MINISTRY OF EDUCATION
MATHEMATICS ASSOCIATION
GHANA (MAG)

MATHEMATICS

for Senior High Schools

Year 3



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FOREWORD

Ghana's new Senior High School Curriculum aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, knowledge, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, progress to further studies and enter the world of work. This is the first time that Ghana has developed a Senior High School Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

The Ministry of Education is proud to have overseen the production of these Learner Materials which can be used in class and for self-study and revision. These materials have been developed through a partnership between the Ghana Education Service, teacher unions (Ghana National Association of Teachers-GNAT, National Association of Graduate Teacher -NAGRAT and the Coalition of Concerned Teachers- CCT) and National Subject Associations. These materials are informative and of high quality because they have been written by teachers for teachers with the expert backing of each subject association.

I believe that, if used appropriately, these materials will go a long way to transforming our Senior High Schools and developing Ghana so that we become a proud, prosperous and values-driven nation where our people are our greatest national asset.

Haruna Iddrisu MP

Minister for Education

SECTION

1

LOGICAL REASONING & VARIATIONS



NUMBERS FOR EVERYDAY LIFE

Proportional Reasoning

Introduction

In this section, you will be learning about logical reasoning and variation — two important mathematical concepts that help us understand how ideas are connected and how values change. These skills are not new to you. In Year 1, when you worked with sets, fractions, and De Morgan's Law, you were already using logical steps to solve problems. In Year 2, you explored ratios, rates, and proportions, which are all examples of variation — they show how one quantity changes in relation to another.

Now, we will build on what you have already learned. You will apply logical thinking to make sound decisions, identify patterns, and solve real-life problems. You will also explore how variables can affect each other, such as how distance and time relate or how changes in price affect cost.

These skills will help you reason better, both in mathematics and in everyday life.

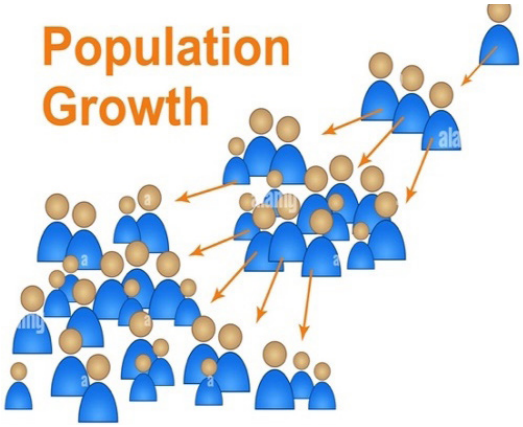
KEY IDEAS

- **Argument Analysis**: Evaluating the structure and validity of arguments to draw sound conclusions.
- **Conjunctions and Disjunctions**: Using logical connectives ("and", "or") to combine or contrast statements.
- **Decision Making Tools**: Applying tables and diagrams to support logical conclusions.
- **Proportional Reasoning**: Understanding and solving problems involving ratios, proportions and variations.
- **Statement Classification**: Identifying types of statements (interrogative, declarative, positive, negative) for logical analysis.
- **Strategic Guessing**: Using logical patterns and elimination methods to make intelligent guesses in uncertain situations.

INTRODUCTION TO SYMBOLIC LOGIC AND SITUATIONAL GROWTH

In our daily lives, we often make decisions based on reasoning. Symbolic logic is the branch of mathematics that allows us to express these reasoning patterns using symbols and variables. By applying symbolic logic, we can analyse and compare relationships

between variables in various situations—such as population growth, investment returns, or business profits—and make sound conclusions or predictions.



Think about this

In a certain town, the population is growing by 15% every year.

But does this mean job opportunities also grow by 15%?

Consider the following statements P and Q below:

P: "The town's population increases by 15% every year."

Q: "Job opportunities increase by 15% every year."

Think about the following real-life situations

- Have you heard of a community or town where the number of people keep increasing, but the number of jobs are not?
- Can you think of any reasons why job opportunities may not grow at the same rate as the population?
- Write at least two real-life reasons in your book. For example:
- Companies are not hiring new people.
- The town has more school graduates than job openings.

Now answer the following:

- If the population increases, does it always mean that job opportunities will increase?
- Can job opportunities increase even if the population does not?
- Can you give an example of when job opportunities might decrease even if the population grows?

Now consider these statements:

- "The number of students in the school increases."
- "The number of textbooks available increases."

Then ask yourself:

- Does one really lead to the other?
- What might affect each one?

Write your pair of statements and explain your thinking.

Conclusion:

Even if two things are growing, they may not grow at the same rate. One does not always cause the other. Logical reasoning helps us examine these differences clearly.

Concept of Logical Reasoning

Take a moment to think about this

Have you ever seen someone with a big belly?

Does it always mean the person is pregnant?

Step 1: Go through this problem step by step

Do only pregnant people have big bellies?

Now, think about other possible reasons someone may have a big belly. Can you name two?

For example:

- Someone has eaten a lot.
- Someone is sick.
- Someone has fat around their stomach.

Step 2: Read this situation

Ama sees a woman with a big belly. She quickly decides she is pregnant.

Ask yourself

- Is Ama correct?
- What mistake could she have made?
- What could she do before drawing a conclusion?

Step 3

Imagine you see someone with a big belly. What would you do to find out the truth?

For example:

"I would ask politely or wait until I have more information. Just looking at someone's belly is not enough to know if they are pregnant."

Learn something here

- A good thinker doesn't rush to conclusions.
- Just because something *looks* a certain way doesn't always mean your guess is correct.
- Logical reasoning means asking questions, checking facts, and being careful with conclusions.

What is Logical Reasoning?

Logical reasoning is the process of using clear steps, facts and evidence to arrive at a valid conclusion. It involves thinking in a clear and orderly way so that the conclusions you reach are based on facts or evidence, not just guesses or feelings. It involves analysing information, eliminating wrong possibilities and making connections between statements.

Interrogative and Declarative Statements

Have a look at the information below:

Scenario: In many Ghanaian communities, school attendance has been improving over the years.

But does this mean that the students attending school always achieve good grades?

Step 1: Identify the Two Statements

Declarative Statement (this is a statement of fact, as you are 'declaring' something)**:**

More children are attending school each year.

Interrogative Statement (this is a question, as you are seeking information – you are 'interrogating' the statement)**:**

Are the students getting better grades every year?

Step 2: Connect to Real-Life Situations

- Have you heard of a town where many children go to school but still perform poorly in exams?
- Can you think of any reasons why school attendance may increase, while grades do not?

Write at least two real-life reasons based on your understanding. For example:

- Some pupils are in school but lack textbooks at home.

- Teachers may not have enough time to give individual attention.

Step 3: Convert Declarative to Interrogative

- Does more school attendance *always* mean better performance? (Interrogative)
- Can pupils perform better even if attendance stays the same? (Interrogative)
- Can you give an example where performance drops even when attendance is high?

Step 4: Make Your Own Example

Now create your own **pair of statements**—one **declarative** and one **interrogative**—about two related things in real life.

For example:

Declarative Statement: The number of shops in the town is increasing.

Interrogative Statement: Does that mean customers are buying more goods?

Then ask yourself:

- Does one fact automatically answer the question?
- What could affect each situation?

Write your pair of statements and briefly explain your thinking.

Declarative statements give us information, while **interrogative statements** help us seek information. By comparing both, we learn to think deeply about everyday situations and the questions we need to ask to understand them better.

What is an Interrogative Statement?

An interrogative statement, also called an interrogative sentence, asks a question. Its main purpose is to gather information, clarify a point or confirm something.

Interrogative sentences usually start with words like who, what, where, when, why, how or a helping (auxiliary) verb (is, are, will, do, can) and end with a question mark (?).

What is a Declarative Statement?

A declarative statement, also known as a declarative sentence, makes a direct statement or provides information. Its purpose is to state facts, opinions, explanations, or observations.

Example 1.1:

Convert these interrogative statements into declarative statements using the table:

a. Is Kumasi located in the Ashanti Region?

b. Do farmers in Ghana rely on the rainy season?

c. Is cocoa a major export crop in Ghana?

d. Does education improve the lives of the youth?

Solution:

Table 1.1

Item	Interrogative Statement	Declarative Statement
A	Is Kumasi located in the Ashanti Region?	Kumasi is located in the Ashanti Region.
B	Do farmers in Ghana rely on the rainy season?	Farmers in Ghana rely on the rainy season.
C	Is cocoa a major export crop in Ghana?	Cocoa is a major export crop in Ghana.
D	Does education improve the lives of the youth?	Education improves the lives of the youth.

Positive and Negative Statements:

To deepen your understanding and appreciation of this concept, carefully analyse this scenario:

Scenario: It is a market day in Techiman. Amma wants to buy some tomatoes and onions for a stew. What she hears from different sellers influences her buying decisions.

Seller A says:

"The tomatoes today are fresh and sweet." (This is a positive statement about the tomatoes.)

Seller B says:

"The onions are not fresh; they are starting to spoil." (This is a negative statement about the onions.)

Amma replies:

"I will buy tomatoes from Seller A because the tomatoes are fresh and sweet."

"I will avoid onions from Seller B because they are not fresh."

Positive statements express a favourable condition or fact (e.g., "The tomatoes are fresh.")

Negative statements express an unfavourable or opposite condition (e.g., “The onions are not fresh.” or “The onions are spoiled.”)

This scenario shows how people make decisions based on positive and negative information.

Now look at these positive statements and how they change when made negative. This will help you understand them better.

1. **Positive statement:** Yaa Asantewaa led the Ashanti in the war against the British.
2. **Negative statement:** Yaa Asantewaa did not lead the Ashanti in the war against the British.
3. **Positive statement:** 5 is a prime number.
4. **Negative statement:** 5 is not a prime number.
5. **Positive statement:** Honesty is an important moral value in Ghanaian society.
6. **Negative statement:** Honesty is not an important moral value in Ghanaian society.

NOTE:

Mostly, we represent statements with capital letters of the English alphabet and we attach “~” to mean negation.

Example:

E: The Volta River is the longest river in Ghana.

~E: The Volta River is not the longest river in Ghana.

F: Students who revise regularly perform better in exams.

~F: Students who revise regularly do not perform better in exams.

G: $2x + 5 = 15$

~G: $2x + 5 \neq 15$

Are declarative statements always true?

Declarative statements can either be negative or positive. A negative declarative statement can be true or false or debatable.

Example1.2:

1. **B:** The capital city of Ghana is Accra.
2. This is a true statement because Accra is indeed the capital city of Ghana.
3. **Q:** All mammals can fly.

4. This is a false statement because most mammals cannot fly—only bats can.
5. **R**: Students perform better when they are taught with digital tools like projectors and GeoGebra. This is a debatable statement. While digital tools can enhance understanding, performance also depends on students' interest, the teacher's skill, and other factors.

Conjunctions and Disjunctions

Instruction:

Think about your daily school life. Write down **two activities** that often happen in your classroom or school environment.

For example:

- Doing homework
- Sweeping the classroom

Now form two simple statements based on your list. For example:

1. **P:** I did my homework.
2. **Q:** I swept the classroom.

Combine the statements using logical connectives: Have you got something like this?:

- I did my homework **and** swept the classroom.
- I did my homework **or** swept the classroom.

Conjunctions

A conjunction connects two statements and is true only if both statements are true.

A: Kofi owns a bicycle, **Q:** He rides it to school

These simple declarative statements can be connected with words (connectives) such as **and**, **or**, **not**, and if **...** then to form compound statements.

We can join two statements to get compound statements which can be a conjunction or a disjunction.

Examples of Conjunctions

1. " Kofi owns a bicycle and he rides it to school." Making the statement a **conjunction**.
2. "The teacher marked the scripts **and** recorded the scores." Making the statement a **conjunction**.
3. "Yaw is a carpenter **and** he lives in Tamale." Making the statement a **conjunction**.

A conjunction is true if both statements are true, otherwise, it is a false conjunction.

Both parts need to be true for the conjunction to be true.

Disjunctions

A disjunction connects two statements and is true if at least one of the statements is true.

Examples of Disjunctions

1. "Kofi will eat rice **or** he will eat yam." This is a disjunction.
2. "It is raining **or** the sun is shining." This is a disjunction.
3. "You can travel by bus **or** you can travel by train." This is a disjunction.

A disjunction is considered true when at least one part of the statement is true. This means that even if only one of the statements is correct, the whole disjunction is still true.

Using the symbols "∧" and "∨"

Consider the use of **"and"** and **"or"** in forming compound statements. We denote "**and**" with "∧" and "**or**" with "**∨**"

Example 1.3

1. Determine whether each combined statement is True or False based on the facts:

   A: Water boils at 100°C. *(True)*

   B: Water freezes at 0°C. *(True)*

Solution

A ∧B: Water boils at 100°C and water freezes at 0°C (True conjunction)

A ∨ B: Water boils at 100°C or water freezes at 0°C (True disjunction)

2. Determine whether each combined statement is True or False based on the facts:

   P: Kumasi is the capital of Ghana

   Q: Accra is the capital of Ghana

Solution

P ∧ Q: Kumasi is the capital of Ghana and Accra is the capital of Ghana (**False** conjunction)

P ∨ Q: Kumasi is the capital of Ghana **or** Accra is the capital of Ghana (True disjunction)

Activity 1.1 Understanding Logical Reasoning

Copy the table below.

1. Read each sentence carefully and fill in the table.

Sentence	Type of Statement (Interrogative / Declarative)	Is it True or False?	Rewrite it in the Opposite Form
Do you attend school every weekday?			
I attend school every weekday.			
We don't learn mathematics in school.			
The headteacher is a pupil.			
Are exams fun for students?			

2. Write two short sentences about your school in your exercise book:
   - One sentence must be **true**.
   - One sentence must be **false**.
   - Underline the sentence if it says something *good (positive)* about the school.
   - Put a star (*) next to the sentence if it says something *bad (negative)*.
3. In one or two sentences, explain how you can use *logical thinking* to tell which sentence is true and which one is false.

Activity 1.2 Investigating How Real-Life Variables Change and Connect Using Logic

You are a data collector in a farming town in Ghana. Farmers say the price of maize goes up when the harvest is poor. But is that always true? Now it's your turn to investigate.

Step 1: Collect or use this sample data

Table 1.2

Year	Amount of Maize Harvested (bags)	Price of Maize per bag (GHS)
2020	150	100
2021	130	110

2022	110	125
2023	170	95

Step 2: Answer these questions (write in full sentences):

1. Does the amount of maize harvested always increase or decrease?
2. Does the price per bag always move in the same direction as harvest amount?
3. Can you say that "If maize harvest increases, price increases"? Is this always true?

Step 3: Use logical connectors to describe what you see.

Complete these statements in your notebook:

a. The maize harvest was low, and the price went up.

b. The maize harvest increased, or the price decreased.

c. The maize harvest was poor, and the price remained the same.

d. The price of maize changed, but the harvest did not.

Which of these sentences show a clear connection? Which ones show that two things can happen together or not**?**

Activity 1.3 Thinking Logically About Real-Life Issues

In a social studies class discussion, two groups shared different opinions about whether plastic bags should be banned in Ghana.

The boys said: *"If the government bans plastic bags, it will help reduce flooding in Accra during the rainy season."*

The girls argued: *"Banning plastic bags will create problems for traders and increase waste in other forms."*

Now it's your turn to judge using logical reasoning

Step 1: Examine each view carefully.

a. Boys' opinion

- What is the *cause* they believe in?
- What *effect* do they expect?
- Can you think of real examples where this has happened?
- Why might it not always work the way they think?

b. Girls' opinion

- What problem do they think the ban might create?

- In what ways could this affect *market women, food sellers, or shoppers*?
- Is their view *certain* to happen, or is it *just a possibility*?

Step 2: Write a short paragraph (4–6 lines):

Provide your answers to the questions below:

- Can banning plastic bags help reduce flooding?
- What challenges might arise if plastic bags are banned?
- Can people switch easily to alternatives?
- Give at least *one real or imagined example from Ghana* (e.g. Makola, Kejetia, Tamale Central Market, etc.) to support your thinking.

Step 3: Final Logical Question

Can two people have different opinions and still be logical?

Write your answer with an explanation. Remember: Logical reasoning doesn't always mean only one answer is correct — it means using evidence and careful thinking to support your view.

Real-World Applications of Logical Reasoning

Logical reasoning helps us think clearly and make good decisions. We use it every day — when solving maths problems, choosing what to buy, planning our time, or solving problems. It helps us understand situations better and find the best solutions in school, at home, and in the community.

Take a close look at the scenario below:

Doing so will help you understand how logical reasoning can be applied in daily life.

Scenario

Ama and her younger brother Kwame need to buy ingredients for a family meal from the local market in Kete - Krachi. They have only GH¢50 to spend and want to buy tomatoes, onions, and some fresh fish.

Situation Description

Budget Constraint

Ama says: "We have GH¢50 total. Tomatoes cost GH¢10 per heap, onions cost GH¢7 per heap, and fresh fish costs GH¢15 a kilo."

Decision Making Using Logical Reasoning

Kwame suggests buying 2 heaps of tomatoes (GH¢20), 3 heaps of onions (GH¢21), and 1 kilo of fish (GH¢15).

Ama calculates: 20 + 21 + 15 = GH¢56, which is over their budget.

Ama then suggests adjusting the quantities. She reasons:

"If we buy 1 heap of tomatoes (GH¢10), 2 heaps of onions (GH¢14), and 2 kilos of fish (GH¢30), we spend exactly GH¢54 — still above budget."

Kwame proposes: "What if we buy 1 heap of tomatoes, 1 heap of onions, and 2 kilos of fish?"

Ama calculates: 10 + 7 + 30 = GH¢47, which fits the budget with 3 cedis remaining.

Logical reasoning here

They decide to compare quantities and prices logically to get the best combination that meets the budget and food needs.

Conclusion: Logical reasoning helps Ama and Kwame figure out the best way to spend their money without running out of money.

Example 1.4

Consider the following statements

M: A person wears a helmet

N: The person is safe while riding a motorbike

Tasks

1. Represent the statements M and N on a Venn diagram.
2. Write an implication from the two statements (e.g., *If M, then N*), and state whether the implication is true or false. Kwame wore a helmet but still got injured in an accident.

Solution

U = {all motorbike riders}

M = A person wears a helmet

N = The person is safe while riding a motorbike

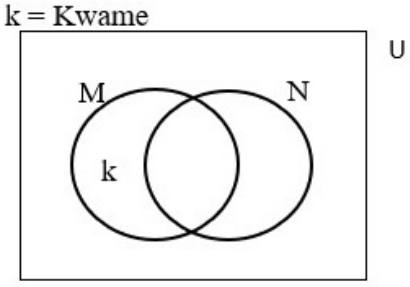


Figure 1.1

M ⇒ N: If a person wears a helmet, then the person is safe while riding a motorbike. This is **false** because M is not a subset of G, ie Wearing a helmet does not necessarily mean a person will be safe while riding a motorbike.

Example 1.5

Consider the following statements:

E: People who speak Twi fluently are Ghanaians.

F: People who attend basic school in Ghana speak Twi fluently.

a. Draw a Venn diagram to illustrate the above statements.

b. Deduce which of the following statements are valid conclusions based on E and F.

State whether each statement is true or false:

i. Kojo attends basic school in Ghana ⇒ Kojo is Ghanaian.

ii. Abena speaks Twi fluently ⇒ Abena attended basic school in Ghana.

iii. Yaw is Ghanaian ⇒ Yaw speaks Twi fluently.

Solution

Let **U** = persons

T = people who speak Twi fluently

B = people who attend basic school in Ghana

G = Ghanaians

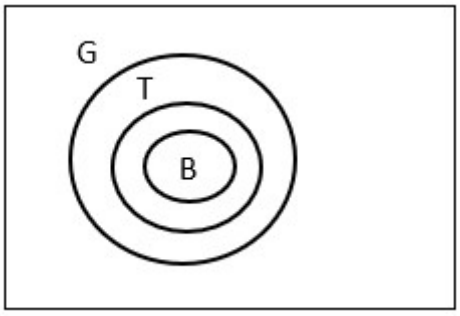


Figure 1.2

i. Valid – True statement because B is a subset of G

ii. Invalid – False statement because while all basic school attendees speak Twi (Q ⊆ P), not all Twi speakers necessarily attended basic school in Ghana. So, T is not a subset of B.

iii. Invalid – False because not all Ghanaians (G) necessarily speak Twi fluently (T). There are Ghanaians who may not speak Twi, so G is not a subset of T. members of G belong to T

Activity 1.4 Investigating the Use of Logical Reasoning in the Real World

Title: ***Be the Change: Investigating How to Save Water and Electricity in My School***

Background

In many schools in Ghana, water and electricity are often wasted without people noticing. Logical reasoning helps us investigate the problem, find out the causes, and choose the best ways to solve it.

Task for you

You are to carry out a short investigation on how water and electricity are used or wasted in your school. You will use a checklist/questionnaire to collect your data, then analyse your findings and write a short report with suggested solutions.

Steps to Follow

Step 1: Observation and Data Collection

Use the checklist below to observe **five (5)** different places in your school. Tick (✓) the boxes that apply.

Table 1.3: Observation Checklist / Questionnaire

Location Observed (e.g., Classroom, Bathroom, Dormitory, Kitchen)	1. Are lights/fans left on when the rooms are not in use?	2. Are taps left running or dripping?	3. Are people using too much water for simple tasks (e.g., washing hands)?	4. Are electric appliances used unnecessarily (e.g., TV/radio always on)?	5. Could this be avoided? (Yes/No)
Classroom					
Bathroom					
Dormitory					
Kitchen					

Step 2: Analyse Your Data

Write your answers to the following questions as part of your findings.

1. Which location showed the most waste?
2. What type of resource was mostly wasted — water or electricity?
3. What was the most common cause of waste you observed?
4. Which things can be changed easily, and which need more planning?

Step 3: Write Your Findings and Suggestions

Write a short report (6–8 lines) including:

- A summary of your findings
- Two logical suggestions to help reduce the waste of water or electricity
- One example of how you will personally help make the change at school

Note: At home, try doing a similar check to find out if there's any waste of water or electricity and think of ways to fix it.

Intelligent guessing

If one ball of banku costs GH¢*5, how much would 2 balls cost?*

Solution: Estimated answer will be GH¢10

Example 1.6

Observe the pattern and make a guess of the next 3 numbers of cocoa seedlings planted each week:

5, 9, 13, 17, 21, ...

A young farmer in Ghana is expanding his cocoa farm by planting more seedlings every week. He starts with 5 seedlings and increases the number each week using a fixed pattern.

Your Task

- Guess the number of seedlings to be planted in the next 3 weeks.
- Justify your guess based on the pattern.
- Provide a general rule that can be used to predict how many seedlings he will plant in any given week.

Solution

Find the difference between consecutive terms:

So, it's a pattern where each term increases by **4**

- $9 - 5 = 4$
- $13 - 9 = 4$
- $17 - 13 = 4$
- $21 - 17 = 4$

Therefore, the number of seedlings for each of the next three weeks will be:

- $21 + 4 = 25$
- $25 + 4 = 29$
- $29 + 4 = 33$

Justification

The farmer adds 4 more seedlings each week. That is, the number of seedlings increases in equal steps.

Therefore, the number of seedlings in the **n-th week** can be expressed by the rule $4n + 1$.

Decision making in solving problems

Decision making is a logical reasoning process that helps individuals choose the best possible solution among available options when solving a problem. It involves evaluating facts, analysing options, predicting outcomes and selecting the most appropriate course of action.

Look at the scenario below carefully to understand how logical reasoning is used to make a logical decision.

Scenario

Mr. Yeboah is a farmer in Ejura who wants to increase his maize yield this year. He has two fertiliser options:

- **Option A:** An *organic fertiliser* made from animal manure — it is *cheaper* and *environmentally friendly*, but takes *longer to work*.
- **Option B:** A *chemical fertiliser* — it acts *faster* and can produce *higher yields*, but it is *more expensive* and may reduce soil quality over time.

To make a logical decision, Mr. Yeboah

1. Checks the facts — how much each fertiliser costs and how fast they work.
2. Analyses his options — which one fits his *budget* and *timeline* before the rainy season starts.
3. Predicts the outcomes — more profit in the short term (Option B) vs healthier soil in the long term (Option A).
4. Chooses the best solution — based on his goal to earn more money quickly and reinvest, he selects Option B for this farming season.

Example 1.7:

Obaa Yaa Akos is at the Dambai market and feels very hungry. She has GH¢ 15 and has only one hour before she must catch her bus to Chinderi. She sees two popular chop bars nearby:

A: Ma Vida Chop Bar is very busy. She hears people say the food is tasty, but she may have to wait 20 –30 minutes to get served.

B: Auntie Esi's Chop Bar is less crowded. There's no queue, but her friend once said the food there gave him a stomach upset.

Using intelligent guessing and logical reasoning, what should Obaa Yaa Akos do — wait at Ma Vidas' chop bar, eat quickly at Auntie Esi's, or find a snack nearby instead?

Support your decision with valid arguments.

Solution:

Eating at Ma Vida's Chop Bar means Obaa Yaa Akos gets quality and satisfying food, which would be to her advantage. However, the long queue poses a risk of delay, and she may miss her bus.

Auntie Esi's Chop Bar offers faster service, but the risk of food poisoning could affect Obaa Yaa Akos's comfort during her journey.

Looking for a snack nearby will save time and reduce the health risk, allowing Obaa Yaa Akos to catch her bus on time.

There are many snack options in a busy place like Dambai, so it is unlikely that she will be unable to find something affordable and satisfying.

Obaa Yaa Akos has a better chance of being safe, on time, and less stressed if she finds a clean and quick snack.

Therefore, Obaa Yaa Akos should decide to buy a snack nearby instead of eating at either chop bar.

Example 1.8

Consider the following statements

x: No plantain sellers are dishonest.

y: Some dishonest traders are very persuasive.

Using logical reasoning and Venn diagrams, determine whether the following statements are valid conclusions or false based on the given statements (***x*** and ***y***):

i. Afia is a plantain seller ⇒ Afia is not persuasive.

ii. Kojo is a plantain seller and persuasive ⇒ Kojo is not dishonest.

iii. Adwoa is not a plantain seller ⇒ Adwoa is dishonest and persuasive.

iv. Which of these people would you prefer to buy plantain from and why?

Solution

Let **U** = {All traders}

Let **D** = {Dishonest traders}

Let **P** = {Plantain sellers}

Let **V** = **{**Persuasive traders}

From P ∩ D = **Ø** (because no plantain seller is dishonest)

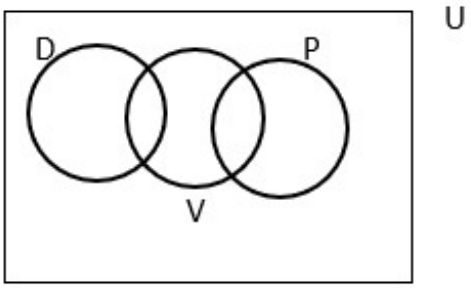


Figure 1.3

i. Afia is a plantain seller ⇒ Afia is not persuasive. We cannot know if this is valid, as some plantain sellers are persuasive, as there is an intersection between P and V.

ii. Kojo is a plantain seller and persuasive ⇒ Kojo is not dishonest. We know that not all plantain sellers are dishonest, so this is true.

iii. Adwoa is not a plantain seller ⇒ Adwoa is dishonest and persuasive. This is not valid, as it is not only plantain sellers who are not dishonest, and Adwoa.

iv. I would buy from Afia as he sells plantain and there is a chance they are not persuasive, whereas, buying from Kojo, I know they are persuasive. I would not buy from Adwoa as he does not sell plantain.

Example 1.9

The school bell rings when it is break time ⇒ It is break time because the bell just rang.

Verify the statement as valid or not.

Solution

- Does the school bell only ring at break time?
- Can the bell ring for other reasons (e.g., assembly, emergency, change of periods)?
- If the bell rang for a reason other than break time, then the conclusion that *"it is break time"* is not valid.

Therefore, if the first part of the statement is not always true (i.e., the bell rings only during break), then we cannot confidently say it is break time just because the bell rang.

So, the statement is **not** valid.

Example 1.10

Every Monday the students wear their school uniforms ⇒ Today is Monday, so the students are wearing their school uniforms. Verify the statement as valid or not.

Solution

- Is it true that students always wear school uniforms on Mondays?
- Is today indeed Monday?
- Are the students wearing their uniforms today as expected?

Since all the conditions are true and consistent with the statement, the conclusion is logically valid. Therefore, the statement is valid.

Investigating Arguments

Investigating arguments involves carefully analysing statements or claims to determine whether they are logically valid, sound or false. It requires examining how conclusions are drawn from given premises and whether those premises support the conclusion.

The steps involved in solving a problem may include:

1. Clearly defining the purpose of the investigation,
2. Collecting relevant information through methods such as interviews or surveys,
3. Examining the data to uncover patterns and insights, and
4. Drawing a conclusion based on the findings.

In a similar vein, Jerome Bruner suggests that effective problem-solving involves first grasping the nature of the problem, then creating a strategy to tackle it, putting that strategy into action, and finally reflecting on the entire process (Okafor, 2019).

Some challenging situations that might require a solution

1. You have run out of data but need to submit an assignment online before midnight.
2. Your roommate plays loud music while you are trying to study for an important test.
3. You missed an important class because of a family emergency and now feel behind.
4. There is a misunderstanding between you and a close friend, and it is affecting group work.
5. You lost your student ID and need it to access the library and exam hall.
6. Your electricity has been disconnected, and you have a major take-home test due.
7. Your class project partner is not contributing to the work, but your grades depend on teamwork.

8. Heavy rains have flooded your area, making it hard to get to school on time.

Example 1.11

It is said that students spend about GH¢2 daily on school lunch.

Investigate the statement that 'Everyone in our class spends exactly GH¢2 on lunch each day'.

Solution

Ask 10 students how much they spent on lunch today:

Amounts (in GH¢): 2, 2.5, 1.5, 2, 2, 2, 2, 1.5, 2.5, 2

Step 1: Find the total amount spent:

$2 + 2.5 + 1.5 + 2 + 2 + 2 + 2 + 1.5 + 2.5 + 2 = 20$

Step 2: Find the mean amount:

Mean $(\pi) = \frac{20}{10}$

The **average amount** spent is GH¢**2**, which matches the general belief.

However, not all students spent exactly GH¢**2** with some spending more, others less.

So, the claim that "***everyone spends exactly GH¢2***" is not valid, but the average spending of GH¢2 is accurate and supported by the data.

VARIATION 1: INTRODUCTION TO VARIATION

In mathematics, variation refers to how one quantity changes in relation to another. It helps us understand relationships between variables—how a change in one affects the other.

In this lesson, we are going to look at four types of variation:

1. Direct Variation
2. Inverse Variation
3. Joint Variation
4. Partial Variation

Direct Variation

Direct variation occurs when one variable increases, or decreases, in direct proportion to another. That is, as one value goes up, the other goes up too—or when one goes down, the other goes down as well.

Mathematical Representation

If y varies directly as x, then:

$y \propto x$ or $y = kx$

where k is the constant of proportionality.

Example 1.12

In your community, a sachet water vendor sells one sachet of water for GH¢0.50. The cost of the water depends directly on the number of sachets bought.

Complete the table below by calculating the cost of different quantities of sachet water.



Figure 1.4

Table 1.5

Number of Sachets (x)	Cost (GH¢) (y)
1	
2	
5	
10	
20	

Use your results to:

i. Identify the constant of variation .

ii. Write the equation that relates cost to quantity .

iii. Work out how much 50 sachets would cost.

iv. Determine how many sachets one can buy with GH¢7.00.

Solution:

Table 1.6: Completed table

Number of Sachets (x)	Cost (GH¢) (y)
1	0.50
2	1.00
5	2.50
10	5.00
20	10.00

i. Finding the constant of variation: $y = kx$

We know that when $x = 1$, $y = 0.5$, so we can substitute these values into the equation to find the constant k.

$k = \frac{y}{x} = \frac{0.5}{1} = 0.5$

ii. $y = 0.5x$

iii. Cost of 50 sachets $y = 0.5 \times 50$

$y = \text{GH¢}\ 25.00$

iv. $x = \frac{7}{0.5} = 14\ sachets$

This example shows how direct variation applies in everyday life

Example 1.13

If y varies directly as x, and $y = 15$ when $x = 5$.

Find the value of y when $x = 8$

Solution

From the direct variation formula: $y = kx$

Step 1: Find the value of the constant k by putting $x = 5\ and\ y = 15$

$y = kx$

$15 = k(5) \Rightarrow 15 = 5k$

$k = \frac{15}{5}\ (Dividing\ both\ sides\ by\ 5) = 3$

Step 2: Use the value of $k = 3$ to find y when $x = 8$

$y = 3 \times 8$

$y = 24$

Inverse Variation

Inverse variation (or inverse proportion) is a relationship between two variables where as one increases, the other decreases in such a way that their product remains constant.

Mathematical Representation

If y varies inversely as x, then:

$y \alpha \frac{1}{x}$ or $y = \frac{k}{x}$

where k is the constant of proportionality.

Example 1.14



Figure 1.5

In your school/community, fetching water from the borehole is a common chore. It usually takes one student 30 minutes to fill 10 buckets. But when more students help, the time it takes reduces.

Use the table below to calculate the time it takes based on different numbers of students working together.

Table 1.7

Number of Students (x)	Time (minutes) (y)
1	30
2	
3	
5	
6	

Use the inverse variation formula:

a. Calculate the constant k, then complete the table.

b. What happens to the time when more students are involved?

c. Predict:

i. How long will it take 10 students to fill the 10 buckets?

ii. How many students are needed to complete the task in 6 minutes?

Solution

a. **Step 1:** find the constant k

$$y = \frac{k}{x}$$

$$k = y \times k$$

$$k = 1 \times 30 = 30$$

Step 2: Fill the table using $y = \frac{30}{x}$

Completed table

Number of Students (x)	Time (minutes) (y)
1	30
2	15
3	10
5	6
6	5

b. When more students are involved, the time taken to complete the task decreases.

This is because the work is shared among more people, so each person does less, and the task is finished faster.

c.

i. For 10 students: $y = \frac{30}{10} = 3\ minutes$

ii. For 6 minutes: $x = \frac{30}{6} = 5\ students$

Example 1.15

It takes 4 students 12 minutes to clean a classroom. If the time taken varies inversely with the number of students, how long will it take 6 students to clean the same classroom?

Solution

Since time *t* varies inversely with the number of students *s*, we use the formula: $t = \frac{k}{s}$

Step 1: Find the constant k:

$$12 = \frac{k}{4}$$

$$\Rightarrow k = 12 \times 4$$

$$k = 48$$

Step 2: Find the time for 6 students:

$$t = \frac{48}{6} = 8\ minutes$$

Therefore, it will take 6 students 8 minutes to clean the same classroom.

Using Proportional Reasoning

Proportional reasoning involves using the concept of ratios and proportions to solve problems. It helps us compare two quantities and find unknown values when one part of a relationship changes.

a. In direct variation, the ratio $\frac{y}{x}$ is constant.

b. In inverse variation, the product xy is constant.

Table 1.8: Table Comparison

Type of Variation	Relationship	General Form	Constant Quantity
Direct Variation	$y \propto x$	$y = kx$	$\frac{y}{x} = k$
Inverse Variation	$y \propto \frac{1}{x}$	$y = \frac{k}{x}$	$xy = k$

Generalisations from Variation

By investigating different scenarios, students can generalise:

a. In direct variation, doubling one variable doubles the other.

b. In inverse variation, if one variable is doubled, the other is halved (assuming a constant product).

c. The graphs of direct variation are straight lines through the origin.

d. The graphs of inverse variation are rectangular hyperbolas, or a 'reciprocal graph'.

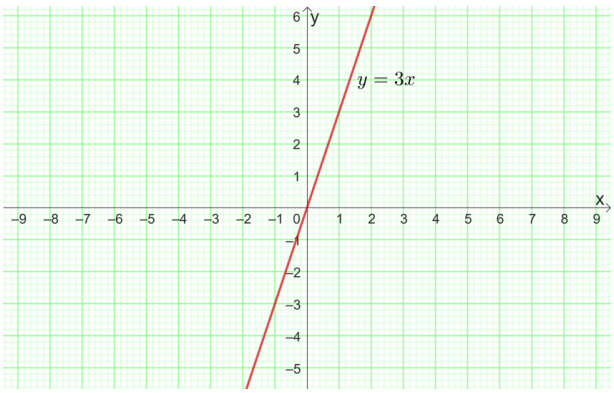


Figure 1.6: Graph of direct variation

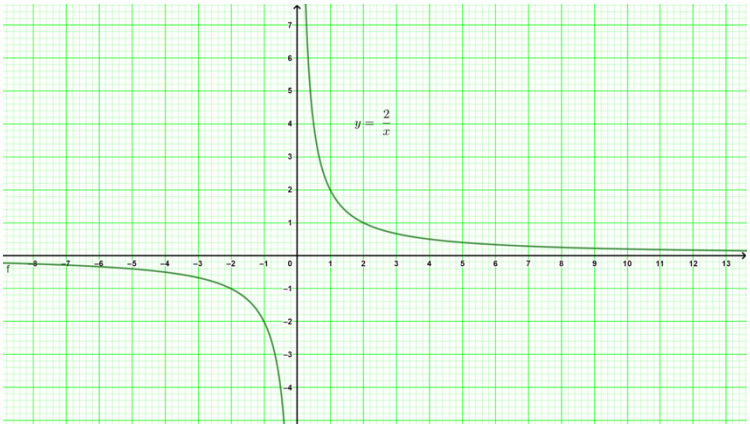


Figure 1.7: Graph of inverse variation

Example 1.16

A textbook costs GH¢ 75.00. The total cost C varies directly with the number of textbooks n.

a. Express the cost C in terms of n.

b. If a school pays GH¢ 525.00, how many textbooks did they buy?

Solution

a. Since cost C varies directly with the number of textbooks n:

$C \propto n$. This implies $C = kn$

$$75 = k(1)$$

$$\Rightarrow k = 75$$

Therefore $C = 75n$

b. Given: $C = 525$

Substitute into the formula:

$525 = 75n$

$n = 7$

The school bought 7 textbooks for GH¢525.00.

Example 1.17

The time (T) taken to fill a water tank is inversely proportional to the rate (R) at which water flows into the tank. If the rate is 10 litres per minute, it takes 30 minutes to fill the tank.

a. Derive the equation and find k (the constant of proportionality).

b. Calculate the time it will take to fill the tank if the water flows at 15litres per minute.

c. **Debate**: *Why might this inverse model fail in areas with low or fluctuating water pressure?*

Solution

a. Deriving the equation

Since time T is inversely proportional to the rate R,

$T \propto \frac{1}{R}$

$T = \frac{k}{R}$

Given:

T = 30 minutes, R=10 litres/min

Substitute into the equation to find k:

$30 = \frac{k}{10} \Rightarrow k = 300$

So, the equation is: $T = \frac{300}{R}$

b. Calculate the length of time it takes to fill the tank when the water flow rate is 15 litres per minute

$T = \frac{300}{15} = 20\ minutes$

It will take 20 minutes to fill the tank at a rate of 15 litres per minute.

c. **Debate**: Why might this inverse model fail in areas with low or fluctuating water pressure?

- The model assumes a constant rate of water flow.
- In reality, water pressure can drop or fluctuate, especially during peak hours or in dry seasons.
- If pressure reduces during filling, the flow rate drops, and the time increases—violating the assumption of a fixed inverse relationship.
- Therefore, in areas with unstable supply, the inverse model may not accurately predict the filling time.

Joint variation

Activity 1.5 Understanding Joint Variation

Objective: Discover how the strength of a paper bridge depends on two variables (width and thickness) to understand joint variation.

Materials needed

- 3 sheets of paper (same size)

- 10–15 coins (or small weights like paper clips)
- Ruler
- Tape (optional, to secure bridge ends)

Instructions

1. **Bridge A (Control)**
   - Fold 1 sheet to 5 cm width.
   - Place between desks (15 cm gap).
   - Add coins until collapse. Record:
   - Max coins = _____
2. **Bridge B (2x Width)**
   - Fold 1 sheet to 10 cm width.
   - Repeat test. Record:
   - Max coins = _____
3. **Bridge C (2x Thickness)**
   - Stack 2 sheets, fold to 5 cm width.
   - Repeat test. Record:
   - Max coins = _____

Table 1.9: Data table

Bridge	Width (cm)	Thickness (sheets)	Max Coins Held
A	5	1	
B	10	1	
C	5	2	

Answer these questions after the experiment

1. Which bridge held the most coins? Why?
2. How did increasing width affect strength? How did increasing thickness affect it?
3. If strength depends on BOTH width and thickness, what maths term could describe this?

In joint variation, a variable depends on two or more other variables simultaneously — combining direct and sometimes inverse variation.

For example, the volume of bags of maize (V) a farmer can harvest varies jointly with the size of the farmland (F) in acres and the amount of rainfall (R) in mm during the season.

Example 1.18

The total wages (W) paid to farm workers vary jointly with the number of workers (n) and the number of days (d) worked. If 10 workers working for 6 days are paid GH¢3,000.00.

a. Find the constant of variation (k).

b. How much will be paid to 8 workers for 5 days?

