [v, \infty)$
2. If $a < 0$
   a. The graph of $f(x)$ is "$\cap$" shaped
   b. It has a maximum point, same as the turning point
   c. It increases in the interval: $x = (-\infty, u)$ while decreasing in the interval: $x = (u, \infty)$
   d. Its range is the interval: $y = (-\infty, v]$

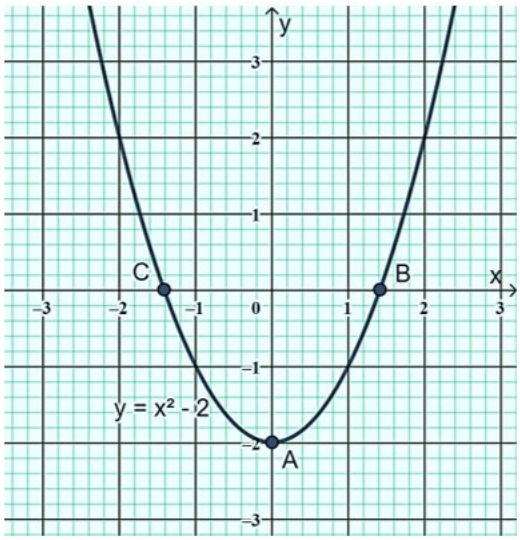


Illustration 1

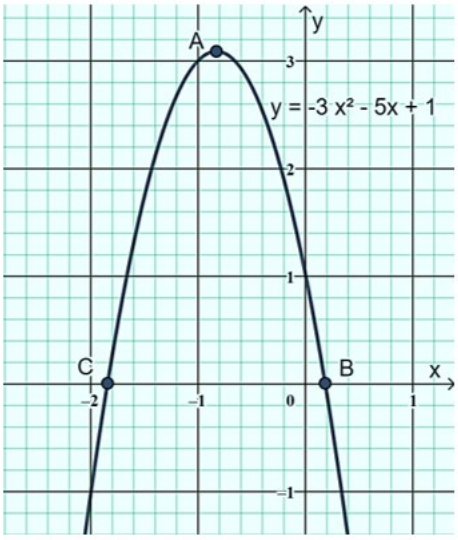


Illustration 2

Figure 5.1: *Parabolic graphs*

Illustration 1 in Figure 5.1 shows the graph of $y = x^2 - 2$. This has a positive number, 1, as the coefficient of the squared term. It has a minimum value at $A(0, -2)$, its turning point. It decreases for values of x in the interval: $(-\infty, 0)$ and increases for $0 < x < \infty$. $y \in [-2, \infty)$ as the values of y would not be less than -2, the y coordinate of the lowest point on the graph.

For $y = -3x^2 - 5x + 1$ (the corresponding graph shown in Illustration 2) however, the coefficient of the squared term, i.e., $-3 < 0$ and thus, is shaped like $\cap$. It has a maximum value at $A(-0.833, 3.083)$, its turning point. It increases for values of x in the interval: $(-\infty, -0.833)$ and decreases for $x \in (-0.833, \infty)$ and has a range between negative infinity to 3.0833 (inclusive) as the values of y cannot be greater than 3.0833.

FOCAL AREA 2: EXPLORING THE GRAPHS OF QUADRATIC FUNCTIONS WITH DIFFERENT PARAMETERS

Before we explore the graphs of quadratic functions, let us simplify the general quadratic equation by completing the square:

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 + a\left[\frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

Let $C = a\left[\frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + C$$

Also, let $A = a$ and $B = \frac{b}{2a}$

We can now simply write

$$f(x) = A(x + B)^2 + C$$

The parameters which determine the shape of the graph of a quadratic function are hereby A, B and C

The basic parabolic function is $f(x) = x^2$. For this function, $a = 1, b = 0$ and $c = 0$ for a, b and c as used in the general quadratic function: $f(x) = ax^2 + bx + c$. A comparison to the simplified form gives $A = 1, B = 0$ and $C = 0$

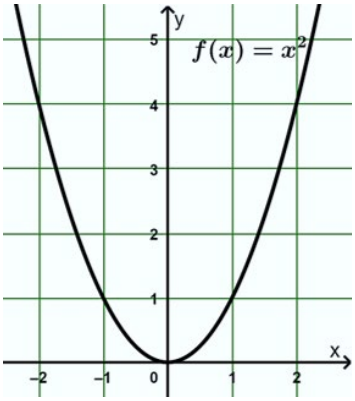


Figure 5.2: *Graph of* $f(x) = x^2$

The graph of $f(x) = x^2$ has the following properties:

1. It is U-shaped.
2. The turning point is located at the origin.
3. The turning point is a minimum point.

Let us consider the graph of $g(x) = 3x^2$. For this function, $a = 3, b = 0$ and $c = 0$ for a, b and c as used in the general quadratic function: $f(x) = ax^2 + bx + c$. A comparison to the simplified form gives $A = 3, B = 0$ and $C = 0$

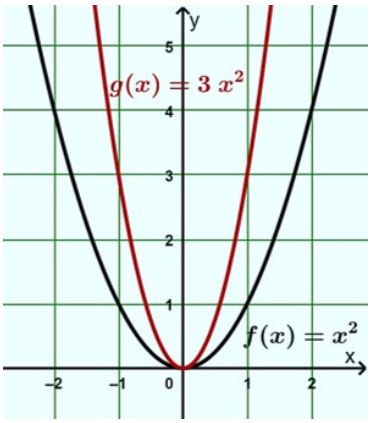


Figure 5.3: *Graphs of* $f(x) = x^2$ *and* $g(x) = 3x^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $g(x) = 3x^2$:

1. They are both U-shaped.
2. Their turning points are both located at the origin.
3. Their turning points are both minimum points.
4. The graph of $f(x)$ has been '*shrunk*' to obtain the graph of $g(x)$. This is actually known as a vertical stretch.

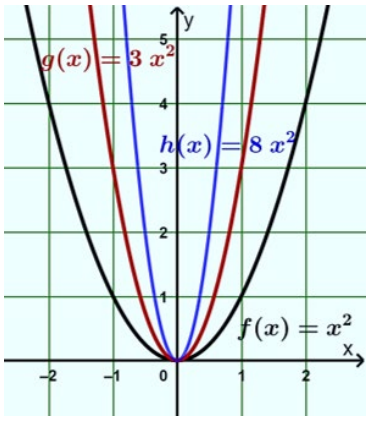


Figure 5.4: *Graphs of* $f(x) = x^2$, $g(x) = 3x^2$ *and* $h(x) = 8x^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$, $g(x) = 3x^2$ and $h(x) = 8x^2$:

1. They are all U-shaped.
2. Their turning points are all located at the origin.
3. Their turning points are all minimum points.
4. The graph of $f(x)$ has been *shrunk* (vertically stretched) to obtain graphs of $g(x)$ and $h(x)$.
5. The graph of $f(x)$ is the widest followed by graph of $g(x)$ then the graph of $h(x)$.

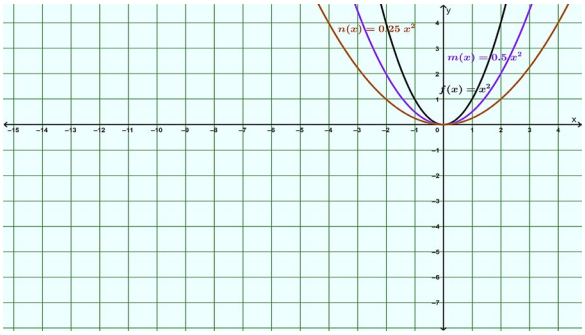


Figure 5.5: *Graphs of* $f(x) = x^2$, $m(x) = 0.5x^2$ *and* $n(x) = 0.25x^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$, $m(x) = 0.5x^2$ and $n(x) = 0.25x^2$:

1. They are all U-shaped.
2. Their turning points are all located at the origin.
3. Their turning points are all minimum points.
4. The graph of $f(x)$ has been stretched vertically (dilated) to obtain graphs of $m(x)$ and $n(x)$.
5. The graph of $f(x)$ is the narrowest followed by graph of $m(x)$ then the graph of $n(x)$.

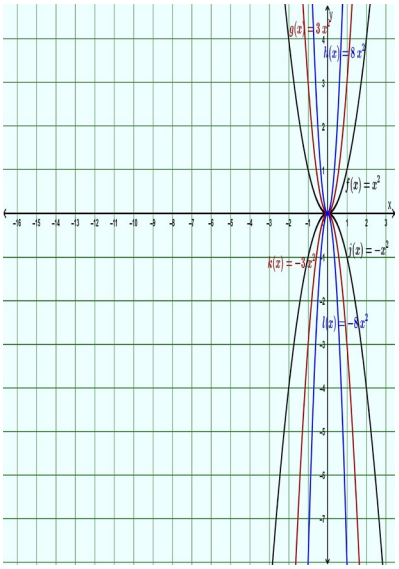


Figure 5.6: *Graphs of* $f(x)$, $g(x)$ *and* $h(x)$ *and their reflections*

It can be observed from Figure 5.6 that when the value of "A" is negative, the resulting graph is merely a reflection of the original graph about the x −axis.

The following conclusions can be made from the graphs in Figure 5.3 through to Figure 5.6:

1. The greater the value of $|A|$, the narrower the curve.
2. The smaller the value of $|A|$, the wider the curve.
3. If A is negative, the resulting graph is a reflection about the x − axis.

Let us now explore the shape of parabolic graphs when the value of "B" is altered.

Consider the function $g(x) = (x + 1)^2$. A comparison to the simplified form: $f(x) = A(x + B)^2 + C$ gives $A = 1, B = 1$ and $C = 0$

Note that $g(x) = x^2 + 2x + 1$. For this version function, a comparison to the general quadratic function: $f(x) = ax^2 + bx + c$ yields $a = 1, b = 2$ and $c = 1$

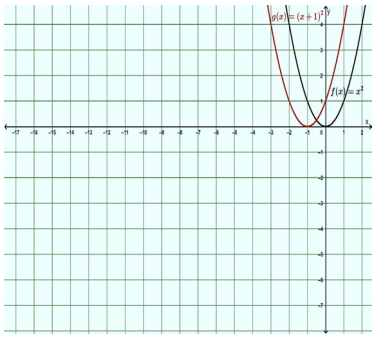


Figure 5.7: *Graphs of* $f(x) = x^2$ *and* $g(x) = (x + 1)^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $g(x) = (x + 1)^2$:

1. They are both U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(-1, 0)$ indicating a horizontal shift of the graph of $f(x)$ one unit to the left.
3. Their turning points are both minimum points.

4. The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrink or dilation (i.e., no stretching), they are merely translated.

Consider also the graph of $h(x) = (x+2)^2$ for which $A = 1, B = 2$ and $C = 0$ or $h(x) = x^2 + 4x + 4$ for which $a = 1, b = 4$ and $c = 2$ in comparison to the graph of $f(x) = x^2$

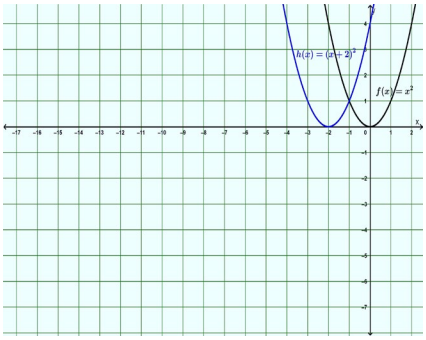


Figure 5.8: *Graphs of* $f(x) = x^2$ *and* $h(x) = (x+2)^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $h(x) = (x+2)^2$:

1. They are both U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $h(x)$ is located at $(-2, 0)$ indicating a horizontal shift of the graph of $f(x)$ two units to the left.
3. Their turning points are both minimum points.
4. The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrink or dilation, they are simply translated.

Let us compare the graphs of $f(x) = x^2$ and $j(x) = (x - 0.5)^2$ or $j(x) = x^2 - x + 0.25$.
Note that $A = 1, B = -0.5$ and $C = 0$ and $a = 1, b = -1$ and $c = 0.25$

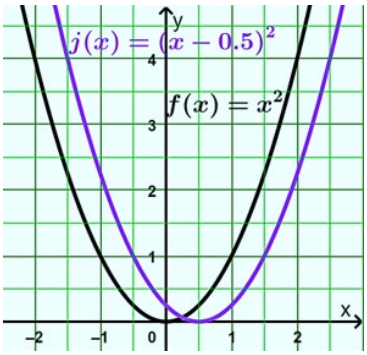


Figure 5.9: *Graphs of* $f(x) = x^2$ *and* $j(x) = (x - 0.5)^2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $j(x) = (x - 0.5)^2$:

1. They are both U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $j(x)$ is located at $(0.5, 0)$ indicating a horizontal shift of the graph of $f(x)$ half a unit to the right.
3. Their turning points are all minimum points.
4. The graphs of $f(x)$ and $j(x)$ do not change in size. There is no shrink or dilation, they are simply translations.

The following conclusions can be made about the effect of changing the value of B in a function defined as $f(x) = A(x + B)^2 + C$ from the graphs in Figure 5.7 through to Figure 5.9:

1. The value of "$|B|$" indicates the distance of horizontal shift of the resulting graph form the graph of $f(x) = x^2$.
2. The greater the value of "$|B|$", the farther the resulting curve is to the graph of $f(x) = x^2$.
3. The smaller the value of "$|B|$", the closer the resulting curve is to the graph of $f(x) = x^2$.
4. When $B < 0$, there is a shift to the right.
5. When $B > 0$, there is a shift to the left.

Finally, let us now explore the shape of parabolic graphs when the value of "C" is changed.

We will first do a comparison of the graph of $f(x) = x^2$ and the function $g(x) = x^2 + 2$ i.e., for the parameters: $A = 1, B = 0$ and $C = 2$ or $a = 1, b = 0$ and $c = 2$.

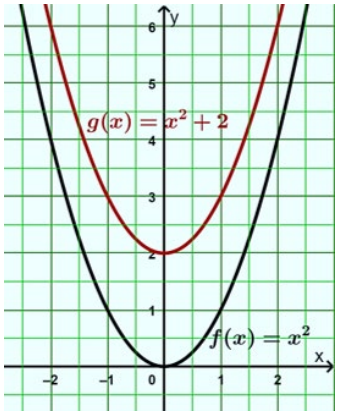


Figure 5.10: *Graphs of* $f(x) = x^2$ *and* $g(x) = x^2 + 2$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $g(x) = x^2 + 2$:

1. They are both U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(0, 2)$ indicating a vertical translation of the graph of $f(x)$ two units upward.
3. Their turning points are both minimum points.
4. Move one unit upward from the vertices of both graphs to 1 and 3 on the $y-$ axis and we find the widths of the curves equal to two units. If we move 2.25 units upward i.e., to 2.25 and 4.25, the width is three units for both curves. If we move four units upward, the width is four units for both curves.

   It is then observable that the graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrink or dilation, they are simply translations.

Consider also the graph of $h(x) = x^2 - 3$ for which $A = 1, B = 2$ and $C = 0$ or $h(x) = x^2 + 4x + 4$ for which $a = 1, b = 4$ and $c = 2$ in comparison to the graph of $f(x) = x^2$.

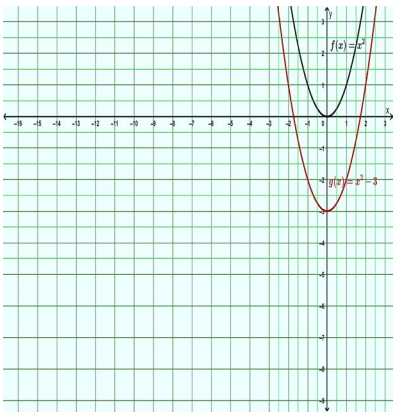


Figure 5.11: *Graphs of* $f(x) = x^2$ *and* $h(x) = x^2 - 3$

The following observations can be made from a comparison of the graphs of $f(x) = x^2$ and $h(x) = x^2 - 3$.

1. They are both U-shaped.
2. While the turning point of $f(x)$ is located at the origin, the turning point of $g(x)$ is located at $(0, -3)$ indicating a vertical shift of the graph of $f(x)$ three units downward.
3. Their turning points are both minimum points.
4. The graphs of $f(x)$ and $g(x)$ do not change in size. There is no shrink or dilation, they are simply translations.

The following conclusions can be made about the effect of changing the value of "C" in a function defined as $f(x) = A(x + B)^2 + C$ from the graphs in Figure 5.10 and Figure 5.11:

1. The value of "$|C|$" indicates the distance of vertical shift of the resulting graph form the graph of $f(x) = x^2$.
2. The greater the value of "$|C|$", the farther the resulting curve is to the graph of $f(x) = x^2$.
3. The smaller the value of "$|C|$", the closer the resulting curve is to the graph of $f(x) = x^2$.
4. When $C < 0$, there is a downward shift.
5. When $C > 0$, there is an upward shift.

FOCAL AREA 3: SKETCHING QUADRATIC CURVES

Example 1

Describe the shape of the graph of $y = x^2 - 4x + 5$ and sketch it.

Solution

$$y = x^2 - 4x + 5$$

$$= x^2 - 4x + 2^2 + 5 - 2^2$$

$$= (x - 2)^2 + 1$$

A comparison of $y = (x - 2)^2 + 1$ to $y = A(x + B)^2 + C$ implies that $A = 1, B = -2$ and $C = 1$.

The graph of $y = x^2 - 4x + 5$ or $y = (x - 2)^2 + 1$ results from shifting the graph of $y = x^2$ to the right by two units and upward by one unit.

The transformation can be represented by the translation vector, $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This translation moves the vertex of the graph of $y = x^2$ from $(0, 0)$ to $(2, 1)$.

Figure 5.12 shows the graphs of $y = x^2$ and $y = x^2 - 4x + 5$ and the translation vector, u.

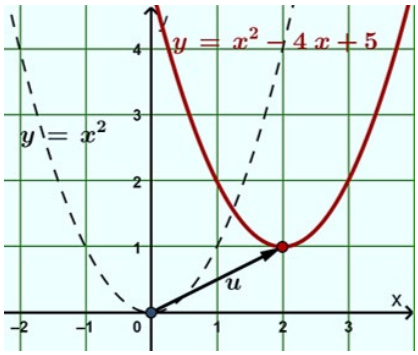


Figure 5.12: *Graphs of* $y = x^2$ *and* $y = x^2 - 4x + 5$

Example 2:

Describe and sketch the graph of $y = -2x^2 + 4$.

Solution

A comparison of $y = -2x^2 + 4$ and $y = A(x + B)^2 + C$ shows that $A = -2, B = 0$ and $C = 4$

To obtain the of graph of $y = -2x^2 + 4$, the graph of $y = x^2$ is shrunk (vertically stretched), reflected about the $x -$ axis and shifted upward by four units.

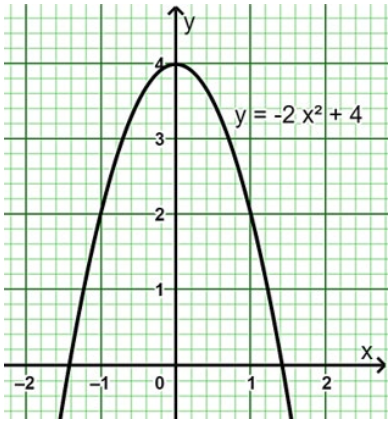


Figure 5.13: *Graph of* $y = -2x^2 + 4$

Learning Tasks

1. Learners discuss in pairs or groups and share observations from real-life parabolas with the class.
2. Learners in groups explore the shapes of a parabola given various quadratic equations.

3. Learners in groups sketch a variety of parabolas using technological tools such as GeoGebra, Geometer's sketchpad, etc., and other available materials or by hand and then investigate how the parabola shape transforms as parameters/coefficients change.
4. Learners in other groups create a quadratics equation for other groups to sketch.

Pedagogical Exemplars

1. Talk-for learning:
   a. Show a series of images (with descriptions) or short video clips (with captions) of real-life parabolas (e.g., bridges, satellite dishes, fountains, and basketball arcs).
   b. Discuss observations from real-life parabolas.
   c. Task learners to share their sketches of quadratic functions and the respective description.
2. Experiential learning:
   a. Learners are given quadratic equations; they are tasked to create tables of values and use graph paper to plot the curves and use colour-coded dots to identify symmetry.
   b. Task learners to sketch and describe the shapes of a parabola given various quadratic equations.
   c. Allow learners to use graphing utilities or their preferred formats.
   d. Use leading questions to support learners to describe their graphs.
3. Collaborative based learning:
   a. Task learners in their small mixed-ability groups to sketch and describe quadratic graphs.
   b. Task learners to take turns to create quadratic functions for their partners to sketch and evaluate them.
   c. Task learners to share their graphs and descriptions with other groups.
4. Provide graph templates with axes marked.
5. Use ICT (e.g., Desmos) with guided prompts.
6. Break tasks into manageable steps with scaffolding.
7. Provide worked examples for reference.
8. Use tactile graphing kits, large print materials, screen readers, and audio instructions.
9. Ensure an accessible classroom layout for mobility aid users to easily reach interactive boards or digital stations.

Key Assessment Level 4: Multiple-Choice Questions

(DOK Level 1, each worth 1 mark)

1. *Which of the following **best** describes the shape of a quadratic function?*
   A. *Circle*
   B. *Straight line*
   C. *U-shaped curve*
   D. *V-shaped curve*

2. *The shape of a quadratic function graph is called a _____*

   A. *ellipse*

   B. *hyperbola*

   C. *line*

   D. *parabola*

3. *Which one of the following graphs shows a quadratic function?*

   A.

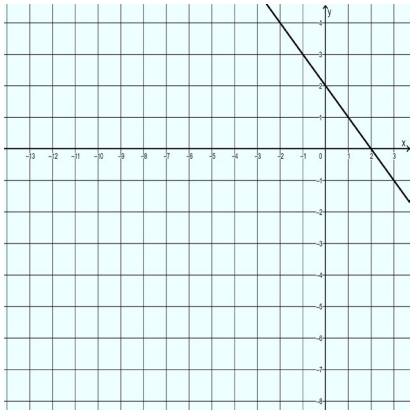


Figure 5.14

   B.

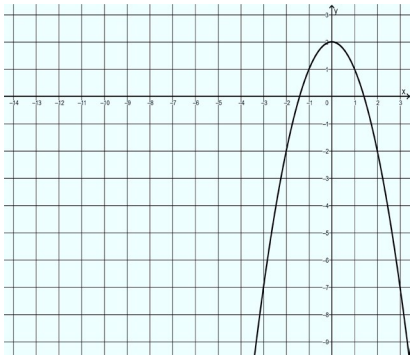


Figure 5.15

   C.

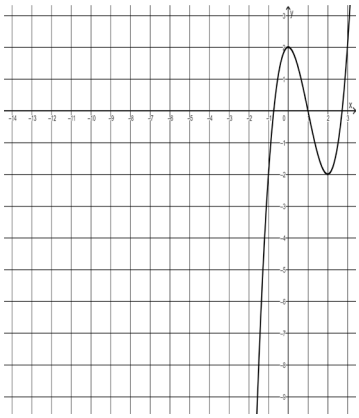


Figure 5.16

D.

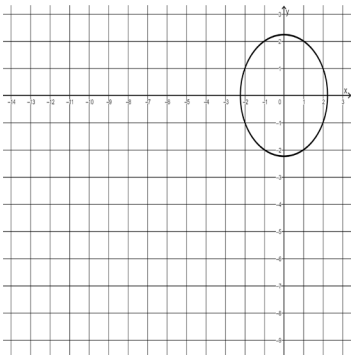


Figure 5.17

4. *The highest or lowest point of a quadratic function graph is called the_____.*

   A. *axis*

   B. *intercept*

   C. *origin*

   D. *vertex*

5. *In the standard form* $y = ux^2 + vx + w$*, which value tells you if the parabola opens upward or downward?*

   A. u

   B. v

   C. w

   D. x

6. *If* $a > 0$ *in the equation* $y = ax^2 + bx + c$*, the parabola opens* _____.

   A. *downward*

   B. *to the left*

   C. *to the right*

   D. *upward*

7. *The graph of* $y = x^2$ *is transformed to* $y = x^2 + 5$*. Which of the following movements represents the transformation?*

   A. *Translated 5 units right.*

   B. *Translated 5 units left.*

   C. *Translated 5 units up.*

   D. *Translated 5 units down.*

8. *The graph of* $y = x^2$ *is transformed to* $y = (x - 3)^2$*. Which of the following movements represents the transformation?*

   A. *Translated 3 units right*

   B. *Translated 3 units left*

   C. *Translated 3 units up*

   D. *Translated 3 units down*

9. *The $y-intercept$ of the quadratic function $y = ax^2 + bx + c$ occurs at the point_____ .*
   A. $(0, c)$
   B. $(c, 0)$
   C. $(a, 0)$
   D. $(0, a)$
10. *How many x-intercepts can a quadratic function have?*
   A. *Exactly one*
   B. *Exactly two*
   C. *Zero, one, or two*
   D. *Any number of intercepts*

Mode: Gamification (DOK LEVEL 4)

Strategy: Quadratic Functions Quest Gallery

1. ***Mission***

   Become a Quadratic Functions Expert by exploring how changing parameters affects graph behaviour through systematic investigation.

2. ***Materials Needed***
   a. *Computer/tablet with GeoGebra or similar graphing software*
   b. *Alternative: Pre-printed graph paper with coordinate grids*
   c. *Poster paper, digital presentation tools, or presentation templates*
   d. *Markers, coloured pencils, rulers*
   e. *Calculator or calculation aids*
   f. *Headphones for students using screen readers/audio assistance*
3. ***Task Instructions***
   a. ***PART 1: Exploration Phase***
      i. *Work with your assigned team*
      ii. *Investigate how changing each parameter (a, b, and c) in $y = ax^2 + bx + c$ affects the graph:*
         - *The vertex position (highest/lowest point)*
         - *Direction of opening (upward/downward)*
         - *Axis of symmetry*
         - *x-intercepts (where the parabola crosses the x-axis)*
      iii. *Multiple Access Points:*
         - *Technology Option: Use GeoGebra with sliders to visualise changes*
         - *Tactile Option: Use manipulatives or raised graph paper*
         - *Structured Option: Follow step-by-step guided worksheets*

iv. *Document your findings using your preferred method:*
- *Screenshots or sketches*
- *Voice recordings*
- *Written observations*
- *Pre-formatted tables*

b. ***PART 2: Gallery Creation***

i. *Prepare your display showing your findings. Choose from these presentation formats:*
- *Visual poster with images and text*
- *Digital slideshow*
- *Audio narration with visual supports*
- *3D model with tactile elements*
- *Video demonstration*

ii. *Every display must include:*
- *Your investigation of parameter effects (in your chosen format)*
- *Explanations suitable for different learning styles (visual, verbal, etc.)*
- *A "Quick Challenge" that can be solved in multiple ways*

c. ***PART 3: Gallery Walk (30 minutes)***

i. *Set up your display in an accessible location*

ii. *Flexible Roles:*
- *Team members can choose roles based on strengths*
- *Options include: presenter, note-taker, question-answerer, visitor*

iii. *Participation Options:*
- *In-person gallery visits*
- *Digital gallery access via tablet*
- *Written response options*

iv. *Switch roles as appropriate for your team*

4. ***Reflection***

a. *Choose your preferred method to reflect:*

i. *Written responses*

ii. *Audio recording*

iii. *One-on-one discussion with teacher*

iv. *Visual representation*

b. *Reflection Prompts:*

i. *What did you discover about how parameters affect quadratic graphs?*

ii. *What strategy helped you understand this topic best?*

iii. *How did your team work together to complete the task?*

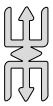
5. ***Support Available***
   a. *Visual step cards for each stage*
   b. *Vocabulary lists with visual references*
   c. *Translation support as needed*
   d. *Additional time accommodations*
   e. *Digital resources with text-to-speech capabilities*

Due Date: Components completed during class with flexible submission options for reflection.

PLC Session 5

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

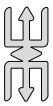
4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***multiple choice questions and gamification***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the assessment (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes

c. the group then works through the problem together step by step

d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 6

Learning Indicators

1. *Describe a given locus as a parabola*
2. *Explore and describe the features of a parabola (focus, directrix, axis of symmetry and vertex) using a technological tool and relate it to real life*

FOCAL AREA 1: DESCRIBING A GIVEN LOCUS AS A PARABOLA

Definition of Terms in a Parabola

1. **A parabola** is defined as the **locus** of a point that moves so that it is always the same distance from a fixed point (called the **focus**) and a given line (called the **directrix**).
2. **The focus** of a parabola is the fixed point located inside the parabola.
3. **Directrix** of a parabola is a line perpendicular to the axis of symmetry.

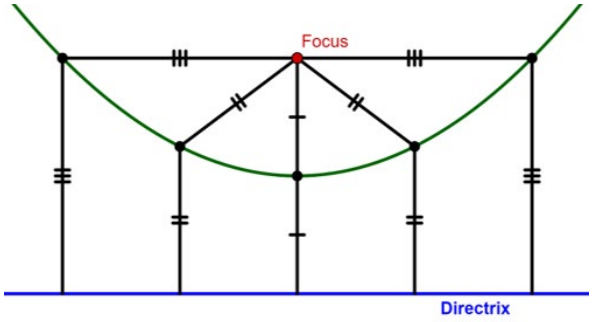


Figure 6.1: *A parabola as a locus of points*

The distance from the focus to the parabola is the same as the distance from the directrix to the parabola

FOCAL AREA 2: DESCRIBING THE FEATURES OF A PARABOLA

In the previous week, we learnt that the basic quadratic function is $f(x) = x^2$ or $y = x^2$. For this function or equation, the independent variable is x while the dependent variable is y.

We could have a quadratic equation where x is dependent on y. This form of quadratic equation is represented by $y^2 = x$. In this week, we will consider the graphs of $y = x^2$ and $x = y^2$. A more general form for $y = x^2$ is $4py = x^2$ and for $x = y^2$ is $4px = y^2$ where p is the distance between the focus and the vertex or the distance between the vertex and the directrix.

Axis of symmetry

All parabolas are bilaterally symmetrical. The axis of symmetry is the line which passes through the turning point (vertex) and the focus thereby splitting the parabola.

Parabolas with equation of the form $y = A(x + B)^2 + C$ open upward. The vertex and focus are located at $V(-B, C)$ and $F\big(-B, (C + p)\big)$ respectively where $p = 1 \div 4A$. The axis of symmetry is the vertical line which passes through V and F and therefore has equation $x = -B$ while the directrix

is the line perpendicular to the axis of symmetry and p units away from the vertex and hence has equation $y = C - p$ as illustrated in Figure 6.2

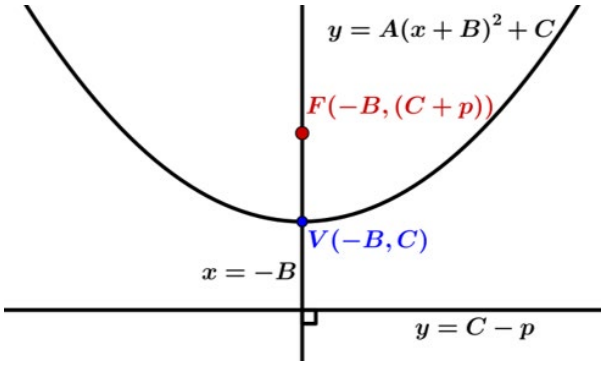


Figure 6.2: *Features of a parabola with equation* $y = A(x + B)^2 + C$

For $y = x^2$, $A = 1, B = 0$ and $C = 0$. $p = 0.25$ hence the vertex and focus are located at $V(0, 0)$ and $F(0, 0.25)$ respectively. The axis of symmetry has equation $x = 0$ and the directrix $y = -0.25$.

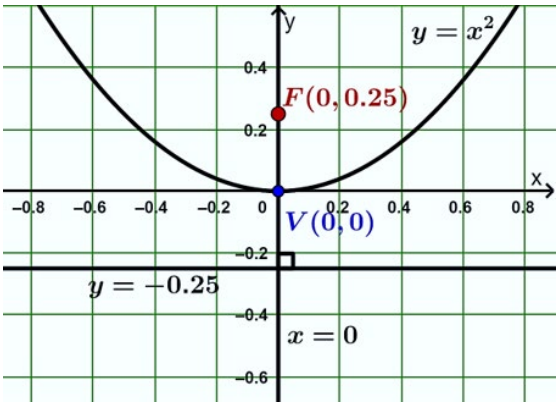


Figure 6.3: *Features of a parabola with equation* $y = x^2$

Parabolas with equation of the form $x = A(y + B)^2 + C$ open rightward. The vertex and focus are located at $V(C, -B)$ and $F(C + p, -B)$ respectively where $p = 1 \div 4A$. The axis of symmetry is the horizontal line which passes through V and F and therefore has equation $y = -B$ while the directrix is the line perpendicular to the axis of symmetry and p units away from the vertex and hence has equation $x = C - p$ as illustrated in Figure 6.4

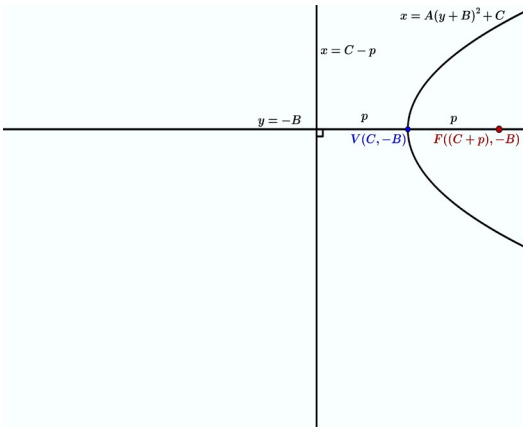


Figure 6.4: *Features of a parabola with equation* $x = A(y + B)^2 + C$

For $x = y^2$, $A = 1, B = 0$ and $C = 0$. $p = 0.25$ hence the vertex and focus are located at $V(0, 0)$ and $F(0.25, 0)$ respectively. The axis of symmetry has equation $y = 0$ and the directrix $x = -0.25$.

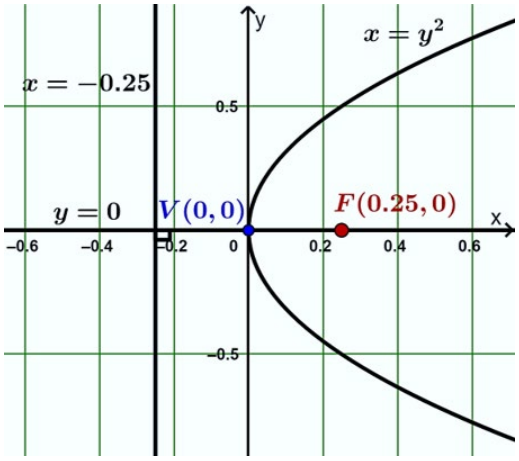


Figure 6.5: *Features of a parabola with equation* $x = y^2$

Example 1

Sketch the curve $x = 2y^2$.

Solution

Comparing $x = 2y^2$ and $x = A(y + B)^2 + C, A = 2, B = 0, C = 0$ and

$p = \frac{1}{4A}$

$p = \frac{1}{4(2)}$

$p = 0.125$

The vertex is located at $(0, 0)$

The focus is located at $F(0.125, 0)$

The directrix has equation: $x = -0.125$

The axis of symmetry has equation $y = 0$

The curve opens towards the right i.e., positive $x-$ axis

The curve is also obtained from a shrink of the curve of $x = y^2$

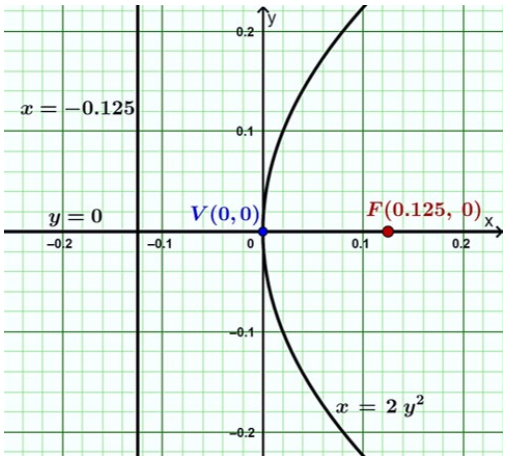


Figure 6.6: *Graph of* $x = 2y^2$

Example 2

Sketch the curve $x = -3y^2$.

Solution

Comparing $x = -3y^2$ and $x = A(y + B)^2 + C, A = -3, B = 0, C = 0$ and

$p = -\frac{1}{12}$

$p \approx -0.08$

The vertex is located at $V(0, 0)$

The focus is located at $F(-0.08, 0)$

The directrix has equation: $x = 0.08$

The axis of symmetry has equation $y = 0$

The curve opens towards the left i.e., negative $x -$ axis

The curve is also obtained from a shrink of the curve of $x = y^2$

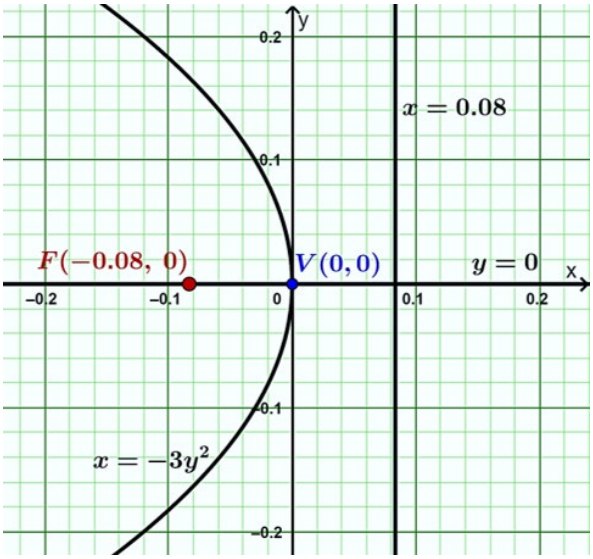


Figure 6.7: *Graph of* $x = -3y^2$

Example 3

Sketch the curve $y = 2x^2 - 12x + 22$.

Solution

$y = 2x^2 - 12x + 22$

$y = 2(x-3)^2 + 4$

Comparing $y = 2(x-3)^2 + 4$ and $y = A(x + B)^2 + C, A = 2, B = -3, C = 4$ and $p \approx 0.13$

The vertex is located at $V = (-B, C) = (3, 4)$

The focus is located at $F = \big(-B, (C + p)\big) = (3, 4.13)$

The directrix has equation: $y = C - p = 3.87$

The axis of symmetry has equation $x = 3$

The curve opens upward

The curve is also obtained from a shrink of the curve of $y = x^2$

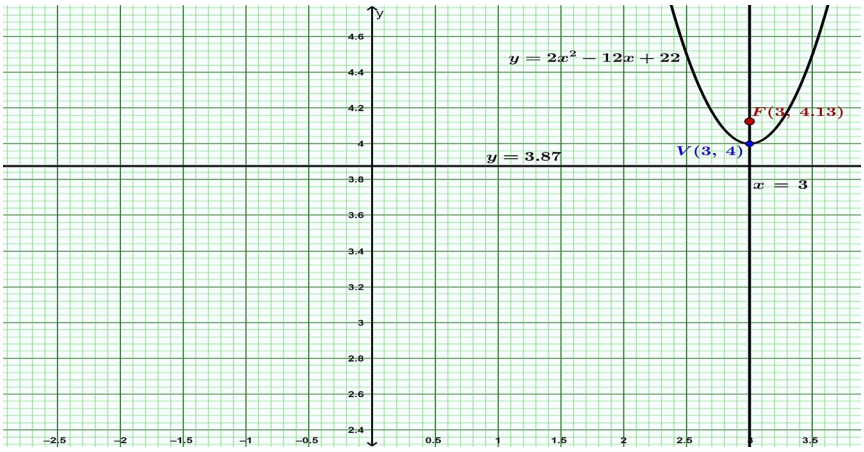


Figure 6.8: *Graph of* $y = 2x^2 - 12x + 22$

Learning Tasks

1. Learners brainstorm to define a parabola using their knowledge in locus and explain some keywords.
2. Learners in groups brainstorm to describe a parabola using visible properties and discuss some keywords in the parabola.
3. Learners in groups brainstorm to discover the relationship between the focus and the directrix.
4. Learners in groups choose an interval to sketch the graph of $y = x^2, y^2 = x, y = -x^2$ and $x = -y^2$ using an ICT tool and by hand and discussing the transformation of the parabolic shapes.
5. Learners in groups compare and contrast the graphs of $y = x^2$ and $y^2 = x$ and analyse how the graphs transform when x is positive or negative.

Pedagogical Exemplars

1. **Experiential learning**
   a. Assist learners to use dynamic geometry software (e.g., GeoGebra) or string constructions to plot points equidistant from a fixed point and a line. They trace and describe the emerging shape.
   b. Given several parabolic diagrams, learners describe how changes in the position of the focus and directrix affect the vertex and curve shape.
   c. Prompt learners to observe the shape formed and discuss their results using terms like “same distance”, “curve”, “focus”, and “directrix”.
2. **Collaborative based learning**
   a. Provide printed parabola diagrams and task learners to identify and label focus, vertex, directrix, and axis of symmetry with peer guidance.

b. Task learners to sketch parabolas given the focus and the directrix.

3. Provide structured worksheets with diagrams.
4. Use tactile materials for visually impaired learners.
5. Use visual aids with consistent colour-coding for each parabola feature.
6. Provide written instructions, captioned video demos, use of diagrams, partner-based activities.

Key Assessment: Mid-Semester Examination

1. *A botanist is studying the natural curvature of a finger of banana. She notices that a cross-section of the banana's curve closely resembles a parabola. For her mathematical model, she places the banana on a coordinate system with the middle at the origin and measures that the banana is 12 centimetres long (along the x-axis) and curves 2 centimetres at its maximum height (along the y-axis).*

   To complete her model, she needs to determine the focus of this parabolic curve. If the banana's curve can be approximated by a parabolic equation, determine the location of its focus. Refer to Figure 6.9

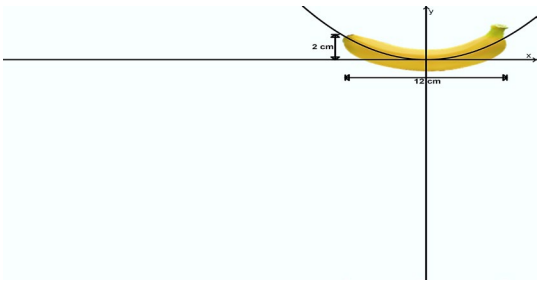


Figure 6.9: *Cross section of a finger of banana*

(10 marks)

2. *Create a collection showing how parabola features appear in different structures.*
   a. *Show the structure with the parabola highlighted (using photos, drawings, or diagrams).*
   b. *Mark and explain these features: focus, directrix, axis of symmetry, and vertex.*
   c. *Explain why this structure benefits from being parabolic.* ***(20marks)***
3. *Investigate how parabolas appear in a sport or game of your choice.*
   a. *Show the parabolic path in your chosen activity.*
   b. *Mark and explain what the vertex, axis of symmetry, and other features mean in this context.*
   c. *Explain how changing the action (throwing harder, changing angle) affects the parabola.*
   d. *Connect the mathematical features to success in the activity.*

(20 marks)

HINT

*Conduct **Mid-semester Examination**. It should cover contents taught from Week 1 to 5. Refer to the **Table of Specification and Structure of the Paper**. Learners' papers should be marked immediately and scores entered into the SAP.*

SUGGESTED STRUCTURE OF THE MID-SEMESTER EXAMINATION

Section A: 25 Multiple-choice questions

Section B: 3 Essay Questions

Table of specification for item construction for section A and B

Week	Learning Indicator	Types of questions	DoK Levels				Total
			1	2	3	4	
1	Form statements including negation statements and comment on them Draw implications from given statements and their converses	MCQs	2	2	-		4
		ESSAY	-	-	-		-
2	Identify simple statements and construct compound statements using the connectives Use conjunctions, disjunctions, implications and negations to construct truth tables of compound statement	MCQs	2	2	2		6
		ESSAY	-	-	1		1
3	Find the equation of the image of a line under a linear transformation Use linear transformation to determine image and object point	MCQs	-	2	1		3
		ESSAY	-	-	-		
4	Find the composition of linear transformations Apply linear transformation in finding translations, reflections, rotations and enlargement of points and plane figures	MCQs	3	2	1		6
		ESSAY			1		

5	Describe the shape of the graphs of quadratic functions Explore the graphs of quadratic functions with different parameters using technological tools such as GeoGebra, Geometer's sketchpads, etc., other available materials, or by hand	MCQs	1	3	2		6
		ESSAY			1		1
		Total	8	11	9		28

Mid-Semester Examination

Note

The Mid-semester examination should consist of 25 multiple choice questions and 3 essay type questions. Candidates will answer all the questions. Some sample questions are provided.

Sample Questions

Part A (15 marks)

The following are multiple-choice questions with options lettered A to D. Read each question carefully and choose the correct answer.

1. *Given the matrix* $D = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, *determine the image of* $R(-3, 8)$ *under the transformation of* D.
   A. $R'(-3, 21)$
   B. $R'(3, 21)$
   C. $R'(-3, 27)$
   D. $R'(3, -27)$

DoK Level 1

2. *If* $T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$ *find the matrix of the transformation* T *followed by* V.
   A. $\begin{bmatrix} -1 & 2 \\ -2 & 14 \end{bmatrix}$
   B. $\begin{bmatrix} 4 & 23 \\ 2 & 9 \end{bmatrix}$
   C. $\begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix}$
   D. $\begin{bmatrix} -3 & -3 \\ 2 & 3 \end{bmatrix}$

DoK Level 2

3. *If M is the statement "3 is a prime number and an odd number" What type of compound statement is M?*

    A. *Biconditional*

    B. *Conjunction*

    C. *Disjunction*

    D. *Implicative*

DoK Level 1

4. *Given the statements "N: 14 is a natural number" and "H: 5 is exactly divisible by 3". Determine the truth value of* $N \Longrightarrow H$.

    A. *False*

    B. *True*

5. *If a quadratic function has its graph entirely above the* x*-axis, which statement* ***could*** *be* ***false****?*

    A. *Its discriminant is negative*

    B. *Its leading coefficient is positive*

    C. *Its constant term is positive*

    D. *It has no real roots*

DoK Level 3

6. *If a parabola has two* x*-intercepts, which of the following statements can be* ***true****?*

    A. *The discriminant equals zero*

    B. *Its equation is of the form* $4py = x^2$

    C. *The vertex is on the x-axis*

    D. *The vertex is located at the origin*

DoK Level 1

7. *Which of these quadratic functions has the* ***widest*** *parabola?*

    A. $y = 4x^2$

    B. $y = 2x^2$

    C. $y = 0.5x^2$

    D. $y = 0.25x^2$

DoK Level 2

8. *When a parabola is reflected across the x-axis, which part of the equation* $y = ax^2 + bx + c$ *changes sign?*

    A. *a only*

    B. *b only*

    C. *c only*

    D. *a, b, and c*

DoK Level 3

9. *The axis of symmetry of a parabola always_____.*

   A. *Passes through the origin*

   B. *Passes through the vertex*

   C. *Is horizontal*

   D. *Crosses the y-axis*

DoK Level 1

10. *If the graph of a quadratic function has x-intercepts at* $x = 1$ *and* $x = 7$*, where is its axis of symmetry?*

    A. $x = 0$

    B. $x = 3.5$

    C. $x = 4$

    D. $x = 8$

DoK Level 2

Part B (35 marks)

Answer all the following questions.

1. *Let P. O. and R be statement variables. Use a truth table to prove whether or not:* $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \equiv [P \Rightarrow R]$.

(10 marks)

DoK Level 2

2. *Given the matrices* $V = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ *and* $P = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$*. Determine*

   a. $V \circ P$

(3 marks)

DoK Level 1

   b. *the image of P when reflected in the y-axis then rotated about the origin at* $70°$ *anticlockwise.*

(7 marks)

DoK Level 2

(Total 10 marks)

3. *In Northern Ghana, solar energy projects are becoming increasingly important as a renewable energy source. One type of solar collector used is a parabolic trough, which focuses sunlight onto a pipe containing oil or water.*

   A solar energy project in Tamale uses mirrored parabolic troughs to heat water for electricity generation. The parabolic trough focuses rays of sunlight onto a pipe that contains water. As the water heats up, it produces steam that turns a turbine to generate electricity. For maximum efficiency, the centre of the pipe must be located at the focus of the parabolic trough.

   The cross-section of the trough is a parabola with equation $y = 0.25x^2$ *(measurements in metres) as shown in Figure 6.10. The sun's rays strike the parabola parallel to the y-axis when the sun is directly overhead.*

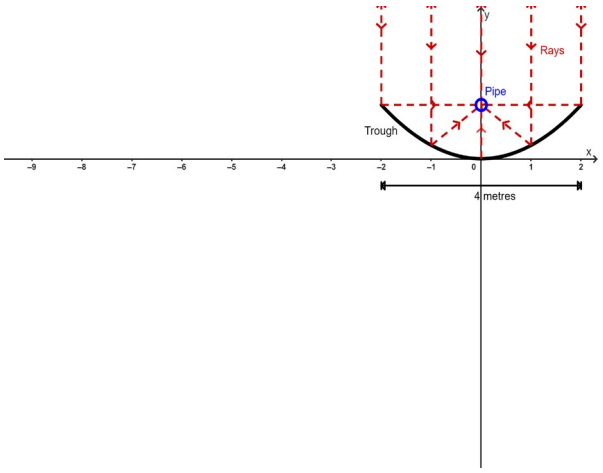


Figure 6.10: *Cross section of trough*

Note

Figure 6.10 is not drawn to scale

Questions:

a. *What is the distance from the vertex to the centre of the pipe?*

(4 marks)

b. *If the width of the trough is 4 metres, what is the depth of the trough?*

(4 marks)

c. *Give two features of the parabolic shape that makes it more effective for solar collection than a semi-circular shape.*

(2 marks)

d. *How would the focus position change if we modified the equation to* $y = 0.4x^2$*?*

(3 marks)

e. *State two ways this technology could benefit local communities in Ghana's Northern Region?*

(2 marks)

(Total 15 marks)

DoK Level 3

PLC Session 6

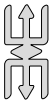
45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous session, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand.
   b. identify three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to

a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.

b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q).

c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Use your app to plan and administer the mode of assessment for the Student Assessment Portal (SAP).

   a. The key assessment for the week is ***mid-semester examination***. Chat with your app to plan this assessment by reviewing or developing the assessment items/tasks and marking schemes/rubrics for the mid-semester examination (NTS 3k-3q).

10 minutes **Content Exploration**

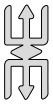
5. As a subject group, select one problem or topic/idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then

   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)

   b. all teachers in the group work through the problem individually for 3–4 minutes

   c. the group then works through it together the problem, step by step

   d. highlight the most common errors made at each step (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).

   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.

   b. After the enactment, ask for structured feedback from the group

      i. was the concept explained in a way that a learner with no prior knowledge could follow?

ii. were the examples grounded in contexts familiar to Ghanaian learners?

iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 7

Learning Indicators

1. *Apply the formula of distance between two points to find the general equation of a parabola*
2. *Deduce the directrix and focus from a parabolic equation and vice-versa*

FOCAL AREA 1: APPLYING THE FORMULA OF DISTANCE BETWEEN TWO POINTS TO FIND THE GENERAL EQUATION OF A PARABOLA

In Week 6, we defined a parabola as a locus of points equidistant from the focus and the directrix. In this lesson, this definition will be relied on to find a general equation for different orientations of parabolas.

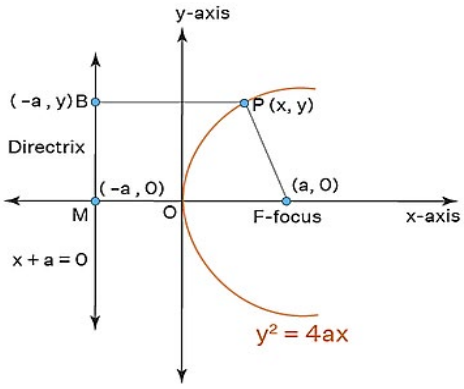


Figure 7.1: *A parabola which opens toward the positive x − axis*

The distance formula is given as $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Using the diagram, the relationship between the focus and the directrix implies that

$|PF| = |PB|$

$\therefore \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x+a)^2 + (y-y)^2}$

$(x-a)^2 + y^2 = (x+a)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$y^2 = 2ax + 2ax$

$y^2 = 4ax$

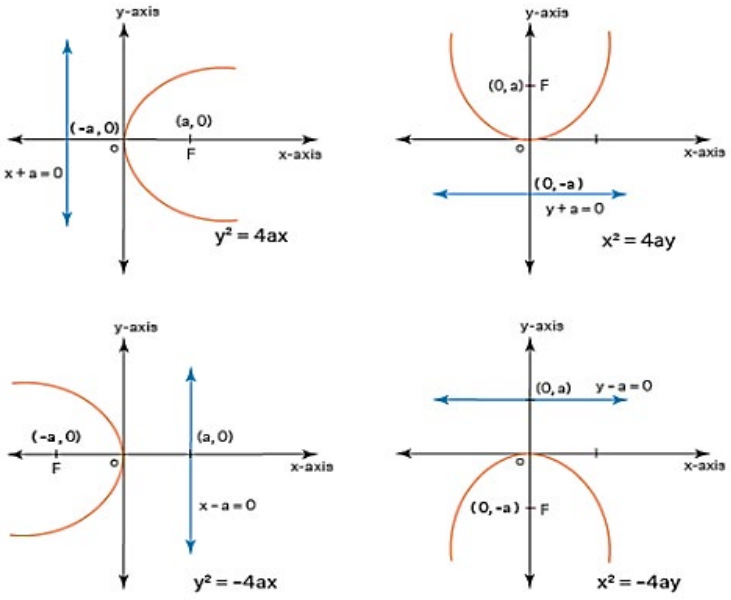


Figure 7.2: *Parabolas with vertex at the origin but with different orientations*

Observe that:

1. If the equation has the term with y^2, then the axis of symmetry is along the $x - axis$, and if the equation has the term with x^2, then the axis of symmetry is along the y-axis.
2. When the equation has the term with y^2, the parabola opens to the right if the coefficient of the x is positive and opens to the left if the coefficient of x is negative.
3. When the equation has the term with x^2, the parabola opens upwards if the coefficient of y is positive and opens downwards if the coefficient of y is negative.

Deriving standard equation of a parabola with vertex not at the origin

1. If the vertex of the parabola is not at the origin, then it is ***translated***.
2. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k).
3. This translation will cause x to be replaced with $(x - h)$ and y replaced with $(y - k)$ in the standard form of the equation.

Example 1

Find the equation of a parabola with a focus at $(0, 4)$ and vertex at $(0, 0)$.

Solution

The vertex is located at the origin, $(0, 0)$ and focus at $(0, 4)$ therefore the parabola opens up on the y–axis.