Solution

a. $W \propto nd$

$W = knd$

Given: $W = 3000,\ d = 6,\ n = 10$

$3000 = k \times 10 \times 6$

$\Rightarrow 3000 = 60k$

$\Rightarrow k = 50$

b. $W = knd$

$W = 50 \times 8 \times 5 = GH¢2,000.00$

Therefore, GH¢2,000.00 will be paid to 8 workers for 5 days

Variation 2: Introduction to Partial Variation

In daily life, there are many situations where the total cost or value does not start at zero but includes both a fixed charge and a variable component. This is known as partial variation. Learning how this works helps manage budgets, plan services, and analyse real-world data. Partial variation is similar to direct variation but includes an added constant value.

Understanding Partial Variation

Partial variation happens when a quantity is made up of two parts.

- A fixed amount (also called a constant or base value), and
- A variable portion that changes directly with another quantity.

This can be described with the equation: $y = kx+c$

Where:

- y is the outcome (dependent variable),
- x is the input (independent variable),
- k is the rate at which y changes with x,

- c is the starting or fixed value.

This formula represents a straight-line graph that does not pass through the origin, showing a linear relationship with a constant starting point.

Example 1.19

In a school in Ghana, a student is charged a fixed amount of GH¢10.00 every week for maintenance, plus an additional GH¢2.00 for each hour they use the school's computer lab.

a. Write a mathematical equation that represents the total amount (T) the student pays in a week in terms of the number of hours (h) they use the computer lab.

b. If the student uses the computer lab for 5 hours in a week, how much will they pay in total?

c. What part of this variation is constant, and what part varies with the number of hours?

Solution

a. The total amount (T) is made up of a fixed charge and a variable charge that depends on hours used.

Fixed charge = GH¢10.00

Variable charge per hour = GH¢2.00

Therefore, the equation is:

$T = 10 + 2h$

b. If the student uses the computer lab for 5 hours:

$T = 10 + 2(5)$

$T = 10 + 10$

$T =$ GH¢20.00

So, the student will pay GH¢20.00 in total.

c. The constant part of the variation is the GH¢10.00 fixed maintenance fee.

The part that varies is the GH¢2.00 per hour, which depends on the number of hours (h) used.

Example 1.20

A mobile money vendor in Ghana charges a fixed fee of GH¢3.00 for every withdrawal transaction, plus GH¢0.50 for each GH¢100.00 withdrawn.

a. Write an equation that models the total charge (C) for a withdrawal of GH¢x.

b. Find the total charge if a customer withdraws GH¢500.00.

c. Identify the constant part and the variable part of the variation.

Solution

a. The total charge (C) includes a fixed fee and a variable fee based on the amount withdrawn.

Fixed fee = GH¢3.00

Variable fee = GH¢0.50 for every GH¢100 withdrawn, which is $0.50 \times \left(\frac{x}{100}\right)$

Therefore, the equation is:

$C = 3 + 0.50\left(\frac{x}{100}\right)$

$C = 3 + 0.005x$

b. For a withdrawal of GH¢500.00:

$C = 3 + 0.005 \times 500$

$C = 3 + 2.5$

$C =$ GH¢5.50

c. The constant part is the GH¢3.00 fixed fee.

The variable part is GH¢0.005 times the amount withdrawn x, which depends on the withdrawal amount.

Activity 1.10 Planning a Three-Day Inter-SHS STEM Fair at Your School

Your school has been selected to host a STEM Fair with participants from 3 different SHS levels. You are in charge of the Planning and Budget Committee.

Use your knowledge from SHS 1 to SHS 3 to solve the tasks and prepare a proposal.

1. You have data on 120 students: 60 study Mathematics, 50 study ICT and 30 study both.

a. Use a Venn diagram to show the distribution.

b. How many study only Mathematics? Only ICT? Neither?

c. Write the set equation using De Morgan's Law to represent the group not studying either subject.

d. Write two declarative and two interrogative statements based on this data.

e. Identify a positive and negative statement from your discussion.

f. Construct one valid and one faulty logical argument about students who study ICT.

2. Out of a GH¢3,000.00 sponsorship package:

   $\frac{1}{3}$ will be spent on materials,

   20% on refreshments,

   The rest on certificates and souvenirs.

   a. Calculate the amount to be spent on each category.

   b. Show connections between the fraction and percentage amounts.

   c. Apply additive and multiplicative inverse to check allocations using model charts.

3. To create a banner, the area is measured as $\sqrt{50}$ m^2

   a. Simplify the area and express it in its simplest surd form.

   b. If the printing company gives a discount using the formula:

   Discount $= 10\log_{10}(x)$, where x is the number of banners.

   Calculate the discount for 100 banners.

4. You are budgeting for 3 items: printing, feeding, and transport in a ratio 4:3:3.

   a. If the total cost is GH¢2,000, calculate each item's share.

   b. Calculate the rate if printing costs GH¢800 for 4 hours of service.

5. During a STEM Fair, water sachets are distributed to guests for GH¢0.50 per sachet. Also, one volunteer can fetch 10 buckets of water in 30 minutes.

Answer the following questions

a. The cost of water sachets varies directly with the number of sachets distributed.

b. Calculate the total cost of 80 sachets of water.

c. The time taken to fetch water varies inversely with the number of students fetching the water.

d. If one volunteer fetches 10 buckets of water in 30 minutes, determine the time it will take 5 students to fetch the same number of buckets.

EXTENDED READING

- Akrong Series: Core Mathematics for Senior High Schools New International Edition (Pages 207 – 217, 788 - 794)
- Aki – Ola series: Core Mathematics for Senior High Schools in West Africa, Millennium edition 5 (Pages 25 – 32, 114 - 124)
- Baffour – Ba Series: Core Maths for Schools and Colleges, (Pages 494 – 515, 760 - 771)

REVIEW QUESTIONS

1. Analyse these statements and label each as True, False, or Debatable.
   - **i.** Nigeria is the most populous country in Africa.
   - **ii.** If x is an even number, then $x + 1$ is odd.
   - **iii.** Ashanti Region borders the Upper East Region.
   - **iv.** Ghana became a republic in 1960.
   - **v.** Science is more useful than Social Studies.
   - **vi.** Every rhombus is a parallelogram.
   - **vii.** All square numbers are even.
   - **viii.** If $y < 0$, then $y - 3 < 0$.
   - **ix.** The angles on a straight line add up to 180°.
2. Write a conjunction and a disjunction for each pair of statements.

   State whether each is "True" or "False"

   - **a.** **P:** Multiples of 5 end in 0 or 5.
   - **b.** **Q:** All multiples of 5 are even numbers.
   - **c.** **R:** All quadrilaterals have four sides.
   - **d.** **S:** Some quadrilaterals have equal angles.
   - **e.** **T:** All mammals breathe through lungs.
   - **f.** **U:** Whales are mammals.
3. A farmer in the Northern Region wants to grow groundnuts and yam on her 12-acre land. She must grow at least 4 acres of groundnuts and at least 3 acres of yam. Groundnuts yield 1.5 tons per acre, and yam yields 2.5 tons per acre. How many acres of each crop should she plant to maximize her total yield and why?
4. Consider these two statements:
   - **Statement A:** Respecting elders is a sign of discipline.
   - **Statement B:** Discipline is a core value in many Ghanaian homes.

   Examine the conclusions and label each as "Definitely True", "Possibly True", or "False".
   - **a.** Efua respects elders ⇒ Can we conclude that she is disciplined?
   - **b.** Kofi is disciplined ⇒ Can we conclude that he respects elders?
   - **c.** Amma is from a Ghanaian home ⇒ Can we conclude that she is disciplined?

5. A farmer buys fertiliser at GH¢60.00 for 3 bags. How much will 8 bags cost, assuming the cost varies directly with the number of bags?

6. The distance a car travels is directly proportional to the time when moving at constant speed. If it covers 120 km in 2 hours, how far will it go in 5 hours?

7. A group of workers can complete a job in 12 days. If more workers join and the job now takes 6 days, how many workers were added, assuming the number of days varies inversely with the number of workers?

8. A pipe fills a tank in 4 hours. If a second pipe is added and both work together to fill the same tank in 2 hours, how long would it take the second pipe alone to fill the tank?

9. The total cost (C) of buying yams varies jointly as the number of tubers (t) and the price per tuber (p). If 10 tubers cost GH¢80.00 when each costs GH¢8.00, find the total cost of 15 tubers at GH¢6.00 each.

10. The work done varies inversely as the number of workers (n) and the number of hours (h) worked. If 5 workers complete a task in 6 hours, how many hours will it take 3 workers to do the same task?

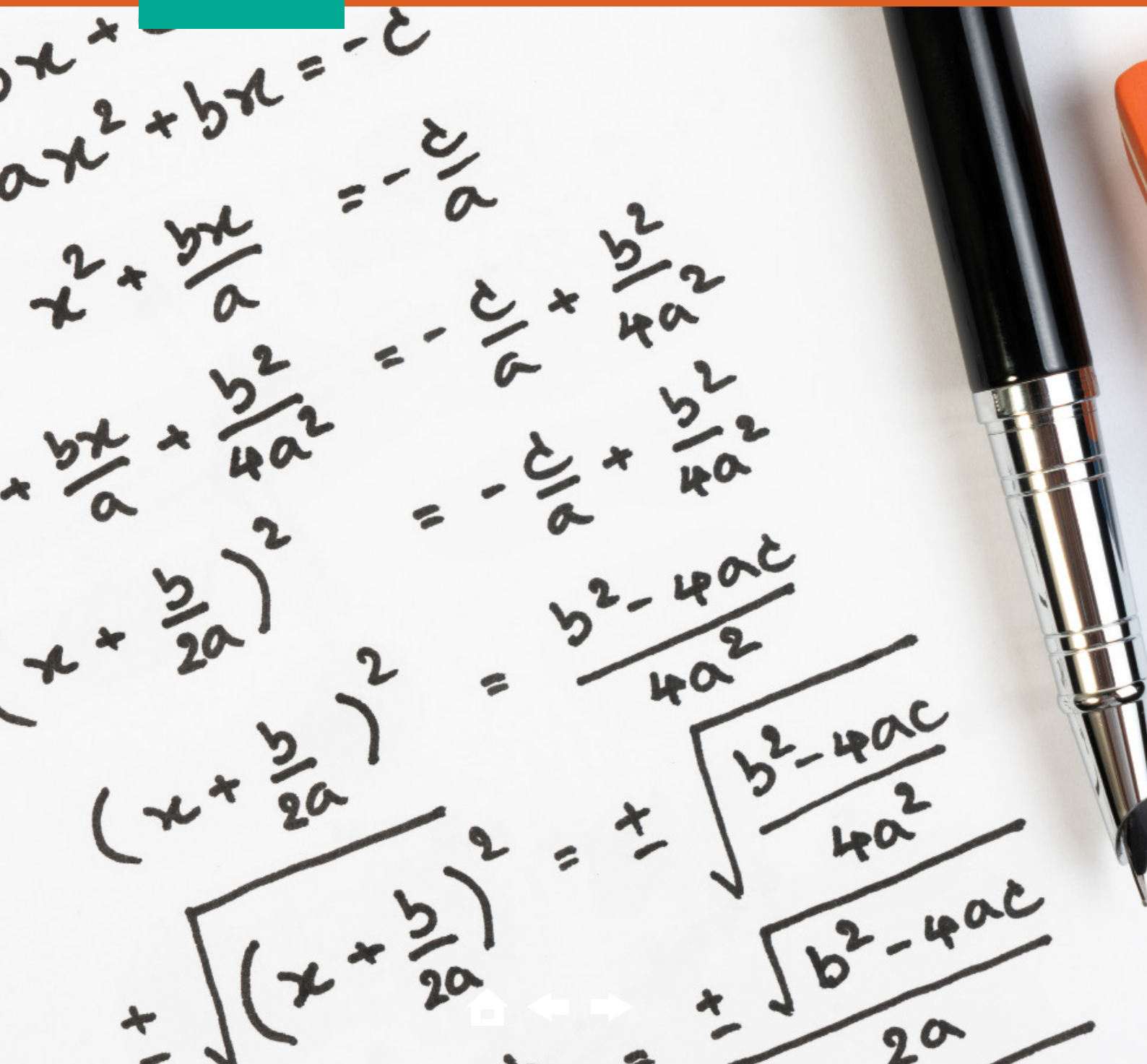
11. A taxi driver in Accra charges a base fare of GH¢10.00, plus GH¢2.00 per kilometre. Write an equation for the total cost (C) of a trip and calculate the cost for a 7 km ride.

12. A data bundle costs GH¢5.00 for activation plus GH¢0.10 per MB used. How much will 250 MB of data cost?

SECTION

2

ALGEBRAIC REASONING



PATTERNS AND RELATIONSHIPS

Identifying And Solving Quadratic Equations

Introduction

Over the past two years, you have developed a strong foundation in core algebraic concepts. In SHS 1, you explored number patterns, algebraic expressions, factorisation, linear equations, and simultaneous equations using both elimination and substitution methods. You also examined functions, graphs, gradients, and line segments. These are skills crucial for analytical thinking.

In SHS 2, you deepened your understanding through the graphical solution of linear equations and practical applications of simultaneous equations. You engaged with sequences and series, enhancing your ability to model and solve real-life problems mathematically.

This year you will review solving quadratic equations using the factorisation method, difference of two squares and the general formula. You will also explore how to solve quadratic equations using graphical methods. Finally, you will interpret quadratic functions, identify maximum and minimum values, equations of the axis of symmetry and integrate both linear and quadratic equations graphically. This will prepare you for advanced problem-solving and real-world mathematical applications.

KEY IDEAS

- Identifying and solving quadratic equations.
- Solving quadratic equations graphically.
- Finding maximum and minimum points.
- Finding equations of axes of symmetry.
- Solving linear and quadratic equations together using graphs.
- Applying quadratic equations to real-life problems.

FACTORISATION OF QUADRATIC EQUATIONS

In Year 1, Section 3, you learned how to solve quadratic equations by factorisation, using algebraic tiles, standard identities, and the general formula.

What Is A Quadratic Function And Equation?

A quadratic function is a mathematical expression of the form: $y = ax^2 + bx + c$, where a, b, and c are real numbers, and $a \neq 0$. The graph of a quadratic function is a parabola, which opens upwards if $a > 0$ or downwards if $a < 0$ as shown below.

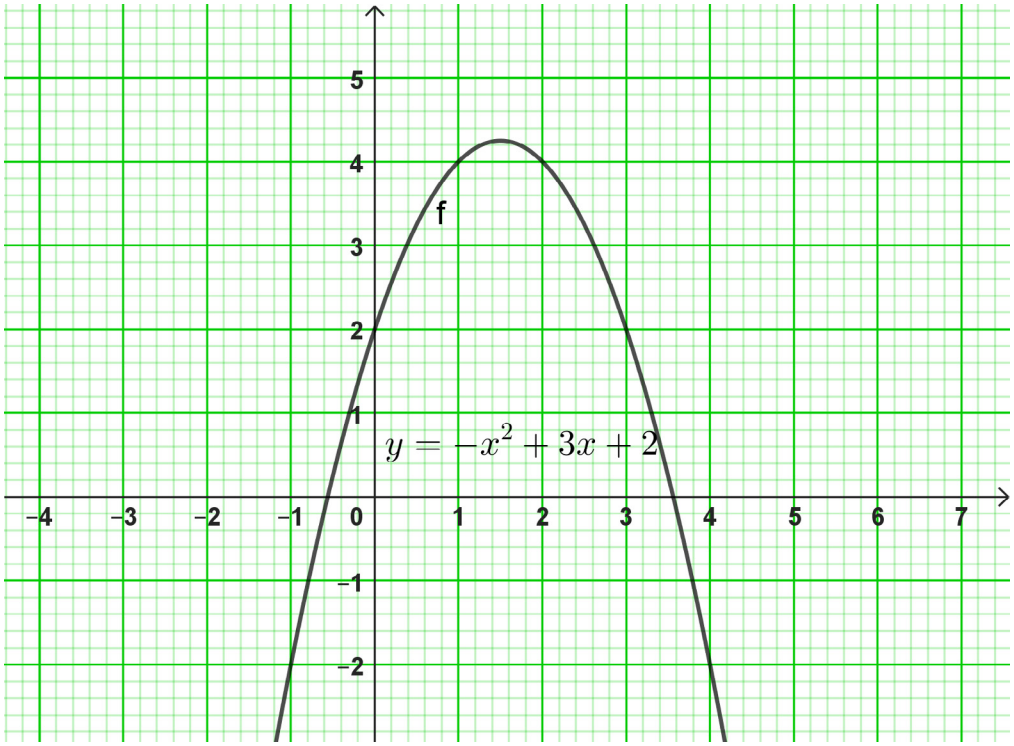


Figure 2.1a: Graph of a parabola (a < 0)

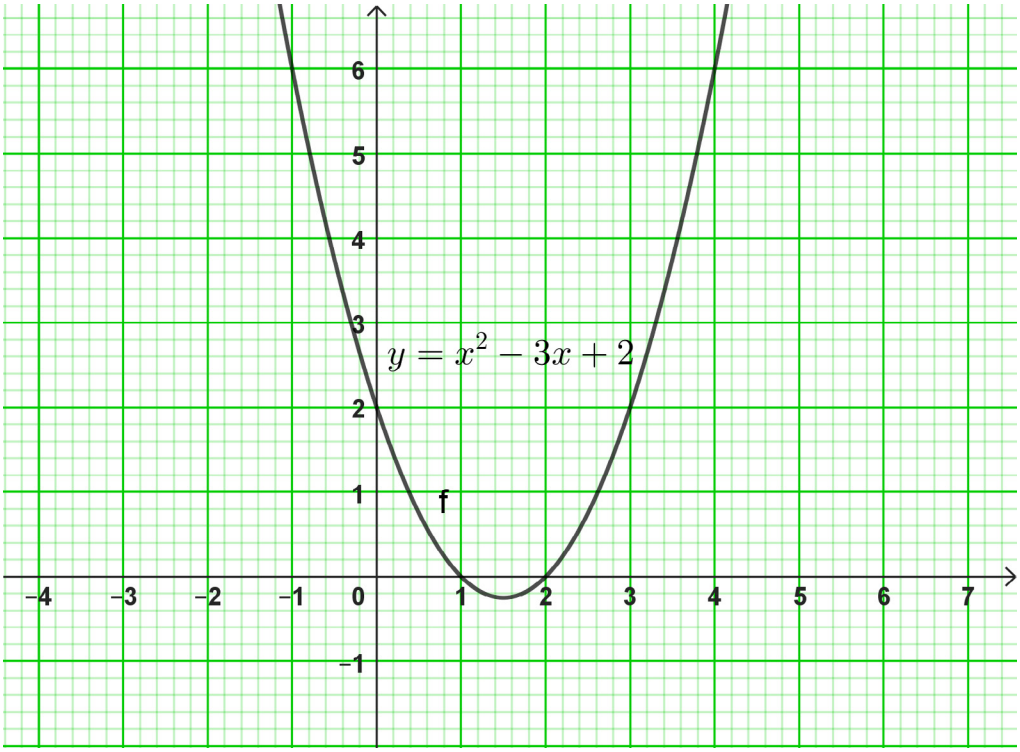


Figure 2.1b: Graph of a parabola (a > 0)

The following are examples of quadratic functions

- $f(x) = 2x^2 + x - 3,$
- $y = -3t^2 + 4t + 5,$
- $y = 3x(x + 2) - 4$

When we set a quadratic function equal to zero, we get a quadratic equation

$$ax^2 + bx + c = 0$$

The following are examples of quadratic equations:

- $2x^2 + x - 3 = 0$
- $3t^2 + 4t + 5 = 0$
- $3x(x + 2) - 4 = 0$

Every quadratic equation has a maximum of two roots or solutions.

Methods for solving quadratic equations

The following methods can solve quadratic equations.

- Factorisation
- Completing the square
- The general quadratic formula
- Graphical method

Finding the roots of a quadratic equation by factorisation

The roots (or solutions) of the quadratic equation, $ax^2 + bx + c = 0$, are the values of x that satisfy $ax^2 + bx + c = 0$

Here are the steps for solving quadratic equations by factorisation.

1. Identify a, b and c
2. Calculate the product of a and c, that is: $a \times c = ac$
3. Find two numbers whose product is equal to ac and whose sum is equal to b.
4. Rewrite the middle term by replacing the term bx with the two numbers you found in step 3.
5. Group the first two terms and the last two terms, then factor out common factors from each group.
6. Factor out the common binomial factor to obtain the factored form of the quadratic equation.
7. Set each factor equal to zero and solve for x to find the solutions of the quadratic equation.

Go through the following examples to refresh your knowledge on how to solve quadratic equations by factorisation.

Example 2.1

Find the roots of the equation $3x2 + 10x - 8 = 0$

Solution

To find the roots of the equation $3x^2 + 10x - 8 = 0$, follow these steps.

1. Multiply the coefficient of 'x^2' (the term) by the constant term '**-8**' (the c term):

   That is $3(-8) = -24$

2. Find two factors that have the product of **–24** and the sum of **10** (that is the coefficient of x, or the term).
3. The numbers are: **12** and **−2**
4. Replace $10x$ with $12x$ and $-2x$:

   $3x^2 + 12x - 2x - 8 = 0 = 0$

5. Group the pair of terms on the left side of the equation and the right-hand side:

   $(3x^2 + 12x) - (2x + 8) = 0$

   Note: The brackets in $(2x + 8)$ has changed to a positive sign because of the negative in front of $2x$ just before the bracket.

6. Factor each group: $3x(x + 4) - 2(x + 4) = 0$
7. Factor the common binomial: $(x + 4)(3x - 2) = 0$
8. Equate each factor to zero and solve for :

   $(x + 4) = 0$ or $(3x - 2) = 0$

   $x =$ -4 or $x = \frac{2}{3}$

9. Therefore, the roots of the quadratic equation:

   $3x^2 + 10x - 8 = 0$ are $x = -4$ or $x = \frac{2}{3}$

Example 2.2

Find the roots of the equation $2x^2 - x - 6 = 0$

Solution

To find the roots of the equation $2x^2 - x - 6 = 0$, replicate the steps in ***Example 2.1*** as follows.

$$2x^2 - 4x + 3x - 6 = 0$$
$$(2x^2 - 4x) + (3x - 6) = 0$$
$$2x(x - 2) + 3(x - 2) = 0$$
$$(x - 2)(2x + 3) = 0$$
$$(x - 2) = 0 \text{ or } (2x + 3) = 0$$
$$x = 2 \text{ or } x = \frac{-3}{2}$$

Therefore, the roots of the quadratic equation, $2x^2 - x - 6 = 0$ are $x = 2$ or $x = \frac{-3}{2}$

Example 2.3

Find the zeros of the function, $y = x^2 - 5x + 6$

Solution

To find the zeros of the function, $y = x^2 - 5x + 6$ apply the steps in ***Example 2.1*** and solve as follows:

$x^2 - 3x - 2x + 6 = 0$

$(x^2 - 3x) - (2x - 6) = 0$

Note: The brackets in $(2x - 6)$ have changed to a negative sign because of the negative multiplier in front of the bracket.

$x(x - 3) - 2(x - 3) = 0$

$(x - 3)(x - 2) = 0$

$(x - 3) = 0$ *or* $(x - 2) = 0$

$x = 3$ *or* $x = 2$

Therefore, the zeros of the function, $y = x^2 - 5x + 6$ are $x = 3$ *or* $x = 2$

Example 2.4

Find the zeros of the function $f(x) = 2x^2 + 7x + 3$

Solution

To find the zeros of the function, $f(x) = 2x^2 + 7x + 3$ equate $f(x)$ to zero and follow the steps below.

$2x^2 + x + 6x + 3 = 0$

$(2x^2 + x) + (6x + 3) = 0$

$x(2x + 1) + 3(2x + 1) = 0$

$(2x + 1)(x + 3) = 0$

$(2x + 1) = 0$ *or* $(x + 3) = 0$

$x = -\frac{1}{2}$ *or* $x = -3$

Therefore, the zeros of the function, $f(x) = 2x^2 + 7x + 3$ are $x = -\frac{1}{2}$ *or* $x = -3$

Example 2.5

Solve the equation: $x^2 - 7x + 12 = 0$

Solution

Given, $x^2 - 7x + 12 = 0$, solve as follows:

$x^2 - 3x - 4x + 12 = 0$

$(x^2 - 3x) - (4x - 12) = 0$

$x(x - 3) - 4(x - 3) = 0$

$(x - 3)(x - 4) = 0$

$(x - 3) = 0$ *or* $(x - 4) = 0$

$x = 3$ *or* $x = 4$

Therefore, the values of x from the equation, $x^2 - 7x + 12 = 0$, are 3 and 4

Example 2.6

Solve the equation: $-x^2 - 7x - 12 = 0$

Solution

Given the equation, $-x^2 - 7x - 12 = 0$.

$-x^2 - 3x - 4x - 12 = 0$

$(-x^2 - 3x) - (4x + 12) = 0$

$-x(x + 3) - 4(x + 3) = 0$

$(x + 3)(-x - 4) = 0$

$(x + 3) = 0$ *or* $(-x - 4) = 0$

$x = -3$ *or* $x = 4$

Therefore, the values of from the equation, $-x^2 - 7x - 12 = 0$ are -3 and 4.

Solving quadratic equations using the quadratic formula

You can also use the quadratic formula to solve any quadratic equation. The formula is derived from completing the square of the general quadratic equation $ax^2 + bx + c = 0$.

You do not need to learn how the formula is derived, but it is helpful to understand where it comes from. What is most important is that you know and can use the quadratic formula correctly.

Given the general quadratic equation: $ax^2 + bx+c = 0$, follow these steps to appreciate where the formula is coming from:

1. Subtract '**c**' from both sides: $ax^2 + bx+c - c = 0 - c$.

   $ax^2 + bx = -c$.

2. Divide both sides of the equation by 'a':

   $\frac{a}{a}x^2 + \frac{b}{a}x = -\frac{c}{a}$

   $x^2 + \frac{b}{a}x = -\frac{c}{a}$

3. Divide the coefficient of x by 2:

   $\frac{b}{a} \div 2 = \frac{b}{2a}$

4. Square $\frac{b}{2a}$ and add it to either side of the equation

   $x^2 + \frac{b}{a}x = -\frac{c}{a}$: [note: $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$]

   $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

5. Complete the squares and simplify the right-hand side:

   $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$

6. Find the square root of both sides:

   $\sqrt{\left(x + \frac{b}{2a}\right)^2} == \pm\sqrt{\frac{b^2-4ac}{4a^2}}$

   $x + \frac{b}{2a} == \pm\sqrt{\frac{b^2-4ac}{4a^2}}$

7. Subtract $\frac{b}{2a}$ from both sides:

   $x + \frac{b}{2a} - \frac{b}{2a} = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$

8. Therefore the roots of the equation, $ax^2 + bx+c = 0$ *is*

   $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

   The general quadratic equation formula is $\boldsymbol{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

   Now, use the general quadratic formula to solve the following questions.

Example 2.7

Solve the quadratic equation, $x^2 + 3x - 10 = 0$

Solution

Observe the following steps.

1. Comparing $x^2 + 3x - 10 = 0$ with the general quadratic equation:

   $ax^2 + bx + c = 0$; $a = 1$, $b = 3$ and $c = -10$

2. Substitute the values of *a, b, and c* into the general formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-10)}}{2(1)}$$

3. Simplifying:

$$x = \frac{-3 \pm \sqrt{9+40}}{2}$$

$$x = \frac{-3 \pm \sqrt{49}}{2}$$

$$x = \frac{-3+\sqrt{49}}{2} \text{ or } x = \frac{-3-\sqrt{49}}{2}$$

$$x = \frac{-3+7}{2} \text{ or } x = \frac{-3-7}{2}$$

$$x = \frac{4}{2} \text{ or } x = \frac{-10}{2}$$

Therefore, $x^2 + 3x - 10 = 0$ is $x = 2$ or $x = -5$

Example 2.8

Solve the quadratic equation, $3x^2 - 2x - 7 = 0$

Solution

1. Compare $3x^2 - 2x - 7 = 0$ with the general quadratic equation $ax^2 + bx + c = 0$; $a = 3$, $b = -2$ and $c = -7$

3. Substitute the values of *a, b, and c* into the general formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-7)}}{2(3)}$$

4. Simplifying:

$$x = \frac{2 \pm \sqrt{4+84}}{6}$$

$$x = \frac{2 \pm \sqrt{88}}{6}$$

Therefore, $x = \frac{2+\sqrt{88}}{6}$ or $x = \frac{2-\sqrt{88}}{6}$

Example 2.9

Solve $-2x^2 + 4x + 6 = 0$

Solution

1. Compare $-2x^2 + 4x + 6 = 0$ with the general quadratic equation:

$ax^2 + bx + c = 0$; $a = -2$, $b = 4$ and $c = 6$

2. Substitute the values of *a, b, and c* into the general formula:

$$x = \frac{-4 \pm \sqrt{16-4(-2)(6)}}{2(-2)}$$

3. Simplifying:

$$x = \frac{-4 \pm \sqrt{16+48}}{-4}$$

$$x = \frac{-4 \pm \sqrt{64}}{-4}$$

$$x = \frac{-4 + \sqrt{64}}{-4} \text{ or } x = \frac{-4 - \sqrt{64}}{-4}$$

Therefore, the values of x for the equation, $-2x^2 + 4x + 6 = 0$ are $x = -1$ or $x = 3$

Example 2.10

Solve $-x^2 + 3x + 4 = 0$

Solution

1. Comparing $-x^2 + 3x + 4 = 0$ with the general quadratic equation:

   $ax^2 + bx + c = 0$; $a = -1$, $b = 3$ and $c = 4$

2. Substitute the values of into the general formula:

$$x = \frac{-3 \pm \sqrt{9-4(-1)(4)}}{2(-1)}$$

3. Simplifying:

$$x = \frac{-3 \pm \sqrt{9+16}}{-2}$$

$$x = \frac{-3 \pm \sqrt{25}}{-2}$$

Therefore, the values of x for the equation, $-x^2 + 3x + 4 = 0$ are $x = -1$ or $x = 4$

Activity 2.1 Identifying and solving quadratic equations

Materials Needed

- Graph book/Graph sheet
- Ruler
- Pencil
- Calculators

1. Given the function, $y = 2x^2 - 3x - 1$, complete the table below with the given interval, $-1 \leq x \leq 4$

x	-1	0	1	2	3	4
y	4			1		19

Using a scale of 2cm to 0.5 units on the x-axis and 2cm to 2 units on the y-axis:

2. Plot the points from the completed table on a graph sheet or graph book.
3. Join all the points with freehand, making a smooth curve.
4. Replace the coefficient of x^2 with -2, so we are sketching $y = -2x^2 - 3x - 1$
5. Create a table using the new function ($y = -2x - 3x - 1$) and the given domain, $-1 \leq x \leq 4$.
6. Draw the graph of the function.
7. Compare the two graphs $y = 2x - 3x - 1$ and $y = -2x - 3x - 1$.
8. What is your observation? Discuss with a classmate.
9. What effect did changing the sign of the coefficient of 2 have on the shape and direction of the graph?

Activity 2.2 Exploring why a ≠ 0 in a quadratic equation

1. Given the function, $y = -3x^2 - 5x - 1$, complete the table below with the given interval, $-2 \leq x \leq 2$

x	-2	-1	0	1	2
y	-3		-1		-23

Using a scale of 2cm to 0.5 units on the x-axis and 2cm to 4 units on the y-axis:

2. Plot the points on a graph sheet or graph paper.
3. Join all the points with freehand, making a smooth curve.
4. Replace the coefficient of x^2 with 0.
5. Write down the new equation.
6. Create a table of values using the function, $y = 0x^2 - 5x - 1$ and the given domain, $-2 \leq x \leq 2$.
7. Draw the graph of the function.
8. Compare the two graphs of $y = -3x^2 - 5x - 1$ and $y = 0x^2 - 5x - 1$.
9. What is your observation? Discuss with a classmate.
10. What happened to the graphs when the coefficient of x^2 was replaced with 0
11. How does the presence or absence of the x^2 term affect the curvature of the graph?
12. In what ways do the graphs of quadratic functions differ from those of linear functions?
13. Can you now predict the shape of a graph just by looking at the equation?

14. Why is it useful to explore different versions of the same equation by changing coefficients?
15. How can these observations help you in solving real-world problems or designing solutions? Discuss with your classmates.

Real-Life Applications of Quadratic Functions and Equations

Take a look at the following diagram in ***Figure 2.2***.

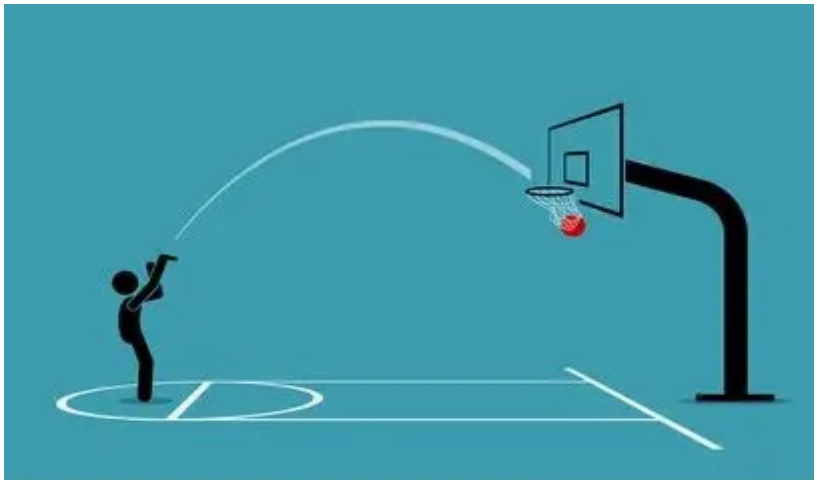


Figure 2.2: Illustration of a basketball shot path creating a parabola. Source: OIP.jwp-0KMvxIvbyO4Ac-c9MPAAAAA (404×280)

Now look at the bridge in ***Figure 2.3*** below.



Figure 2.3: Adomi Bridge showing parabolic shape. Source: maxresdefault.jpg (1280×720)

As you can see, both can be explained and studied using quadratic functions and equations.

A quadratic function shows a special kind of relationship between two variables that makes a curve called a **parabola**. A quadratic equation helps us find the exact points where the curve meets the horizontal axis, or the ground, in real life. Here are some examples of how quadratic equations can help us in real life:

- Solve problems about curved paths, finding the highest or lowest points, and many other real-life situations.
- Indicate precision in sporting activities (for example, when throwing a basketball).
- Plan the best shape for bridges and buildings.
- Predict how things move (like water flowing or cars braking).
- Work with patterns in farming, business, and even fashion.

Study the following real-life examples.

Example 2.11

A rectangular garden in Kumbungu Senior High School has an area of 60m^2. The length is 4 metres more than the width. Find the dimensions of the garden.

Solution

1. Let the **width** be x metres. Then the **length** will be $x+4$ metres.
2. Set up the equation:
   $x(x + 4) = 60$
3. Solve:
   $x^2 + 4x - 60 = 0$
   $(x + 10)(x - 6) = 0$
   Either $x = -10$ or $x = 6$

Distances cannot be negative; therefore, must be 6, so the dimensions are 6 metres and 10 metres for width and length respectively.

In many Ghanaian communities, sports like football or athletics are common. When a ball is kicked (as in ***Figure 2.4***), its path follows a parabola.

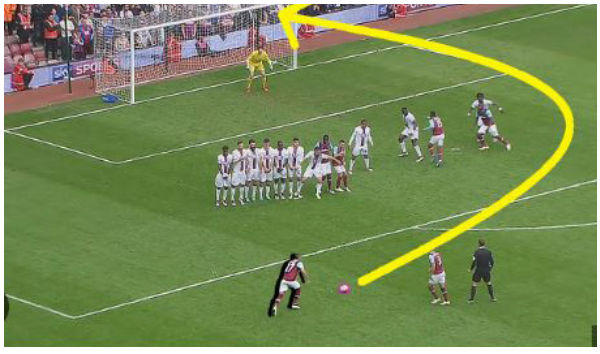


Figure 2.4: Illustration of football shot paths creating a parabola

Example 2.12

Consider the table of values for the equation $y = -x^2 + 5x$ which represents a basketball's path, where is the height in metres and is the horizontal distance in metres.

x	0	1	2	3	4	5
y	0	4	6	6	4	0

1. Plot the points and join them smoothly to form the parabola.

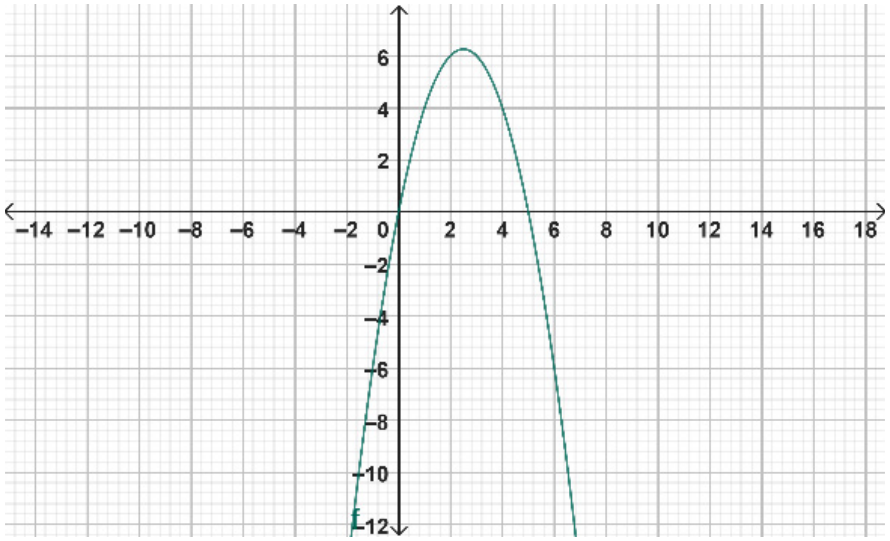


Figure 2.5: Graph of $y = -x^2 + 5x$

5. Draw the line: $y = 3$ on the same graph sheet.

This represents the height of the basketball hoop.

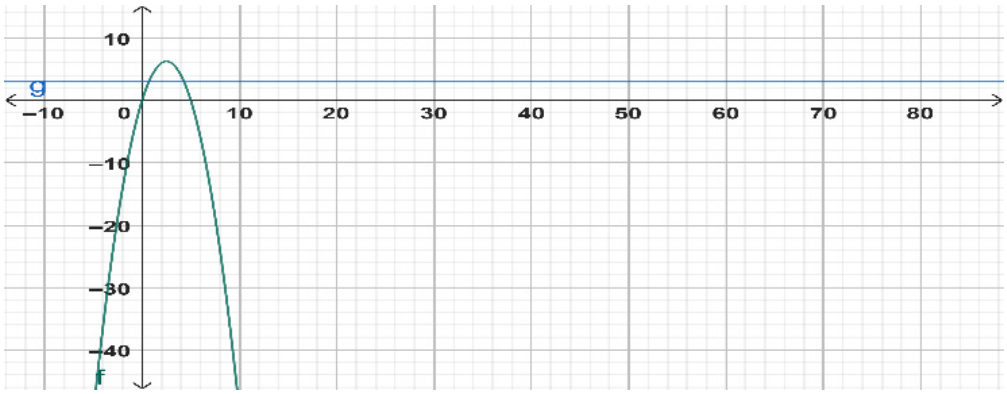


Figure 2.6: Graphs of $y = -x^2 + 5x$ *and* $y = 3$

6. Where does the ball's path meet the hoop?

The ball meets the hoop when $-x^2 + 5x = 3$

This implies $-x^2 + 5x - 3 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute $a = -1$, $b = 5$ and $c = -3$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(-1)(-3)}}{2(-1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{-2}$$

$$x = \frac{-5 \pm \sqrt{13}}{-2}$$

$$x = \frac{-5+\sqrt{13}}{-2} \text{ or } x = \frac{-5-\sqrt{13}}{-2}$$

Therefore, the ball's path meets the hoop at $x = \frac{-5+\sqrt{13}}{-2}$ or $x = \frac{-5-\sqrt{13}}{-2}$

7. What shape does the graph of a quadratic equation make? The graph makes a parabola.

8. Why does a basketball shot follow a curved path instead of a straight one?

- Because of gravity.
- When a ball is kicked, it has both horizontal and vertical motions. Gravity pulls it downwards, creating a curved (parabolic) path.
- This type of motion is called projectile motion, modeled by a quadratic equation.

9. How can we find the highest point of the ball's path?

Use the vertex formula for a parabola:

$\frac{-b}{2a}$, where a $= -1$ and b $= 5$

Substituting into the formula, we have; $\frac{-5}{2(-1)} = \frac{-5}{-2} = \frac{5}{2} = 2.5$metres.

Therefore, the highest point of the ball's path is $x = 2.5$m when and the height at this point is 6.25m

10. At what horizontal distance(s) did the ball reach the hoop height?

The ball reaches three metres high at $x = \frac{-5+\sqrt{13}}{-2}$ or $x = \frac{-5-\sqrt{13}}{-2}$

Solving Quadratic Equations Using Graphs

A graphical method can also be used to solve quadratic functions and equations. This method helps you get approximate answers to quadratic equations.

How to draw a quadratic graph given the domain

1. Find -values - Plug each -value into the equation to get y.

2. Plot the points (x, y) on the graph.

3. Draw the curve - Sketch a smooth U- shaped curve or ∩-shaped curve through the points. Use a ruler for the axes, and freehand for the curve.

Example 2.13

Given the quadratic equation, $y = -0.5x^2 + 3x + 2$ and the domain, $x = -1,0,1,2,3,4,5,6,7$

Use the information to answer the following questions:

1. Find the corresponding y-values from the x-values.

Substitute the values of x into the given function, $y = -0.5x^2 + 3x + 2$

For $x = -1$; $y = (-0.5)(-1)^2 + 3(-1) + 2 = -1.5$

For $x = 0$; $y = (-0.5)(0)^2 + 3(0) + 2 = 2$.

Create a table for x and y values as shown below.

x	−1	0	1	2	3	4	5	6	7
y	−1.5	2	4.5	6	6.5	6	4.5	2	−1.5

2. Using a scale of 1cm to 1 unit, on the x-axis and 1cm to 5 units on the y-axis, plot the points on a graph and sketch the curve.

Draw the graph as shown below:

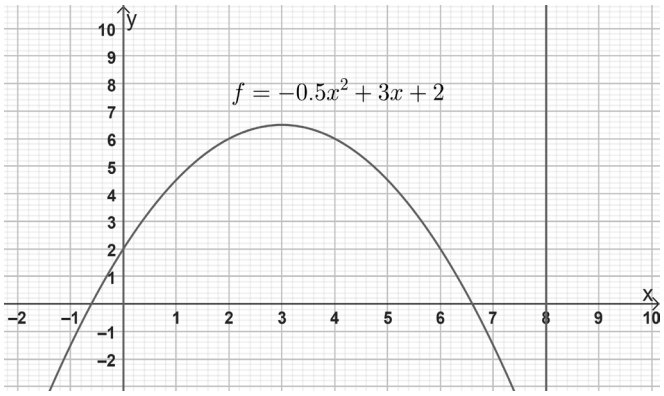


Figure 2.7: Graph of $y = -0.5x^2 + 3x + 2$

From looking at the graph we can see that the roots of the equation (where the graph cuts the x-axis) are approximately $x = -0.6$ and $x = 6.6$

Finding the Minimum and Maximum Value of a Quadratic Graph

The general quadratic function is in the form $y = ax^2 + bx + c$, where *a, b, and c* are constants with . The graph's turning point is called the **vertex**. If the graph opens **upward**, the vertex is a **minimum point**, which happens when $a > 0$.

For example, $y = 2x^2 - 4x + 1$ has a **minimum turning point** because the $2x^2$ term is positive. Refer to ***Figure 2.8*** on the next page.

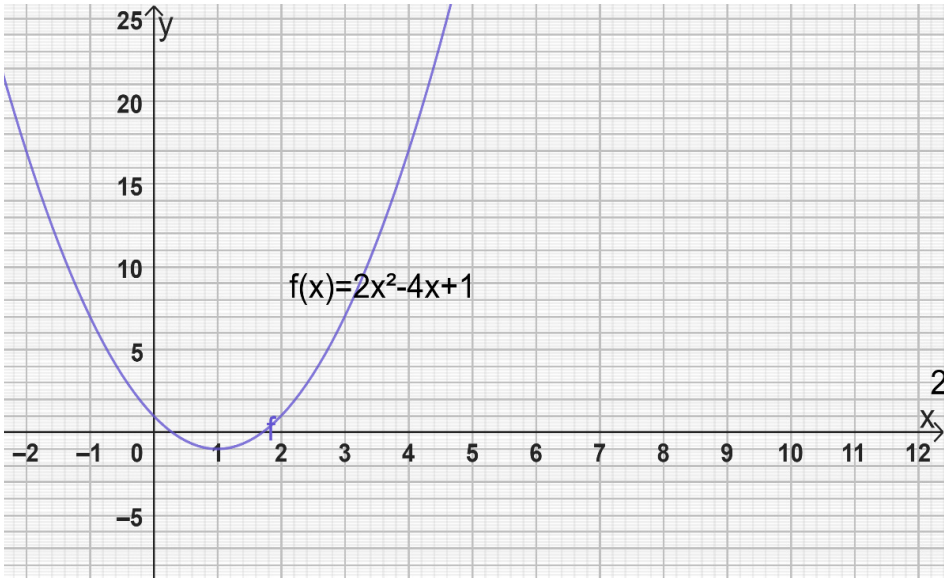


Figure 2.8: Graph of $f(x) = 2x^2 - 4x + 1$

On the other hand, if a quadratic graph opens downward, its vertex is a **maximum point**. This happens when the coefficient of x^2 is negative ($a < 0$).

For example, $y = 2x^2 - 4x + 1$ has a maximum turning point because the x^2 term is negative. Refer to the diagram in ***Figure 2.9.***

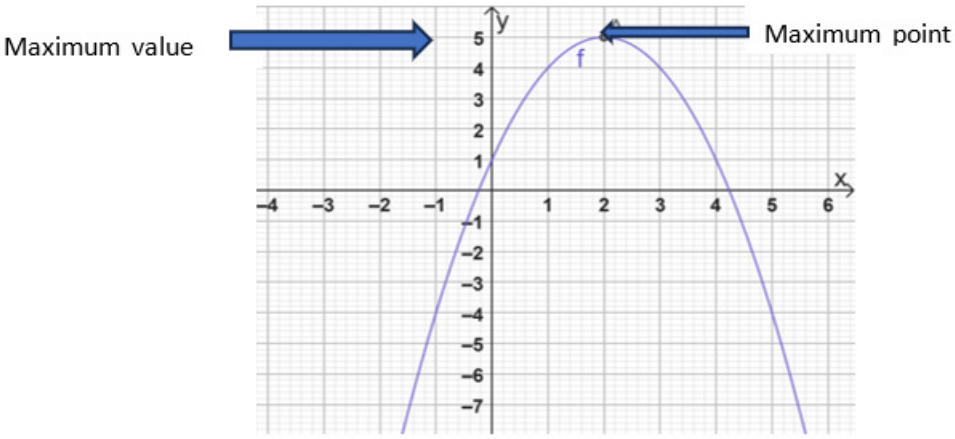


Figure 2.9: Graph of $f(x) = 2x^2 - 4x + 1$

Nature of roots of quadratic functions and equations

The nature of a quadratic function or equation can be 'real', 'repeated' or 'complex' (no roots). When the graph touches the axis at two different points, it has real (distinct) roots; if it touches the axis at only one point it has repeated roots, and if it does not touch the axis, it has complex (no real) roots. The following diagrams show the various kinds of roots:

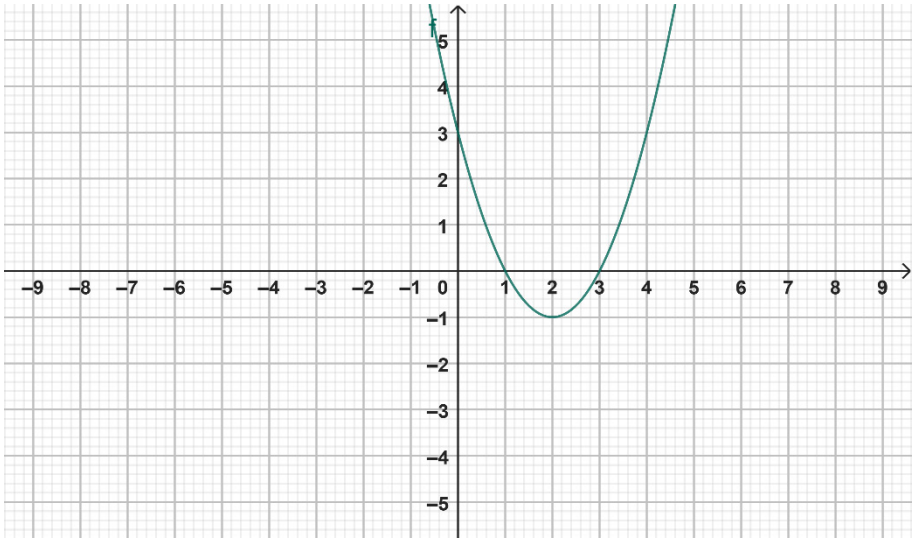


Figure 2.10a: real (distinct) roots

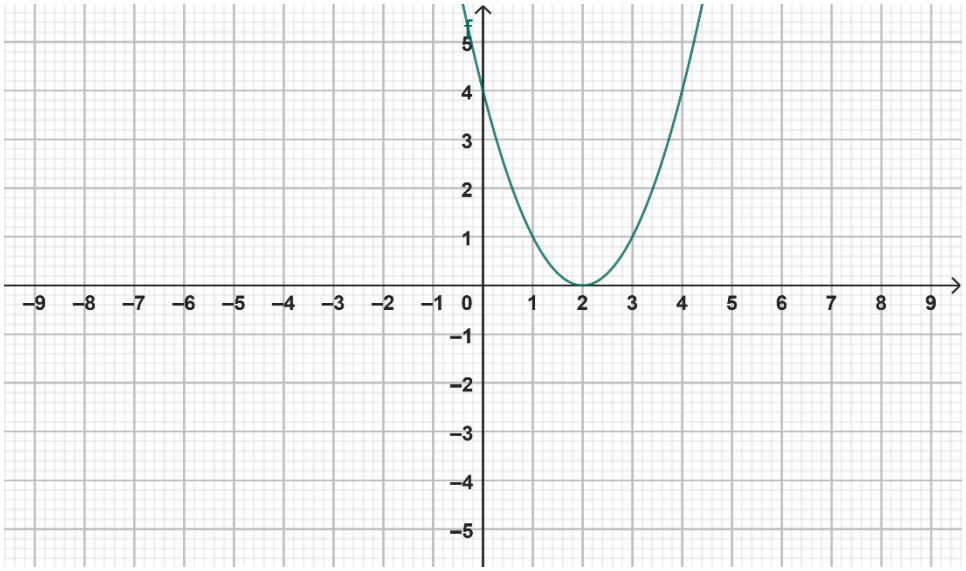


Figure 2.10b: repeated (equal) roots

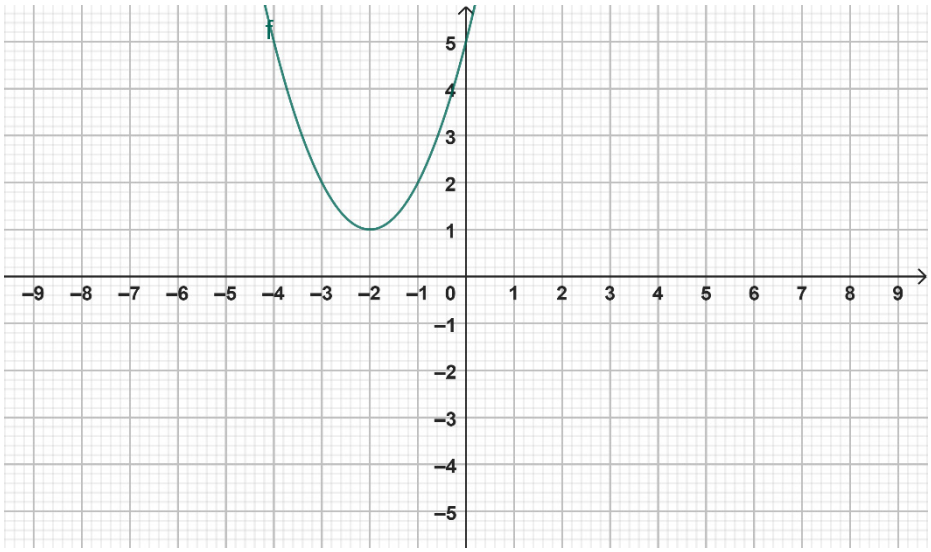


Figure 2.10c: complex (no real) roots

Example 2.14

1. Draw the graph of the function, $y = x^2 - 6x + 8$ for the interval $0 \leq x \leq 6$ using a scale of 1 cm to 1 unit on the x-axis and 1 cm to 5 units on the y-axis.

Given the function, $y = x^2 - 6x + 8$ for the interval $0 \leq x \leq 6$, create a table as shown below.

For $x = 0$, $y = (0)2 - 6(0) + 8 \Rightarrow y = 8$

For $x = 1$, $y = (1)2 - 6(1) + 8 \Rightarrow y = 3$

For $x = 2$, $y = (2)2 - 6(2) + 8 \Rightarrow y = 0$

For $x = 3$, $y = (3)2 - 6(3) + 8 \Rightarrow y = -1$

For $x = 4$, $y = (4)2 - 6(4) + 8 \Rightarrow y = 0$

For $x = 5$, $y = (5)2 - 6(5) + 8 \Rightarrow y = 3$

For $x = 6$, $y = (6)2 - 6(6) + 8 \Rightarrow y = 8$

x	0	1	2	3	4	5	6
y	8	3	0	−1	0	3	8

The graph of the function, $y = x^2 - 6x + 8$ is shown below.

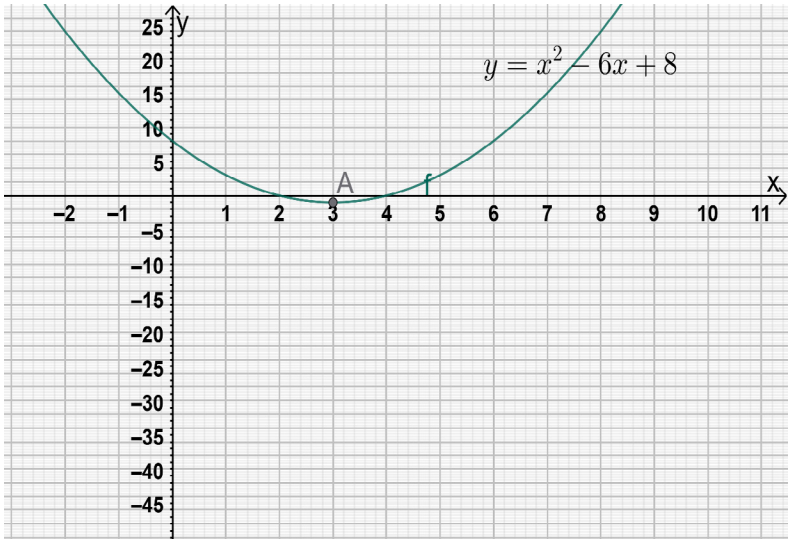


Figure 2.11: Graph of y = x2 – 6x + 8

2. From the graph, determine the roots of the equation.

The roots of the equation, $y = x^2 - 6x + 8$ are 2 and 4, as these are the points when the graphs cuts the x-axis.

3. Determine the minimum value and the point at which it occurs.

The minimum value is −1 and it occurs at (3, −1).

Example 2.15

Copy and complete the table of values below for the relation $y = -0.5x^2 + 3x + 2$

x	0	1	2	3	4	5	6
y	2			6.5			2

Using a scale of 2 cm to 2 units on both *x and* y-axis. Draw the graph of the relation.

Solution

x	0	1	2	3	4	5	6
y	2	4.5	6	6.5	6	4.5	2

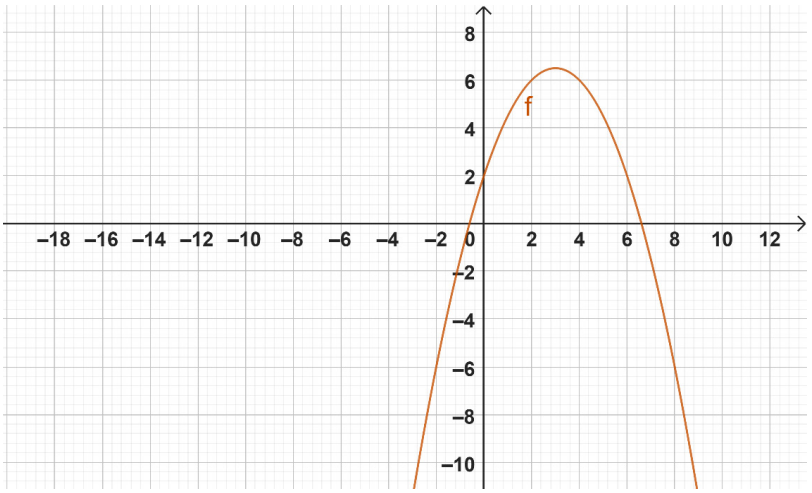


Figure 2.12: Graph of y = $-0.5x^2 + 3x + 2$

Example 2.16

1. Copy and complete the table of values below for the relation $y = -x^2 + 4x$ with the interval $-1 \leq x \leq 5$

	-1	0	1	2	3	4	5

Using a scale of 1cm to 2 units on *x-axis* and 1cm to 2 units on y-axis. Draw the graph of the relation.

For $x = -1, y = -(-1)2 + 4(-1) \Rightarrow y = -5$

For $x = 0, y = -(0)2 + 4(0) \Rightarrow y = 0$

For $x = 1, y = -(1)2 + 4(1) \Rightarrow y = 3$

For $x = 2, y = -(2)2 + 4(2) \Rightarrow y = 4$

For $x = 3, y = -(3)2 + 4(3) \Rightarrow y = 3$

For $x = 4, y = -(4)2 + 4(4) \Rightarrow y = 0$

For $x = 5, y = -(5)2 + 4(5) \Rightarrow y = -5$

x	-1	0	1	2	3	4	5
y	-5	0	3	4	3	0	-5

The graph is shown below.

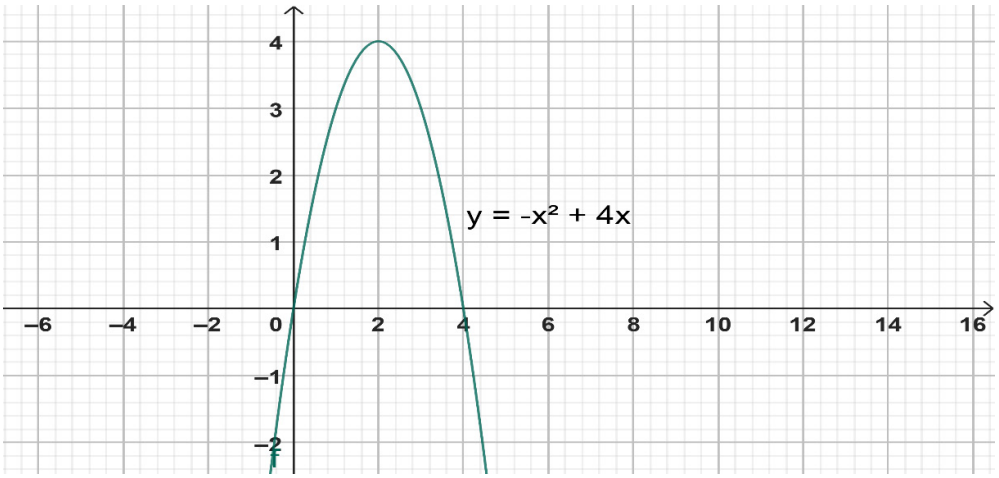


Figure 2.13: *Graph of $y = -x^2 + 4x$*

2. Use this graph to find the x-values where the graph cuts the x-axis.

The values of x where the graph cuts the x-axis are 0 and 4.

3. Find the greatest value of y and the value of x at which it occurs.

The greatest value of y is 4, and it occurs at $x = 2$.

4. Find the coordinates of the turning point.

The coordinates are (2, 4).

Example 2.17

The diagram in ***Figure 2.14*** shows the graph of a quadratic function, $y = 0.1x^2 + 3x - 1$.

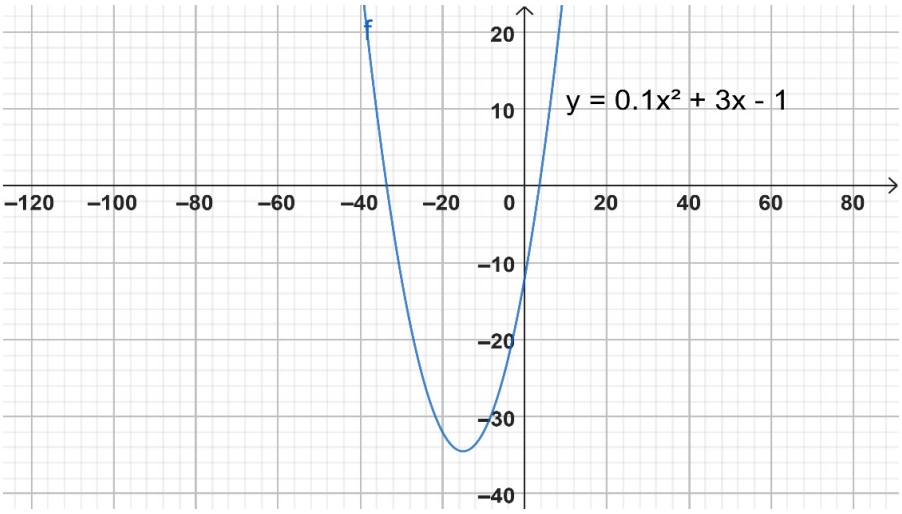


Figure 2.14: *Graph of $y = 0.1x^2 + 3x - 1$*

1. Find the minimum value of $y = 0.1x^2 + 3x - 1$.
2. Find the minimum point.

Solution

1. The minimum value of y is -23.5
2. The minimum point is $(-15, -23.5)$

Example 2.18

Figure 2.15 is the graph of the function $f(x) = -0.2x^2 + 20$.

1. Write down the maximum value of $f(x)$.
2. What value of x does the maximum value occurs.
3. What is the maximum point?

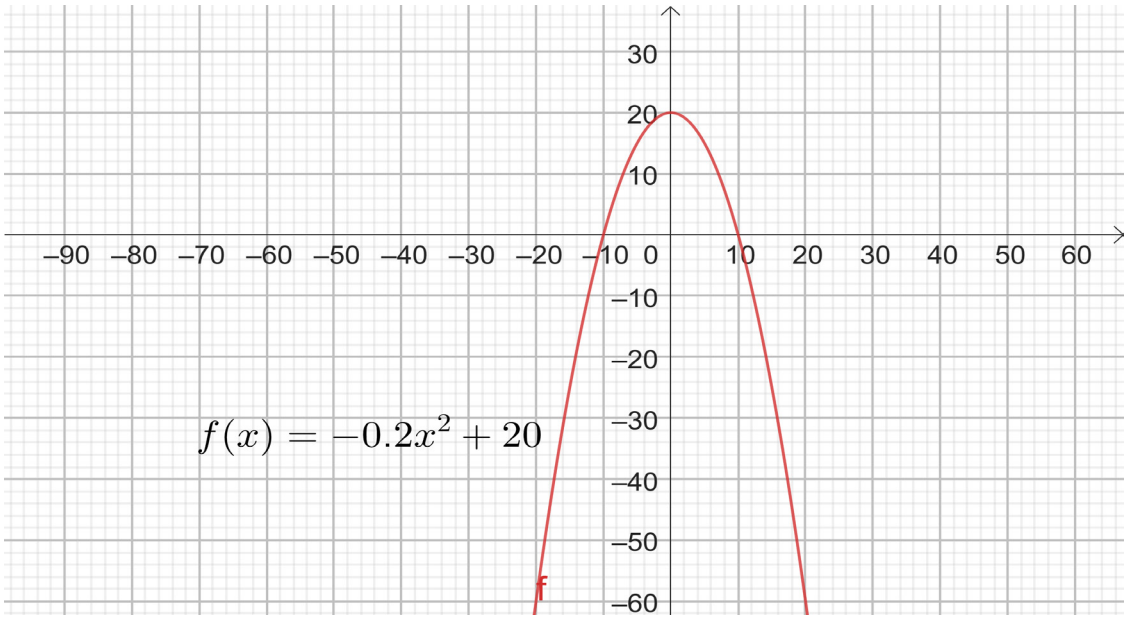


Figure 2.15: Graph of $y = -0.2x^2 + 20$

Solution

1. The maximum value of $f(x)$ is 20.
2. The maximum value occurs at $x = 20$.
3. The maximum point is $(0, 20)$

Real-Life Applications of Quadratic Graphs

Example 2.18

During inter-school games in Tamale, a striker kicks a penalty, and the ball follows a parabolic path. The ball's height is modelled by, $h(t) = -6t^2 + 12t + 1$. Where $h(t)$ is the height of the ball (in metres) and is the time in seconds.

After how many seconds does the ball reach the goalkeeper's height of 2 metres?

Solution

Given, $h(t) = -6t^2 + 12t + 1$, where $h(t) = 2 \Rightarrow -6t^2 + 12t + 1 = 2$

Rearranging; $-6t^2 + 12t + 1 - 2 = 0 \Rightarrow -6t^2 + 12t - 1 = 0$

Solving using the general quadratic formular where $a = -6$, $b = 12$ and $c = -1$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-12 \pm \sqrt{(12)^2 - 4(-6)(-1)}}{2(-6)}$$

$$t = \frac{-12 \pm \sqrt{144 - 24}}{-12}$$

$$t = \frac{-12 \pm \sqrt{120}}{-12}$$

$$t = \frac{6 + \sqrt{30}}{6} \text{ or } t = \frac{6 - \sqrt{30}}{6}$$

$$t \approx 1.91 \text{ or } t \approx 0.87$$

Therefore, the time taken for the ball to reach 2 metres is 0.087 seconds and again at 1.91 seconds when it is on its way down again.

Example 2.19:

Kwabena, a farmer in the Ashanti Region, wants to build a parabolic arch for a greenhouse using bamboo. The shape of the arch can be modelled by the equation, $h(x) = -0.5x^2 + 4x$, where: $h(x)$ is the height of the arch in metres at any point, meters from one end of the base.

If the base of the arch is 8 metres long, what is the maximum height of the arch, and at what point along the base does it occur?

Solution

The graph can be modelled as shown below.

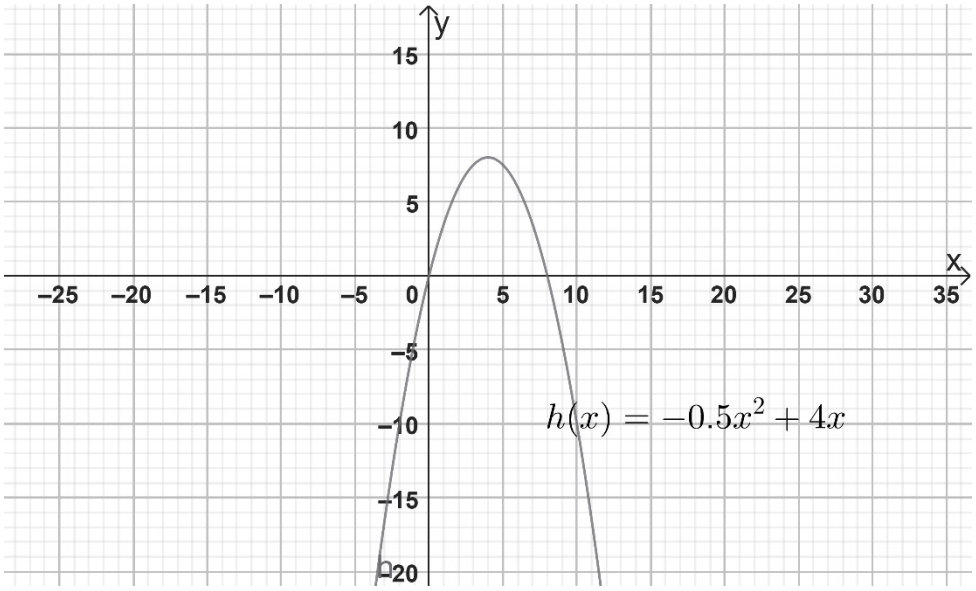


Figure 2.16: Graph of $y = -0.5x^2 + 4x$

From the graph, the maximum height occurs at $x = 4$ and the maximum height is 8 metres.

Alternatively, using the vertex formula, where $a = -6$, $b = 12$ and $c = -1$:

$$x = \frac{-b}{2a} = \frac{-4}{2(-0.5)}$$

$$x = \frac{-4}{-1} = 4$$

Now, finding the maximum height:

$h(x) = -0.5(4)^2 + 4(4)$

$h(x) = -8 + 16$

$h(x) = 8$

Therefore, the maximum height of the arc is 8 metres.

Activity 2.3 Exploring quadratic graphs with real-life situations

A ball is thrown into the air. Its height (in metres) above the ground at time, seconds is given by the equation, $h(x) = -x^2 + 4x + 5$.

1. On a graph sheet or paper, using an appropriate scale on the x and y-axes, label the x-axis from -1 to 5 and the y-axis from -1 to 9.
2. Create a table of values for both x and y, where $y = h(x)$.

3. Draw the graph using the values of x and y.
4. From the graph, determine the maximum height the ball reaches?
5. After how many seconds does the ball reach the maximum height?
6. How would the graph and the motion of the ball change if the initial speed or direction of the throw were different?
7. What real-life factors could affect the shape of this parabola?
8. In each of the above steps, discuss the results with your classmates.

Graphs of Parabolic Functions

The graph of a parabolic function of the form $f(x) = y = -0.5x^2 + 4x$ takes the shape of either of the diagrams below. These types of graphs are symmetrical, with the line of symmetry occurring at $x = -\frac{b}{2a}$.

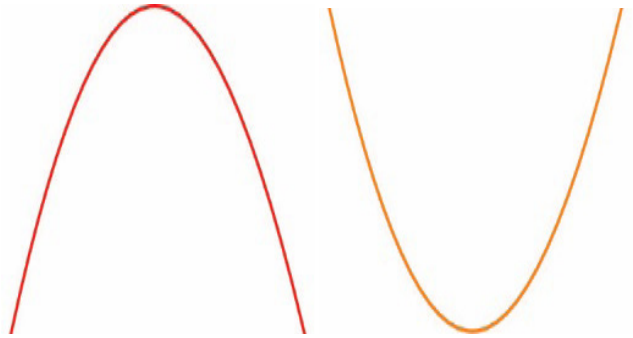


Figure 2.17: Graphs of parabolic shapes

For graphs with a *maximum turning point* (shaped like this ∩, when $a < 0$:

The coordinates of the turning point is $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$ and the maximum value is $f\left(-\frac{b}{2a}\right)$

The line of symmetry is $x = -\frac{b}{2a}$

The graph is increasing when $x < -\frac{b}{2a}$, i.e. the gradient is positive.

The graph is decreasing when $x > -\frac{b}{2a}$, i.e. the gradient is negative.

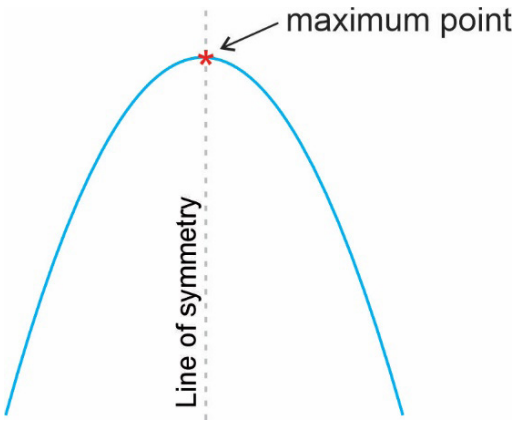


Figure 2.18: Line of symmetry on a maxima parabola

For graphs with a *minimum turning point* (shaped like this ∪, when $a > 0$:

The coordinates of the turning point are $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$ and the minimum value is $f\left(-\frac{b}{2a}\right)$.

The line of symmetry is $x = -\frac{b}{2a}$.

The graph is decreasing when $x < -\frac{b}{2a}$, i.e. the gradient is negative.

The graph is increasing when $x > -\frac{b}{2a}$, i.e. the gradient is positive.

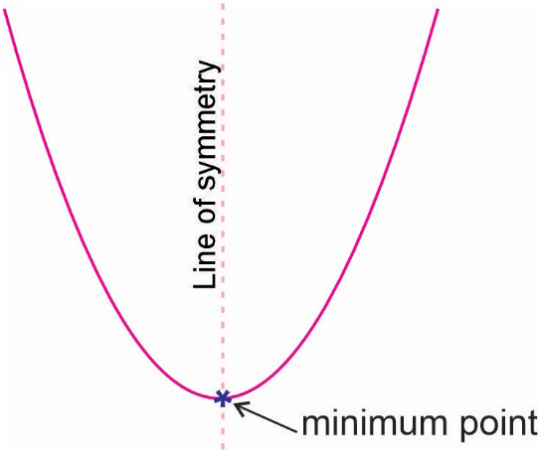


Figure 2.19: Line of symmetry on a minima parabola

Therefore, for a quadratic equation of the form, $f(x) = ax^2 - bx + c$, the maximum or minimum value is $f\left(-\frac{b}{2a}\right)$.

If the graph is written in the form $f(x) = a(x-h)^2 + k$ the minimum or maximum value is k and the line of symmetry is $x = h$ or $x - h = 0$ and the turning point is at (h, k).

Example 2.20:

Find the line of symmetry and turning point of the quadratic function,

$f(x) = 2x^2 - 4x + 1$

Solution

The line of symmetry is $x = -\frac{b}{2a}$ where $a = 2$ and $b = -4$

Substituting; $x = -\frac{(-4)}{2(2)} \Rightarrow x = \frac{4}{4} = 1$

To find the turning point, first find the minimum value which is $f\left(-\frac{b}{2a}\right) = f(1)$

$\Rightarrow f(1) = 2(1)^2 - 4(1) + 1$

$f(1) = 2 - 4 + 1 = -1$

Therefore, the turning point $= \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right) = (1, -1)$.

Example 2.21:

Determine whether the function, $f(x) = -x^2 + 6x - 5$ has a maximum or minimum point, and find its value.

Solution:

Since $a = -1$, which is < 0, the graph opens downwards, hence, it has a maximum point.

Maximum point $= \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, where $x = -\frac{b}{2a} = -\frac{6}{2(-1)} = \frac{-6}{-2} = 3$

Now, $f\left(-\frac{b}{2a}\right) = f(3) = -(3)^2 + 6(3) - 5 = 4$

Therefore, the maximum point is (3, 4) and the maximum value is 4.

Example 2.22:

The graph of $f(x) = 3(x - 2)^2 - 7$ is given in vertex form. Write the:

1. Line of symmetry
2. Turning point
3. State if it has a maximum or minimum value
4. At what interval is the graph
   a. increasing?
   b. decreasing?

Solution

1. Since $a = 3 > 0$, the graph opens upwards, it has a minimum point.

   Comparing $f(x) = 3(x - 2)^2 - 7$ with $f(x) = a(x - h)^2 + k$

   $\Rightarrow$h $= 2$ and k $= -7$

   Line of symmetry is $x = 2$.
2. The turning point is $(h, k) = (2, -7)$
3. It has a minimum value since $a > 0$.
4.
   a. The graph is increasing when $x < 2$.
   b. The graph is decreasing when $x > 2$.

Example 2.23:

Study the graph below and answer the questions that follow.

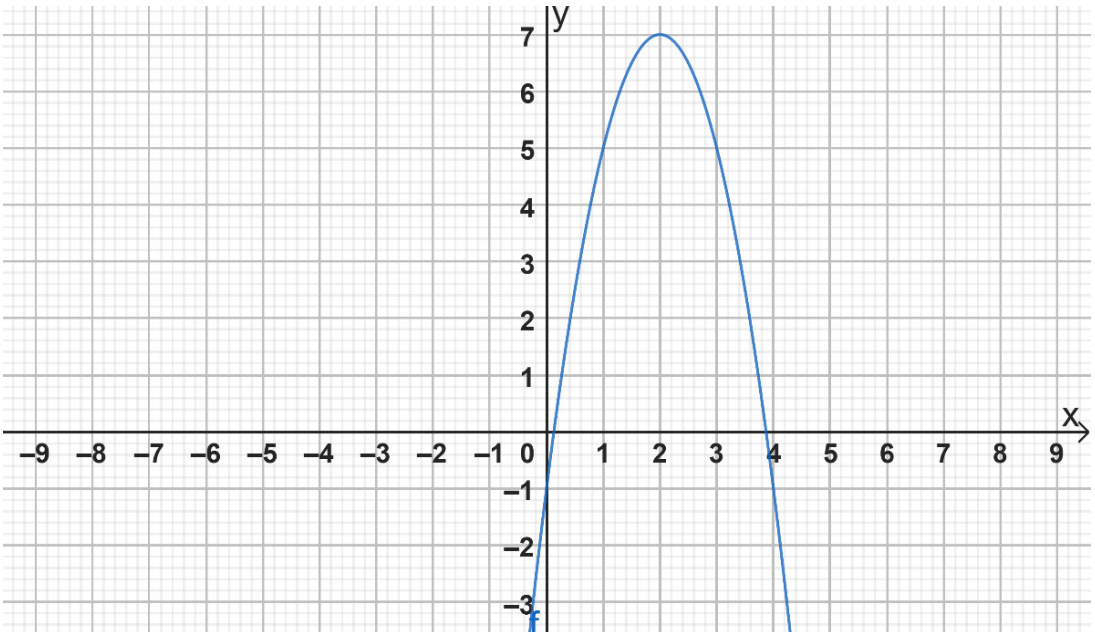


Figure 2.20: Graph of a parabola

Find:

i. The coordinates of the maximum point,

ii. The coordinates of the x intercept.

iii. The equation of the line of symmetry.

iv. The coordinates of the y intercept

Solution

i. The coordinates of the maximum point is (2, 7).

ii. The coordinates of the x intercept are (0.2, 0) and (3.8, 0).

iii. The equation of the line of symmetry is $x = 2$.

iv. The coordinates of the y intercept is $(0, -1)$.

Solving Linear and Quadratic Equations Using Graphs

A linear and a quadratic equation can have up to two solutions. When drawn together on a graph, these are where the straight line and the curve meet. They can meet at two points (two solutions), touch at one point (one solution), or not meet at all (no solution).

Observe the diagram in ***Figure 2.21***.

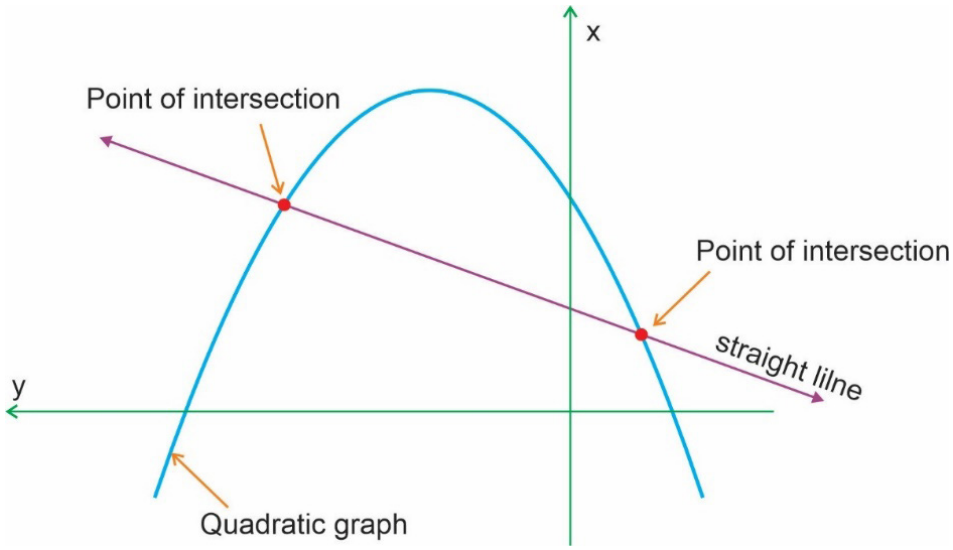


Figure 2.21: Linear and quadratic graphs

Example 2.24:

Solve the following equations simultaneously using graphs.

$y = x + 2$ and $x^2 + 2$

Solution

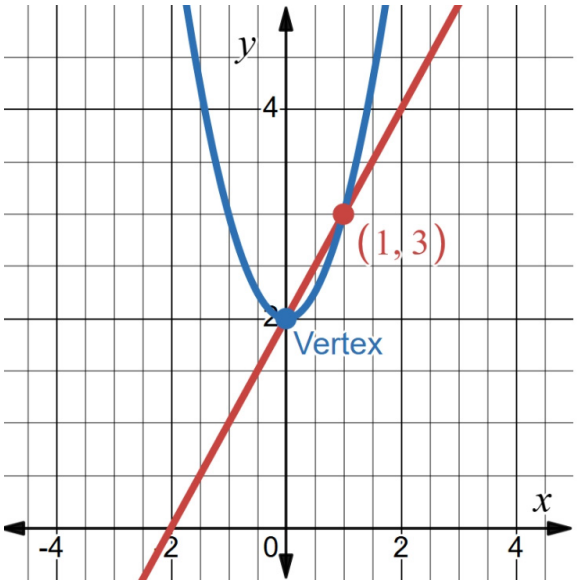


Figure 2.22: Graph of $x + 2$ and $x^2 + 2$

The simultaneous equations of a linear and quadratic equations is given by the point or points of intersection of the line and parabola representing the equations.

Therefore, the solutions are (1, 3) and (0, 2)

Example 2.25:

Solve the following simultaneous equations:

$y = x + 2$

$y = x^2$

Solution

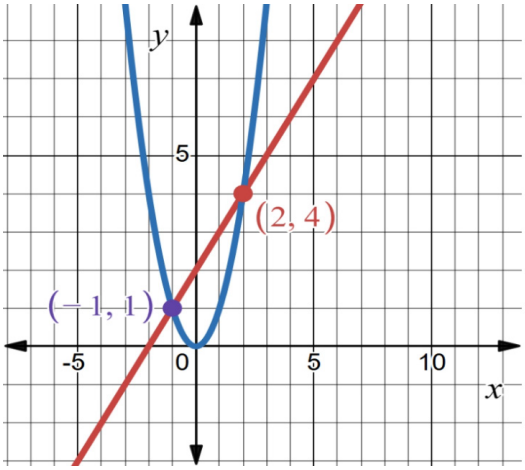


Figure 2.23: Graph of $y = x + 2$ and $y = x^2$

Therefore, the solutions are when $x = 2, y = 4$ and when $x = -1, y = 1$.