The equation of the parabola is given by

$x^2 = 4py$

$Distance\ between\ the\ vertex\ and\ focus = 4\ units$

The required equation is obtained from

$x^2 = 4(4)y$

$x^2 = 16y$

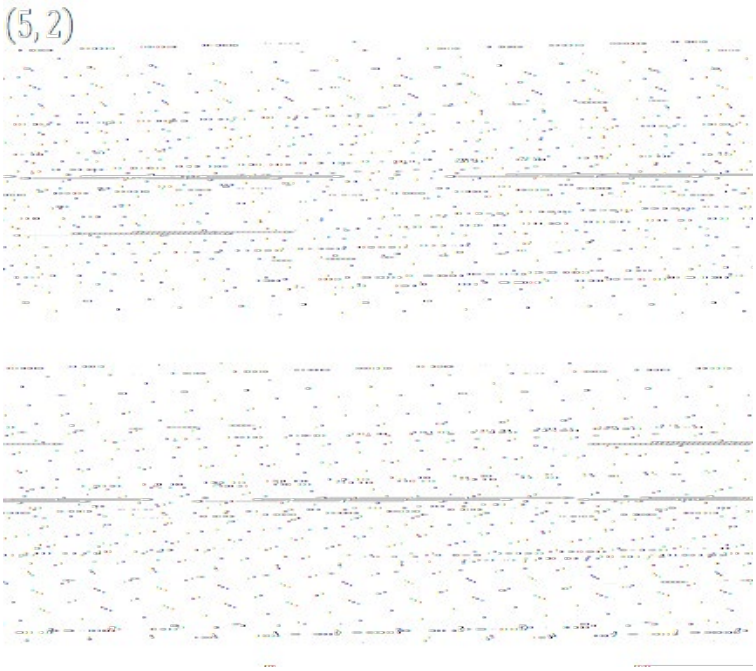


Figure 7.3: *Translated parabolas with different orientations*

Example 2

Find the equation of a parabola with a focus at $(5, 2)$ and vertex at $(-3, 2)$.

Solution

The parabola with vertex at $(-3, 2)$. and focus at $(5, 2)$ parabola opens to the right.

Since the vertex is not located at the origin, the equation of the parabola is given by

$(y - k)^2 = 4p(x - h)$

$h = -3, k = 2$ and

$Distance\ between\ the\ vertex\ and\ focus, p = 5 - (-3) = 8\ units$

The required equation is

$(y - 2)^2 = 32(x + 3)$

FOCAL AREA 2: DEDUCING THE DIRECTRIX AND FOCUS FOR A PARABOLIC EQUATION AND VICE VERSA

Example 1

Find the focus and directrix of $y^2 = -8x$.

Solution

Compare the equation given to the standard equation of a parabola, i.e. $y^2 = 4px$

$\therefore 4p = -8$

$p = -2$

Therefore, the focus of the parabola is $(-2, 0)$, and the directrix is $x = 2$.

Example 2

Find the directrix and focus of $(y-1)^2 = -16(x+3)$.

Solution

Since $(y-1)^2 = -16(x+3)$ is not of the form $y^2 = 4px$ or $x^2 = 4py$, it represents a parabola whose vertex is not located at the origin.

Comparing $(y-1)^2 = -16(x+3)$ to $(y-k)^2 = 4p(x-h)$, $h = -3$ and $k = 1$

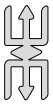
Also $4p = -16$, and $p = -4$

Since $p < 0$, the parabola opens left, $\therefore$ *the cordinates of the focus is* $(h+p, k)$

$\therefore focus = (-3-4, 1)$

$= (-7, 1)$

Directrix is $y = k$, so in this case it is $y = 1$



Note

In general, the equation of the parabola with focus (a, b) *and directrix* $y = c$ *is* $(x-a)^2 + b^2 - c^2 = 2(b-c)y$.

Learning Tasks

1. Learners in mixed-ability groups derive the standard equations of a parabola with the vertex at the origin.
2. In mixed-ability groups, learners work to resolve the expression of the relationship between directrix and focus $(a, 0)$ to obtain the equation of a parabola that is $y^2 = 4ax$.
3. Learners in their groups brainstorm and share ideas on how to derive the other three standard equations of a parabola and state their observations.
4. Learners in their mixed-ability groups brainstorm and explore the equation of a parabola with the vertex not at the origin.

5. Learners in mixed-ability groups deduce the directrix and focus of a parabola as well as write the equation of a parabola given its focus and directrix.
6. Learners in their mixed-ability group brainstorm to deduce the directrix and focus of a given parabola with vertex at the origin.
7. Learners in their mixed-ability groups create and solve questions on deducing the directrix and focus of a given parabola with vertex at the origin.
8. Learners in their mixed-ability groups to brainstorm to deduce the directrix and focus of a given parabola with the vertex not at the origin.
9. Learners in their mixed-ability groups create and solve questions on deducing the directrix and focus of a given parabola with vertex not at the origin.
10. Learners in their mixed-ability groups brainstorm to make generalisations on how to derive the equation of a parabola given its focus and directrix.
11. Learners in mixed-ability draw a parabola and label the coordinates of the focus and directrix.

Pedagogical Exemplars

1. Experiential learning:
   a. Draw a diagram on the board with a line (directrix), a point (focus), and an arbitrary point on the plane (moving point on the parabola).
   b. Derive the equation for a parabola with vertex at the origin using: Distance from a variable point (x, y) to the focus $(0, p)$ = Distance from (x, y) to the directrix $y = -p$.
   c. Use a step-by-step scaffolded worksheet where learners fill in missing parts of the derivation with teacher support.
   d. Use visual organisers or templates to structure the derivation.
   e. Provide both symbolic and verbal instructions.
   f. Task learners to deduce the focus and the equation of the directrix from a given equation for a parabola.
   g. Provide frequent positive reinforcement and feedback during graphing and solving tasks.
2. Talk-for learning:
   a. Initiate a whole-class discussion on what it means for a point to be equidistant from a fixed point and a line.
   b. Recall the definition of a parabola as a locus of points equidistant from the focus and directrix.

Key Assessment Level 2: Classwork

1. Find the general equation of a parabola with focus at $F(0, 3)$ and directrix at $y = -3$. Find out if the point $P(7, 4)$ lies on this parabola. **(5 marks)** *DoK Level 2*
2. The focus of a parabola is at $F(2, 0)$ and the directrix is the vertical line $x = -2$.

   Use the distance formula to find the general equation of this parabola.

   Identify the axis of symmetry and vertex.

   (5 marks) *DoK Level 2*

3. A designer is creating a curved canopy with a parabolic cross-section. The highest point of the canopy is 3 metres above ground, and the focus of the parabola is 0.75 metres below the highest point.

   Use the distance formula to find the equation of the parabolic curve, with the axis of symmetry on the $y-$ axis. **(5 marks)** *DoK Level 2*

4. A civil engineer needs to design a parabolic arch for a bridge. The arch must have its highest point 20 metres above water level and must span 60 metres across the river.

   a. Find a parabolic equation that satisfies these constraints.

   (5 marks)

   b. Determine the focus and directrix of this parabola.

   (3 marks)

   c. If the focus needs to be at least 25 metres above water level for structural integrity, how must the design be modified?

   (2 marks)

   (Total 10 marks)

   DoK Level 3

5. A student claims that the equation $x^2 + y^2 - 6x + 2y + 5 = 0$ represents a parabola with focus at $(3, -1)$.

   a. Analyse this claim and determine whether it is correct.

   (2 marks)

   b. Modifv the equation to create a true parabola with focus at $(3, -1)$ and a directrix at $y = -5$.

   (3 marks)

   (Total 5 marks)

   DoK Level 2

PLC Session 7

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).

Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***classwork***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the class work (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).

8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
    a. transfer your chats into your learning planner template.
    b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
    c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 8

Learning Indicators

1. *Describe the appropriate orientation of the parabola to the given the given equation*
2. *Use directrix and focus to sketch a parabola*

In the previous weeks in this section, learners have considered the four orientations of a parabola. They have learnt to deduce the equation of parabolas given the focus and directrix. In this week's lesson, they will be required to apply all that they learnt to sketch parabolas.

FOCAL AREA 1: DESCRIBING THE ORIENTATION OF A PARABOLA WITH THE GIVEN EQUATION

Before we zoom into sketching parabolas, we will try to describe various parabolas with the information obtained from their equations.

Example 1

Describe the orientation of the parabola with equation $(y+4)^2 = -(x-1)$.

Give the parabola's vertex and focus.

Solution

$(y+4)^2 = -(x-1)$

$\Rightarrow x = -(y+4)^2 + 1$

The resulting graph therefore has the following properties:

1. The vertex is located at $(1, -4)$.
2. The focus is located at $(0.75, -4)$.
3. The parabola thus has a horizontal axis of symmetry and opens leftward.

Example 2

Describe the orientation of the parabola with equation $(x-3)^2 = 2(y+1)$.

Give the parabola's vertex and focus.

Solution

$(x-3)^2 = 2(y+1)$

$\Rightarrow y = 0.5(x-3)^2 - 1$

The resulting graph therefore has the following properties:

1. The vertex is located at $(3, -1)$.

2. The focus is located at $(3, -0.5)$.
3. The parabola thus has a vertical axis of symmetry and opens upward.

FOCAL AREA 2: SKETCHING PARABOLAS GIVEN DIRECTRIX AND FOCUS

Example 1

Sketch the parabola with focus located at $(3, -3)$ and directrix $x = 4$.

Solution

Since the directrix is a vertical line and the focus is located at its left side, we expect the parabola to open leftward.

The vertex should be located midway between the focus and the directrix and will have its x coordinate as $\frac{3+4}{2} = 3.5$. Hence the vertex has is located at $(3.5, -3)$.

$Distance\ between\ the\ focus\ and\ the\ vertex, p = 3.5 - 3 = 0.5$

The parabola opens to the left as the vertex is located to the right of the focus

The equation of the parabola is of the form: $(y - k)^2 = -4p(x - h)$

Where $h = 3.5$ and $k = -3$

$(y + 3)^2 = -2(x - 3.5)$

When $x = 0, y = -0.35$ or $y = -5.65$

The y- intercepts are $(0, -0.35)$ and $(0, -5.65)$

When $y = 0, x = -1$

The x- intercept is $(0, -1)$

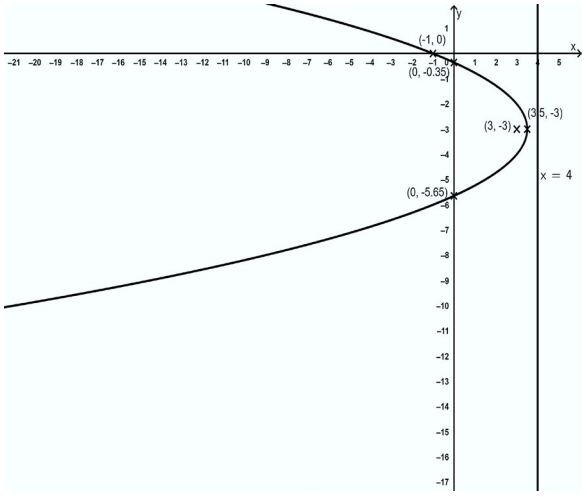


Figure 8.1: *Parabola with focus at* $(3, -3)$ *and directrix* $x = 4$

Example 2

Sketch the parabola whose focus is located at $(3, 4)$ and has a directrix with equation $y = -3$.

Solution

The directrix is a horizontal line and the focus is located at above it. It is therefore expected that the parabola opens upwards.

$Vertex = \left(3, \frac{4-3}{2}\right) = (3, 0.5)$

$Distance\ between\ the\ focus\ and\ the\ vertex, p = 4 - 0.5 = 3.5$

The equation of the parabola is of the form: $(x - h)^2 = 4p(y - k)$

Where $h = 3$ and $k = 0.5$

$(x - 3)^2 = 14(y - 0.5)$

When $x = 0, y = 1.14$

The y- intercept is $(0, 1.14)$

The parabola does not intercept the x-axis the vertex is located above the x-axis and the parabola opens upwards

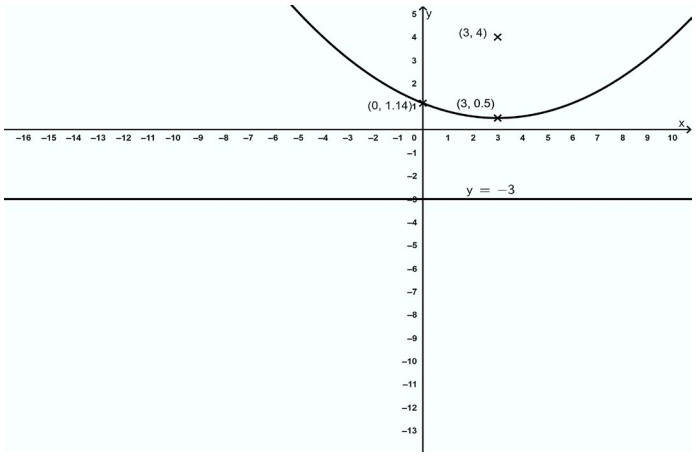


Figure 8.2: *Parabola with focus at* $(3, 4)$ *and directrix* $y = -3$

Learning Tasks

1. Learners in the various groups recall the equation of a parabola, its directrix, vertex, and focus.
2. Learners in their mixed-ability groups associate the correct orientation of a parabola with its equation.
3. Learners work in groups to create about 5 parabolas for a project in class. Learners, in groups, play a simple game of mapping given curves to given parabolic equations.
4. Learners in groups through Talk for Learning apply their knowledge in a parabola to establish how they can use the directrix and focus to draw a parabola.
5. Learners work in groups to create about 5 parabolas for a project in class. Learners, in groups, play a simple game of mapping given curves to given parabolic equations.

Pedagogical Exemplars

1. Talk-for learning
    a. Task learners to share with their peers the orientations of given parabolic graphs and share the features that they based on to make the decision.
    b. Learners take turns explaining how they deduced the focus and directrix to a partners
    c. Rotate between groups, guiding reasoning with probing questions.
    d. Provide parabolic equations and ask learners to determine the orientation of the respective graphs.
    e. Use tactile shapes or 3D parabolic models for visually impaired learners.
    f. Allow learners to use their preferred accessible means of communication.
2. Experiential learning:
    a. Allow use of digital graphing software to manipulate parabola equations.
    b. Provide touchscreen or voice-activated tools for exploring different orientations and components of a parabola.
    c. Use tactile graph sheets and raised-line diagrams to show orientation and key features.
    d. Present steps for finding directrix/focus in visually sequenced slides with colour codes.
    e. Provide repetitive practice with increasing independence.
3. Collaborative game based learning: Engage learners in the "Parabola Orientation Battle" game. Refer to key assessment for details on this game and rubrics for assessing same.
4. Problem-based learning:
    a. Provide learners with various parabolic equations and task them in small mixed-ability groups to find the vertices, determine the orientation and determine the location of the focus and equation of the directrix.
    b. Describe real-life applications of parabolic functions or shapes and task learners to determine the features (vertices, foci and directrix) of the parabola used and predict the equations which model them.

Key Assessment Level 3: Gamification – "Parabola Orientation Battle"

Game Description

Students compete in teams in this strategic card game. Each team has a deck of "equation cards" and "orientation cards." Teams take turns playing an equation card, and opponents must quickly match it with the correct orientation card and identify vertex, focus, directrix and explain their reasoning. Strategic play involves choosing equations that might confuse opponents.

Setup

1. *Equation cards containing various forms of parabola equations.*
2. *Orientation cards showing "Opens Up," "Opens Down," "Opens Left," "Opens Right".*
3. *Bonus cards for more complex orientations (parabolas rotated by specific angles).*
4. *Score tracker for team points.*

Game Rules

1. *Teams take turns playing equation cards*
2. *Opposing team has 60 seconds to:*
    a. *Match with correct orientation card*
    b. *Identify vertex, focus, and directrix positions*
    c. *Explain how they determined the orientation*
3. *Points awarded for:*
    a. *Correct orientation*
    b. *Correct identification of features*
    c. *Complete mathematical justification*
4. *Strategic elements*
    a. *"Challenge" option to dispute opponents' answers*
    b. *"Transform" cards that allow teams to rewrite equations in different forms*
    c. *"Real-world" bonus cards worth double points*

Homework

1. *Design a logo for a technology company that incorporates at least three different parabolas with specific focus-directrix relationships.*

    Requirements:

    a. *Create a design that includes:*
        i. *One parabola opening upward with focus* $(0, 4)$ *and directrix* $y = -4$
        ii. *One parabola opening rightward with focus* $(3, 0)$ *and directrix* $x = -3$
        iii. *One parabola of your choice with focus and directrix clearly indicated*
    b. *Use the focus-directrix method to accurately sketch each parabola.*
    c. *Explain how the parabolas intersect and identify the coordinates of intersection points.*
    d. *Write a reflection on how the focus-directrix approach helped you create precise curves.*
    e. *Explain how your design represents concepts related to the technology company (such as reflection, focus, or precision).* ***(20marks)***

2. *You are designing a parabolic solar cooker that must focus sunlight onto a cooking vessel 15 cm above the vertex of the parabola.*

    Requirements:

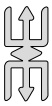
    a. *Determine the focus and directrix for your design.*
    b. *Create a precise scale drawing showing the parabola with focus-directrix method.*
    c. *Choose at least 8 strategic points and show the equidistance property at each point.*
    d. *Calculate the diameter needed at the opening to capture a significant amount of sunlight.*
    e. *Explain how changing the focus-directrix distance would affect cooking efficiency.*
    f. *Design a method to physically construct this parabola using string, tacks, and a straightedge.*

    (20 marks)

PLC Session 8

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
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   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



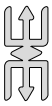
Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***gamification***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the assessment (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
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6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
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5 minutes Reflection

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 9

Learning Indicators

1. *Describe the equation of lines and derivatives of functions*
2. *Determine the equations of tangents and normal to a parabola*
3. *Find the point of intersection for a line and a parabola*

During the lessons in week 13 of the year two teacher manual, learners learnt to find the equation of parallel and perpendicular lines. In week 18, they learnt to determine the equation and normal to curves at specific points. This week's lessons will build on this to assist learners to describe the equation of lines and derivatives of function and determine the equations of tangents and normals to parabolas.

FOCAL AREA 1: DESCRIBING THE EQUATION OF LINES AND DERIVATIVES OF FUNCTIONS

For a linear equation, $y = mx + c$,

1. The value of the y intercept is the y coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $x = 0$.
2. The value of the x intercept is the x coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $y = 0$.
3. m is the slope or gradient of the line. It is the ratio of the increment / decrement of values of the dependent variable to the increment / decrement of values of the independent variable. It is often written as $m = \frac{\Delta y}{\Delta x}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$ where (x_1, y_1) and (x_2, y_2) are points on the line. Note that, $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ since x_1 and y_2 are not corresponding coordinates just as x_2 and y_1 are not coordinates of the same point.
4. If $m > 0$, the graph strictly increases and when $m < 0$, it strictly decreases.

The derivative of a function provides the gradient function. The derivative can be used to obtain the gradient at any point on the graph of original function.

For parabolas (quadratic functions), the derivative will be a linear equation. The gradient function also provides the gradient of the tangent line.

FOCAL AREA 2: DETERMINING AND DESCRIBING THE EQUATIONS OF TANGENTS AND NORMALS TO A PARABOLA

Equation of a tangent to a parabola

The slope of a tangent is given by $\frac{dy}{dx}$.

Differentiating $y^2 = 4ax$ on both sides with respect to x will be

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

Therefore, the slope (m) of the tangent to the parabola at the point (x_1, y_1) on it is $\frac{2a}{y_1}$.

Now, the equation of the tangent to the parabola at the point (x_1, y_1) on it is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Longrightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

But $y_1^2 = 4ax_1$

$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

$\therefore yy_1 = 2a(x + x_1)$ is the equation of the tangent at a point (x_1, y_1) on the parabola.

For a parabola with equation of the form $x^2 = 4ay$,

Differentiating $x^2 = 4ay$ on both sides with respect to x will be

$$2x = 4a\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

Therefore, the slope (m) of the tangent to the parabola at the point (x_1, y_1) on it is $\frac{x_1}{2a}$.

Now, the equation of the tangent to the parabola at the point (x_1, y_1) on it is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - y_1 = \frac{x_1}{2a}(x - x_1)$$

$$\Longrightarrow xx_1 - x_1^2 = 2ay - 2ay_1$$

But $x_1^2 = 4ay_1$

$$\Rightarrow xx_1 - 4ay_1 = 2ay - 2ay_1$$

$\therefore 2ay = xx_1 - 2a(2y_1 - y_1)$ is the equation of the tangent at a point (x_1, y_1) on the parabola.

Equation of a normal to a parabola

Recall that the relationship between the gradients of two lines which a perpendicular, in this case, a tangent and a normal, given that their gradients are m_1 and m_2 is $m_1 \times m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

Since the gradient of the tangent to the parabola with equation $y^2 = 4ax$ at (x_1, y_1) is $\frac{2a}{y_1}$, the slope (m) of the normal is $-\frac{y_1}{2a}$ since normal and tangent are perpendicular to each other.

$\therefore$ The equation of the normal at a point (x_1, y_1) on the parabola becomes

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$\Longrightarrow y_1x - 2ay_1 - x_1y_1 + 2ay = 0$$

Similarly, the slope (m) of the normal to the parabola with equation $x^2 = 4ay$ at (x_1, y_1) is $-\frac{2a}{x_1}$ since the gradient of the tangent is $\frac{x_1}{2a}$.

$\therefore$ The equation of the normal at a point (x_1, y_1) on the parabola becomes

$$y - y_1 = -\frac{2a}{x_1}(x - x_1)$$

$$\Rightarrow x_1 y + 2ax - x_1 y_1 - 2ax_1 = 0$$

Example 1

A Parabola $y^2 = 8x$ passes through the point $P(2, 4)$. Find:

1. equation of the tangent
2. equation of the normal to the parabola at the point P

Solution

If $y^2 = 8x$, then comparing it to $y^2 = 4ax$

$a = 2$

1. Equation of the tangent to the parabola is given as

   $yy_1 = 2a(x + x_1)$

   $\Rightarrow y(4) = 2 \cdot 2(x + 2)$

   $y = x + 2$

   Alternatively,

   $y^2 = 8x$

   $\Rightarrow 2y\frac{dy}{dx} = 8$

   $\frac{dy}{dx} = \frac{4}{y}$

   The gradient of the tangent line at P, $\left.\frac{dy}{dx}\right|_{y=4} = \frac{4}{4} = 1$

   The equation of the tangent line that passes through P:

   $y - 4 = 1(x - 2)$

   $y = x + 2$

2. The equation of the normal is given by

   $y_1 x - 2ay_1 - x_1 y_1 + 2ay = 0$

   $\Rightarrow 4x - 2(2)4 - (2)(4) + 2(2)y = 0$

   $y = -x + 6$

   Alternatively,

   $The\ gradient\ of\ the\ normal\ line\ at\ P = -\frac{1}{1} = -1$

   The equation of the tangent line that passes through P:

   $y - 4 = -1(x - 2)$

   $y = -x + 6$

FOCAL AREA 3: FINDING THE POINT OF INTERSECTION BETWEEN A LINE AND A PARABOLA

Example

The line $2y + 3x - 6 = 0$ intersects the parabola $y^2 = 6x$ at K and M respectively.

1. Find the coordinates of K and M.
2. Find the equation of the normal to the parabola at the point $\left(\frac{2}{3}, 2\right)$.

Solution

$2y + 3x - 6 = 0 \ldots..(1)$

$y^2 = 6x \ldots.\ldots\ldots\ldots\ldots..(2)$

From (1), $x = \frac{6-2y}{3}$ and from (2), $x = \frac{y^2}{6}$

Since the line and parabola intersect, (1)=(2)

$\frac{6-2y}{3} = \frac{y^2}{6}$

$y^2 + 4y - 12 = 0$

$y = 2$ or $y = -6$

When $y = 2,\ x = \frac{2}{3}$ and when $y = -6, x = 6$

$\therefore K = \left(\frac{2}{3}, 2\right)$ and $M = (6, -6)$

1. $2ay - 2ay_1 + xy_1 - x_1y_1 = 0$

   Since $y^2 = 6x$, $a = \frac{2}{3}$

   The point is $\left(\frac{2}{3}, 2\right)$ therefore, the equation of the normal is

   $2\left(\frac{3}{2}\right)y - 2\left(\frac{3}{2}\right)\cdot 2 + (2)x - 2\cdot\left(\frac{2}{3}\right) = 0$

   $9y + 6x - 22 = 0$

Learning Tasks

1. Learners in pairs recall how to find derivatives of functions and create and pose problems in derivates.
2. Learners in mixed-ability groups investigate the equation of the tangent and the equation of the normal to the parabola.
3. Learners in their mixed-ability groups apply their knowledge in finding derivatives and the equation of a line to investigate how to derive the equation of a tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1).
4. Learners establish that the slope (m) of the normal is $-\frac{y}{2a}$ since normal and tangent are perpendicular to each other.

5. Learners in mixed-ability groups apply their knowledge in finding derivatives and the equation of a line to explore and investigate how to derive the equation of normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1).
6. Learners in groups create and pose problems for other groups to solve and share their solutions.

Pedagogical Exemplars

1. Talk-for learning:
   a. Use probing questioning to extend learners' reasoning.
   b. Allow oral explanation for those with writing difficulties.
2. Experiential learning
   a. Guide learners to derive the tangent and normal lines to parabolas in general form.
   b. Learners work in pairs to find the equation of tangents and normal.
   c. Task learners to find the intersection between linear and parabolic equations.
   d. Provide extra time and verbal instructions for learners who may need it.
   e. Use tactile parabola templates for visually impaired learners.
   f. Use graphing software (GeoGebra) for learners who need visual scaffolding.
3. Problem-based learning: Guide learners to choose points on a parabolic curve and find the equation of the tangent and normal that passes through the point and sketch all three (curve, tangent and normal).

Key Assessment Level 4: Group Work & Presentations

Mode: Group Work

Strategy: Presentation on group project of parabola assigned in week 6

1. Design and make a slingshot. Discuss the parabolic properties of the mathematical model used and the factors that affect the slingshot's performance. Use Figure 9.1 as a guide.

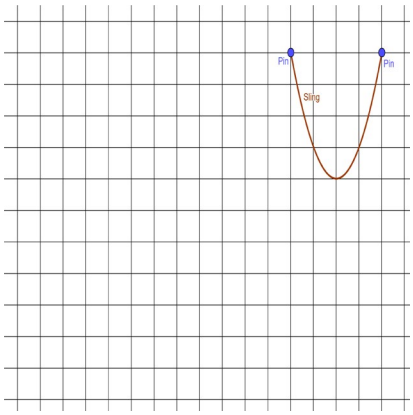


Figure 9.1

Materials:

a. Graph paper

b. Drawing instruments

c. Project report template

d. Nails

e. Wooden board

f. A small ball

Procedure

Part A: Preparation of materials

a. Draw a grid with an appropriate scale on the board.

b. Sketch a parabola with equation $y = ax^2$ for your slingshot frame on the grid board, where you will determine an appropriate value for 'a' and the location of the axes.

c. Identify and label the focus, directrix, axis of symmetry, and vertex on your sketch.

d. Explain how the focal property of a parabola makes it ideal for a slingshot design.

e. Design and sketch a complete slingshot model using your parabolic frame.

f. Locate a point on your parabola (the frame of the slingshot) and determine:

   i. the equation of the tangent line at this point

   ii. the angle at which the projectile would leave the slingshot

g. Locate two points on the grid and pin a nail at each point.

h. Tie the string around the two nails such that the sling lies on the frame already drawn as shown in Figure 9.1.

i. Shoot the ball from the sling and trace its path as shown in Figure 9.2 and Figure 9.3

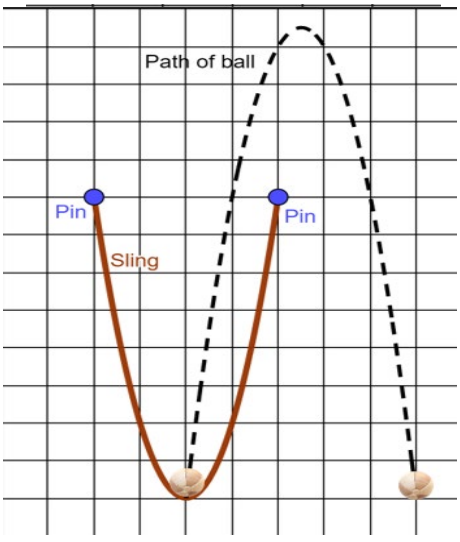


Figure 9.3

j. Explain how changing the parameter 'a' in your equation affects:

   i. The range of the projectile

   ii. The force required to launch

   iii. The accuracy of the slingshot

k. Propose modifications to your design that would improve performance based on parabolic properties.

(**20 marks**). DoK 4

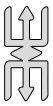
2. Design a cape or collar that uses parabolic curves to create a striking silhouette. Using technological tools or by sketching, create a mathematical pattern model identifying the focus, directrix, axis of symmetry, and vertex of your parabolic design. Explain how these mathematical features contribute to the drape, structure, and visual impact of your garment element

(**20 marks**).

PLC Session 9

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***group work & presentations***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the presentations (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step

d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 3 Review

This section discussed all the lessons to be taught from week five to week nine. This section reviewed the shape of quadratic function and introduced the parabola as a locus. The general and standard equations of parabolas were discussed and the features of a parabola investigated. Learning and assessment tasks for this section emphasised the development of skills and competencies required for modelling real-life scenarios or objects with parabolic equations and graphs. Assessment tasks which required research and collaboration among learners were suggested to reinforce learning from year one and year two. It is intended that leaners will get the opportunity to revise lessons taught in the previous years as they work through the learning and assessment tasks. Teachers are strongly encouraged to foster inclusive learning environments for all learners

MARKING SCHEMES FOR THE KEY ASSESSMENTS

WEEK 5

1. *C* *(1 mark)*
2. *D* *(1 mark)*
3. *B* *(1 mark)*
4. *D* *(1 mark)*
5. *A* *(1 mark)*
6. *D* *(1 mark)*
7. *C* *(1 mark)*
8. *A* *(1 mark)*
9. *A* *(1 mark)*
10. *C* *(1 mark)*

Table 2: *Rubric for Scoring Gamification Task*

Criteria	Levels of Performance			
	Excellent (4 marks)	Very Good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 mark)
Mathematical Understanding (max 4 marks)	*Correctly identifies and explains all parameter effects with specific examples. Makes connections between parameters and graph features with precise mathematical vocabulary. Shows insight into why these relationships occur.*	*Correctly identifies most parameter effects with supporting examples. Uses accurate mathematical vocabulary to describe relationships between parameters and graph features.*	*Identifies some parameter effects correctly but misses others. Uses some mathematical terms correctly. Explanations contain minor errors or omissions.*	*Identifies few parameter effects correctly. Makes significant errors when explaining relationships. Minimal use of mathematical vocabulary.*
Parameter Analysis (max 4 marks)	*Documents 8 or more functions with clear patterns identified for each parameter. Shows how changing multiple parameters simultaneously affects graphs. Creates organised comparison charts/visuals.*	*Tests all three parameters (a, b, c) systematically with 6–7 functions. Shows specific examples for each parameter with clear documentation of effects.*	*Tests two parameters with 4–5 functions. Documents some effects correctly but analysis lacks detail or contains inconsistencies.*	*Tests only one parameter or creates fewer than 3 functions. Documentation is incomplete or contains major errors.*

Communication Clarity *(max 4 marks)*	*Presents information in a logical, step-by-step sequence. Uses visuals that clearly show parameter changes and effects. Labels all parts correctly with precise terminology. Easy for others to follow and learn from.*	*Presents information in an organised manner. Uses helpful visuals that demonstrate key concepts. Labels most parts correctly. Others can follow the presentation without confusion.*	*Presents information with some organisation. Uses basic visuals that partially demonstrate concepts. Some labels or explanations are unclear or missing.*	*Presents information in a disorganised manner. Visuals are minimal or unrelated to concepts. Labels are missing or incorrect. Others struggle to follow the presentation.*
Collaboration (max 4 marks)	*All team members contribute equally and have assigned roles they fulfil completely. Members actively help each other understand concepts. Team makes decisions together and resolves disagreements constructively.*	*Most team members contribute actively with defined roles. Members assist each other when needed. Team works together to complete all parts of the task with minimal conflicts.*	*Some team members contribute more than others. Roles are partially defined. Team completes most tasks but may need teacher intervention for conflicts or work distribution.*	*One or two members do most of the work. Roles are unclear or ignored. Task completion is incomplete or required significant teacher intervention.*
Engagement with Others' Work (max 4 marks)	*Visits all other displays and asks specific questions about their findings. Completes all peer feedback forms with detailed, helpful comments. Offers specific suggestions that help other teams improve.*	*Visits most displays and asks relevant questions. Completes peer feedback forms with useful comments. Offers some constructive suggestions to other teams.*	*Visits some displays and asks general questions. Completes peer feedback forms with brief comments. Feedback tends to be general rather than specific.*	*Visits few displays and asks minimal questions. Partially completes or provides very brief feedback. Comments tend to be superficial (e.g., "good job").*
Total	*20 marks*			

WEEK 6

Marking Scheme

Homework

1. B1 (Correct labelling of axes)

   B2 (1 mark for each correct point)

   B1 (Parabola passing through the points)

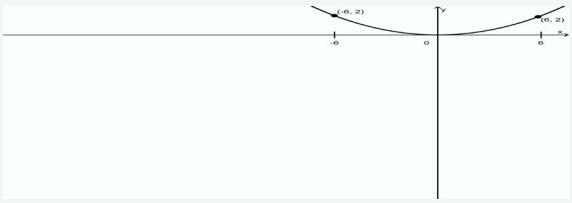


Figure 6.11: *Parabola representing cross section of banana*

Figure 6.11 shows the cross section of the banana in a rectangular coordinate system with the vertex of the parabola at $(0, 0)$ and the axis of symmetry of the parabola on the y-axis.

Half of the length of the banana $= \frac{12}{2} = 6\ cm$ **B1**

The parabola has an equation of the form $4py = x^2$ since it opens upwards. **B1**

Since the height of the banana is $2\ cm$, the point $(6, 2)$ lies on the parabola **B1**

Hence, $(6, 2)$ satisfies $4py = x^2$

$\Rightarrow 4(p)(2) = 6^2$ **B1**

$8p = 36$ **M1**

$p = 4.5$

$\therefore$ The focus is located 4.5 cm above the vertex **A1**

2. **Table 3:** A sample Rubric for Scoring Question 2 of Homework Project

Criteria	Levels of Performance			
	Excellent (4 marks)	**Very good (3 marks)**	**Satisfactory (2 marks)**	**Needs improvement (1 mark)**
Identification of Parabola Features	*Correctly identifies all 4 features (focus, directrix, axis of symmetry, vertex) with exact locations. Includes accurate definition for each feature using proper mathematical terms. Shows how features relate to each other.*	*Correctly identifies all 4 features with mostly accurate locations. Includes correct definition for 3 or 4 features using some mathematical terms.*	*Correctly identifies 2 or 3 features. Contains 1 or 2 errors in location or definition. Uses some mathematical terms.*	*Correctly identifies only one feature or none. Contains more than three errors in location or definition. Uses few or incorrect mathematical terms.*
Real-World Connection	*Directly links each parabola feature to a specific function in the real-world example. Gives more than 3 concrete examples of how the mathematical properties create practical benefits. Uses specific details to support connections.*	*Explains how at least 3 features relate to the real-world example. Gives 2 concrete examples of practical benefits. Includes some specific details.*	*Explains how 1 or 2 features relate to the real-world example. Gives 1 example of a practical benefit. Provides general statements with limited details.*	*Mentions a real-world example but doesn't connect specific features to it. Provides vague statements without examples.*

Benefits of Parabolic Shape	*Provides more than 3 specific reasons why the parabolic shape is ideal for the structure's purpose. Uses mathematical properties to explain these benefits. Compares with at least one alternative shape and why it would be less effective.*	*Provides 2 specific reasons why the parabolic shape benefits the structure. Connects mathematical properties to these benefits. Mentions how another shape would be less suitable.*	*Provides 1 specific reason why the parabolic shape benefits the structure. Makes basic connection to mathematical properties. May mention alternatives briefly.*	*States that the parabolic shape is beneficial but with little or no explanation why. No comparison to alternatives.*
Analysis of Changing Aspects	*Shows more than 3 specific examples of how changing equation values affects the parabola shape. For each change, correctly identifies what happens to all features. Explicitly states why these changes matter in the real-world context.*	*Shows 2 specific examples of how changing equation values affects the parabola shape. For each change, identifies what happens to most features. States how these changes affect the application.*	*Shows 1 example of how changing an equation value affects the parabola shape. Identifies some changes to features but may have minor errors. Makes a basic connection to the application.*	*Attempts to discuss changes to the parabola but with significant errors. Fails to show how equation changes affect features. No clear connection to real-world implications.*
Identification of Parabola Features	*Correctly identifies all 4 features (focus, directrix, axis of symmetry, vertex) with exact locations. Includes accurate definition for each feature using proper mathematical terms. Shows how features relate to each other.*	*Correctly identifies all 4 features with mostly accurate locations. Includes correct definition for 3 or 4 features using some mathematical terms.*	*Correctly identifies 2 or 3 features. Contains 1 or 2 errors in location or definition. Uses some mathematical terms.*	*Correctly identifies only 1 feature or none. Contains more than 3 errors in location or definition. Uses few or incorrect mathematical terms.*
Total Score	*20 marks*			

3. **Table 4:** A sample Rubric for Scoring Question 3 of Homework Project

Criteria	Levels of Performance			
	Excellent (4 marks)	**Very good (3 marks)**	**Satisfactory (2 marks)**	**Needs improvement (1 mark)**
a. Parabolic Motion Analysis	*Accurately represents parabolic motion in the chosen sport/game with precise diagrams or models. Includes multiple positions/ time points on the path. Uses appropriate mathematical notation to represent the motion. Shows exceptional attention to realistic details.*	*Accurately represents parabolic motion in the chosen sport/ game. Includes several positions on the path. Uses some mathematical notation. Shows good attention to realistic details.*	*Represents basic parabolic motion with some minor inaccuracies. Includes limited positions on the path. Mathematical representation may have minor errors. Some realistic details included.*	*Attempts to represent parabolic motion but with significant inaccuracies. Few or no positions marked on path. Limited mathematical representation. Lacks realistic details.*
b. Feature Identification & Explanation	*Correctly identifies and precisely marks all key features (vertex, axis of symmetry, focus, directrix) on the parabolic path. Provides thorough explanation of what each feature represents physically in the context of the sport/ game. Makes insightful connections between mathematical and physical meanings.*	*Correctly identifies and marks most key features on the parabolic path. Provides clear explanation of what most features represent physically in the context. Makes logical connections between mathematical and physical meanings.*	*Identifies some key features with minor errors in marking or explanation. Provides basic explanation of what some features represent in the physical context. Makes some connections between mathematical and physical meanings.*	*Identifies few features correctly. Explanations are missing or contain significant misconceptions. Makes minimal connections between mathematical and physical meanings.*
c. Variable Effects Demonstration	*Shows 3 or more specific examples of how changing variables (initial velocity, angle, height, etc.) affects the parabolic path. For each variable change, precisely describes the mathematical effect on the parabola's equation and shape. Uses proper terminology (e.g., parameter, coefficient) consistently.*	*Shows 2 specific examples of how changing variables affects the parabolic path. For each variable change, describes the mathematical effect on the parabola with mostly correct terminology. Some minor errors may be present.*	*Shows 1 example of how changing a variable affects the parabolic path. Description of mathematical effect may be partially accurate. Some terminology used but with inconsistencies.*	*Attempts to discuss variable changes but with significant mathematical errors. Limited or incorrect description of effects on the parabola. Minimal use of proper terminology.*

d. Application to Sport/Game	*Provides detailed analysis of how parabolic properties directly impact success in the chosen activity. Includes specific examples from the sport/game with numerical data or precise observations. Explains how players/participants intuitively use parabolic principles. Makes connections to optimisation (e.g., finding ideal angle for maximum distance).*	*Provides clear analysis of how parabolic properties impact success in the chosen activity. Includes examples from the sport/game with some specific details. Explains some ways players/participants use parabolic principles. References optimisation concepts.*	*Provides basic analysis of how parabolic motion relates to the chosen activity. Examples may be general rather than specific. Limited explanation of how players use these principles. Some mention of optimisation but may lack detail.*	*Provides minimal analysis of how parabolic motion relates to the activity. Few or no specific examples. Little or no explanation of how players use these principles. No meaningful discussion of optimisation.*
e. Parabolic Motion Analysis	*Accurately represents parabolic motion in the chosen sport/game with precise diagrams or models. Includes multiple positions/time points on the path. Uses appropriate mathematical notation to represent the motion. Shows exceptional attention to realistic details.*	*Accurately represents parabolic motion in the chosen sport/game. Includes several positions on the path. Uses some mathematical notation. Shows good attention to realistic details.*	*Represents basic parabolic motion with some minor inaccuracies. Includes limited positions on the path. Mathematical representation may have minor errors. Some realistic details included.*	*Attempts to represent parabolic motion but with significant inaccuracies. Few or no positions marked on path. Limited mathematical representation. Lacks realistic details.*
Total Score	*20 marks*			

Answers to Sample mid-semester questions

Part A (1.5 marks for each)

1. *C*
2. *A*
3. *B*
4. *A*
5. *D*
6. *B*
7. *D*
8. *D*
9. *B*
10. *C*

Part B

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$	$P \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	F	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

B$\frac{1}{4}$ for each correct resulting entry for $P \Rightarrow Q$, $Q \Rightarrow R$, $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ and $P \Rightarrow R$.

(8 marks)

From the entries of the table, $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ is **not** logically equivalent to $P \Rightarrow R$ **B2**

4. *(a)* $V = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$

$$V \circ P = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$V \circ P = \begin{bmatrix} 1(-1)+3(4) & 1(2)+3(-3) \\ 0(-1)+2(4) & 0(2)+2(-3) \end{bmatrix}$$ ***M1***

$$V \circ P = \begin{bmatrix} 11 & -7 \\ 8 & -6 \end{bmatrix}$$ ***A2*** *($\frac{1}{2}$ for each correct entry)*

(b) To determine image of P when reflected in the y-axis, compute

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$$

M1 *for the correct matrix for reflection in the x axis*

M1 *for multiplying the matrices in the correct order*

$$\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$ ***A1***

Rotating the reflected image about the origin at $70°$ anticlockwise.

Using $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ ***M1*** *for the correct matrix for $70°$ anticlockwise*

$$\begin{bmatrix} \cos 70° & -\sin 70° \\ \sin 70° & \cos 70° \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$ ***M1*** *for multiplying the matrices in the correct order*

$$\begin{bmatrix} -3.4168 & 2.1350 \\ 2.3078 & -2.9054 \end{bmatrix}$$

The image of P when reflected in the y-axis then rotated about the origin at $70°$ anticlockwise is $\begin{bmatrix} -3.4168 & 2.1350 \\ 2.3078 & -2.9054 \end{bmatrix}$. ***A2*** *($\frac{1}{2}$ for each correct entry)*

a. *For a parabola with equation $y = ax^2$, the focus is located at $\left(0, \frac{1}{4a}\right)$*

By comparing $y = 0.25x^2$ to the general equation $y = ax^2$, $a = 0.25$ ***B1***seen or implied

Substitute $a = 0.25$ *into the focus formula:* $Focus = \left(0, \frac{1}{4\times0.25}\right)$ **M1**

$Focus = (0,1)$ **B1**

The vertex is at the origin $(0, 0)$*, so the distance from vertex to pipe (focus) is 1metre.* **A1**

b. *If* $width = 4\ metres$*, then* x *ranges from* -2 *to 2* **B1**

When $x = 2$,

$y = 0.25(2)^2$ **M1**

$y = 0.25 \times 4$ **M1**

$y = 1$ **A1**

Since the vertex (lowest point of the trough) is located at the origin, the depth of the trough is 1 metre. **A1**

Alternatively,

When $x = -2$,

$y = 0.25(-2)^2$ **M1**

$y = 0.25 \times 4$ **M1**

$y = 1$ **A1**

Since the vertex (lowest point of the trough) is located at the origin, the depth of the trough is 1 metre. **A1**

c. ***B2*** *for any correct response, for example:*

i. *Parabolas have the special property that all rays parallel to the axis of symmetry reflect to a single point (the focus) OR*

ii. *Semi-circular shapes reflect parallel rays to different points along a line (rather than a single point) OR*

iii. *This focusing property of parabolas allows for maximum concentration of solar energy at the pipe, resulting in higher temperatures and greater efficiency, etc.*

d. $y = 0.4x^2$

$New\ Focus = \left(0, \frac{1}{4\times0.4}\right)$ **M1**

$= (1, 0.625)$ **A1**

The focus would move closer to the vertex (from 1 metre to 0.625 metres distance) ***A1***

e. ***B2*** *for any correct response, for example:*

i. *Reliable power source in regions of Ghana with abundant sunshine and frequent power outages OR*

ii. *Reduced electricity costs for communities after initial installation OR*

iii. *Can power essential services like water pumping for agriculture, schools, and health centres OR*

iv. *Creates local jobs and develops technical skills within the community*

WEEK 7

1. Focus, $F(0,3)$ and directrix $y=-3$:

   One suggested method to solve, but any correct method will gain the method/accuracy marks where applicable.

   Distance from $P(x,y)$ to focus = distance from P to directrix

   $\sqrt{x^2+(y-3)^2} = y+3$ **M1**

   $x^2 + (y-3)^2 = (y+3)^2$

   $x^2+y^2-6y+9 = y^2+6y+9$ **M1**

   $x^2 = 12y$

   Therefore, $x^2=12y$ is the equation of the parabola **A1**

   When $x=7$ and $y=4$, $7\times 7 \neq 12\times 4$ **M1**

   Therefore the point $(7,4)$ does not lie on the parabola **A1**

2. Focus $F(2,0)$ and directrix $x=-2$

   $\sqrt{(x-2)^2+y^2} = x+2$ **M1**

   $(x-2)^2+y^2=(x+2)^2$

   $x^2-4x+4+y^2 = x^2+4x+4$ **M1**

   $y^2=8x$ **A1**

   The axis of symmetry is $y=0$ **A1**

   The vertex is at $(0,0)$ **A1**

3. For a parabolic canopy with vertex at $(0,3)$ and focus at $(0,2.25)$:

   The directrix has equation $y=3.75$ **B1**

   $\sqrt{x^2+(y-2.25)^2} = y-3.75$ **M1**

   $x^2+(y-2.25)^2 = (y-3.75)^2$

   $x^2+y^2-4.5y+5.0625 = y^2-7.5y+14.0625$ **M1**

   $x^2 = 3y+9$

   $y=\frac{1}{3}x^2-3$ **A1**

   Since the graph opens downwards, it is a reflection of the graph obtained about the $x-$ axis.

   Therefore, $y=-\frac{1}{3}x^2+3$ is the equation of the required parabola. **A1**

4. With highest point at $20\ m$ and spanning $60\ m$

   a. Let the origin be located on the river surface and directly below the highest point of the arch.

      Parabola passes through $(0,20)$, $(-30,0)$, and $(30,0)$ **B3**

      Using $y=Ax^2+C$ with vertex at $(0,20)$

      When $x=30$, $y=0$:

      $\Rightarrow 0 = a(30)^2+20$ **M1**

      $a=-\frac{20}{900}$

$= -\frac{1}{45}$ **A1**

Parabolic equation: $y = -\frac{1}{45}x^2 + 20$

b. For $y = -\frac{1}{45}x^2 + 20$ to $y = A(x+B)^2 + C$, $A = \frac{1}{45}$

$p = 11.25$ **B1**

Focus: $(0, 20 - 11.25) = (0,\ 8.75)$ **B1**

Directrix: $y = 20 + 11.25 = 31.25$ **B1**

c. Since the focus is at $8.75m, (< 25m)$, modification is needed for structural integrity. Therefore, the focus height must be adjusted by changing parameter a, with the relationship $p = \frac{1}{4|a|}$. We want our p to be <5m Therefore, -5 $= \frac{1}{4a}$, $a = -\frac{1}{20}$ or greater.