Example 2.26:

a. Copy and complete the table of values for the relation, $y = x^2 + 2$ for the interval $-2 \leq x \leq 4$

x	-2	-1	0	1	2	3
y	6			3		

b. Using a scale of 1 cm to 2 units on the x-axis and 1cm to 5 units on y-axis, draw the graph of the relation for $y = x^2 + 2$ the interval $-2 \leq x \leq 3$

c. Use the graph to find the solution set of

i. $x^2 + 2 = 0$

ii. $x^2 + 2 = x + 2$

Solution

a. Substitute the values into the equation $y = x^2 + 2$ to find the corresponding values of y.

For $x = -2, y = (-2)2 + 2 \Rightarrow y = 4 + 2 = 6$

For $x = -1, y = (-1)2 + 2 \Rightarrow y = 1 + 2 = 3$

For $x = 0, y = (0)2 + 2 \Rightarrow y = 0 + 2 = 2$

For $x = 1$, y $= (1)2 + 2 = 1 + 2 = 3$

For $x = 2$, y $= (2)2 + 2 = 4 + 2 = 6$

For $x = 3$, y $= (3)2 + 2 = 9 + 2 = 11$

Now, complete the above table with the calculated values of y

x	-2	-1	0	1	2	3
y	6	3	2	3	6	11

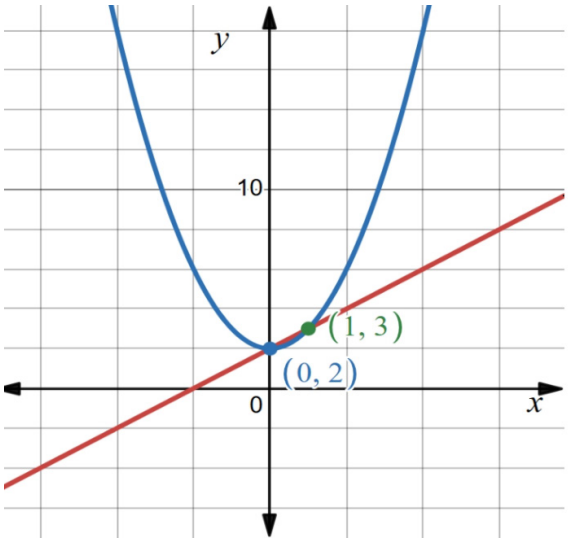


Figure 2.24: Graph of $x^2 + 2$ and $x + 2$

Now, plot the points and draw the quadratic graph as shown below.

c.

i. Since the graph of $y = x^2 + 2$ does not touch the x-axis, it has no solution (That is, it has complex roots).

ii. When $x = 0, y = 2$ and when $x = 1, y = 3$.

Finding the range of values of *x* for which y is increasing or decreasing

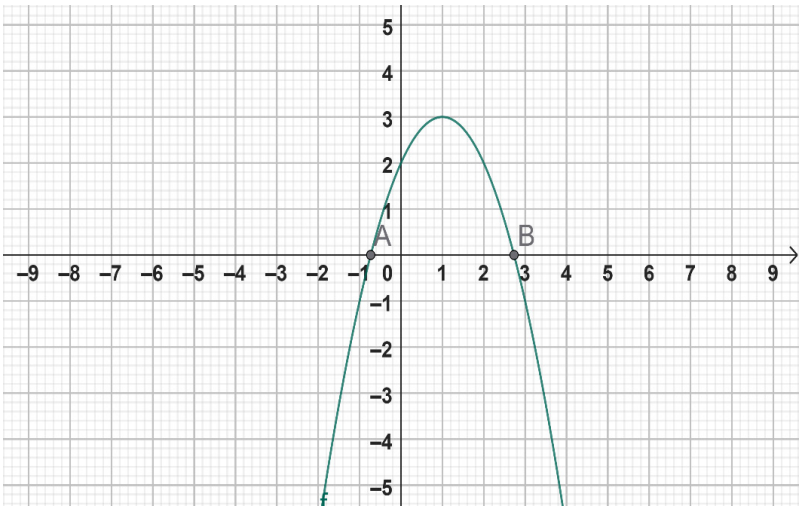


Figure 2.25: Maximum graph

From the graph:

- The range of values of x for which y increases as x increases is $x < 1$.
- The range of values of x for which y decreases as x increases is $x > 1$.
- The range of values of x for which y is positive (greater than zero) is $A < x < B$.
- The range of values of x for which y is negative (less than zero) is or $x < -0.8$ or $x > 2.8$.

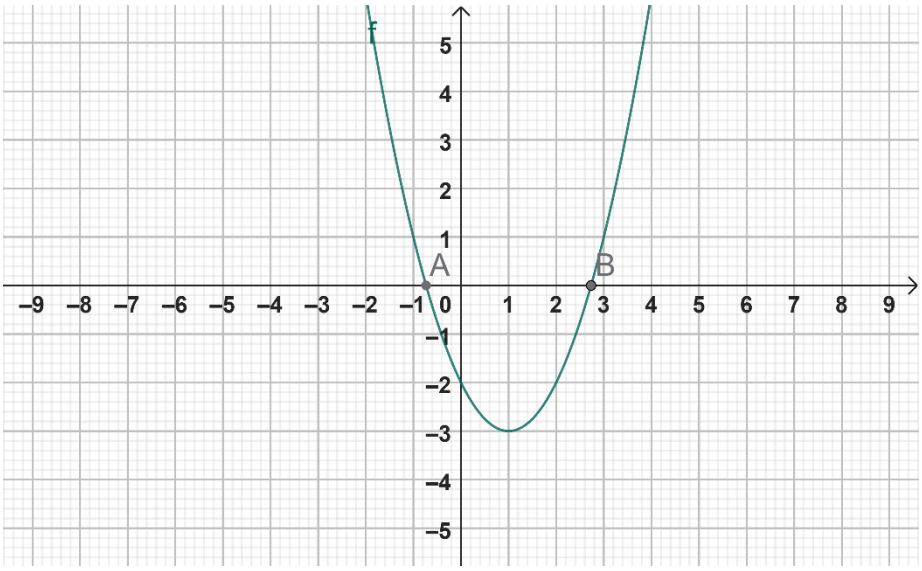


Figure 2.26: Minimum graph

From the graph:

- The range of values of x for which y increases as x increases is $x > 1$.
- The range of values of x for which y decreases as x increases is $x < 1$.
- The range of values of x for which y is positive (greater than zero) is $x < A$ or $x > B$.
- The range of values of x for which y is negative (less than zero) is $A < x < B$.

Example 2.27:

a. Copy and complete the following table for the relation $y = x^2 + 2x - 1$ for the interval $-3 \leq x \leq 2$

x	−3	−2	−1	0	1	2
y						

b. Using a scale of 1 cm to 1 unit on the x-axis and 1cm to 1 unit on y-axis, draw the graph of the relation.

c. From the graph, find:

i. The range of values of x for which y decreases as x increases.

ii. The range of values of x for which y increases as x increases.

iii. The range of values of x for which y is negative.

iv. The range of values of x for which y is positive.

Solution

a. Substitute the values into the equation $y = x^2 + 2x - 1$ to find the corresponding values of y.

For $x = -3, y = (-3)2 + 2(-3) - 1 \Rightarrow y = 2$

For $x = -2, y = (-2)2 + 2(-2) - 1 \Rightarrow y = -1$

For $x = -1, y = (-1)2 + 2(-1) - 1 \Rightarrow y = -2$

For $x = 0, y = (0)2 + 2(0) - 1 \Rightarrow y = -1$

For $x = 1, y = (1)2 + 2(1) - 1 \Rightarrow y = 2$

For $x = 2, y = (2)2 + 2(2) - 1 \Rightarrow y = 7$

The completed table is shown below.

x	-3	-2	-1	0	1	2
y	2	-1	-2	-1	2	7

b. The graph is shown below.

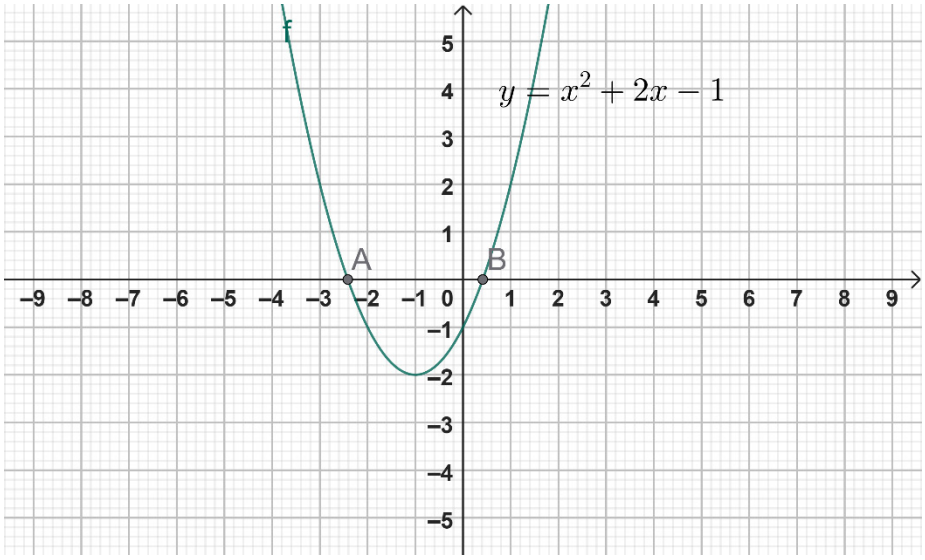


Figure 2.27: Graph of $x^2 + 2x - 1$

c.

i. $-3.8 \leq x < -1$

ii. $-1 < x \leq 1.8$

iii. $-2.4 < x < 0.4$

iv. $x < -2.4$ or $x > 0.4$

Example 2.28:

c.

i. $x < -1$

ii. $x > -1$

iii. $A < x < B$

iv. $x < A$ or $x > B$

Real-Life applications of simultaneous equations

Example 2.29:

A ball is thrown upward and its height after t seconds is given by: $h = t^2 + 4t + 2$

A drone is hovering at a constant height (h) of 5m.

Find the time when the ball is at the same height as the drone by drawing both graphs.

Complete the following table of values to help you solve this:

t	0	1	2	3	4	5
h						

Solution

Substitute the values into the equation $h = t^2 + 4t + 2$ to find the corresponding values of h.

For $t = 0$, $h = (0)2 + 4(0) + 2 \Rightarrow y = 2$

For $t = 1$, $h = (1)2 + 4(1) + 2 \Rightarrow y = 7$

For $t = 2$, $h = (2)2 + 4(2) + 2 \Rightarrow y = 14$

For $t = 3$, $h = (3)2 + 4(3) + 2 \Rightarrow y = 23$

For $t = 4$, $h = (4)2 + 4(4) + 2 \Rightarrow y = 34$

For $t = 5$, $h = (5)2 + 4(5) + 2 \Rightarrow y = 47$

The completed table is shown below:

t	0	1	2	3	4	5
h	2	7	14	23	34	47

The graph is shown below.

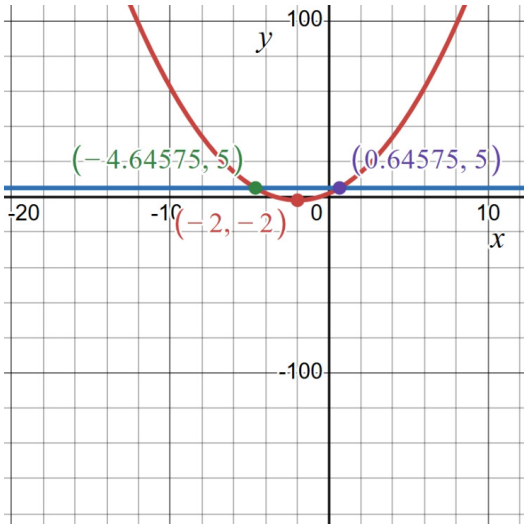


Figure 2.28: Graph of $h = t^2 + 4t + 2$ and $f(x) = 5$

Therefore, the drone and the ball are at the same height when $t = 0.65$ seconds. As time cannot be negative we ignore the negative solution.

Activity 2.4 Drawing Graphs

1. Copy and complete the following table for the relation, $y = x^2 - 4x + 3$ for $-1 \leq x \leq 5$.

x	-1	0	1	2	3	4	5
y	8			-1		3	

2. Using a scale of 2cm to 1 unit on both x and y axes, draw the graph of the relation using the completed table.

3. From your graph:

i. Write down the equation of the line of symmetry.

ii. State the range of values of x for which the curve is increasing.

4. State the range of values of x for which $y > 0$.

Activity 2.5 Real-Life Simultaneous Equation

Two taxi companies charge according to the equations:

- **Company A:** $y = 2x + 4$
- **Company B:** $y = x^2 - 2x + 8$

where y is the cost in cedis and x is the number of kilometres travelled.

1. Draw both graphs for $0 \leq x \leq 6$ on the same set of axes using a table of values.
2. From your graph:
   - **a.** Find the number of kilometres at which both companies charge the same amount.
   - **b.** For what range of distances is Company B cheaper?
   - **c.** Discuss the results with your classmates.

REVIEW QUESTIONS

1. Draw the graph of the quadratic function $y = x^2 - 4x + 3$, for $-1 \leq x \leq 5$
2. From the graph of $y = -x^2 + 6x - 5$, determine its maximum value and the x-coordinates where it occurs.

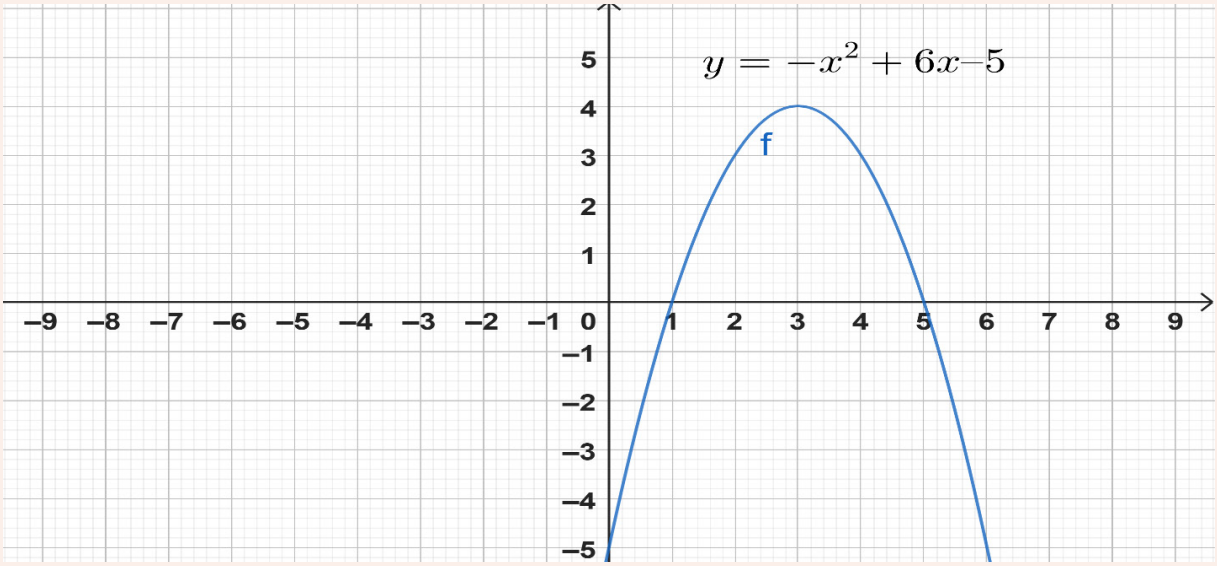


Figure 2.29: Graph of $y = -x^2 + 6x - 5$

3. Draw the graph of $y = x^2 - 2x - 8$ and use it to determine the roots of the equation. Explain how the graph shows these roots.
4. Without solving algebraically, explain how you can determine whether the equation, $2x^2 - 3x + 4 = 0$ has complex roots by using its graph.
5. A ball is thrown upward and its height (in metres) after t seconds is given by $h(t) = -5t^2 + 20t + 2$. Draw the graph of h(t) for and use it to determine the maximum height of the ball and the time at which this occurs.
6. Draw and compare the graphs of $y = x^2 - 4x + 3$ and $y = -x^2 + 4x - 3$. Describe in detail how their shapes, maximum/minimum points, and roots differ, and explain why.
7. Solve the quadratic equation: $x^2 - 5x + 6 = 0$
8. Solve the quadratic equation $2x^2 + 3x - 2 = 0$ using the quadratic formula.
9. Given the graph of the quadratic function $y = -x^2 + 4x + 1$, determine the coordinates of the maximum point.
10. Sketch the graph of $y = x^2 - 6x + 8$. Identify and label its vertex and intercepts.
11. Write the equation of the axis of symmetry for $y = 2x^2 - 8x + 3$.
12. A ball is thrown into the air, and its height is given by the equation

$$h(t) = -5t^2 + 20t + 1$$

a. What is the maximum height?

b. After how many seconds does it hit the ground?

13. Using the graph of $y = -x^2 + 4x - 3$, determine the values of x for when y is positive.

14. For the function $y = x^2 - 6x + 5$, use the graph to determine the x value interval(s) where:

a. y is decreasing

b. y is increasing

15. A company's profit P(x) in thousands of dollars is modelled by the equation $P(x) = -2x^2 + 12x - 5$, where x is the number of items produced in hundreds.

a. What number of items will maximize profit?

b. What is the maximum profit?

c. For what values of x is the profit positive?

16. A rectangular garden is to be fenced on three sides with a fixed length of fencing: 60 metres, with one side left open along a wall.

Let the width of the garden be x meters.

a. Write an expression for the length of the garden in terms of x.

b. Write the area of the garden as a function of x.

c. Determine the value of x that will maximize the area and find the maximum area.

d. Interpret your solution in the context of the problem.

17. A projectile is launched from a platform, and its height (in metres) after t seconds is modelled by the equation: $h(t) = -4.9t^2 + 14t + 3$

a. Determine the time when the projectile reaches its maximum height and find the maximum height.

b. Find the total time of flight before it hits the ground.

c. A sensor is placed at a height of 10 metres.

d. Explain how you can verify your answers graphically and describe the shape and key features of the graph.

18. Two companies offer different payment plans, where x represents the hours worked:

- **Company A** charges a flat rate: $y = 25x + 100$
- **Company B** charges based on a quadratic model: $y = 5x^2 + 10x$

a. Graph both functions on the same set of axes.

b. Determine the number of hours for which both companies charge the same amount.

c. For what range of hours is Company B cheaper than Company A?

d. Evaluate the practicality of each plan depending on the number of hours and explain which option is better in different scenarios.

SECTION

3

CIRCLE THEOREMS & GEOMETRIC CONSTRUCTION



CIRCLES

Circle Theorems

Introduction

In Year 1, you studied angles, their types and constructions, parallel and perpendicular lines, polygons and how to find the perimeter, area and volume of shapes. Skills acquired from these concepts are useful for tasks like designing floor plans, fencing land, and building structures.

In Year 2, you learnt trigonometric ratios and their inverses, applying them to solving real-life situations such as finding building heights, estimating distances, and calculating angles of elevation and depression, which are applicable in areas such as surveying, navigation and construction.

In Year 3, you will explore exciting topics like circles, circle theorems, and the construction of circles, triangles, and quadrilaterals. Circle theorems reveal important relationships between angles, lines and points in and around a circle. This will build on your earlier knowledge of constructions, polygons and trigonometry. These topics are valuable in engineering, architecture and technical drawing. These concepts help greatly in sharpening your mathematical reasoning and equipping you to apply mathematics confidently in schoolwork and everyday problem-solving.

KEY IDEAS

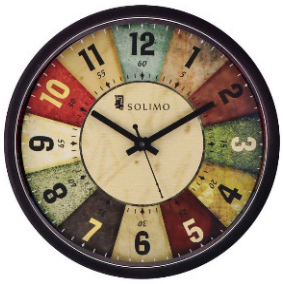
- In geometry, a point, line or object is said to be equidistant from two or more objects if it is the same distance from each of them.
- In classical constructions, use only a compass and a straight edge/ruler. This emphasises logical reasoning and geometric relationships.
- Each step in a construction must be accurate and based on geometric principles (e.g., perpendicular bisector, angle bisector, etc.)
- Basic constructions include constructing perpendicular bisectors, angle bisectors, perpendicular lines from a point and copying or constructing given angles and triangles.
- Many constructions are justified using theorems (e.g., the angle in a semicircle is a right angle, or isosceles triangle properties).
- Constructions are also based on the idea of a locus; a set of points satisfying certain conditions (e. g. all points equidistant)

When you look around, what circular objects do you see? Probably objects like bicycle tyres, the rim of a tank or a round bowl, a wall clock, etc. With a classmate, carefully

observe the images in ***Figure 3.1*** below and identify the part(s) of each image that is circular.



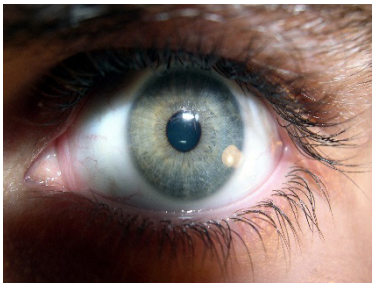
Bowl



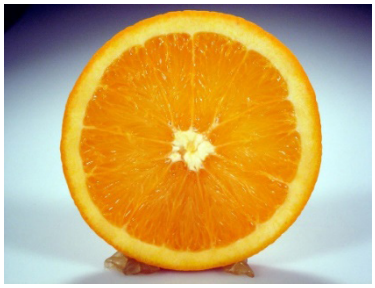
Wall clock



Roundabout



Iris in a human eye



Orange



Bicycle

Figure 3.1: Real-life objects that have circular shapes.

From your observation and discussion, describe a circle in your own words.

At Junior High School, you learnt about circles and the parts of a circle. In this section, we will explore additional interesting parts of the circle and the relationships among them.

Before we delve into this lesson, write down the "parts of a circle" that you were taught in junior high school. Parts like: the circumference of a circle, radius, diameter, chord and many others. Discuss what you have remembered about these parts of a circle with a classmate. This will help you to better understand the new lesson.

Circles and Circle Theorems

A circle: A circle is a plane (two-dimensional) figure bounded by a circumference such that all the points on the circumference are equidistant from a fixed point within it.

In other words, a circle is a two-dimensional figure (plane figure) formed by a set of points that are at a constant or at a fixed distance (radius) from a fixed point (centre) on the plane.

Figure 3.2. is an example of a circle in which A, B, D and E are points on the circumference. Point O is at the centre of the circle.

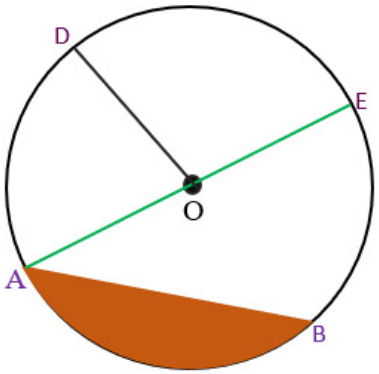


Figure 3.2: A diagram illustrating parts of a circle.

Parts of a circle

1. **Centre:** The centre of the circle is the fixed point from which all points on the boundary of the circle (circumference) are equidistant. In ***Figure 3.2*** and ***Figure 3.3***, the centre is the point labelled **O** in the circle.

   With this description, you can identify the centre of the wall clock as well.

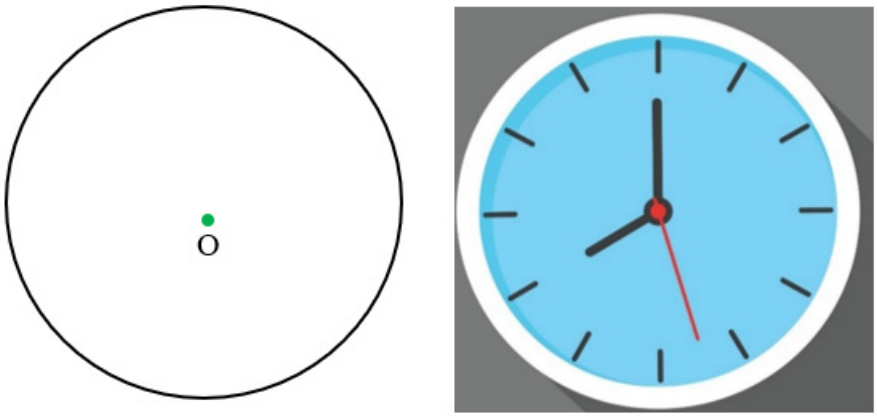


Figure 3.3: Shows the centre of a circle

2. **Circumference:** The distance around the circle is called the circumference. It is the area around the circle or the perimeter of the circle as shown in ***Figure 3.4***.

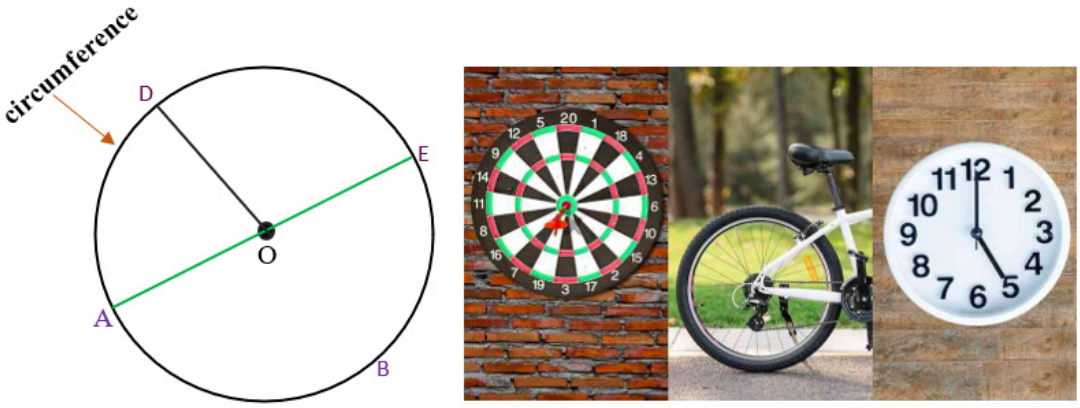


Figure 3.4: Circumference of a circle

With a classmate, identify the part of each object in Figure 3.4. that represents the circumference.

3. **Arc:** A part of the circumference is called an **arc**, as shown in ***Figure 3.5***.

   Major arc: A major arc is greater than half the circumference.

   Minor arc: A minor arc is less than half the circumference.

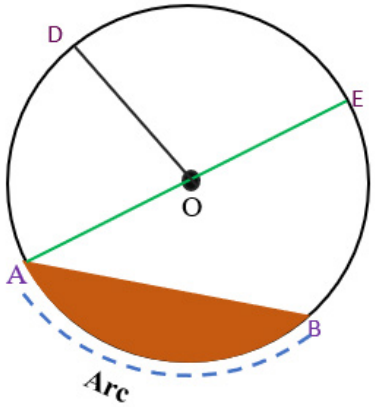


Figure 3.5: Illustration of an arc of a circle

4. **Radius:** A radius is the shortest distance between the centre of a circle and any point on the circumference of the circle.

   Any line drawn from the centre of a circle to any part of the circumference is a radius. The plural form of radius is radii.

   In ***Figure 3.5***, $|OD|$,$|OA|$ and $|OE|$ are radii. The radius (r) of a circle can be used to find the area, circumference and volumes of circular shapes.

$$|OD|=|OA|=|OE|= Radius\ (r)$$

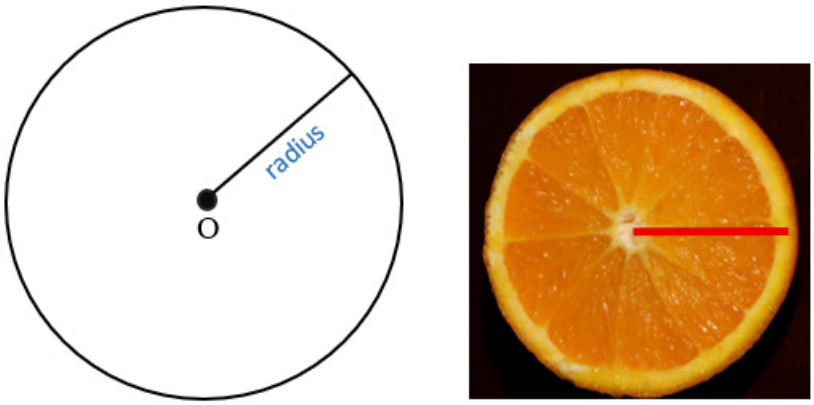


Figure 3.6: illustration of a radius

5. **Diameter:** The diameter of a circle is any straight-line segment that passes through the **centre** of the circle and whose endpoints lie on the circumference of the circle.

   The diameter is special and the longest chord of the circle. The diameter of a circle is twice the length of the radius of that circle.

In ***Figure 3.7,*** it can be observed that |AE| is the diameter and it is formed by two radii

Since $|OA| = |OE|$ = radius (r), then, diameter $= 2 \times$ radius $= 2r$.

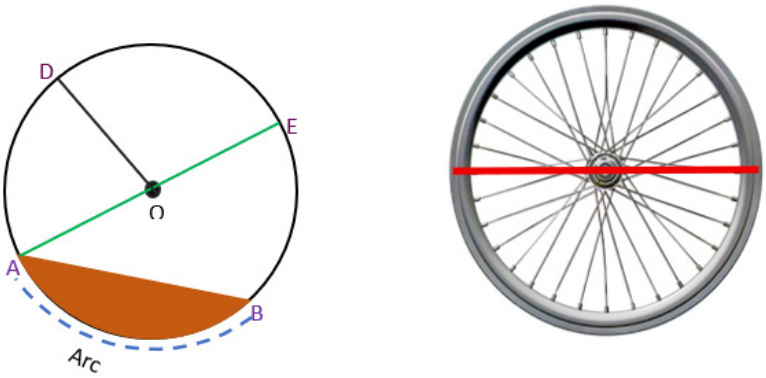


Figure 3.7: illustrations of the diameter of a circle

6. **Sector:** The region bounded by two radii and an arc. Point O is the centre of the circle from which the sector was obtained, and lines OA and OB are radii.

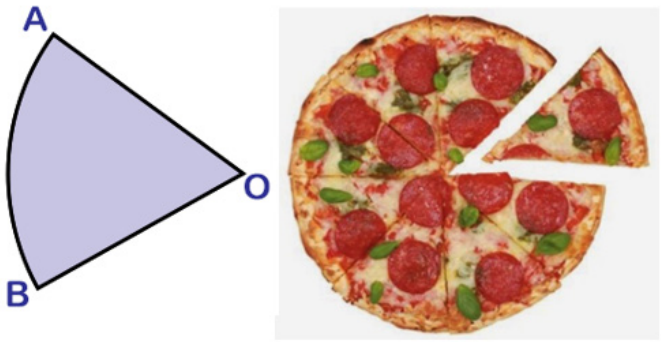


Figure 3.8: Sector of a circle.

Major sector: A major sector has a central angle which is more than 180°.

Minor sector: A minor sector has a central angle which is less than 180°.

Therefore, considering the pizza in the figure above, the portion that has been taken out represents the minor sector and the remaining portion represents the major sector.

7. **Semi-circle:** Semi circle is exactly half of a circle. The central angle of a semicircle is 180°. The diameter divides the circle into two equal parts (halves/ semi-circles).

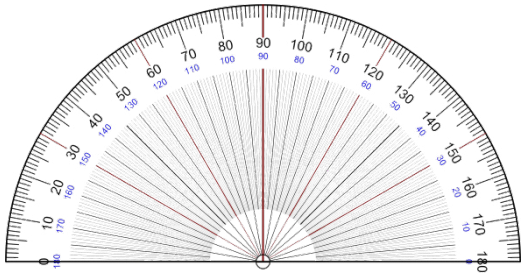


Figure 3.9: An illustration of a semi- circle

8. **Quadrant:** A quarter of a circle created by two radii that are perpendicular to each other is called quadrant. From ***Figure 3.10***, |AD| and |AB| and are radii and are perpendicular to each other.

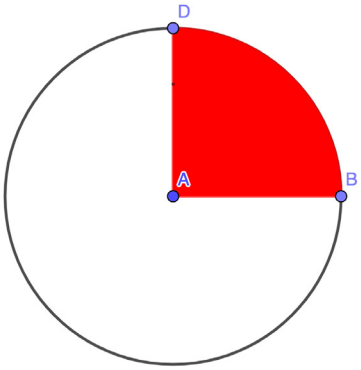


Figure 3.10: Quadrant of a circle

9. **Chord:** A straight line connecting two points on the circumference of a circle is called a chord. In the figure below A, B, P and Q are points on the circumference. Lines |AB| and |PQ| are chords.

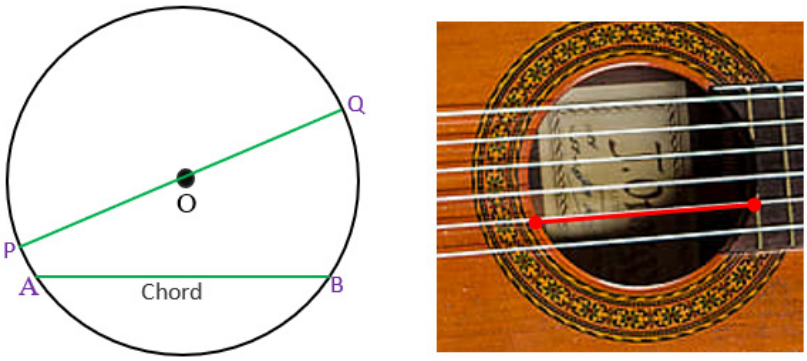


Figure 3.11: Illustration of a chord

Have you observed from the ***Figure 3.11*** that |PQ| is also a diameter? The diameter is also a chord and it is the longest chord in a circle.

10. **Segment:** A segment of a circle is a region bounded by a chord and the corresponding arc. The space shaded is called a minor **segment.** This is the region bounded by a chord |AB| and an arc AB.

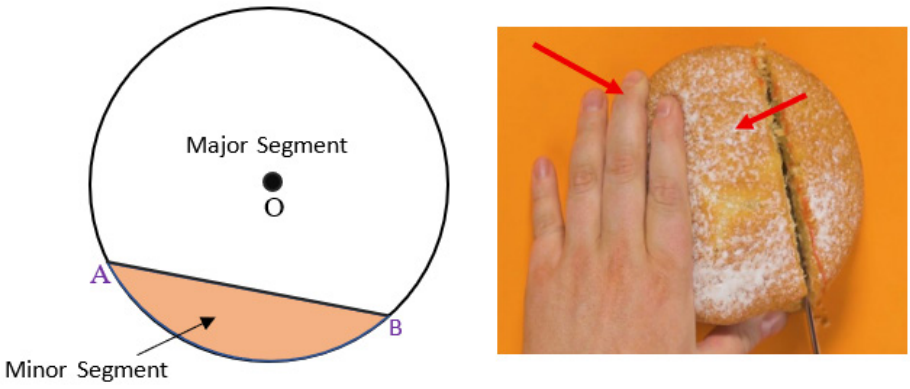


Figure 3.12: A segment of a circle

- **Major segment**: a segment where the arc is greater than half the circumference.
- **Minor segment**: a segment where the arc is less than half the circumference.

From your observation, discuss with a classmate which part of the cake in ***Figure 3.13*** represents the major segment and minor segment.

11. **A Secant:** A secant to a circle is a line that crosses or passes through the circle at exactly two distinct points. The line QP is called the secant.

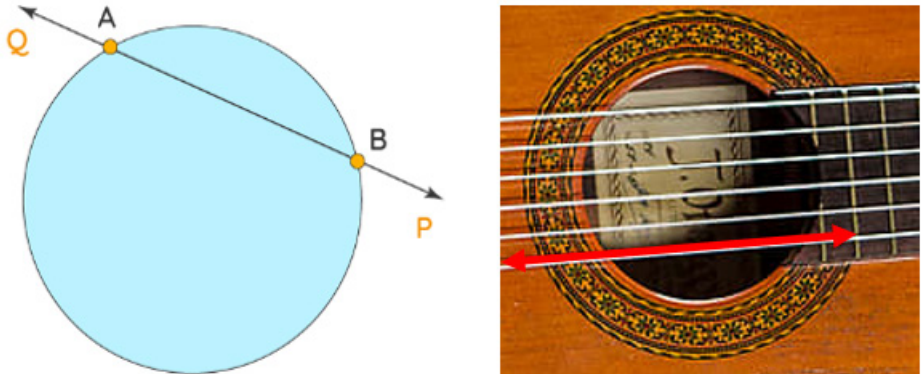


Figure 3.13: Illustrations of a secant line

12. **Tangent:** A tangent to a circle is a straight line that intersects the circle at exactly one point, i.e. a tangent touches the circumference of the circle at just one/ a single point.

The point of contact of the circle and the tangent line is called the **point of tangency.**

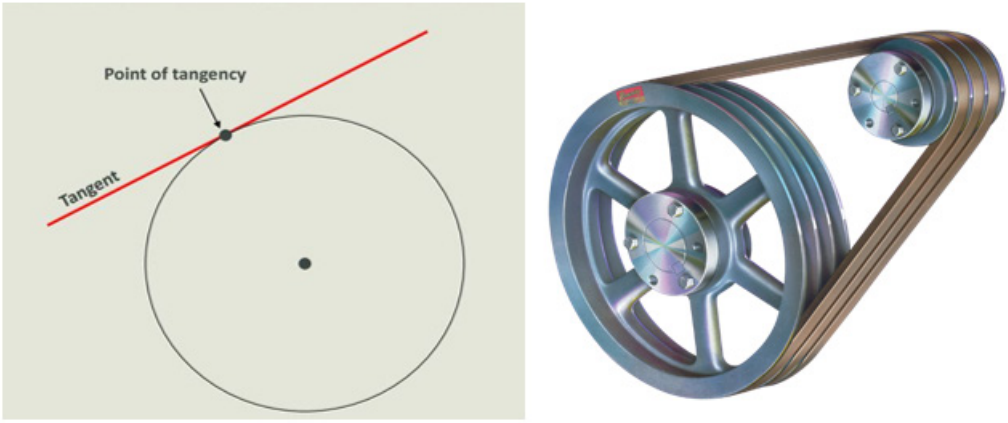


Figure 3.14: Tangent to a circle

Hopefully this has revised the parts of a circle for you and you have also seen the relationship between them.

In Year 1, you learnt how to construct and measure angles, so revise this quickly by discussing the following questions with a classmate.

1. What are the instruments used in constructing angles?
2. How are angles constructed? Try at least two examples (60° and 90°)

3. Which instrument(s) are used for measuring angles?
4. How are angles measured?

Construction of Circles

Circles can be constructed when the centre and the radius are known..

Materials needed: A ruler and a pair of compasses.

Construction of a circle of a given radius

To construct a circle with a radius of 6cm, follow the steps below:

Steps

1. Use your ruler to measure a distance of 6cm between the compass tip and the pencil.
2. Mark a point and label it O. This point O will be the centre of your circle.
3. Place the compass pin firmly on point O and slowly rotate the compass 360° to draw a full circle.

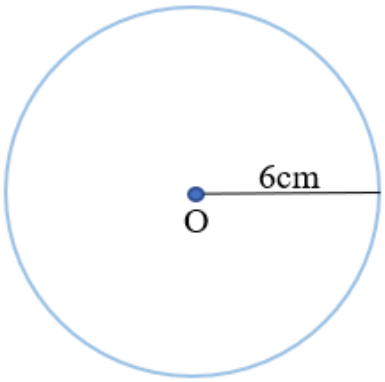


Figure 3.15: Construction of a circle of radius 6cm

4. Use a ruler to draw at least three (3) straight lines from point O to different points on the circumference of your circle. Label each line drawn from the centre to the circle.
5. Measure the length of each of the lines (radii) with your ruler and record the measurement of each line.
6. Discuss your observations:
   - Do all the lines from the centre to the circle have the same length?
   - Record your answer and explain briefly.
7. To extend your learning:
   - Repeat the above steps using different radii such as 3cm, 6cm, or 8cm with a classmate.

- Compare your findings and discuss what happens to the circle when the radius changes.

Constructing an infinite number of circles from the same centre

Steps

1. Mark the centre point and label this point as O
2. Place the pin of the compass firmly on point O and choose a starting radius (e.g., 2cm) and draw the first circle.
3. With the compass pin still at point O, increase the radius slightly (e.g., by 1 cm) and draw the second circle around the first one.
4. Repeat the process by gradually increasing the radius by equal or chosen intervals (e.g., 1cm each time). For each new radius, keep the compass pin fixed at point O and draw a new circle.
5. Observe the pattern:
   - After drawing several circles, observe how the circles become larger as the radius increases.
   - Notice how all the circles share the same centre but have different sizes.

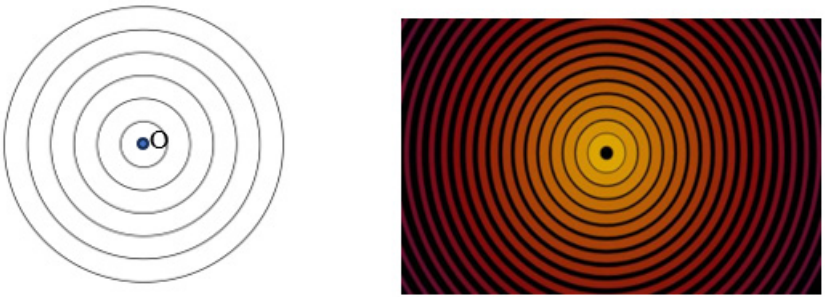


Figure 3.16: Many circles with the same centre

An infinite number of circles can be drawn around a single point, as illustrated in ***Figure 3.16***. These circles are called **concentric circles**.

Constructing an infinite number of circles that pass through a single point

Steps

1. Mark a fixed-point P on your paper (this is the point through which all the circles will pass).
2. Choose a centre point for a circle - not at P. The centre can be anywhere else on the paper.
3. With the compass pin at the chosen point, set the radius to the point P and draw the circle through P.

4. Repeat Steps 2 and 3 by selecting a new point as the centre (not coinciding with P).
5. Construct new circles with the radius equal to the distance from the new centres to point P.
6. Draw the circles as many as you can.

   As there are infinitely many points you can choose as centres (not at P) on your drawing sheet. You can construct infinitely many different circles, each passing through point P.
7. Discuss your observations with a classmate.

   Note: Each circle will have a different radius and centre, but will share the same point P on their circumferences.

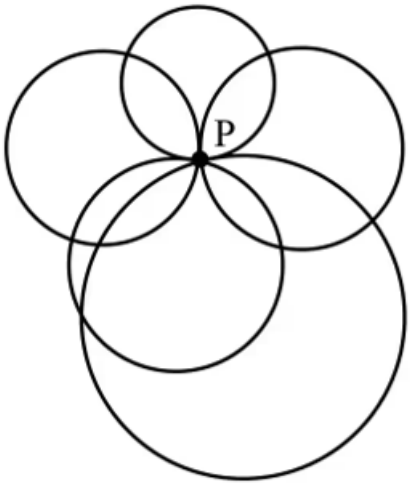


Figure 3.17: Some circles all passing through point P

Example 3.1

Draw three circles with centres A, B and C respectively passing through a single point D. Expected result

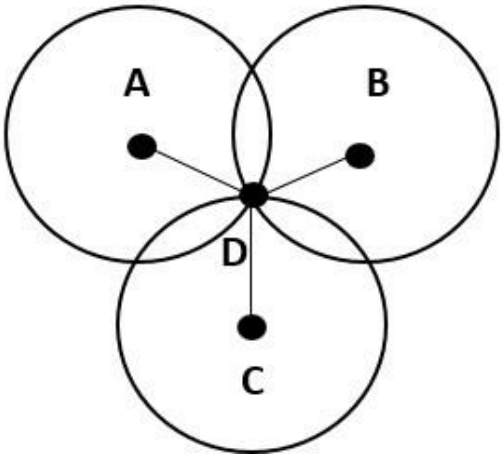


Figure 3.18: Three circles passing through a single point

Likewise, infinitely many circles can pass through two points. How can these different circles be constructed such that they all pass through the two fixed points?

Circles passing through two fixed points

Click on the link below to watch a video on how to construct infinitely many circles passing through two fixed points.

https://youtu.be/NqSjPJZ3O1E

From the video you have watched, try this example in your workbook and show it to your teacher or compare your work with that of a classmate.

Example 3.2

Construct circles passing through two fixed points A and B

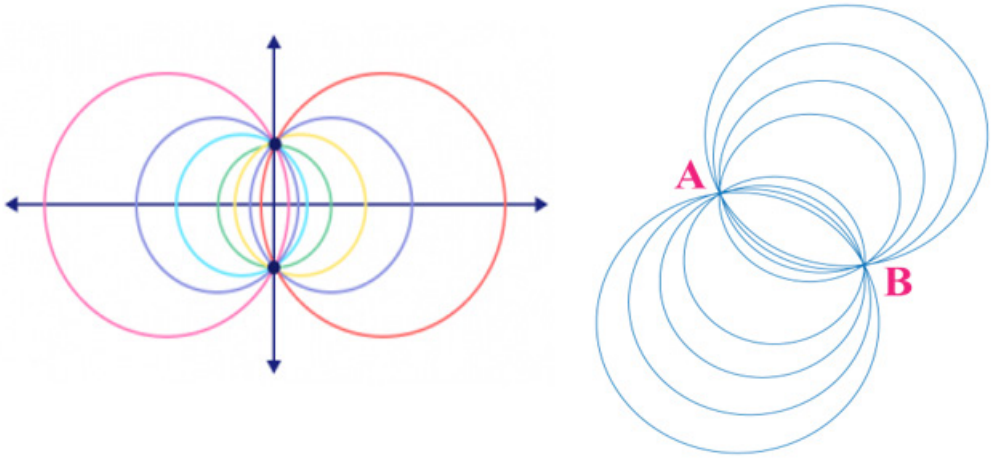


Figure 3.19: Circles passing through two points (A and B)

Have you ever tried drawing a circle through three points that are not on the same line?

Can a circle be drawn through three points that are not on a straight line (non-collinear)?

Discuss this with your classmate and see if you can construct one.

Click on the link below to watch a video on how to draw a circle through three non-collinear points https://www.youtube.com/shorts/heHjglruyIM?feature=share

From the video, you have observed that only ONE circle can pass through three points which are NOT on a straight line (non-collinear). When such a circle is created through the vertices of a triangle, it is called a circumscribed circle and its centre is called a circumcentre. The circumcentre is equidistant from all three vertices.

Construction of a circle through three points

Example 3.3

Construct a circle to intersect three points X, Y, Z which are not on the same line in a plane. Label the centre of the circle, O.

Solution

Steps

1. Draw three points (the points should not be on the same line) and label them X, Y and Z respectively.
2. With a ruler, join the points X to Y with a straight line
3. Draw a perpendicular bisector of |XY| and label the bisector **d**.

   This is the same as constructing the locus of all points equidistant from points X and Y.

   Any point on **d** will be equidistant from X and Y.
4. Join points X to Z with a straight line and draw a perpendicular bisector of |XZ| and label it **f**.

   This is the same as constructing the locus of all points equidistant from points X and Z. Any point on **f** will be equidistant from X and Z.
5. Indicate the point of intersection of the lines as O.

   Point O is equidistant from X, Y, and Z. This means that |OX|=|OY|=|OZ|
6. With your compass pin at O, draw a circle with centre at O and radius |OX| or |OY| or |OZ|.

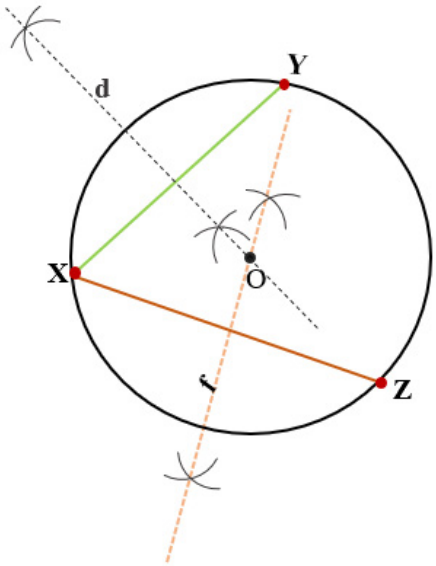


Figure 3.20: A circle passing through three points

What are your observations from the diagram drawn?

Note: All three points X, Y and Z must be equidistant from the circle's centre, O. This means that we seek to find a point, O, such that |OX|=|OY|=|OZ|

Can you recall how to copy a given angle? Try one in your sketch book and then click on the link to watch this video to help revise on how to copy a triangle https://youtu.be/2DVI5xmIhkI

CIRCLE THEOREMS

A circle theorem is a geometric rule that explains special relationships between angles, lines and points in and around a circle.

These theorems describe how angles and lengths behave when they are connected to the centre of a circle, circumference, chords, tangents and arcs. The theorems are also used in geometry to answer questions relating to circles.

Circle Theorem Statements

Statement One

The angle subtended by a chord or arc at the centre of a circle is twice the angle it subtends at the circumference of the circle.

In ***Figure 3.21***, A, B and C are points on the circle. O is the centre of the circle. The angle subtended by the chord |AB|, at the centre is and at the circumference is y.

The circle theorem statement one means that:

$x = 2y$ or $y = \frac{1}{2}x$

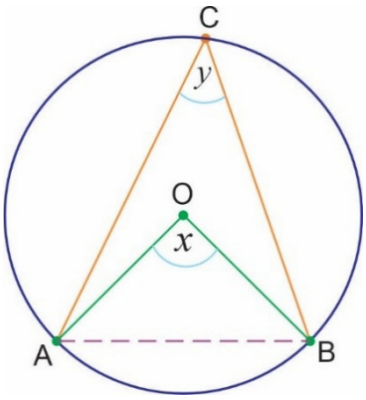


Figure 3.21: Cyclic triangle ABC with centre O

Example 3.4

In the figure below is a circle with centre O, ∠MON=96°, ∠MRN= 2x.Calculate the value of x

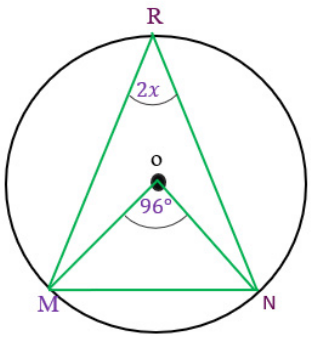


Figure 3.22: Cyclic triangle MNR with centre O

Solution

You know that the angle subtended at the centre is twice the angle subtended at the circumference: $\Rightarrow 2\angle MRN = \angle MON$ or $\angle MRN = \frac{1}{2}(\angle MON)$

$$2(2x) = 96°$$
$$\frac{4x}{4} = \frac{(96^\circ)}{4}$$
$$\therefore x = 24°$$

Statement Two

The angle subtended by the diameter at the circumference is a right angle.

In ***Figure 3.23***, O, A and B are points on the circumference of the circle with centre V. |AB| represents the diameter of the circle.

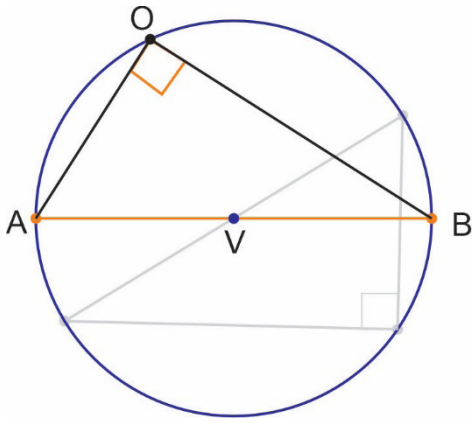


Figure 3.23: Cyclic triangle OAB with centre V $\angle AOB = 90°$

Example 3.5

In the figure below, O is the centre of the circle and |AE| is a diameter.

Given that $\angle AED = 56°$. Determine the value of $\angle DAE$

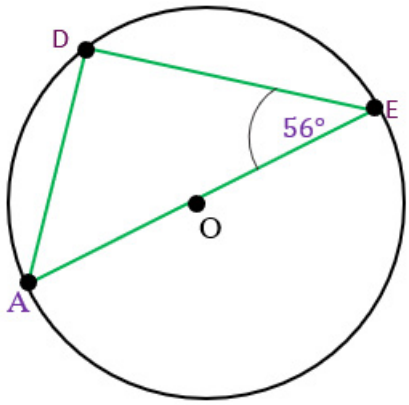


Figure 3.24

Solution

In ***Figure 3.24***, it can be observed that $\angle DAE$ *is an angle subtended by the diameter* (Theorem 2)

Therefore, $\angle ADE = 90°$
$\Rightarrow \angle DAE + \angle ADE + \angle AED = 180°$ (sum of interior angles of a triangle)
$\angle DAE + 90° + 56° = 180°$
$\angle DAE + 146° = 180°$
$\angle DAE = 180° - 146°$
$\angle DAE = 34°$

Statement Three

The angles subtended at the circumference by the same chord or arc in the same segment are equal.

In ***Figure 3.25*** M, N, O and P are points on the circumference of the circle.

If you consider |MN| as the chord ,then $\angle MPN = \angle MON$ since they are subtended at the circumference by the same chord |MN|.

This implies $a = b$

Again, if |OP| is the chord ,then $\angle PMO = \angle PNO$.

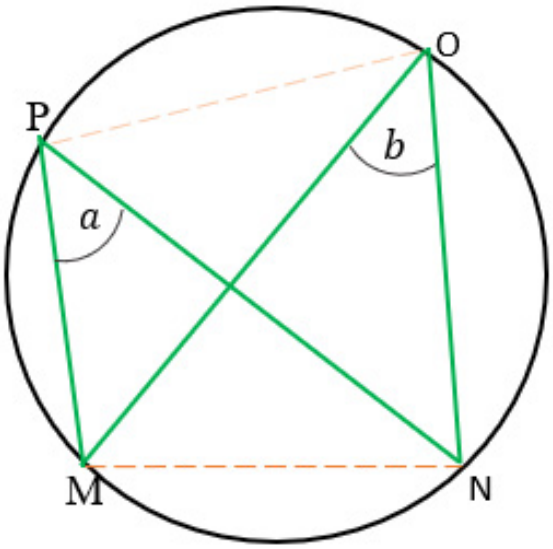


Figure 3.25: A circle with points M, N, O, P on its circumference

Example 3.6

In the figure below $\angle MON = 40°$, $\angle MPN = y$.

Determine the value of y

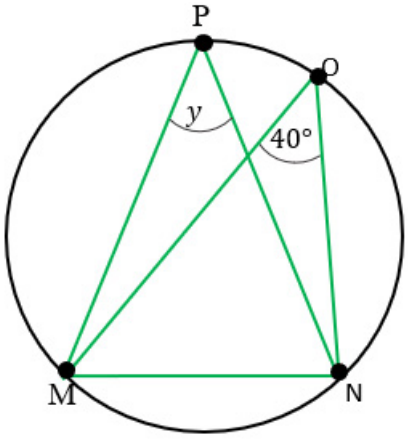


Figure 3.26:

Solution

From the diagram, chord |MN| subtends ∠MON and ∠MPN at the circumference

$\Rightarrow \angle MPN = \angle MON = 40°$

$\therefore y = 40°$

Statement Four

Two equal chords subtend equal angles at the centre of the circle.

If chords |XY| = |AB|, then ∠XOY = ∠AOB

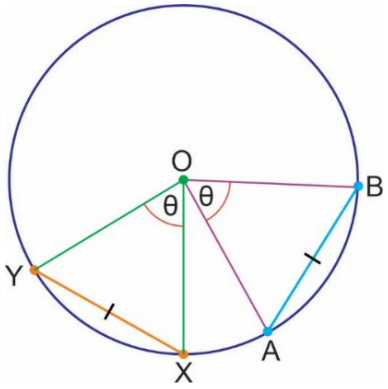


Figure 3.27: A circle of two equal chords

Statement Five

If the angles subtended by two chords at the centre are equal, then the two chords are equal.

Statement five is similar to statement four. One implies the other.

In ***Figure 3.28***, point O is the centre of the circle and |XY| and |AB| are chords in the circle.

From statement four, if |XY| = |AB|, then $\angle XOY = \angle AOB$.

From statement five, if $\angle XOY = \angle AOB$, then |XY|=|AB|

In short, we can say that "equal chords subtend equal angles at the centre of a circle."

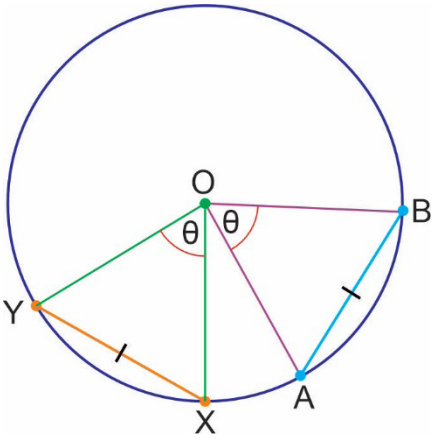


Figure 3.28: Circle with points Y, X, A and B on its circumference with centre O.

Example 3.7

In the figure below, ABCD are points on the circle whose centre is O. |AB| = |BC|.

Given that $\angle BOC = 52°$

Find the value of:

i. $\angle AOB$

ii. $\angle ADC$

iii. $\angle OBC$

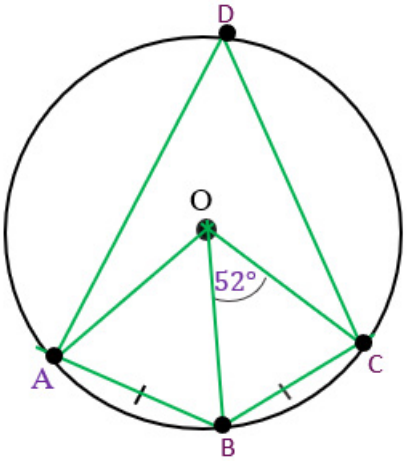


Figure 3.29

Solution

From the ***Figure 3.29***, since |AB| = |BC| then $\angle BOC = \angle AOB$ (equal chords subtend equal angles at the centre of a circle)

i. $\Rightarrow \angle BOC = \angle AOB = 52°$

ii. From theorem statement 1, it implies $2\angle ADC = (\angle AOC)$

Where $\angle ADC = \frac{1}{2}(\angle AOC)$

But $\angle AOC = \angle AOB + \angle BOC = 52° + 52° = 104°$

Substituting, $\angle ADC = \frac{1}{2}(104°) = 52°$

$\therefore \angle ADC = 52°.$

iii. Triangle OBC is isosceles as |OB|=|OC|, both radii. Therefore the base angles are equal, so $\angle OBC = \frac{180 - 52}{2} = 64°$.

Statement Six

Opposite angles in a cyclic quadrilateral are supplementary (add up to 180°)

Figure 3.30 shows a cyclic quadrilateral with interior angles ***a, b, c,*** and ***d***. Angles ***a*** and ***c*** are opposite to each other. Likewise, and .

This means that: $\boldsymbol{a + c = 180°}$ and $\boldsymbol{b + d = 180°}$

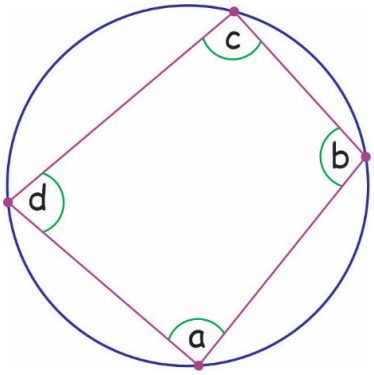


Figure 3.30: A cyclic quadrilateral with interior angles *a, b, c* and *d.*

Example 3.8

Figure 3.31 below is a circle with points Q, R, S and T on the circumference of the circle.

Given that $\angle RST = 78°$, find the value of $\angle TQR$

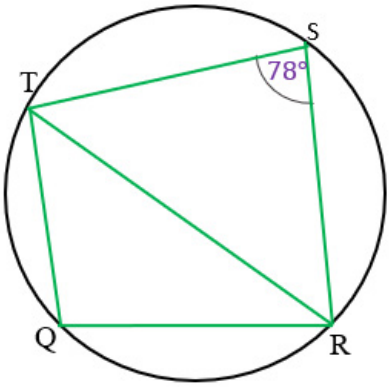


Figure 3.31

Solution

From the diagram, $\angle RST + \angle RQT = 180°$ (Opposite angles in a cyclic quadrilateral are supplementary)

$\angle RQT + 78° = 180°$

$\angle RQT = 180° - 78°$

$\therefore \angle RQT = 102°$

Statement Seven

The angle between a tangent and a chord at a point of contact is equal to the angle in the alternate segment.

In ***Figure 3.32***, ABC is a circumscribed triangle. The circle is tangent to Line DAE at point A. This means that $\angle BAE$ is equal to $\angle ACB$. This is the same as saying $x = y$.

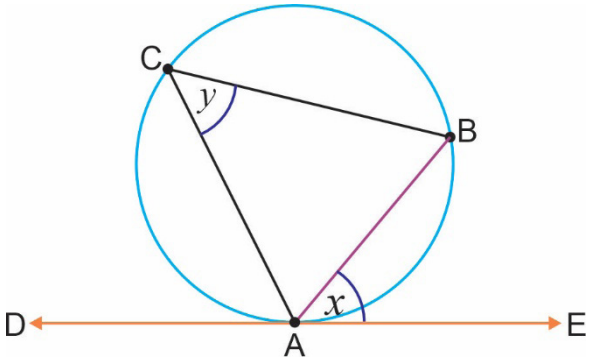


Figure 3.32: Line DAE tangent to a circle at point A

Example 3.9

In ***Figure 3.33*** |TP| is a tangent to the circle at point Q. Given that $\angle SQT = 70°$ and $\angle QSR = 30°$, work out for the value of the angle marked ***a***

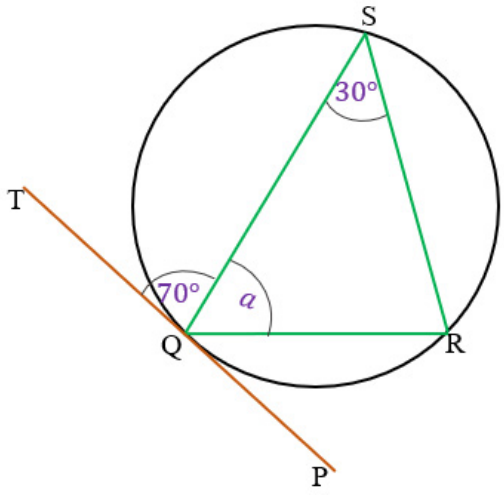


Figure 3.33

Solution

From theorem statement seven, $\angle SRQ = \angle SQT = 70°$(angle between tangent and a chord)

From the diagram, $\angle SQR + \angle QSR + \angle SRQ = 180°$ (sum of interior angles of a polygon)

$\Rightarrow a + 30° + 70° = 180°$

$a + 100° = 180°$

$a = 180° - 100° = 80°$

$\Rightarrow a = 80°$

Proof of Circle Theorem Statements

It is now time to prove some of these statements.

Proof of Statement One

Statement

The angle subtended by a chord at the centre is twice the angle subtended by it at the circumference.

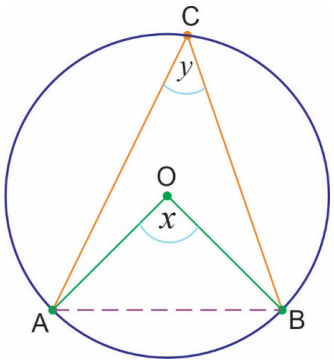


Figure 3.34: Cyclic triangle ABC with centre O.

Proof

From ***Figure 3.34***, we want to prove that $x = 2y$.

Draw a straight line from point C, through point O, to intersect the circle at point D.

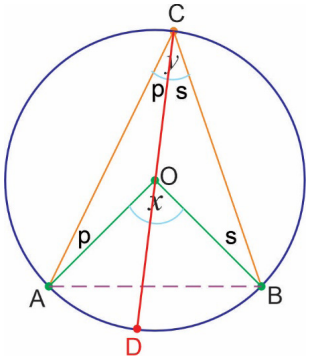


Figure 3.35: Cyclic triangle ABC with centre O and CD as a diameter.

This will generate two triangles: AOC and BOC.

Both triangles are isosceles since two of the sides of each triangle are radii. For instance, in ΔAOC,<OAC = <OCA = p and for ΔBOC, $\angle OBC = \angle OCB = s$

Remember that the "*exterior angle of a triangle theorem*" states that, for any triangle, the measure of the exterior angle of a triangle is equal to the sum of the measures of its two non-adjacent interior angles. Therefore, from this theorem, we have:

$\angle AOD = 2p$ and $\angle BOD = 2s$

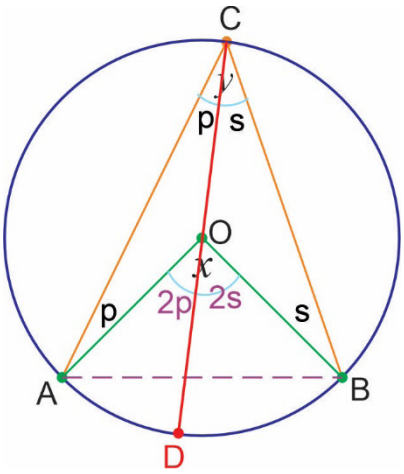


Figure 3.36: Cyclic triangle ABC with centre O and CD as a diameter.

This means that the angle at the centre

$x = 2p + 2s$

$x = 2(p + s)$ .....(1)

but $y = p + s$

substitute $y = p + s$ into equation (1)

This gives $x = 2y$ which proves the theorem.

Proof of Statement Two

Statement: The angle subtended by the diameter at the circumference is a right angle.

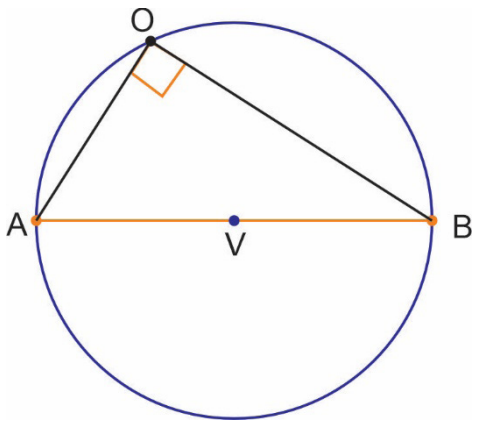


Figure 3.37: Cyclic triangle OAB with centre V.

Proof:

To prove that $\angle AOB = 90°$:

From **Statement 1**, *'The angle subtended by a chord at the centre is twice the angle subtended by it at the circumference.'*

So, you have . $\angle AVB = 2 \times \angle AOB$

You remember also that the angle at a point on a straight line is equal to 180°.

This implies that $\angle AVB = 180°$ as is a diameter.

So, we have which implies that $\angle AOB = 90°$.

Proof of Statement Three

Statement: The angles subtended at the circumference of a circle by the same arc or chord are equal.

Proof

Consider the figure 3.38 below, which shows an arc AB subtending and at two arbitrary points, C and D, on the circumference. O is the centre of the circle.

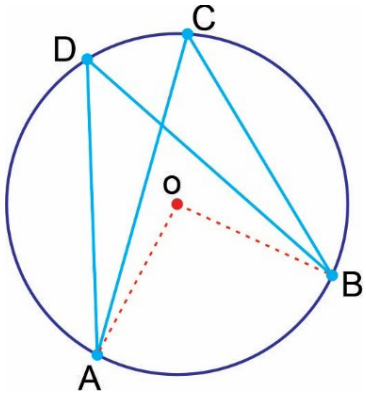


Figure 3.38: Circle ABCD with centre O.

To prove that $\angle ACB = \angle ADB$.

Using the circle theorem statement one; *'The angle subtended by a chord at the centre of a circle is twice the angle subtended by it at the circumference of the circle'*

$\angle ACB = \frac{1}{2} \times \angle AOB$ ..... (1)

Like wise $\angle ADB = \frac{1}{2} \times \angle$AOB ..... (2)

From equations (1) and (2), we get $\angle ACB = \angle ADB$ which proves theorem three.

Proof of Statement Four

Two equal chords subtend equal angles at the centre of the circle.

Proof

Consider the circle given below with centre O and two chords |AB| and |XY|, such that |AB| = |XY|.

To prove that $\angle AOB = \angle XOY$, consider the ***Figure 3.39*** below

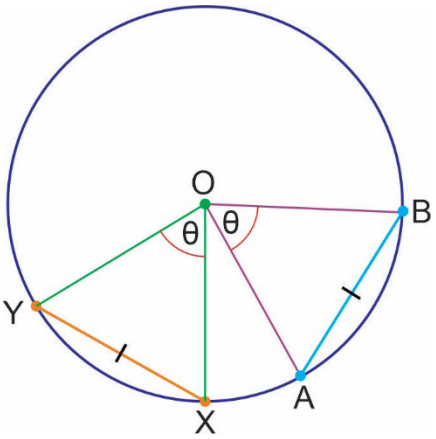


Figure 3.39: Circle with centre O with points Y, X, A and B on its circumference.

In triangles XOY and AOB, you have |OA|=|OB| (Radii), |OX|=|OY| (Radii) and |AB|=|CD| (Given)

Triangles XOY and AOB are congruent by the Side-Side-Side (SSS) congruence rule.

So, you have $\angle XOY = \angle AOB$ since corresponding sides and angles are the same.

Proof of Statement Five

Statement: If the angles subtended by two chords at the centre are equal, then the two chords are equal.

Proof

Consider ***Figure 3.40***; It is a circle with centre O and two chords, |AB| and |CD|, such that ∠AOB = ∠COD.

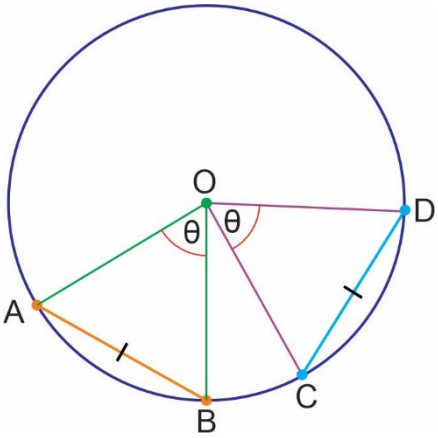


Figure 3.40: Circle ABCD with centre O

To prove that |AB| = |CD|

Consider triangles AOB and COD, you have the radius |OA| = |OC|=|OB| = |OD|= radii and ∠AOB = ∠COD (Given)

This means that triangles AOB and COD are congruent by the Side-Angle-Side congruence rule.

So, you have |AB| = |CD| (Corresponding parts of congruent triangles).

We will explore the other proofs in later lessons.

The tangent of a circle

Remember that a tangent is a straight line drawn from an external point that touches a circle at exactly one point at the circumference of the circle. There can be an infinite number of tangents to a circle. These tangents follow certain properties/ theorems that can be used as identities to perform mathematical computations on circles.

Consider the theorems of the tangent to a circle and we will then look at how to prove them.

Theorem 1

The tangent at any point of a circle is perpendicular to the radius at the point of contact (tangency). In other words, a tangent is perpendicular to a radius at the point of contact (tangency) from the centre of a circle.

To prove that the tangent at any point of a circle is perpendicular to the radius at the point of contact, consider the ***Figure 3.41*** below.

O is the Centre of the circle and |XL| is the tangent to the circle at point A.

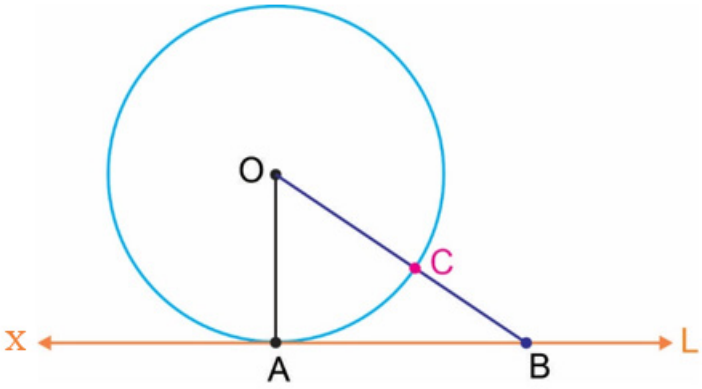


Figure 3.41: A tangent to a circle at point A

In ***Figure 3.41***, |OA|=|OC| as they are radii of the circle.

In addition, |OB| =|OC| + |CB|.

It can be observed clearly that, |OC| < |OB| and |OA| < |OB|

This logic holds for every point B on the tangent L, except point A.

So |OA| is shorter than any other line segment joining O to any point on |XL|.

Hence, OA is perpendicular to |XL|

With your drawing tools and materials, let us perform the activity on the next page.

Activity 3.1 Tangent to a Circle

The tangent at any point of a circle is perpendicular to the radius at the point of contact.

1. Draw a circle with centre O and radius OA.
2. Draw a tangent line |XL| through point A.
3. Choose a point B on |XL|, other than A.
4. Join O to B, intersecting the circle at C.
5. Measure |OA| and |OB|
6. Write down your observations, discuss them with your classmate(s) and compare your findings with the conclusion below.

Since every point on |XL|, other than A, lies outside the circle and OB > OA. This implies OA is the shortest line segment from O to L. The shortest line segment drawn to a line from a point outside it, is perpendicular to the line. Therefore, |OA| is perpendicular to line L.

Mathematically, |OA| ⊥ |XL|

Therefore, it is true that, the tangent at any point of a circle is perpendicular to the radius passing through the point of contact.

With this understanding, let us look at the examples below.

Example 3.10

In the diagram below, O is the centre of the circle, |NP| is tangent to the circle at R and $\angle MOR = 110°$.

Calculate the value of $\angle MRP$

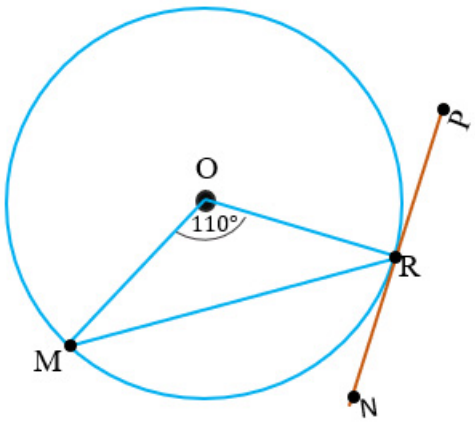


Figure 3.42: A tangent to a circle at point R

Solution

In ***Figure 3.42***, the triangle *MOR* is isosceles, since |OM|=|OR|= radius r.

$\Rightarrow \angle OMR = \angle ORM$

Since $|OR|$ (radius) meets $|NP|$ (tangent) at R (point of contact)

$\Longrightarrow \angle ORP = 90°$ (a tangent is perpendicular to a radius at the point of contact)

To find the value of $\angle MRP$, find the value of ∠ORM first.

So, let $\angle ORM = x$

$\angle OMR + \angle ORM + \angle MOR = 180°$

$x + x + 110° = 180°$

$2x = 180° - 110°$

$2x = 70°$

$\frac{2x}{2} = \frac{70°}{2}$

$x = 35°$

$\therefore \angle OMR = \angle ORM = 35°$

Now, $\angle MRP = \angle ORM + \angle$ ORP

But ∠ ORP = 90°

$\angle MRP = 90° + 35° = 125°$

Theorem 2

Alternate Segment Theorem

The Alternate Segment Theorem states that "*For any circle, the angle formed between the tangent and the chord at the point of contact of the tangent and the circle is equal to the angle formed by the chord in the alternate segment*".

The alternate segment theorem is also known as the tangent-chord theorem.

Let us use ***Figure 3.43***; to break down the theorem.

In the ***Figure***, ABC is a cyclic triangle with centre O. The circle is tangent to Line DAE at point A. The shaded part is a minor segment. Its alternative segment is the unshaded part of the circle. According to the alternative segment theorem, ∠BAE is equal to ∠ACB. i.e. $x = y$

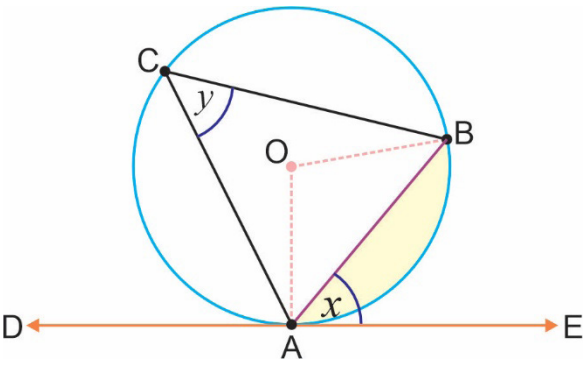


Figure 3.43: Tangent to a circle at point A

Proof:

From our previous proof, we established that the angle the radius or the diameter makes with the tangent at the point of tangency is a right angle (90°)

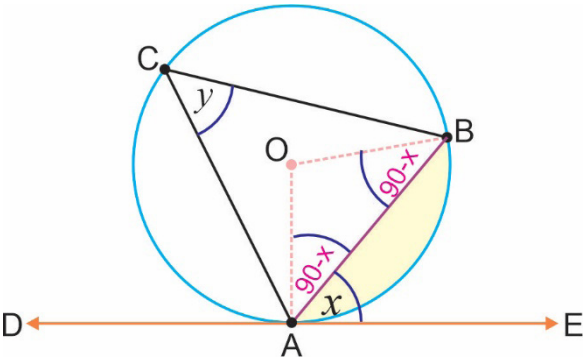


Figure 3.44: Tangent to a circle at point A

Therefore, $\angle OAE = 90°$

This implies that $\angle OAB + x = 90°$

solving for $\angle OAB$, we have $\angle OAB = 90° - x$

Since triangle OAB is isosceles,

$\angle OAB = \angle OBA = 90° - x$

Also, $\angle AOB + (90° - x) + (90° - x) = 180°$

solving for AOB, we have $\angle AOB = 180° - 90° - 90° + 2x$

This implies $\angle AOB = 2x$ .....(1)

Remember that *the angle at the centre of the circle is twice the angle at the circumference of the circle.*

This means that:

$\angle AOB = 2y$..... (2)

Comparing (1) to (2), we have $2x = 2y$

This means $x = y$

Example 3.11

In the figure below,|AB |= |AC| and the tangent |BT| touches the circumcircle of triangle ABC at B. Given that $\angle ABC = 70°$, find:

i. $\angle BCT$

ii. $\angle ATB$

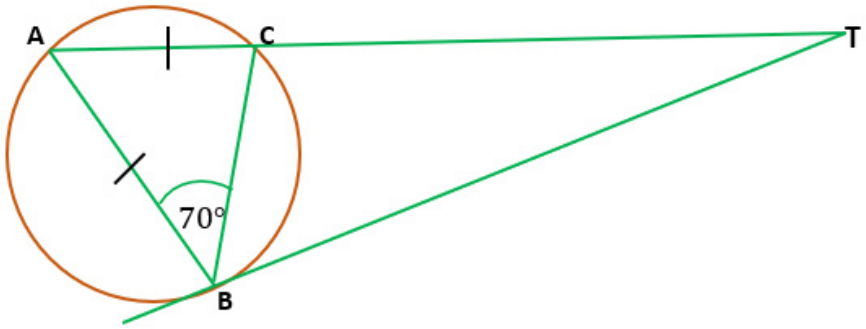


Figure 3.45: Tangent to a circle at point A

Solution

i. $\angle ABC = \angle BCA = 70°$ (base $\angle$s of an Isosceles Δ)

$\angle ABC + \angle BCA + \angle CAB = 180°$

$70° + 70° + \angle CAB = 180°$

$140° + \angle CAB = 180°$

$\angle CAB = 180° - 140°$

$\angle CAB = 40°$

$\angle CAB = \angle CBT = 40°$

$\angle BCT = \angle TAB + \angle ABC$

$\angle BCT = 40° + 70° = 110°$

ii. $\angle ABT + \angle TAB\ ° + \angle ATB = 180°$

$(70° + 40°) + 40° + \angle ATB = 180°$

$150° + \angle ATB = 180°$

$\angle ATB = 180° - 150° = 30°$

Alternate Segment Theorem for Quadrilaterals

In ***Figure 3.46***, line FAE is a tangent to the circle at point A. B, C and D are points on the circumference of the circle and ABCD is a cyclic quadrilateral.

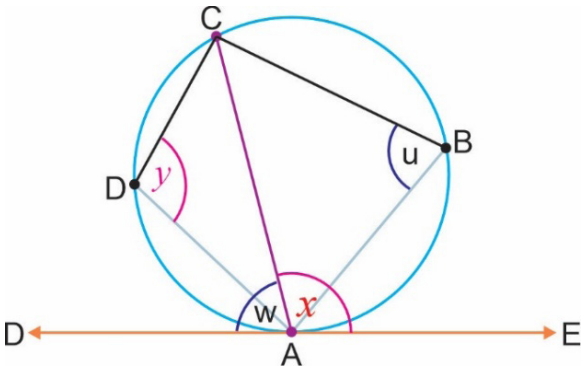


Figure 3.46: Line DAE tangent to cyclic quadrilateral ABCD at point A

From the alternate segment theorem, $\angle DAF = \angle CBA$

$\Rightarrow w = u$……..…(1)

We need to prove that $\angle CAE = \angle CDA \Rightarrow x = y$

Although the alternate angle theorem holds for this, we can try another theorem to prove that this is true.

From the previous theorems we know that opposite angles in a cyclic quadrilateral add up to 180°.

$\Rightarrow \angle y + \angle u = 180°$ ……(2)

Since $\angle w$ and $\angle x$ are adjacent at point A on a straight line, we can conclude that

$\angle w + \angle x = 180°$ …….(3)

From equations (2) and (3), we have $\angle y + \angle u = \angle w + \angle x$

but from (1), we have $w = u$

Therefore, $\angle y + \angle w = \angle w + \angle x$

Eliminating $\angle w$,

$\Rightarrow \angle y + \angle w - \angle w = \angle x$

we have $\angle y = \angle x$ or $\angle x = \angle y$

Theorem 3

If two segments from the same external/exterior point of a circle are tangents to the circle, then they are congruent.