B2

a. $x^2 + y^2 - 6x + 2y + 5 = 0$ has both x^2 and y^2 terms and therefore, it is a circle **B1**

Therefore, the claim is incorrect – this is not a parabola. **A1**

b. To create a parabola with focus at $(3, -1)$:

If we locate the vertex at $(3, -3)$,

Distance from vertex to focus=2 units **B1**

Standard form: $(x-h)^2 = 4p(y-k)$

With $h = 3, k = -3$ and $p = 2$

Equation: $(x-3)^2 = 8(y+3)$ **M1A1**

This is a parabola with focus at $(3, -1)$ and directrix at $y = -5$

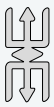
WEEK 8

Question 1

Table 5: *Rubric for Scoring Parabola Orientation Battle*

Criteria	Levels of Performance			
	Excellent (4 marks)	Very good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 mark)
Orientation Identification	*Correctly identifies orientation for all equation types including non-standard forms. Never hesitates on rotated parabolas or mixed terms.*	*Correctly identifies orientation for most equation types with occasional hesitation on complex forms.*	*Correctly identifies orientation for standard forms but struggles with non-standard equations.*	*Only identifies basic orientations (up/down) and makes frequent errors.*

Mathematical Justification	*Provides precise mathematical reasoning, referencing coefficient signs, squared terms, and standard forms. Connects equation structure directly to geometric meaning.*	*Explains mathematical reasoning with mostly correct references to equation features that determine orientation.*	*Provides basic justification with some mathematical terminology but lacks precision in connecting equation to orientation.*	*Offers minimal or incorrect mathematical reasoning for orientation choices.*
Feature Identification	*Precisely locates vertex, focus and directrix for all equation types, using correct calculations and formulas.*	*Correctly identifies most features with minor calculation errors for complex equations.*	*Identifies some features correctly but shows inconsistent understanding of relationship between equation and features.*	*Struggles to identify features beyond vertex or makes significant conceptual errors.*
Strategic Game Play	*Strategically selects equations that challenge opponents while demonstrating deep understanding. Effectively uses game mechanics to maximise learning and points.*	*Makes good strategic choices that demonstrate solid understanding. Uses game mechanics effectively most of the time.*	*Makes some strategic choices but often plays predictably. Limited use of advanced game mechanics.*	*Shows minimal strategy. Plays equations randomly without considering mathematical complexity.*
Response Speed & Accuracy	*Responds within time limit with complete accuracy for all equation forms. Handles time pressure without sacrificing mathematical precision.*	*Responds within time limit with few errors. Occasionally needs full time allowance for complex equations.*	*Often approaches time limit and makes some errors under pressure. Speed affects accuracy.*	*Frequently exceeds time limit or makes hasty errors. Struggles to balance speed and accuracy.*
Total Score	*20 marks*			



Assessment Notes

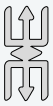
1. *Game play should be observed over multiple rounds to ensure reliable assessment*
2. *Both individual contributions and team dynamics should be considered*
3. *Students should receive feedback on mathematical content and game strategy*
4. *For students with accommodations, the time element may be modified while maintaining high expectations for mathematical reasoning*

Homework

Table 6: *Rubric for scoring Homework*

Criteria	Levels of Performance			
	Excellent (4 marks)	Very good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 mark)
a. Mathematical Accuracy of Parabola Construction	*Precisely constructs all three parabolas using focus-directrix method. Each parabola exactly matches its mathematical definition with correct orientation and shape. Shows construction steps clearly.*	*Constructs all parabolas with minor inaccuracies. Focus-directrix relationship is maintained but with small measurement errors. Construction steps visible.*	*Constructs parabolas with noticeable errors in shape or orientation. Basic understanding of focus-directrix method is evident despite inaccuracies.*	*Significant errors in parabola construction. Shapes don't match mathematical definitions. Limited evidence of focus-directrix method use.*
b. Intersection Analysis	*Correctly identifies all intersection points with exact coordinates and mathematical verification. Explains how different orientations create these intersections with reference to equations.*	*Identifies most intersection points with reasonably accurate coordinates. Some verification calculations with minor errors.*	*Identifies some intersection points but with estimation rather than calculation. Basic explanation of how parabolas interact.*	*Few or no intersection points identified correctly. Little understanding of how different parabolas interact geometrically.*
c. Design Integration & Creativity	*Creates a cohesive, professional logo where parabolas are meaningfully integrated. Design shows creativity while maintaining mathematical accuracy. Third parabola choice enhances overall concept.*	*Creates a unified logo with good integration of parabolas. Design shows some creativity and third parabola complements required parabolas.*	*Creates basic logo with parabolas arranged in a simple pattern. Limited creativity but requirements are met. Third parabola may seem disconnected.*	*Logo appears disjointed or incomplete. Parabolas seem randomly placed without design consideration. Missing requirements.*

d. Technical Precision & Presentation	*Executes design with excellent precision. Clear labels for all focus points, directrices, and key features. Presentation is polished with appropriate use of colour/shading to distinguish elements.*	*Good technical execution with most elements clearly labelled. Presentation is neat with some use of colour/shading to distinguish elements.*	*Basic execution with some labels for mathematical elements. Presentation is adequate but may lack clarity in some areas.*	*Poor execution with few or no labels. Presentation is messy or difficult to interpret.*
e. Reflection & Conceptual Connection	*Provides insightful reflection on how focus-directrix method enabled precise construction. Makes specific connections between mathematical properties and design concepts (reflection, focus, precision) with examples from the logo.*	*Reflects on focus-directrix method with some specific examples from their work. Makes general connections between mathematical properties and design concepts.*	*Basic reflection on the method with limited specific examples. Makes simple connections between math and design without detailed explanation.*	*Minimal reflection lacking specific examples. Little or no connection made between mathematical properties and design concepts.*
Total Score	*20 marks*			



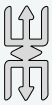
Assessment Notes

1. *Both mathematical accuracy and design quality should be considered equally important*
2. *Students should clearly label the focus and directrix for each parabola in their design*
3. *The choice of the third parabola should demonstrate understanding of how orientation is determined by focus-directrix relationship*
4. *Look for evidence that students understand how the mathematical properties contribute to the visual impact of their design*
5. *For higher marks, students should demonstrate how the focus-directrix method specifically enabled greater precision than other construction methods*

1. **Table 7:** *Rubric for Scoring Homework*

Criteria	Levels of Performance			
	Excellent (4 marks)	**Very good (3 marks)**	**Satisfactory (2 marks)**	**Needs improvement (1 mark)**
Mathematical Calculation & Setup	*Correctly determines that focus is at* $(0, 15)$ *and directrix at* $y = -15$. *Shows clear mathematical reasoning connecting the* $15\ cm$ *height requirement to focus–directrix placement.*	*Determines focus and directrix with minor calculation errors. Shows mostly correct reasoning for placement based on requirements.*	*Attempts to determine focus and directrix with conceptual understanding but makes significant errors in calculations.*	*Incorrectly places focus and/or directrix, showing fundamental misunderstanding of their relationship to vertex.*
Precision in Scale Drawing	*Creates precise scale drawing with appropriate scale clearly indicated. Parabola is smooth and accurate. Labels key elements with exact coordinates.*	*Creates scale drawing with minor inaccuracies. Scale is appropriate and most elements are correctly labeled.*	*Creates basic drawing with some scale errors. Curve lacks precision but shows general parabolic shape.*	*Drawing lacks appropriate scale and shows significant inaccuracies in parabolic shape.*
Equidistance Property Demonstration	*Selects more than eight strategic points that effectively demonstrate property across different regions. For each point, clearly shows that distance to focus equals distance to directrix with precise measurements.*	*Selects six to eight appropriate points and demonstrates equidistance property with minor measurement errors for some points.*	*Selects four or five basic points and attempts to show equidistance but has calculation errors or inconsistent demonstration.*	*Shows equidistance for fewer than four points or fails to demonstrate the property correctly.*
Analysis of Design Parameters	*Correctly calculates optimal diameter with mathematical justification. Provides insightful analysis of how focus–directrix distance affects efficiency with specific examples and calculations.*	*Calculates reasonable diameter with adequate justification. Explains focus–directrix relationship to efficiency with some specific examples.*	*Attempts to calculate diameter with limited justification. Provides basic explanation of efficiency relationship without specific calculations.*	*Provides diameter without calculations or gives incorrect analysis of efficiency relationship.*

Construction Method Design	*Designs a practical, detailed construction method showing deep understanding of geometric principles. Includes precise steps, measurements, and materials. Method guarantees accurate parabola.*	*Designs workable construction method with mostly correct steps. Some minor details missing but approach would produce accurate parabola.*	*Designs basic construction method that shows some understanding but lacks precision or has gaps in procedure.*	*Proposes method that would not produce an accurate parabola or lacks critical understanding of focus–directrix relationship.*
Total Score	*20 marks*			



Assessment Notes

1. *The key focus is on strategic application of the focus-directrix relationship, not just procedural fluency*
2. *Students should demonstrate understanding of how mathematical properties translate to practical design considerations*
3. *For the equidistance property, look for strategic point selection that shows understanding, not just mechanical verification*
4. *Construction method should reflect understanding of the defining property of a parabola, not just memorised steps*

WEEK 9

Rubrics

1. 20 marks

Table 8: *Rubric for Scoring Group project Question 1*

Criteria	Levels of Performance			
	Excellent (4 marks)	Very good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 mark)
Mathematical Model and Feature Identification	*Parabola drawn with high precision; value of 'a' determined with clear justification; all features (focus, directrix, axis, vertex) precisely identified with accurate measurements*	*Parabola drawn accurately; value of 'a' determined with adequate justification; all features correctly identified and properly labelled*	*Parabola drawn with reasonable accuracy; value of 'a' determined with limited justification; most features identified with minor errors*	*Parabola drawn with minimal accuracy; value of 'a' inappropriate or unexplained; limited features identified with significant errors*

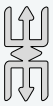
Explanation of Focal Property and Application	*Explains all 3 focal properties with specific examples showing how each directly improves slingshot performance*	*Explains at least 2 focal properties with direct links to slingshot functionality*	*Explains 1 focal properties with basic connections to slingshot design*	*Mentions focal property but fails to connect it correctly to slingshot function*
Mathematical Calculations and Analysis	*Tangent and angle calculations precise with elegant approach; clear interpretation of mathematical significance to projectile motion*	*Calculations accurate with appropriate working shown; good interpretation of mathematical significance*	*Calculations completed with minor errors; basic interpretation of results*	*Calculations attempted with significant errors; limited interpretation*
Physical Construction and Implementation	*Exceptional construction with attention to precision; string and nail arrangement perfectly matches mathematical model; highly functional design*	*Well-constructed model accurately reflecting mathematical design; good alignment between theory and physical model*	*Reasonable construction following design specifications; some discrepancies between model and theory*	*Basic construction with limited adherence to design; significant differences between mathematical model and physical construction*
Parameter Effects and Design Modifications	*Correctly describes specific effects of changing 'a' on all 3 factors (range, force, accuracy) with numerical examples; proposes 2-3 practical improvements*	*Describes effects of 'a' on all 3 factors with some numerical support; proposes at least 2 reasonable modifications*	*Describes effects of 'a' on 2 factors with limited detail; proposes 1 basic modification*	*Describes effect of 'a' on only one factor with minimal detail; modification suggested is impractical or missing*
Total	*20 marks*			

2. 20 marks

Table 9: *Rubric for Scoring Group project Question 2*

Criteria	Levels of Performance			
	Excellent (4 marks)	Very good (3 marks)	Satisfactory (2 marks)	Needs improvement (1 mark)
a. Pattern Design & Mathematical Parameters (4 marks)	*Creates pattern with exact parabolic equation ($y = ax^2 + bx + c$) correctly applied to design. All measurements match calculations.*	*Creates pattern with correct parabolic shape and most measurements matching calculations. Equation has minor errors.*	*Creates basic curved pattern with attempted calculations. Shape resembles a parabola but measurements don't fully match.*	*Uses a single basic curve with no calculations or incorrect equation that doesn't match the design.*
b. Technical Representation & Features (4 marks)	*Labels all four key features (focus, directrix, axis, vertex) with exact measurements. Shows how pattern pieces align with mathematical points.*	*Labels at least three key features with mostly correct measurements. Shows how some pattern pieces align with math points.*	*Labels two key features but measurements have errors. Connection between pattern and math points is unclear.*	*Labels only one feature or none. Pattern lacks mathematical reference points.*
c. Mathematical Justification & Fashion Application (4 marks)	*Explains step-by-step how each mathematical property directly affects specific design elements (e.g., how focus location determines drape point).*	*Explains how most mathematical properties connect to design choices with clear examples.*	*Explains basic math-design connections but examples are vague or only partly relevant.*	*States basic math terms without explaining how they connect to design decisions.*
d. Functional & Aesthetic Analysis (4 marks)	*Shows with fabric samples or diagrams how focus point affects drape, how axis creates symmetry, and how vertex shapes silhouette. Includes fabric weight considerations.*	*Shows how 2-3 parabolic features affect garment appearance and structure. Mentions fabric considerations.*	*Shows how one parabolic feature affects the garment. Limited mention of how fabric interacts with the shape.*	*Describes garment appearance without connecting it to specific mathematical features.*

e. Peer Evaluation & Reflection (4 marks)	*Identifies specific mathematical errors or strengths in peer's work. Suggests exact improvements to both calculations and design application.*	*Points out general mathematical accuracy in peer's work. Offers helpful suggestions for design improvement.*	*Comments mainly on design appearance with only brief mention of mathematical elements.*	*Comments only on personal preferences without addressing mathematical accuracy or design application.*
Total Score	*20 marks*			



Assessment Notes

1. *Each level describes specific, observable student work rather than subjective quality judgments*
2. *Mathematical accuracy and practical application are equally valued*
3. *Students should be able to clearly understand what is expected at each performance level*

Alternative presentation formats (physical models, verbal explanations, etc.) can be assessed using the same specific criteria.

SECTION 4: TRIGONOMETRY

STRAND: GEOMETRIC REASONING AND MEASUREMENT

Sub-Strand: Measuring triangles

Learning Outcomes

1. **Week 10:** *Draw and analyse basic trigonometric graphs using values of the unit circle, maximum values and minimum values*
2. **Week 11:** *Apply understanding of compound angles to verify harmonic identities and solve related problems*

Content Standards

1. **Week 10:** *Demonstrate understanding of graphs of trigonometric functions*
2. **Week 11:** *Demonstrate understanding of compound angles and multiple angles and solve related problems*

INTRODUCTION AND SECTION SUMMARY

Trigonometric graphs, transformations, and harmonic identities are key elements of the Year Three Additional Mathematics curriculum in Ghanaian Senior High Schools. This section enhances learners' understanding of sine, cosine, and tangent functions through their graphical representations. Students will explore how transformations—like changes in amplitude, period, phase shift, and vertical shift—affect these graphs. They will also study harmonic identities, which express relationships between trigonometric functions and are essential for simplifying expressions and solving complex equations. This topic deepens learners' conceptual grasp of trigonometry while providing tools to model real-world phenomena such as sound waves and tides. An interactive approach will connect algebraic manipulation with graphical interpretation, making learning both meaningful and applicable.

The weeks and the indicators covered by the section are:

- *Week 10*

1. *Sketch basic trigonometric graphs by hand and/or by technological tools, e.g., GeoGebra, and analyse their properties in the unit circle.*
2. *Identify the maximum and minimum values of trigonometric graphs.*

- *Week 11*

1. *Review compound identities.*
2. *Use knowledge in compound angle identities to derive harmonic identities and find the maximum and minimum values of a trigonometric function.*

Recap weeks

Year 1	Week 15 Recall basic trigonometric ratios and use the knowledge to solve problems relating to triangles. Use special triangles and the unit circle to determine the geometrical and functional values of trigonometric ratios including special angles. Identify the coordinates of the quadrantal angles in a unit circle and use it to find the trigonometric values of quadrantal angles.
Year 2	Week 14: Prove and apply compound angles to derive the identities for multiple angles and half angles. Week 21: Use the second derivative of a function to classify the maximum, minimum and saddle point of that function and perform curve sketching. Week 22: Find the maximum and minimum values and points of a function, sketch the functions and solve some real-life problems.

SUMMARY OF PEDAGOGICAL EXEMPLARS

Teaching trigonometric graphs, transformations, and harmonic identities to final-year senior high school learners requires varied pedagogies for deep understanding and solving real-life problems. Talk-for-learning encourages verbalising mathematical thinking, while experiential learning, through sketching or physical modelling, helps visualise amplitude and frequency. Project-based learning applies trigonometric functions to real-life situations, and inquiry-based learning explores how parameter changes affect graph shapes. Problem-based learning focuses on contextual tasks that reinforce critical thinking. To support mixed-ability learners, use scaffolded tasks, visual aids, and graphing templates, along with extension problems for advanced students. In resource-limited classrooms, chalkboard sketches and simple tools can substitute for digital resources. These methods foster active participation and conceptual clarity for all learners.

ASSESSMENT SUMMARY

To evaluate third-year senior high learners' understanding of trigonometric graphs, their transformations, and harmonic identities, a balanced combination of practical and oral assessment methods is recommended. Project administered on group basis provides learners to review previous learning and apply skills and concepts to create innovative solutions to problem while assisting each other's disabilities.

Classwork offers daily opportunities to assess learners' comprehension of key concepts like amplitude, period, phase shift, and identity transformations. Teachers can assign differentiated tasks that range from basic graph sketching to more complex identity proofs, catering to varied learning abilities.

Questioning during lessons—both open-ended and targeted—encourages active participation and assesses conceptual understanding in real time. Teachers can scaffold questions to help below-average learners build confidence, while offering extension questions to challenge above-average learners.

A practice mock examination in week ten serves as a summative assessment, simulating formal testing conditions. Including a variety of question types (multiple-choice, sketching, explanation, and application) ensures comprehensive assessment coverage. Group reviews of the mock results can further reinforce learning. Together, these methods create an inclusive and adaptive assessment structure that is suited to a range of classroom contexts and learner needs.

Indicators	DOK Level	Assessment Strategy and/ or Mode
Week 10		
Sketch basic trigonometric graphs by hand and / or by technological tools, e.g., GeoGebra, and analyse their properties in the unit circle	4	Mode: Project Strategy: Group work Mode: End of semester examination Strategy: Practice Mock
Identify the maximum and minimum values of trigonometric graphs	2	
Week 11		
Review compound identities	1	Mode: Classwork Strategy: Questioning
Use knowledge in compound to derive harmonic identities and find the maximum and minimum values of a trigonometric function	2	

WEEK 10

Learning Indicators

1. *Sketch basic trigonometric graphs by hand and / or by technological tools, e.g., GeoGebra, and analyse their properties in the unit circle*
2. *Identify the maximum and minimum values of trigonometric graphs*

This week's lessons are dedicated to exploring trigonometric graphs. We will revise the properties of the unit circle and quadrantal angles as they form a basis for sketching the basic trigonometric functions.

Recall that:

1. The quadrantal angles are $0, 90°, 180°, 270°$ and $360°$.
2. The unit circle centred at the origin, $(0, 0)$ is the circle with radius equal to one unit. It has equation $x^2 + y^2 = 1$. It intersects the x and y axes at $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.

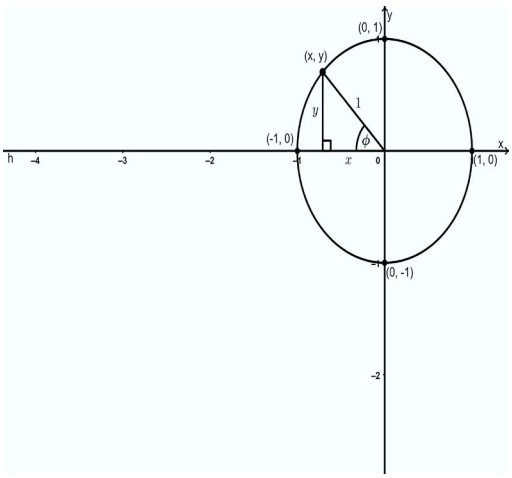


Figure 10.1: *Unit circle*

From Figure 10.1, $\sin(\phi) = y$ and $\cos(\phi) = x$. This means that for any point on the unit circle, the $x -$ coordinate is the value of the cosine value of the angle between line that joins the point to the origin and the $x - axis$. The $y -$ coordinate of a point on the unit circle is the value of the sine value of the angle between line that joins the point to the origin and the $x - axis$.

Let us now consider a unit circle with the quadrantal angles indicated.

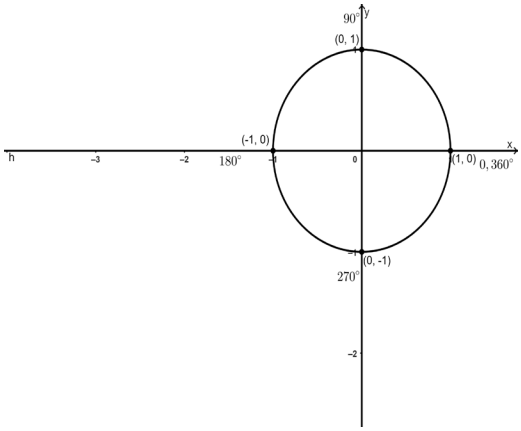


Figure 10.2: *Unit circle with quadrantal angles*

Table 10.1: *Functional values of quadrantal angles for sine and cosine*

	Quadrantal Angles				
	0	90°	180°	270°	360°
Point	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$	$(1,0)$
sine	0	1	0	-1	0
cosine	1	0	-1	0	1

FOCAL AREA 1: SKETCHING BASIC TRIGONOMETRIC GRAPHS

Graphs of basic trigonometric functions

The graphs of trigonometric functions can be described as sinusoidal. The shape looks like a stretch of mountains and valleys.

Let us now consider the graph of $y = \sin(x)$

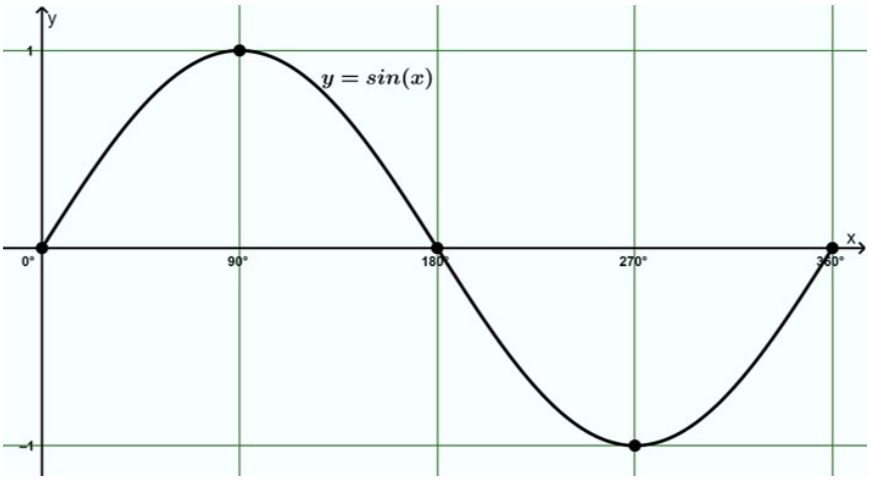


Figure 10.3: *The graph of* $sin(x)$

The features of the graph as shown in Figure 10.3 are as follows:

1. The range of the graph is $[-1, 1]$. The graph has a maximum value of 1 and minimum value of -1.
2. The graph increases before decreasing when starting from the origin.
3. The graph cuts the $x-$ axis at $0, 180°$ and $360°$ i.e., values of $x = 180n°$ for integer values of n.
4. The graph is symmetric about the origin, confirming that it is an odd function.
5. The $x-$ axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.

The graph of $y = \cos(x)$ is shown in Figure 10.4

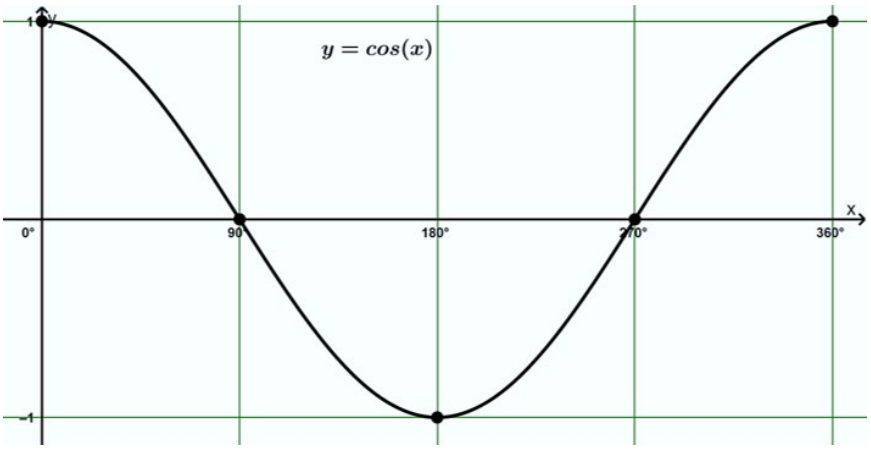


Figure 10.4: *The graph of* $cos(x)$

The features of the graph as shown in Figure 10.4 are as follows:

1. The range of the graph is $[-1, 1]$. The graph has a maximum value of 1 and minimum value of -1.
2. The graph decreases before increasing when starting from the origin.
3. The graph cuts the $x-$ axis at $90°$ and $270°$ i.e., values of $x = 90° + 180n°$ for integer values of n.
4. The graph is symmetric about the $y-$ axis, confirming that it is an even function.
5. The $x-$ axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.

Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. We will use this identity to obtain quadrantal angle values for $y = \tan(x)$ and consider the graph.

Table 10.2: *Functional values for sine, cosine and tangent*

	0	90°	180°	270°	360°
sine	0	1	0	-1	0
cosine	1	0	-1	0	1
tangent	0	Undefined	0	Undefined	0

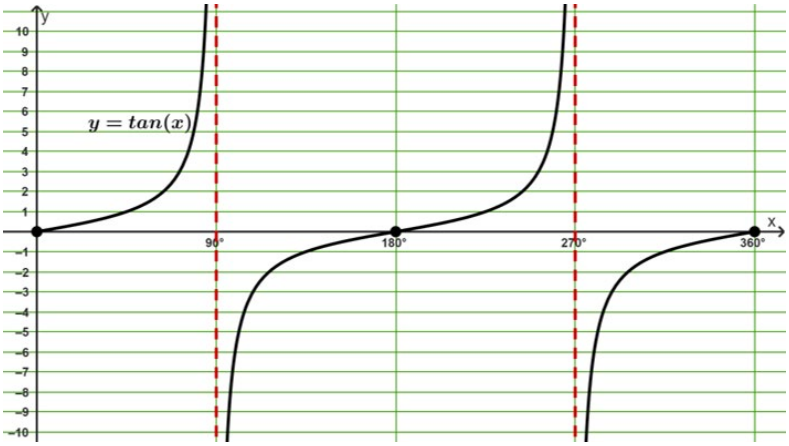


Figure 10.5: *The graph of* $tan(x)$

The features of the graph as shown in Figure 10.5 are as follows:

1. The range of the graph is $(-\infty, \infty)$.
2. The graph cuts the $x-$ axis at $0, 180°$ and $360°$ i.e., values of $x = 180n°$ for integer values of n.
3. There are asymptotes at $x = 90°$ and $x = 270°$ i.e., values of $x = 90° + 180n°$ for integer values of n.
4. The graph has no maximum or minimum values.
5. The graph is symmetric about the origin, confirming that it is an odd function.
6. The $x-$ axis, with equation $y = 0$ is the midline. It divides the two halves horizontally.

Transformation of graphs of trigonometric functions

The basic trigonometric functions can be stated in a more general way to allow for transformation.

1. The sine function can be written as $f(x) = A\sin(Bx - C) + D$
2. The cosine function can be written as $f(x) = A\cos(Bx - C) + D$

Transformations of the form *A sin (x)* or *A cos (x)*

When:

1. $A > 1$, the graph of the function stretches vertically.
2. $0 < A < 1$, the graph of the function compresses or shrinks vertically.
3. $A < 0$, the graph of the function reflects across the $x - axis$.

For these graphs, the amplitude, $|A|$ is the largest value the functions attain. $|A|$ is the distance between the midline and the highest or lowest point on the graph.

Example 1

Draw the graphs of the following and examine the effect of the change in A when the equations are compared to $y = A\sin(x)$

1. $y = \sin(x)$
2. $y = 2\sin(x)$
3. $y = 4\sin(x)$
4. $y = -3.5\sin(x)$

Solution

If we are drawing the graphs by hand, we will need a series of data points. The simplest way to obtain this dataset will be to use multiples of the sine values of the quadrantal angles as shown in Table 10.3

Table 10.3: Functional values for $y = \sin(x), y = 2\sin(x), y = 4\sin(x)$ and $y = -3.5\sin(x)$

x	o	$90°$	$180°$	$270°$	$360°$
$\sin(x)$	o	1	o	-1	o
$2\sin(x)$	o	2	o	-2	o

$4\sin(x)$	0	4	0	-4	0
$-3.5\sin(x)$	0	-3.5	0	3.5	0

If a graphing utility of software is available, we will simply enter the functions.

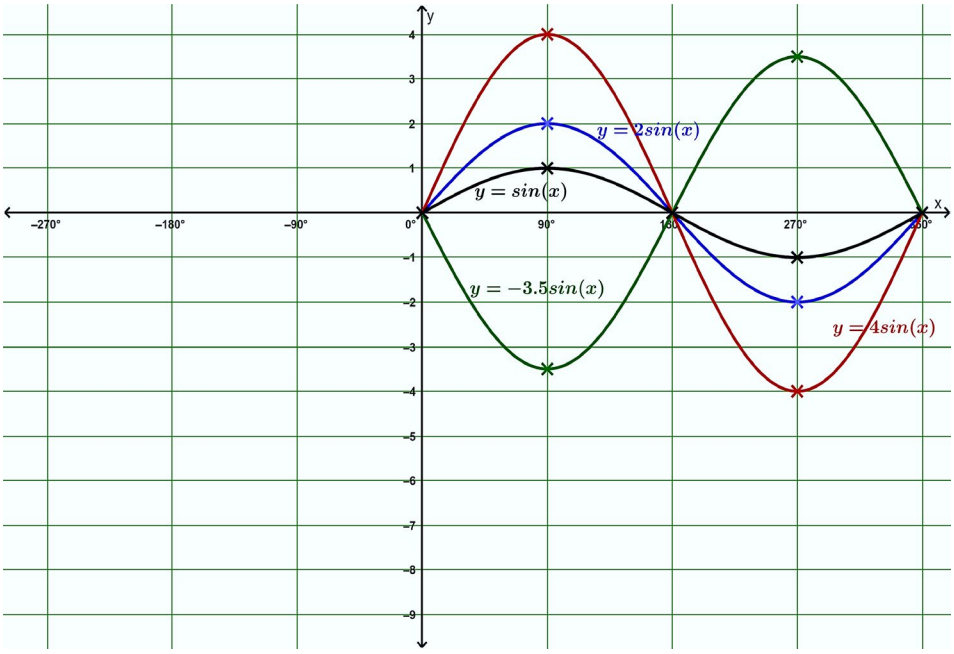


Figure 10.6: *Transformations of* $y = A\ sin(x)$

It can be observed from Figure 10.6 that;

1. the graph of $y = A\sin(x)$ stretches or shrinks vertically as the value of $|A|$ changes.
2. when the value of $|A|$ gets larger, the graph is stretched more and as the value of $|A|$ gets smaller, it is compressed.
3. when $A > 0$, the graph increases before decreasing.
4. when $A < 0$, the graph decreases before increasing.
5. the $x-$ intercepts remain the same.
6. The width (domain of the function for a full revolution) do not change.
7. The midlines for all three graphs remain at $y = 0$.

It must be noted that similar transformations apply for $y = A\cos(x)$

Transformations of the form *sin(Bx)* or *cos(Bx)*

The graphs of these functions have amplitude $|A|$ and period $\frac{2\pi}{B}$ or $\frac{360°}{B}$. The appropriate interval of one complete period is $\left[0, \frac{2\pi}{B}\right]$ or $[0, 360°]$.

Consider the graphs of $y = \cos(2x)$ and $y = \cos(3x)$ in comparison to $y = \cos(x)$

For $y = \cos(2x)$, $A = 1$ and $B = 2$

$Amplitude\ (A) = 1$ so, there will be no changes in the height of the graph. The $y-$ coordinates of the five points which divide the graph into quadrants will be no different from that of the original function. They will remain as $1, 0, -1, 0$ and 1

$$Period = \frac{360°}{B} = \frac{360°}{2} = 180°$$

Note, the period is the domain (horizontal distance) it takes for the curve to complete one full cycle before it starts repeating again.

$$Interval = [0, 180°]$$

We need to divide this interval by 4, because we have 4 points where we need to know the x values.

i.e., $180° \div 4 = 45°$

To find all points start with the lower boundary of the interval: 0 and keep adding $45°$

i.e., $(0, 1), (45°, 0), (90°, -1), (135°, 0)$ and $(180°, 1)$.

In other words, we have compressed our graph horizontally by a scale factor of $\frac{1}{2}$.

For $y = \cos(3x)$, $A = 1$ and $B = 3$

$Amplitude = 1$ hence there would be no changes in the height of the graph. The $y-$ coordinates of the five points which divide the graph into quadrants would be no different from that of the original function. They will remain as $1, 0, -1, 0$ and 1.

$$Period = \frac{360°}{B} = \frac{360°}{3} = 120°$$

$$Interval = [0, 120°]$$

We need to divide this interval by 4.

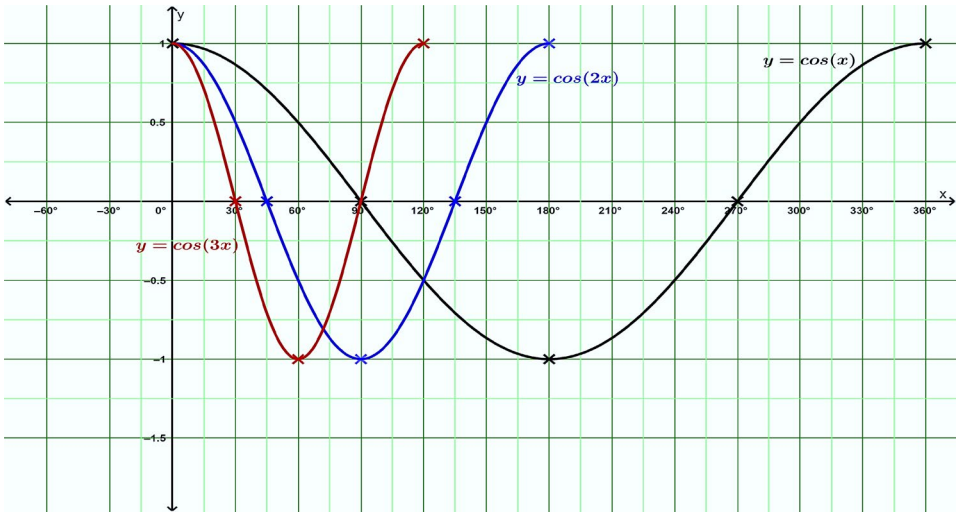
i.e., $120° \div 4 = 30°$

To find all points start with the lower boundary of the interval: 0 and keep adding $45°$

i.e., $(0, 1), (30°, 0), (60°, -1), (90°, 0)$ and $(120°, 1)$

In other words, we have compressed our graph horizontally by a scale factor of $\frac{1}{3}$.

The graphs (showing one period each of the functions) are shown in in Figure 10.7



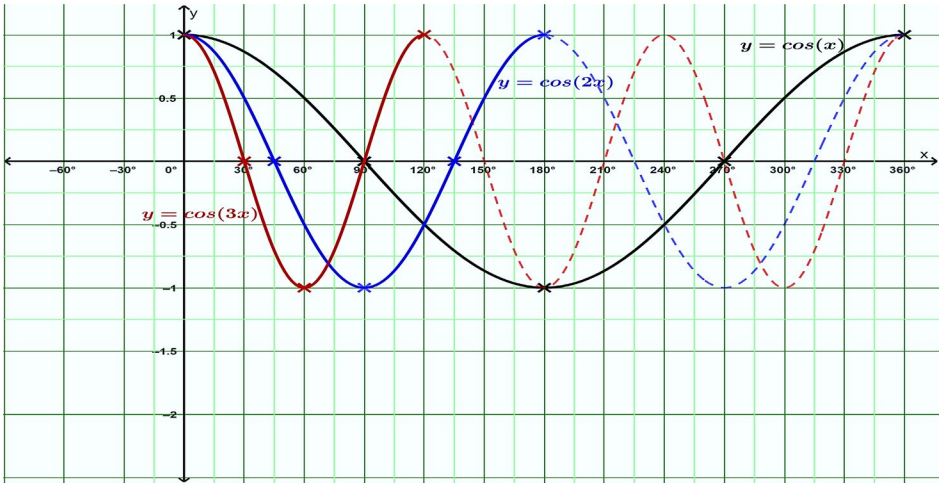


Figure 10.7: *Graphs of* $y = \cos(x)$, $y = \cos(2x)$ *and* $y = \cos(3x)$

Observations from the graphs

1. The period of $y = \sin(2x)$ is $\frac{360°}{2} = 180°$. The graph of $y = \sin(2x)$ is obtained from a horizontal shrink from the graph of $y = \cos(x)$. Its width is half of that of $y = \cos(x)$.
2. The period of $y = \sin(x)$ is $\frac{360°}{3} = 120°$. The graph of $y = \sin(3x)$ is obtained from a horizontal shrink from the graph of $y = \cos(x)$. Its width is a third of that of $y = \cos(x)$.
3. The minimum and maximum values do not change.
4. The midlines for all three graphs remain at $y = 0$.

Conclusions

1. If $0 < B < 1$, the period is greater than $360°$ and represents a horizontal stretch.
2. If $B > 1$,the period is less than $360°$, and a horizontal shrink.
3. The scale factor for the stretch is $\frac{1}{B}$
4. If B is negative, use $\sin(-x) = -\sin x$ or $\cos(-x) = \cos(x)$ to rewrite the function.
5. For these graphs, the amplitude (A) is the largest value the functions attain, however B either stretches or shrinks the graph.

Figure 10.8 shows the graph of $y = \cos(x)$ and $y = 3\cos(0.5x)$. We would compare them to describe the transformations which have taken place.

Note that for $y = 3\cos(0.5x)$, $A = 3$ and $B = 0.5$. The amplitude is thus 3 units and we expect a vertical stretch of scale factor 3 such that the $y-$ coordinates of the five points which divide the graph into quadrants would be three times that of the original function. They will become as $3, 0, -3, 0$ and 3.

The value of B which is 0.5 or $\frac{1}{2}$ causes a horizontal stretch of the graph of $y = \cos(x)$ by a scale factor of 2. The period therefore increases to $[0, 720°]$. Recall that the period of $y = \cos(x)$ is $360°$ hence the new period $360° \times 3 = 720°$

$720° \div 4 = 180°$

The five points required to graph $y = 3\cos(0.5x)$ are therefore $(0, 3)$, $(180°, 0)$, $(360°, -3)$, $(540°, 0)$ and $(720°, 3)$

These transformations can be confirmed from the graphs in Figure 10.8 (showing one period each of the functions).

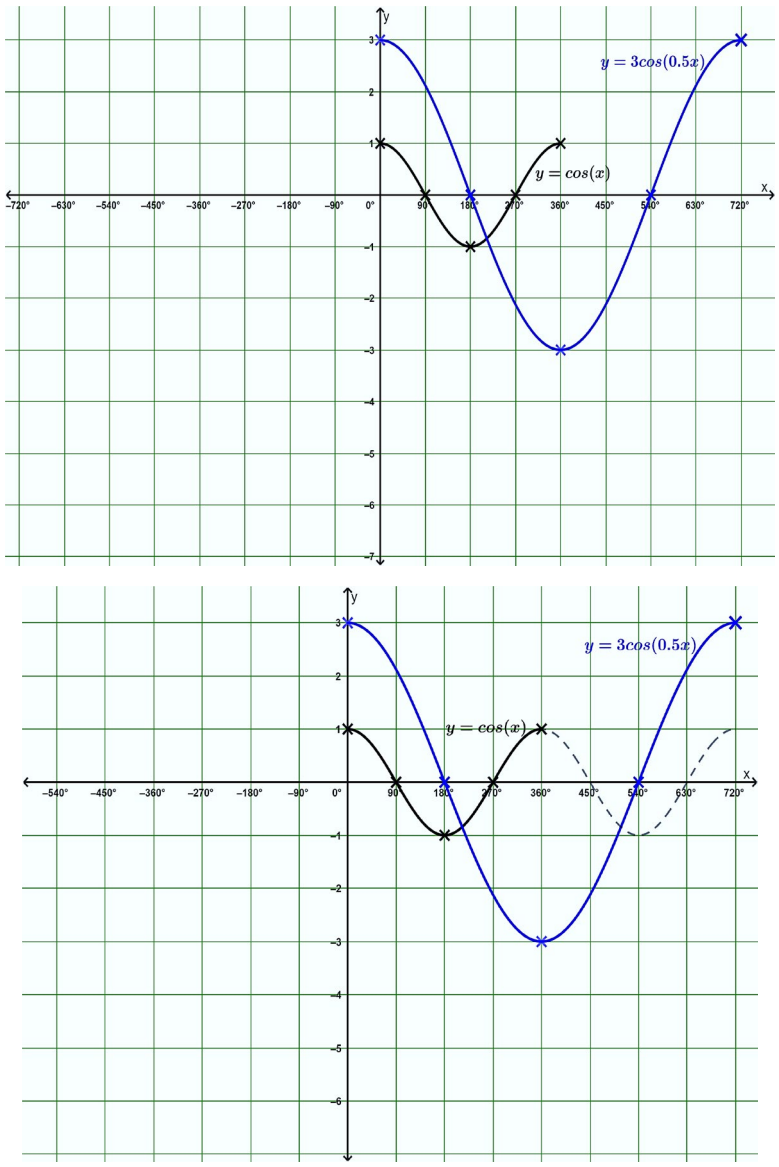


Figure 10.8: *The graph of* $y = \cos(x)$ *and* $y = 3\cos(0.5x)$

Transformations of the form *A sin(Bx −C)* or *A cos(Bx −C)*

The graph of $A\sin(Bx - C)$ has

1. $Amplitude = |A|$
2. Period of $\frac{2\pi}{B}$ or $\frac{360°}{B}$
3. The interval of one cycle is $\frac{C}{B} \leq x \leq \frac{C}{B} + \frac{360°}{B}$, where;
   a. $\frac{C}{B}$ is the left endpoint, phase shift or the amount of shift from the original position and
   b. $\frac{C}{B} + \frac{360°}{B}$ is the right endpoint for the first cycle

Note

"$Bx - C$" in the equation represents a shift/translation to the right while "$Bx + C$" represents a shift/translation to the left.

Example 1

Describe the transformations performed on the graph of $y = \sin(x)$ to obtain the graph of one cycle of $y = \sin(2x - 180°)$. Refer to Figure 10.9 for the graphs.

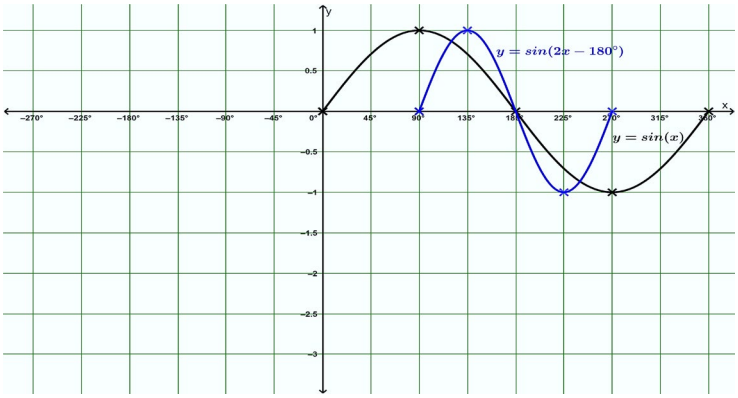


Figure 10.9: *Graphs of* $y = \sin(x)$ *and* $y = \sin(2x - 180°)$

Solution

The graph of $y = \sin(2x - 180°)$ has a period of $180°$ i.e., $270° - 90°$ which is a half of the period of $y = \sin(x)$. This implies a horizontal shrink of the graph of $y = \sin(x)$ by a scale factor of 2 (or a horizontal stretch of scale factor $\frac{1}{2}$).

The graph of $y = \sin(2x - 180°)$ starts from $= 90°$ while the graph of $y = \sin(x)$ starts from the origin, i.e., $x = 0$. This depicts a horizontal shift to the right by $90°$ confirmed numerically by $\frac{C}{B} = \frac{180°}{2}$.

The minimum and maximum values for y do not change hence leaving the amplitude at 1 and implying that there has not been a vertical stretch or shrink.

Example 2

Draw the graph of $y = 4\cos\left(x + \frac{\pi}{4}\right)$ and describe the transformations performed on the graph of $y = \cos(x)$ to obtain the graph of one cycle of $y = 4\cos\left(x + \frac{\pi}{4}\right)$.

Note, here we have the angles in radians. To remind the learners of this, look back to Week 15 of the Year One of the teacher Manual.

Solution

$y = 4\cos\left(\boldsymbol{x} + \frac{\pi}{4}\right)$

$Amplitude = 4$

$Period = 2\pi$

Step $= 2\pi \times \frac{1}{4} = \frac{1}{2}\pi$

Left endpoint: $x + \frac{\pi}{4} = 0$, $x = -\frac{\pi}{4}$

Right endpoint: $x + \frac{\pi}{4} = 2\pi$,

$$x = \frac{7\pi}{4}$$

$$\therefore \textit{The interval} = \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$$

Points: $\left(-\frac{\pi}{4}, 4\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, -4\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 4\right)$

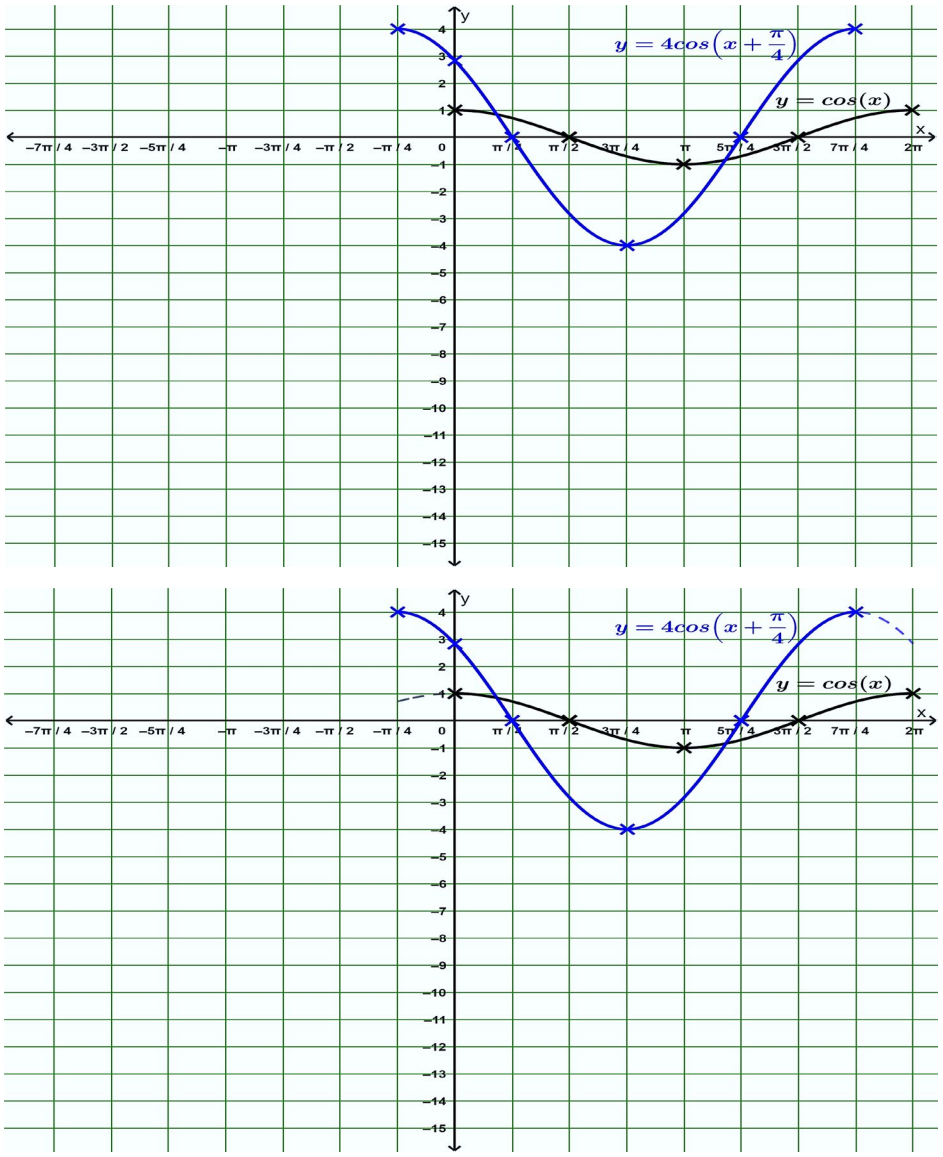


Figure 10.10: *Graphs of* $y = cos(x)$ *and* $y = 4\,cos\left(x + \frac{\pi}{4}\right)$

A comparison of the graphs of $y = \cos(x)$ and $y = 4\cos\left(x + \frac{\pi}{4}\right)$ show a horizontal shift of $\frac{\pi}{4}$ to the left. The graph of $y = \cos(x)$ is stretched vertically by a scale factor of 4 to obtain the graph of $y = 4\cos\left(x + \frac{\pi}{4}\right)$.

Transformations of the form $A\sin(Bx - C) \pm D$ or $A\cos(Bx - C) \pm D$

D represents a vertical translation. The vertical translation causes a vertical shift in the graph. If $D > 0$, the shift is upward and if $D < 0$, the shift is downwards.

Example 1

Describe the transformations done to $y = \cos(x)$ to obtain the graph of $y = -3\cos\left(\frac{1}{2}x\right) + 4$. Refer to Figure 10.11 for their graphs

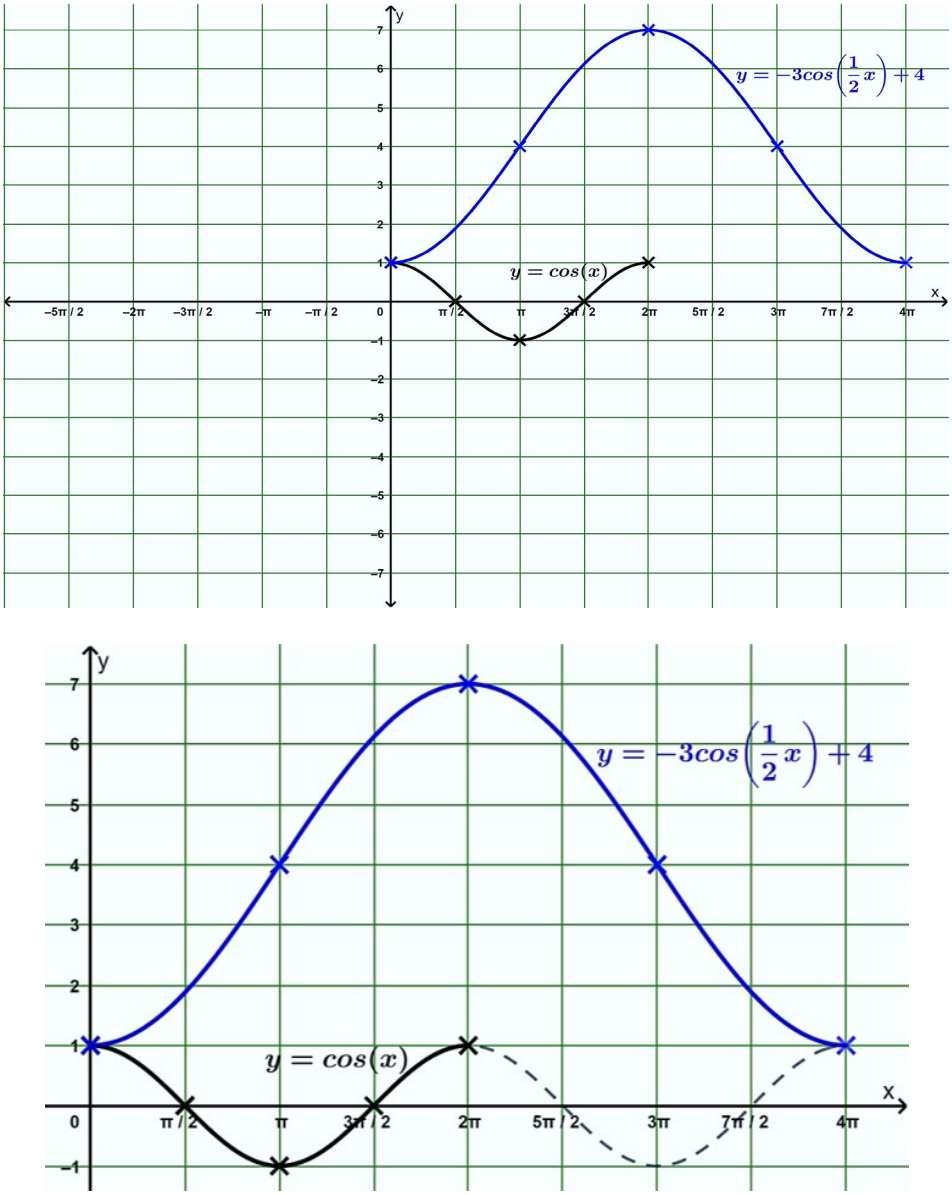


Figure 10.11: *Graphs of* $y = \cos(x)$ *and* $y = -3\cos\left(\frac{1}{2}x\right) + 4$

Solution

The graph of $y = \cos(x)$ is reflected in the $x-$ axis, stretched vertically by a scale factor of 3, stretched horizontally by a scale factor of 2 and shifted upward by 4 units.

Example 2

1. Copy and complete the table for $y = \sin(x) - 2\cos(x)$, correct to 1 decimal place

x	o	30°	60°	90°	120°	150°	180°	210°	240°
y									

3. Using a scale of 2 cm to 30° on the $x-$ axis and 2 cm to 0.5 units on the $y-$ axis, draw the graph of $y = \sin(x) - 2\cos(x)$ for $0 \leq x \leq 240°$.
4. Use your graph to solve the equations;
   a. $\sin(x) - 2\cos(x) = 0$
   b. $\sin(x) = 1.4 + 2\cos(x)$
5. Find y when $x = 42°$

Solution

x	o	30°	60°	90°	120°	150°	180°	210°	240°
y									

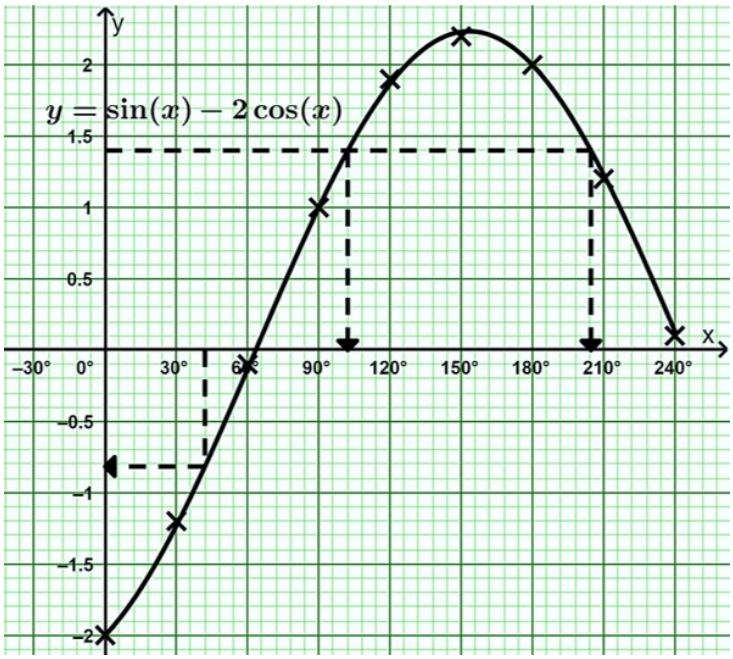


Figure 10.12: *Graphs of* $y = sin(x) - 2\,cos(x)$

a. A comparison of $y = \sin(x) - 2\cos(x)$ and $\sin(x) - 2\cos(x) = 0$ reveals that $y = 0$.

   We will therefore find the point at which $y = \sin(x) - 2\cos(x)$ cuts the line, $y = 0$ (the $x - axis$).

   From the graph, $x = 66°$

b. $\sin(x) = 1.4 + 2\cos(x)$

   $\Rightarrow \sin(x) - 2\cos(x) = 1.4$

   A comparison of $y = \sin(x) - 2\cos(x)$ and $\sin(x) - 2\cos(x) = 1.4$ reveals that $y = 1.4$.

   From the graph, when $y = 1.4, x = 102°$ or $x = 198°$

7. From the graph, when $x = 42°$, $y = -0.8$

FOCAL AREA 2: MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRIC GRAPHS

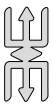
The maximum and minimum heights for the graph of trigonometric functions can be read from their graphs by inspecting their sketches, however there is a way to find out the maxima and minima values without a sketch.

To find the maxima and minima of a trigonometric expressions of the form $A\sin(B\theta \pm C) + D$ or $A\cos(B\theta \pm C) + D$ multiply A by 1 and -1 and add the D.

Mathematically,

$Minimum\ or\ maximum\ value = A + D$ or

$Minimum\ or\ maximum\ value = -A + D$



Note

- D *could be negative or positive.*
- *The smaller of the values of "*$A + D$*" and "*$A + D$*" is the minimum value of the expression.*
- *The larger of the values of "*$A + D$*" and "*$A + D$*" is the maximum value of the expression.*

Example 1

If $y = 2\sin(\theta)$, then $A = 2$ and $D = 0$

$A(-1) + D = 2(-1) + 0$

$= -2$

$A(1) + D = 2(1) + 0$

$= 2$

Since $-2 < 2$, -2 is the minimum value while 2 is the maximum value of $y = 2\sin(\theta)$

Example 2

Find the maximum and minimum values of $-3\cos\left(\frac{1}{2}x\right) + 4$

Solution

For $-3\cos\left(\frac{1}{2}x\right) + 4$, $A = -3$ and $D = 4$

$A(-1) + D = -3(-1) + 4$

$= 7$

$A(1) + D = -3(1) + 4$

$= 1$

Since $1 < 7$, 1 is the minimum value while 7 is the maximum value of $-3\cos\left({}^{1}/_{2}\,x\right) + 4$

It must be noted that the highest or lowest points of a trigonometric graphs are the same as the turning points.

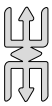
Learning Tasks

1. Learners in mixed-ability groups explore properties of trigonometric graphs using GeoGebra.
2. Learners in groups use GeoGebra to sketch trigonometric graphs.
3. Learners in groups recollect the values of the quadrantal angles for sine and cosine.
4. Learners use these values to sketch trigonometric graphs with a specified interval.
5. Learners in mixed-ability identify the maximum and minimum values of the trigonometric graphs.
6. Learners in mixed-ability groups explore sketch trigonometric graphs using the quadrantal angles.
7. Learners in groups discuss their observations as the graphs transform.
8. Learners in groups discuss how the graphs transform as the trigonometric function transforms.
9. Learners in groups explore the turning point of trigonometric graphs.
10. Learners in groups create and sketch the trigonometric graphs and identify the minimum and maximum points.

Pedagogical Exemplars

1. **Experiential Learning**
   a. **Graph Walk:**
      i. Learners use string and pegs to physically plot the sine or cosine curve on the classroom floor.
      ii. Label peaks, troughs, and intercepts with cards.
   b. **Change with Sliders**: Use interactive tools like GeoGebra or printed grids with moveable sliders to demonstrate transformations in $A\sin(Bx \pm C) + D$ and $A\cos(Bx \pm C) + D$
   c. Task learners in their groups to sketch graphs manually or digitally and reflect on how each term affected the graph.
   d. Use structured worksheets with fill-in-the-blank sections.
   e. Allow alternative response formats (drawing, digital tools).
   f. Offer guided peer pairing with more confident learners.
   g. Use physical props and tactile graph lines for learners with visual impairments.
   h. Allow oral explanation or use of pictorial expressions for learners with writing difficulties.
   i. Use simplified language.
2. Collaborative problem and inquiry-based learning:
   a. Task learners explain to peers how to apply transformations or express harmonic identities.
   b. Provide a scenario and task learners to model data using a sine or cosine function.
   c. Task learners to share their graphs and descriptions with other groups.

Key Assessment Level 4: Group work



Note

The following are sample tasks. It is required that each group completes just one project of these of similar projects which will be suitable.

1. *Investigate the relationship between the graphs of* $y = \sin(\theta)$, $y = \sin(\theta + 90°)$, *and* $y = \sin(\theta - 90°)$.

   a. *Sketch all three functions on the same coordinate system (by hand or using technology) for* $0 \leq \theta \leq 360°$.

   Ensure you mark on all the key points where the graphs intersect the x-axis and reach maximum/ minimum values.

   9 marks

   b. *On a unit circle, identify and label the angles that correspond to these key points.*

   4 marks

   c. *Analyse how the phase shifts relate to positions on the unit circle.*

   4 marks

   d. *Write a mathematical argument explaining why* $\sin(\theta + 90°) = \cos(\theta)$ *using your graphs and the unit circle.*

   3 marks

   (20 marks)

   DoK level 3

2. *Trigonometric functions create patterns found throughout art, architecture, and design. This task challenges learners to analyse these patterns and create original designs based on mathematical principles.*

   a. *Find examples of patterns in local architecture, textiles, or crafts.*

   b. *Using grid paper and compass or graphical tools:*

      i. *Recreate these patterns by hand using straight-edge and compass*

      ii. *Identify where circular or repeating elements appear*

      iii. *Show how these patterns relate to sine and cosine curves*

   c. *Create your own pattern incorporating:*

      i. *At least two different trigonometric-based elements*

      ii. *Careful hand measurement of angles using protractor*

      iii. *Clear connection to unit circle properties*

   d. *Provide an explanation of the mathematical principles in your design.*

   e. *Display your work as a classroom exhibition with mathematical explanations.*

   (20 marks)

   DoK Level 4

3. *Solar and wind energy production follows patterns that can be modelled with trigonometric functions. This task challenges learners to design an optimisation system for renewable energy collection based on mathematical modelling.*
   a. *Track the sun's position at the same times each day for one week*
   b. *Using simple tools (stick shadow, cardboard angle measurer) or graphing tools and software:*
      i. *Record shadow lengths and angles*
      ii. *Plot the sun's path*
      iii. *Create a trigonometric model of daily patterns*
   c. *Design a paper-based sun predictor or digital visualisation showing:*
      i. *How the sun's path relates to the unit circle*
      ii. *Predictions for optimal sun angles during different times of the day*
      iii. *Connection between trigonometric functions and solar position*
   d. *Build a simple solar demonstration using a flashlight and protractor.*
   e. *Write an analysis of how your model could be used for practical applications.*

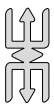
(20 marks)

DoK Level 4

PLC Session 10

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.

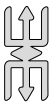
Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.

4. Plan the week's key assessment with your app

a. The key assessment for the week is ***group work***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the group work (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 11

Learning Indicators

1. *Review compound identities*
2. *Use knowledge in compound to derive harmonic identities and find the maximum and minimum values of a trigonometric function*

In this week's lesson, learners will review some trigonometric identities, derive harmonic identities and use them to solve problems

FOCAL AREA 1: REVIEWING TRIGONOMETRIC IDENTITY

Recall that in Week 14 of the Year 2 Teacher Manual, learners were guided to derive various trigonometric identities. Compound-angle identities and multiple-angle identities were part of the identities discussed. The compound-angle identities are as follows:

1. $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
2. $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
3. $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
4. $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
5. $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$
6. $\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

We will use these identities to derive harmonic identities.

The multiple-angle identities are also as follows:

1. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
2. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$
3. $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

FOCAL AREA 2: DERIVING HARMONIC IDENTITIES

There are situations when writing the sum of two different trigonometric ratios as a single ratio of the sum or difference of two angles makes solving the problem easier. For instance, to sketch the graph of $y = 4\cos(x) + 3\sin(x)$ or determine the maximum and minimum values, it will be more appropriate to write the equation in terms of just one function. By this, the phase shift, amplitude and the vertical translation can be found by inspection.

Expression of *a cos(o) + b sin (o)* as *R cos(o−a)*

To express $a\cos(\theta) + b\sin(\theta)$ as $R\cos(\theta - \alpha)$,

1. Equate the two expressions as such:

$$a\cos(\theta) + b\sin(\theta) = R\cos(\theta - \alpha)$$

Note

$\cos(\theta - \alpha)$ is selected instead of $\cos(\theta + \alpha)$ because the LHS is a sum and the signs (plus or minus) are alternated during the expansion of cosine functions of two angles.

i.e.: $\cos(\theta \pm \alpha) = \cos(\theta)\cos(\alpha) \mp \sin(\theta)\sin(\alpha)$

2. Expand the RHS

$$a\cos(\theta) + b\sin(\theta) = R\cos(\theta)\cos(\alpha) + R\sin(\theta)\sin(\alpha)$$

3. Find the values of R and α by comparing and equating expressions on both sides of the equations

$$R\cos(\alpha) = a \text{ and } \quad R\sin(\alpha) = b$$

$$a^2 + b^2 = R^2\cos^2(\alpha) + R^2\sin^2(\alpha)$$

$$R^2[\cos^2(\alpha) + \sin^2(\alpha)] = a^2 + b^2$$

$$R^2 = a^2 + b^2 \text{ since } \cos^2(\alpha) + \sin^2(\alpha) = 1$$

$$R = \sqrt{a^2 + b^2}, \qquad R > 0$$

$$\frac{R\sin(\alpha)}{R\cos(\alpha)} = \frac{b}{a} = \tan(\alpha)$$

Alternatively,

A right triangle can be used to find the values of R and α thus

Since $R\cos(\alpha) = a$ and $R\sin(\alpha) = b$, for α, the lengths of the adjacent side, opposite side and the hypotenuse are a, b and R respectively

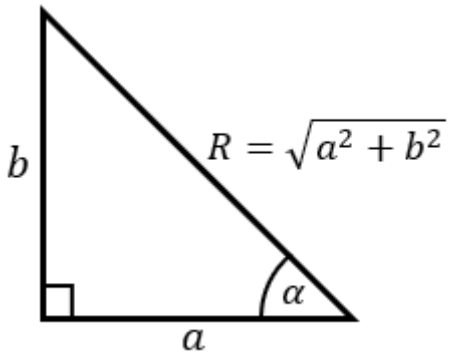


Figure 11.1: *Right triangle with lengths: "a", "b" and "$R = \sqrt{a^2 + b^2}$"*

From Figure 11.1, $\alpha = \sin^{-1}\left(\frac{b}{R}\right)$, $\alpha = \cos^{-1}\left(\frac{a}{R}\right)$ or $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$\tan^{-1}\left(\frac{b}{a}\right)$ is more accurate for finding α especially when R results in a non-terminating decimal

The other harmonic identities are shown in Table 11.1

Table 11.1: *Harmonic Identities*

Trigonometric Expression	Harmonic identities
$a\cos(x) + b\sin(x) + c$	$R\cos(x - \alpha) + c$
$a\cos(x) - b\sin(x) + c$	$R\cos(x + \alpha) + c$

$a\cos(x)+b\sin(x)+c$	$R\sin(x+\alpha)+c$
$a\cos(x)-b\sin(x)+c$	$R\sin(x-\alpha)+c$

Example 1

^

Solution

A comparison of $4\cos(x)+3\sin(x)$ and $a\cos(x)+b\sin(x)+c$ reveals that $a=4, b=3$ and $c=0$

Let $a\cos(x)+b\sin(x)+c=R\cos(x-\alpha)+c$

$R=\sqrt{4^2+3^2}$

$=5\,units$

$\tan(\alpha)=\frac{3}{4}$

$\alpha=\tan^{-1}\left(\frac{3}{4}\right)$

$=36.8699°$

$4\cos(x)+3\sin(x)=5\cos(x-36.8699°)$

Example 2

Rewrite $y=5\cos(x)+12\sin(x)$ as an equation in terms of cosine only and indicate the features of its graph.

Solution

A comparison of $y=5\cos(x)+12\sin(x)$ and $a\cos(x)+b\sin(x)+c$ reveals that $a=5, b=12$ and $c=0$

Let $a\cos(x)+b\sin(x)+c=R\cos(x-\alpha)+c$

$R=\sqrt{5^2+12^2}$

$=13\,units$

$\tan(\alpha)=\frac{12}{5}$

$\alpha=\tan^{-1}\left(\frac{12}{5}\right)$

$=67.3801°\approx 67°$

$5\cos(x)+12\sin(x)=13\cos(x-67°)$

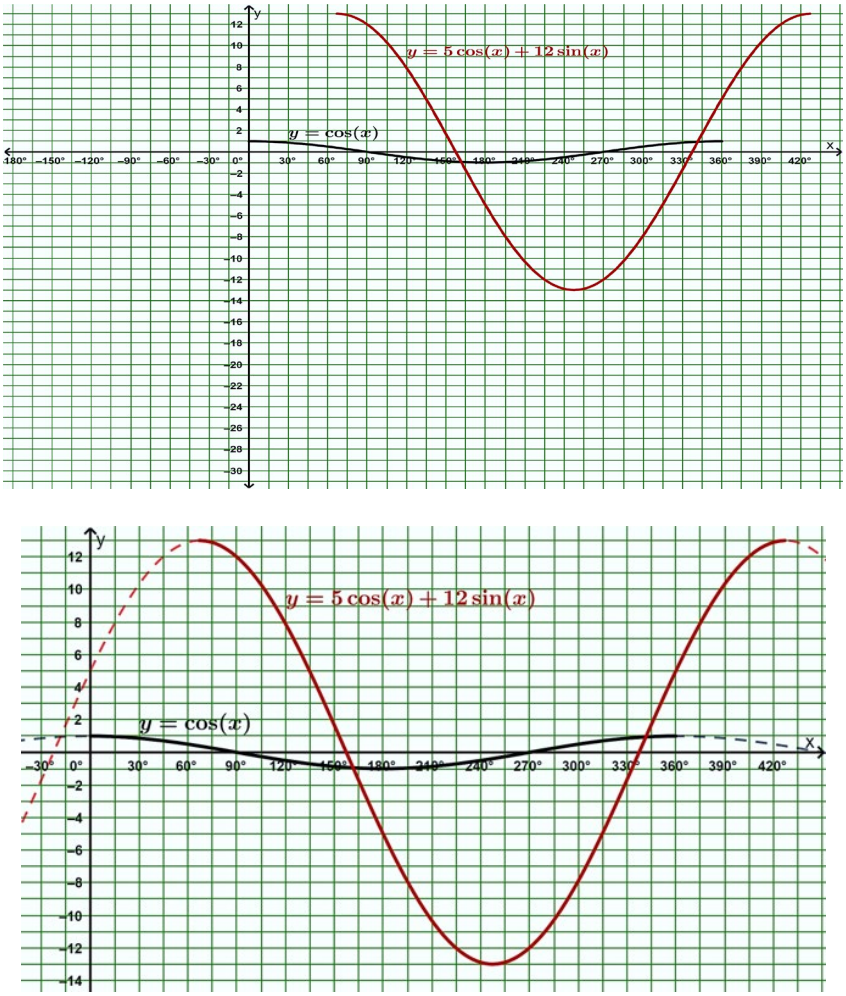


Figure 11.2: *Graphs of* $y = 5\cos(x) + 12\sin(x)$ *and* $y = \cos(x)$

As shown in Figure 11.2, the graph $y = 5\cos(x) + 12\sin(x)$ has an amplitude of 13 units. It is therefore a result of a vertical stretch by a scale factor of 13. The graph has a period of $360°$ and has a horizontal shift of $67°$ to the right. Its midline lies on the $x-$axis as there is no vertical shift. It has a maximum and minimum as 13 and -13 respectively.

Example 3

The motion of a particle in the $x - y$ plane can be modelled by $y = 6\sin(x) - 2\cos(x)$. Find the greatest height that can be achieved by the graph of the motion.

Solution

$6\sin(x) - 2\cos(x) = -2\cos(x) + 6\sin(x)$

Let $-2\cos(x) + 6\sin(x) = a\cos(x) + b\sin(x) + c$

$a = -2, b = 6$ and $c = 0$

Also let $-2\cos(x) + 6\sin(x) = R\sin(x + \alpha)$

$R = \sqrt{a^2 + b^2}$

$R = \sqrt{(-2)^2 + 6^2}$

$= 2\sqrt{10}$

Since $c = 0$, the greatest height that can be achieved by the graph of the motion is $R = 2\sqrt{10}$

Example 4

Given that $f(\theta) = 4\sin(\theta) - 3\cos(\theta) + 2$ find the

a) the minimum and maximum values of $f(\theta)$

b) the minimum and maximum points of $f(\theta)$

Solution

a) $f(\theta)$ can be expressed in the form $R\sin(\theta - \beta) + 2$

$4\sin(\theta) - 3\cos(\theta) + 2 = R\sin(\theta - \beta) + 2$

$4\sin(\theta) - 3\cos(\theta) + 2 = R[\sin\theta\cos\beta - \cos\theta\sin\beta] + 2$

$4\sin(\theta) - 3\cos(\theta) + 2 = R\sin\theta\cos\beta - R\cos\theta\sin\beta + 2$

Comparing the terms

$R\sin\theta\cos\beta = 4\sin(\theta)$

$R\cos\beta = 4$.....(1)

$R\cos\theta\sin\beta = 3\cos(\theta)$

$R\sin\beta = 3$..... (2)

Eliminate β by **squaring** both equations and **add**

$R^2\cos^2\beta = 16$ ..... (3)

$R^2\sin^2\beta = 9$ ..... (4)

$R^2\cos^2\beta + R^2\sin^2\beta = 16 + 9$

$R^2(\cos^2\beta + \sin^2\beta) = 25$ but $[\cos^2\beta + \sin^2\beta] = 1$

$R^2 = 25$

$R = 5$

The maximum and minimum values of $f(\theta)$ occur at 1 and -1 respectively

$R(\pm 1) + D$

Maximum value $= 5(1) + 2 = 7$

Minimum value $= 5(-1) + 2 = -3$

b) Find the value of β by dividing equation 2 and 1

$\frac{R\sin\beta}{R\cos\beta} = \frac{3}{4} \rightarrow \tan\beta = \frac{3}{4}$

$\therefore \beta = \tan^{-1}\frac{3}{4}$

$\beta = 36.87°$

The maximum value of F is 7. The value of θ for which it occurs is 1.

Therefore, for maximum $\sin(\theta - 36.87°) = 1$.

$\theta - 36.87° = \sin^{-1}(1)$

$\theta - 36.87° = 90°$

$\theta = 126.87°$

Maximum point $= (7, 126.87°)$

Now for minimum, $\sin(\theta - 36.87°) = -1$

$\theta - 36.87° = \sin^{-1}(-1)$

$\theta - 36.87° = 270°$

$\theta = 306.87°$

Minimum point $= (-3, 306.87°)$

Learning Tasks

1. Learners in mixed-ability recollect compound angles
2. Learners in groups use the addition formula for cosine to obtain the expansion of $R\cos(\theta - \alpha)$
3. Learners in mixed-ability groups brainstorm and learn how to express $a\cos(\theta) + b\sin(\theta)$ as $R\cos(\theta - \alpha)$
4. Learners brainstorm to find the values of R and α by comparing and equating expressions on both sides of the equations
5. Learners in mixed-ability groups brainstorm how to identify the maximum or minimum value

Pedagogical Exemplars

1. Experiential learning
   a. **Harmonic Form Structure Aid**
      i. Provide scaffolds where $R = \sqrt{a^2 + b^2}$ is already calculated
      ii. Let learners colour code components (e.g., a, b, R, a) in the harmonic form
2. Collaborative based learning:
   a. Task learners to graph and compare $y = a\sin(x) + b\cos(x)$ and $y = R\sin(x + \alpha)$ to verify equivalence.
   b. Task learners to solve for unknowns using inverse trigonometric functions
   c. Provide structured worksheets with diagrams
   d. Use tactile materials for visually impaired learners
   e. Provide written instructions, captioned video demos, use of diagrams, partner-based activities

Key Assessment Level 2: Questioning

1. *Use an appropriate harmonic identity to evaluate* $\sin(75°) + \sin(15°)$ *without using a calculator*

5 marks

2. *Find the maximum and minimum values of* $g(\theta) = 2\sin(\theta) - 3\cos(\theta) + 4$

10 marks

3. *A pendulum's displacement from equilibrium is modelled by* $d(t) = 5\sin^2(2t)$*, where t is time in seconds and d is in centimetres.*

   a. *Use the double-angle formula to rewrite* $\sin^2(2t)$ *in terms of* $\cos(4t)$

   2 marks

   b. *Express d(t) in the form* $d(t) = a + b\cos(4t)$

   2 marks

   c. *Determine the maximum and minimum values of* $d(t)$

   4 marks

Total: 8 marks

Section 4 Review

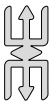
This section discussed all the lessons to be taught in week ten and week eleven. It focused on trigonometric graphs and their transformations, reviewed compound identities and introduced harmonic identities. Learning and assessment tasks for this section emphasised the development of skills and competencies required for modelling real-life scenarios or objects with trigonometric equations and graphs. Assessment tasks which required collaboration among learners and provide the opportunity for learners to practice were suggested to reinforce learning from year one and year two. It is intended that leaners will get the opportunity to revise lessons taught in the previous years as they work through the learning and assessment tasks. Teachers are strongly encouraged to foster inclusive learning environments for all learners.

PLC Session 11

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)

c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***questioning***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the assessment (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
    a. transfer your chats into your learning planner template.
    b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
    c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

MARKING SCHEMES AND RUBRICS FOR THE KEY ASSESSMENTS

WEEK 10

a. *Graph of functions (9 marks)*

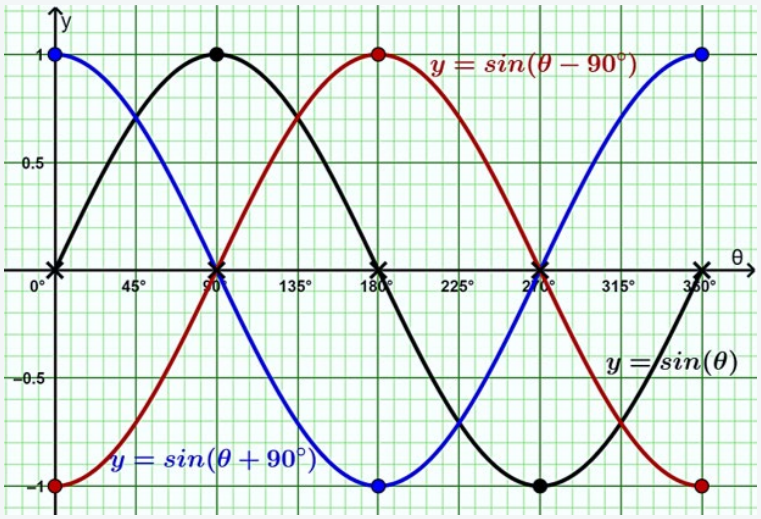


Figure 10.13: *Graphs of* $y = \sin(\theta)$, $y = \sin(\theta + 90°)$ *and* $y = \sin(\theta - 90°)$

Correct graph of $y = sin(\theta)$ *with proper scale*

(3 marks, 1 mark for correct intersections, 1 mark for correct turning points and 1 mark for a smooth curve drawn)

Correct graph of $y = sin(\theta + 90°)$ *with proper shift*

(3 marks, 1 mark for correct intersections, 1 mark for correct turning points and 1 mark for a smooth curve drawn)

Correct graph of $y = sin(\theta - 90°)$ *with proper shift*

(3 marks, 1 mark for correct intersections, 1 mark for correct turning points and 1 mark for a smooth curve drawn)

Note, for key points:

For $y = sin(\theta)$

x-intercepts: $0°, 180°, 360°$

Maximum value (+1) occurs when $\theta = 90°$

Minimum value (-1) occurs when $\theta = 270°$

For $y = sin(\theta + 90°)$*:*

x-intercepts: 90°, 270°

Maximum value (+1) occurs when $\theta = 0$ *and* $\theta = 360°$

Minimum value (-1) occurs when $\theta = 180°$

For $y = sin(\theta - 90°)$*:*

x-intercepts: 90°, 270°

Maximum value (+1) occurs when $\theta = 180°$

Minimum value (-1) occurs when $\theta = 0$ *and* $\theta = 360°$

b. *Unit Circle Representation (4 marks)*

For $y = \sin(\theta)$*:*

At 0°, 180°, 360°: $\sin(\theta) = 0$

At 90°: $\sin(\theta) = 1$

At 270°: $\sin(\theta) = -1$

For $y = \sin(\theta + 90°)$*:*

At 0, $\sin(0 + 90°) = \sin(90°) = 1$

At $90°$*:* $\sin(90° + 90°) = \sin(180°) = 0$

At $180°$*:* $\sin(180° + 90°) = \sin(270°) = -1$

At $270°$*:* $\sin(270° + 90°) = \sin(360°) = 0$

At $360°$*:* $\sin(360° + 90°) = \sin(450°) = 1$

For $y = \sin(\theta - 90°)$

At $0°$*:* $\sin(0 - 90°) = \sin(-90°) = -1$

At $90°$*:* $\sin(90° - 90°) = \sin(0) = 0$

At $180°$*:* $\sin(180° - 90°) = \sin(90°) = 1$

At $270°$*:* $\sin(270° - 90°) = \sin(180°) = 0$

At $360°$*:* $\sin(360° - 90°) = \sin(180°) = -1$

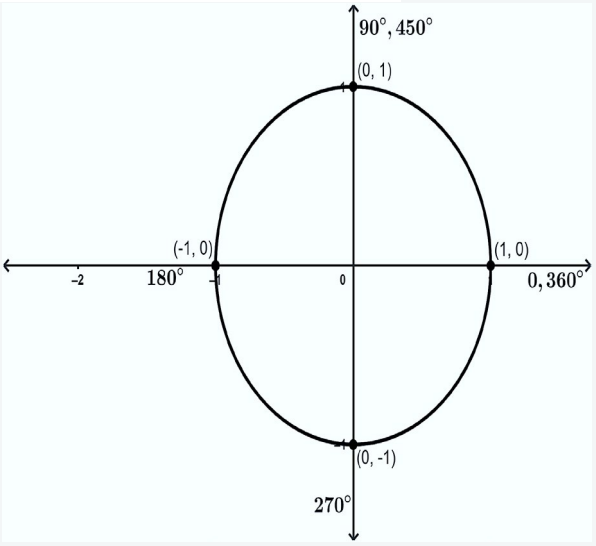


Figure 10.14: *Unit circle*

Accurate labelling of quadrantal angles (2 marks, 0.5 mark each)

Accurate plotting of key points (1 mark, 0.25 mark each)

Correct drawing of unit circle passing through key points (1 mark)

c. *Phase Shift Analysis (4 marks)*

The phase shift of 90° corresponds to a quarter rotation around the unit circle:

A phase shift of $+90°$ *(in* $y = \sin(\theta + 90°)$*) means we start to evaluate the sine function at a point 90° anticlockwise i.e.,* $-90°$ *or* $270°$

A phase shift of $-90°$ *(in* $y = \sin(\theta - 90°)$*) means we start to evaluate the sine function at a point 90° clockwise i.e.,* $90°$

For any angle θ *on the unit circle:*

$\sin(\theta + 90°)$ *gives the y-coordinate of the point 90° anticlockwise from* θ

$\sin(\theta - 90°)$ *gives the y-coordinate of the point 90° clockwise from* θ

Mark allocation

Correct explanation of physical meaning of phase shifts (1 mark, 0.5 mark for each phase shift)

Accurate connection between phase shifts and unit circle rotation (2 marks, 1 mark for each direction)

Clear explanation of how phase shifts affect the function values (1 mark for any correct explanation)

d. ***Mathematical Argument (3 marks)***

On the unit circle. $\sin(\theta)$ *represents the y-coordinate of the point at angle* θ*, while* $\cos(\theta)$ *represents the* $x-$*coordinate of the same point.*

When we rotate 90° counterclockwise from any angle θ *to position* $(\theta + 90°)$*:*

The $y-$*coordinate of the point at* $(\theta + 90°)$ *equals the* $x-$ *coordinate of the original point at* θ

This is because a 90° rotation maps the $x-$*axis to the* $y-$*axis*

$\sin(\theta + 90°) = the\ y - coordinate\ at\ angle\ (\theta + 90°) = the\ x - coordinate\ at\ angle, \theta\ = \cos(\theta)$

This can also be seen in our graphs, where $y = \sin(\theta + 90°)$ *exactly matches the graph of* $y = \cos(\theta)$*.*

Marks allocation*:*

Provide a connection between the $x-$ *coordinate of a point and the cosine value of an acute angle (1 mark)*

Provide a connection between the $y-$ *coordinate of a point and the sine value of an acute angle (1 mark)*

Connect the phase change to rotation about the origin (1 mark)

Assessment Notes

Full marks require both accurate calculations and clear conceptual understanding.

Partial credit should be awarded for correct reasoning with minor computational errors.

Students using technology should still demonstrate understanding of mathematical relationships.

Hand-drawn graphs should be assessed for qualitative accuracy rather than perfect precision.

	Levels of Performance			
Criteria	**Excellent (4 marks)**	**Very Good (3 marks)**	**Satisfactory (2 marks)**	**Needs improvement (1 mark)**
Pattern Identification & Analysis	*Identifies 4 or more distinct mathematical patterns in cultural examples. Measures and documents at least 3 different angle relationships (±2° precision). Makes explicit connections to specific trigonometric functions. Analyses at least 2 different cultural contexts.* *Manual example: "Islamic star uses 8-fold symmetry with 45° angles".* *Digital example: Documents pattern measurements with digital protractor to 1° precision*	*Identifies 3 distinct mathematical patterns. Measures and documents 2 angle relationships (±5° precision). Makes general connections to trigonometric concepts. Analyses at least 1 cultural context in depth.* *Manual example: "Textile has elements at 60° intervals"* *Digital example: Uses basic measurement tools in software*	*Identifies 2 basic patterns. Attempts angle measurements with limited precision (±10°). Makes vague connections to trigonometry. Superficial cultural context analysis.* *Manual example: "Pattern looks like a sine wave"* *Digital example: Limited use of measurement tools*	*Identifies only 1 simple pattern. No specific angle measurements. No clear mathematical connections. No cultural context analysis.* *Manual example: "Design has curved shapes"* *Digital example: Takes photos without analysis*
Mathematical Reconstruction	*Reconstructs patterns with maximum error of 2mm or 2°. Uses appropriate tools with precision (compass, protractor, ruler). Documents at least 5 specific steps in construction process. Demonstrates understanding of underlying geometry.* *Manual example: Precisely drawn with visible construction lines.* *Digital example: Multi-layered reconstruction showing each construction step*	*Reconstructs patterns with maximum error of 5mm or 5°. Uses appropriate tools effectively. Documents 3-4 steps in construction process• Shows good understanding of geometry.* *Manual example: Well-drawn with most construction lines.* *Digital example: Clear digital construction with most steps visible*	*Reconstructs with errors of 5-10mm or 5-10°. Inconsistent tool use. Documents only 1-2 construction steps. Basic understanding of geometry.* *Manual example: Attempted reconstruction with some inaccuracies.* *Digital example: Basic digital drawing with few construction references*	*Reconstruction errors exceed 10mm or 10°. Poor tool use or inappropriate tools. No documented construction steps. Limited geometric understanding.* *Manual example: Rough sketch without precision.* *Digital example: Simple shapes without construction method*

Original Design Creation	*Creates pattern incorporating at least 3 trigonometric elements. All angles precisely measured and documented (±2°). Design shows at least 8 repetitions with consistent spacing (±1mm). Includes specific unit circle references at multiple angle measures.* *Manual example: Hand-drawn with precise measurements.* *Digital example: Parametric design with exact mathematical relationships*	*Creates pattern with 2 trigonometric elements. Most angles measured and documented (±5°). Design shows 5 to 7 repetitions with consistent spacing (±3mm). Includes basic unit circle references.* *Manual example: Hand-drawn with good precision.* *Digital example: Digital design with defined mathematical bases*	*Creates pattern with 1 clear trigonometric element. Some angles measured (±10°). Design shows 3 or 4 repetitions with variable spacing. Limited unit circle connection* *Manual example: Hand-drawn with visible attempt at precision.* *Digital example: Basic digital curves with approximate measurements*	*Creates simple pattern with no clear trigonometric elements. Unmeasured or inaccurate angles· Fewer than 3 repetitions or inconsistent pattern. No unit circle connection.* *Manual example: Freehand drawing without measurement.* *Digital example: Digital drawing without mathematical basis*
Mathematical Explanation	*Uses 5 or more correct trigonometric terms (amplitude, period, etc.). Explains pattern using specific angle measurements (30°, 45°, 60°, etc.). Provides exact mathematical relationships for at least 2 elements. Connects pattern features to specific trigonometric functions.* *Manual example: Written explanation with diagrams and formulas* *Digital example: Documented function parameters with explanatory notes*	*Uses 3 or 4 correct trigonometric terms. References specific angle measurements in explanation. Provides general mathematical relationships. Makes clear connections to trigonometry* *Manual example: Written explanation with some diagrams.* *Digital example: Function descriptions with basic parameter explanation*	*Uses 1 or 2 correct mathematical terms. General reference to angles without specifics. Vague mathematical relationships. Basic trigonometric connections.* *Manual example: Brief explanation with limited terminology.* *Digital example: Basic description of functions used*	*Uses no correct mathematical terminology. No reference to angle measurements. No mathematical relationships described. No clear trigonometric connections.* *Manual example: Minimal or incorrect explanation.* *Digital example: Missing or incorrect function descriptions*

Presentation & Communication	*Creates comprehensive visual display with all elements labelled. Includes side-by-side comparison with cultural example. Provides unit circle diagram with angles marked in 15° increments. Includes Organises information in logical, easy-to-follow sequence.* *Manual example: Carefully mounted displays with clear handwriting.* *Digital example: Professional-quality presentation with dynamic elements*	*Creates clear visual display with most elements labelled. References cultural example with comparison. Includes unit circle with 30° increment markings. Organises information effectively.* *Manual example: Neat display with readable handwriting.* *Digital example: Well-organised digital presentation*	*Creates basic visual display with some labels. Mentions cultural connection without clear comparison. Includes simple unit circle reference with 90° quadrants. Basic organisation of information.* *Manual example: Functional display with inconsistent quality.* *Digital example: Basic digital presentation with limited design*	*Creates minimal display with few/ no labels• No clear cultural reference. Missing or incorrect unit circle reference. Poor organisation of information.* *Manual example: Disorganised or messy display.* *Digital example: Incomplete digital files or poor formatting*
Total Score	*20 marks*			

	Levels of Performance			
Criteria	**Excellent** **(4 marks)**	**Very Good** **(3 marks)**	**Satisfactory** **(2 marks)**	**Needs improvement** **(1 mark)**
Data Collection & Documentation	*Manual: Records sun position at 3 specific times (morning, noon, afternoon) for 5 consecutive days with precise shadow measurements (32cm at 8am) and angles (27° east, 0°, 24° west). Includes weather notes affecting readings.* *Digital: Creates spreadsheet logging sun positions 3 times daily for 5 days, with automated calculations for angle change rates (averaging 15° per hour) and comparative analysis between days.*	*Manual: Records sun position at 2 specific times daily for 4 days with shadow lengths and angles measured to nearest 5°.* *Digital: Uses sun position app to collect data twice daily for 4 days, exports to organised spreadsheet showing daily patterns.*	*Manual: Records sun position once daily for 3 days with inconsistent measurement times.* *Digital: Uses basic app to capture some sun positions but with gaps or missing angle measurements.*	*Manual: Records only 1 or 2 days of data with minimal measurements.* *Digital: Takes scattered screenshots from sun app without organised data recording method.*

Mathematical Modelling	*Creates precise trigonometric model with defined variables• Formula includes daily variation and hints at seasonal component. Tests model against data with calculated error margins (±3° or better). Explains meaning of each coefficient.* *Example (Manual): "Daily sun path follows model: Elevation angle (α)=50°−40°×cos(15°×t) where t is the hours from solar noon. Verified accurate to $\pm 3°$ across all measurements."* *Example (Digital): Creates regression model in spreadsheet that accurately predicts sun position throughout day with $R^2 > 0.95$, using elevation angles in degrees with clear labelling of variables.*	*Creates appropriate trigonometric model• Formula correctly represents daily variation. Tests model against some data points (±7° accuracy). Explains general meaning of model component s.* *Example (Manual): "Sun elevation angle = 45° − 35° × cos(hour × 15°)" with verification showing accuracy within $\pm 7°$.Example (Digital): Creates spreadsheet model using trigonometric functions with clearly labeled angle measurements in degrees, showing reasonable fit to collected data.*	*Creates simplified model with basic trigonometric elements. Formula has conceptual errors or missing parameters. Limited testing against actual data (±15° accuracy). Minimal explanation of components.* *Example (Digital): Attempts function fitting but with limited accuracy or missing proper angle references.*	*Creates oversimplified model without proper trigonometric functions. Formula fails to capture pattern correctly. No testing against collected data. Cannot explain mathematical components.* *Example (Manual): "Sun is highest at noon, lower in morning and afternoon" without quantification.* *Example (Digital): Plots data points without meaningful function fitting or angular analysis.*
Sun Predictor Design	*Creates functional, accurate predictor from one week's data. Example (Manual): Cardboard dial showing hourly sun positions from sunrise to sunset with angles marked in 5° increments, shadow length predictions for 1m vertical object, and ability to show seasonal adjustment based on calendar date.* *Example (Digital): Interactive spreadsheet where time input returns sun elevation to nearest degree, azimuth direction, and shadow projection with visual representation.*	*Creates effective predictor.* *Example (Manual): Paper-based sun finder with hourly marks and 10° angle increments that reasonably predicts sun position throughout current week.* *Example (Digital): Spreadsheet calculator that outputs sun elevation angles when time is entered, with accuracy to nearest 5°.*	*Creates basic predictor.* *Example (Manual): Simple cardboard tool showing approximate sun positions for morning, noon, and afternoon with basic angle markings in 15° increments.* *Example (Digital): Basic diagram showing sun path with limited angle references or calculation ability.*	*Creates minimal predictor.* *Example (Manual): Rough diagram showing general sun path without specific angle measurements.* *Example (Digital): Static digital image with minimal functionality or angle references.*

Physical/Digital Demonstration	*Creates comprehensive demonstration despite time constraints.* *Example (Manual): Setup with flashlight, meter stick, and protractor that demonstrates daily sun path with precise angle measurements every $15°$. Includes handout explaining how to interpret angle measurements.* *Example (Digital): Animation showing daily sun path with clearly labelled elevation angles in 5° increments, shadow lengths for standard object, and explanation of how to extend model to different dates.*	*Creates effective demonstration.* *Example (Manual): Lamp and protractor setup showing sun angles at 3–4 key times of day with angles marked in $10°$ increments.* *Example (Digital): Digital model showing sun path with elevation angles marked at $30°$ intervals and approximate shadow projections.*	*Creates basic demonstration.* *Example (Manual): Simple setup showing sun at high/mid/low positions with approximate angle measurements.* *Example (Digital): Pre-made sun path visualisation with basic angle labels but limited explanation.*	*Creates minimal demonstration.* *Example (Manual): Basic light source pointed at unmarked surface.* *Example (Digital): Static sun diagram with no specific angle measurements or minimal interactivity.*
Analysis & Application	*Provides insightful analysis from limited dataset.* *Example (Manual): Report identifying daily pattern accuracy (±4°) with specific applications: "Window awning should extend 40cm to block midday summer sun but admit winter sun at 30° elevation" with supporting calculations.* *Example (Digital): Analysis with charts comparing predicted vs. actual sun positions, explaining how current week's pattern would change seasonally with specific angle adjustments (65° summer, 35° winter) for optimal solar applications.*	*Provides solid analysis.* *Example (Manual): Report with reasonably accurate angle predictions and practical application such as optimal times for photography with specific angle recommendations.* *Example (Digital): Presentation with visualisation showing how sun angles affect practical applications like solar panel placement at 20–25° tilt.*	*Provides basic analysis.* *Example (Manual): Report with general observations about sun patterns and basic suggestion for practical use.* *Example (Digital): Simple digital report with general recommendations but limited specific angle measurements or calculations.*	*Provides minimal analysis.* *Example (Manual): Brief statement about sun movement without specific angles or applications.* *Example (Digital): Very basic digital documentation without meaningful analysis of patterns or specific recommendations.*
Total Score	*20 marks*			

WEEK 11

1. $\sin(75°) + \sin(15°) = 2\sin\left(\frac{75°+15°}{2}\right)\cos\left(\frac{75°-15°}{2}\right)$ **M1**

$= 2\sin(45°)\cos(30°)$ **M1**

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$ **M1**

$= \frac{\sqrt{6}}{2}$ **M1A1**

2. $g(\theta) = 2\sin(\theta) - 3\cos(\theta) + 4$

Let $2\sin(\theta) - 3\cos(\theta) + 4 = R\sin(\theta + \alpha) + d$ ***B1, or equivalent***

$2\sin(\theta) - 3\cos(\theta) + 4 = R\sin(\theta)\cos(\alpha) + R\cos(\theta)\sin(\alpha) + d$

By comparison,

$R\cos(\alpha) = 2$, $R\sin(\alpha) = -3$ *and* $d = 4$

$(R\sin(\alpha))^2 + (R\cos(\alpha))^2 = (-3)^2 + 2^2$

$R^2\sin^2(\alpha) + R^2\cos^2(\alpha) = 9 + 4$

$R^2(\sin^2(\alpha) + \cos^2(\alpha)) = 13$

$R^2 = 13$

$R = \sqrt{13}$ **M1A1**

$R + d = 4 + \sqrt{13}$

$-R + d = 4 - \sqrt{13}$

The minimum value of $g(\theta)$ *is* $4 - \sqrt{13}$ **A1**

The maximum value is $4 + \sqrt{13}$ **A1**

a. $\sin^2(2t) = \frac{1-\cos(2(2t))}{2}$ **M1**

$= \frac{1-\cos(4t)}{2}$ **A1**

b. $d(t) = 5\sin^2(2t)$

$= 5\left(\frac{1-\cos(4t)}{2}\right)$ **M1**

$= \frac{5}{2} - \frac{5\cos(4t)}{2}$

$= 2.5 - 2.5\cos(4t)$ **A1**

c. *By comparing* $d(t) = 2.5 - 2.5\cos(4t)$ *to* $y(t) = A\cos(Bt + C) + D$

$A = -2.5$ *and* $D = 2.5$

$A + D = -2.5 + 2.5$ **M1**

$= 0$

$-A + D = -(-2.5) + 2.5$ **M1**

$= 5$

The maximum value of $d(t)$ *is* $5\ cm$ **A1**

The minimum value is 0 **A1**

SECTION 5: INTEGRATION

STRAND: CALCULUS

Sub-Strand: Principles of Calculus

Learning Outcome: *Identify and apply the integration rules to evaluate integrals.*

Content Standard: *Demonstrate conceptual understanding of the rules and techniques of integration to select and apply them appropriately to a function.*

INTRODUCTION AND SECTION SUMMARY

Integration is more than a mathematical procedure; it is a powerful tool for solving real-world problems in physics, engineering, economics, and beyond. This section focuses on identifying and applying fundamental integration rules, using substitution to evaluate complex expressions, and integrating transcendental functions such as natural logarithms and exponential functions. These techniques allow us to analyse rates of change, compute accumulated quantities, and model exponential growth and decay. Not all functions, however, have exact antiderivatives. In such cases, we apply numerical methods like the Trapezium Rule to approximate areas under curves; a key application of definite integration. Whether calculating distance from velocity, estimating areas, or analysing non-elementary functions, mastering these concepts builds essential problem-solving skills. Through practice and application, learners will develop a deeper understanding of integration as both a theoretical and practical tool for interpreting and navigating the world around them.

The weeks and the learning indicators covered by this section are:

- *Week 12*

1. *Identify and apply the integration rules to evaluate integrals.*
2. *Identify and apply appropriate techniques for integrating a function.*

- *Week 13*

1. *Identify and apply appropriate techniques of integration to integrate transcendental functions.*
2. *Solve problems involving integrals with no exact antiderivative (Trapezium rule).*

Recap weeks

Year 2	**Week 19** 1. Distinguish between partitioning an interval for a given step size and the number of subintervals 2. Partition intervals for a given step size or number of sub-intervals and find area under curves through graphics. 3. Compare and judge the effect of reduction in step size and increase in the number of subintervals in an interval for a given function on the area under a curve.

	Week 20 1. Identify and write a definite integral notation and its connection to the limits of the partial sum of areas. 2. Identify integration as a reverse process of differentiation.

SUMMARY OF PEDAGOGICAL EXEMPLARS

Teaching integration techniques and the trapezium rule to final-year senior high school learners requires strategic pedagogies for conceptual mastery and exam success. Demonstrations effectively illustrate substitution methods and step-by-step trapezium rule application, particularly for transcendental functions lacking elementary antiderivatives. Collaborative work encourages learners to tackle complex integrals together, fostering mathematical discourse when selecting appropriate techniques and verifying solutions. Experiential learning, through graphical area approximation activities and physical representations of accumulated quantities, helps visualise definite integrals and numerical approximation. Provide structured integration templates and pre-worked examples alongside extension problems exploring accuracy in numerical integration to support learners. Hand-drawn graphs with counted grid squares can demonstrate numerical approximation principles where technological resources are unavailable.

ASSESSMENT SUMMARY

For third-year senior high learners studying integration, combine homework assignments and essay-type questions for comprehensive assessment. Homework should progress from guided practice to independent application, featuring both analytical integration of transcendental functions and numerical approximation using the trapezium rule. Essay questions should require learners to explain their reasoning, justify technique selection, and analyse when numerical methods become necessary for functions lacking elementary antiderivatives. Differentiate tasks to accommodate various abilities, provide structured templates for struggling learners while challenging advanced students to explore how subinterval quantity affects approximation accuracy. Include procedural calculations, explanations, and real-world applications to ensure balanced assessment.

Indicators	DOK Level	Assessment Strategy and/or Mode
Week 12		
Identify and apply the integration rules to evaluate integrals.	4	Strategy: Homework Mode: Individual Classwork
Identify and apply appropriate techniques for integrating a function.		
Week 13		
Identify and apply appropriate techniques of integration to integrate transcendental functions.	4	Strategy: Essay-type Mode: Individual Classwork
Solve problems involving integrals with no exact antiderivative (Trapezium rule).		

WEEK 12

Learning Indicators

1. *Identify and apply the integration rules to evaluate integrals*
2. *Identify and apply appropriate techniques for integrating a function*

FOCAL AREA 1: IDENTIFYING AND APPLYING THE INTEGRATION RULES TO EVALUATE INTEGRALS

Integration is the reverse of differentiation. For instance, given the function $y = 5x^3$, differentiating y with respect to x results in $\frac{dy}{dx} = 15x^2$. Now, the integral of $15x^2$ with respect to x has to result in $5x^3$.

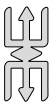
There are different ways to assist in integrating functions hence the term "rules of integration".

Identifying the rules of integration

1. Power Rule

   Given a function $f(x) = ax^n$, the power rule of integration posits that
   $\int f(x)dx = \int ax^n dx$ becomes $\frac{ax^{n+1}}{n+1} + c$, where c is the **arbitrary constant**.

   If $a = 1$, then $\int (1)x^n dx = \frac{x^{n+1}}{n+1} + c$.



Note

The power rule is not applicable if $n = -1$. *Thus,* $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c,\ n \neq -1$.

2. Sum rule

   Given the functions $f(x)$ and $h(x)$ the sum rule of integration posits that
   $\int [f(x) + h(x)]\, dx = \int f(x)dx + \int h(x)\, dx$

3. Difference rule

   Given the functions $f(x)$ and $h(x)$ the difference rule of integration posits that
   $\int [f(x) - h(x)]\, dx = \int f(x)dx - \int h(x)\, dx$

4. Multiplication by constant: Given a function $f(x)$, and a constant k then,
   $\int kf(x)dx = k \int f(x)dx$.

Example 1

Which rule of integration is applicable and why?

1. $\int 3x\, dx$
2. $\int (36x^5 + 5x)\, dx$
3. $\int (7x^3 - 25)\, dx$
4. $\int (36x^5 + 5x - 15x^2)\, dx$

Solution

1. Power rule because x in $3x$ has an exponent of one.
2. Power and sum rule because the functions have exponents and are being added.
3. Power and difference rule because the functions have exponents and are being subtracted.
4. Power, sum and difference rule because the functions have exponents, two of them are being added and then subtracted.

Definite Integral

Definite integrals are integrals that have upper and lower limits and can be evaluated as a numerical value. The Fundamental Theorem of Calculus covered in Year Two, Week twenty establishes a profound connection between the operations of differentiation and integration. Specifically, the second part of the theorem provides a method for evaluating definite integrals using antiderivatives, thus bridging the conceptual gap between the process of integration and differentiation.

Let f be a continuous real-valued function defined on a closed interval $[a, b]$ and let F be an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$. Then, the Second Fundamental Theorem of Calculus states that: $\int_a^b f(x)dx = F(b) - F(a)$. This result implies that the value of the definite integral of f over the interval $[a, b]$ can be computed by evaluating any one of its antiderivatives at the endpoints of the interval and taking the difference. Thus, rather than relying on the limiting process of Riemann sums to calculate the area under the curve $y = f(x)$, one can obtain the same result more efficiently by identifying an antiderivative of f. Some limitations that come with using the Riemann sums is that it is an approximation and not the exact value, more sub-intervals need to be used for a better accuracy, etc. On the other hand, usage in antiderivatives provide exact results.

Theorem (Integral Theorems)

If F and G are two antiderivatives of the same function, then F and G differ by a constant. In other words, there is a constant such that: $F(x) = G(x) + C$

If $F'(x) = f(x)$ for all x, then:

$$\int f(x)dx = F(x) + C$$

Let f be a continuous non-negative function for an interval $[a, b]$. Let $A(x)$ be the area of the region under the graph of the function from a to the number x. Then:

$$A(x) = \int_a^x f(x)\, dx$$

The function $A(x)$ is an antiderivative of $f(x)$.

FOCAL AREA 2: IDENTIFYING AND APPLYING APPROPRIATE TECHNIQUES FOR INTEGRATING A FUNCTION

Method of substitution

If an integral could be written in the form $\int f[g(x)]g'(x)\,dx$ then we can set $u = g(x)$ and integrate $\int f(u)\,du$.

Example 1

Rewrite each of the integrals in the form $\int f[g(x)]g'(x)\,dx$. Identify $g(x)$ and $g'(x)$.

1. $\int \sin(4x)\,dx$
2. $\int v\sqrt{v^2+4}\,dv$
3. $\int (x^2+3)\cdot 2x dx$
4. $\int \frac{4x}{\sqrt{1+x^2}}\,dx$

Solution

1. Let $u = g(x) = 4x$
   $\frac{du}{dx} = g'(x) = 4$
   Now, $\int sin(4x)\,dx = \frac{1}{4}\int sin(4x)\cdot 4\,dx$
2. Let $u = g(v) = v^2+4$
   $\frac{du}{dv} = g'(v) = 2v$
   Now, $\int v\sqrt{v^2+4}\,dv = \frac{1}{2}\int \sqrt{v^2+4}\cdot 2v\,dv$
3. Let $u = g(x) = x^2+3$
   $\frac{du}{dx} = g'(x) = 2x$
   Now, $\int (x^2+3)\cdot 2x dx = \int (x^2+3)\cdot 2x dx$ which is same as the original.
4. Let $u = g(x) = 1+x^2$
   $\frac{du}{dx} = g'(x) = 2x$
   Now, $\int \frac{4x}{\sqrt{1+x^2}}\,dx = 2\int \frac{1}{\sqrt{1+x^2}}\cdot 2x\,dx$

Example 2

Evaluate the following:

1. $\int_{-2}^{3}(1+x^3)\,dx$
2. $\int_{-1}^{0} x(1+x^2)^4\,dx$
3. $\int_{2}^{4} \frac{4x}{\sqrt{1+x^2}}\,dx$

Solution

1. $\int_{-2}^{3}(1 + x^3)\,dx = \int_{-2}^{3} 1\,dx + \int_{-2}^{3} x^3 dx$

$$= [x]_{-2}^{3} + \left[\frac{x^4}{4}\right]_{-2}^{3}$$

$$= (3+2) + \frac{(3)^4}{4} - \frac{(-2)^4}{4}$$

$$= 21\frac{1}{4}$$

2. Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{1}{2x}du$$

$$\frac{1}{2}\int_{-1}^{0} x\,(u^4)\frac{1}{x}du = \frac{1}{2}\int_{u=2}^{u=1} u^4\,du = 0, u = 1$$

$$= -\frac{1}{2}\int_{1}^{2} u^4\,du$$

$$= -\frac{1}{2}\left[\frac{u^5}{5}\right]_{1}^{2}$$

$$= -\frac{1}{2}\left(\frac{2^5}{5}\right) + \frac{1}{2}\left(\frac{1^5}{5}\right)$$

$$= \frac{-31}{10}$$

$$= -3\frac{1}{10}$$

Thus $\int_{-1}^{0} x(1 + x^2)^4\,dx = -3\frac{1}{10}$

3. Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{1}{2x}du$$

$$\int_{2}^{4} \frac{4x}{u^{\frac{1}{2}}} \cdot \frac{1}{2x}du = 2\int_{5}^{17} u^{-\frac{1}{2}}\,du \; ; = 4, u = 17$$

$$= 2\left[\frac{u^{\frac{1}{2}}}{0.5}\right]_{5}^{17}$$

$$= 2\left(\frac{17^{\frac{1}{2}}}{0.5}\right) - 2\left(\frac{5^{\frac{1}{2}}}{0.5}\right)$$

$$= 4\sqrt{17} - 4\sqrt{5}$$

$$= 7.548\ (to\ 4\ sf)$$

Indefinite integrals

The two parts of the Fundamental Theorem of Calculus establish connections between antiderivatives and definite integrals. The first part posits that if f is continuous, then $\int_a^x f(t)dt$ is an antiderivative of f while the second part shows that $\int_a^b f(x)dx$ can be determined by simplifying $F(b) - F(a)$ where F is an antiderivative of f. A convenient and widely accepted notation for antiderivatives is $\int f(x)\,dx$. An indefinite integral is an integral without limiting boundaries. In such instances, we need to introduce an arbitrary constant "C" as mentioned earlier in the section.

Example 3

Simplify $\int (5x^4 - 2\sec^2(x))\,dx$.

Solution

$$\int (5x^4 - 2\sec^2(x))\,dx = 5\int x^4\,dx - 2\int \sec^2(x)\,dx$$

$$= 5\left(\frac{x^5}{5}\right) - 2\tan(x) + C$$

$$= x^5 - 2\tan(x) + C$$

Example 4

Find $\int x^3 \sin(x^4 + 2)\,dx$.

Solution

Let $u = x^4 + 2$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{1}{4x^3}du$$

Now, $\int x^3 \sin(x^4 + 2)\,dx = \int x^3 \sin(u) \cdot \frac{1}{4x^3}du$

$$= \frac{1}{4}\int \sin(u)\,du$$

$$= -\frac{1}{4}\cos(x^4 + 2) + C$$

Example 5

A balloon is rising straight up into the air. Its velocity at time t seconds after release is given by $v(t) = 4t + 2$ metres per second. At the moment of release, the bottom of the balloon is 5 metres above the ground. Find the height of the balloon (measured from the bottom of the balloon to the ground) at time t.

Solution

At $t = 0$, the **bottom of the balloon** is **5 metres above** the ground.

Height, $h(t)$ of the balloon at any time (t) can be found by integrating the function of the velocity.

$$h(t) = \int (4t + 2)\ dt$$

$$= \frac{4t^2}{2} + 2t + C$$

$$= 2t^2 + 2t + C$$

When $t = 0$, $h(t) = 5$ thus,

$$5 = 2(0)^2 + 2(0) + C$$

$$C = 5$$

Thus, height of the balloon at time t is $2t^2 + 2t + 5$

Example 6

A rock is thrown upwards from a cliff. Its acceleration due to gravity is constant and given by $a(t) = -9.8\ m/s^2$ (The negative sign indicates that the acceleration due to gravity is acting downwards). At time $t = 0$, the rock has an initial velocity of $4\ m/s$ upward, and its height above the ground is 120 metres.

a. Find an expression for the rock's velocity as a function of time.

b. Find an expression for the rock's height as a function of time.

c. How high is the rock after 3 seconds?

Solution

a. Since acceleration is the derivative of velocity: $a(t) = \frac{dt}{dv} = -9.8$

Now, $v(t) = \int -9.8\ dt$

$= -9.8t + C$

From the initial condition $v(0) = 4$,

$$4 = -9.8(0) + C$$

$$C = 4$$

The rock's velocity as a function of time is $v(t) = -9.8t + 4$

b. To determine the height, integrate the velocity function of the rock.

$$h(t) = \int (-9.8t + 4)\ dt$$

$$h(t) = \frac{-9.8t^2}{2} + 4t + C$$

From the initial condition, when $t = 0$, $h(t) = 120m$.

$$120 = \frac{-9.8(0)^2}{2} + 4(0) + C$$

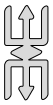
$$C = 120$$

Thus, the height function is $h(t) = -4.9t^2 + 4t + 120$.

c. To determine $h(t)$ when $t = 3$

$h(t) = -4.9(3)^2 + 4(3) + 120 = 87.9$

The rock's height is 87.9m when t = 3.



Note

Assist learners to:

1. *derive the integrals of trigonometric functions.*
2. *explore integrals of odd powers of* $\sin(x)$ *and* $\cos(x)$.
3. *explore integrals of even powers of* $\sin(x)$ *and* $\cos(x)$
4. *powers of* $\tan(x)$.

Learning Tasks

1. Provide a set of integrals and a set of rule cards. Ask learners to match each integral with the correct rule that will assist in solving the integral.
2. Guide learners to come up with the corresponding integrals of a given list of derivatives.
3. Provide learners with a list of integrals and assist them to match the integrals with the appropriate technique (algebraic manipulation, substitution) needed for solving.

Pedagogical Exemplars

1. Review previous learning
2. Problem-Based Learning – present learners with real-world problems modelled by complex integrals (areas, velocity from acceleration, etc.) to choose and justify the appropriate integration technique and then solve the problem. This helps build deeper understanding through the context and applications.
3. Inquiry-Based learning – provide learners with integrals without any indication of the method to use. Guide them to experiment with algebraic manipulation, substitution, etc., and then reflect on which strategy worked best and why.

Key Assessment: Practice Paper 1

1. *Draw the graph of the function* $y = \sqrt{4 - x^2}$.

   Deduce a relationship between the area under the curve and the x-axis and areas of sectors.

   (8 marks)

2. *A spacecraft is performing a complex orbital movement near a distant planet. The spacecraft's acceleration* $(in\ m/s^2)$ *is modelled by the function:* $a(t) = 5 - 0.4t^2,\ 0 \leq t < 10$.