This means that two tangents drawn from the same external point/exterior point to a circle are equal when measured from their point of contact.

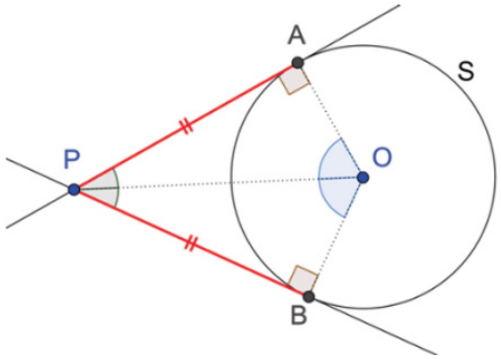


Figure 3.47: |PA| and |PB| are tangent to a circle with centre O at points A and B.

In ***Figure 3.47***, and are tangents to circle, S, at points A and B. Point O is the centre of the circle. According to the equal tangent theorem, |PA| = |PB|.

Also, $\angle POA = \angle POB$ and $\angle APO = \angle BPO$

Note: APBO can also be called a tangent–kite and has axis of symmetry OP.

Two tangents to a circle from a point outside the circle are equal in length. |AP|=|BP|.

Thus, $\angle APO = \angle BPO$ and $\angle POA = \angle POB$

Proof

All three parts will be proved if we show that ΔPAO is congruent to ΔPBO.

Comparing the two triangles, we see that:

- OA = OB (radii of the same circle)
- OP = hypotenuse of both triangles.
- From our first theorem, $\angle OAP = \angle OBP = 90°$

By the Side-Angle-Side (SAS) property of triangles, triangle OAP is congruent or the same as triangle OBP.

This follows that the lengths of the tangents |AP| and |BP| are equal. Also, |OP| bisects BPA and AOB.

Hence, $\angle BOP = \angle AOP$ and $\angle BPO = \angle APO$.

Example 3.12

A belt ABCD moves round a shaft D (whose radius is negligible) and a pulley of radius 1.2m. O is the centre of the pulley and |OD| = 3.0m.

The straight portions AD and CD of the belt are tangents at A and C.

i. To the nearest degree, determine the value of $\angle AOC$

ii. The total length of the belt (ABCD) to the nearest metre (Take $\pi = 3.142$)

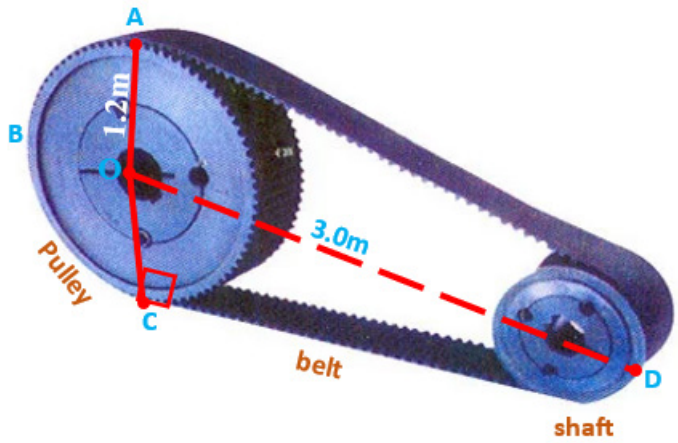


Figure 3.48: A belt around a pulley and a shaft

To solve the above, the figure can be sketched and since the radius of the shaft is negligible, the figure will look like this:

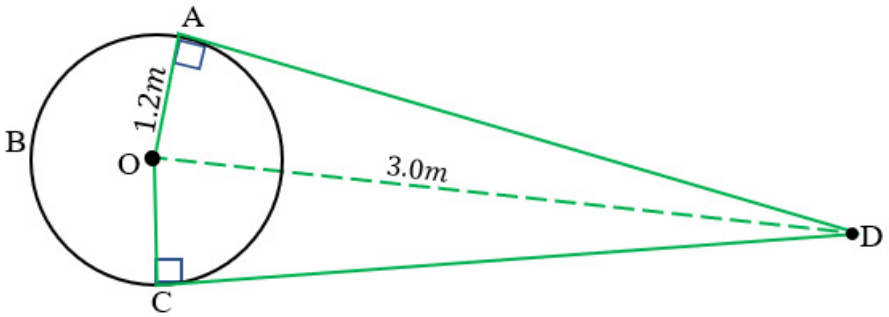


Figure 3.49: Sketch of the pulley system

To find $\angle AOC$, take any of the right triangles ($\triangle AOD$ or $\triangle COD$)

Considering $\triangle AOD$

i. Let ∠AOD be θ

$$Cos\theta = \frac{Adjacent}{hypotenuse} = \frac{|OA|}{|OD|}$$

$$Cos\theta = \frac{1.2}{3.0} = \frac{2}{5}$$

$$\theta = cos^{-1}\left(\frac{2}{5}\right) = 66.4218°$$

$$\angle AOC = \angle AOD + \angle COD$$

By the tangent theorem, $\angle AOD = \angle COD = 66.4218°$

$$\Longrightarrow \angle AOC = 2(66.4218°) = 132.8436°$$

$$\therefore \angle AOC = 133°(to\ the\ nearest\ degree)$$

ii. To find the total length of the belt, you need to find the length of the belt from the external point D to the circle at the points of tangency (A and B) and around the circle.

Considering $\triangle AOD$ again,

By Pythagoras theorem, $|OD|^2 = |OA|^2 + |AD|^2$

$$|AD|^2 = |OD|^2 - |OA|^2 = (3.0)^2 - (1.2)^2$$

$$|AD|^2 = 9 - 1.44 = 7.56$$

$$|AD| = \sqrt{7.56} = 2.7495m$$

By the tangent theorem, $|AD| = |CD| = 2.7495m$

Finding the length of the belt around the circle

$$\text{Length of major Arc } ABC = \frac{360° - \angle AOC}{360°} \times 2\pi r$$

$$= \frac{360° - 133°}{360°} \times 2 \times 3.142 \times (1.2)$$

$$= 4.7549m$$

Total length of the belt= $|AD| + |CD|$ + Length of major Arc ABC

$$2.7495m + 2.7495m + 4.7549m = 10.2539m$$

$\therefore$ The total length of the belt ABCD $= 10m$ (to the nearest metre)

Perform Activity 3.2 below with your classmate.

Activity 3.2 Proof of Some Circle Theorems

Part 1: Angles at the centre and the circumference

i. Draw a circle and label the centre O.

ii. Mark two different points A and B on the circumference and join them to form a chord AB.

iii. With the ruler, join O to A and O to B. This forms a central angle $\angle AOB$.

iv. Mark a point C on the same arc AB and join A to C and B to C with a ruler.

v. Measure $\angle AOB$ and $\angle ACB$ with a protractor and record them.

vi. Discuss what have observed with your classmate.

This verifies that $\angle AOB \approx 2\angle ACB$.

Part 2: Angles in the same segment

i. With the same circle and chord , choose another point D on the same arc AB as point D.

ii. Join A to D and B to D using a ruler.

iii. Measure $\angle ACB$ and $\angle ADB$ and record each.

Verify that they are equal.

Part 3: Cyclic quadrilateral theorem

i. Draw another circle

ii. Mark four points A, B, C, D on the circumference so they form a quadrilateral.

iii. Join them in this order: A to B, B to C, C to D, D to A.

iv. Measure opposite angles $\angle ABC$ and $\angle ADC$, find their sum and record the value.

v. Verify that their sum is about 180°. Do the same for the other pair of opposite angles.

Part 4: Tangent–Radius Theorem

i. Draw a new circle with centre O.

ii. Choose a point P on the circumference.

iii. Join O to P to form a radius.

iv. At P, draw a tangent to the circle (make sure the line touches the circle at one point P only).

v. With the protractor, measure the angle between OP and the tangent.

vi. Verify that the angle is 90°.

- What patterns do you notice for each theorem?
- Do the results change if you choose different points?

Understanding Geometric Constructions

In Year 1, we learnt how to construct various angles such as 75°, 105°, 135° and 150°. We will revise this with a reminder on the construction of angles .

Remember that to have an accurate outcome in construction:

- Make a rough sketch of the task assigned. This will help anticipate challenges associated with the construction.

- Do not clean anything that contributes to the final result. Leave all construction lines visible.
- For lines and points to be as clear and accurate as possible, use hard pencils with sharp tip/point.

1. Construction of angle 105°

Materials needed: pair of compasses, ruler, pencil

Steps:

i. With your ruler and pencil, draw a straight horizontal line

ii. Choose a point on the line as your centre and any suitable radius,

iii. Construct angle 90°, see the diagram below to remind you how to do this.

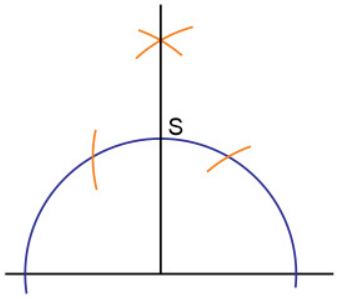


Figure 3.50: Construction of 90°

iv. Construct 120° adjacent to 90°

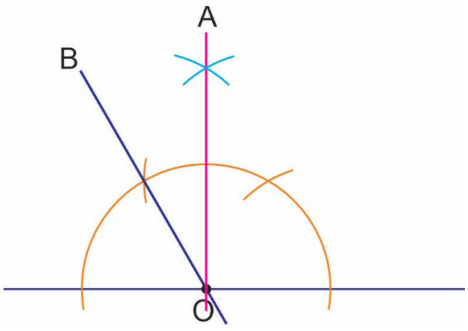


Figure 3.51: Construction of 90° adjacent to 120°

v. Bisect AOB

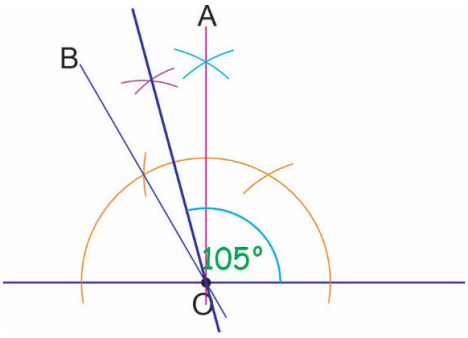


Figure 3.52: Construction of 105°

Use your protractor to measure the angle to verify

2. **Construction of 135°**

   You can achieve this if you construct 90° and bisect the angle between the other 90°.

Steps:

i. With your compass and ruler, construct angle 90°, as above.

ii. Bisect the adjacent to 90° obtain angle 45, 90°+ 45° = 135°

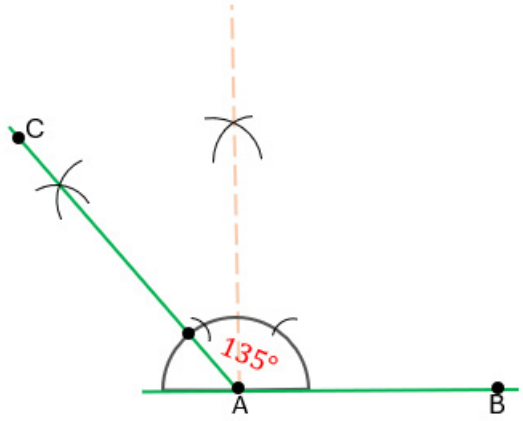


Figure 3.53: Construction of 135°

Click on this link www.math-only-math.com to watch more tutorials on how to construct angles for better understanding.

Construction of quadrilaterals: Concept, examples and visualisation of surds

We have learnt how to construct lines, angles, circles and some triangles. We will now continue with the construction of other interesting shapes, including revising how to construct triangles.

Construction of Triangles

A triangle can be constructed using a ruler, pencil and compass only if the following information is provided:

1. All three sides of the triangle are provided, or
2. Two sides and the included angle are given, or
3. One side and two angles are given.

Use the following examples to perform some activities on constructing triangles.

Materials needed: pair of compasses, ruler, pencil, eraser and workbook.

Example 3.13

Using a ruler and a pair of compasses only, construct triangle ABC such that:

|AB|=10cm, |AC|=8cm, and |BC|=6cm

Solution

We can choose to construct any of the lines first.

Steps

To construct line |AB|, first.

1. Draw a line of 10cm and label the line AB
2. Open your compass to 8cm on the ruler and with the compass pin at A, draw an arc
3. Open the compass to a length of 6cm on your ruler and with your compass point now at B, draw an arc to intersect the other arc you have drawn earlier. Indicate the point of intersection of the two arcs as C.
4. Finally, with your ruler, join the points A to C and B to C with straight lines to get |AB| and |BC|

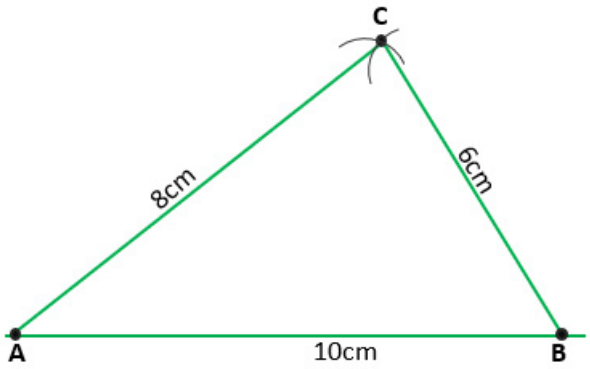


Figure 3.54: Triangle ABC

Example 3.14

Using a ruler and a pair of compasses only, construct triangle ABC such that: |AB|= 7cm, |AC| = 5cm and ∠CAB= 90°. Measure |BC|

Solution

Steps:

1. Construct line |AB|, indicating the points A and B, which are 7cm apart.
2. At point A, construct a 90° angle.

3. Measure 5cm along this line and mark point C.

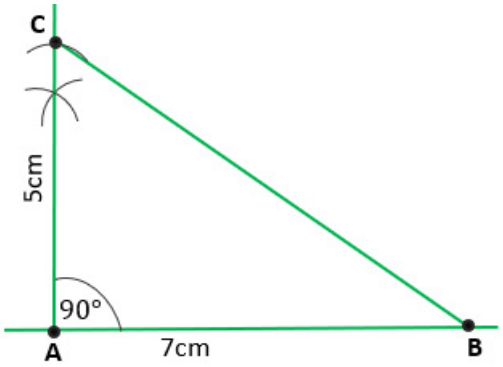


Figure 3.55: Triangle ABC (Not drawn to scale)

4. With a ruler, measure line |BC| ≈ 8.5cm

Example 3.15

Using a ruler and a pair of compasses only, construct triangle XYZ such that:

|XY| = 5.5cm, $\angle YXZ = 30°$, XYZ = 105°.

Solution

Steps:

1. First construct |XY| and indicate clearly the points X and Y which are 5.5cm apart.
2. Construct ∠30° at X and ∠105° at Y.
3. Name the point of intersection of the two lines Z. This gives you the triangle XYZ

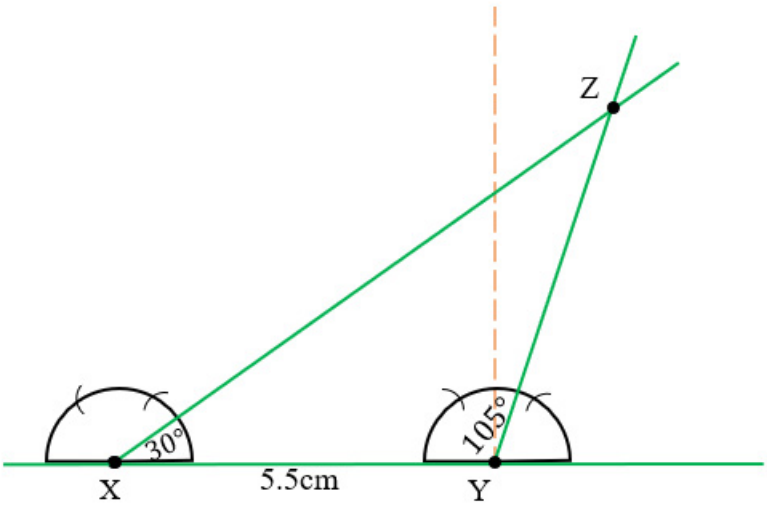


Figure 3.56: Triangle XYZ

Construction of a circumscribed circle when given a triangle

To construct a circumscribed circle when given a triangle (i.e., a circle where the circumference touches the vertices of the triangle), you have to determine the centre of the triangle. The centre cannot be obtained by just observing and guessing. Below are the steps to guide you to locate the exact centre that will help you draw the circle accurately.

Example 3.16

Copy the triangle below and use it to construct a circumscribed circle.

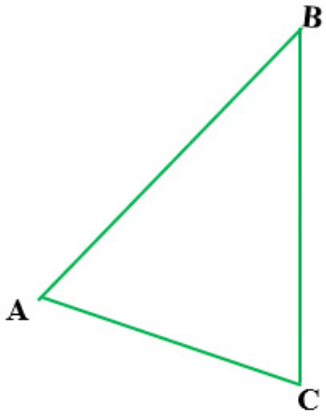


Figure 3.57: Triangle ABC

Steps:

1. Construct a copy of the triangle.
2. Construct a perpendicular bisector of any two sides of the triangle.
3. Label the point of intersection of the bisectors as O
4. With the compass pin at O and using |OA| or |OB| or |OC| as the radius, construct a circle to pass through A, B and C

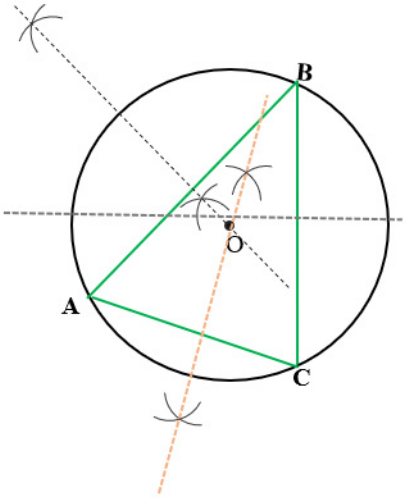


Figure 3.58: A cyclic triangle ABC with centre O.

Construction of quadrilaterals

We know that a quadrilateral is a four-sided polygon bounded by four straight lines and it has four sides, four vertices and four angles. We know that common quadrilaterals include rectangles, squares, rhombuses, trapeziums, kites. These figures can be constructed using a pair of compasses and a ruler and are easy to construct just as the triangles.

Follow the steps in the examples below on how to construct some of them.

Example 3.17

Using ruler and a pair of compasses only, construct a square ABCD of side 5cm.

Solution

To construct a square, all you need to know is the length of a side and then apply the properties of a square to get the expected outcome.

Steps:

i. Using a ruler and pencil, draw line AB of length 5cm.

ii. With a compass, construct perpendicular lines (angles 90°) at points A and B, as the corner angles of a square are right angles.

iii. Measure 5cm with your compass. With the pin at A, draw an arc on the perpendicular line and label the point D.

iv. With the pin at B, draw another arc (same radius) on the perpendicular line and label it C, sides of a square are equal and parallel.

v. Join C to D to form square ABCD.

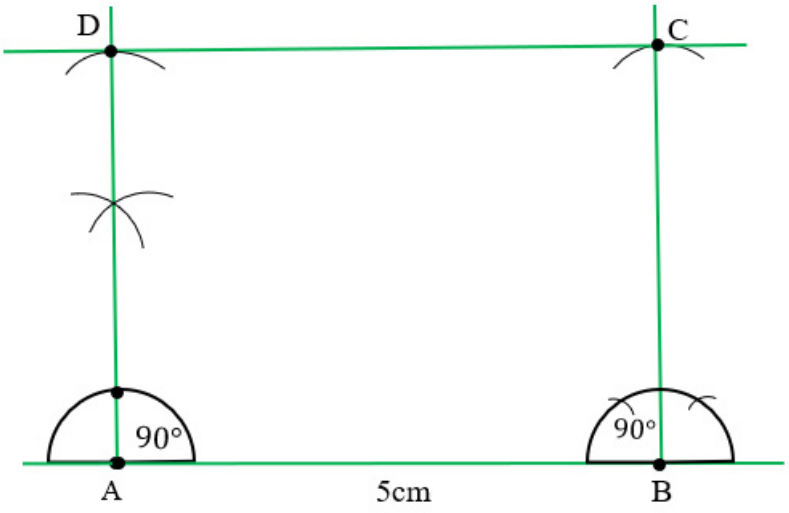


Figure 3.59: Construction of square ABCD

Example 3.18

Using a ruler and a pair of compasses only, construct a trapezium MNOP, with

|MN| = 7.5cm, |NO|=5.5cm, ∠PMO = 60°, ∠MNO = 90°.

Solution

Steps:

1. With your ruler and pencil, construct line |MN| which is 7.5cm
2. With your compass pin at point M, construct angle 60°
3. Again, with your compass pin at point N, construct angle 90°
4. Measure on the ruler with your compass and with your compass pin at N, draw an arc to intersect the perpendicular line at point O
5. Since it is a trapezium, a pair of opposite sides must be parallel. Therefore, |MN| || |PO|.

   With the compass pin again at point M and with the same measure, draw an arc to cut the perpendicular line at point Q
6. Join point Q to point O. Mark the point of intersection of the 60° line and |OQ| as point N.

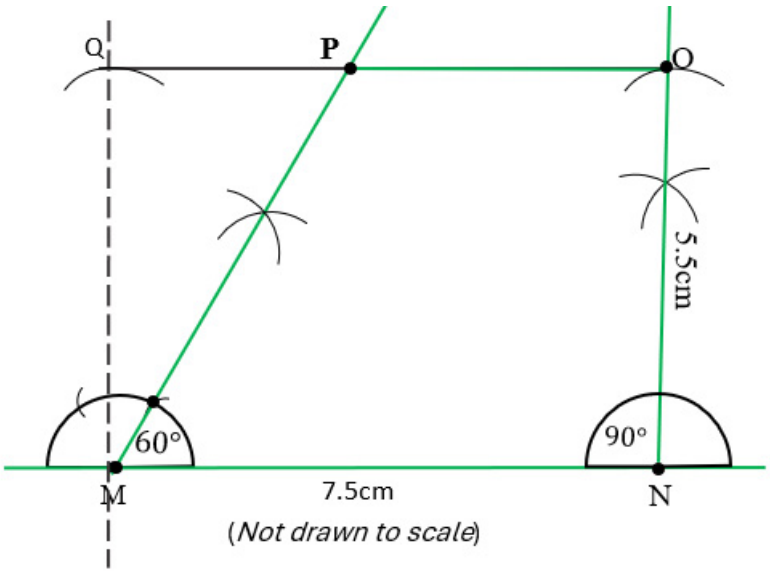


Figure 3.60: Construction of trapezium MNOP

Example 3.19

Using a ruler and a pair of compasses only, construct a parallelogram ABCD such that:

|*AB*| = 6cm, |*BC*| = 5cm ∠*DAB* = 60°.

Measure |CD|

Solution

To construct a parallelogram, remember that it has its pairs of opposite sides are equal and parallel.

Steps:

1. Draw the line AB, 6cm
2. With the compass pin at point A, construct angle 60°
3. At point B construct angle 60° (parallel to point A) but from the other direction (anticlockwise)
4. Open your compass to 5cm on your ruler and with your compass pin at point B, draw an arc to intersect the line at C
5. With the same radius and with your compass pin at point now at A, draw an arc to intersect the line at D
6. Join the point C to D
7. At point C, open your compass to point D and measure the radius on the ruler.
8. |CD| ≈ 6cm

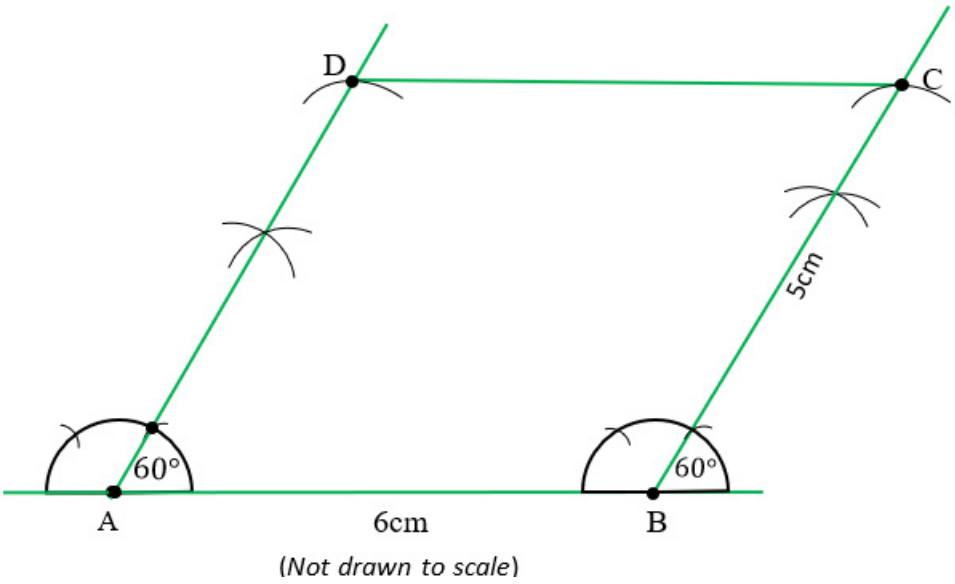


Figure 3.61: Construction of a parallelogram ABCD

Have you ever thought of determining a point or a line that is of equal distance from a point, points, line or lines in a plane?

Earlier we constructed a circle through three points that are non-collinear (not on the same line). This circle was achieved using the idea of the construction of a locus (plural is loci). Now, we will look at how to construct a locus of a point or points, a line or lines in a plane.

Locus

A locus is a set of all points that satisfy a given condition or set of conditions. Think of it as a path traced by a point moving according to some specified rules.

Here we will explore the construction of locus in two ways:

- *Theorems and conditions of locus*
- *Construction of locus*

Locus theorem 1

Locus of points equidistant from a single point

Method: Using the point as the centre and the given distance as the radius, construct a circle. The locus is the circle.

Example 3.20

Draw a locus of all points equidistant from a point O.

Solution

Steps:

1. in your workbook, draw a point and label it O
2. With your compass pin at O and opened to a suitable radius, for example, 4cm, and draw a circle.

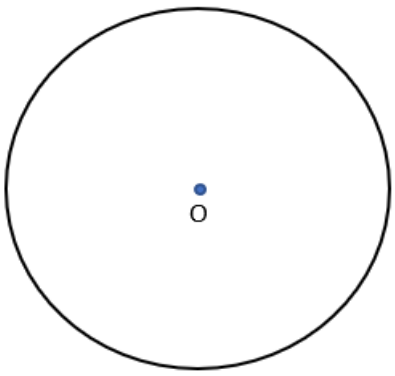


Figure 3.62: A circle equidistant from O

The circle is the locus of all points equidistant from O. Any point drawn on any part on the circumference of the circle above, will be the same distance away from the centre O.

Example 3.21

Construct a locus of all points that are 3cm from A on the quadrilateral ABCD in the figure below.

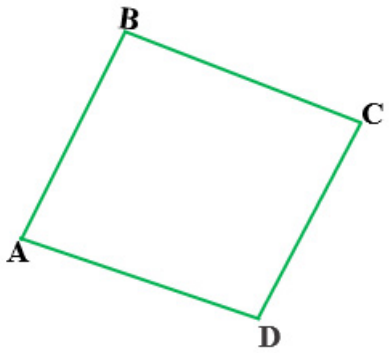


Figure 3.63: A quadrilateral ABCD

Solution

With the aid of a ruler and a pair of compasses only, make a copy of the quadrilateral ABCD.

Open your compass to a radius of 3cm and with your compass pin at A, construct a complete circle.

The circle is the locus of the point A.

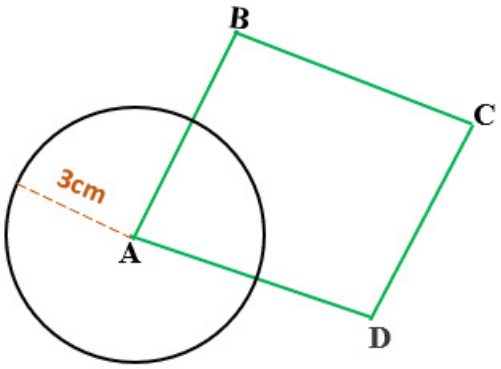


Figure 3.64: A quadrilateral ABCD and a circle with centre A

Locus Theorem 2

Locus of points equidistant from two points.

Method: Construct a perpendicular bisector of the line segment determined by the two points.

The locus is the perpendicular bisector of the line.

Example 3.22

Draw a locus of points which are equidistant from A and B, with $|AB| = 8$cm.

Solution

The solution is drawing a perpendicular bisector of line AB.

Draw a straight line and two points on the line 8cm apart and label them A and B

Using a ruler and a pair of compasses, construct a bisector of AB.

The bisector is the locus, of points equidistant from A and B as shown in ***Figure 3.65***

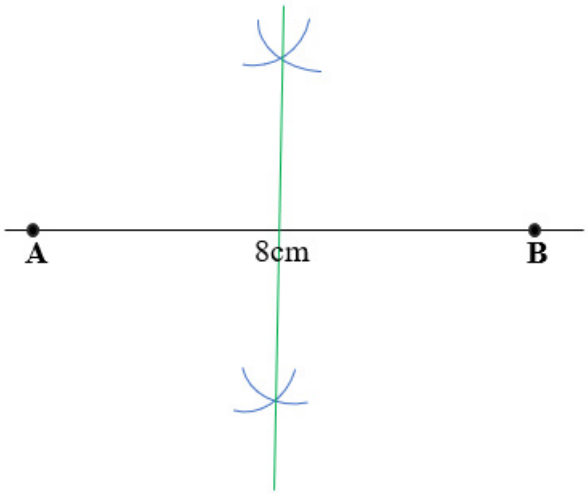


Figure 3.65: Locus of two points A and B

You would observe that the locus of points equidistant from the points A and B is the perpendicular bisector of |AB|.

Example 3.23

Construct a locus of points equidistant from A and C in ***Figure 3.66***

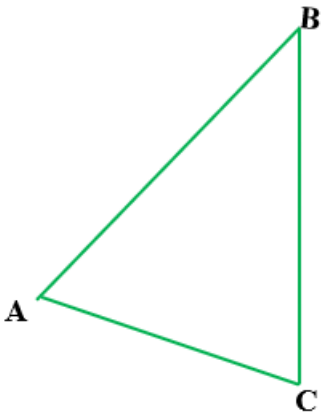


Figure 3.66: A triangle ABC

Solution

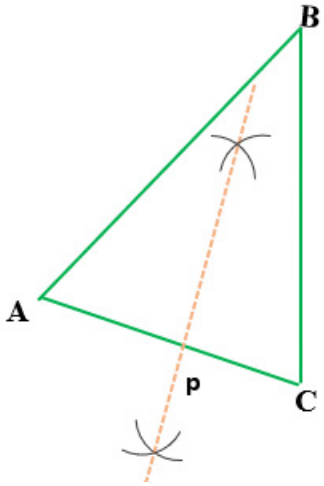


Figure 3.67: Triangle ABC and line *p* bisecting line AC

Line *p* is the locus of the point A and C

Locus theorem 3

Locus of points equidistant from two intersecting lines.

Method: Bisect the angle of intersection of the two lines. The locus is the line bisecting the angle. It can also be called the angle bisector.

To construct an angle bisector of any two segments |AB| and |BC|.

Steps:

1. Draw any two intersecting lines (|AB| and |BC|)
2. Locate the point of intersection of the two lines which is B
3. With your compass point at the B and with a suitable radius, draw an angle between the two lines.
4. With your compasses again, bisect the angle between the lines (|AB| and |BC|)

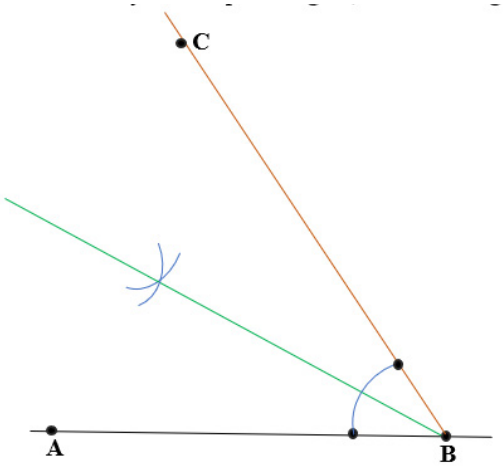


Figure 3.68: Angle bisector of ABC

The bisector of the angle at B (the line that bisects the angle) is the locus, which is equidistance from lines AB and BC

Example 3.24

Construct the locus of points equidistant from AB and CD as shown in ***Figure 3.69*** below.

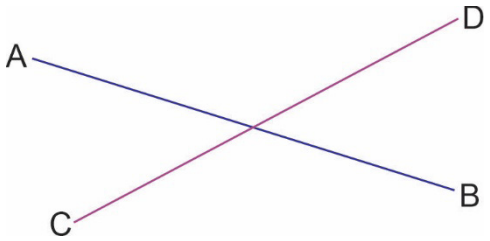


Figure 3.69: lines AB and CD

Solution

Using a ruler and a pair of compasses only, make a copy of the figure

Construct the locus P, of points equidistant from AB and CD

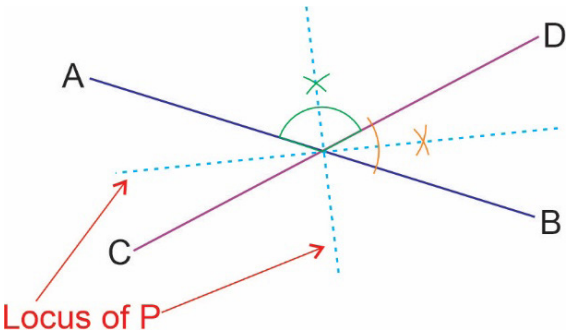


Figure 3.70: Construction of loci equidistant from AB and CD

Locus theorem 4

Locus of points equidistant from two parallel lines

Method: The locus equidistant from two parallel lines, |AB| and |CD| is a line parallel to both AB and CD and halfway between them.

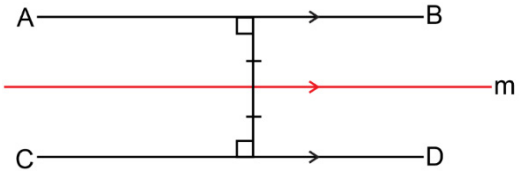


Figure 3.71: line m equidistant from AB and CD

In the diagram above, the locus is line m. Line m is parallel to and equidistant from lines AB and CD. Line m is also parallel to AB and CD.

Example 3.25

The diagram below represents a construction of two parallel lines m_1 and m_2.

Construct the locus, d equidistant from m_1 and m_2 .

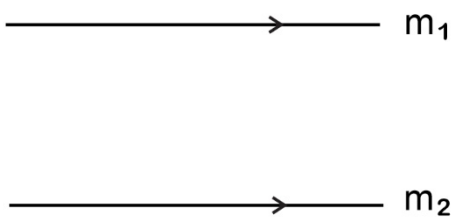


Figure 3.72: parallel lines m_1 amd m_2

Solution

Steps:

1. Construct a perpendicular at any point O on line m_1 to intersect line m_2 at point A.

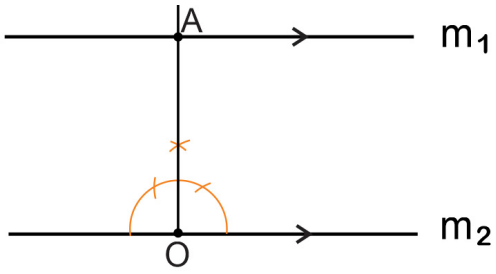


Figure 3:73: A perpendicular transversal intersecting parallel lines m_1 and m_2 at point O and A

2. Construct a perpendicular bisector of line |OA|

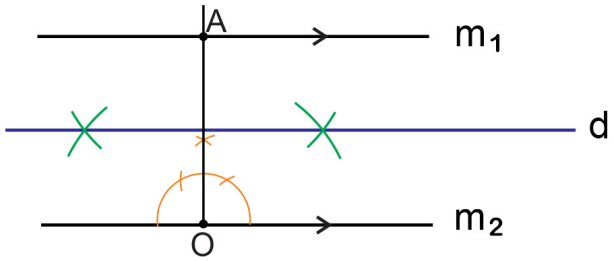


Figure 3.74: Line **d** equidistant from parallel lines m_1 and m_2

3. Line **d** is the locus of points parallel to lines and .

Locus theorem 5

Locus of points equidistant from a line.

Method: This is a pair of parallel lines that are a distance away from the given line.

This theorem is illustrated in the ***Figure 3.75***.

The figure illustrates the locus of points, p, of points m units from line AB.

The locus, p, represents the dotted lines on opposite sides of line AB.

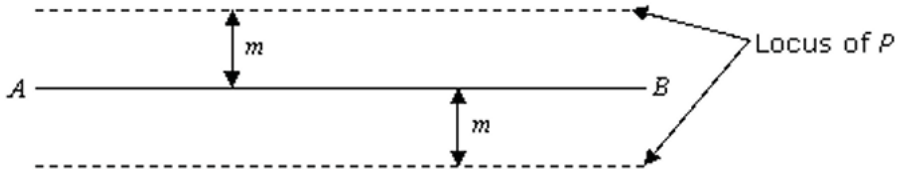


Figure 3.75: Loci p equidistant from line AB

Locus theorem 6

Constructing a perpendicular from a point *O* to a line.

Method: Open the compass to a distance more than the perpendicular distance from the point *O*, to the line.

Example 3.26

Construct a perpendicular bisector of |AB| from a point C

Solution

Steps:

1. Draw a line.
2. Locate a point C anywhere above the and with your compass pin at point C, draw an arc to intersectline at the two points and label the points A and B.

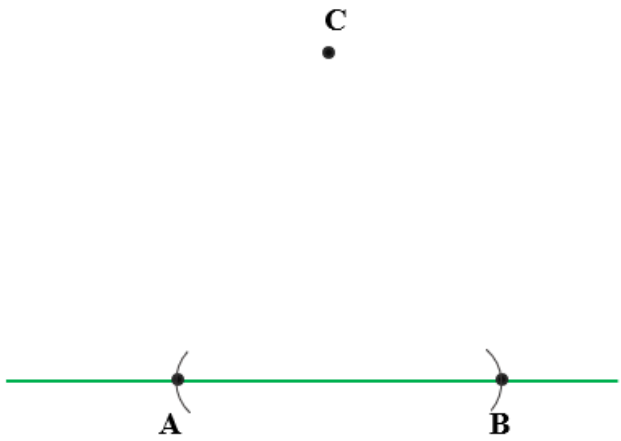


Figure 3.76: Construction of an arc from point C to |AB|

3. With the compass point at A and using the radius |AC|, draw an arc on the other side of the line.
4. With the compass point at B and using the same radius as used in step 3, draw an arc to intersect the one drawn in step 3.

5. Finally, join point C with a straight line to the intersection of the two arcs.

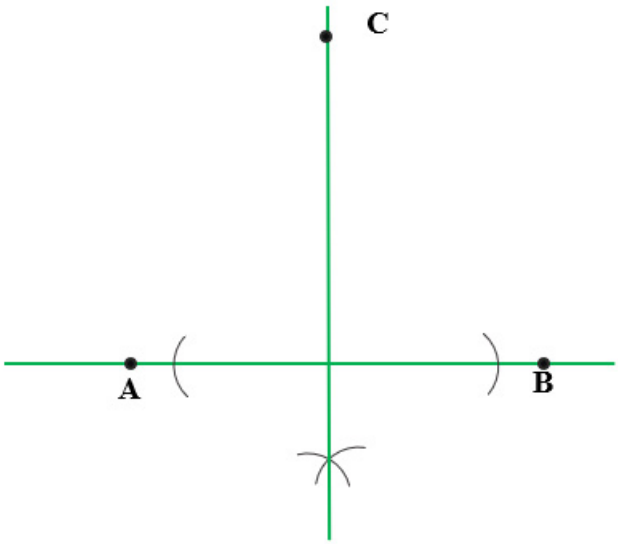


Figure 3.77: Construction of a perpendicular from point O to line AB

Locus Theorem 7

Locus of points equidistant from three fixed points.

Method: First, join the three points to make a triangle.

Construct the perpendicular bisectors of any two sides of the triangle.

The intersection of the two perpendicular bisectors is the locus. This locus is a single point.

NOTE: This locus is a single point called the circumcentre. This point can be used to construct a circumscribed circle.

Example 3.27

Three points X, Y and Z are positioned at different positions in a plane. Given that all these points are non-collinear (not on the same line), determine the precise centre of the three points in the plane.

Solution

Steps:

1. Draw three points that are far apart from one another and are not on the same line (non-collinear).

   Label the points X, Y and Z.