At time $t = 0$, the spacecraft has an initial velocity of $-30\ m/s$ (negative indicates motion toward the planet) and is positioned 8000 metres from the planet's surface along its trajectory path.

a. *Derive expressions for the spacecraft's;*

i. *velocity $v(t)$.* ***(3 marks)***

ii. *position $p(t)$.* ***(3 marks)***

b. *At what time does the spacecraft reach its maximum velocity away from the planet?*

(2 marks)

c. *Engineers want to ensure the spacecraft maintains a velocity between $-50\ m/s$ and $50\ m/s$ throughout the entire movement.*

Analyse whether this constraint is satisfied, providing mathematical justification for your answer. ***(6 marks)***

(Total 14 marks)

HINT

*Conduct **Practice Paper 1**. It should cover the contents taught from Year 1 to 3. Refer to the **Table of Specification and Structure of the Paper** at the end of this section. Learners' papers should be marked immediately and scores entered into the SAP.*

PRACTICE PAPER 1

Tables of Specifications

Test Specification (MCQ)

STRAND	SUB-STRAND	DoK LEVELS			TOTAL
		1	2	3	
Modelling with Algebra	*1. Number and algebraic patterns* *2. Application of Algebra*	*5*	*6*	*5*	*16*
Geometric Reasoning and Measurement	*1. Spatial Sense* *2. Measurement of triangles*	*2*	*4*	*2*	*8*
Calculus	*1. Principles of calculus* *2. Application of calculus*	*2*	*4*	*2*	*8*
Handling Data	*1. Organising, representing and interpreting data* *2. Making predictions with data*	*3*	*2*	*3*	*8*
TOTAL		*12 (30.0%)*	*16 (40.0%)*	*12 (30.0%)*	*40 (100.0%)*

MCQ Test Specification (with indicators)

LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Describe and interpret the characteristics of commutative, associative, distributive and closure properties of binary operations.		1		
Use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion.		1		
Recognise the relationship between surds and indices and apply laws of indices to simplify expressions.	1			
Find the nth term of linear and exponential sequences.		1		
Identify Relations and Functions and describe their differences.	1			
Establish, describe and determine bijective functions and composite functions.	1			
Use the remainder and factor theorems to find the factors and remainders of a polynomial of degree not greater than 4.			1	
Establish De Morgan's laws of set theory and use it to solve related problems.	1			
Use the expansion for (1−x)^n or (1+x)^n to approximate exponential numbers.	1			
Generate the terms of a recurrence sequence and find an explicit formula for the sum of the sequence.			1	
Use the properties of logarithms and indices to solve equations involving logarithms, including changing the base.			1	
Determine the maximum/minimum values within the given constraints.		1		
Distinguish between 'singular' and 'non−singular' square matrices (2 × 2 and 3 × 3) and evaluate determinants.			1	
Form statements including negation statements and comment on them.		1		
Use linear transformation to determine image and object points.		1		
Apply linear transformation in finding translation, reflections, rotations and enlargement of points and plane figures.			1	16
Describe the properties of lines, including parallel, perpendicular and midpoints.	1			

Perform algebraic and graphical operations (addition, subtraction, scalar multiplication) and their geometrical interpretation.			1	
Apply knowledge of the distance between two points and the Pythagoras theorem to describe a circle in the algebraic form.		1		
Apply the knowledge of operations of vectors to solve simple geometric problems, including the position vector of a point that divides a vector internally and externally in a given ratio.	1	1		
Describe a given locus as a parabola.		1		
Determine the equations of tangent and normal to a parabola.			1	
Prove and apply compound angles to derive the identities for multiple angles and half angles.		1		8
Use the limits of a function to find its derivative.		1		
Identify the rules of differentiation.	1			
Distinguish between partitioning an interval for a given step size and the number of subintervals.		1		
Identify and apply the integration rules to evaluate integrals.		2		
Use knowledge of differentiation to determine the equation of tangents and normal to curves at a given point.	1			
Find the maximum and minimum values and points of a function, sketch the functions and solve some real life problems.			1	
Find the area under a constant function, curves and distinguish between total, net and signed area.			1	8
Organise data into appropriate frequency distribution tables manually, and with Microsoft Excel.		1		
Compare various statistical representations and justify why a particular representation is more suitable than others for a given situation.		1		
Perform analysis on a data set using appropriate techniques and tools and interpret findings.			1	
Distinguish between univariate and bivariate data and give examples and explain the concept of correlation.	1			
Describe the concept of regression and distinguish between correlation and regression, and describe situations that call for the use of regression.	2			
Describe the number of ways objects can be arranged using fundamental counting principles.			2	8

TOTAL	12 (30.0%)	16 (40.0%)	12 (30.0%)	40 (100.0%)

Paper 2

Section A

STRAND	SUB-STRAND	DoK LEVELS			TOTAL	
		1	2	3		
Modelling with Algebra	Number and Algebraic Patterns	1	1		2	3
	Application of Algebra		1		1	
Geometric Reasoning and Measurement, and Calculus	Spatial Reasoning		1		1	3
	Measurement of Angles	1			1	
	Principles of Calculus	1			1	
	Application of Calculus				0	
Handling Data	Organising, Representing and Interpreting Data	1			1	2
	Making Prediction with Data		1		1	
TOTAL		4 (50.0%)	4 (50.0%)	0	8 (100%)	8 (100%)

Test Specification (with indicators)

LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Explain binary operations and apply that knowledge in solving problems.	1			
Expand binomial expressions for positive integer indices using Pascal's triangle.		1		
Determine the arithmetic and geometric means of linear and exponential sequences and apply linear and exponential sequences to solve real life problems.		1		3
Determine the projection of one vector on a given vector.		1		
Recall basic trigonometric ratios and use the knowledge to solve problems relating to triangles.	1			
Apply the product and quotient rules to differentiate functions.	1			3
Work out simple measures of dispersion (range, quartile and inter-quartile, etc.) for raw data and interpret them in context.	1			

Simplify permutation and combination expressions and solve related problems.		*1*		*2*
TOTAL	*4 (50.0%)*	*4 (50.0%)*	*0*	*8 (100%)*

Section B

Test Specification

STRAND	SUB-STRAND	DOK LEVELS			TOTAL	
		1	2	3		
Modelling with Algebra	*Number and Algebra*		*1*		*1*	*3*
	Application of Algebra			*2*	*2*	
Geometric Reasoning and Measurement, and Calculus	*Spatial Reasoning*		*1*		*1*	*2*
	Measurement of Angles				*0*	
	Principles of Calculus				*0*	
	Application of Calculus			*1*	*1*	
Handling Data	*Organising, Representing and Interpreting Data*		*1*		*1*	*2*
	Making Prediction with Data			*1*	*1*	
TOTAL		*0*	*3 (42.9%)*	*4 (57.1%)*	*7 (100%)*	*7 (100%)*

Test Specification (With indicators)

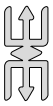
LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Establish the relationship between indices and logarithm and use the properties to solve related problems in one base.		*1*		
Solve real life problems involving linear inequalities.			*1*	
Use conjunction, disjunctions, implications and negations to construct truth tables of compound statements.			*1*	*3*
Apply knowledge of properties of lines to derive the equations of a circle, a tangent and normal to a circle.		*1*		
Find the area between two curves.			*1*	*2*
Construct a scatter plot using given data sets and use it to describe the relationship between two variables.		*1*		
Model and solve real life problems involving permutation and combination.			*1*	*2*

TOTAL	*0*	*3 (42.9%)*	*4 (57.1%)*	*7 (100%)*

PLC Session 12

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous session, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand.
   b. identify three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q).
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.

Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.

4. Use your app to plan and administer the mode of assessment for the Student Assessment Portal (SAP).
   a. The key assessment for the week is ***practice paper 1***. Chat with your app to plan this assessment by reviewing or developing the assessment items/tasks and marking schemes/rubrics for the practice paper (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or topic/idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through the problem individually for 3–4 minutes
   c. the group then works through it together the problem, step by step
   d. highlight the most common errors made at each step (NTS 1c, 1e, 2c, 2e, 3e-3j).

Note

- *This part is not 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 13

Learning Indicators

1. *Identify and apply appropriate techniques of integration to integrate transcendental functions*
2. *Solve problems involving integrals with no exact antiderivative (Trapezium rule)*

FOCAL AREA 1: IDENTIFYING AND APPLYING APPROPRIATE TECHNIQUES OF INTEGRATION TO INTEGRATE TRANSCENDENTAL FUNCTIONS.

Transcendental functions are functions that cannot be obtained in a finite number of operations as a solution of an algebraic equation. For instance, an algebraic function such as $y^2 - x = 0$ can be solved to obtain $y = \pm\sqrt{x}$. However, for a transcendental function such as $y = e^x$, no finite degree polynomial equation exists. Examples of transcendental functions include trigonometric, exponential, natural logarithms, etc. However, the focus of this week's discussion will be on integration of exponential functions as well as natural logarithms.

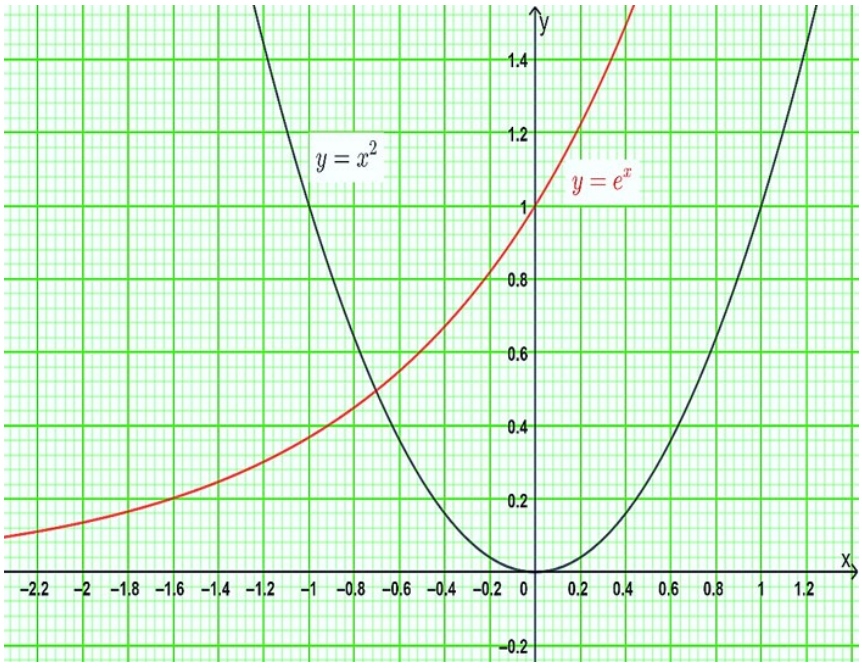


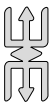
Figure 13.1: *Graphical illustration of* $y = x^2$ *and* $y = e^x$.

We can observe from Figure 13.1 that the polynomial function $y = x^2$ has its finite algebraic solution because it intersects with the x −axis at a finite point $(0, 0)$. However, the function $y = e^x$ is asymptotic to the x −axis and never touches or intersects it. It has no algebraic solution. This gives us a visual insight into the differences between polynomial and transcendental functions.

Integration of exponential functions

In integrating exponential functions;

1. $\int e^x dx = e^x + C$
2. $\int e^{kx} dx = \frac{1}{k} e^{kx} + C, k \neq 0$ and k is a constant.
3. $\int e^{h(x)}\, dx = \frac{1}{h'(x)} e^{h(x)} + C$ where $h'(x) \neq 0$.
4. $\int e^{-x}\, dx = -e^{-x} + C$
5. $\int \frac{1}{x} e^{-x}\, dx = Ei(-x)$ or $E_1(x)$



Note

this is a special function called exponential integral and cannot be integrated using analytical solutions.

Example 1

Evaluate $\int x\, e^{\frac{x^2}{2}} dx$.

Solution

Using the substitution method, let $u = \frac{x^2}{2}$,

$\frac{du}{dx} = x$

$dx = \frac{1}{x} du$.

Now, $\int x\, e^{\frac{x^2}{2}} dx = \int x\, e^u \frac{1}{x} du$

$= \int e^u du$

$= e^u + C$ but $u = \frac{x^2}{2}$ thus,

$\int x\, e^{\frac{x^2}{2}} dx = e^{\frac{x^2}{2}} + C$.

Example 2

Evaluate $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\, dx$

Solution

Using the substitution method, let $u = \frac{3}{x} = 3x^{-1}$.

$dx = -\frac{x^2}{3} du$

$$\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\,dx = \int_1^3 \frac{e^u}{x^2}\left(-\frac{x^2}{3}\right)du$$

$$-\frac{1}{3}\int_1^3 e^u du$$

When $x = 1, u = 3$ and when $x = 3, u = 1$.

Thus $-\frac{1}{3}\int_1^3 e^u du = \frac{1}{3}\int_1^3 e^u du$

Note

the operation became positive because there was a direct switch in the direction.

Now, $\frac{1}{3}\int_1^3 e^u du = 1$

$$= \frac{1}{3}e^3 - \frac{1}{3}e^1$$

$$= 5.7891$$

Thus $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\,dx = 5.7891$.

Integrating natural logarithmic functions

The integral of a natural logarithmic function in the form:

$\int \ln|x|dx = x\ln|x| - x + C$.

Also,

1. $\int \frac{1}{x}dx = \ln|x| + C$.
2. $\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C$

Example 3

A small social media platform is gaining users according to the equation: $\frac{dN}{dt} = 0.2N$. Where $N(t)$ is the number of users at time t (measured in months) and $\frac{dN}{dt}$ represents the rate of change of users per month.

a. If we rearrange this equation to separate variables, we get $\frac{dN}{N} = 0.2dt$. Integrate both sides of this equation using the formula $\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C$.

b. Using your result from part (a), find an expression for $N(t)$ in terms of the initial number of users N_0 at time $t = 0$.

c. If the platform had 500 users when it launched how many users will it have after 6 months?

d. After how many months will the platform reach $2\,000$ users?

e. The company predicts that once they reach $10\,000$ users, their growth will slow down and follow a new equation: $\frac{dN}{dt} = 0.1N(1 - \frac{N}{50\,000})dt$.

Explain in simple terms why this new model might be more realistic than the original equation for long-term growth.

Solution

a. $\int \frac{dN}{N} = \int 0.2dt$

$\ln|N(t)| + C_1 = 0.2t + C_2$ let $C = C_2 - C_1$

$\ln|N(t)| = 0.2t + C$

b. To find $N(t)$, exponentiate both sides of the equation:

$e^{\ln|N|} = e^{0.2t + C}$

$N = e^{0.2t} \cdot e^{C}$

At $t = 0$, we have $N = N_0$: $N_0 = e^{C}$

Therefore: $N(t) = N_0 e^{0.2t}$

c. Initial users $N_0 = 500$

Time $t = 6$ months

Substituting into our formula:

$N(6) = 500e^{0.2 \times 6}$

$= 500e^{1.2}$

$= 500 \times 3.32$

$\approx 1\,660$ users.

After 6 months, the platform will have approximately 1 660 users.

d. To find when $N(t) = 2000$,

$2000 = 500e^{0.2t}$

$\frac{2000}{500} = e^{0.2t}$

$4 = e^{0.2t}$

Taking the natural logarithm of both sides,

$\ln(4) = 0.2t$

$t = \frac{\ln(4)}{0.2}$

$= \frac{1.386}{0.2} \approx 6.93$ months

It will take approximately 7 months to reach 2000 users.

e. The new model $\frac{dN}{dt} = 0.1N\left(1 - \frac{N}{50000}\right)dt$ is more realistic because:

i. The original model represents unlimited exponential growth, which isn't possible in the real world.

ii. The new model includes a "limiting factor" that slows growth as the user base approaches 50 000 users.

iii. This matches real-world platforms where growth eventually slows due to a limited number of potential users, market saturation, competition from other platforms or technical constraints like server capacity.

Example 4

Find $\int \frac{1}{x(\ln x)^3}dx$.

Solution

Using the method of substitution, let $u = \ln x, \frac{du}{dx} = \frac{1}{x}, dx = x\,du$.

Now, $\int \frac{1}{x(\ln x)^3}dx = \int \left(\frac{1}{x}\right)\left(\frac{1}{u^3}\right)x du.$

$$= \int \frac{1}{u^3}du$$

$$= \int u^{-3}du$$

$$= \frac{u^{-2}}{-2} + C = -\frac{1}{2u^2} + C \text{ , but } u = \ln x.$$

$$= -\frac{1}{2(lnx)^2} + C.$$

Thus $\int \frac{1}{x(\ln x)^3}dx = -\frac{1}{2(lnx)^2} + C.$

Evaluation of exponential function

We shall use knowledge obtained on integrating exponential function to evaluate some exponential definite integrals.

Example 5

Compute $\int_1^4 \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right)dx.$

Solution

$$\int_1^4 \left(x^{-3} + 7e^{5x} + \frac{4}{x}\right)dx = \int_1^4 x^{-3}dx + \int_1^4 7e^{5x}dx + \int_1^4 \frac{4}{x}dx$$

$$= \left[\frac{x^{-2}}{-2} + \frac{7}{5}e^{5x} + 4\ln|x|\right]_1^4$$

$$= \left[\frac{(4)^{-2}}{-2} + \frac{7}{5}e^{5(4)} + 4\ln(4)\right] - \left[\frac{(1)^{-2}}{-2} + \frac{7}{5}e^{5(1)} + 4\ln(1)\right]$$

$$= 679\,231\,071.81$$

Example 6

Evaluate the integral $\int_0^5 (2e^x + 4\cos x)dx.$

Solution

$$\int_0^5 (2e^x + 4\cos x)dx = \int_0^5 2e^x dx + \int_0^5 4\cos x\, dx$$
$$= 2[e^x]_0^5 + 4[\sin x]_0^5$$
$$= 2(e^5 - e^0) + 4(\sin 5 - \sin 0)$$
$$= 295.17$$

Thus, $\int_0^5 (2e^x + 4\cos x)dx = 295.17$

Example 7

If $v'(t)$ is the rate of growth of a child in kilograms per year. What does $\int_5^{10} v'(t)dt$ represent?

Solution

The increase in the child's weight (in kilograms) between the ages of 5 and 10

FOCAL AREA 2: SOLVING PROBLEMS INVOLVING INTEGRALS WITH NO EXACT ANTIDERIVATIVE

Some functions cannot be integrated using analytical solutions because they do not have exact derivatives. Examples of such functions are the error function $\left(\int e^{-x^2} dx\right)$, sine integral $\left(\int \frac{\sin x}{x} dx\right)$, logarithmic integral $\left(\int \frac{1}{\ln x} dx\right)$, etc. In such situations, the use of numerical solutions become pertinent.

Numerical solutions of integrals

Here, approximate values of a definite integral are computed because the exact derivative cannot be easily determined. Such approximate values may be computed using rules such as the midpoint rule or trapezium rule. Here we focus on the use of the trapezium rule in providing numeric solutions to definite integrals.

Deriving the trapezium rule

Recall that we learnt how to determine sub-intervals in year 2 week 19. We will make use of that in deriving the trapezium rule.

The trapezium rule (also called trapezoidal rule) approximates the definite integral $\int_a^b f(x)dx$ by replacing the curve with straight line segments, creating trapezoids whose areas can be easily calculated.

Using geometry to derive the rule;

1. Divide $[a, b]$ into n equal subintervals such that the endpoints will be;
   $x_0 = a, x_1, x_2, x_3, \dots, x_{n-1}, x_n = b$
2. Determine the width of each sub-interval as $h = \frac{b-a}{n}$ such that $x_i = a + ih$ for $i = 0, 1, 2, 3, \dots, n$.
3. Now, create trapezoids for each sub-interval such that the left height is $f(x_{i-1})$, the right height is $f(x_i)$ and the width is h.

4. Calculate the area of each trapezoid formed either manually $\left(\frac{h}{2}(f(x_{i-1}+f(x_i)\right)$ or with the help of technological tools.
5. Find the sum of the areas of the trapezoid. Using intervals such that,
   $\int_a^b f(x)dx = \sum_{i=1}^{n} \frac{h}{2}(f(x_{i-1}) + f(x_i)$
   When simplified results in:
   Area $= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$.

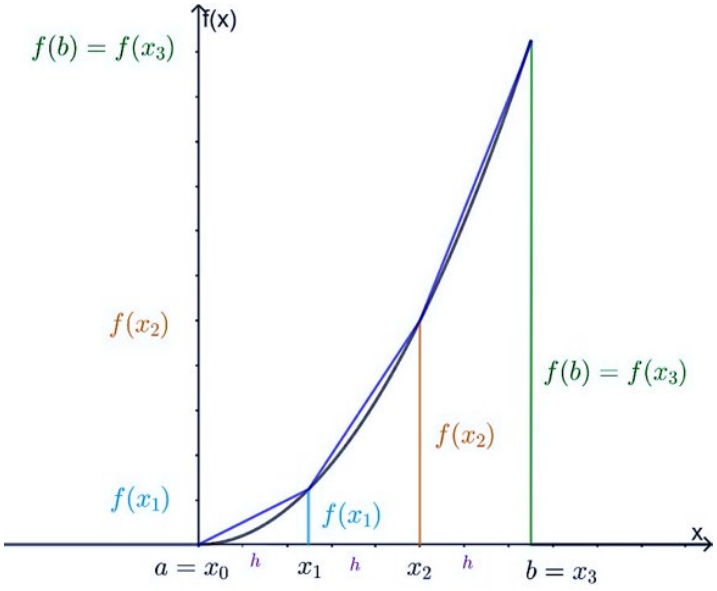


Figure 13.2: *Graphical illustration of creating trapeziums to determine area under the curve (overestimate)*

Figure 13.2 shows a graphical representation of creating trapeziums using three sub-intervals. An observation shows that the trapeziums created have some portions above the curve. This helps us realise that the area obtained from summation of the areas of the trapeziums will be an overestimate of the area under the curve.

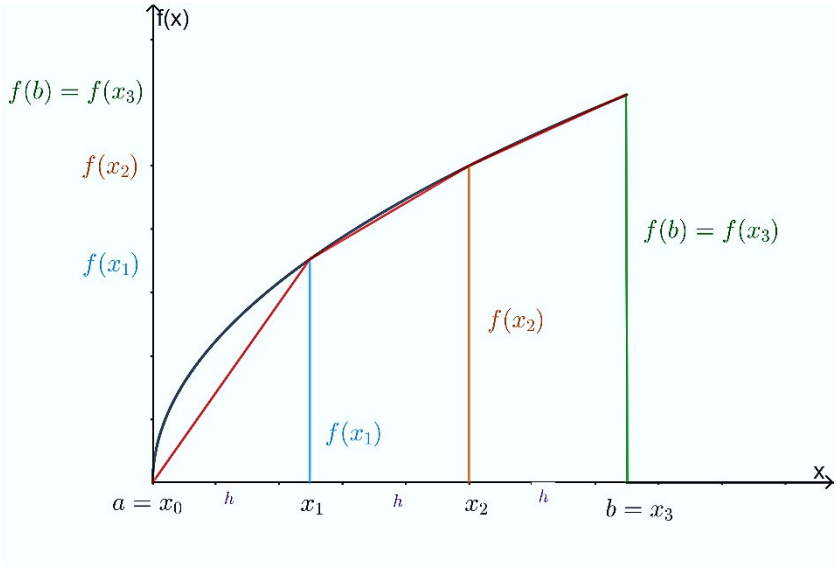


Figure 13.3: *Graphical illustration of creating trapeziums to determine area under the curve (underestimate).*

Figure 13.3 shows a graphical representation of creating trapezoids using three sub-intervals. An observation shows that some portions below the curve are not covered by the trapeziums. This reveals that the area obtained using a summation of the area of the trapezoids will be an underestimate of the area under the curve.

Observation: The use of the trapezoidal rule is an approximation of the actual area under the curve and can be an underestimation or overestimation. This reveals that there is a margin of error when using the trapezoidal rule in finding the area under a curve.

Note

Guide learners to:

- *Find the calculus-based derivation of the trapezium rule.*
- *Research on Liouville's theorem.*

Using the trapezium rule

Now, let us delve into details on how the trapezium rule is used in determining the area under a given curve. $\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$

Where h = number of sub-intervals (step size).

Note

The greater the number of sub-intervals used, the more accurate the area estimation.

Example 1

Use the trapezium rule with 5 partitions to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

Solution

$h = \frac{2-1}{5} = 0.2$ and $f(x) = \frac{1}{x}$

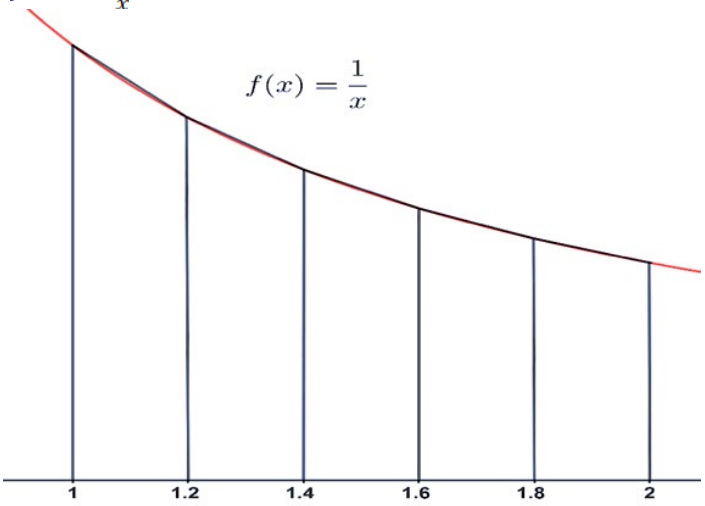


Figure 13.4: *Pictorial representation of* $f(x) = \frac{1}{x}$ *in from* $x = 1$ *to* $x = 2$ *in 5 partitions.*

$$\int_1^2 \frac{1}{x}\,dx = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$\int_1^2 \frac{1}{x}\,dx = \frac{0.2}{2}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$\int_1^2 \frac{1}{x}\,dx = 0.6956$ to four decimal places.

From the graph it can be seen that this is an overestimate of the actual area.

Example 2

Estimate $\int_0^1 \cos x^2\,dx$ using the trapezoidal rule with 4 intervals. From the graph of the integrand, determine whether the answer obtained are underestimates or overestimates.

Solution

$h = \frac{1-0}{4} = 0.25$ and $f(x) = \cos x^2$.

$$\int_0^1 \cos x^2\,dx = \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.50) + 2f(0.75) + f(1)]$$

$$\int_0^1 \cos x^2\,dx = 0.125[\cos(0)^2 + 2\cos(0.25)^2 + 2\cos(0.50)^2 + 2\cos(0.75)^2 + \cos(1)^2]$$

$\int_0^1 \cos x^2\,dx = 0.8750$ to four decimal places.

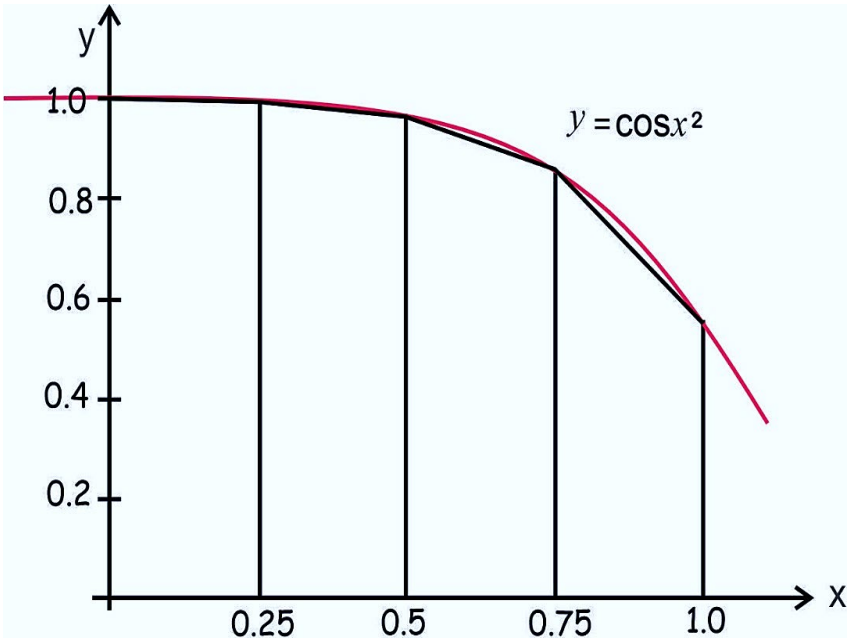


Figure 13.5: *Graphical representation of* $y = \cos x^2$

From Figure 13.5 it can be seen that the area calculated is an underestimate of the actual.

Learning Tasks

1. Assist learners to research on transcendental functions.
2. Guide learners in groups to distinguish among techniques of integration of exponential and natural logarithm functions.

3. Assist learners in small groups to solve problems on techniques of integration and present to the class.
4. Guide learners to draw a curve $y = f(x)$, divide the interval into subintervals and draw the trapezoids.
5. Assist learners to compare the total area of trapezoids with the actual area under the curve.

Pedagogical Exemplars

1. Collaboration – Place learners in appropriate groups to undertake discussions on solving transcendental functions and present to the class.
2. Demonstrations - Use technological software (e.g., GeoGebra, Desmos) to draw curves, divide into given sub-intervals, create trapeziums based on the divisions, measure areas of the trapeziums and compare with the area under the curve.
3. Experiential learning -Assist learners to perform their own demonstrations using technological tools to deepen their understanding. Where technological tools are unavailable, hand drawn graphs may be used to solidify learners' understanding.
4. Talk for learning – encourage learners to share their observations from the demonstrations.

Key Assessment Level 4: Essay

1. A population of bacteria grows according to the differential equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ where P is the population at time t, k is the growth rate, and M is the maximum sustainable population.

   a. Show that this equation can be rewritten as: $\int \frac{dP}{P\left(1-\frac{P}{M}\right)} dt = \int k\, dt$

   (2 marks)

   b. Using an appropriate substitution technique, solve this integral to find P as a function of t, assuming $P(0) = P_0$.

   (9 marks)

   c. Analyse and explain how the solution behaves as t increases for different initial values of P_0. Consider cases where $P_0 < M$ and $P_0 > M$.

   (5 marks)

   Total = 16 marks

2. The temperature at position x along a heat-conducting rod is governed by:

   $\frac{dT}{dx} = \frac{a}{x} e^{-\frac{b}{x}}\left(1 - \frac{b}{x}\right)$ where a and b are positive constants.

   Find an expression for temperature $T(x)$ by integrating this equation, using an appropriate substitution.

   (5 marks)

3. Consider the region bounded by the curve $y = \ln(x) \cdot e^{-x}$, the x–axis, and the lines $x = 1$ and $x = 4$.

   a. Explain why this integral has no elementary antiderivative.

   (3 marks)

b. Use the Trapezium rule with 3 subintervals to approximate the area of this region.

(6 marks)

c. Use the Trapezium rule again with 6 subintervals to approximate the same area.

(7 marks)

d. Compare your results from parts (b) and (c). Based on the pattern of change, make a prediction about what would happen if you continued increasing the number of subintervals.

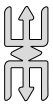
(2 marks)

Total (18 marks)

PLC Session 13

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

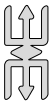
- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***essay***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the presentations (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step

d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 5 Review

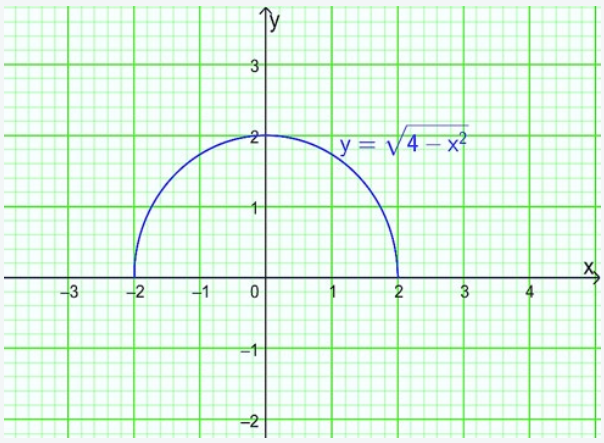
The section focused on lessons to be delivered in weeks twelve and thirteen of the year three Teacher Manual. It sought to provide guidance to teachers in assisting learners to identify and apply the integration rules to evaluate integrals, identify and apply appropriate techniques for integrating a function and to integrate transcendental functions and solve problems involving integrals with no exact antiderivative using the trapezium rule. Learners were to be assessed using homework and essay-type questions. Learning from year two was to be reinforced on antiderivatives. It was intended that learners got the opportunity to revise lessons taught in the previous years as they worked through the learning and assessment tasks. Teachers were encouraged to be accommodative of the learning needs of learners.

MARKING SCHEMES FOR THE KEY ASSESSMENTS

WEEK 12

1. ***B1*** *for correct labelling of axes and use of scale*

   B2 *for plotting any four points correctly (-* $\frac{1}{2}$ ***each error****)*



Learner recognises the equation is that of a semi-circle of radius 2 thus;

Area $= \frac{1}{2}\pi(2)^2 = 2\pi.$ ***B1***

$\int_{-2}^{2} \sqrt{4-x^2}\ dx = 2\pi$ ***B1***

Area of sector $= \frac{\frac{\pi}{2}}{2\pi} \times \pi(2)^2$ ***M1***

$= \pi$ ***A1***

Relationship: *The area between the curve and the x-axis is same as twice the area of a sector of radius 2 and angle* $\frac{\pi}{2}$ *because the function represents a semi-circle.* ***B1***

2. *a.*

   i. *To find velocity, integrate the acceleration function;*

   $v(t) = \int 5 - 0.4t^2\ dt = 5t - \frac{0.4t^3}{3} + C$ ***M1***

   Given that at $t = 0, v = -30,$

   $-30 = 5(0) - \frac{0.4(0)^3}{3} + C$ ***M1***

   $C = -30$

   Thus, the velocity function $v(t) = 5t - \frac{0.4t^3}{3} - 30$ ***A1***

   ii. *To find the position function, integrate the velocity function.*

   $$p(t) = \int \left(5t - \frac{0.4t^3}{3} - 30\right) dt$$

$p(t) = \frac{5t^2}{2} - \frac{0.4t^4}{12} - 30t + C$ **M1**

From the initial conditions, when $t = 0,\ p(t) = 8000$

$8000 = \frac{5(0)^2}{2} - \frac{0.4(0)^4}{12} - 30(0) + C$ **M1**

$C = 8000$

Thus, the position function $p(t) = \frac{5t^2}{2} - \frac{0.4t^4}{12} - 30t + 8000$ **A1**

b. *Maximum velocity occurs when acceleration becomes zero thus;*

$5 - 0.4t^2 = 0$ **M1**

$t = 3.54$ *seconds to 2 decimal places*

The spacecraft reaches its maximum velocity away from the planet at 3.54 seconds. **A1**

c. *Simplify the function for clarity:*

$v(t) = 5t - \frac{0.4t^3}{3} - 30 = 5t - \frac{2}{15}t^3 - 30$ *Find the minimum and maximum values of* $v(t)$ *on* $[0, 10)$ *and compare them with the constraint:* $-50 \leq v(t) \leq 50$.

Take derivative of $v(t)$ *to find turning points, i.e., when acceleration = 0.*

From part (b)we know this is $t = \sqrt{\frac{25}{2}} = 3.54$ **B1**

Evaluate $v(t)$ *at endpoints and turning point:*

At turning point $t \approx 3.54$,

$v(3.54) = 5(3.54) - \frac{2}{15}(3.54)^3 - 30$ ***M1(for correct substitution)***

So:

$v(3.54) = 17.7 - 5.91 - 30$

$= -18.21$ **A1**

At $t = 10$

$v(10) = 5(10) - \frac{2}{15}(10)^3 - 30$ ***M1(for correct substitution)***

$= 50 - \frac{2}{15}(1000) - 30 = -113.33$ **A1**

Minimum velocity: $v(10) \approx -113.33\ m/s$

Maximum velocity: occurs around $t \approx 3.54$

The velocity constraint $-50 \leq v(t) \leq 50$ *is* ***not*** *satisfied, because: At* $t = 10$, *velocity reaches below* $-50\ m/s$ *(approximately* $-113.33\ m/s$*)* **B1**

WEEK 13

1. *a.* $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

$\frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = k\,dt$

$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = \int k\,dt$ **M1**

$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = \int k\,dt$ **A1**

a.

Now, $\frac{1}{P\left(1-\frac{P}{M}\right)} = \frac{1}{\frac{P(M-P)}{M}} = \frac{M}{P(M-P)}$ **M1**

Using partial fractions $\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$

$M = A(M-P) + BP$ **M1**

If $P = 0$, $A = 1$

If $P = M$, $B = 1$

$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$

$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = \int \frac{1}{P} + \frac{1}{M-P}\,dP = \int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP$ **M1**

$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = ln|P| - ln|M-P| + C_1$ **A1**

$= ln\left|\frac{P}{M-P}\right| + C_1$ **M1**

Using $\int k\,dt = kt + C_2$

Now, $ln\left|\frac{P}{M-P}\right| + C_1 = kt + C_2$. *Let* $C_2 - C_1 = C$.

$ln\left|\frac{P}{M-P}\right| = kt + C$ **A1**

$\frac{P}{M-P} = e^{kt+C}$

From indices, $\frac{P}{M-P} = e^{kt} \cdot e^{C}$ **M1**

Let e^{C} *be a constant*

R

$\frac{P}{M-P} = Re^{kt}$

$P = Re^{kt}(M-P)$

$P(t) = \frac{RMe^{kt}}{1+Re^{kt}}$ **M1**

Using the condition $P(0) = P_0$, $P_0 = \frac{RM}{1+R}$.

$$P(t) = \frac{MP_0e^{kt}}{M-P_0+P_0e^{kt}}$$ **A1**

b. *From* $P(t) = \frac{MP_0e^{kt}}{M-P_0+P_0e^{kt}}$ *as* $t \to \infty, e^{kt} \to \infty$ *with* P_0e^{kt} *as the dominant term.* **B1**

Thus $P(t) = \frac{MP_0e^{kt}}{P_0e^{kt}} = M.$ **B1**

If $P_0 < MP_0$ *: The population grows toward the carrying capacity M, following an S-shaped (sigmoid) logistic curve.*
B1

If $P_0 > MP_0$*: The model still gives the same form, but the math implies a decrease toward M. This reflects a population that exceeds the environmental limit and then decreases to stability.* **B1**

Hence in all cases, $P(t) \to M$ *as* $t \to \infty$. **B1**

2. $\frac{dT}{dx} = \frac{a}{x}e^{-\frac{b}{x}}\left(1 - \frac{b}{x}\right)$

$T(x) = \int \frac{a}{x}e^{-\frac{b}{x}}\left(1 - \frac{b}{x}\right)dx$ **B1**

Let $u = \frac{b}{x}$

$dx = -\frac{b}{u^2}du$

$\int \frac{a}{x}e^{-\frac{b}{x}}\left(1 - \frac{b}{x}\right)dx = \int \frac{au}{b}e^{-u}(1 - u)\cdot\left(-\frac{b}{u^2}\right)du$ **M1**

$= \int -a\,e^{-u}\left(\frac{1-u}{u}\right)du$

$= -a\int e^{-u}\left(\frac{1-u}{u}\right)du$ **M1**

$= -a\left[\int \frac{1}{u}e^{-u}du - \int e^{-u}du\right]$

$= -a[Ei(-u) + e^{-u}] + C$ **A1**

Substitute $u = \frac{b}{x}$

$\int \frac{a}{x}e^{-\frac{b}{x}}\left(1 - \frac{b}{x}\right)dx = -a\left[Ei\left(-\frac{b}{x}\right) + e^{-\frac{b}{x}}\right] + C$

Therefore $T(x) = -a\left[Ei\left(-\frac{b}{x}\right) + e^{-\frac{b}{x}}\right] + C.$ **A1**

3. *a. The function is a* ***product of a logarithmic function and an exponential decay function****.* **B1**
Although both $ln(x)$ *and* e^{-x} *are elementary functions, their* ***product*** *does not have an elementary antiderivative.* **B1**

This is a known result from the theory of ***non-elementary integrals****, often proven using* ***Liouville's theorem****, which states that some combinations of elementary functions cannot be integrated in terms of elementary functions.* ***B1***

b. *Using intervals* $[1, 4]$ *and 3 sub-intervals;*

$h = \frac{4-1}{3}$ ***M1***

$h = 1.$ ***A1***

$f(x) = ln(x) \cdot e^{-x}$

Using the trapezium rule,

$Area = \frac{1}{2}[(ln(1) \cdot e^{-1}) + 2(ln(2) \cdot e^{-2}) + 2(ln(3) \cdot e^{-3}) + (ln(4) \cdot e^{-4})]$

M2 (-0.5 each error)

$= \frac{1}{2}(0.3224)$

$= 0.1612$ ***M1***

Area of the region is 0.1612. ***A1***

c. *Using intervals* $[1, 4]$ *and 6 sub-intervals;*

$h = \frac{4-1}{6}$ ***M1***

$h = \frac{1}{2}$ ***A1***

$f(x) = ln(x) \cdot e^{-x}$

Using the trapezium rule,

$Area = \frac{1}{4}[(ln(1) \cdot e^{-1}) + 2(ln(1.5) \cdot e^{-1.5}) + 2(ln(2) \cdot e^{-2}) + 2(ln(2.5) \cdot e^{-2.5}) + 2(ln(3) \cdot e^{-3}) + 2(ln(3.5) \cdot e^{-3.5}) + (ln(4) \cdot e^{-4})]$ ***M3 (-0.5 each error)***

$= \frac{1}{4}(0.7294)$

$= 0.1824$ ***M1***

Area of the region is 0.1824. ***A1***

d. *Comparison: As the number of sub-intervals increases, the area approximation increases.* ***B1***

Prediction: If the number of sub-intervals keep increasing, the estimated area will converge to the true area. The margin of error that comes with area approximations using the trapezium rule will decrease. ***B1***

SECTION 6: APPLICATIONS OF INTEGRATION

STRAND: CALCULUS

Sub-Strand: Application of Calculus

Learning Outcome: *Determine distance, area under curve and solid of volumes formed under revolution*

Content Standard: *Demonstrate conceptual understanding of integration to find distances, area under curve, volumes and other related real-life problems*

INTRODUCTION AND SECTION SUMMARY

In Year Two, learners were introduced to basic techniques of integration and learnt how to estimate areas under curves using rectangles and trapezia. Building on this foundation, this section in Year Three extends their understanding to the practical applications of integration in solving real-world and mathematical problems. Learners will apply definite integrals to calculate areas under constant functions and curves more accurately, moving beyond approximation methods. They will also learn to distinguish between total area, net area, and signed area, and explore how transformations affect these measures.

Additionally, learners will study how to determine the area between two curves by integrating the difference of the functions over a given interval. These ideas will be applied to practical problems involving distances, displacement, and rates of change, helping learners interpret integration in meaningful contexts such as motion and growth.

Finally, learners will explore how integration is used to calculate the volume of a solid formed by rotating a region about a horizontal or vertical axis. The topic will be taught using visual representations, digital tools such as GeoGebra, and real-life examples, ensuring learners develop strong conceptual understanding and analytical skills needed for advanced problem-solving.

The weeks and the indicators covered by the section are:

- *Week 14*

1. *Find the area under a constant function, curves and distinguish between totals, net and signed area*
2. *Find the area between two curves*

- *Week 15*

1. *Apply the techniques of integration to solve real-life problems involving distances and rates of change*
2. *Find the volume of a solid formed after rotation about horizontal or vertical axis*

Recap weeks

Year 1	Week 8 1. Recognise and construct linear and parabolic functions by hand and with the aid of technology, where appropriate show the intercepts and investigate the turning points. 2. Recognise and use appropriate algebraic notions, properties of linear and non-linear functions, linear and non-linear equations and solve linear and non-linear simultaneous systems. 3. Recognise linear equations in two variables, draw its graph by hand and by using the appropriate technology (e.g., GeoGebra, Desmos, Phet Simulations, Geometers Sketch Pad), find the solution using the graph and determine the area enclosed by the graphs
Year 2	Week 20 1. Identify and write a definite integral notation and its connection to the limits of the partial sum of areas. 2. Identify integration as a reverse process of differentiation.

SUMMARY OF PEDAGOGICAL EXEMPLARS

The effective learning of applications of integration in the final-year of senior high school requires diverse and inclusive teaching methods. These methods should foster a deep understanding of the concepts while preparing students for examinations.

One effective approach is "talk-for-learning," which encourages students to explain the steps of integration, compare different solutions, and discuss real-life contexts, such as motion or the accumulation of quantities. Another approach is experiential learning, where students engage in hands-on activities like estimating areas using cut-out shapes or constructing physical models (e.g., cylinders or cones) to visualise the volumes formed by rotation.

Collaborative problem-based learning also plays a vital role, as it promotes peer interaction. In groups, students work together to tackle integration problems, helping each other apply formulas and interpret results. Inquiry-based learning allows students to explore how definite integrals relate to physical quantities, discover rules through guided questioning, and investigate how rotating curves create three-dimensional shapes.

To accommodate varying abilities, teachers can provide simplified tasks, visual aids, and oral instructions for those needing additional support while offering abstract or more challenging problems to others. In resource-limited settings, alternatives like chalkboard illustrations, string curves, hand-drawn graphs, and measuring cylinders can effectively replace digital tools.

These strategies ensure active engagement, promote conceptual clarity and build confidence, helping all students prepare well for their final assessments.

ASSESSMENT SUMMARY

The recommended assessment modes and strategies for this section aim to promote both conceptual understanding and practical application. Classwork will serve as a continuous formative assessment tool, helping track individual progress through routine problem-solving tasks. These tasks can be tiered to match learners' abilities, providing simpler problems for foundational understanding and more complex, real-life application problems for advanced learners.

Practical assessments, such as simulations, displays, or exhibitions, offer opportunities for learners to demonstrate their understanding in hands-on and creative ways. For instance, learners can use cardboard or string to model solids of revolution, or draw large-scale graphs to approximate areas under curves. These methods support kinaesthetic and visual learners and encourage collaboration among mixed-ability groups.

In resource-limited settings, teachers can encourage students to reuse materials, such as graph sheets from old notebooks or bottle caps as markers, and use chalkboard demonstrations for simulations. Group presentations during exhibitions allow learners with varying strengths; whether verbal, artistic, or analytical, to contribute meaningfully. This combination of classwork and practical assessment ensures an inclusive, engaging and adaptable evaluation of learners' mastery of integration applications.

Indicators	DOK Level	Assessment Strategy and/or Mode
Week 14		
Find the area under a constant function, curves and distinguish between totals, net and signed area.	4	Mode: Classwork Strategy: Problem-based assessment
Find the area between two curves.	4	
Week 15		
Apply the techniques of integration to solve real-life problems involving distances and rates of change.	3	Mode: Practical assessment Strategy: Simulation, displays and exhibition
Find the volume of a solid formed after rotation about horizontal or vertical axis.	4	

WEEK 14

Learning Indicators

1. *Find the area under a constant function, curves and distinguish between totals, net and signed area*
2. *Find the area between two curves*

FOCAL AREA 1: FINDING THE AREA UNDER A CONSTANT FUNCTION, CURVES AND DISTINGUISHING BETWEEN TOTALS, NET AND SIGNED AREA

In finding areas under functions, different area measurements can be computed. We shall focus on total and net area in this week's lesson.

Concept of "total" and "net"

Total area refers to the sum of the absolute values of all areas between the function and the x −axis. It is always positive or zero and represents the actual physical area irrespective of the position relative to the x −axis. It is denoted by $\int |f(x)|dx$.

To find the total area;

Given an interval $[a, b]$, total area $= \int_a^b |f(x)|dx$ or

1. Divide the interval into subintervals where $f(x)$ is entirely above or below the x −axis.
2. Add the absolute values of all areas.

Net area is the algebraic sum of areas between the function and the x −axis. It can be positive, negative or zero. It represents the balance between positive and negative contributions.

To find the total area;

Given an interval $[a, b]$, total area $= \int_a^b f(x)dx$.

Areas above the x −axis contribute positively while areas below the x −axis contribute negatively.

Signed area is equivalent to net area but emphasises that the result carries a sign indicating position relative to the x −axis.

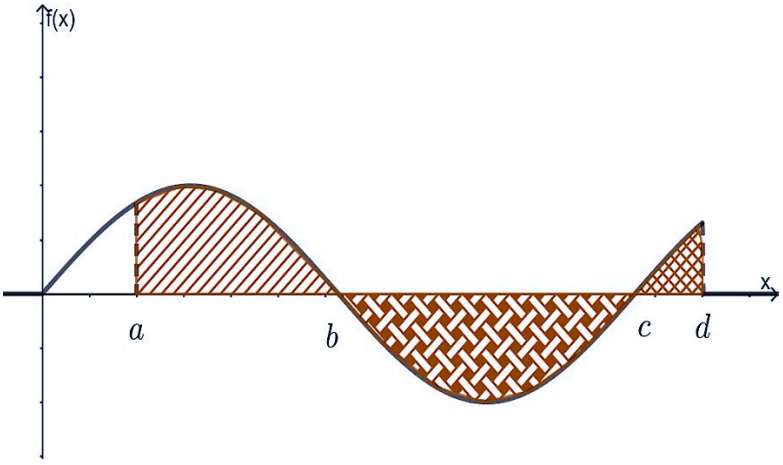


Figure 14.1: *Graphical illustration of the function* $f(x) = sin(x)$

Let us see how we can make use of the various area measurements discussed earlier.

Suppose we want to determine the area covered by the region in Figure 14.1 using the interval $[a, d]$, we need to compute the respective areas of the differently shaded regions and realise the differences.

1. Total Area

   $\int_a^d |f(x)|\,dx = \int_a^b |f(x)|\,dx + \int_b^c |f(x)|dx + \int_c^d |f(x)|\,dx$

   Recall total area is always positive.
2. Net Area $= \int_a^d f(x)\,dx$

Note

The shaded region below the x - axis will have a negative area and affect the final result.

Prompt: If $f(x)$ is a trigonometric function, analyse how $f(x)$ behaves in specific quadrants and apply the signs needed to make them positive when calculating the total area.

Area under constant function

A constant function is any function that has the same value as its input regardless of the input in its domain. It is usually denoted by $f(x) = k$ where k is a constant and x is the independent variable.

Note

If $f(x) = k$ where k is a constant,

1. $\frac{d}{dx}(k) = 0$.
2. $\int k\,dx = kx + C$.

To determine the area under a constant function $f(x) = k$ from $x = a$ to $x = b$, the area will

$$\int_a^b f(x)\,dx = \int_a^b k\,dx$$
$$= k\int_a^b 1\,dx$$
$$= k(b - a)$$

Note

- *If $k > 0$ the area is above the $x - axis$*
- *If $k < 0$ the area is below the $x - axis$*

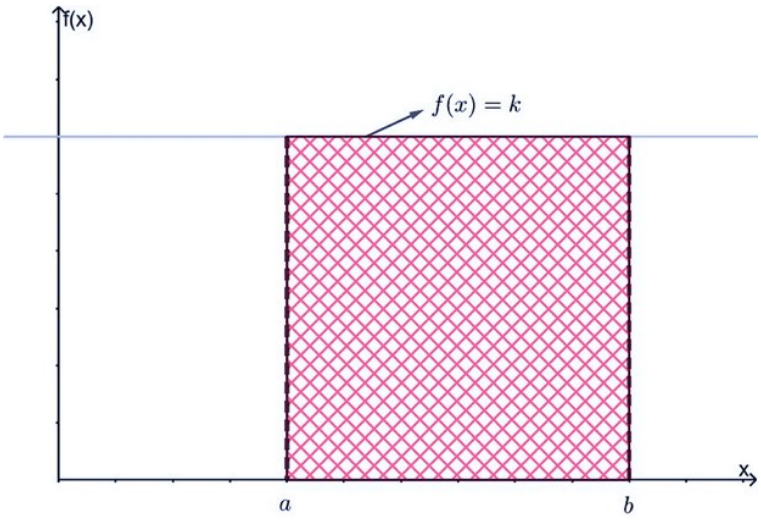


Figure 14.3: *Graphical illustration of area covered by* $f(x) = k, k > 0$.

Figure 14.3 shows the area covered by $f(x) = k, k > 0$ with intervals $[a, b]$. It can be observed that the area lies above the x −axis.

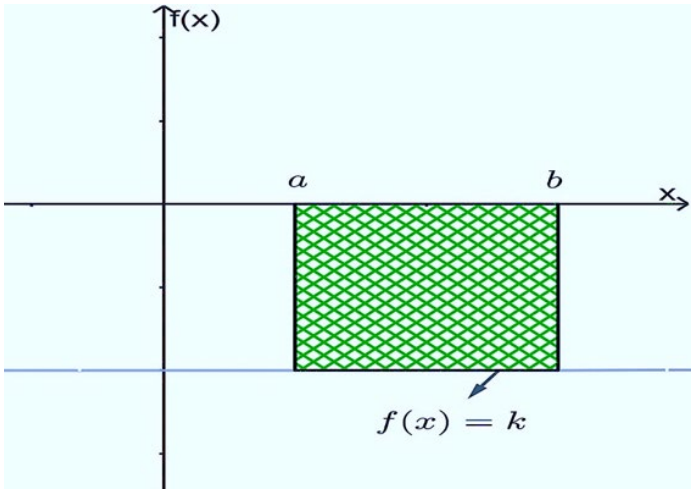


Figure 14.4: *Graphical illustration of area covered by* $f(x) = k, k < 0$.

Figure 14.4 shows the area covered by $f(x) = k, k < 0$ with intervals $[a, b]$. It can be observed that the area lies below the x −axis.

Example 1

Determine the area covered by $f(x) = 5$ over the interval $[2, 8]$.

Solution

Area covered $= \int_2^8 5\,dx$.

$$\int_2^8 5\,dx = [5x]_2^8$$

$$= 5(8) - 5(2)$$

$$= 30 \text{ units squared}$$

Area under curves

In section 5, we discussed how to determine the area covered by functions with no exact antiderivatives using the trapezium rule. Here, we shall look at how definite integrals are used to determine the area under a curve.

To determine the area under a curve over a given interval $[a, b]$, find the definite integral of the function representing the curve and evaluate. It is important to note that the concept of 'total' and 'net' area will be applied in this.

Example 1

A function $f(x) = x^2 - 4$ stretches on the interval $[-3, 3]$.

Draw the graph of $f(x)$ and calculate the:

i. Total area

ii. Net area

Solution

From the graph in Figure 14.5, the areas under the curve are between the intervals $[-3, -2]$, $[-2, 2]$ and $[2, 3]$.

The portions covered by intervals $[-3, -2]$ and $[2, 3]$ are above the x −axis while the portion covered by the interval $[-2, 2]$ is below the x −axis.

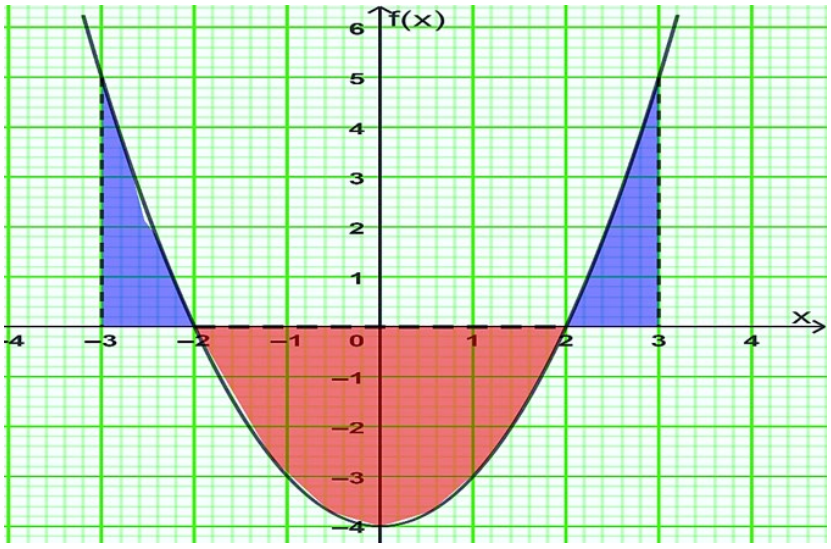


Figure 14.5: *Graphical illustration of area covered by* $f(x) = x^2 - 4, [-3, 3]$

i. Total Area $= \int_{-3}^{3} |f(x)|\, dx$

$$\int_{-3}^{3} |f(x)|\, dx = \int_{-3}^{-2} |f(x)|\, dx + \int_{-2}^{2} |f(x)| dx + \int_{2}^{3} |f(x)|\, dx$$

$$\int_{-3}^{3} |x^2 - 4|\, dx = \int_{-3}^{-2} |x^2 - 4|\, dx + \int_{-2}^{2} |x^2 - 4|\, dx + \int_{2}^{3} |x^2 - 4|\, dx$$

$$= \left[\left| \frac{x^3}{3} - 4x \right|\right]_{-3}^{-2} + \left[\left| \frac{x^3}{3} - 4x \right|\right]_{-2}^{2} + \left[\left| \frac{x^3}{3} - 4x \right|\right]_{2}^{3}$$

$$= \left(\left|\left(\frac{(-2)^3}{3} - 4(-2)\right) - \left(\frac{(-3)^3}{3} - 4(-3)\right)\right|\right)$$
$$+ \left(\left|\left(\frac{(2)^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)\right|\right)$$
$$+ \left(\left|\left(\frac{(3)^3}{3} - 4(3)\right) - \left(\frac{(2)^3}{3} - 4(2)\right)\right|\right)$$

$$= \left(\frac{16}{3} - 3\right) + \left(\left|-\frac{16}{3} - \frac{16}{3}\right|\right) + \left(-3 + \frac{16}{3}\right)$$

$$= \frac{7}{3} + \frac{32}{3} + \frac{7}{3}$$

$$= \frac{46}{3} = 15.\dot{3}$$

Therefore, the total area is $15.\dot{3}$ units squared.

ii. Net Area $= \int_{-3}^{3} f(x)\,dx$

$$\int_{-3}^{3} f(x)\,dx = \int_{-3}^{3} (x^2 - 4)\,dx$$

$$\int_{-3}^{3} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-3}^{3}$$

$$\int_{-3}^{3} (x^2 - 4)\,dx = \left(\frac{(3)^3}{3} - 4(3)\right) - \left(\frac{(-3)^3}{3} - 4(-3)\right)$$

$$= -6 \text{ units squared}$$

The net area is -6 units squared.

Note

If the graph of the function is not drawn, factorise the expression to know if it intersects with the x −axis and at which point it does. A quick sketch of the graph is always helpful once you have done this too.

Example 2

The Environmental Protection Agency is analysing rainfall data from a severe storm that occurred last month. The rainfall rate (in millimetres per hour) was recorded over a 15-hour period and can be modelled by the function: $R(t) = 5 + 15\sin\left(\frac{\pi t}{12}\right) + 8e^{-\frac{(t-15)^2}{50}}$, where t represents time in hours after midnight $(0 \leq t \leq 15)$. As a hydrologist tasked to design appropriate drainage systems for a project in the area, analyse the rainfall event by:

a. Calculating the total rainfall accumulation during this 15-hour period. Use 6 intervals where a numerical solution is needed.

b. Determining the 2-hour period the rainfall was most intense. Justify your answer using the concept of average value over an interval.

c. Determining if a local reservoir can safely process 250mm of rainfall in a period of 18 hours. Based on your calculations, would the current drainage system be adequate?

d. identifying periods of "flash flood risk" i.e., any period where the rainfall rate exceeds 19 mm/ hour. Determine these time intervals.

Solution

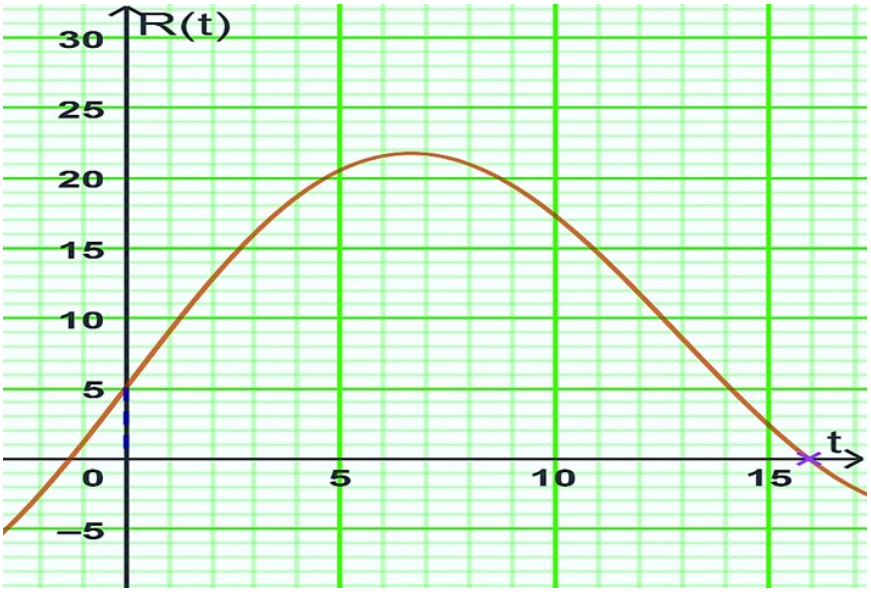


Figure 14.6: *Graphical illustration of* $R(t) = 5 + 15\sin\left(\frac{\pi t}{12}\right) + 8e^{-\frac{(t-15)^2}{50}}, [0, 15]$.

a. For time interval given, only one portion of area can be obtained under the curve and it is above the x −axis. Hence the total area will be $\int_0^{15} |R(t)|\, dt$.

$$\int_0^{15} |R(t)|\, dt = \int_0^{15} \left| 5 + 15\sin\left(\frac{\pi t}{12}\right) + 8e^{-\frac{(t-15)^2}{50}} \right| dt$$

$$= \int_0^{15} |5|\, dt + \int_0^{15} \left| 15\sin\left(\frac{\pi t}{12}\right) \right| dt + \int_0^{15} \left| 8e^{-\frac{(t-15)^2}{50}} \right| dt$$

Solving them separately,

$$\int_0^{15} |5|\, dt = [5t]_0^{15} = 75.$$

$$\int_0^{15} \left| 15\sin\left(\frac{\pi t}{12}\right) \right| dt = 15\int_0^{15} |\sin u\,| \frac{12}{\pi}\, du, u = \frac{\pi t}{12}.$$

When $t = 0, u = 0$ and when $t = 15, u = \frac{3}{2}\pi$

Now, $\frac{180}{\pi}\int_0^{\frac{3}{2}\pi} |\sin(u)\,|\, du = \frac{180}{\pi}[|-\cos(u)|]_0^{\frac{3}{2}\pi}$

$$\int_0^{15}\left|15\sin\left(\frac{\pi t}{12}\right)\right|dt = \frac{180}{\pi}\int_0^{\frac{3}{2}\pi}|\sin(u)|\,du$$

$$\frac{180}{\pi}\int_0^{\frac{3}{2}\pi}|\sin(u)|\,du = \frac{180}{\pi}[|-\cos(u)|]_0^{\frac{3}{2}\pi}$$

$$= \frac{180}{180}\left|-\cos\left(\frac{3}{2}\pi\right)+\cos(0)\right|$$

$$\int_0^{15}\left|15\sin\left(\frac{\pi t}{12}\right)\right|dt = \frac{180}{\pi}$$

Using $h = \frac{15-0}{6} = 2.5$

$$\int_0^{15}\left|8e^{-\frac{(t-15)^2}{50}}\right|dt$$

$$= \frac{2.5}{2}\left[\left|\left(8e^{-\frac{(0-15)^2}{50}}\right)+\left(8e^{-\frac{(15-15)^2}{50}}\right)\right.\right.$$

$$+2\left(8e^{-\frac{(2.5-15)^2}{50}}+8e^{-\frac{(5-15)^2}{50}}+8e^{-\frac{(7.5-15)^2}{50}}+8e^{-\frac{(10-15)^2}{50}}\right.$$

$$\left.\left.+8e^{-\frac{(12.5-15)^2}{50}}\right)\right|\Bigg]$$

$$\int_0^{15}\left|8e^{-\frac{(t-15)^2}{50}}\right|dt = \frac{2.5}{2}(39.976) = 49.970$$

Now, $\int_0^{15}|R(t)|\,dt = 75 + \frac{180}{\pi} + 49.970 = 182.266\ mm^2$.

a. Rainfall was most intense between 5 to 8 hours after midnight.

b. The reservoir can safely process 250mm of rainfall because the net accumulation was 182.266 which is **less than** the reservoir capacity. Hence the current drainage system will be adequate.

c. The rainfall rate exceeds 19mm/h between 4 – 9.2 hours after midnight. Thus, flash flood risks are likely to occur between such period.

Note

When a problem suggests that you should determine the area under the curve without stating if it is total area or net area, calculate the total area.

FOCAL AREA 2: FINDING THE AREA BETWEEN TWO CURVES

The area between two regions

We have explored areas under curves in relation to the x –axis. Now, we will take a look at determining the area between two curves.

If $y = f(x)$ lies above $y = g(x)$ from $x = a$ to $x = b$, the area of the region between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is;

$A = \sum_{i=1}^{n}[f(c_i) - g(c_i)]h$, where $h = b - a$.

$$A = \int_a^b [f(x) - g(x)]dx$$

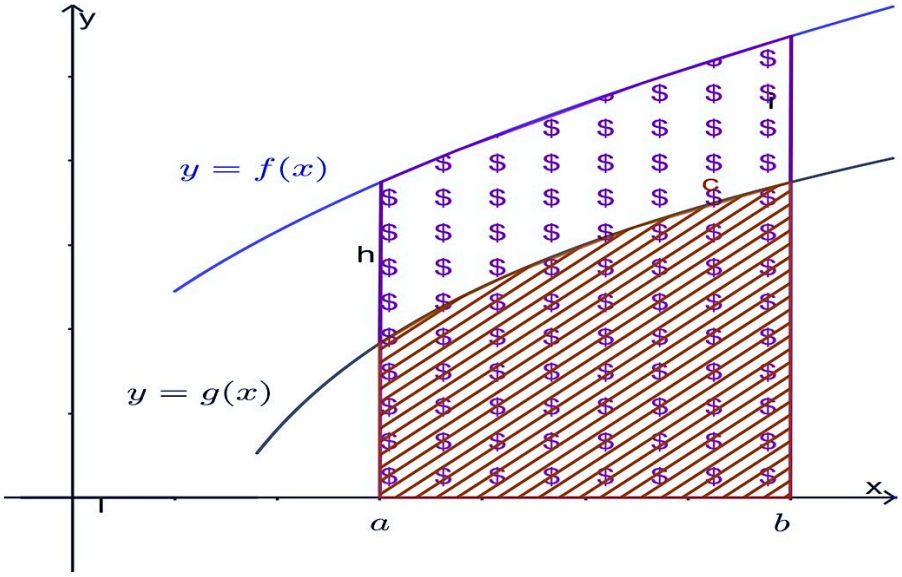


Figure 14.7: *Graphical illustration of area between* $y = f(x)$ *and* $y = g(x)$

From Figure 14.7, we can observe that, area between the two curves is the portion where the two kinds of shading do not intersect. Thus, if we find the area under $y = g(x)$ and subtract it from $y = f(x)$. We shall get the required area.

Example 1

Find the area of the region between $y = 2x^2 - 5x + 6$ and $y = -x^2 + 2x + 1$ from $x = 0$ to $x = 2$.

Solution

$$Area = \int_0^2 [(2x^2 - 5x + 6) - (-x^2 + 2x + 1)]dx$$

$$= \left[\frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x + \frac{x^3}{3} - x^2 - x\right]_0^2$$

$$= \left[x^3 - \frac{7}{2}x^2 + 5x\right]_0^2$$

$$= (2)^3 - \frac{7}{2}(2)^2 + 5(2)$$

$$= 4$$

Note that the two equations could be subtracted from each other as shown below and then integrated to arrive at the same answer.

$$\int_0^2 (3x^2 - 7x + 5)dx = \left[x^3 - \tfrac{7}{2}x^2 + 5x\right]_0^2 = \text{Etc}$$

Thus, area between the two regions is 4 units squared from $x = 0$ to $x = 2$.

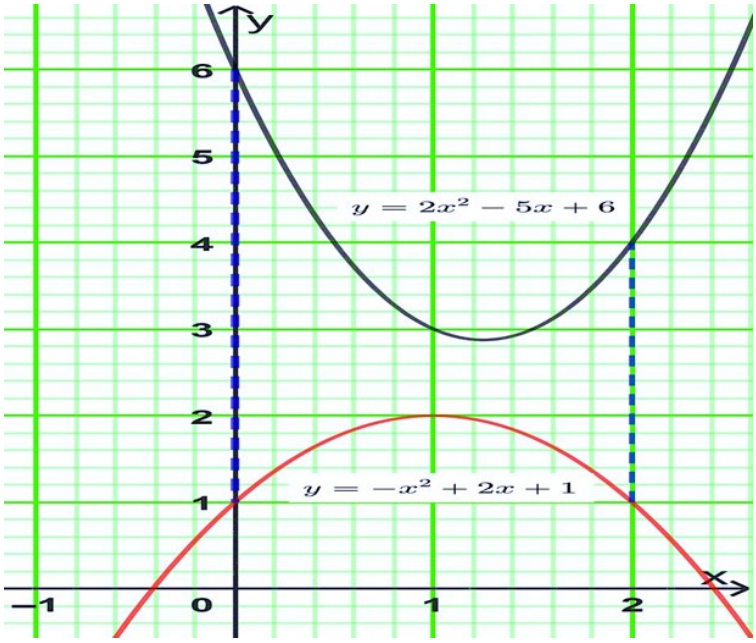


Figure 14.8: *Graphical illustration of area between* $y = 2x^2 - 5x + 6$ *and* $y = -x^2 + 2x + 1$.

The area enclosed by two graphs

In some instances, two curves may intersect at some point(s), creating a region enclosed (bounded) by such graphs. To determine the area enclosed by the curves use:

$A = \int_a^b |f(x) - g(x)|dx$ where $f(x)$ lies above $g(x)$ in the given interval, just as illustrated in Figure 14.9.

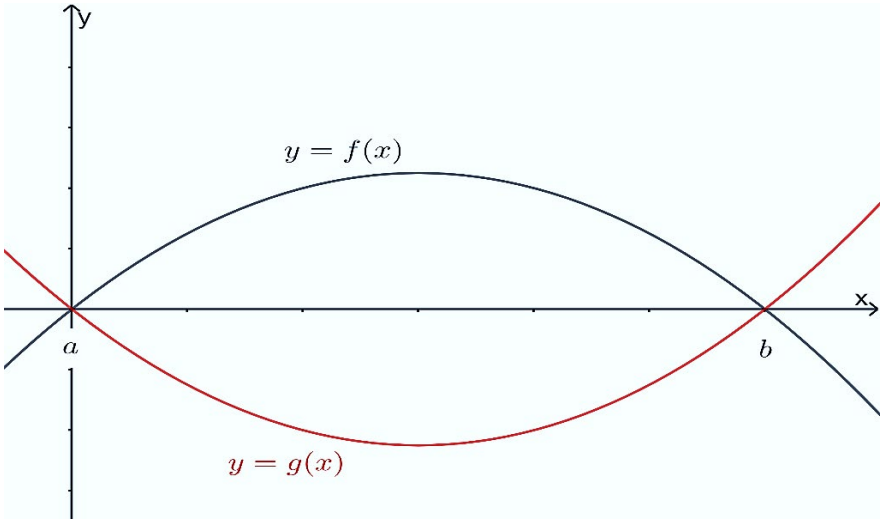


Figure 14.9: *Graphical illustration of area between* $y = f(x)$ *and* $y = g(x)$.

Example 2

Draw the graphs of the curves $y = x^2$ and $y^2 = 8x$. From your graph:

i. Determine their points of intersection and sketch the two graphs on the same axes.

ii. Calculate the area enclosed by the two curves.

Solution

Figure 14.10 possesses the details of the graphs of the two functions.

i. From the graph in Figure 14.10, the points of intersection of the curves are $(0, 0)$ and $(2, 4)$.

ii. The area enclosed by the two curves will be determined using $\int_0^2 \left(\sqrt{8x} - x^2\right)\, dx$.

$$\int_0^2 \left(\sqrt{8x} - x^2\right)\, dx = \int_0^2 \sqrt{8x}\, dx - \int_0^2 x^2\, dx$$

$$= \frac{4\sqrt{2}}{3}\left[x^{\frac{3}{2}}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{4\sqrt{2}}{3}(2\sqrt{2}) - \frac{8}{3}$$

$$= \frac{8}{3}$$

$$= 2\frac{2}{3} \text{ units squared}$$

Therefore, the area of the region enclosed by $= x^2$ and $y^2 = 8x$ is $2\frac{2}{3}$ units squared.

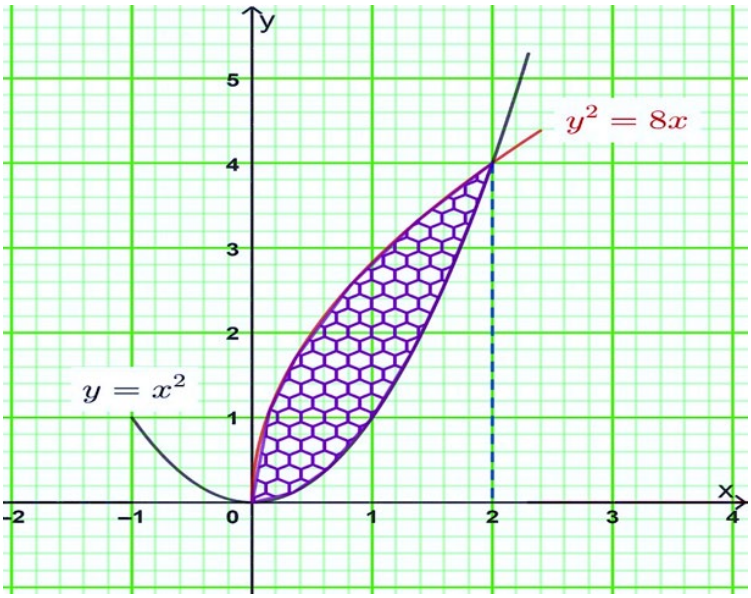


Figure 14.10: *Graphical illustration of area between* $y = x^2$ *and* $y^2 = 8x$.

Example 3

Find the area bounded by the graphs $y = x^2$ and $y = 2 - x^2$ for $0 \leq x \leq 2$.

Solution

An observation from the graph in Figure 14.11 shows that within the interval [0, 2], the curves intersect at point (1, 1). Thus, the area enclosed will be in 2 parts [0, 1] and [1, 2]. To find the enclosed area, calculate the total area of both parts and add.

$$\text{Enclosed Area} = \int_0^1 [(2 - x^2) - x^2]\, dx + \int_1^2 [x^2 - (2 - x^2)]\, dx$$

$$= \int_0^1 [2 - 2x^2]\, dx + \int_1^2 [2x^2 - 2]\, dx$$

$$= \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2$$

$$= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right)$$

$$= 4$$

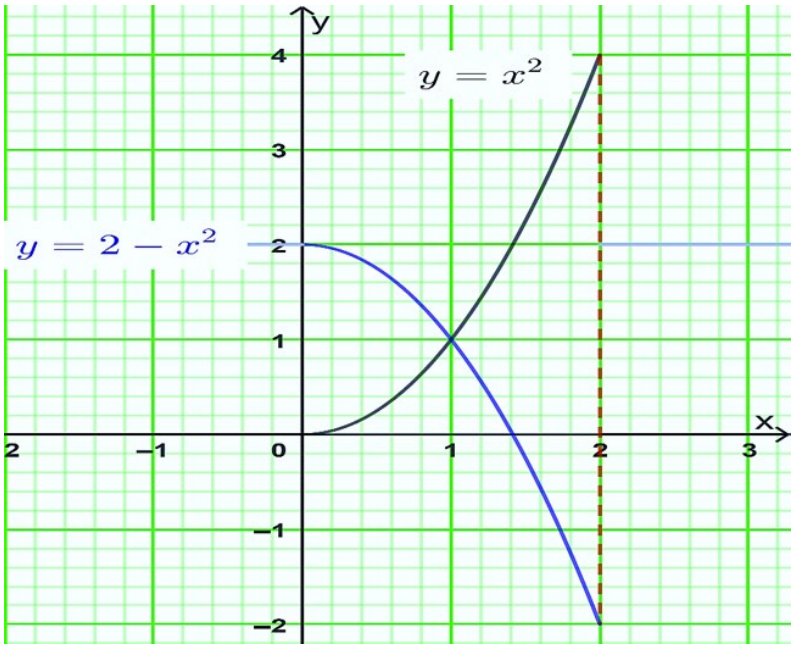
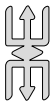


Figure 14.11: *Graphical illustration of area between* $y = x^2$ *and* $y = 2 - x^2$.



Note

If the graphs of the functions are not required, solve the given equations simultaneously to determine the points of intersection but it does always help to see things graphically.

Learning Tasks

1. Guide learners in mixed-ability groups to shade regions under simple functions and categorise the areas as above or below the x −axis. Where technological tools (GeoGebra, Desmos, etc.) are available, make use of them in the exploration.
2. Using GeoGebra or other graphing software, let learners visualise areas under different curves. Then create sliders to dynamically change functions and observe how areas change.
3. Present functions with equal net areas but different total areas and assist learners to calculate both and explain the difference.
4. Present complex arrangements of multiple curves, have learners identify and label all enclosed regions, identifying which curve is "on top" of the other at specific intervals.
5. Assist learners calculate areas of specific regions using appropriate techniques.

Pedagogical Exemplars

1. Experiential Learning

   a. Provide pre-built GeoGebra or Desmos file with functions with adjustable parameters (with help of sliders), dynamic visualisation of total and net area, area between function and adjustable horizontal line to learners.



https://www.geogebra.org/m/pBYT7BRx

Figure 14.12: *Definite Integral*



https://www.geogebra.org/m/aJBApdr5

Figure 14.13: *Using rectangles to find definite integrals*

   b. Provide learners with prompts to analyse how parameters affect total and net area, determine when net area is zero and total area is large as well as how intersection points change as parameters change.

   c. In the absence of technological resources, assign different functions to different groups to graph by hand and use the different drawings to assist learners analyse how total and net area are affected.

2. Collaborative problem and inquiry-based learning

   With the help of cardboard sheets, rulers, markers, sand and small containers task learners in groups to;

   a. create two different function curves on cardboard.

   b. fill the area between curves with sand.

   c. use weight or volume of sand to compare areas.

   d. develop proportional relationships between sand quantity and area.

   e. share their graphs and observations with other groups.

Key Assessment Level 4: Classwork

1. *A civil engineer is modelling the cross-section of a river as:* $h(x) = 10 - (x-5)^2$, *for* x *in* $[2, 8]$.*This represents the depth (in metres) of the river at various points across its 6-metre-wide cross-section.*

   c. *Find the total area of the cross-section of the river.* ***(4 marks)***

   d. *The river flow rate is proportional to the cross-sectional area. If the river can safely carry* $90m^3/s$ *per unit area, determine the maximum discharge rate.* ***(2 marks)***

   e. *If climate models predict a* 20% *increase in runoff, evaluate whether the current channel can handle it.* ***(5 marks)***

   Total: 11 marks)

   DoK Level 4

2. *Find the total area of the regions contained by* $y = x^3 + 2x^2 - 3x$ *and the* $x - axis$. ***(6 marks)***

   DoK Level 2

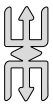
3. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent? **(7 marks)**

   DoK Level 4

PLC Session 14

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

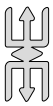
- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app

a. The key assessment for the week is ***classwork***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the classwork (NTS 3k-3q).

10 minutes Content Exploration

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes Enactment

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes Reflection

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 15

Learning Indicators

1. *Apply the techniques of integration to solve real-life problems involving distances and rates of change*
2. *Find the volume of a solid formed after rotation about horizontal or vertical axis*

In the previous week, learners learnt how to find the area bounded by a curve and the axes or other curves. This week's lesson will focus on using integration to solve problems involving distances and rates of change and finding the volume of solids generated by the rotation of curves about the Cartesian axes.

FOCAL AREA 1: USING THE TECHNIQUES OF INTEGRATION TO SOLVE REAL-LIFE PROBLEMS

Example 1

Suppose the downward velocity of a sky diver is given by $v(t) = 30(1 - e^{-t})ft/s$ for the first 5 seconds of a jump. Compute the distance fallen.

Solution

$v(t) = 30(1 - e^{-t})ft/s$

Let $Displacement = s(t)$

Since $v(t) = \frac{d}{dt}\big(s(t)\big)$

$s(t) = \int v(t)dt$

$= \int_0^5 30(1 - e^{-t})dt$

$= [30t + 30e^{-t}]_0^5$

$= (150 + 30e^{-5}) - (0 + 30e^{0})$

$= 120 + 30e^{-5}$

$\approx 120.2\ feet$

Therefore, the sky diver fell through a distance of approximately 120.2 feet

Example 2

Suppose that water can flow in and out of a storage tank. If the net rate of change of water in the tank is $f(t) = 200(t^2 - 1)$ litres per minute,

1. for $0 \leq t \leq 3$,determine when the water level is increasing and when the water level is decreasing.
2. if the tank has 2 000 litres of water at time $t = 0$, determine how many litres are in the tank at time $t = 3$.

Solution

1. Water level decreases if $f(t) < 0$.

   $200(t^2 - 1) < 0$

   $200(t^2 - 1) = 0$

   $t^2 - 1 = 0$

   $(t + 1)(t - 1) = 0$

   $t = -1, 1$

   Since $0 \leq t \leq 3$ the range of t must fall within the intervals $0 < t < 1$ or $1 < t < 3$

   For $0 < t < 1$, take $t = 0.5$,

   $200(0.5^2 - 1) = -150$

   $\Rightarrow 200(t^2 - 1) < 0$

   Therefore, the water level decreases between zero and one second.

   Water level increases if $f(t) > 0$.

   When $t = 2$ i.e., $t \in (1, 3)$,

   $200(2^2 - 1) = 600$

   $\Rightarrow 200(t^2 - 1) > 0$

   Therefore, the water level increases between one second and three seconds.

2. $Amount\ of\ water\ added\ in\ the\ tank = \int (The\ net\ rate\ of\ change)\, dt$

$$= \int_0^3 f(t)dt$$
$$= \int_0^3 200(t^2 - 1)dt$$
$$= \left[200\left(\frac{t^3}{3} - t\right)\right]_0^3$$
$$= \left[200\left(\frac{3^3}{3} - 3\right)\right] - \left[200\left(\frac{0^3}{3} - 0\right)\right]$$
$$= 1\,200\ litres$$

   Since there were already 2 000 litres of water in the tank, the amount of water in the tank after three seconds is given by

   $2\,000 + 1\,200 = 3\,200$ litres

Example 3

Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred cedis per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred cedis per year.

1. For how many years does the rate of profitability of the second investment exceed that of the first?
2. Compute the net excess profit for the time period determined in part (1).

Solution

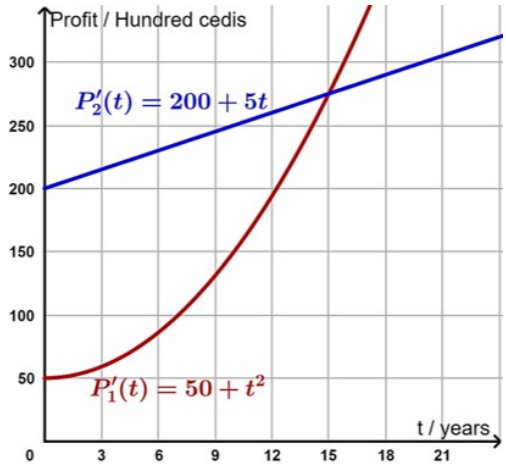


Figure 15.1: *Graph of* $P_1'(t)$ *and* $P_2'(t)$

1. From Figure 15.1, the graph of $P_2'(t)$ is above the graph of $P_1'(t)$ in the interval $0 \leq t < 15$. This implies that the rate of profitability of the second investment exceed that of the first for 15 years.

   Alternatively,

   Since $P_2'(0) > P_1'(0)$ i.e., $200 > 50$

   The rate of profitability of the second investment exceeds that of the first till t years when

   $P_1'(t) = P_2'(t)$

   $50 + t^2 = 200 + 5t$

   $t^2 - 5t - 150 = 0$

   $(t - 15)(t + 10) = 0$

   $t = -10, 15$

   Since $t \geq 0, t = 15$

   $\therefore$ the rate of profitability of the second investment exceeds that of the first for 15 years

2. The profit generated by the first investment and second investment are $P_1(t)$ and $P_2(t)$ respectively.

   In the time interval $t \in [0, 15]$

   $P_1(t) = \int_0^{15} P_1'(t)dt$ and $P_2(t) = \int_0^{15} P_2'(t)dt$

   $\Rightarrow P_1(t) = \int_0^{15}(50 + t^2)dt$ and $P_2(t) = \int_0^{15}(200 + 5t)dt$

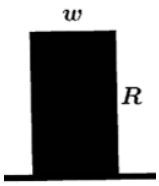
   Net excess profit for $t \in [0, 15]$ is given $P_2(t) - P_1(t)$ since $P_2'(t) > P_1'(t)$ for $t \in [0, 15]$

   $$\text{Net excess profit} = \int_0^{15}(200 + 5t)dt - \int_0^{15}(50 + t^2)dt$$
   $$= \int_0^{15}\left(200 + 5t - (50 + t^2)\right)dt$$
   $$= \int_0^{15}(150 + 5t - t^2)dt$$
   $$= \left[150t + \frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^{15}$$
   $$= \left[150(15) + \frac{5}{2}(15)^2 - \frac{1}{3}(15)^3\right] - \left[150(0) + \frac{5}{2}(0)^2 - \frac{1}{3}(0)^3\right]$$
   $$= 1687.50 \text{ hundred cedis}$$

   $\therefore$ The net excess profit is $168\,750$ cedis

FOCAL AREA 2: FINDING THE VOLUME OF A SOLID FORMED AFTER ROTATION ABOUT THE HORIZONTAL OR VERTICAL AXIS

When two-dimensional figures are rotated about an axis. A solid is formed. Take for example, a rectangle on the x-axis as shown in Figure 15.2.



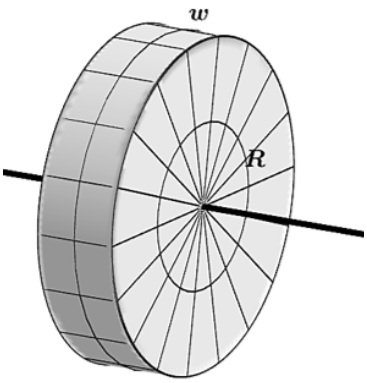


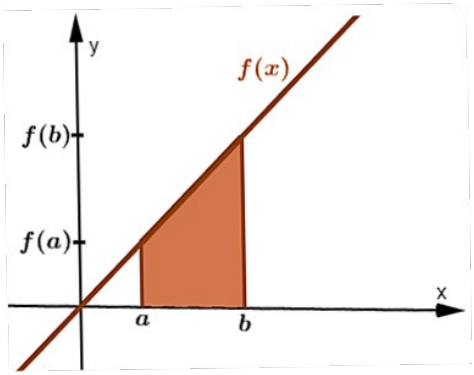
Figure 15.2a **Figure 15.2b**

Figure 15.2: *Rectangle and its resulting solid of evolution*

The width of the rectangle is given by w. Its height is simply represented by R. When the rectangle is rotated about the axis, it creates a cylinder whose volume, V can be found as the product of the area of the circle formed from the rotation (πR^2) and the thickness of the cylinder (same as the width of the rectangle, w) i.e., $V = \pi R^2 w$

Method of discs

More interesting solids can be obtained from revolving the curve represented by $f(x)$ between some interval of x, say $[a, b]$ about the x-axis or an interval of y about the y-axis.



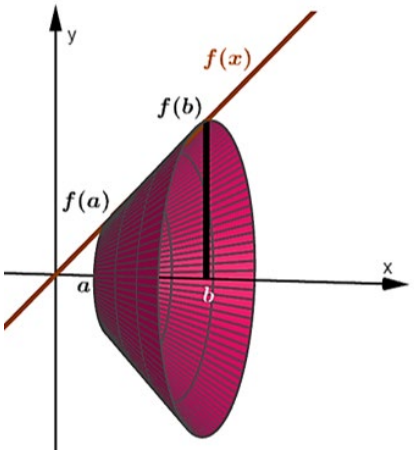


Figure 15.3a **Figure 15.3b**

Figure 15.3: *Graph of $f(x)$ and solid of evolution about the x-axis*

Since the curve may not be horizontal (in the case of revolutions about the x-axis) or vertical (in the case of revolutions about the y-axis), the heights of the curve at the extremities of the intervals i.e., $f(a)$ and $f(b)$ may be unequal i.e., $f(a) \neq f(b)$ and thus, R cannot be used to represent the height. This means we cannot simply infer that the volume of the solid obtained is $V = \pi R^2 w$, where, $w = b - a$.

A way to take care of this problem is to divide the interval, $[a, b]$ into n smaller intervals of width Δx or Δy so small that the heights of the curve at the ends of those intervals (x_i, for $i = 0, 1, 2, \ldots, n$ or y_i, for $i = 0, 1, 2, \ldots, n$) are the significantly the same. This would mean that many rectangles of height $f(x_i)$, for $i = 0, 1, 2, \ldots, n$ and width, Δx or Δy are created. The revolutions of these rectangles would produce n circles (c_i, for $i = 0, 1, 2, \ldots, n$) of radii, $f(x_i)$ and thereby n cylinders (discs) of thickness, Δx or Δy.

The volume of the resulting solid is thus the summation of all the volumes of the n cylinders (discs) obtained i.e.,

$$V = \sum_{i=1}^{n} [A(c_i)]\,\Delta x$$

This process is the method of discs.
Finding the definite integral over the interval $[a, b]$ eliminates the burden of having to iterate.
Concretely, the method of discs can be described as follows:

Suppose that $f(x) \geq 0$ and f is continuous on the interval $[a, b]$, take the region bounded by the curve $y = f(x)$ and the x-axis, for $a \leq x \leq b$ and revolve it about the x- axis generating a solid (Figure 15.4a and Figure 15.4b). We can find the volume of this solid by slicing it perpendicular to the $x - axis$ and recognising that each cross section is a circular disk of radius $r = f(x)$. We then have that the volume of the solid as

$$V = \sum_{i=1}^{n} [A(c_i)]\,\Delta x = \int_a^b \pi[f(x)]^2 dx$$

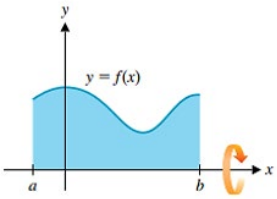


Figure 15.4a

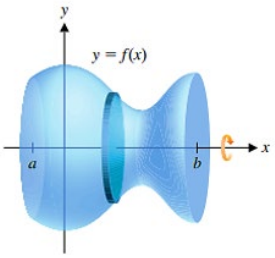


Figure 15.4b

Figure 15.4: *Graph of* $y = f(x) \geq 0$ *and the resulting solid of revolution*

Example 1

Revolve the region under the curve $y = \sqrt{x}$ on the interval $[0, 4]$ about the $x - axis$ and find the volume of the resulting solid of revolution.

Solution

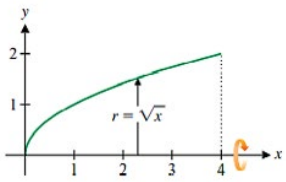


Figure 15.5a

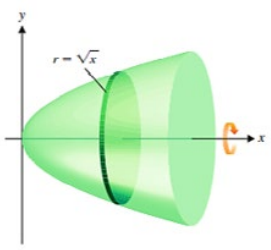


Figure 15.5b

Figure 15.5: *Graph of* $y = \sqrt{x}$ *and the resulting solid of revolution*

$$The\ circular\ area\ of\ a\ disc = \pi y^2$$

$$= \pi\left(\sqrt{x}\right)^2\ squared\ units$$

$$Volume\ of\ solid = \int_0^4 \pi\left(\sqrt{x}\right)^2 dx$$

$$= \pi \int_0^4 \left(\sqrt{x}\right)^2 dx$$

$$= \pi \int_0^4 x dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4$$

$$= 8\pi\ cubic\ units$$

Note

In a similar way, suppose that $g(y) \geq 0$ *and* g *is continuous on the interval* $[c, d]$*. Then, revolving the region bounded by the curve* $x = g(y)$ *and the* $y-$axis*, for* $c \leq y \leq d$*, about the* $y-$axis *generates a solid. (See Figures 15.6a and Figure 15.6b.) Once again, notice from Figure 15.6b that the cross sections of the resulting solid of revolution are circular disks of radius* $r = g(y)$*. All that has changed here is that we have interchanged the roles of the variables* x *and* y*. The volume of the solid is then given by*

$$V = \int_c^d \pi[g(y)]^2 dy$$

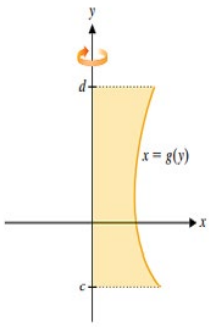


Figure 15.6a

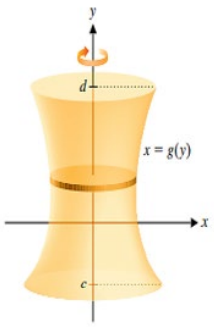


Figure 15.6b

Figure 15.6: *Graph of* $y = f(x) \geq 0$ *and the resulting solid of revolution*

Revolve about the y-axis

Example 2

Find the volume of the solid resulting from revolving the region bounded by the curve $y = 4 - x^2$ from $y = 1$ to $y = 4$.

Solution:

$y = 4 - x^2$, $y = 1$, $y = 4$

By solving for x, we have

$x = \sqrt{4 - y}$

$$Volume = \int_1^4 \pi\left(\sqrt{4-y}\right)^2 dy$$

$$= \int_1^4 \pi(4 - y)\, dy$$

$$= \pi\left[4y - \frac{y^2}{2}\right]_1^4$$

$$= \pi\left[(16 - 8) - \left(4 - \frac{1}{2}\right)\right]$$

$$= 4.5\pi \ cubic\ units$$

Example 3

Find the volume of the solid resulting from revolving the portion of the curve $y = 2 - \frac{x^2}{2}$ from $x = 0$ to $x = 2$ about the $y - axis$.

Solution

$y = 2 - \frac{x^2}{2}$

$\Rightarrow x = \sqrt{4 - 2y}$

When $x = 0$, $y = 2$

When $x = 2$, $y = 0$

Note that area and volume are scalar quantities and hence it would not matter which limit is used as the lower or upper limit.

$$\int_2^0 \pi\left(\sqrt{4-2y}\right)^2 dy = \int_0^2 \pi\left(\sqrt{4-2y}\right)^2 dy$$

$$\int_0^2 \pi\left(\sqrt{4-2y}\right)^2 dy = \pi \int_0^2 (4 - 2y) dy$$

$$= \pi[4y - y^2]_0^2$$

$$= \pi[(4(2) - 2^2) - (4(0) - 0^2)]$$

$$= 4\pi$$

$$Required\ volume = 4\pi \ cubic\ units$$

Example 4

Suppose that a ruptured pipe on an offshore oil rig in Ghana's Jubilee Field (located in the Gulf of Guinea) has produced a circular oil slick. Environmental protection officials from the Ghana Environmental Protection Agency (EPA) need to assess the volume of oil spilled to determine the appropriate clean-up response and potential impact on coastal communities from Axim to Half Assini.

$T(r) = 2 + \sqrt{r}, r \geq 0$

At the time the spill is contained by the Petroleum Commission's emergency response team, the radius of the slick is 4 metres. We wish to find the volume of oil that has been spilled to help the EPA assess the environmental impact on Ghana's western coastline and fishing communities.

Solution

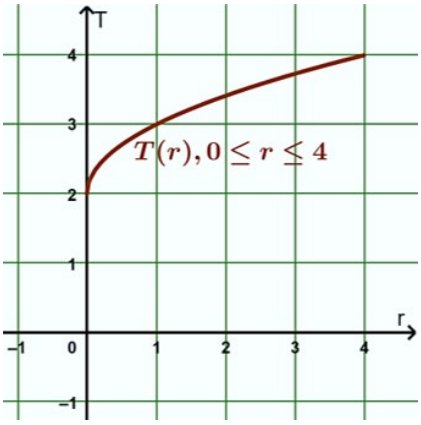


Figure 15.7: *Graph of* $T(r)$

A revolution of the curve about the T axis results in the solid in Figure 15.8

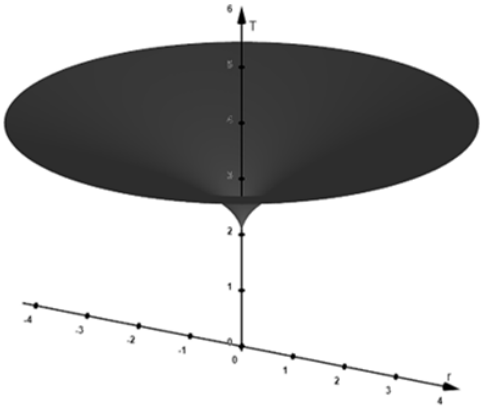


Figure 15.8: *Solid of revolution of* $T(r)$ *about the* T *axis*

Notice that the volume we want is obtained by rotating the curve $T(r)$ about the T axis (vertical axis)

$T(r) = 2 + \sqrt{r}, r \geq 0$

$T(0) = 2 + \sqrt{0}$

$= 2$

$T(4) = 2 + \sqrt{4}$

$= 4$

$Volume = \int_2^4 \pi r^2 dT$

Let $T(r) = T$

$T = 2 + \sqrt{r}$

$\sqrt{r} = T - 2$

$r(T) = (\mathrm{T} - 2)^2$

$= T^2 - 4T + 4$

$= T^2 - (4T - 4)$

$Volume = \int_2^4 \pi[T^2 - (4T - 4)]^2 dT$

$= \pi \int_2^4 (T^4 - 2T^2(4T - 4) + (4T - 4)^2) dT$

$= \pi \int_2^4 (T^4 - 8T^3 + 8T^2 + 16T^2 - 32T + 16) dT$

$= \pi \int_2^4 (T^4 - 8T^3 + 24T^2 - 32T + 16) dT$

$= \pi \left[\frac{T^5}{5} - 2T^4 + 8T^3 - 16T^2 + 16T\right]_2^4$

$= \pi \left[\left(\frac{4^5}{5} - 2(4)^4 + 8(4)^3 - 16(4)^2 + 16(4)\right) - \left(\frac{2^5}{5} - 2(2)^4 + 8(2)^3 - 16(2)^2 + 16(2)\right)\right]$

$= (12.8 - 6.4)\pi$

$= 6.4\pi\ m^3$

Learning Tasks

1. Learners revise the techniques of integration
2. Learners solve problems involving rates of change
3. Learners find areas below curves and generate solids of revolutions
4. Learners cut circles of different radii to model solids of revolution
5. Learners explain the meaning of each term in the integral formula and find volumes of solids of revolution

Pedagogical Exemplars

1. Experiential learning
   a. Use grid paper and simple curves to estimate areas visually
   b. Shade regions and approximate integrals using rectangles

c. Provide guided integration problems with missing steps

d. Encourage peer tutoring and provide structured sentence stems for learners with reading or language difficulties

e. Engage learners in GeoGebra activities to find areas below curves, generate solids of revolutions and find volumes of solids of revolution. Some examples are given:



https://www.geogebra.org/m/u8KtPdqf

Figure 15.7: *Volume of revolution: Disk method (3D)*



https://www.geogebra.org/m/mzWq2Cet

Figure 15.8: *Volume of solid of revolution about x-axis*

2. Problem-based learning:

   a. Use integration to find total volume of water at specific times in a tank given the rate of change

   b. Given a curve and interval, learners calculate the volume formed by rotating the area about the x-axis and y-axis

   c. In small heterogeneous groups, learners solve integration problems then rotate to explain their method to the next group

   d. Solve real-life problems involving motion (distance from velocity graphs), cost accumulation, or biological volume models

   e. Guide learners to design a rotational solid (e.g., tank or silo) by rotating a function and calculate volume and present rationale and sketch in 2D and 3D

   f. Solve real-life problems involving motion (distance from velocity graphs), cost accumulation, or biological volume models

3. Provide calculator or computer access for numerical integration

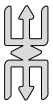
4. Use digital tools to plot graphs, define limits of integration, and visualise rotated solids

5. Allow manipulation of 3D visuals on screen in place of physical drawing or modelling

6. Provide audio-supported graphing software and step-by-step verbal instructions

7. Describe real-life integration applications through auditory examples
8. Use captioned videos and animations to explain integration techniques and volume rotation

Key Assessment Level 4: Simulation, displays and/or exhibition



Note

The following are sample tasks. It is required there is a blend of written tasks as in task 1 and 2 and a practical assessment as in task 3. The practical assessment may be adapted to suit learning environments.

1. *A cocoa* farm in the Western Region of Ghana has a production rate (in tonnes/month) that changes throughout the year. The rate of production is modelled by the function:

   $R(t) = 0.5t^2 - 0.1t^3$

   where t is measured in months from January ($t = 1$) to December ($t = 12$).

   a. Set up and evaluate a definite integral to find the total cocoa production from April to August inclusive. Interpret your result in context. **(6 marks)**

   b. Using differentiation find the month when the production rate is at its maximum.

   (5 marks)

   c. Determine whether the farm is profitable by calculating if the annual production exceeds 15 tonnes and explain your conclusion.

   (4 marks)

   Total: 15 marks

   DoK Level 3

2. In a village near Bolgatanga in northern Ghana, traditional huts are constructed with a cylindrical base and a conical thatched roof. A master builder wants to calculate the exact volume of a traditional hut to determine the amount of air circulation needed for comfort during hot weather.

   The hut consists of two parts: A cylindrical base with radius 3 metres and height 2.5 metres, and a conical thatched roof.

   When viewing the hut's cross-section with the vertical axis as the axis of revolution, the profile of the conical roof can be described by the function: $h(d) = 2.5 + 2(3 - d)$, for $0 \leq d \leq 3$

   Where:

   d is the horizontal distance from the central axis (in metres)

   $h(d)$ is the height from the ground (in metres)

   The function gives $h(3) = 2.5$ at the edge where the roof meets the cylindrical wall

   The function gives $h(0) = 8.5$ at the peak of the roof

   a. Find the volume of the cylindrical base of the hut.

   (2 marks)

   b. Find the volume of the conical roof by setting up and evaluating an appropriate integral.

   (6 marks)

c. Calculate the total volume of the traditional hut.

(1 mark)

d. If the average person requires 12 cubic metres of air space for comfort, how many people can comfortably occupy this hut?

(1 mark)

Total: 10 marks

DoK level 3

3. Design a pencil holder by rotating a polynomial function around the y-axis.

   a. Create a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ that will generate an aesthetically pleasing shape when rotated. Justify your choice of coefficients.

   b. Use the disc method to calculate the exact volume of your pencil holder, showing all working.

   c. Create a cross-sectional model using cardboard discs, with at least 6 precisely sized discs showing key points along the curve.

   d. Test if your design can hold at least 10 standard pencils while remaining stable. Document modifications needed.

   e. Present your mathematical model alongside your physical model, explaining how the polynomial coefficients influence the shape and functionality.

20 marks

DoK level 4

PLC Session 15

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).

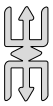
Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
    a. The key assessment for the week is ***simulation, displays and/or exhibition***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the assessment (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
    a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
    b. all teachers in the group work through a similar problem individually for 3–4 minutes
    c. the group then works through the problem together step by step
    d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
    a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
    b. After the enactment, ask for structured feedback from the group
        i. was the concept explained in a way that a learner with no prior knowledge could follow?
        ii. were the examples grounded in contexts familiar to Ghanaian learners?
        iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
    a. transfer your chats into your learning planner template.
    b. read your Teacher Manual and related Learner Material in preparation for the next session.

Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).

c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 6 Review

This section discussed all the lessons to be taught in week 14 and week 15. This section focused on applications of integration to find areas under and between curves, solve real life problems and find volumes of solids of revolution. Learning and assessment tasks for this section emphasised the enforcement of skills and competencies regarding the techniques of integration and the applications of integration. Assessment tasks which required collaboration among learners and provide the opportunity for learners to practice were suggested to reinforce learning from year one and year two. It is intended that learners will get the opportunity to revise lessons taught in the previous years as they work through the learning and assessment tasks. Teachers are strongly encouraged to foster inclusive learning environments for all learners.

MARKING SCHEMES FOR THE KEY ASSESSMENNTS

WEEK 14

1. *a.* $\int_2^8 h(x)\,dx = \int_2^8 [10 - (x-5)^2]\,dx$

$\int_2^8 [10 - (x-5)^2]\,dx = \int_2^8 10\,dx - \int_2^8 (x-5)^2 dx$

M1, or correct use of an alternate method

Let $u = x - 5$, $du = dx$

$\int_2^8 [10 - (x-5)^2]\,dx = 10(8) - 10(2) - \int_{-3}^3 u^2\,du$ ***M1***

$= 60 - \left[\frac{(3)^3}{3} - \frac{(-3)^3}{3}\right]$ ***M1***

$= 42\,m^2$ ***A1***

a. Maximum discharge rate $= 42 \times 90$ ***M1***

$= 3780 m^3/s$ ***A1***

b. 20% of width $= 20\% \times 6$ ***M1***

$= 1.2$

Required capacity $= 1.2 \times 3780$ ***M1***

$= 4536 m^3/s$ ***A1***

Extra needed will be $4536 - 3780 = 756 m^3/s$ **A1**

This means that the current channel could not handle it. ***A1***

2. If $x^3 + 2x^2 - 3x = 0$

$x(x-1)(x+3) = 0$ ***M1***

Thus y *cuts the* x *–axis at* $0, 1$ *and* -3. ***A1***

Total area $= \int_{-3}^0 |x^3 + 2x^2 - 3x|\,dx + \int_0^1 |x^3 + 2x^2 - 3x|\,dx$ ***M1***

$= \left|\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^0\right| + \left|\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_0^1\right|$ ***M1***

$= 11\frac{1}{4} + \frac{7}{12}$ ***M1***

$= 11\frac{5}{6}$ *squared units.* ***A1***

3. *If* $b(t) = 2200e^{0.024t}$ *and* $d(t) = 1460e^{0.018t}$ *then, area between curves is:*

Area $= \int_0^{10} |2200e^{0.024t}|\,dt - \int_0^{10} |1460e^{0.018t}|\,dt$ ***M1***

$= 2200\left[\frac{1}{0.024}e^{0.024t}\right]_0^{10} - 1460\left[\frac{1}{0.018}e^{0.018t}\right]_0^{10}$ ***M1***

$= 2200\left[\frac{1}{0.024}(1.2712 - 1)\right] - 1460\left[\frac{1}{0.018}(1.1972 - 1)\right]$ ***M1***

$= 24860 - 15995$ ***M1***

$= 8865$ ***A1***

This area represents the net population increase over the 10-year period due to the difference between the birth rate and the death rate. In other words, between year 0 and year 10, the population increased by approximately 8,865 people due to natural causes (births minus deaths).