2. With your ruler, join the three points to form a triangle.

3. Construct the perpendicular bisectors of any two sides of the triangle or for even the three lines.
4. Locate the point of intersection of the perpendicular bisectors and label it O

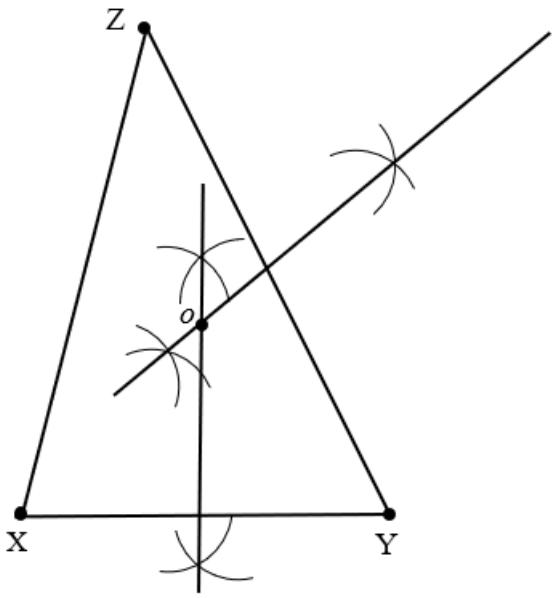


Figure 3.78: Perpendicular bisectors of sides XY and ZY of triangle XYZ

The point of intersection of the two perpendicular bisectors is the locus. This locus is a single point (O) and it is equidistant from the three points X, Y and Z

Now have a go at summarising the locus theorems in your own words for easy recall. Write them in your jotter and discuss them with a classmate or compare it with the ones below.

In summary, the Locus theorems can be considered under these five (5) rules.

Five Rules of Locus Theorems

Rule 1: Given a point, the locus of points is a circle.

Rule 2: Given two points, the locus of the points is a straight line midway between the two points.

Rule 3: Given a straight line, the locus of points is two parallel lines.

Rule 4: Given two parallel lines, the locus of points is a line midway between the two parallel lines.

Rule 5: Given two intersecting lines, the locus of points is a pair of lines that cut the intersecting lines in half.

Read through the example provided below carefully and perform the activity that follows, either with a classmate or on your own.

Example 3.28

Using a ruler and a pair of compasses only, construct

a. triangle XYZ with |XY| = 6cm, |XZ| = 8cm and $\angle YXZ = 75°$

b.

i. the locus d_1 of the points equidistant from |XY| and |XZ|

ii. the locus of d_2 points equidistant from X and Y

c. Locate

i. the point of intersection O of d_1 and d_2 inside the triangle

ii. with your compass pin at point O and a radius |OX| construct a circle.

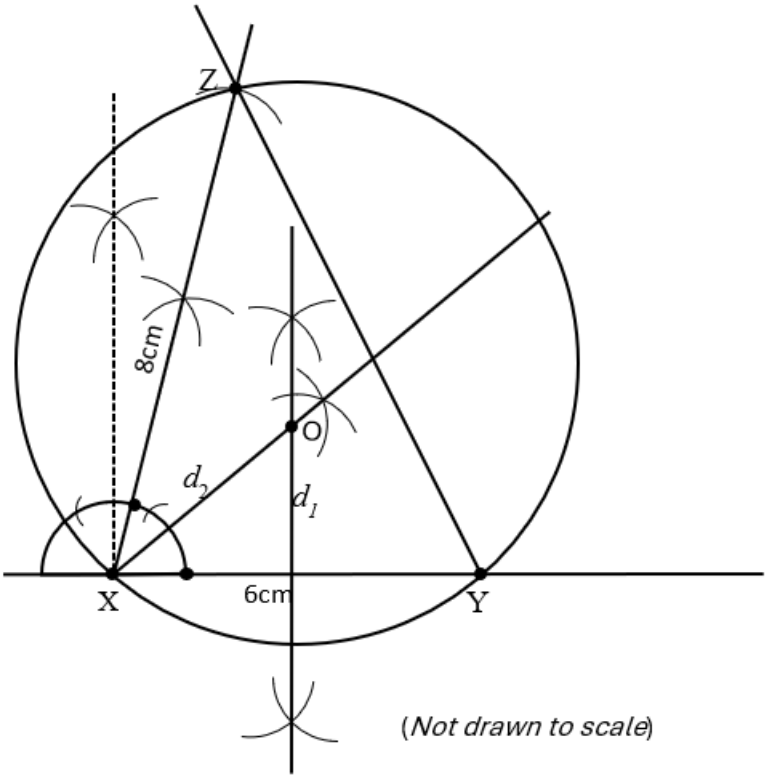


Figure 3.79: Solution to example 3.28

Example 3.29

Using a ruler and a pair of compasses only, construct:

i. A quadrilateral PQRS, |PQ| = 8cm, |PS| = 6cm, |QR| = 10cm, $\angle QPS = 60°$ and $\angle PSR = 135°$

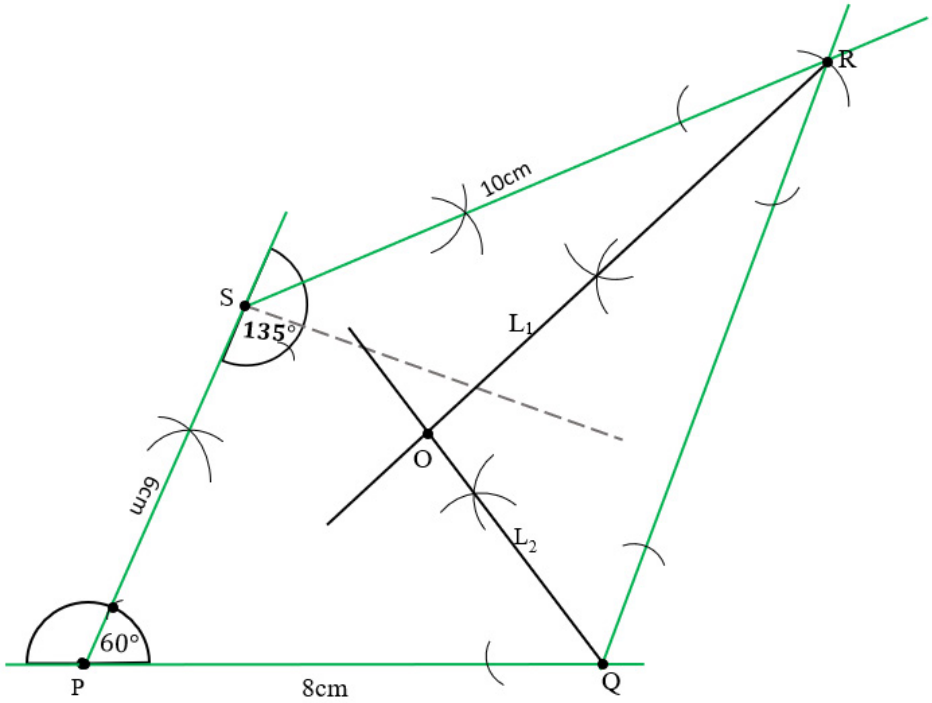
ii. The locus L_1, of points equidistant from |QR| and |RS|

iii. The locus L_2, from Q, perpendicular to L_1

Locate O, the point of intersection of L_1 and L_2.

Solution

Given |PQ| = 8cm, |PS| = 6cm,|RS| = 10cm, $\angle QPS = 60°$ and $\angle PSR = 135°$



(Figure not drawn to scale)

Figure 3.80: Solution to example 3.29

Applications of Construction

The idea of construction especially the concept of locus is very important and applicable in areas like road construction. It helps to check:

- **Safety**: Proper median and lane divisions to reduce accidents.
- **Efficiency**: Parallel drainage systems to prevent erosion and flooding.
- **Traffic Flow:** Roundabouts and bisector-aligned intersections minimise congestion in the urban areas.

Remember, construction can be applicable in many areas. Look for other areas where these concepts are applicable.

Activity 3.2 Application of Locus

Discuss the following scenarios with a classmate. For each of the scenarios, investigate which theorem(s) of locus construction will be applicable.

1. Mr Abotsi instructs students to move around a point, keeping a distance of 3 metres away at all times.

2. Your constituency's MP wants to site a market equally accessible to three separate towns by placing it at their central point.
3. Students are to construct a straight horizontal line through the exact centre of an A4 sheet without folding it.
4. A contractor plans to build gutters equally spaced on both sides of a 1500m road to control flooding.
5. At a 60° walkway intersection in a school, the headmistress wants lights positioned to brighten both paths equally.

Expected Responses

1. Locus of points equidistant from a single point.
2. Locus of points equidistant from three fixed points.
3. Locus of points equidistant from two parallel lines
4. Locus of points equidistant from a line.
5. Locus of points equidistant from two intersecting lines.

The activity below will help you practice what you have learnt.

Activity 3.3 Constructing a circle through a triangle using the idea of locus

Steps

1. Draw a line segment XY of length 6cm using your ruler.
2. With your compass pin at X, construct an angle 75°.
3. From point X, construct Z of length 8 cm.
4. Join points Y and Z to form triangle XYZ.
5. For locus L_1, Construct the angle bisector of ∠YXZ to find the locus of points equidistant from |XY| and |XZ|.
6. For Locus L_2, construct the perpendicular bisector of to find the locus of points equidistant from X and Y.
7. Locus L_3, with your compass pin at Y as the centre and a radius of 4.5 cm, draw a circle to find the locus of points 4.5cm from Y.
8. Identify the point of intersection of L_1 and L_2 inside the triangle and label it as d_1
9. Identify the point of intersection of L_1 and L_3 inside the triangle and label it d_2
10. Identify the point of intersection of L_3 and L_2 inside the triangle and label it as d_3

With your protractor, measure angle $d_2\ d_1\ d_3$ (the angle at d_1)

11. Compare your diagram and also the angle value of d_1 to that of your classmate(s).

12. Compare your diagram with the one in the figure below.

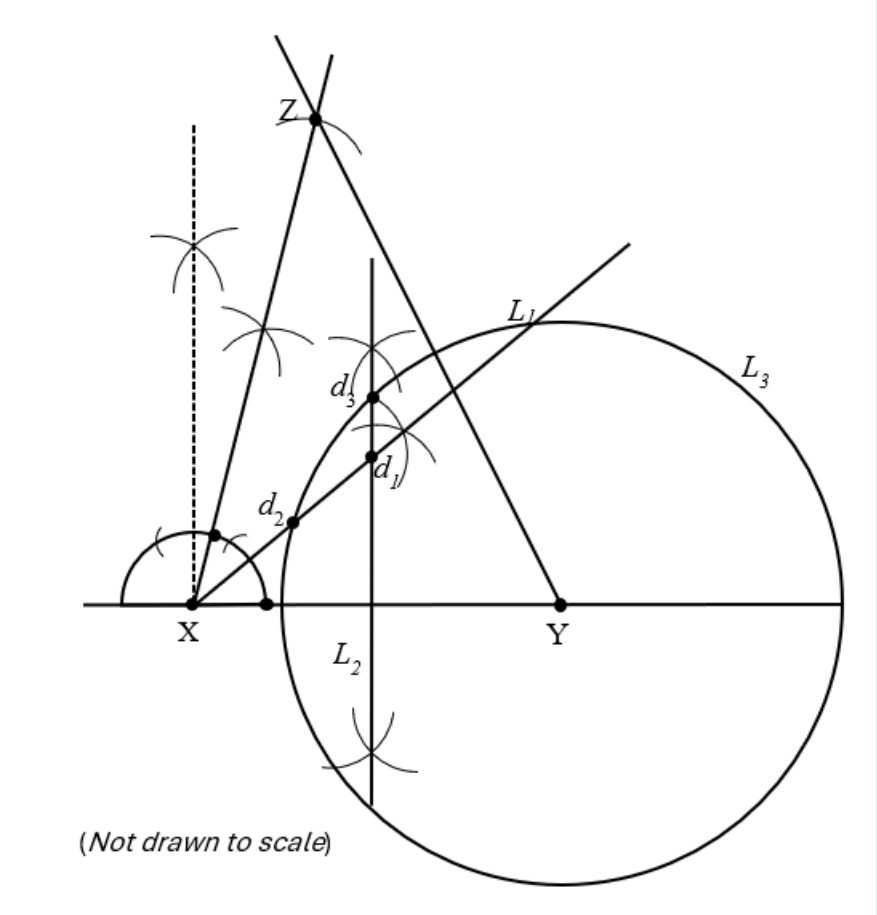


Figure 3.81: Solution to activity 3.3

EXTENDED READING

- Baffour Ba Series, Core Mathematics for Schools and Colleges. Pages 718-730
- *PowerPoint Series Core Mathematics for West Africa Senior High Schools and Technical institutions,* Page 397- 428, 543-558
- https://youtu.be/NqSjPJZ3O1E

REVIEW QUESTIONS

1. In the figure below, ABCD is a cyclic quadrilateral. |EF| is tangent to the circle at point C.

$\angle DCE = 46°$ and $\angle ABC = 98°$

a. Calculate:

i. $\angle DAC$

ii. $\angle ADC$

iii. $\angle ACE$

iv.

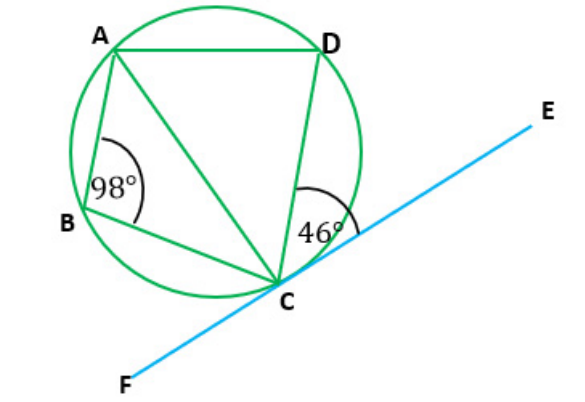


Figure 3.82: A cyclic quadrilateral

2.

a. Using a ruler and a pair of compasses only, construct triangle ABC such that: |AB| = 6cm, |AC| = 6cm and $\angle CAB = 60°$.

b. Construct a perpendicular from C to meet |AB|.

c. Construct |CD|= 6cm, parallel to |AB| through C.

d. With a ruler, join B to D.

e. What type of quadrilateral is formed?

3.

a. Using a ruler and a pair of compasses only, construct triangle ABC such that: $\angle ABC = 30°$ and $\angle CAB = 60°$ and |AB| = 10cm

b. Construct a perpendicular from the point C to meet the line AB at P.

c. Extend the line CP to meet point D such that BC = BD.

Join A to D and B to D.

d. What type of quadrilateral is ADBC?

4. In the figure below, |AD| and |CD| are tangents to a circle centre O.

If $\angle ABC = 70°$, calculate:

i. $\angle AEC$

ii. $\angle AOC$

iii. $\angle ADC$

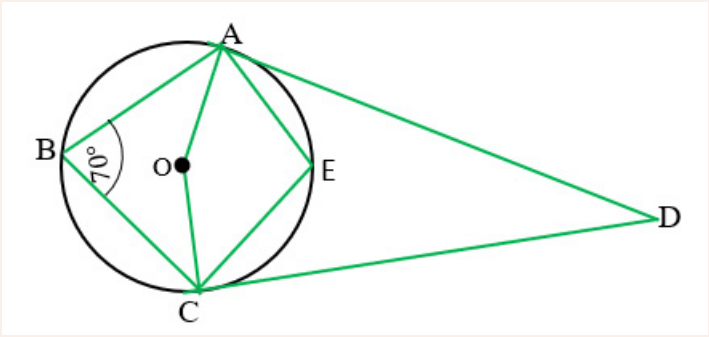


Figure 3.83: Diagram for question 4

5. In the diagram below, O is the centre of the circle, BD is a tangent to the circle, $\angle CBD = 26°$ and AO is parallel to BC

i. Find, $\angle AOB$

ii. Calculate the value of the reflex angle AOC

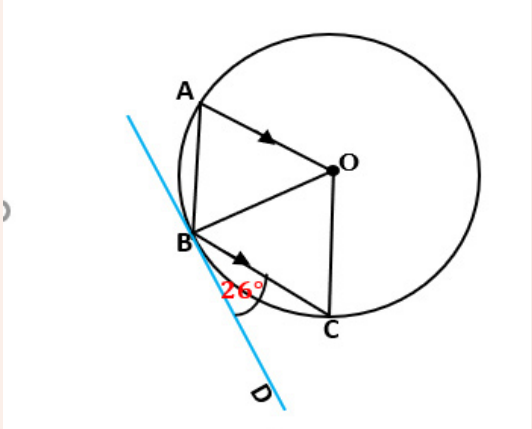


Figure 3.84: Diagram for question 5

SECTION

4

TRIGONOMETRY GRAPHS



GEOMETRY AROUND US

Measurement

Introduction

In this section, we will learn how to draw and interpret graphs of trigonometric functions such as sine, cosine and tangent. These graphs help us understand how angles and lengths relate in real-life and mathematical situations.

We have already studied important topics in Year 1 and Year 2 that will help us here. In Year 1, we worked with linear equations, functions and gradients of straight lines, which introduced us to plotting points and understanding graph shapes. In Year 2, we explored trigonometric ratios and vectors, which involved working with angles and direction. These are key ideas in trigonometric graphs.

Now, we will build on these skills to sketch trigonometric graphs and use them to solve problems. This will deepen our understanding of patterns, waves and movement in both mathematics and the real world.

KEY IDEAS

- Understanding Trigonometric Functions: Recognising sine, cosine and tangent as functions that relate angles to ratios in right-angled triangles.
- Graphing sine, cosine and tangent functions: Drawing and interpreting the basic shapes of their graphs over given intervals (e.g. 0°–360°).
- Amplitude, Period and Phase Shift: Exploring how changes in the graphs reflect variations in height (amplitude), horizontal length (period) and horizontal movement (phase shift).
- Real-life applications of trigonometric graphs: Connecting the wave-like behaviours to real-world contexts such as sound, tides and circular motion.
- Interpreting values from trigonometric graphs: Using the graphs to estimate function values, identify maximum/minimum points, intercepts and key features.
- Comparing trigonometric graphs: Understanding similarities and differences among sine, cosine and tangent graphs and how transformations affect their shapes.

TRIGONOMETRIC GRAPHS

A trigonometric graph provides a visual representation of how a trigonometric function behaves across a range of angle values. In these graphs:

- The horizontal axis (x-axis) displays the angle measurement, typically denoted as θ (theta).

- The vertical axis (y-axis) shows the corresponding output value of the trigonometric function at each angle.

In this section, we will focus solely on the graphs of trigonometric functions using angles measured in degrees, although they may also be expressed in radians.

Here are the key characteristics of the three primary trigonometric graphs.

1. Sine Graph ($y = \sin x$)

   Domain: All real numbers

   Range: $(-1, 1)$ (inclusive)

   Key Features: A smooth, continuous wave that starts at the origin (0,0) and oscillates between -1 and 1 and 1 with a period of 360° (2π radians)

2. Cosine Graph ($y = \cos x$)

   Domain: All real numbers

   Range: $(-1, 1)$ (inclusive)

   Key Features: Similar wave shape to sine but starts at (0,1), showing how it has phase-shifted by 90° from the sine function

3. Tangent Graph ($y = \tan x$)

   Domain: All real numbers except where $x = 90°, 270°$, etc.

   Range: All real numbers (unbounded)

The Graph of y = Sinθ

Take a close look at the scenario below

Scenario

In many parts of Ghana, especially in the villages, children enjoy swinging on ropes tied to strong tree branches. When you swing back and forth, your movement follows a smooth and regular pattern, you go up, come down, pass the middle, then go up again on the other side.

If someone were to measure your height from the ground as you swing over time, they would see your motion go up and down like a wave and that wave shape is exactly what the sine graph shows.

In this lesson, we will learn how to sketch this kind of graph and use it to solve problems that involve wave-like movements.

This shows a diagram of the sine graph for $-360° \leq \theta \leq 360°$

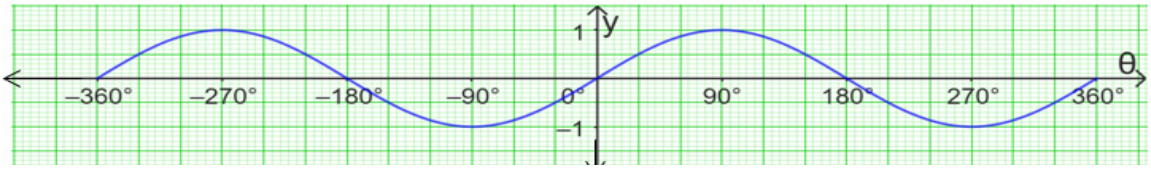


Figure 4.1: Graph of $y = \sin$ for

Properties of the graph of

1. The graph does not start at the origin, but it does pass through it.
2. The graph is continuous and repeats every 360.
3. The maximum value is +1.
4. The minimum value is −1.
5. The zeros/roots are where is an integer. The zeros = {..., -360, -180, 0, 180, 360, ...}
6. The domain is all real numbers. Domain = {θ:θ ∈ R}
7. The range is $-1 \leq \theta \leq 1$.
8. The sine and cosine graphs have the same shape which is a smooth wave. The only difference is that the **cosine** graph is just a shifted version of the sine graph.
9. The sine graph passes through the origin (0,0), while the cosine graph starts at (0,1). If you move the sine graph a little to the left, it becomes the cosine graph.

So, based on what we saw with the sine graph, we can now easily understand the cosine graph.

Now look at the shape of the cosine graph and its properties.

The graph of y = Cosθ

The diagram shows a graph of $y = \text{Cos}\theta$ for $-360° \leq \theta \leq 360°$

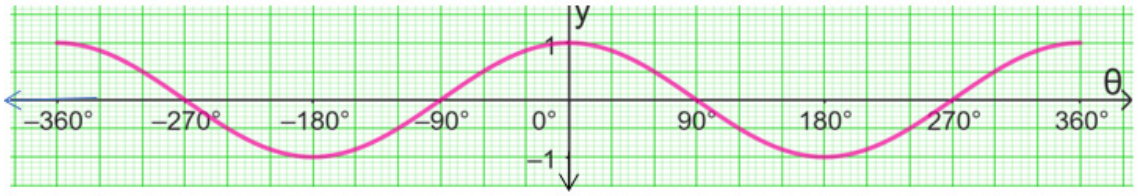


Figure 4.2: Graph of $y = \text{Cos}\theta$ *for* $-360° \leq \theta \leq 360°$

Properties of the graph of y = Cosθ

1. The graph does NOT pass through the origin, unlike the Sine and the Tangent graphs
2. The graph is continuous and repeats every 360°.
3. The maximum value is +1.
4. The minimum value is −1.
5. The zeros/roots are where is an integer. Zeros = {...,-270, -90, 90, 270, ...}
6. The domain is all real numbers. Domain = {θ:θ ∈ R}

   The range is $-1 \leq \theta \leq 1$.

Hopefully you can now understand the relationship between the sine and cosine graphs, as shown or described in ***Figure 4.3*** *below.*

The graphs of $y = \sin\theta$ **and** $y = \text{Cos}\theta$

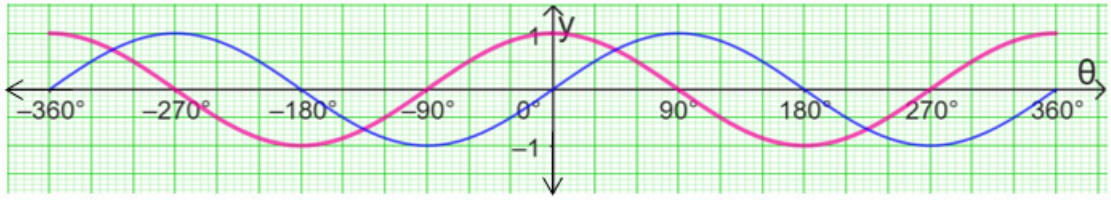


Figure 4.3: Graph of y = sinθ and y = Cosθ for $-360° \leq \theta \leq 360$

Relationship between the sine and the cosine graph:

1. The graphs have the same shape.
2. If you shift any of the graphs horizontally, either to the left or right by 90°, you will get the other graph. This means $Cos(90 - \theta) = Sin\theta$. Likewise, $Sin(90 - \theta) = \text{Cos}\theta$.

So, we also conclude that if $Cos\boldsymbol{x} = \text{Sin}y$, it means $x + \mathbf{y} = 90°$

The graph of y=tanθ

This graph has a unique shape compared with the sine and cosine graphs

Figure 4.4 shows a diagram of the tan graph for $-360° \leq \theta \leq 360°$.

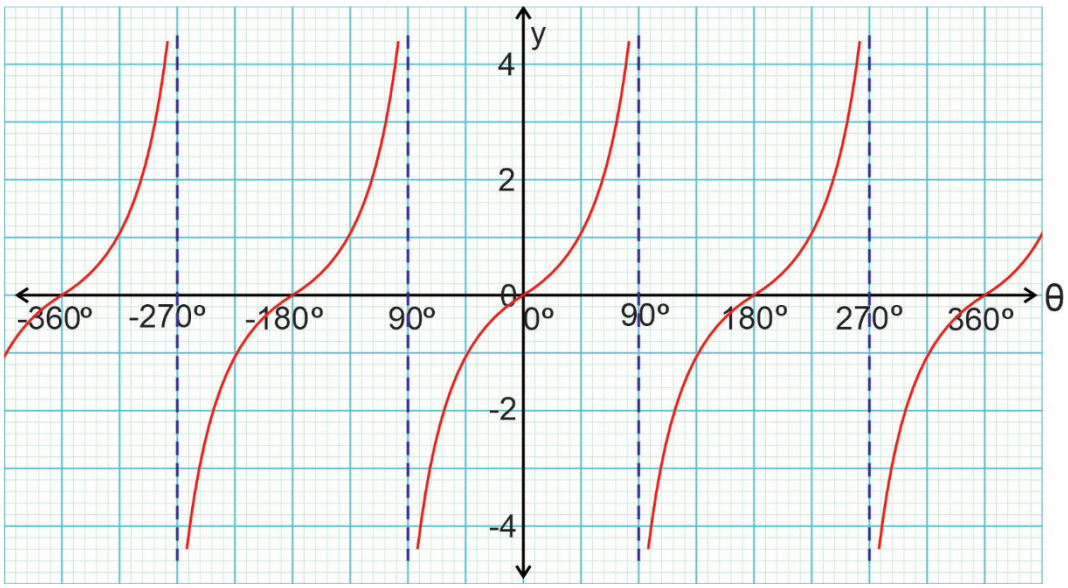


Figure 4.4: Graph of $y = tan\theta$ *for* $-360° \leq \theta \leq 360°$

Properties of the graph of y = tanθ

1. The graph passes through the origin.
2. The graph is NOT continuous and repeats every 180.

3. It has no maximum value (maximum value approaches ∞).
4. It has no minimum value (minimum value approaches $-\infty$).).
5. The zeros/roots are where is an integer. Zeros = {...,−360, −180, 0, 180, 360,..}
6. Domain = {θ:θ ∈ R, except θ = 90° +180°n where n is an integer}
7. The range is $-\infty \leq \theta \leq \infty$ or y∈R.
8. Vertical asymptotes at θ = 90° + 180°n where n is an integer. (Remember that an asymptote is a line that the graph will approach but never meet.)

Since the graphs of sine, cosin, and tangent have been explored, it is now time to learn how to draw trigonometric graphs. This lesson will focus only on the sine graph and the cosine graph

Drawing Trigonometric Graphs

Drawing a trigonometric graph follows the same procedure as drawing a quadratic graph

Example 4.1

a. Copy and complete the ***Table 4.1***.

Table 4.1: *Partially completed table for the relation* y = 1.5 sin*x*

x (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y = 1.5sinx	0				1.3		0			-1.5			0

b. Using a scale of 2 cm to 30° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 1.5\sin x$ for $0° \leq \theta \leq 360°$

c. Use your graph to find:

i. the maximum point

ii. the minimum point

Solution:

Table 4.2: Completed table for the relation $y = 1.5 \sin x$

x (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y = 1.5sinx	0	0.75	1.3	1.5	1.3	0.75	0	−0.75		−1.5	−1.3	−0.75	0

For example:

When $x = 30$

$= 1.5 \sin 30 = 0.75$

When $x = 60°$

$$y = 1.5 \sin 60 = 1.3$$

Now we plot the above table values

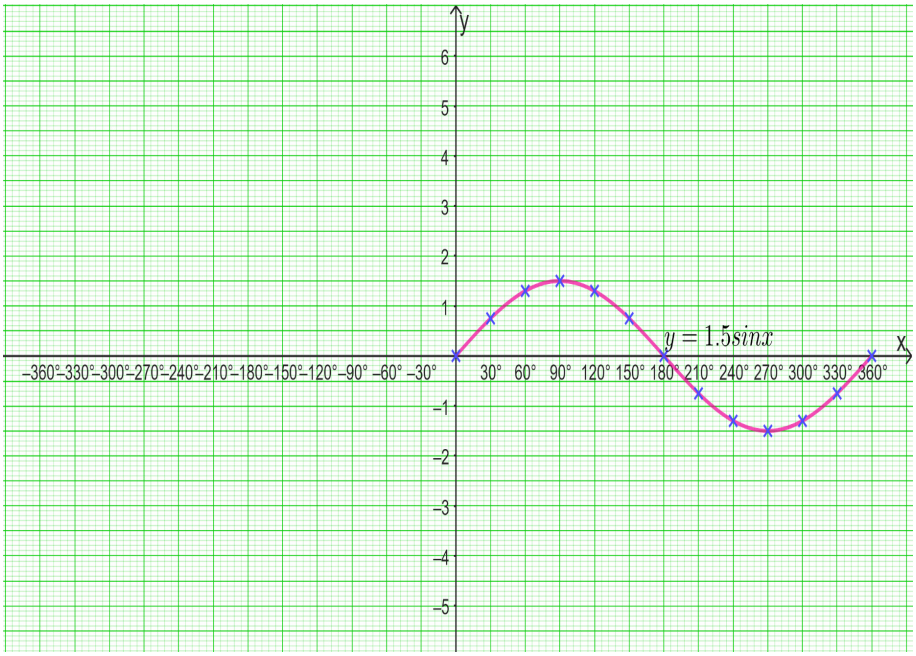


Figure 4.5: Graph of y = 1.5sin for $0° \leq x \leq 360$

c. From the graph below:

i. The maximum point is (90°, 1.5)

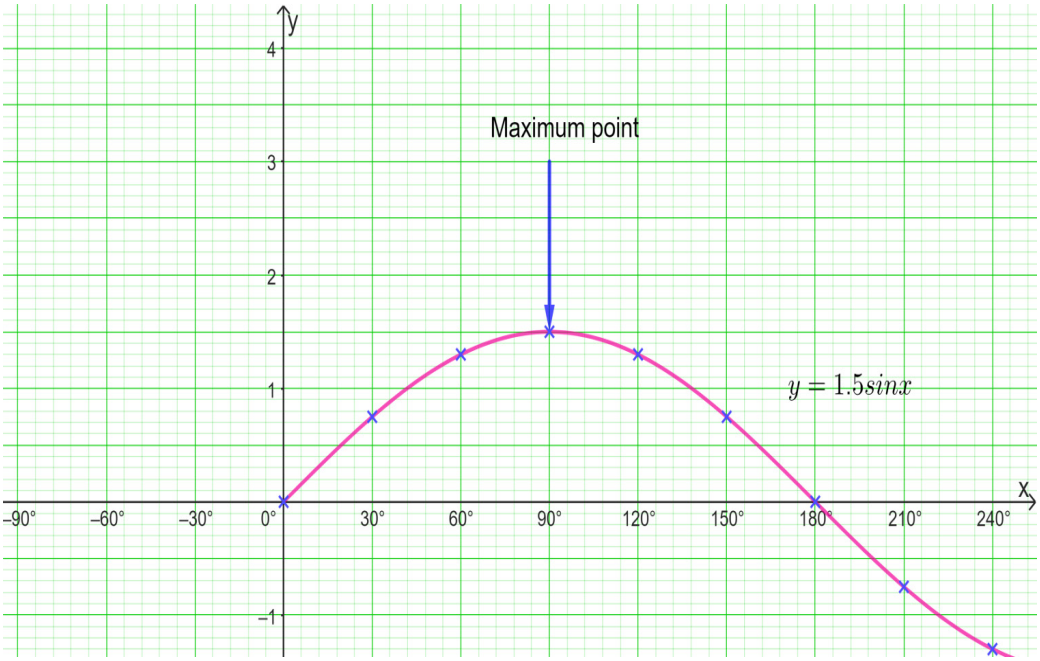


Figure 4.6: A zoom in on the maximum part of the graph of $y = 1.5\sin x$

iii. The minimum point is (270°, 1.5)

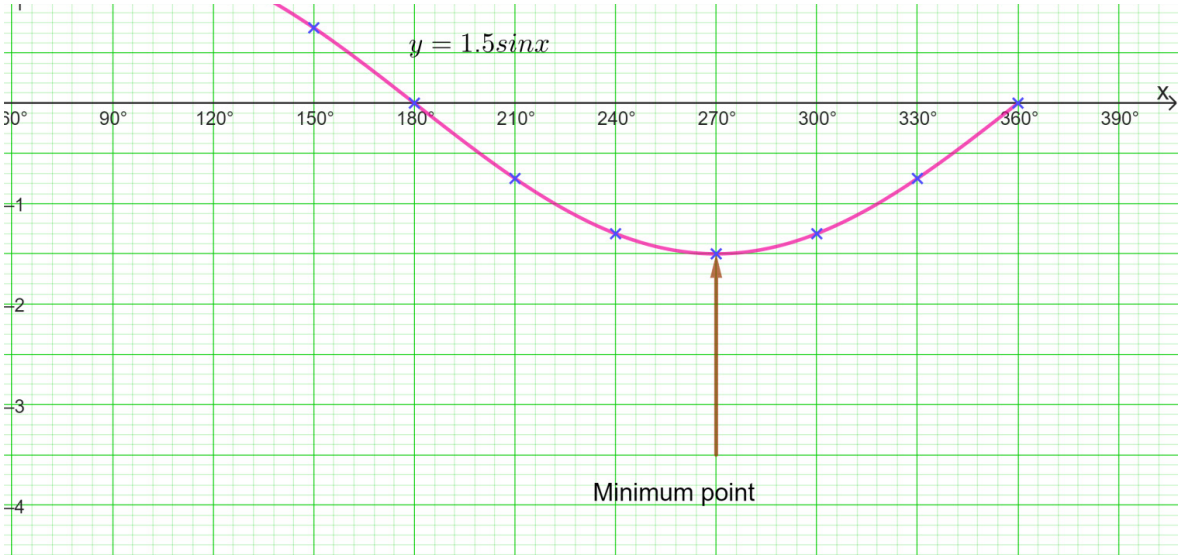


Figure 4.7: A zoom-in on the minimum part of the graph of y=1.5sin*x*

Example 4.2:

a. Copy and complete the ***Table 4.3***.

Table 4.3: *Partially completed table for the relation* $y = 2\cos x$

x (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y = 2cosx	2			0	-1		-2			0	1		2

b. Using a scale of 2 cm to 30° on the and 2 cm to 1 unit on the , draw the graph of $y = 2\cos x$ for $0° \leq \theta \leq 360°$

c. Use your graph to find the minimum point

Solution:

Table 4.4: Completed table for the relation $y = 2\cos x$

x (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y = 2cosx	2	1.73	1	0	-1	-1.73	-2	-1.73	-1	0	1	1.73	2

When $x = 30°$

$y = 2\cos(30) = 1.73$

When $x = 60°$

$y = 2\cos(60) = 1$

When $x = 150°$

$y = 2\cos(150) = -1.73$

When $x = 210°$

$y = 2\cos(210) = -1.73$

When $x = 240°$

$$y = 2\cos(240) = -1$$

When $x = 300°$

$$y = 2\cos(300) = 1$$

When $x = 330°$

$$y = 2\cos(330) = 1.73$$

Plot the points from the completed table

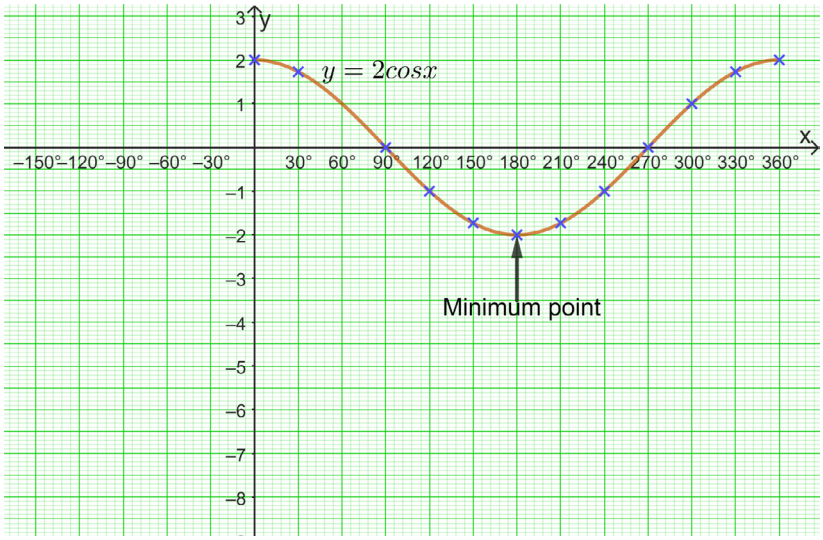


Figure 4.8: Graph of $y = 2\cos x$ for $0° \leq x \leq 360°$

c. From the graph, the minimum point is $(180°, -2)$

Example 4.3:

Copy and complete the table of the function

$y = 3\sin(x)\ \ 2\cos(2x)$ for the interval $0° \leq x \leq 180°$

Table 4.5: Partially completed table for the relation $y = 3\sin(x) - 2\cos(2x)$

x (degrees)	0°	30°	60°	90°	120°	150°	180°
y = 3sin(x) - 2cos(2x)		0.5					-2.0

Using a scale of 2 cm to 30° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 3\sin(x) - 2\cos(2x)$ for the interval $0° \leq x \leq 180°$

Solution:

Table 4.6: Completed table for the relation $y = 3\sin(x) - 2\cos(2x)$

x (degrees)	0°	30°	60°	90°	120°	150°	180°
y = 3sin(x) - 2cos(2x)	-2.0	0.5	3.60	5	3.60	0.5	-2.0

When $x = 0°$

$y = 3\sin(0) - 2\cos(2 \times 0) = -2$

When $x = 60°$

$y = 3\sin(60) - 2\cos(2 \times 60) = 3.60$

When $x = 90°$

$y = 3\sin(90) - 2\cos(2 \times) = 5$

When $x = 120°$

$y = 3\sin(120) - 2\cos(2 \times = 3.60$

When $x = 150°$

$y = 3\sin(150) - 2\cos(2 \times = 0.5$

Plotting the points on the graph

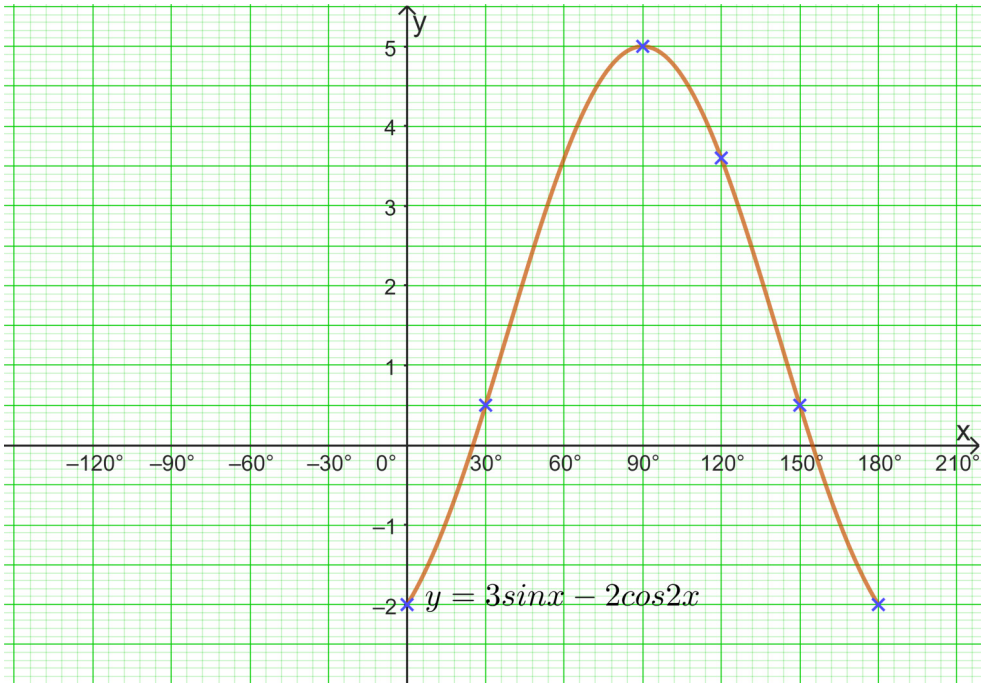


Figure 4.9: Graph of $y = 3\sin x - 2\cos 2x$ for $0° \leq \theta \leq 180°$

Example 4.4:

a. Copy and complete the table of values for the relation $y = 2 + 3\sin\theta$

Table 4.7: Partially completed table for $y = 2 + 3\sin\theta$

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$y = 2 + 3sin\theta$	2	3.5		3.5		0.5		0.5	2

b. Graph Plotting Instructions

Using: Scale: 2 cm = 30° on the θ-axis, 2 cm = 1 unit on the y-axis for the interval . Draw the graph of $y = 2 + 3\sin\theta$

c. Use your graph to find:

i. The minimum value of y.

ii. The coordinates where the minimum occurs.

iii. The maximum point

iv. The values of θ where $y = 2$, correct to 1 decimal place.