B1 for net population, or equivalent

B1 for an explanation of this

WEEK15

1. $R(t) = 0.5t^2 - 0.1t^3$

g. *For April,* $t = 4$

For August, $t = 8$

$Total\ cocoa\ production\ from\ April\ to\ August = \int_4^8 R(t)\,dt$

$= \int_4^8 (0.5t^2 - 0.1t^3)\,dt$ ***M1A1***

$= \left[\frac{t^3}{6} - \frac{t^4}{40}\right]_4^8$ ***M1***

$= \left(\frac{8^3}{6} - \frac{8^4}{40}\right) - \left(\frac{4^3}{6} - \frac{4^4}{40}\right)$ ***M1***

$= -\frac{256}{15} - \frac{64}{15}$

$= -\frac{64}{3}$

$\approx -21.33\ tonnes$ ***A1***

The total cocoa production is $-21.33\ tonnes$ *and it represents the net cocoa production (new growth minus harvested cocoa) between April and August.* ***A1***

h. $R'(t) = t - 0.3t^2$ ***B1***

At the maximum production rate, $R'(t) = 0$

$t - 0.3t^2 = 0$ ***M1***

$t(1 - 0.3t) = 0$

$t = 0$ *or* $1 - 0.3t = 0$

$0.3t = 1$

$t = \frac{1}{0.3} = 3\frac{1}{3}$

$t = 0$ *or* $t = 3\frac{1}{3}$ ***B1***

Since $1 \le t \le 12, t = 3\frac{1}{3}$ ***A1***

The maximum production rate occurs in March ***A1***

i. *For January,* $t = 1$

For December, $t = 12$

$Annual\ cocoa\ production = \int_1^{12} R(t)\,dt$

$= \int_1^{12} (0.5t^2 - 0.1t^3)\,dt$ ***B1***

$= \left[\frac{t^3}{6} - \frac{t^4}{40}\right]_1^{12}$ **M1 (Awarded once)**

$= \left(\frac{12^3}{6} - \frac{12^4}{40}\right) - \left(\frac{1^3}{6} - \frac{1^4}{40}\right)$ **M1**

$= -\frac{1152}{5} - \frac{17}{120}$

$= -\frac{5533}{24}$

$\approx -230.54\ tonnes$ **A1**

The farm is not profitable since the annual cocoa production is less than 15 tonnes **A1**

2. *For the conical top,* $h(d) = 2.5 + 2(3 - d)$, *for* $0 \leq d \leq 3$

For the cylinder, $radius, r = 3\ m$ *and* $height, h = 2.5\ m$

a. $Volume\ of\ the\ cylinder = \pi r^2 h$

$= \pi \times 3^2 \times 2.5$ **M1**

$= 22.5\pi\ m^3$ **A1**

b. $h = 2.5 + 2(3 - d)$

$h(3) = 2.5$ *and* $h(0) = 8.5$

$h = 2.5 + 6 - 2d$

$2d = 8.5 - h$

$d = 4.25 - 0.5h$ **B1**

$Volume\ of\ conical\ top = \int_{2.5}^{8.5} \pi d^2 dh$

$= \int_{2.5}^{8.5} \pi(4.25 - 0.5h)^2 dh$ **M1**

$= \pi \int_{2.5}^{8.5} (0.25h^2 - 4.25h + 18.0625) dh$

$= \pi \left[\frac{h^3}{12} - \frac{17h^2}{8} + \frac{289h}{16}\right]_{2.5}^{8.5}$ **M1**

$= \pi \left[\left(\frac{(8.5)^3}{12} - \frac{17(8.5)^2}{8} + \frac{289(8.5)}{16}\right) - \left(\frac{(2.5)^3}{12} - \frac{17(2.5)^2}{8} + \frac{289(2.5)}{16}\right)\right]$ **M1**

$= \left(\frac{4913}{96} - \frac{3185}{96}\right)\pi$ **M1**

$= 18\pi\ m^3$ **A1**

c. $Total\ volume\ of\ hut = (22.5 + 18)\pi$

$= 40.5\pi\ m^3$ **B1**

d. $Number\ of\ person\ who\ can\ get\ comfortably\ occupy\ the\ hut = 40.5\pi \div 12$

$= 3.375\pi$

≈ 10 **B1**

	Levels of Performance			
Criteria	*Excellent* *(4 marks)*	*Very Good* *(3 marks)*	*Satisfactory* *(2 marks)*	*Needs improvement* *(1 mark)*
Polynomial Function Creation	*Creates a cubic polynomial* $y = ax^3 + bx^2 + cx + d$ *with purposefully chosen coefficients.* *Provides explicit mathematical justification for each coefficient (e.g., "*a *is negative to create inward curve at top").* *Function domain is clearly specified and appropriate (e.g.,* $x \in [0, 10]$*).* *Shape is original and includes specific functional features (e.g., wider opening, stable base).* *Includes graph of function with axis labels and scale*	*Creates a cubic polynomial with appropriate coefficients.* *Provides justification for 3 coefficient choices.* *Function domain is specified.* *Shape is suitable for purpose with at least one intentional design feature.* *Includes basic graph of function*	*Creates a basic cubic polynomial.* *Limited justification for only 1 or 2 coefficient choices.* *Function domain may be unclear.* *Shape is functional but lacks intentional design features.* *Graph may be present but lacks detail*	*Creates incorrect polynomial (wrong degree or non-functional).* *No justification for coefficient choices.* *No domain specified or inappropriate domain.* *Shape would not function as pencil holder.* *Missing or severely inaccurate graph*
Volume Calculation (Disc Method)	*Sets up integral perfectly:* $V = \pi \int_a^b r^2 dx$ *with correct radius expression.* *Shows all algebraic steps including expansion of polynomial terms.* *All integration rules correctly applied with detailed working.* *Final answer includes exact form (with* π*) and decimal approximation.* *Correct units stated (*cm^3 *or* ml*).* *Includes verification step or alternative calculation method*	*Sets up integral correctly with appropriate limits.* *Shows most algebraic steps with minor omissions.* *Integration rules correctly applied with occasional minor errors.* *Final answer in either exact or decimal form with correct units.* *Working is organised and legible*	*Sets up basic integral but may have errors in radius expression.* *Shows some algebraic steps but significant steps missing.* *Integration rules applied inconsistently with computational errors.* *Final answer may lack units or have minor conceptual errors.* *Working is somewhat disorganised*	*Incorrect integral setup (wrong radius or limits).* *Major conceptual errors in application of disc method.* *Few or no algebraic steps shown.* *Answer significantly incorrect or missing.* *Working is illegible or highly disorganised*

Cross-sectional Model	*Creates more than 6 precisely sized discs with radius accuracy of* $\pm 1\ mm$ *.* *Positions discs at mathematically significant points (e.g., zeros, extrema, inflection points)* *Each disc clearly labelled with* x*-coordinate and radius.* *Central axis is straight with correct scale markings.* *Model is sturdy and precisely assembled.* *Excellent visual correspondence to mathematical function*	*Creates 6 well-sized discs with radius accuracy of* $\pm 3\ mm$*.* *Positions discs at reasonable intervals.* *Most discs labelled with coordinates.* *Central axis is straight with some scale markings.* *Model is relatively sturdy.* *Good visual correspondence to function*	*Creates 4 or 5 discs with radius accuracy of* $\pm 5\ mm$*.* *Positioning shows basic understanding of function.* *Some basic labelling present.* *Central axis may be slightly curved.* *Model is functional but lacks precision• Basic visual correspondence to function*	*Creates fewer than 4 discs with poor accuracy (*$> 5\ mm$ *error).* *Random or illogical positioning of discs.* *No labelling of discs.* *Central axis missing or severely misaligned.* *Model falls apart or cannot stand.* *Poor visual correspondence to function*
Functional Testing & Documentation	*Tests with at more than 3 different types of writing implements (e.g., pencils, pens, markers)• Tests different arrangements (10+ pencils).* *Provides clear photographs from multiple angles.* *Includes specific measurements of stability (e.g., tipping angle).* *Identifies at least 3 specific design strengths and weaknesses.* *Documents modifications with mathematical explanation*	*Tests with standard pencils in 3 arrangements.* *Provides at least one clear photograph.* *Includes some basic measurements.* *Identifies 1 or 2 design strengths and weaknesses.* *Suggests reasonable modifications.* *Links some modifications to mathematical properties*	*Tests with standard pencils in 2 arrangements.* *Limited or poor-quality documentation.* *Few or no measurements included.* *Identifies only obvious design issues.* *Suggests modifications without clear rationale.* *Minimal connection to mathematical properties*	*Basic testing with one type of pencil or no evidence of testing.* *No visual documentation• No measurements recorded.* *Design issues not identified.* *No modifications suggested.* *No connection to mathematical properties*

Mathematical & Physical Model Presentation	*Presentation shows side-by-side comparison of mathematical model and physical construction.* *Explicitly explains effect of each coefficient (a, b, c and d).* *Demonstrates how changing specific coefficients would affect specific design features* *Uses correct mathematical terminology throughout.* *Includes discussion of how polynomial degree impacts design possibilities.* *Considers practical constraints (e.g., stability, material use)*	*Presentation shows clear connection between mathematical and physical models.* *Explains effect of most coefficients.* *Demonstrates how changing one coefficient would affect the design.* *Uses some correct mathematical terminology.* *Considers at least one practical constraint*	*Basic presentation showing some connection between models.* *General explanation of coefficient effects without specifics.* *Limited discussion of how changes would affect design.* *Some mathematical terminology but with errors.* *Minimal consideration of practical constraints*	*Poor or missing presentation.* *No explanation of coefficient effects.* *No discussion of how changes would affect design.* *Incorrect mathematical terminology.* *No consideration of practical constraints*
Total Score	*20 marks*			

SECTION 7: STATISTICS

STRAND: HANDLING DATA

Sub-Strand: Organising, representing and interpreting data

Learning Outcomes

1. **Week 16:** *Model and solve problems using regression analysis*
2. **Week 17 & 18:** *Conduct research on a given situation, collect, organise, and represent the data graphically and explain the findings of the research using the graphs*

Content Standards

1. **Week 16:** *Demonstrate an understanding of relationship arising from cause and effect or contribution of one or more variables to another*
2. **Week 17 & 18:** *Apply techniques for collecting data, including census, sampling and observation in real life.*

INTRODUCTION AND SECTION SUMMARY

During lessons in Section 9 (Weeks 19 to 21) of Year One Additional Mathematics Teacher Manual, learners were engaged in activities to develop skills in collecting, representing and interpreting data. Learners considered the types of data representations (tabular and graphical forms) and the justifications for using which type of representation for a particular type of data. Learners were also introduced to measures of central tendencies and dispersion to provide them with the tools (statistics) to describe and make interpretation of data.

In Year Two of the Additional Mathematics Manual, lessons in Weeks 11 and 12 (under Section 7) were dedicated to guiding learners to analyse the relationship between data of variables. Spearman's rank correlation was introduced.

The lessons in the weeks of this section are prepared to introduce learners to modelling data with linear functions (where appropriate) and to guide learners to conduct a statistics project.

The weeks and the indicators covered by the section are:

- *Week 16*

1. *Describe the concept of regression and distinguish between correlation and regression and describe situations that call for the use of regression*
2. *Fit a linear function and use the line of best fit to solve problems in the context of the data*

- *Week 17 & 18: Undertake research to gather and analyse data using appropriate tools*

Recap weeks

Year 1	Week 19 1. Identify and present appropriate ways of collecting and representing data 2. Categorise data and determine which scale of measurement describes the data 3. Organise data into appropriate frequency distribution tables manually and with Microsoft Excel Week 20 1. Present data using appropriate graphs by hand and/or by technology and justify why a particular representation is more suitable than others for a given situation 2. Present grouped data using appropriate graphs by hand and/or by appropriate technology and justify why a particular representation is more suitable than others for a given situation 3. Compare various statistical representations and justify why a particular representation is more suitable than others for a given situation Week 21 1. Calculate measures of central tendencies (mode, mean and median) for a given data by formulas or other techniques and establish which is appropriate to report on a given data 2. Work out simple measures of dispersion (range, quartile and, inter-quartile etc) for raw data and interpret them in context.
Year 2	Week 11 1. Distinguish between univariate and bivariate data and give examples and explain the concept of correlation 2. Construct a scatter plot using given dataset and use it to describe the relationship between two variables Week 12 1. Analyse and describe visual data in a scatter plot by interpreting the relationship between given bivariate datasets 2. Describe the Spearman's rank correlation coefficient and interpret the result within a given situation

SUMMARY OF PEDAGOGICAL EXEMPLARS

The following pedagogies are suggested for teaching the lessons in this section to enhance understanding and exam preparation. Talk-for-learning fosters mathematical dialogue, allowing students to explain variable relationships, interpret lines of best fit, and justify conclusions. Experiential learning engages them in collecting real-life data, like height versus arm span and plotting points to identify patterns.

Problem-based learning challenges students to solve real-world issues using regression, enhancing critical thinking. Inquiry-based learning encourages exploration of how correlation strength affects prediction accuracy. Project-based learning allows students to gather data, apply regression analysis, interpret results, and link mathematics to real-life scenarios.

To support mixed-ability learners, teachers can provide structured guides, visual aids, and peer support while offering open-ended investigations for advanced students. In low-resource settings, alternatives like graph paper and chalkboard plots can replace technology. These approaches create an inclusive environment that develops statistical reasoning and readiness for exams.

ASSESSMENT SUMMARY

For the lessons in the week, the recommended mode of assessment is classwork. It is recommended that demonstration be used as a strategy for week 16 while presentation on statistics project be employed in week 17 and 18. It is expected that the tasks administered in this section will provide the opportunity for learners to review lessons from year one through to three.

Indicators	DOK Level	Assessment Strategy and/or Mode
Week 16		
Describe the concept of regression and distinguish between correlation and regression and describe situations that call for the use of regression	3	Mode: Classwork Strategy: Demonstration
Fit a linear function and use the line of best fit to solve problems in the context of the data	3	
Week 17 & 18		
Undertake research to gather data using appropriate tools	3	Mode: Classwork Strategy: Presentation on group statistics project assigned in week 3
Perform analysis on a data set using appropriate techniques and tools and interpret findings	3	

WEEK 16

Learning Indicators

1. *Describe the concept of regression and distinguish between correlation and regression and describe situations that call for the use of regression*
2. *Fit a linear function and use the line of best fit to solve problems in the context of the data*

During the lessons in Week 11 of the Year Two Additional Mathematics Manual, learners discussed the types of data, the scatter plot and lines of best fit. They also described and interpreted the types of relationship between bivariate datasets. Understanding the relationship between the data will help us draw conclusions of a given data.

FOCAL AREA 1: DESCRIBING THE CONCEPT OF REGRESSION

One of the aims of studying the relationship between bivariate data is to allow for predictions. For example, exploring the relationship between the scores obtained in a mid-semester exam and the scores in the final semester exam in mathematics, as we did in year two, allow us to predict the score a candidate will obtain in the final exam if we consider what was scored in the mid-semester exams.

The process of estimating the value of a dependent variable for a value of the independent variable is termed as regression.

As was discussed in year two, several functions can be used to represent the relation between bivariate data. The simplest of them is the linear function i.e., of the form, $y = mx + c$.

Simple linear regression is a statistical measurement used to describe the relationship between the independent variable and the dependent variables where change in the independent variable affects the dependent variable.

Table 16.1 shows the differences between correlation and regression.

Table 16.1: *Differences between correlation and regression.*

Correlation	Regression
Correlation is used to determine whether two variables relate in any manner or not.	Regression is used to numerically describe how a dependent variable changes with a change in an independent variable.
Correlation is used to tell the strength (strong, weak) and direction (positive, negative) of the relationship between the two variables.	Linear regression uses an equation to express the amount of effect (contribution) that one variable has on another in a relationship.
The variables can be used interchangeably.	The variables cannot be interchanged.

Linear regression is not suitable for every situation/relationship. To use linear regression to predict the values of variables it must be established that the relation is linear. The stronger the correlation between variables in a dataset, the more suitable the linear regression and the better the prediction will be.

Independent and dependent variables

Before we start fitting lines to data and relying on those lines for predictions, we will review independent and dependent data

1. The value of an independent variable does not change due to the impact of any other variable.
2. The dependent variable is the variable that changes due to a change in another variable.

Consider a situation where we want to determine what happens to students' test scores when students increase the number of hours they spend studying.

Independent variable: Hours students spent studying

Dependent variable: Test scores

FOCAL AREA 2: FITTING A LINEAR FUNCTION AND USING THE LINE OF BEST FIT TO SOLVE PROBLEMS IN THE CONTEXT OF THE DATA

If the relationship between two variables is linear, we can describe the relationship with the equation of a straight line. That linear equation is the linear model, or simple regression model, as we intend to use it for predictions. The greatest level of accuracy should be sought. The best estimates for decision making and planning are those estimates which produce the minimum error in any approximation.

It is possible to obtain several linear models to represent the relationship in a dataset. The best line is the line which yields the smallest (least) error margins when it used for predictions.

A line of best fit can be roughly determined using an eyeball method. This is done by drawing the straight line on a scatter plot that passes through, or at least, is closest to the greatest number of points. This is the **graphical method**.

Another way is to used numerical methods to determine the equation of the line of best fit. We will focus on the numerical method of *least squares*.

Method of least squares

Consider Figure 16.1 which shows a scatter plot and line drawn to model the relationship between two variables: x and y. The graph shows the actual points and the points obtained from tracing to the line.

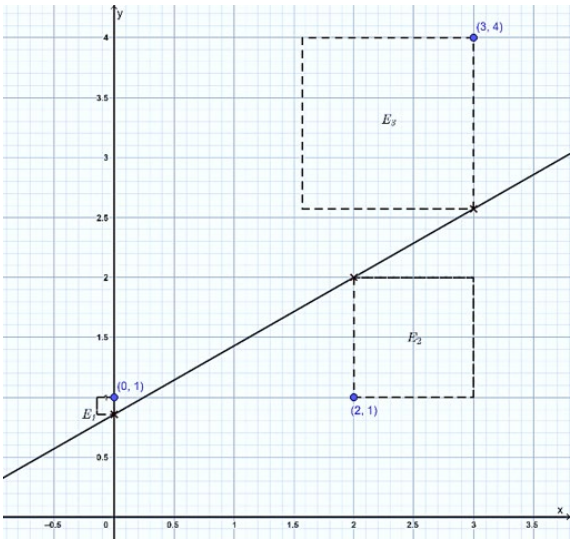


Figure 16.1: *Error squares from obtaining points from a fitted line*

From Figure 16.1, it can be observed that there are two different values of y for $x = 0, x = 2$ and $x = 3$. The observed value of y i.e., the values from the original data are $1, 1$ and 4 respectively. If the line in the graph is used for predicting the values of y at $x = 0, x = 2$ and $x = 3$, (if the line is regarded as the regression line), the results would be $0.85, 2$ and 2.6 respectively, as can be read from the graph.

The errors committed by the predictions, E_1, E_2 and E_3 produce squares as shown in Figure 16.1. The squares of the errors are as follows:

$(E_1)^2 = \big(0 - (-0.15)\big)(1 - 0.85)$,

$(E_2)^2 = (3 - 2)(2 - 1)$ and

$(E_3)^2 = (3 - 1.6)(4 - 2.6)$

Generally, the squared of an error committed by predicting / estimating, $E^2 = (\Delta x)(\Delta y)$

The aim of the least squares method is to obtain the line for which the total area of these squares is the smallest.

Recall that measures of central tendency (mean included) discussed in Year 1, were described as statistics which provide single values to represent data. It has been established (beyond the scope of this lesson) that the line which consists the point $(\overline{x}, \overline{y})$ where $\overline{x}$ is the mean of the x values while $\overline{y}$ is the mean of the y values, is the best line to fit a dataset.

Given a dataset with n points: (x_i, y_i) for $i = 1, 2, 3, \ldots, n$, a line with equation $y = mx + c$ can be found to model the data and predict other data points. Note that:

$$m = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

The value of c can be found by substituting $(\overline{x}, \overline{y})$ into the general linear equation.

Example 1

Use the least squares method to determine the equation of the line of best fit.

x	8	2	11	6	5	4	12	9	6	1
y	3	10	3	6	8	12	1	4	9	14

Solution

Let the equation of the line of best fit be $y = mx + c$

$Mean\ of\ x\ values, \overline{x} = \frac{8+2+11+6+5+4+12+9+6+1}{10} = 6.4$

$Mean\ of\ y\ values, \overline{y} = \frac{3+10+3+6+8+12+1+4+9+14}{10} = 7$

$m = \frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sum_{i=1}^{n}(x_i-\overline{x})^2}$

x	y	$(x_i - \overline{x})$	$(y_i - \overline{y})$	$(x_i - \overline{x})(y_i - \overline{y})$	$(x_i - \overline{x})^2$
8	3	1.6	-4	-6.4	2.56
2	10	-4.4	3	-13.2	19.36
11	3	4.6	-4	-18.4	21.16

6	6	-0.4	-1	0.4	0.16
5	8	-1.4	1	-1.4	1.97
4	12	-2.4	5	-12	5.76
12	1	5.6	-6	-33.6	31.36
9	4	2.6	-3	-7.8	6.76
6	9	-0.4	2	-0.8	0.16
1	14	-5.4	7	-37.8	29.16
				$\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})=-131$	$\sum_{i=1}^{n}(x_i-\bar{x})^2=118.41$

$$m=\frac{-131}{118.41}$$

$$=-1.1063$$

$$\approx -1.1$$

Since $(\bar{x},\bar{y})$ lies on the line of best fit, $\bar{y}=m\bar{x}+c$

$$\therefore 7=(-1.1063\times 6.4)+c$$

$$c=14.104$$

$$c\approx 14.1$$

Therefore, the equation of the best-fit line is $y=-1.1x+14.1$

Figure 16.2 shows the graph of $y=-1.1x+14.1$

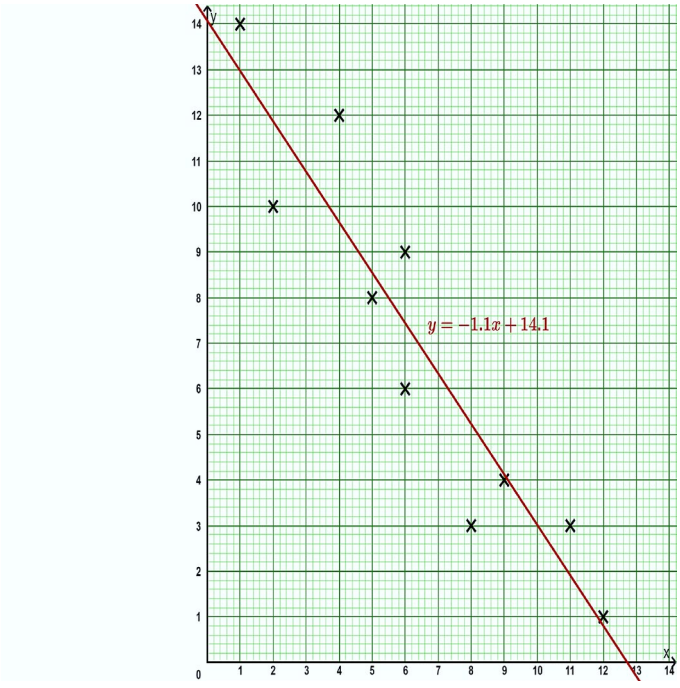


Figure 16.2: *Scatter plot for the data showing line of best-fit*

Example 2

Use least square approximation to fit a straight line to the data

x	1	2	3	4	5	6
y	120	90	60	70	35	11

Solution

Let the equation of the line of best fit be $y = mx + c$

$Mean\ of\ x\ values, \bar{x} = \frac{1+2+3+4+5+6}{6} = 3.5$

$Mean\ of\ y\ values, \bar{y} = \frac{120+90+60+70+35+11}{6} = 64.33$

$m = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	120	-2.5	55.67	-139.175	6.25
2	90	-1.5	25.67	-38.505	2.25
3	60	-0.5	-4.33	2.165	0.25
4	70	0.5	5.67	2.835	0.25
5	35	1.5	-29.33	-43.995	2.25
6	11	2.5	-53.33	-133.325	6.25
				$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = -350$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 17.5$

$m = \frac{-350}{17.5}$

$= -20$

Since $(\bar{x}, \bar{y})$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$\therefore 64.33 = (-20 \times 3.5) + c$

$c = 134.33$

Therefore, the equation of the best-fit line is $y = 134.33 - 20x$

Figure 16.3 shows the graph of $y = 134.33 - 20x$

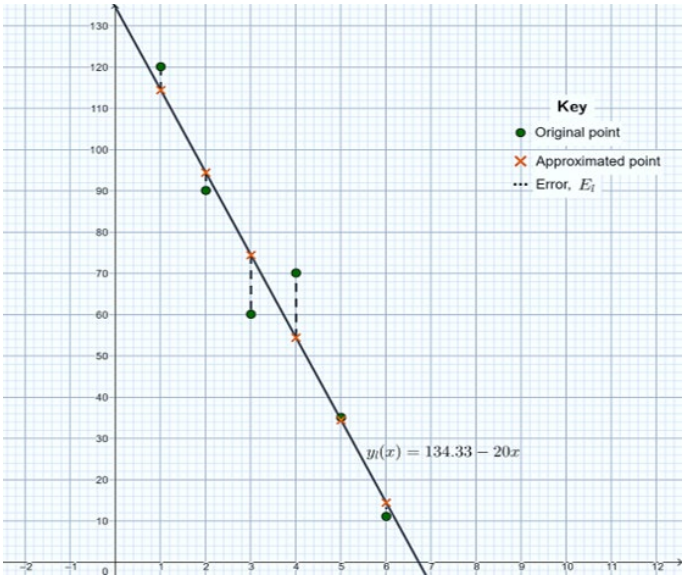


Figure 16.3: *Graph of* $y = 134.33 - 20x$

Example 3

A class is conducting an experiment on the effect of different watering frequencies on tomato plant growth. Seven identical tomato seedlings were planted in identical soil conditions and received the same amount of sunlight. The only variable changed was the number of days each plant was watered during the week.

After 4 weeks, students measured the growth (in centimetres) of each plant compared to its original height, represented by y and the number of days each plant was watered, represented by x as shown in the table.

x	4	3	0	1	8	5	3
y	5	3	-1	2	7	6	4

1. Construct a scatterplot for the data
2. What does the scatterplot suggest about the relationship between x and y?
3. Using the method of least squares, fit a linear model to the data.
4. If the margin of error allowed for predicting the height of a tomato seedling is ± 0.8 , would the model be appropriate for predicting the height if a seedling was watered for only four days?

Note

$y = -1$ for $x = 0$ indicates that the seedling which had no days of watering shrunk by a centimetre

Solution

1. The scatterplot is shown in Figure 16.4

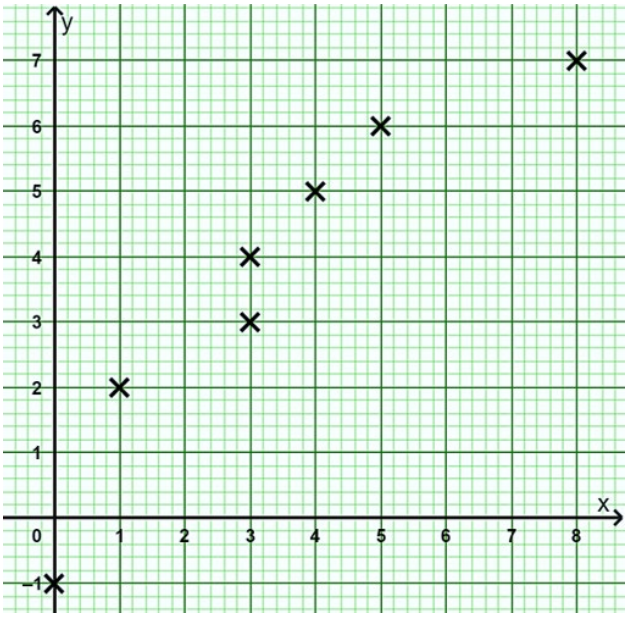


Figure 16.4: *Scatter for the data*

2. As the points on the scatterplot are observed from left to right, it can be noticed that as the values of x increase, the values of y increase. There is therefore a positive correlation between the values of x and y. We can predict that the linear model of best fit would have a positive gradient. If the number of days for watering a seedling increases, we expect an increase in the height

3. Let the linear model of best fit be $y = mx + c$

x	y	$(x_i - \overline{x})$	$(y_i - \overline{y})$	$(x_i - \overline{x})(y_i - \overline{y})$	$(x_i - \overline{x})^2$
0	-1	-3.43	-4.71	16.16	11.76
1	2	-2.43	-1.71	4.16	5.90
3	3	-0.43	-0.71	0.31	0.18
3	4	-0.43	0.29	-0.12	0.18
4	5	0.57	1.29	0.73	0.33
5	6	1.57	2.29	3.59	2.47
8	7	4.57	3.29	15.02	20.90
$\sum x = 24$	$\sum y = 26$			$\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = 39.86$	$\sum_{i=1}^{n}(x_i - \overline{x})^2 = 41.71$

$$Mean\ of\ x\ values, \overline{x} = \frac{\sum x}{7}$$
$$= \frac{24}{7}$$
$$= 3.43$$
$$\approx 3.4$$
$$Mean\ of\ y\ values, \overline{y} = \frac{\sum y}{7}$$
$$= \frac{26}{7}$$

$$= 3.71$$

$$\approx 3.7$$

$$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$m = \frac{39.86}{41.71}$$

$$= 0.9556$$

$$\approx 1$$

Since $(\bar{x}, \bar{y})$ lies on the line of best fit, $\bar{y} = m\bar{x} + c$

$$\therefore 3.71 = (0.9556 \times 3.43) + c$$

$$c = 0.4323$$

$$\approx 0.43$$

Therefore, the linear model of the best-fit $y = x + 0.43$

When $x = 0, y \approx 0.4$

Figure 16.5: *Graph of* $y(x) = x + 0.43$

4. From the data obtained from the actual measurement, the seedling which was watered for only four days was 5 centimetres.

   From the model, when a seedling is watered for only four days, its height should be approximately 4.4 centimetres i.e., from reading on the graph or substituting $x = 4$ into the model.

   The error committed in using the model is $5 - 4.4 = 0.6$.

   Since 0.6 falls between -0.8 and 0.8. the model is appropriate for the prediction.

Learning Tasks

1. Learners in groups discuss regression and distinguish between correlation and regression.
2. Learners brainstorm and share ideas on the situations that require the use of simple linear regression analysis.

3. Learners work in pairs to describe events where correlation and regression occur and distinguish between them in tabular form
4. Learners brainstorm to identify independent and dependent variables and describe examples or situations where both variables are used for regression analysis.
5. Learners in groups brainstorm to find the line of best fit for better accuracy using least square method
6. Learners work in groups to solve given examples and use the equation of best fit to estimate values within the domain and the range of the data.
7. Learners to model their own data and use the idea of line of best fit to arrive at estimated values.
8. Learners review each other's regression models using established criteria

Pedagogical Exemplars

1. **Talk-for learning**
    a. **Think-Pair-Share**
        i. Have learners discuss the difference between correlation and regression in pairs before sharing with the class
        ii. Present examples and non-examples of regression situations for learners to discuss and identify patterns.
    b. **Vocabulary Development:** Create a word wall with key terms (regression, correlation, dependent/independent variables) with visual representations
    c. Provide learners with prompts such as "What does the slope of the line mean here?" and "What happens if the data changes?" to aid them in interpreting the relationship in bivariate data.
    d. Allow alternative response formats (drawing, digital tools).
    e. Offer guided peer pairing with more confident learners.
    f. Use simplified language
2. **Experiential learning**
    a. Learners measure their height and arm span to explore the relationship between these variables
    b. Collect data on prices of goods over time in local markets to visualise regression concepts
    c. Learners calculate the mean point, draw the regression line through it using estimation, then check with technology
    d. Use structured worksheets with fill-in-the-blank sections.
    e. Allow extended time for learners with special needs
    f. Use physical props and tactile graph lines for learners with visual impairments.
    g. Allow oral explanation or use of pictorial expressions for learners with writing difficulties.
    h. Break tasks into simpler steps
3. **Problem-based learning**
    a. Present learners with raw data like farming yields and rainfall in Ghana and challenge them to determine the relationship
    b. Collaborate with other teachers to develop case studies analysing socio-economic data

c. Guide learners to apply regression analysis to local issues

d. Offer structured rubrics and guidance checklists

e. Use local contexts that are familiar to students from diverse backgrounds

f. Use spreadsheet software or graphing calculators to plot data points and fit lines for visual learners

4. **Enquiry-based learning**

a. Present learners with scatterplots without regression line and task them to use the eyeballing method to obtain a line of best fit

b. Task learners to make predictions based on fitted models and test their accuracy

c. Encourage learners to analyse the appropriateness of linear models for predictions beyond the dataset used to obtain the line of best fit

Key Assessment Level 3: Demonstration

1. Anthropologists use a linear model that relates femur (the bone that joins the hip to the knee) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analysing the data on femur length and height for the eight males given in the table

Table 16.2: *Femur lengths and body heights*

Femur length / cm	Height / cm
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

a. Make a scatter plot of the data. **(4 marks)**

b. Using the method of least square, find and graph (on the same grid) a linear function that models the data. **(16 marks)**

Total: 20 marks

DoK Level 3

2. **Task Description:** Examine three news articles that present statistical relationships between variables.

Task Components

a. For each article, identify whether the authors are describing correlation, implying causation, or using regression for prediction. Support your analysis with specific evidence from the texts. Include discussion of any statistical misconceptions present in the reporting. For each article, provide a detailed explanation of:

i. The variables being studied and their relationship

ii. The statistical claims being made (explicit or implicit)

iii. Whether the authors appropriately distinguish between correlation and causation

iv. How the reporting could be improved for statistical accuracy

(12 marks)

b. For one article where regression would be appropriate, sketch a possible scatter plot of the data and explain why regression analysis would add value beyond simple correlation. Include:

i. A clearly labelled scatter plot showing the potential relationship

ii. An explanation of what specific questions regression could answer that correlation analysis cannot

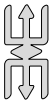
iii. A description of how regression results could be used for prediction in this specific context

iv. An assessment of what additional information would be needed to develop a reliable regression model

(8 marks)

Total: 20 marks

DoK Level 3



NOTE

- *Teachers should provide a variety of articles relevant to learners' context.*
- *The articles should be categorised in two different groups.*
- *One of the categories must have a minimum of three misconceptions in reporting in each article. The criterion for the grouping should note be shared with learners.*
- *Learners should select two articles with misconception and one from the other category.*

3. ***Task Description:*** *Our bodies respond to physical activity by changing heart rate. In this demonstration, you will investigate how walking speed affects heart rate, create a simple mathematical model, and apply it to everyday situations*

Task Components

a. *Data collection:*

i. *Work in pairs - one person walks while the other records data and ensures safety.*

ii. *Set up a straight walking path in the school yard or corridor with clear distance markers (e.g., chalk marks at 10m intervals).*

iii. *Measure heart rates using the simple method of counting pulse for 15 seconds and multiplying by 4.*

iv. *Collect data at 4 different walking speeds:*

1. *Very slow (casual stroll)*
2. *Normal walking pace*
3. *Brisk walking*
4. *Fast walking (but not running)*

i. *For each speed, record:*

1. *Time taken to walk 20 metres*
2. *Heart rate immediately after walking*

ii. *Create a clear data table showing:*

1. *Walking speed (calculated as metres per minute)*
2. *Heart rate (beats per minute)*

(4 marks)

b. *Modelling the data*

i. *Plot your data points on graph paper (walking speed on x-axis, heart rate on y-axis).*

ii. *Draw your "line of best fit" by hand, trying to have an equal number of points above and below your line.*

iii. *Choose two points on your line and calculate the slope using the rise/run method.*

iv. *Find the y-intercept by extending your line to where it crosses the y-axis.*

v. *Write your equation in the form* $H = mx + c$*, where:*

1. *H = heart rate in beats per minute*
2. *x = walking speed in metres per minute*
3. *m = slope (how much heart rate increases for each additional metre per minute)*
4. *c = y-intercept (estimated resting heart rate)*

(4 marks)

c. *Using the model*

i. *Use your equation to predict heart rate at a new walking speed (e.g., 70 metres per minute).*

ii. *Calculate what walking speed would give a heart rate of 100 BPM according to your model.*

iii. *Find the difference between actual and predicted heart rates for each of your data points.*

iv. *Explain in simple terms what the slope of your line tells you about how your body responds to faster walking.*

v. *Discuss one reason why someone might have a different slope than yours (e.g., fitness level, age).*

(4 marks)

d. *Practical applications*

i. *Create a colourful poster showing*

1. *Your graph with the line of best fit*
2. *Your equation written clearly*
3. *A simple heart rate zone chart showing:*
   a. *Resting zone (60-80 BPM)*
   b. *Light activity zone (80-100 BPM)*
   c. *Moderate activity zone (100-120 BPM)*
   d. *Vigorous activity zone (120+ BPM)*

   ii. *Mark on your graph what walking speeds would put you in each zone.*

   iii. *Explain how someone could use your findings to plan a walking exercise routine.*

   iv. *Suggest one way to make your model more accurate with the simple resources available.*

   v. *Describe how your findings connect to what you've learned in science about the circulatory system.*

(4 marks)

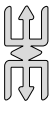
Total: 16 marks

DoK Level 3

PLC Session 16

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



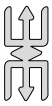
Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***demonstration***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the demonstration (NTS 3k-3q).

10 minutes Content Exploration

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes Enactment

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes Reflection

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEKS 17 & 18

Learning Indicators

1. *Undertake research to gather data using appropriate tools*
2. *Perform analysis on a data set using appropriate techniques and tools and interpret findings*

FOCAL AREA: UNDERTAKE RESEARCH TO GATHER AND ANALYSE DATA USING APPROPRIATE TOOLS

Learners should be guided to conduct a mini project to collect and represent data. A sample research plan has been prepared to be adopted or adapted to suit learners' environments and prevailing situations.

Note that learners should have been assigned to groups by week 3. Research topics should have been selected and data collection and represented already started as prompted in week 3 in this manual.

Suggested Research Topics

1. Study Habits & Academic Performance: Hours spent studying per week vs. Mathematics test scores
2. Physical Activity & Health: Weekly exercise hours vs. Resting heart rate (or BMI)
3. Reading Habits & Language Performance: Books read per month vs. English test scores
4. Transportation & Punctuality: Distance from home to school vs. Number of late arrivals per term
5. Extracurricular Participation & Social Skills: Number of clubs/activities joined vs. Leadership roles held

Week 17: Statistical Analysis & Regression Application

Days 1–2: Complete Statistical Calculations

- Finalise measures of central tendency and dispersion from collected data
- Verify all calculations and create summary tables
- Prepare data for regression analysis

Days 3–4: Apply Regression Analysis

- Plot scatter diagrams using their collected data
- Calculate correlation coefficient to determine relationship strength
- Find equation of line of best fit using regression techniques learned in Week 16
- Make predictions using their regression equation

Day 5: Analysis & Interpretation

- Interpret correlation results and regression equation
- Draw conclusions about their research question

- Identify limitations in their data/methodology

Week 18: Final Report & Presentations

Days 1-2: Report Compilation

- Organise all work: data tables, graphs, calculations, analysis
- Write conclusions and recommendations
- Format final project report

Days 3-5: Group Presentations

- 10-15 minute presentations per group
- Display charts, graphs, and regression analysis
- Answer questions from classmates and teacher

Week 17 Deliverables

- Complete statistical summary (all measures calculated)
- Scatter plot with line of best fit
- Regression equation and correlation coefficient
- Written interpretation of results

Week 18 Deliverables

- Final project report (including all phases from Week 3 onwards)
- Group presentation
- Individual reflection on learning process

	Levels of Performance			
Criteria	**Excellent (4 marks)**	**Very Good (3 marks)**	**Satisfactory (2 marks)**	**Needs improvement (1 mark)**
1. Research Design & Planning	Research question is clearly defined, measurable, and original. Comprehensive methodology with detailed data collection plan. Ethical considerations fully addressed with consent procedures. Sample size justified (30+ participants).	Research question is clear and measurable with good methodology. Data collection plan is well-structured. Basic ethical considerations addressed. Adequate sample size (20-29 participants)	Research question is present but may lack clarity. Basic methodology with some gaps in planning. Limited ethical considerations. Smaller sample size (15-19 participants).	Research question is unclear or poorly defined. Weak or missing methodology. No ethical considerations mentioned. Very limited sample size (under 15 participants)

2. Data Collection & Quality	Data collected systematically using appropriate instruments. High data quality with minimal missing values. Multiple data collection methods used effectively. Clear documentation of collection process with dates and sources.	Data collected using suitable methods with good quality. Few missing values handled appropriately. Good documentation of collection process. Sources clearly identified.	Data collected adequately but with some quality issues. Some missing values present. Basic documentation of collection process. Limited source identification.	Poor data collection with significant quality issues. Many missing values or unreliable data. Minimal documentation. Sources unclear or missing.
3. Data Organisation & Presentation	All data tables are professionally formatted, clearly labelled with appropriate headings. Frequency tables show grouped and ungrouped data correctly. Raw data organised systematically with proper coding.	Data tables are well-formatted and clearly labelled. Frequency tables mostly correct with minor formatting issues. Raw data reasonably organised.	Basic data tables with adequate labelling. Some errors in frequency table construction. Raw data organisation needs improvement.	Poorly formatted tables with unclear labelling. Significant errors in frequency tables. Raw data poorly organised or missing.
4. Graphical Representation	Excellent variety of appropriate graphs (histograms, scatter plots, box-and-whisker plots). Perfect scaling, labelling, and professional presentation. Creative use of colour and design enhances understanding.	Good variety of appropriate graphs with accurate construction. Proper scaling and labelling with minor presentation issues. Good visual appeal.	Adequate graphs with basic construction. Some scaling or labelling errors. Limited variety or creativity in presentation.	Limited or inappropriate graphs. Poor scaling, labelling, or construction. Unprofessional presentation that hinders understanding.
5. Statistical Calculations - Central Tendency	All measures calculated correctly for both variables with clear working shown. Appropriate selection of best measure for each dataset with detailed justification. Correct interpretation in context of research.	Most measures calculated correctly with good working. Reasonable selection of appropriate measures with basic justification. Generally correct interpretations.	Some measures calculated correctly with basic working shown. Limited justification for measure selection. Some interpretation errors.	Few measures calculated correctly. Poor or missing working. No justification for measure selection. Significant interpretation errors.

6. Statistical Calculations - Dispersion	All dispersion measures calculated accurately with clear working. Excellent understanding of data spread demonstrated through detailed interpretations. Standard deviation calculated and explained correctly.	Most dispersion measures calculated correctly with good working. Good understanding of data spread with reasonable interpretations. Minor errors in standard deviation.	Some dispersion measures calculated with basic working. Limited understanding of data spread. Errors in calculations or interpretations.	Few dispersion measures calculated correctly. Poor understanding of data spread. Significant calculation errors or missing working.
7. Correlation Analysis	Correlation coefficient calculated correctly using appropriate method. Excellent interpretation of strength and direction of relationship. Clear understanding of correlation vs. causation. Proper use of statistical language.	Correlation coefficient mostly correct with good interpretation of relationship. Generally, understands correlation concepts with minor misconceptions.	Basic correlation analysis attempted with some correct calculations. Limited understanding of relationship interpretation. Some confusion between correlation and causation.	Poor or incorrect correlation analysis. Significant errors in calculations. Little understanding of correlation concepts or interpretations.
8. Regression Analysis	Regression equation derived correctly using proper method. Line accurately plotted on scatter diagram. Meaningful predictions made with confidence intervals considered. Excellent understanding of regression limitations.	Regression equation mostly correct with line reasonably placed. Some predictions attempted with basic accuracy. Good understanding of regression concepts.	Basic regression analysis with approximate equation and line placement. Limited predictions made with some accuracy issues. Basic understanding of regression.	Poor or missing regression analysis. Incorrect equation or line placement. No meaningful predictions made. Little understanding of regression concepts.
9. Analysis & Interpretation	Insightful, comprehensive analysis linking all statistical results to research question. Clear, evidence-based conclusions with acknowledgment of limitations. Thoughtful discussion of outliers and data patterns. Recommendations are practical and well-supported.	Good analysis connecting most results to research question. Reasonable conclusions drawn with some limitations noted. Basic discussion of data patterns. Practical recommendations provided.	Adequate analysis with basic connections to research question. Simple conclusions with few limitations acknowledged. Limited discussion of patterns. Basic recommendations given.	Limited analysis with weak connections to research. Unclear or unsupported conclusions. No limitations discussed. Minimal or inappropriate recommendations.

10. Mathematical Communication	Excellent written communication using precise mathematical vocabulary. Report is well-organized with logical flow. Perfect integration of statistical results with explanatory text. Professional formatting and presentation.	Good written communication with appropriate mathematical language. Well-organized report with clear structure. Good integration of results with text. Professional appearance.	Adequate communication with basic mathematical language. Report is organized but may lack clear structure. Some integration of results with text. Acceptable presentation.	Poor written communication with incorrect or limited mathematical language. Poorly organized report with unclear structure. Little integration of results. Unprofessional presentation.
Total Score	40 marks			

Learning Task

Learners in their small heterogeneous groups collaborate to complete mini project

Pedagogical Exemplars

Project-Based learning

1. Facilitate the creation of small heterogeneous learner groups for the completion of the mini project
2. Provide guidance and assistance through all the stages of the project
3. Promote the use of computer devices for calculations, presentations and to aid learners with disabilities

Key Assessment – Practice Paper 2

Classwork

The accompanying table gives the peak power load for a power plant and the daily high temperature for a random sample of 10 days.

High Temperature / °F	Peak load
95	214
82	152
80	156
81	129
99	254
100	266
93	210
95	204

93	213
87	150

1. Draw the scatter plot for the data. **(4 marks)**
2. Compute the value of Spearman's rank correlation coefficient r_s. **(7 marks)**
3. Using the method of least squares, find the equation of the regression line. **(17 marks)**
4. Considering the value of r_s obtained, is it appropriate to model the data with a linear equation as obtained? Why? **(2 marks)**

Total: 30 marks

HINT

*Conduct **Practice Paper 2**. It should cover the contents taught from Year 1 to 3. Refer to the **Table of Specification and Structure of the Paper** at the end of this section. Learners' papers should be marked immediately and scores entered into the SAP.*

PRACTICE PAPER 2

Tables of Specification

Test Specification (MCQ)

STRAND	SUB-STRAND	DoK LEVELS			TOTAL
		1	2	3	
Modelling with Algebra	1. Number and algebraic patterns 2. Application of Algebra	5	6	5	16
Geometric Reasoning and Measurement	1. Spatial Sense 2. Measurement of triangles	2	4	2	8
Calculus	1. Principles of calculus 2. Application of calculus	2	4	2	8
Handling Data	1. Organising, representing and interpreting data 2. Making predictions with data	2	4	2	8
TOTAL		11 (27.5%)	18 (45.0%)	11 (27.5%)	40 (100.0%)

MCQ Test Specification (with indicators)

LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Determine the identity element and use it to find the inverse of a given element.	1			
Establish the operations on sets, including commutative, associative and distributive laws, set algebra and apply them to solve problems.		1		
Establish the relationship between indices and logarithm and use the properties to solve related problems in one base.		1		
Determine the composite of two given functions.	1			
Carry out the basic arithmetic operations on rational functions.			1	
Use recursive and explicit formulae of sequences to model situations and translate between the two forms.			1	
Determine the arithmetic and geometric means of linear and exponential sequences and apply them to solve real life problems.	1			
Solve quadratic inequalities involving real life problems.		1		
Find the factors and zeros of a polynomial function using conventional and personal strategy.		1		
Resolve rational functions into partial fractions, specifically quadratic denominators, and improper rational functions.	1			
Multiply an m × n matrix by an n × 1 matrix.	1			
Model and solve problems based on real life situations using matrices.			1	
Draw implications from given statements and their converses.			1	
Find the composition of linear transformations.		1		
Apply linear transformation in finding translation, reflections, rotations and enlargement of points and plane figures.		1	1	16
Use standard algebraic manipulations to find the equation of parallel and perpendicular lines, including the equation of the perpendicular bisector of a line.		1		
Deduce relations of various loci under given conditions.			1	
Derive the rule for scalar (dot) product and use it to solve problems relating to angles between two vectors.	1			

Deduce the directrix and focus from a parabolic equation and vice versa.		1		
Derive the sine and cosine rules and apply them to solve problems.		1		
Use correct algebraic techniques to isolate the trigonometric functions and find the values for the angle that makes the equation true.			1	
Identify the maximum and minimum values of the trigonometric graphs.	1			
Find the point of intersection for a line and a parabola.		1		8
Describe and interpret the meaning of limit of a function through graphical and algebraic approaches.		1		
Distinguish between continuous and discontinuous functions near an input value on its domain and investigate them with the use of technology or any other appropriate means.			1	
Apply the product and quotient rules to differentiate functions.		1		
Generalise the behaviour with respect to the slope of a moving object along a curve.	1			
Identify and apply appropriate techniques of integration to solve transcendental functions.	1			
Apply differentiation to find the rate of change.		1		
Use the second derivative of a function to classify the maximum, minimum and saddle point of that function and perform curve sketching.			1	
Find the volume of a solid formed after rotation about horizontal or vertical axis.		1		8
Categorise data and determine which scale of measurement describes the data.		1		
Work out simple measures of dispersion (range, quartile and inter-quartile, etc.) for raw data and interpret them in context.			1	
Describe the Spearman's Rank correlation coefficient and interpret the results within a given situation.	1			
Fit a linear function to given data and use the line of best fit to solve problems in the context of the data.		1		
Recall the basic terminologies such as experiment, events, outcome, trial, sample space, etc. as used in probability and give examples.	1			

Work collaboratively on an experiment (e.g. tossing coins or dice) to determine the relative frequencies of events and interpret them.		1		
Create and solve problems using the axioms and laws of probability.		1		
Model and solve real life problems involving conditional probabilities.			1	8
TOTAL	11 (27.5%)	18 (45.0%)	11 (27.5%)	40 (100.0%)

PAPER 2

Section A

Test Specification

STRAND	SUB-STRAND	DoK LEVELS			TOTAL	
		1	2	3		
Modelling with Algebra	Number and Algebraic Patterns	1			1	3
	Application of Algebra	1	1		2	
Geometric Reasoning and Measurement, and Calculus	Spatial Reasoning		1		1	3
	Measurement of Angles	1			1	
	Principles of Calculus				0	
	Application of Calculus		1		1	
Handling Data	Organising, Representing and Interpreting Data		1		1	2
	Making Prediction with Data		1		1	
TOTAL		3 (37.5%)	5 (62.5%)	0	8 (100%)	8 (100%)

Test Specification (with indicators)

LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion.	1			
Find the inverse of simple functions including one-to-one functions.	1			
Find the equation of the image of a line under linear transformation.		1		3

Determine the acute angles between two intersecting lines with the aid of technological tools e.g. GeoGebra.		1		
Determine radian measure and apply the knowledge to solve practical problems of arc length.	1			
Find the area under a constant function, curves and distinguish between total, net and signed area.		1		3
Describe the Spearman's Rank correlation coefficient and interpret the results within a given situation.		1		
Model and solve real life problems involving binomial probability.		1		2
TOTAL	3 (37.5%)	5 (62.5%)	0	8 (100%)

Section B

Test Specification

STRAND	SUB-STRAND	DOK LEVELS			TOTAL	
		1	2	3		
Modelling with Algebra	Number and Algebra		1		1	3
	Application of Algebra			2	2	
Geometric Reasoning and Measurement, and Calculus	Spatial Reasoning			1	1	2
	Measurement of Angles				0	
	Principles of Calculus			1	1	
	Application of Calculus				0	
Handling Data	Organising, Representing and Interpreting Data		1		1	2
	Making Prediction with Data			1	1	
TOTAL		0	3 (42.9%)	4 (57.1%)	7 (100%)	7 (100%)

Test Specification (With indicators)

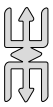
LEARNING INDICATORS	DoK 1	DoK 2	DoK 3	TOTAL
Determine the identity element and use it to find the inverse of a given element.		1		
Solve real life problems involving linear inequalities.			1	
Describe set theory as a foundation for many subfields of mathematics, and create and model set problems in areas pertaining to industry, commerce, sports, etc.			1	3
Use directrix and focus to sketch a parabola.		1		

Solve problems involving integrals with no exact antiderivative (Trapezium rule).			1	2
Present grouped data using appropriate graphs by hand and/or by appropriate technology and justify why a particular representation is more suitable than others for a given situation.		1		
Describe the number of ways objects can be arranged using fundamental counting principles.			1	2
TOTAL	0	3 (42.9%)	4 (57.1%)	7 (100%)

PLC Session 17 & 18

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous session, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand.
   b. identify three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q).
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Use your app to plan and administer the mode of assessment for the Student Assessment Portal (SAP).
   a. The key assessment for the week is ***practice paper 2***. Chat with your app to plan this assessment by reviewing or developing the assessment items/tasks and marking schemes/rubrics for the practice paper (NTS 3k-3q).

10 minutes **Content Exploration**

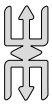
5. As a subject group, select one problem or topic/idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then

a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)

b. all teachers in the group work through the problem individually for 3–4 minutes

c. the group then works through it together the problem, step by step

d. highlight the most common errors made at each step (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 7 Review

This section discussed all the lessons to be taught in weeks sixteen to eighteen. It focused on regression and the completion of a mini project on statistics. Learning and assessment tasks for this section emphasised the development of skills and competencies required for applying statistical measures and procedures to real life data. Assessment tasks which

required collaboration among learners and provide the opportunity for learners to review learning from year one to three were suggested to reinforce learning and help them prepare for their final examination. It is intended that leaners will get the opportunity to revise lessons taught in the previous years as they work through the learning and assessment tasks. Teachers are strongly encouraged to foster inclusive learning environments for all learners.

MARKING SCHEMES AND RUBRICS FOR THE KEY ASSESSMENTS

WEEK 16

a. *Scatterplot* ***(4 marks)***

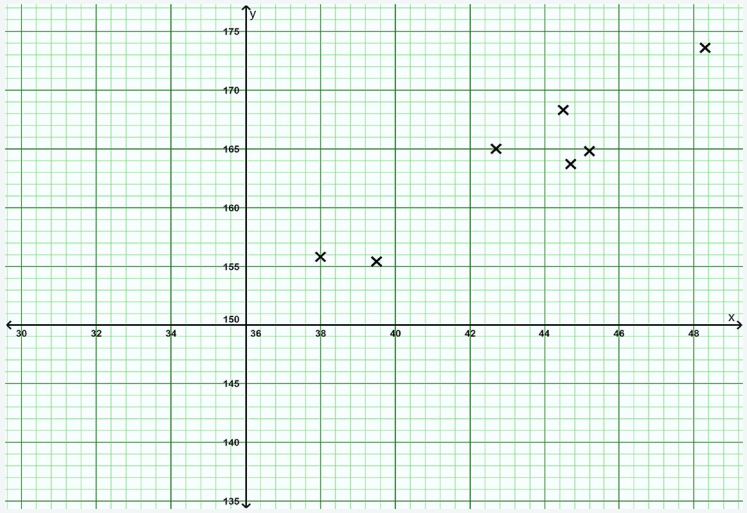


Figure 16.5 : *Scatter for the data*

Axes with correct labelling: (Max **B1**, 0.5 for each axis)

Appropriate scale used on axes: (Max **B1**, 0.5 for each axis)

Correct plotting of points: (Max **B2**, 0.25 for each correct point)

b. $\overline{x} = \frac{\sum x}{8} = \frac{353}{8}$

$= 44.13 \approx 44.1$ ***B1***

$\overline{y} = \frac{\sum y}{8} = \frac{1325.10}{8}$

$= 165.64 \approx 165.6$ ***B1***

Table of values (Max 10 marks)

x	y	$(x_i - \overline{x})$	$(y_i - \overline{y})$	$(x_i - \overline{x})(y_i - \overline{y})$	$(x_i - \overline{x})^2$
50.1	178.5	5.98	12.86	76.85	35.70
48.3	173.6	4.18	7.96	33.24	17.43
45.2	164.8	1.08	-0.84	-0.90	1.16
44.7	163.7	0.58	-1.94	-1.11	0.33
44.5	168.3	0.38	2.66	1.00	0.14
42.7	165.0	-1.43	-0.64	0.91	2.03
39.5	155.4	-4.63	-10.24	47.35	21.39

38.0	155.8	-6.13	-9.84	60.25	37.52
$\sum x = 353$	$\sum y = 1325.1$			$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 217.59$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 115.70$

Correct $(x_i - \bar{x})$ *values:* (Max **B2**, 0.25 for each value)

Correct $(y_i - \bar{y})$ *values:* (Max **B2**, 0.25 for each value)

Correct $(x_i - \bar{x})(y_i - \bar{y})$ *values:* (Max **B2**, 0.25 for each value)

Correct $(x_i - \bar{x})^2$ *values:* (Max **B2**, 0.25 for each value)

$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 217.59$ **B1**

$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 115.70$ **B1**

Let the linear model of best fit be $y = mx + c$

$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

$m = \frac{217.59}{115.70}$

$= 1.8806$ ***B1***

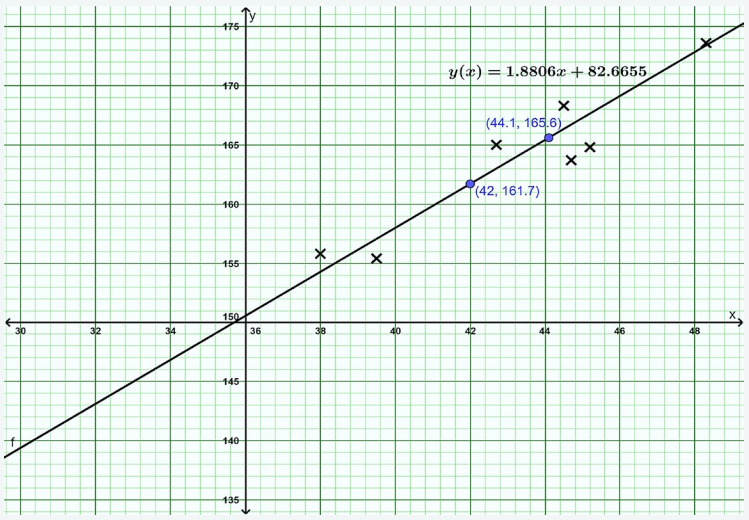
Since $(\bar{x}, \bar{y})$ *lies on the line of best fit,* $\bar{y} = m\bar{x} + c$

$\therefore 165.6 = (1.8806 \times 44.1) + c$

$c = 82.6655$ ***B1***

Therefore, the linear model of the best-fit $y(x) = 1.8806x + 82.6655$ ***A1***

When $x = 42, y \approx 161.7$



Line drawn through $(\bar{x}, \bar{y})$ *and another point obtained from the model* ***B1***

	Levels of Performance			
Criteria	*Excellent* *(4 marks)*	*Very Good* *(3 marks)*	*Satisfactory* *(2 marks)*	*Needs improvement* *(1 mark)*
Article 1 & 2 Analyses *(Articles with misconceptions)*	*Correctly identifies the exact statistical relationship. Includes 2 or more specific quotations as evidence. Identifies 3 or more specific misconceptions in the reporting. Uses correct statistical terminology throughout.*	*Correctly identifies the statistical relationship. Includes 1 specific quotation as evidence. Identifies 2 misconceptions. Uses mostly correct terminology.*	*Partially identifies the statistical relationship. Includes general references rather than specific quotations. Identifies 1 basic misconception. Uses some correct terminology.*	*Incorrectly identifies the relationship. Provides no direct quotations. Fails to identify misconceptions. Uses incorrect terminology.*
Article 3 Analysis *(Article without misconceptions)*	*Correctly identifies the exact statistical relationship. Includes 2 or more specific quotations as evidence. Justifies that there are no misconceptions in reporting with 2 or more quotations. Uses correct statistical terminology throughout.*	*Correctly identifies the statistical relationship. Includes 1 specific quotation as evidence of relationship. Justifies that there are no misconceptions in reporting with 1 quotation. Uses mostly correct terminology.*	*Partially identifies the statistical relationship. Includes general references rather than specific quotations.* *Provides a general justification of the absence of misconceptions in reporting with no quotations*	*Incorrectly identifies the relationship. Provides no direct quotations. Suggests misconception in reporting. Uses incorrect terminology.*
Further Analysis and Recommendations	*Clearly explains whether regression is needed or misused. Clearly explains whether correlation is mistaken for causation with specific examples. Identifies more than 2 flaws in the reporting. Lists 2 or more specific improvements for the reporting.*	*Explains whether regression is appropriate. Explains correlation/ causation distinction. Identifies 2 flaws in reporting. Lists at least 1 specific improvement.*	*Makes basic assessment of regression appropriateness. Makes basic distinction between correlation/ causation. Identifies 1 obvious flaw. Suggests general improvements.*	*Misunderstands regression application. Confuses correlation and causation. Identifies no reporting flaws. Suggests no improvements.*

Scatter Plot Creation	*Creates plot with all required labels (title, x-axis, y-axis, units). Points clearly show the described relationship. Plot includes all points. Scale is appropriate and consistent.*	*Creates plot with most required labels. Points show a relevant relationship. Plot includes a minimum of 80% of the data points. Scale is mostly appropriate.*	*Creates plot with some labels missing. Points show a basic relationship. Plot includes a minimum of 50% of the data points. Scale has minor issues.*	*Creates plot missing most labels. Points show incorrect or no clear relationship. Plot has fewer than 50% of the data points. Scale is inappropriate.*
Regression Value Explanation	*Lists 3 or more specific questions regression could answer that correlation cannot. Correctly explains how regression enables prediction. Identifies 3 or more additional data needs for reliable modelling. Uses correct statistical terms.*	*Lists 2 specific questions regression could answer. Explains prediction capability with minor errors. Identifies 1 or 2 additional data needs. Uses mostly correct terms.*	*Lists 1 basic question regression could answer. Provides simple explanation of prediction. Identifies 1 basic data need. Uses some correct terms.*	*Lists no specific questions regression could answer. Provides incorrect explanation of prediction. Identifies no data needs. Uses incorrect terms.*
Total Score	*20 marks*			

3.

	Levels of Performance			
Criteria	**Excellent (4 marks)**	**Very Good (3 marks)**	**Satisfactory (3 marks)**	**Needs improvement (1 mark)**
a. Simple Data Collection	*Collects data for all 4 walking speeds. Records both time and heart rate accurately. Calculates walking speeds correctly in metres/minute. Creates clear, organised data table with proper units. Shows evidence of repeated measurements or careful technique. Follows all safety procedures*	*Collects data for 3 walking speeds. Records measurements with less than 3 errors. Calculates speeds with less than 3 errors. Creates complete data table with some organisational issues. Follows basic safety procedures*	*Collects data for 2 walking speeds. Records measurements with 3–5 errors. Makes some calculation errors. Creates basic data table with missing labels or units. Follows most safety procedures*	*Collects data for 1 speed. Records measurements with more than 5 errors or provides incomplete records. Makes major calculation errors. Creates disorganised or incomplete data table. Disregards safety procedures*

b. Modelling the data	*Plots all points accurately on appropriately scaled graph. Draws line of best fit with approximately equal points above/ below. Calculates slope correctly. Accurately identifies y-intercept Writes correct equation in form* $H = mx + c$. *Labels all parts of graph clearly*	*Plots not less than 80% of points accurately. Draws reasonable line of best fit. Calculates slope with minor errors. Identifies y-intercept with slight inaccuracy. Writes equation with correct format but minor errors. Labels most parts of graph*	*Plots 50% to 80% points accurately. Draws line of best fit that doesn't balance points well. Makes errors in slope calculation. Approximates y-intercept poorly. Writes equation with format errors. Labels some parts of graph*	*Plots less than 50% of points accurately. Draws inappropriate line of best fit. Makes major errors in slope calculation. Fails to identify y-intercept. Unable to write equation correctly. Missing most graph labels*
c. Using the model	*Correctly predicts a new heart rate using equation. Accurately calculates walking speed for target heart rate. Finds differences between actual and predicted values for all points. Clearly explains meaning of slope in physiological terms. Provides thoughtful discussion of individual differences. Shows all calculations clearly*	*Predicts new heart rate with minor calculation errors. Calculates walking speed for target heart rate with slight errors. Finds differences for most data points. Explains basic meaning of slope. Mentions at least one individual difference factor. Shows most calculations*	*Attempts prediction with significant errors. Attempts to calculate walking speed with errors. Finds differences for some data points. Provides limited explanation of slope. Mentions individual differences vaguely. Shows some calculations*	*Unable to make predictions. Unable to calculate walking speed for target heart rate. Doesn't calculate differences. Misinterprets meaning of slope. No mention of individual differences. Missing calculations*
d. Practical Applications	*Creates clear, colourful poster with all required elements (line of best fit, equation of line of best fit and zone chart). Graph includes accurate line of best fit and equation. Heart rate zone chart is complete with 4 zones clearly marked. Walking speeds correctly marked for each zone. Provides specific, practical exercise application. Suggests feasible improvement to model. Makes clear connection to circulatory system*	*Creates neat poster with 2 of the required elements required elements. Graph includes line of best fit and equation. Heart rate zone chart shows all zones. Most walking speeds marked for zones. Provides general exercise application. Suggests basic improvement to model. Makes general connection to circulatory system*	*Creates basic poster with one required element. Graph includes line with some errors. Heart rate zone chart incomplete. Some walking speeds marked incorrectly. Provides vague exercise application. Suggests impractical improvement. Makes minimal connection to circulatory system*	*Creates incomplete or messy poster. Graph missing or severely flawed. Heart rate zone chart missing or incorrect. Walking speeds not marked for zones. No exercise application provided. No improvement suggested. No connection to circulatory system*
Total Score	*16 marks*			

WEEK 17 & 18

1. *Scatterplot* ***(4 marks)***

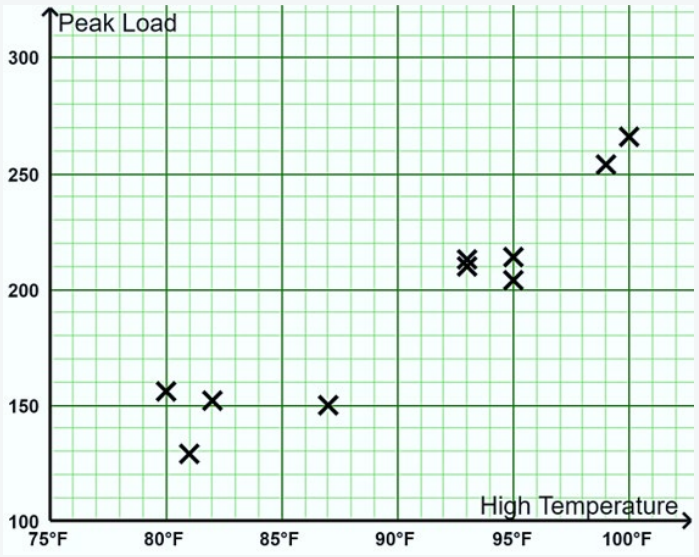


Figure 17.1

Axes with correct labelling: (Max **B1**, 0.5 for each axis)

Appropriate scale used on axes: (Max **B1**, 0.5 for each axis)

Correct plotting of points: (Max **B2**, 0.2 for each correct point)

2. *Let High temperature be* x *and Peak load be* y

95	3	214	3	0	0
82	8	152	8	0	0
80	10	156	7	3	9
81	9	129	10	-1	1
99	2	254	2	0	0
100	1	266	1	0	0
93	5	210	5	0	0
95	3	204	6	-3	9
93	5	213	4	1	1
87	7	150	9	-2	4

Correct R_x *values:* (Max **B1**, 0.1 for each value)

Correct R_y *values:* (Max **B1**, 0.1 for each value)

Correct d values: (Max **B1**, 0.1 for each value)

Correct d^2 values: (Max **B1**, 0.1 for each value)

$\Sigma d^2 = 24$ ***A1***

$r_s = 1 - \frac{6(\sum d^2)}{n(n^2-1)}$

$= 1 - \frac{6(24)}{10(10^2-1)}$ ***M1***

$= 0.8545$ ***A1***

4. *12 marks for table*

Let the regression line be $y = mx + c$

Mean of x values, $\bar{x} = \frac{\Sigma x}{10} = \frac{905}{10} = 90.5$ **B1**

Mean of y values, $\bar{y} = \frac{\Sigma y}{10} = \frac{1948}{10} = 194.8$ **B1**

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
95	214	4.5	19.2	86.4	20.25
82	152	-8.5	-42.8	363.8	72.25
80	156	-10.5	-38.8	407.4	110.25
81	129	-9.5	-65.8	625.1	90.25
99	254	8.5	59.2	503.2	72.25
100	266	9.5	71.2	676.4	90.25
93	210	2.5	15.2	38	6.25
95	204	4.5	9.2	41.4	20.25
93	213	2.5	18.2	45.5	6.25
87	150	-3.5	-44.8	156.8	12.25
$\sum x = 905$	$\sum y = 1948$			$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 2944$	$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 500.5$

Correct $(x_i - \bar{x})$ values: (Max **B2.5**, 0.25 for each value)

Correct $(y_i - \bar{y})$ values: (Max **B2.5**, 0.25 for each value)

Correct $(x_i - \bar{x})(y_i - \bar{y})$ values: (Max **B2.5**, 0.25 for each value)

Correct $(x_i - \bar{x})^2$ values: (Max **B2.5**, 0.25 for each value)

$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 2944$ **B1**

$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 500.5$ **B1**

$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

$m = \frac{2944}{500.5}$

$= 5.88$ ***A1***

Since $(\bar{x}, \bar{y})$ *lies on the line of best fit,* $\bar{y} = m\bar{x} + c$

$\therefore 194.8 = (5.88 \times 90.5) + c$

$c = -337.34$ ***A1***

Therefore, the linear model of the best-fit $y = 5.88x - 337.34$ ***A1***

5. *The value of* r_s*, 0.85 shows a strong positive relationship between high temperature and peak load. The high value (close to 1), confirms the closeness of the points to the line of best fit. It is therefore appropriate to use a straight line to model the data* ***2 marks (1 for partial explanation)***

SECTION 8: PROBABILITY

STRAND: HANDLING DATA

Sub-Strand: Making Prediction

Learning Outcomes

1. **Week 19 & 20:** *Solve problems involving conditional probability using permutations and combinations.*
2. **Week 21:** *Use the concept of permutation and combination to solve real-life problems.*

Content Standard: *Apply and extend the knowledge of the laws of probability and the concept of combination and permutation to solve real-life problems involving conditional and binomial probability*

INTRODUCTION AND SECTION SUMMARY

Probability and counting are essential for solving real-world problems. This section introduces conditional probability and explains how the likelihood of an event can change based on prior outcomes. Learners also explore equally likely events and their probabilities. By applying binomial and conditional probabilities, learners model situations involving risk, predictions, and informed decisions. Permutations and combinations help in counting and arranging objects or people. These are useful in areas like school leadership selection, business planning, and industrial processes. Learners will solve problems and identify practical uses of these concepts in everyday life. Through this, they develop strong analytical skills and understand how mathematics supports fairness, organisation, and efficiency.

The weeks and the indicators covered by the section are:

- *Week 19*

1. *Explain the concepts that give rise to conditional probabilities*
2. *Describe the probability of two equally likely events occurring from given experiments.*

- *Week 20*

1. *Model and solve real-life problems involving conditional probabilities.*
2. *Model and solve real-life problems involving binomial probabilities*

- *Week 21*

1. *Model and solve real-life problems involving combination and permutation.*
2. *Determine areas in businesses, commerce, industry, etc., where permutations and combinations can be applied to improve the systems*

Recap weeks

Year 1	**Week 22** Use the fundamental counting principle to identify and determine the number of ways an event can occur Understand the concept of permutation and combination and use it to solve related problems. **Week 23** Distinguish between the concepts of permutation and combination and establish the relation between them. Simplify permutation and combination expressions and solve related problems. **Week 24** Recall the basic terminologies such as experiment, events, outcome, trial, sample space etc. as used in the context of probability and give examples Work collaboratively on an experiment (e.g., tossing of coins or dice) to determine the relative frequencies of events and interpret them
Year 2	**Week 23** Use De Morgan's law in set theory to establish the addition and multiplication laws of probability and apply them to solve real life problems. Investigate the axioms of probability: For any event A, there is $0 \leq P(A) \leq 1$; that is, the probability of an event A is a number between 0 and 1 inclusive. Create and solve problems using the axioms and the laws of probability. **Week 24** Describe the number of ways objects can be arranged using fundamental counting principles. Solve real life problems involving permutations. Model and solve real life problems involving combinations.

SUMMARY OF PEDAGOGICAL EXEMPLARS

Effectively teaching conditional probability, binomial probability, and permutations and combinations to final-year senior high learners calls for deliberate use of learner-centred pedagogies that promote both understanding and application. Collaborative learning enables learners to share strategies and evaluate outcomes as they work through challenging probability scenarios. Problem-based and inquiry-driven tasks prompt learners to explore real-world situations, formulate questions, and justify the use of specific counting or probability rules. Gamified challenges make abstract ideas more tangible, encouraging engagement through interactive simulations, probability games, and combinatorics puzzles. Project-based learning allows learners to investigate and present how probability and combinatorics are applied in fields like marketing, health sciences, or logistics. Use of visual aids such as Venn diagrams, and grids as well as structured scaffolds and targeted extension tasks ensures learners at all levels can access, practise, and extend their understanding.

ASSESSMENT SUMMARY

For third-year senior high learners studying probability, use a blend of individual work and research to support deep understanding. Learners should begin with guided class activities that develop skills in solving problems involving conditional and binomial probabilities, then engage in research tasks where they explore real-life applications in fields like business, health, and games of chance. Differentiate tasks by scaffolding support for learners who need it and extending investigations for advanced learners to analyse how changes in conditions affect probability outcomes.

Indicators	DOK Level	Assessment Strategy and/or Mode
Week 19		
Explain the concepts that give rise to conditional probabilities	3	Strategy: Questioning Mode: Classwork
Describe the probability of two equally likely events occurring from given experiments.	3	
Week 20		
Model and solve real-life problems involving conditional probabilities.	2	Strategy: Case study Mode: Classwork
Model and solve real-life problems involving binomial probabilities		
Week 21		
Model and solve real-life problems involving combination and permutation.	4	Strategy: Research Mode: Individual Research
Determine areas in businesses, commerce, industry, etc., where permutations and combinations can be applied to improve the systems	2	

WEEK 19

Learning Indicators

1. *Explain the concepts that give rise to conditional probabilities*
2. *Describe the probability of two equally likely events occurring from given experiments*

FOCAL AREA 1: EXPLAINING THE CONCEPTS THAT GIVE RISE TO CONDITIONAL PROBABILITIES

Review learners' knowledge on:

1. Experiments: An operation or process performed to observe a result or an outcome or to obtain measurements like flipping a coin (experiment) to see if it lands on heads or tails (outcome). The outcome of an experiment is not certain and thus cannot be predicted.
2. Dependent Events (Sampling without replacement): these are events where the occurrence of one affects the occurrence of another. For instance, a group leader and secretary need to be selected from 5 learners where one person cannot occupy both positions.

   Event A: Learner A is selected as the group leader.

   Event B: Learner B is selected as the group secretary after group leader has been selected.

   The choice of the secretary is now dependent on who becomes the leader.
3. Independent Events: these are events where the occurrence of one **does not** affect the occurrence of another.
4. Multiplication Rule, for dependent events, $\mathrm{P(A} \cap B) = \mathrm{P(A)} \cdot P(B|A)$

   For independent events, $P(B \cap A) = P(B) \cdot P(A)$.
5. For mutually exclusive events, $P(B \cap A) = 0$.
6. Law of total probability: For a partition of the sample space into events $A_1, A_2, \ldots, A_n$: $P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$

How conditional probabilities arise

In an attempt to effectively analyse the relationship between dependent events, conditional probability emerges. Conditional probability arises in situations where the probability of an event occurring is influenced by the occurrence of another related event.

Consider a situation where a learner draws two cards successively from a standard deck of 52 playing cards without replacement. The probability of drawing a heart on the second draw is to be calculated given that a heart was drawn on the first draw. Challenge learners to find out if this probability is conditional or not, with justification. A justification could be that since the first card is not replaced, the sample space for the second draw is affected by the result of the first thus the situation involves dependent events. Also, the total number of cards and the number of hearts change depending on the outcome of the first draw. Thus, conditional probability must be applied to reflect the changing probabilities.

Note

Guide learners to discuss two events from the same experiment and also from two different experiments. Find the probability of two events both from the same experiment and also from two different experiments using the idea of combination or selection.

Example 1

A bag contains 5 black and 2 white identical marbles. Another bag contains 3 white and 6 black of the same marbles. A boy selects one marble at random from a randomly selected bag.

Find the probability that a black marble is selected.

Solution

P(of choosing either bag) $= \frac{1}{2}$

P(a black marble in the first bag) $= \frac{{}^5C_1}{{}^7C_1} = \frac{5}{7}$

P(a black marble in the second bag) $= \frac{{}^3C_1}{{}^9C_1} = \frac{1}{3}$

P(choosing a black marble) $= \frac{1}{2} \times \left(\frac{5}{7} + \frac{1}{3}\right) = \frac{11}{21}$

Therefore, the probability that a black marble is selected is $\frac{11}{21}$.

Example 2

A bag contains 15 identical cards. 6 are blue and the rest are green. Two cards are drawn at random from the bag without replacement.

What is the probability of getting one red card and one green card?

Solution

Number of ways of choosing two cards from the bag $= {}^{15}C_2 = 105$

P(a red and a green card from the bag) $= \frac{{}^6C_1 \times {}^9C_1}{{}^{15}C_2} = \frac{54}{105} = \frac{18}{35}$

Therefore, the probability that one red card and one green card is selected is $\frac{18}{35}$.

FOCAL AREA 2: DESCRIBING THE PROBABILITY OF TWO EQUALLY LIKELY EVENTS OCCURRING FROM GIVEN EXPERIMENTS

Conditional Probabilities

For any two events, A and B, the probability of an event B occurring given that event A has occurred is a conditional probability denoted $P(B|A)$ such that:

$P(B|A) = \frac{P(B \cap A)}{P(A)}, P(A) > 0$.

Note

For mutually exclusive events, $P(B|A) = 0$ *because* $P(B \cap A) = 0$.

Example 1

There are a total of 35 students in a Home Economics class, out of which 20 are boys and 20% of the girls wear spectacles. Find the probability that a student selected at random from the class is a girl who wears spectacles.

Solution

Let P(selecting a girl)= P(G)

$n(G) = 35 - 20 = 15$

Number of girls who wear spectacles $= \frac{20}{100} \times 15 = 3$

$\text{P(girls who wear spectacles)} = \frac{3}{35}$

The probability that a student selected at random from the class is a girl who wears spectacles is $\frac{3}{35}$.

Example 2

A paper bag contains 3 red counters and 5 blue counters. Two counters are drawn randomly from the bag one after the other, without replacement.

a. What is the probability of drawing a red counter first?

b. Determine the probability of drawing a blue counter given that the first one was red?

c. Describe how removing the first counter affect the chances of the second event.

Solution

a. Number of red counters $= 3$

Total number of counters $= 8$

$\text{P(red on 1st draw)} = \frac{3}{8}$

b. If red is drawn first without replacement, then 7 counters are left.
Number of blue counters $= 5$

$\text{P(blue on 2nd draw|red on 1st draw)} = \frac{5}{7}$

c. Removing the first counter without replacement reduces the available total number of counters from which to draw the 2nd counter. That is, the probability of the second event depends on the outcome of the first event.

Learning Tasks

1. Create a "Probability Quest" board game where each card presents a real-life scenario requiring learners to calculate a conditional probability to move forward.
2. Provide a deck of situation cards (e.g., student data with gender and subject choice); learners earn points by correctly identifying the probability of an event given another.

3. Organise a "Conditional Probability Challenge" where teams compete to solve progressively complex problems, unlocking clues to a larger mystery scenario.
4. Assign pairs or small groups to analyse two-way tables or tree diagrams and present conditional probability findings to the class.
5. Provide real-life data sets (e.g., school club membership) and have groups generate conditional probability questions and swap with other groups to solve.
6. Ask groups to simulate experiments (e.g., drawing coloured balls from a bag without replacement) and record outcomes to compute empirical conditional probabilities.
7. Present a surprising statistical scenario (e.g., false positives in medical tests) and ask learners to investigate and explain the probability of the condition given a test result.
8. Allow learners to explore how changing the sample space in an experiment affects conditional probability, using tools like Venn diagrams or frequency tables.
9. Guide learners to collect their own classroom data (e.g., preferences vs gender) and explore how probabilities change when one variable is known to occur.

Pedagogical Exemplars

1. Review previous learning
2. Gamification – Present learners with cards, spinners, or dice to simulate random experiments where learners calculate the probability of certain outcomes. Incorporate conditional aspects (e.g., draw without replacement)
3. Collaborative learning – encourage learners to first attempt a problem individually then discuss in pairs and finally share with the class.
4. Inquiry-Based learning – provide learners with a real-world scenario that prompts them to ask questions and explore relationships using tools like Venn diagrams or two-way tables. Encourage learners to design simple experiments or collect data to investigate how conditions affect outcomes, then guide discussions where they justify their reasoning and compare approaches. Finally, help learners formalise their findings into the conditional probability formula, reinforcing understanding through discovery.

Key Assessment Level 3: Questioning

1. A school is running a survey to determine how students use the library. Out of 135 students who responded to the survey, 75 said they read books regularly, 45 said they use the library regularly and 35 said they do both.

   a. Represent this data on a Venn diagram **(3 marks)**

   b. Using the scenario above, explain the difference between independent and dependent events.

   (3 marks)

   c. What is the probability that a student uses the library given that they read books regularly?

   (2 marks)

   Total (8 marks)

2. Two fair six-sided dice are rolled at the same time. Let two events be defined as follows:
   Event A: The sum of the two dice is greater than 7

Event B: At least one of the dice shows a 4.

If the two dice are rolled at the same time;

a. What is the probability of Event A occurring? **(3 marks)**

b. What is the probability of Event B occurring? **(2 marks)**

c. Determine $P(A|B)$. **(3 marks)**

d. What is the relationship between $P(A|B)$ and $P(A)$? **(1 mark)**

Total (9 marks)

3. A school committee of 7 students is to be selected from a class of 10 boys and 6 girls.

   The committee must meet these two conditions:

   It must contain more girls than boys.

   No student can serve on more than one committee.

   a. In how many distinct ways can such a committee be formed? **(4 marks)**

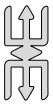
   b. What is the probability that a randomly selected 7-member committee from the class meets these conditions? **(4 marks)**

Total (8 marks)

PLC Session 19

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)
   c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

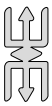
- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app

a. The key assessment for the week is ***questioning***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the assessment (NTS 3k-3q).

10 minutes Content Exploration

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes Enactment

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes Reflection

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 20

Learning Indicators

1. *Model and solve real-life problems involving conditional probabilities*
2. *Model and solve real-life problems involving binomial probabilities*

FOCAL AREA 1: MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING CONDITIONAL PROBABILITIES

Solving problems involving conditional probability

We shall explore real-life problems that can be solved using knowledge on conditional probabilities.

Recall:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \; P(A) > 0.$$

Example 1

At a senior high school, 60 students took a final exam in Mathematics. Out of these, 35 students were present during all revision sessions before the exam, and 25 were not. Among those present, 28 passed the exam. However, for those who were absent, only 10 passed.

a. Create a table to represent the data.

b. If a student randomly selected from the group passed the exam, what is the probability that they were present during all the revision sessions?

c. Interpret your result in part (b) and explain how it could help the school decide whether to continue holding revision sessions in the future.

Solution

	Passed	Failed	Total
Present during revision	28	7	35
Absent during revision	10	15	25
Total	38	22	60

c. Let P(who were present and passed) $= \frac{28}{60} = \frac{7}{15}$

Let P(students who passed) $= \frac{38}{60} = \frac{19}{30}$

P(present in all sessions given that they passed) $= \frac{7}{15} \div \frac{19}{30} = \frac{14}{19}$

d. Given that the probability that a student who passes is about 70% likely to have attended all the revision sessions, this means that the revision sessions really helped and the school should continue holding revision sessions to improve students' performance.

Example 2

In a certain town, 30% of the households own a dog. Of those that own a dog, 80% also have a fenced yard. Among households that do not own a dog, only 25% have a fenced yard.

If a household is selected at random and is found to have a fenced yard, what is the probability that it owns a dog?

Solution

Let:

D: Event that a household owns a dog

D′: Event that a household does not own a dog

F: Event that a household has a fenced yard

Now, $P(D \mid F) = \frac{P(D \cap F)}{P(F)}$

Also, $P(D) = 0.30$

$P(D') = 1 - P(D)$

$= 0.70 \quad P(F \mid D) = 0.80 \quad P(F \mid D') = 0.25$

We determine the probability that a household owns a dog and is fenced

$P(D \cap F) = P(D) \times P(F \mid D)$

$= 0.30 \times 0.80$

$= 0.24$ For probability that a household does not own a dog and is fenced, $P(D' \cap F)$

$P(D' \cap F) = P(D') \times P(F \mid D')$

$= 0.70 \times 0.25$

$= 0.175$ To find total Probability of Fenced Yard, $P(F)$

$P(F) = P(D \cap F) + P(D' \cap F)$

$= 0.24 + 0.175$

$= 0.415$

Now, we calculate the probability that a household owns a dog given that it is fenced

$$P(D \mid F) = \frac{P(D \cap F)}{P(F)}$$

$$= \frac{0.24}{0.415}$$

$= 0.5783$ Thus, the probability that a household owns a dog given that it is fenced is 0.58 to 2 d. p.

FOCAL AREA 2: MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING BINOMIAL PROBABILITIES

Binomial probability

The probability of obtaining a fixed number of successes (x) in a fixed number (n) of independent trials of a two possible outcome experiment, where each trial has the same probability of success is termed as Binomial Probability.

Example 1

A factory produces electronic components with $a\ 16\%$ defect rate. An inspector randomly selects 20 components for testing.

What is the probability of finding non defective components?

Solution

Let $\mathrm{P}(\text{defective component}) = p$

Let $\mathrm{P}(\text{non} - \text{defective component}) = q$

$p = 0.16$

$q = 1 - 0.16 = 0.84$

$n = 20$

$P(\mathrm{X} = 0) = {}^{20}C_0(0.16)^0(0.84)^{20-0}$

$= 0.0306$

The probability of finding no defective components in the 20 components is 0.0306.

The idea of combinations in binomial probability

Remember that combinations focus on the number of ways an arrangement can be carried out when the order is not important. Combination in binomial probability enables us to know how many ways we can obtain (x) successes in a fixed number (n) of independent trials with a two possible outcome (success or failure).

Assuming we want to find the probability of getting exactly 3 heads in 5 coin tosses, our focus is not about one sequence like, HHTTH. We are interested in all possible ways that 3 heads could occur in 5 tosses. The number of such different ways is given by the combination: ${}^nC_x = \binom{n}{x}$. That is, from n trials, how many ways can I choose x trials to be successful?

Suppose p is the probability that an event will occur (success) and q is the probability that the event will not occur (failure) then $q = 1 - p$.

The probability function for a binomial discrete variable X is given by: $\mathrm{P}(X = x) = {}^nC_x p^x q^{n-x}$.

Example 2

A fair coin is tossed five times. Find the probability of getting:

a. Exactly two tails.
b. At least two tails.
c. At most two tails.

Solution

If getting a tail is a success, probability of success for each trial, $p = \frac{1}{2}$.

Probability of a failure (not getting a tail), $q = 1 - \frac{1}{2} = \frac{1}{2}$.

Number of trials, $n = 5$

a. When $x = 2$,

$$P(X = 2) = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2}$$
$$= {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3$$
$$= \frac{5}{16}$$

b. When $x \geq 2$

$$P(X \geq 2) = 1 - P(X < 2)$$
$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - \left[{}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} + {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1} \right]$$
$$= 1 - \left(\frac{1}{32} + \frac{5}{32}\right)$$
$$= \frac{13}{16}$$

c. When $x \leq 2$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} + {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1} + {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2}$$
$$= \frac{1}{32} + \frac{5}{32} + \frac{5}{16}$$
$$= \frac{1}{2}$$

Solving real-life problems in binomial probability

Example 3

A new drug has a 70% chance of curing a certain disease. If 10 patients are given the drug, what is the probability that exactly 7 of them are cured?

Solution

Let $p = 0.70, q = 0.30, n = 10$.

$$P(X = 7) = {}^{10}C_7 (0.70)^7 (0.30)^{10-7}$$
$$= 120(0.0823543)(0.30)^3$$
$$= 9.882516(0.027)$$
$$= 0.267 \text{ to 3dp}$$

Example 4

In a production line, the probability that a light bulb is defective is 0.03. If a random sample of 15 bulbs is taken, what is the probability that at most 2 are defective?

Solution

Let $p = 0.03, q = 0.97, n = 15.$

$$P(X \leq 2) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2)$$

$$= {}^{15}C_0(0.03)^0(0.97)^{15-0} + {}^{15}C_1(0.03)^1(0.97)^{15-1} + {}^{15}C_2(0.03)^2(0.97)^{15-2}$$

$$= 0.6333 + 0.2938 + 0.0636$$

$$= 0.991 \; to \; 3dp$$

Modelling real-life problems in binomial probability

Ask learners to create scenarios, based on real-life examples, of binomial distribution and solved using probabilities.

To be able to model such problems, guide learners to recall the scenarios must:

1. Have a fixed number of trials.
2. Have only two possible outcomes per trial (success or failure).
3. Be independent trials.
4. Have a fixed number of successes.

Steps in modelling

Assist learners to:

1. Define the random variable.
2. Identify, the number of trials (n), successes (p) and failures (q).
3. Apply the binomial probability formula: $\mathrm{P}(X = x) = {}^{n}C_x p^x q^{n-x}$.
4. Interpret the results in the given context.

Learning Tasks

1. Provide learners with different real-life scenarios (e.g., quality control in a factory or success rates in a medical trial) and ask them to identify the trials, success criteria, and compute binomial probabilities.
2. Assign small groups to design and simulate binomial experiments (e.g., flipping coins, rolling dice), record frequencies of outcomes, and compare theoretical probabilities with empirical results.
3. Ask each group to develop and present a binomial word problem based on a real-life context (e.g., sports performance, quiz guessing), then swap problems with another group to solve and explain the steps involved.

Pedagogical Exemplars

1. Project-Based Learning – Assign learners to investigate a real-world situation (e.g., predicting customer satisfaction from survey data) that fits a binomial model. Guide them to collect or simulate data, model it using binomial probability, and present their findings using visual aids.
2. Collaborative Learning – Give learners a practical binomial problem (e.g., success rate of a product passing quality checks), have them first attempt individually, then discuss in small groups, and present their conclusions as a team to the class.
3. Inquiry-Based Learning – Pose a problem involving repeated trials with two outcomes (e.g., chances of picking a red button from a box with a number of green and red buttons). Let learners ask their own questions, identify variables, and explore outcomes before formally introducing the binomial formula to validate their findings.

Key Assessment Level 2: Case Study

1. A school has a debate club with 60 members: 30 boys and 30 girls. Out of the boys, 18 have leadership experience; out of the girls, 12 have leadership experience. The club wants to select a president from those with leadership experience.

   If a member is randomly selected from those with leadership experience, what is the probability that the person is a boy in its simplest form? **(3 marks)**

2. A factory produces light bulbs, and each bulb has a 2% chance of being defective. The quality control officer randomly selects 10 bulbs from a day's production line.

   What is the probability that:

   a. exactly 1 bulb is defective? **(2 marks)**

   b. none are defective? **(2 marks)**

   Total (4 marks)

3. A teacher gives a short online quiz. Based on past behaviour, there is a 70% chance a student will complete the quiz. Out of a class of 8 students, what is the probability that;

   a. exactly 6 students complete it? **(2 marks)**

   b. at least 6 will complete it? **(3 marks)**

   Total (5 marks)

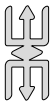
PLC Session 20

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.

b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)

c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app

   a. The key assessment for the week is ***case study***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the case study (NTS 3k-3q).

10 minutes **Content Exploration**

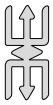
5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then

   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)

   b. all teachers in the group work through a similar problem individually for 3–4 minutes

   c. the group then works through the problem together step by step

   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).

   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.

   b. After the enactment, ask for structured feedback from the group

      i. was the concept explained in a way that a learner with no prior knowledge could follow?

      ii. were the examples grounded in contexts familiar to Ghanaian learners?

      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
    a. transfer your chats into your learning planner template.
    b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
    c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

WEEK 21

Learning Indicators

1. *Model and solve real-life problems involving combination and permutation*
2. *Determine areas in businesses, commerce, industry, etc., where permutations and combinations can be applied to improve the systems*

FOCAL AREA 1: MODELLING AND SOLVING REAL-LIFE PROBLEMS INVOLVING COMBINATION AND PERMUTATION

Review Fundamental Counting Principle

Conduct experiments and state the number of ways in which a particular counting or arrangement can be done.

Example 1

If a sports club decides to elect a chairperson and a secretary from 5 members, assuming the same person cannot hold both positions, find the number of ways this can be done.

Solution

Number of people to choose a chairperson from = 5

Number of people to choose a secretary from given that a chairperson is already selected = 4

Number of ways to choose a chairperson then a secretary $5 \times 4 = 20$.

Therefore, there are 20 ways in which this can be done.

Example 2

A school is organising a cultural performance, and 12 students have volunteered to participate; 7 girls and 5 boys. The teacher wants to form a 5-member dance group such that the group must contain at least 2 girls and the order in which students perform on stage matters.

a. In how many different ways can the teacher select and arrange 5 students to form the dance group under the given condition?

b. Explain how your model accounts for both selection and order, and why a combination or permutation (or both) is appropriate in this context.

Solution

a. Ways to select 5 students $= ({}^{7}C_2 \times {}^{5}C_3) + ({}^{7}C_3 \times {}^{5}C_2) + ({}^{7}C_4 \times {}^{5}C_1) + ({}^{7}C_5 \times {}^{5}C_0)$
$= 756$ ways
Once selected the 5 can be arranged in $5 \times 4 \times 3 \times 2 \times 1$ ways $(= 5!)$
Therefore, to select and arrange $= 756 \times 5! = 90720$ ways

b. Combinations are used to select the required number of girls and boys for each valid case, because selection does not depend on order. Once selected, the students must perform in a specific sequence, so permutation is applied (arranging 5 students in $5!$ ways) to account for order.

FOCAL AREA 2: DETERMINING AREAS IN REAL-LIFE WHERE PERMUTATIONS AND COMBINATIONS CAN BE APPLIED TO IMPROVE THE SYSTEMS

Application of Combination and Permutation

Engage learners in research where they investigate areas in which permutations and combinations can be applied in real-life. Areas can include commerce, industry, factory and games. Guide learners to present the findings from the research in class.

Areas in business

In business, success often depends on making the right choices and organising resources effectively.

1. Permutations are used in designing product display arrangements or calculating different ways to position products in advertisements.
2. Combinations are used in selecting items for combination deals or determining the number of ways to form a business team from a larger group.

These applications help businesses create attractive offers, increase customer interest, and optimise internal planning.

Areas in industry

In industries such as packaging, assembly, or logistics, permutations and combinations can help in:

1. Determining different ways to arrange items on production lines.
2. Choosing team members for specific technical tasks.
3. Planning the most efficient combinations of machines and operators.

These calculations improve productivity, reduce waste, and ensure efficient use of resources.

Areas in games

Permutations and combinations are used to make fair and exciting game arrangements:

1. Permutations help schedule tournaments, matches, and seating arrangements in competitions.
2. Combinations are used to select teams, draw lots, or set up group stages.

Understanding these concepts ensures fairness, efficient planning, and balanced competitions.

Areas in Education

In schools, permutations and combinations are applied in areas such as:

1. Designing examination timetables and seating plans.
2. Selecting prefects, team members, or award winners from a group.
3. Planning lesson schedules for teachers and classrooms.

These tools help to manage limited resources effectively and make the school environment more organised.

Areas in transport

Efficient transportation depends on good planning and decision-making:

1. Permutations help identify different routes and delivery sequences.
2. Combinations help in choosing transport schedules and loading arrangements.

This reduces travel time, saves fuel, and improves service delivery.

Other applications include communication networks, cryptography and network security, computer architecture, electrical engineering, computational molecular biology, languages, both natural and computer and pattern analysis.

Learning Tasks

1. Present learners with scenarios (e.g., assigning roles to a group of students, or selecting a committee) and ask them to decide whether to apply permutation or combination, explaining their reasoning.
2. Provide datasets from school (e.g., club memberships, sports team rosters) and have learners determine how many possible groups or arrangements can be made using permutations or combinations.
3. Ask learners to explore job sectors such as logistics, inventory management, or event planning, and prepare a presentation on how permutations or combinations can improve efficiency or outcomes.
4. Give students role cards (e.g., Head Girl, Sports Prefect, Class Rep) and a pool of names, and challenge them to compute all possible arrangements based on specific selection rules.
5. Assign research tasks where learners investigate and document at least two real-life business or industrial processes that rely on permutations and combinations (e.g., generating security codes, packaging designs).

Pedagogical Exemplars

1. Problem-Based Learning – Present learners with a real-life scenario, such as planning seating arrangements for an event or forming committees from a group, and challenge them to identify whether to apply permutation or combination and justify their method.
2. Collaborative Learning – Organise learners into groups to investigate specific industries (e.g., transport scheduling or manufacturing) and report on how permutations or combinations could be used to optimise operations and decision-making. Use examples from business and commerce (e.g., how different product codes can be generated or how teams are selected) to help learners relate the concept of permutations and combinations to practical applications in daily systems.
3. Gamification – Develop card or board games where learners must use permutations or combinations to unlock tasks or win points, reinforcing their understanding through fun, competitive play.

Key Assessment Level 4: Research

1. Investigate how permutations and combinations are used to create match schedules for tournaments, especially in Ghanaian contexts such as inter-house or inter-school competitions.
   In your research, write a 2-page report and clearly explain:

a. What permutations and combinations mean, using 4 relevant examples. **(4 marks)**

b. When and why, each is used in planning tournament fixtures. **(4 marks)**

c. How they are applied in a real-life Ghanaian tournament. **(4 marks)**

d. How you would organise a fixture, and give 3 mathematical reasoning behind your schedule. **(4 marks)**

e. Any 2 challenges you may face when using maths to design a schedule, and suggest possible solutions. **(4 marks)**

f. Support your work with sources such as books, websites, or interviews with teachers or sports organisers. **(4 marks)**

g. Present your findings neatly with appropriate headings, diagrams, or tables. **(4 marks)**

(Total 28 marks)

2. Carry out research on how permutations and/or combinations are used in the selection or arrangement of school prefects. In your 2-page report, clearly explain:

a. What permutations and combinations mean, with examples relevant to leadership selection in schools. **(4 marks)**

b. When and why each is used in choosing or arranging prefects (e.g., choosing a team of prefects vs. assigning specific roles like Head Boy or Head Girl). **(4 marks)**

c. How mathematics helps ensure fairness or structure in the selection process. **(4 marks)**

d. Any real-life examples from your school or other Ghanaian schools. **(4 marks)**

e. Any possible challenges (e.g., too many qualified candidates, role duplication) and how maths can help solve them. **(4 marks)**

Support your findings with real examples, interviews, or sources. Use tables or diagrams where appropriate to present your ideas clearly. **(8 marks)**

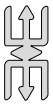
(Total 28 marks)

PLC Session 21

45 minutes **Planning & Reviewing the Learning Plan**

1. Begin the session by recapping lessons from the previous year/session/week, focusing on how learning was applied.
2. Use your subject specific app to explore the content before planning. Ask the app to
   a. explain the week's concepts in a practical way that will make it easier for learners to understand
   b. identify two or three common misconceptions learners may have on the week's content and how to address them using contexts familiar to Ghanaian learners. (NTS 3a, 3g and 3m).
3. Use your subject specific app to plan your lesson. Ask the app to
   a. suggest the most effective method for teaching the week's concepts to learners with diverse backgrounds.
   b. suggest learning activities, differentiation and essential questions based on the method of teaching (NTS 3a-3q)

c. review your prepared learning plan and address any challenging issues you may have identified in your subject area (NTS 3a-3c).



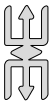
Note

- *Take note of the recap topics linked to Years 1 and 2 while preparing for your lesson.*
- *Refer to the Values Learning Community Handbook and link the values for the week in your lesson to ensure that learners are applying what they learn.*

4. Plan the week's key assessment with your app
   a. The key assessment for the week is ***research***. Chat with your app to plan this assessment, indicating what you and your learners will do before, during and after the research work (NTS 3k-3q).

10 minutes **Content Exploration**

5. As a subject group, select one problem or idea/term in the week's content that is challenging (not a routine procedural question, but one that requires conceptual understanding) and then
   a. use the app, ask it to solve the problem step by step and explain the reasoning behind each step (you may be assisted by a teacher who is strong in the content area)
   b. all teachers in the group work through a similar problem individually for 3–4 minutes
   c. the group then works through the problem together step by step
   d. highlight the most common errors made at each step and discuss (NTS 1c, 1e, 2c, 2e, 3e-3j).



Note

- *This part is not just about 'How do I teach this?' but how do I teach it for learners to understand?*
- *If the group still shares the same content gap, use the app: ask it to solve the problem step by step and explain the reasoning behind each step.*
- *Problems/ideas discussed should be sourced from the Teacher Manual content, not invented separately. This keeps the content exploration directly connected to what teachers will deliver.*

30 minutes **Enactment**

6. Enact a teaching activity based on your learning plan (NTS la-1b, 1e-lg, 2c, 2e, 2f, 3c-3j and 3o).
   a. Colleagues observe with specific attention to clarity of concept explanation, use of familiar Ghanaian examples and how misconceptions are handled if they arise.
   b. After the enactment, ask for structured feedback from the group
      i. was the concept explained in a way that a learner with no prior knowledge could follow?
      ii. were the examples grounded in contexts familiar to Ghanaian learners?
      iii. did any misconceptions surface and if so, how were they addressed?

5 minutes **Reflection**

7. Reflect and share your views on the session. One thing that worked well, what did not work well and what you will do differently. (NTS 1a, 1b).
8. Identify a critical friend to observe and provide feedback on your lesson in relation to the PLC Session conducted (NTS 1f, 1g).
9. Remember to
   a. transfer your chats into your learning planner template.
   b. read your Teacher Manual and related Learner Material in preparation for the next session. Take note of the recap weeks, which are linked to Years 1 and 2 (NTS 3a).
   c. integrate National Values, GESI, SEL, and 21st century skills into your learning activities, assessment tasks, and rubrics/mark schemes.

Section 8 Review

The section focused on lessons to be delivered in weeks nineteen, twenty and twenty-one of the year three Teacher Manual. It sought to provide guidance to teachers in applying and extending the knowledge of the laws of probability and the concept of combinations and permutations to solve real-life problems involving conditional and binomial probability. Learners were to be assessed using individual classwork and research. It is intended that learners will get the opportunity to revise lessons taught in the previous years as they work through the learning and assessment tasks. Teachers are encouraged to be accommodative of the learning needs of learners.

MARKING SCHEMES FOR THE KEY ASSESSMENTS

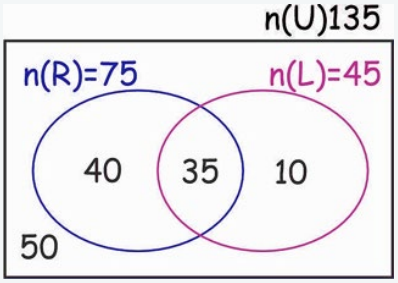
WEEK 19

1. a. Let total number of students $(U) = 135$

Let number of students who read books regularly $(R) = 75$

Let number of students who use library regularly $(L) = 45$

Then $n(R \cap L) = 35$



B1 for $n(R \cap L) = 35$ and $n(R \cap L') = 40$

B1 for $n(L \cap R') = 10$

B1 for $n(L \cup R)' = 50$

b. Independent events are events where the outcome of one event does not affect the outcome of another. **B1**, or for an equivalent statement about dependent events.

In this case it would mean that the likelihood of being a regular reader is not impacted by the likelihood of being a library user. **B1** for putting the above into context

If the events are independent then $P(R \cap L) = P(R) \times P(L)$. This is not the case here. Therefore, the events are not independent but dependent, so if you are a regular reader impacts on the likelihood of being a library user. **B1** for concluding that this event is not independent, or equivalent

c. $P(L|R) = \frac{P(L \cap R)}{P(R)}$

$= \frac{35}{135} \div \frac{75}{135} = \frac{35}{75}$ **M1**

$= \frac{7}{15}$ **A1**

2. a. Produces some form of sample space diagram to show the sum of the dice, or does equivalent to find the number of ways of getting a sum of greater than 7. For example:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

B2

$n(S) = 36$

$n(of\ ways\ of\ getting\ a\ sum\ of\ 7 = A) = 15$

$P(A) = \frac{15}{36}$ **A1**

b. $n(having\ at\ least\ one\ 4 = B) = 11$

$P(B) = \frac{11}{36}$ **M1 A1**

c. $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = \frac{5}{36}$ **M1**

$P(A|B) = \frac{5}{36} \div \frac{11}{36}$ **M1**

$P(A|B) = \frac{5}{11}$ **A1**

d. $P(A|B) > P(A)$, this means that Events A and B are dependent. **B1**

3. a. Combination of 7-member committee with more girls

Girls	Boys	Total
6	1	7
5	2	7
4	3	7

B1 for table, or equivalent

Number of ways $= \left({}^{6}C_6 \times {}^{10}C_1\right) + \left({}^{6}C_5 \times {}^{10}C_2\right) + \left({}^{6}C_4 \times {}^{10}C_3\right)$ **M1**

$= 10 + 270 + 1800$ **M1**

$= 2080$

A committee meeting the conditions can be formed in 2080 ways. **A1**

b. Total number of ways of forming a 7-member committee $= {}^{16}C_7$ **M1**

$= 11440$ **A1**

Probability that committee meets the conditions $= \frac{2080}{11440}$ **M1**

$= \frac{26}{143}$ **A1**

WEEK 20

1. Let boys be represented by B

Let leaders be represented by L

A boy and a leader $(B \cap L) = 18$

Total leaders $= 18 + 12 = 30$ **B1**

$P(B|L) = \frac{P(B \cap L)}{P(L)}$

$P(B|L) = \frac{18}{30}$ **M1**

$P(B|L) = \frac{3}{5}$ **A1**

2. Let number of bulbs $= n = 10$

Let $p = 0.02$

Let $q = 0.98$

a. Let X $= 1$P$(X = 1) = {}^{10}C_1(0.02)^1(0.98)^{10-1}$ **M1**

$= {}^{10}C_1(0.02)^1(0.98)^9$

$= 10(0.02)(0.8337477 \ldots)$

$= 0.1667$ **A1**

b. $P(X = 0) = {}^{10}C_0(0.02)^0(0.98)^{10}$ **M1**

$= 1(1)(0.8171)$

$= 0.8171$ **A1**

3. Let number of bulbs $(n) = 8$

Let $p = 0.7$

Let $q = 0.3$

a. Let X $= 6$P$(X = 6) = {}^8C_6(0.7)^6(0.3)^{8-6}$ **M1**

$= 28(0.1176)(0.3)^2$

$= 0.296$ **A1**

b. $P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8)$ **M1**

M1

$= 0.2964 + 0.1978 + 0.0576$

$= 0.552$ to 3dp **A1**

WEEK 21

Criteria	Excellent (4 marks)	Very Good (3 marks)	Satisfactory (2 marks)	Needs Improvement (1 mark)
Definitions and Examples of Permutations and Combinations	*Gives accurate, clear definitions of both; 4 includes relevant Ghanaian examples (e.g., inter-house events).*	*Mostly accurate definitions; includes at least 3 relevant examples.*	*Basic definitions; 2 examples, may lack local context or clarity.*	*Incomplete or incorrect definitions; 1 example unclear or absent.*
Identifying and Explaining Usage in Scheduling	*Accurately identifies and explains when permutations or combinations are used in the tournament process, with clear reasoning.*	*Identifies both, but explanation may lack depth or clarity.*	*Attempts to identify but confuses or oversimplifies usage.*	*Confused or no attempt to differentiate permutation from combination.*

Application to a Ghanaian Tournament Context	*Clearly connects math concepts to real-life tournaments in Ghana (e.g., school sports, quiz contests); explains logic behind pairings and scheduling formats.*	*Connection to context is good; some details may be missing or general.*	*Makes a basic link to Ghanaian tournaments; explanation may lack clarity or completeness.*	*Little or no relevance to Ghanaian examples; application is weak or unclear.*
Organising a fixture, and the mathematical reasoning behind such a schedule.	*Provides an example showing how to organise a fixture, and detailing at least 3 mathematical reasoning behind the schedule.*	*Provides an example showing how to organise a fixture, and detailing at least 2 mathematical reasoning behind the schedule.*	*Provides an example showing how to organise a fixture and details at least 1 mathematical reasoning behind the schedule.*	*Provides an example showing how to organise a fixture Or details at least 1 mathematical reasoning behind the schedule.*
	Clearly identifies at least two relevant, challenges (e.g., uneven number of teams, avoiding repeat matches, limited time slots) and offers feasible, well-thought-out solutions to each challenge.	*Identifies two relevant challenges but with limited detail or clarity* *and provides reasonable solutions, though one may lack full development*	*Identifies one general or partially relevant challenge and its corresponding solution*	*Mentions a challenge with little relevance or clarity and no proposed solution*
Research Quality and Use of Sources/ Examples	*Uses at least 2–3 sources (books, interviews, online, teachers); integrates real examples from a specific Ghanaian tournament or school event.*	*Uses 1–2 sources; includes at least one realistic local example.*	*Limited or unclear sources; example may be generic or unverified.*	*No sources cited; no identifiable example used.*
Organisation, Language, and Presentation	*Well-structured, clear and neat; uses correct English; includes headings, subheadings, tables/diagrams where appropriate.*	*Generally clear; some issues with organisation or minor grammar errors.*	*Some disorganisation; errors may affect readability.*	*Poorly presented; hard to follow; many languages or formatting issues.*
Total Score	*28 Marks*			

Criteria	Excellent (4 marks)	Very Good (3 marks)	Satisfactory (2 marks)	Needs Improvement (1 mark)
Understanding of Permutations and Combinations (Definition + Examples)	*Clear and accurate definitions with at least 2 relevant examples related to leadership selection*	*Mostly accurate definitions with 1 relevant example*	*Basic or partially accurate definitions; examples unclear or not related*	*Definitions missing or incorrect; no relevant examples*
Application to Prefect Selection (When and Why)	*Clear explanation of when to use permutations vs. combinations with realistic and relevant scenarios (e.g., selecting vs assigning roles)*	*Explanation is mostly clear but some scenarios may lack precision*	*General explanation with some confusion between the two concepts*	*Little to no understanding of their application; scenarios missing or irrelevant*
Use of Mathematics to Ensure Fairness or Structure	*Provides specific mathematical reasoning (e.g., fairness in role assignment, balanced representation) with clarity*	*Mentions fairness or structure but without detailed mathematical explanation*	*Vague reference to fairness or structure; math reasoning not evident*	*No mention of fairness, structure, or mathematics' role*
Real-Life Examples (School or Ghanaian Context)	*Cites at least one real-life example from school or other Ghanaian schools with details or interview/ source*	*Cites example but lacks sufficient details or source support*	*Example mentioned but unclear or not well explained*	*No real-life example provided*
Challenges and Mathematical Solutions	*Identifies at least one valid challenge and explains how math helps solve it (e.g., role duplication, qualified candidates)*	*Identifies challenge but gives only general or unclear solution*	*Challenge mentioned but no solution proposed*	*No mention of challenges or solutions*
Use of Tables, Diagrams, and Organisation	*Report is well-organised, with clear tables/diagrams that enhance understanding*	*Report is organised with some visual support, though not always clear*	*Report shows basic organisation but visuals are missing or unclear*	*Disorganised report with no tables or diagrams*
Overall Presentation and Originality	*Report is coherent, original, and free from grammatical errors; follows the 2-page limit well*	*Mostly coherent with minor grammar issues; within page limit*	*Some grammar/ spelling errors; content may be below or above page limit*	*Unclear, disjointed, or plagiarised content; poorly written*
Total Score	*28 Marks*			

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