Solution:

Table 4.8: Completed table for the relation

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$y = 2 + 3sin\theta$	2	3.5	5	3.5	2	0.5	−1	0.5	2

When $x = 90°$

$y = 2 + 3\sin 90° = 5$

When $x = 180°$

$y = 2+3\sin 180 = 2$

When $x = 270°$

$y = 2 + 3\sin 270° = -1$

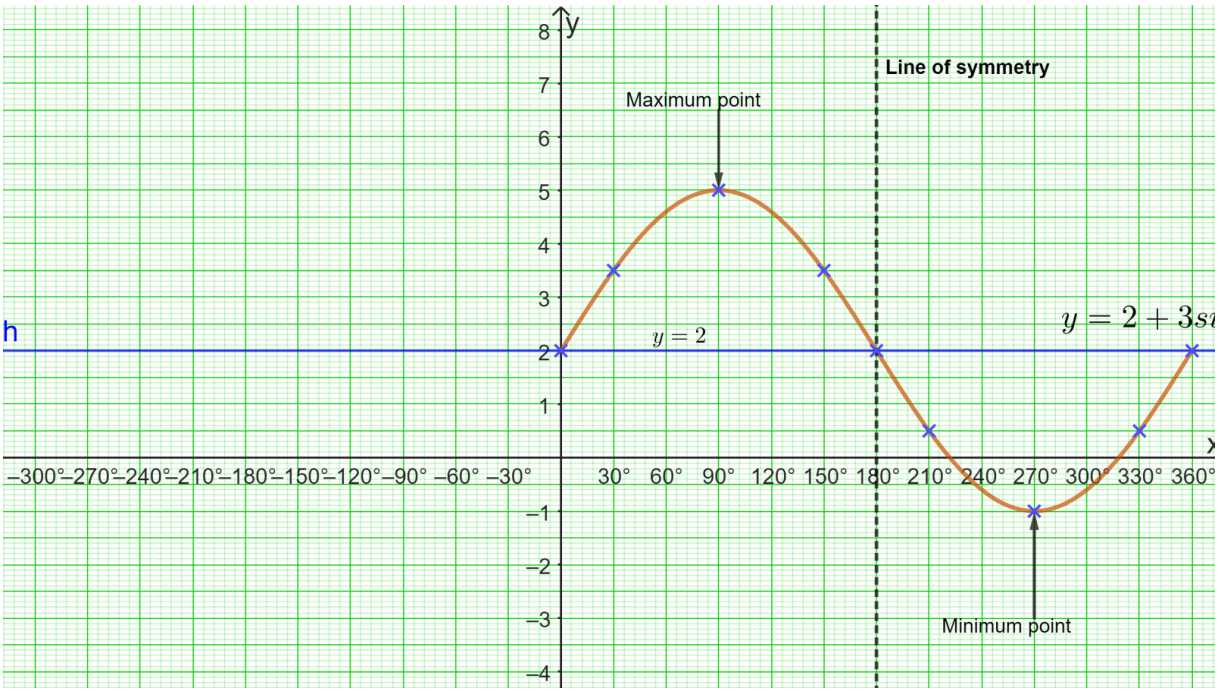


Figure 4.10: Graph of $y = 2 + 3\sin\theta$ for $0° \leq \theta \leq 360°$

c. From the graph above, the

i. Minimum value of y is −1

ii. Minimum point (270°, −1).

iii. Maximum point (90°, 5).

iv. $\theta = 0°, 180°, 360°$, From the graph, these are the points where the curve intersects the line $y = 2$ (Indicated by the blue horizontal line on the graph).

Determining Equations Using Trigonometric Graphs

Use trigonometric graphs to determine equations and solve related problems

Trigonometric graphs are useful tools for identifying and understanding the equations of periodic functions such as sine, cosine and tangent. By observing key features of these graphs — including amplitude, period, phase shift and vertical shift — we can determine their corresponding equations. Once obtained, these equations can then be applied to solve a variety of real-life and mathematical problems involving periodic behaviour.

Finding the solution set from the trigonometry graph

The following examples will guide you in learning how to solve trigonometric equations using graphs.

Example 4.5:

a. Copy and complete the table below for the relation $y = \sin x - \cos x$ for values of x from 0° to 180° at intervals of 30°.

Table 4.9: Partially completed table for the relation $y = \sin x - \cos x$.

x	0°	30°	60°	90°	120°	150°	180°
$y = \sin x - \cos x$	−1.00	−0.37	0.37			1.37	

b. Using a scale of 2 cm to on the x-axis and 2 cm to 1 unit on the y-axis draw a graph of y.

c. Use your graph to estimate, in the given interval, the truth set of the equations.

i. $\sin x - \cos x = 0$

ii. $\sin x - \cos x = 1$

Solution

Table 4.10: Completed table of $y = \sin x - \cos x$.

x	0°	30°	60°	90°	120°	150°	180°
$y = \sin x - \cos x$	−1.00	−0.37	0.37	1.00	1.37	1.37	1.00

When $x = 90°$

$y = \sin 90° - \cos 90° = 1$

When $x = 120°$

$y = \sin 120° - \cos 120° = 1.37$

When $x = 180°$

$y = \sin 180° - \cos 180° = 1$

b. Graph of $y = \sin x - \cos x$

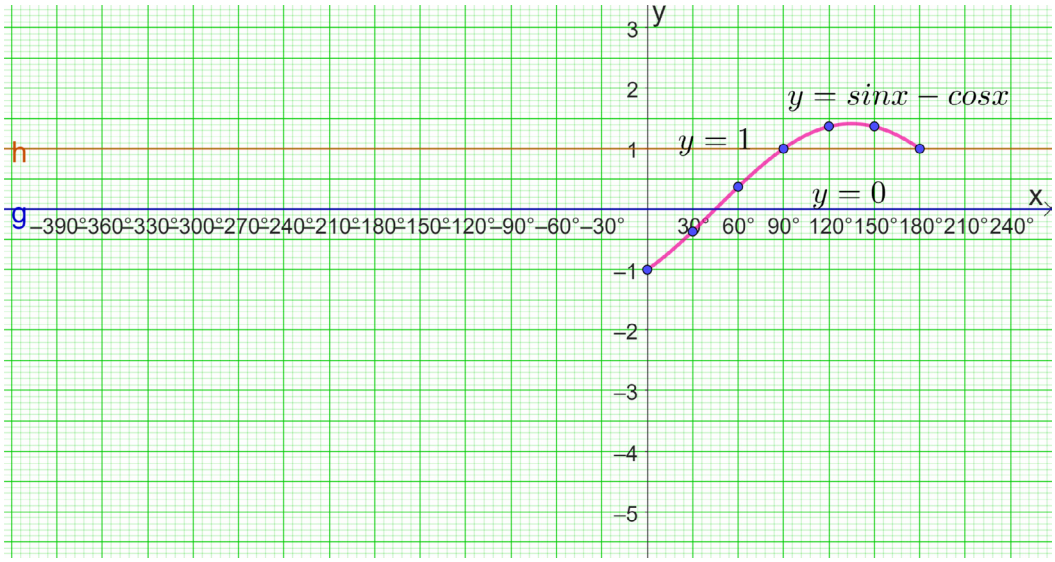


Figure 4.11: *Graph of the relation* $y = \sin x - \cos x$

c.

i. From the graph, the curve intersects the line $y = 0$ at $x \approx 45°$

Hence, the solution set of $\sin x - \cos x = 0$ is given by: $\{x : x \approx 45°\}$.

ii. From the graph, the curve, and the line y = 1 intersect at $x = 90°$ and $180°$

Hence, the solution set of $\sin x - \cos x = 1$ is given by: $\{x : x = 90° \text{ and } 180°\}$.

Example 4.6:

a. Copy and complete the table below for the relation $y = 3 \cos x$ for values of x from 0° to 360° at intervals of 45°.

Table 4.11: Partially completed table for the relation $y = 3 \cos x$

x	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = 3 \cos x$			0		−3		0		

b. Using a scale of 2 cm to on the x-axis and 2 cm to 1 unit on the y-axis, draw a graph of y.

c. Use your graph to estimate, in the given interval, the truth set of the equations

i. $2 + 3\cos x = 0$

ii. the minimum point of $y = 3\cos x$.

iii. equation of the line of symmetry.

Solution:

Table 4.12: Completed table for the relation

x	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = 3\cos x$	3	2.12	0	−2.12	−3	−2.12	0	2.12	3

When $x = 0°$

$y = 3\cos(0) = 3$

When $x = 45°$

$y = 3\cos(45) = 2.12$

When $x = 135°$

$y = 3\cos(135) = -2.12$

When $x = 225°$

$y = 3\cos(225) = -2.12$

When $x = 315°$

$y = 3\cos(315) = 2.12$

When $x = 360°$

$y = 3\cos(360) = 3$

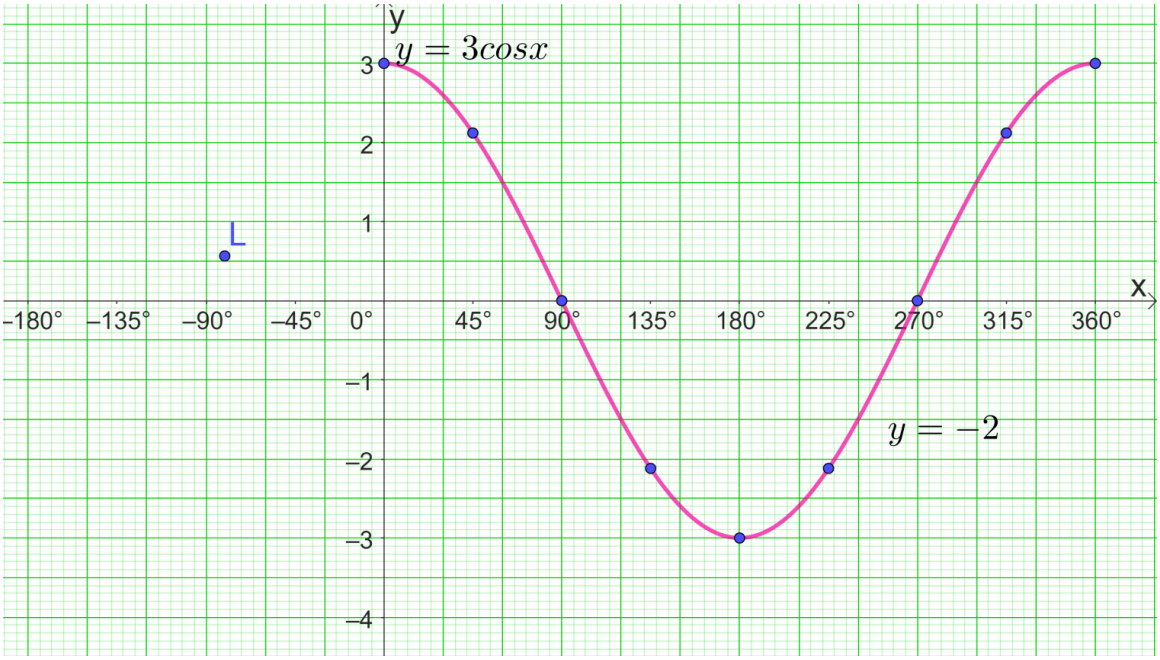


Figure 4.12: Graph of $y = 3\cos x$

c.

i. Comparing the two equations $y = 3\cos x$ and $2 + 3\cos x = 0$, we obtained $y = -2$.

Draw the line $y = -2$ on the graph and find its truth set. That is the values of x for which the line and the curve intersect.

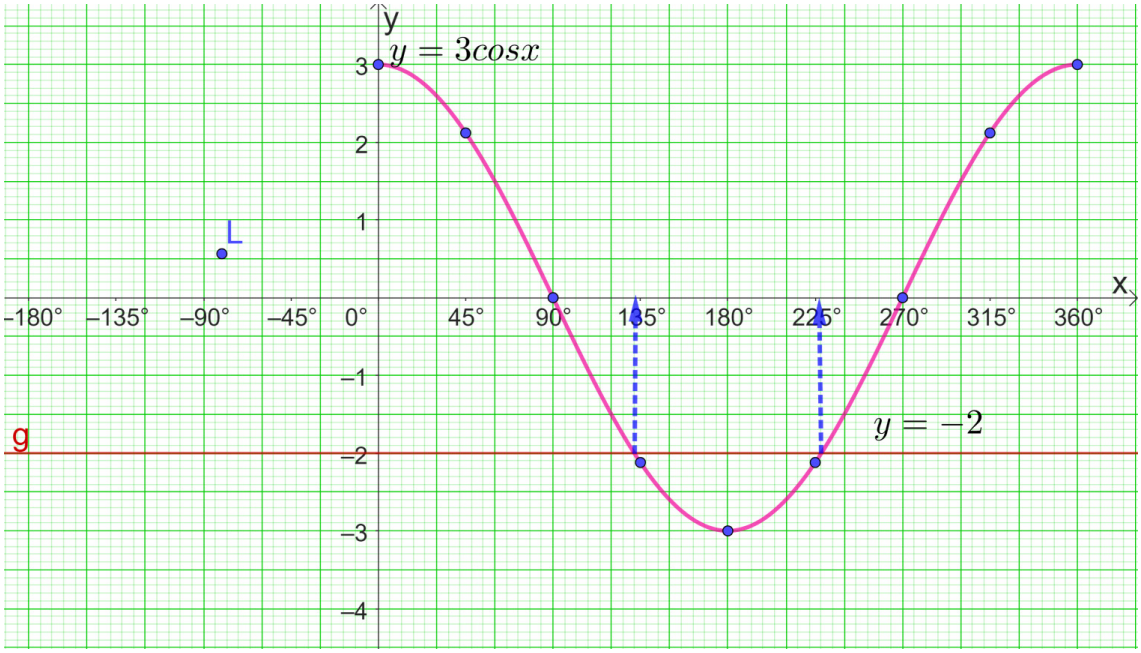


Figure 4.13: Graph of and line $y = 3\cos x$ and line $y = -2$

From the graph, the line $y = -2$ intersects the curve at $x \approx 132.02°$ and $x \approx 229.5°$ as shown in the graph above.

ii. The minimum point is the turning point of the graph.

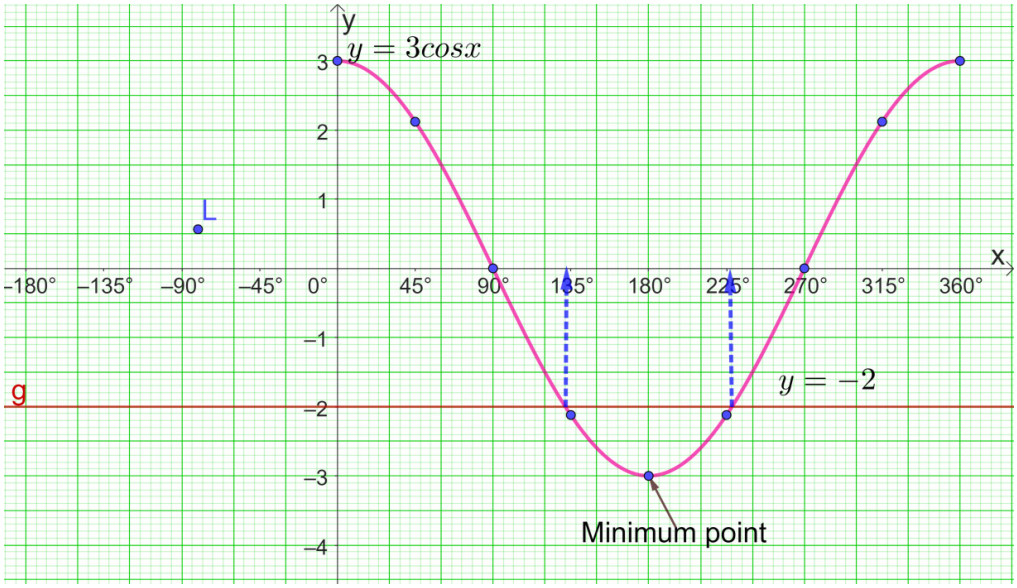


Figure 4.14: Graph of $y = 3\cos x$ showing the minimum point

From the graph, the minimum point as indicated is $(180°, -3)$.

The y-coordinate of this point is called the minimum value, which is −3. The minimum value occurs when $x = 180°$.

iii. The line of symmetry divides the curve within the range given into equal halves.

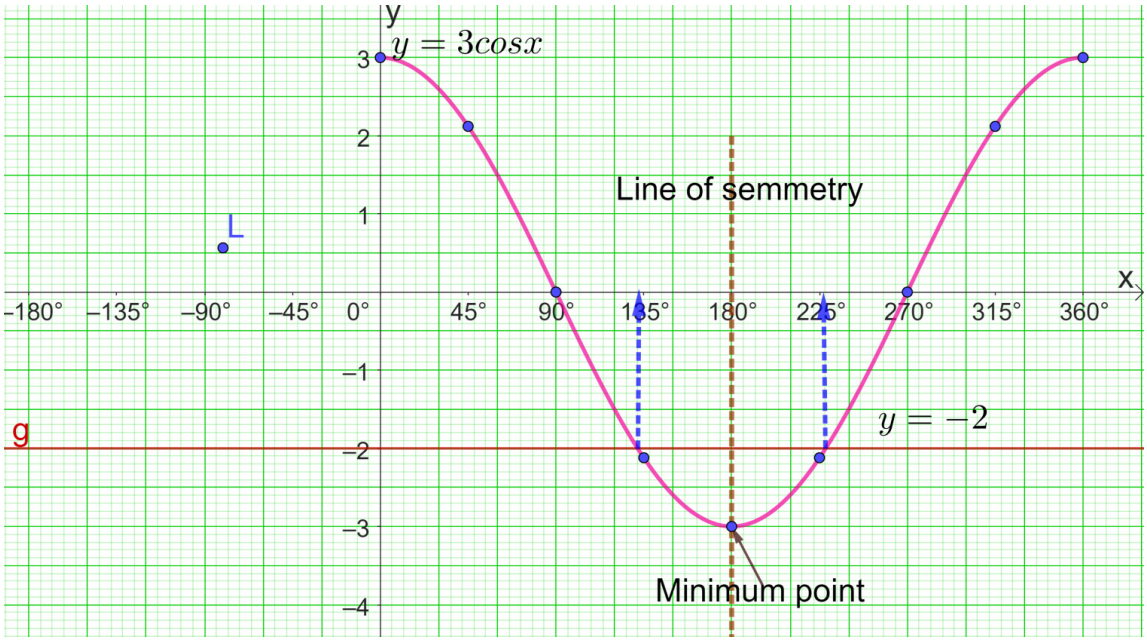


Figure 4.15: Graph of $y = 3\cos x$ showing the equation of the line of symmetry

From the graph, the line of symmetry is indicated by the broken line. It intersects the x-axis at 180°. Hence, the line of symmetry is x = 180°.

Activity 4.1 Comparing Sine and Cosine graphs

Materials Needed

- Graph paper
- Pencil and ruler
- Calculator
- Coloured pens or markers (e.g. blue and red)
- Table of values below

Instructions

Step 1: Complete the Table

Table 4.13

θ (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = sin\theta$													
$y = cos\theta$													

Step 2: Draw the Graphs

- Draw a set of axes on your graph paper or in your graph book.
- Let the horizontal axis (x-axis) represent angles from 0° to 360°, using a scale like:1 cm = 30°
- Let the vertical axis (y-axis) represent values from −1 to 1, using: 1 cm = 0.5 unit

Plot:

- $y = sin\theta$ using a blue pen
- $y = cos\theta$ using a red **pen**

Step 3: Reflect and Respond

On the same page (or at the back of your graph), answer these questions

1. Which graph starts at zero? Which one starts at one?
2. Do both graphs have the same height (amplitude)?
3. Do the graphs repeat after 360°?
4. How are the graphs similar?
5. How are they different?
6. If you were to draw a vertical line to divide each graph into two mirror parts, where would you draw it? Write the equation of the line of symmetry.

Learning Outcome: You will understand how sine and cosine graphs behave, and how they can be used to model real-life patterns that repeat.

EXTENDED READING

- Akrong Series: Core Mathematics for Senior High Schools New International Edition (Pages 506 – 514)
- Baffour – Ba Series: Core Maths for Schools and Colleges, (Pages 772 – 774)

REVIEW QUESTIONS

1.

a. Copy and complete below for the relation $y = 2\sin x$.

Table 4.14: Partially completed table for $y = 2\sin x$.

x (degrees)	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = 2\sin x$	0		2		0		-2		0

b. Using a scale of 2 cm to 45° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 2\sin x$ for $0° \leq x \leq 360°$.

c. Use your graph to find

i. The maximum point.

ii. The minimum point.

2.

a. Copy and complete the table below for the relation $y = 3\cos x + 2\sin x$ for values of x from 0° to 180° at intervals of 30°

Table 4.15: Partially completed table of the relation $y = 3\cos x + 2\sin x$

x	0°	30°	60°	90°	120°	150°	180°
$y = 3\cos x + 2\sin x$	3.00		3.23				-3.00

b. Using a scale of 2 cm to on the x-axis and 2 cm to 1 unit on the y-axis draw a graph of $y = 3\cos x + 2\sin x$.

c. Use your graph to estimate, in the given interval, the truth set of the equations. Correct your answer to one decimal place.

i. $6\cos x + 4\sin x = 1$

ii. $3\cos x + 2\sin x = -1$

3. The table below shows values of $y = 2 + \sin\theta$ for $0° \leq \theta \leq 360°$ for at intervals of 30°

Table 4.16

θ (degrees)	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	2.00	2.50	2.87	3.00	2.87	2.50	2.00	1.50	1.13	1.00	1.13	1.50	2.00

a. Using a scale of 2 cm = 30° on the x-axis and 2 cm = 1 unit on the y-axis, plot the graph of $y = 2 + \sin\theta$ for $0° \leq \theta \leq 360°$

b. From your graph, determine:

i. The maximum point.

ii. The minimum point.

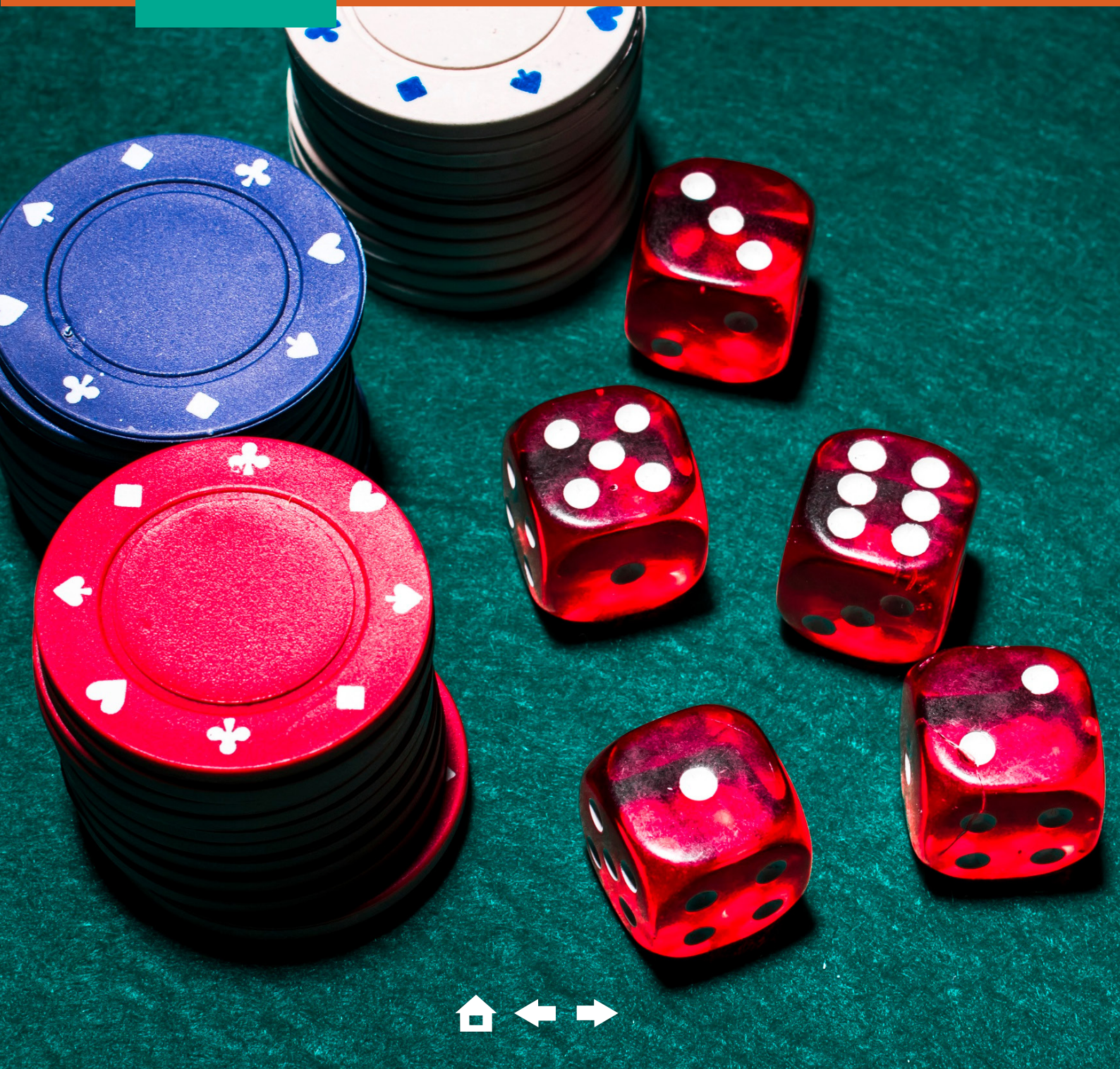
c. Using your graph, find the truth set for each equation below, correct to one decimal place:

i. $y = 2.5$

ii. $y = 1.5$

SECTION

5 DATA HANDLING AND PROBABILITY



MAKING SENSE OF AND USING DATA

Statistical Reasoning And Its Application In Real Life

Introduction

In Year One Section 8, we learnt about the types of data and the methods for collecting data. We went further to learn how to organise and present data to make it more meaningful. Finally, we learnt about the sample space for simple and compound events.

In Year Two Section 8, we studied the different types of data, the measures of central tendency and dispersion and more ways of data presentation, including drawing a cumulative frequency curve. In all of these activities, we were interested in studying the effect of one variable, say, age, mass or height, on given data.

In this section, we will be interested in the relationship between two variables for a given set of data. That is, we will study bivariate data, rather than univariate data. As in the first two years, we will work on a project to collect and analyse data collected from our communities.

KEY IDEAS

- Distinguishing between univariate and bivariate data.
- Investigating the relationship between two variables.
- Drawing a scatter diagram to determine the relationship between two variables.
- Using the scatter diagram to determine the relationship between the variables.
- Applying the skills we have learnt to real-life situations, for example, solving problems in our communities.

REVIEW OF DATA HANDLING AND ANALYSIS

We have learnt that data handling means collecting, organising, and interpreting information (data) to make it easier to understand. Data can be qualitative (descriptive, like colours or names) or quantitative (numerical, like test scores or ages).

To make sense of raw data, we usually:

1. Organise it in a frequency table to show how often each score appears.
2. Represent it using graphs such as bar charts or histograms.

3. Summarise it with measures of central tendency (mean, median, mode) and spread (range, standard deviation).
    - Mean is the average value.
    - Median is the middle value when data is ordered.
    - Mode is the most frequent value.
    - Range is the difference between the highest and lowest scores.
    - Standard deviation tells us how spread out the scores are around the mean (a small standard deviation = data is close together, large = more spread out).

By doing this, we can judge whether the average (mean) truly reflects the data.

We have also learnt that probability is the branch of mathematics that deals with the likelihood of an event happening. It is measured as a number between 0 and 1.

- 0 means the event is impossible.
- 1 means the event is certain.
- Values in between represent different levels of chance.

To calculate probability:

$$\text{P(Event)} = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ possible\ outcomes}$$

For example, if we have a bag with 4 red, 3 blue and 5 green bottle tops (total = 12), the probability of drawing a red one is: $\frac{4}{12} = \frac{1}{3}$

When **two events** are considered, we need to decide if they are:

- **Independent events** (the outcome of the first does not affect the second), for example, tossing a coin twice.
- **Dependent events** (the outcome of the first affects the second), for example, drawing one bottle top without replacing it, then drawing another.

In this activity, we will calculate probabilities for one draw and for two draws, then decide whether the events are dependent or independent.

Activity 5.1 Data Handling and Analysis

Instructions

The table below shows the scores of 15 students in a mathematics test out of 20.

Scores

12, 15, 14, 13, 16, 12, 18, 15, 17, 13, 14, 20, 12, 16, 19

Tasks

1. Identify the type of data represented.
2. Construct a frequency table for the data.
3. Draw a bar chart to represent the data.
4. Calculate the:
   - Mean
   - Median
   - Mode
5. Calculate the range and standard deviation of the scores.
6. Briefly explain whether the mean is a reliable measure of central tendency in this case.

Activity 5.2 Probability Concepts

Instructions

A bag contains **4 red**, **3 blue** and **5 green** bottle tops. One bottle top is drawn at random.

Tasks

1. What is the probability that the bottle top drawn is:
   a. Red?
   b. Blue?
   c. Not green?
2. If one bottle top is drawn and not replaced, and a second one is drawn:
   a. What is the probability that both bottle tops are red?
   b. What is the probability that the first is red and the second is blue?
3. State whether the events are dependent or independent and explain your reasoning.

ESTABLISHING RELATIONSHIPS BETWEEN TWO VARIABLES

In our everyday life, we observe that certain quantities are related. For example:

- The amount of rainfall and crop yield
- A student's study time and exam scores

- The speed of a car and the distance it covers

Here, we will learn how to identify, analyse and represent relationships between two variables using tables and graphs.

Identifying Variables in a Given Data Set

A variable is any characteristic, number or quantity that can be measured or counted and can vary from one individual or observation to another. A variable is any quantity that can change or take different values. When doing a study or investigation, we often look at how one variable affects another.

Types of Variables

1. **Independent Variable** (**Explanatory Variable**): This is the variable that is deliberately changed or controlled in an experiment. The independent variable is the cause. **Think of it as: "*What I change on purpose*."**

   Example: In an experiment to study plant growth, the amount of water given to the plant is the independent variable.

2. **Dependent Variable** (**Response Variable**): This is the outcome that is measured based on changes in the independent variable.

   Example: The height of the plant (dependent variable) depends on the amount of water (independent variable).

3. **Control Variables:** These are kept constant to ensure a fair test.

   Example: Soil type, sunlight and pot size in the plant growth experiment.

Use these scenarios to further your understanding of the types of variables

Scenario 1: Investigating the Effect of Study Time on Mathematics Exam Scores

Mrs. Antwi, a teacher in a senior high school in Accra, wants to know if the amount of time learners spend studying mathematics affects their exam scores.

- **Independent Variable**: Time spent studying mathematics (e.g., 2 hours, 4 hours, 6 hours)
- **Dependent Variable**: Mathematics exam score (e.g., 65%, 75%, 90%)

The teacher changes the amount of study time (independent variable), and then measures the exam score (dependent variable).

Scenario 2: Studying the Impact of Feeding on Learners' Concentration in Class

At a boarding school in Kumasi, a research group investigates whether learners who eat breakfast are more attentive in morning lessons.

- **Independent Variable**: Whether or not the learner ate breakfast
- **Dependent Variable**: Level of concentration in class (measured by participation, responses, or test)

In conclusion

a. The independent variable is like the input or cause.

b. The dependent variable is the output or result.

c. We study how changing one variable (independent) affects another (dependent).

d. Always ask yourself:

- "What is being changed?" → Independent
- "What is being measured?" → Dependent

Activity 5.3 Identify the Variables

Read each of the scenarios below and identify the following.

- The independent variable
- The dependent variable

Scenarios

1. A health researcher is investigating whether the number of hours teenagers sleep affects their academic performance.
2. A PE teacher observes whether the number of laps run during practice affects a learner's heart rate.
3. School cafeteria staff are studying whether the number of learners eating from the canteen affects the amount of food wastage.
4. A scientist studies how temperature affects the rate of a chemical reaction.
5. A teacher investigates whether more practice questions lead to higher test scores.

Follow-Up Questions

- What is being changed in each scenario?
- What is being measured?

Univariate and Bivariate Data

Univariate Data

Univariate data involves the study of only one variable at a time. The prefix "uni-" means one.

We collect univariate data when we are interested in just one characteristic, such as:

- Number of siblings
- Ages of learners in your class
- Exam scores for a subject
- Weight of babies at birth
- Salary of teachers.

Common ways to analyse univariate data:

- Frequency tables
- Bar graphs or histograms
- Pie charts
- Measures of central tendency (mean, median, mode)
- Range

The table below shows an example of univariate data. It shows the test scores of 10 students.

Table 5.1: A teacher in Cape Coast records the mathematics scores of 10 students in a class test.

Student	A	B	C	D	E	F	G	H	I	J
Score (%)	55	70	65	80	75	60	85	90	55	65

This is univariate data because we are looking at only one variable: the mathematics scores.

Bivariate Data

Bivariate data involves two related variables. The prefix "bi-" means two.

We collect bivariate data when we want to study the relationship between two variables, for example, how study time affects exam scores, or how age relates to height.

Common ways to analyse bivariate data

- Scatter plots
- Correlation (to measure strength of relationship)
- Line of best fit

- Two-variable frequency tables

A teacher at a senior high school in Ho wants to find out whether the number of hours students spend studying affects their math exam scores.

Table 5.2: Study Time vs. Math Scores

Student	Study Time (hours)	Math Score (%)
Kekeli	1	45
Eyram	2	50
Etse	3	55
Mawuli	4	65
Etornam	5	70

This is bivariate data because two variables are involved:

- Study time (independent variable)
- Math score (dependent variable)

Difference between Univariate and Bivariate data

Table 5.3: Distinguishing features of univariate and bivariate data

Univariate data	Bivariate data
Involves a single variable	Involves two variables
Does not deal with cause and relationship	Deals with cause and relationship
The purpose is to describe	The purpose is to explain
Does not have any dependent variable	Contains only one dependent variable

Activity 5.4 Class Survey and Univariate Analysis

Instructions

1. Conduct a classroom survey by asking each learner their favourite fruit (e.g., mango, banana, orange, pineapple, apple).
2. Record the responses in a tally and frequency table.
3. Represent the data using a bar graph or a pie chart.
4. Calculate:
   - The mode (most preferred fruit)
   - The total number of learners surveyed
   - Is this univariate or bivariate data? Why?

Example 5.1

The ages of 6 SHS students are: 16, 17, 18, 17, 16, 19.

Find the:

a. Mode

b. Median

c. Mean

Solution

a. **Mode** = 16 and 17 (both appear twice)

b. **Arrange**: 16, 16, 17, 17, 18, 19

Median $= \frac{17+17}{2} = 17$

c. **Mean** $(\bar{x}) = \frac{16+17+18+17+16+19}{6} = 17.17$ years

Plotting and Interpreting Scatter Graphs

A **scatter diagram** (**graph** or **plot**) is a type of graph used to display the relationship between two variables. It consists of a collection of points, where each point represents a pair of values, one from each variable.

- The horizontal axis (x-axis) represents one variable.
- The vertical axis (y-axis) represents the other variable.

Each plotted point shows the value of both variables for a single observation. By looking at how the points are arranged, we can see if there is a pattern or relationship (called a **correlation**) between the two variables.

The graph below shows a scatter diagram that represents a distribution centre. It shows that workers who pick more lines (items or orders) tend to work longer hours.

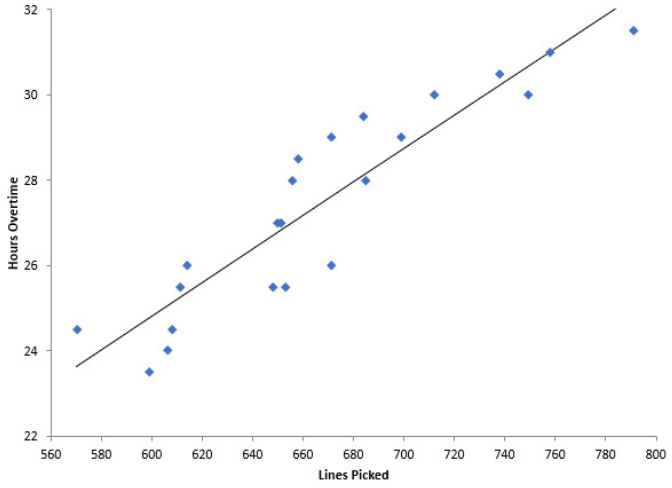


Figure 5.1: Relationship Between Lines Picked and Hours of Overtime

Why do we use scatter diagrams?

- To find out if there is a relationship between two variables.
- To see how strong the relationship is.
- To identify patterns, trends, or unusual data points (outliers).

Let us imagine you are studying whether the amount of time learners spend revising for mathematics affects their test scores.

- The ***x*-axis** shows the number of hours spent revising. This is the independent variable.
- The ***y*-axis** shows the marks scored in a test. This is the dependent variable.

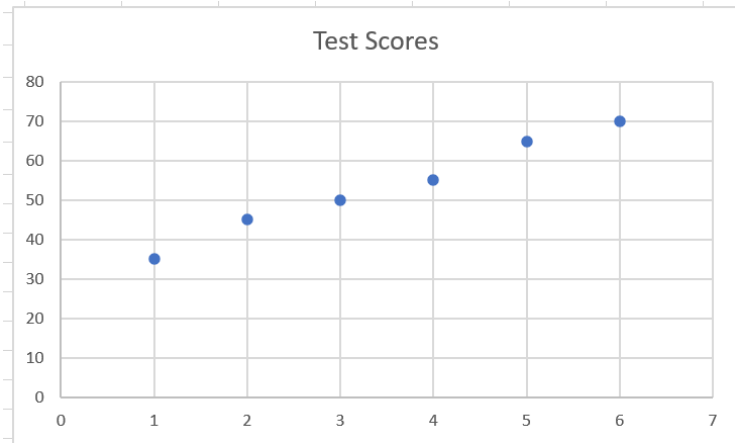


Figure 5:2: Scatter graph of hours study and test score

From the graph you can see that learners who revise more tend to score higher

Direction of the Relationship

The direction of the relationship could be:

1. **Positive correlation**: As one variable increases, the other also increases
2. (e.g., Hours studied vs. Test score)

A positive correlation

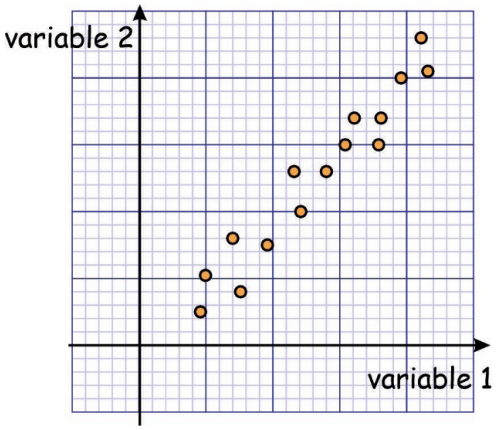


Figure 5.3: A positive correlation

A positive perfect correlation

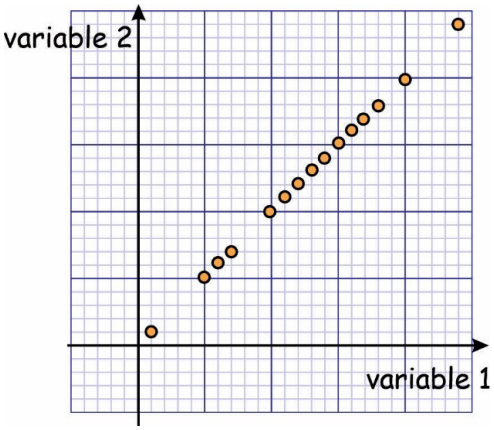


Figure 5.4: A positive perfect correlation

3. **Negative correlation**: As one variable increases, the other decreases
4. (e.g., Alcohol consumption vs. Health level)

A negative correlation

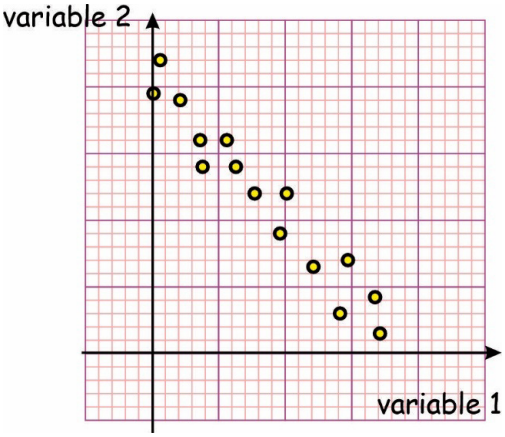


Figure 5.5: A negative correlation

A negative perfect correlation

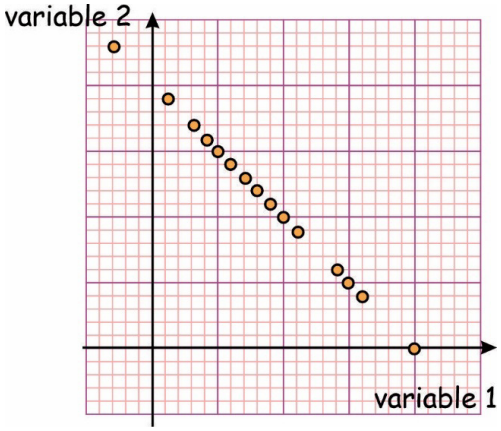


Figure 5.6: A negative perfect correlation

5. **No correlation**: No clear pattern or relationship
6. (e.g., Shoe size vs. Intelligence)

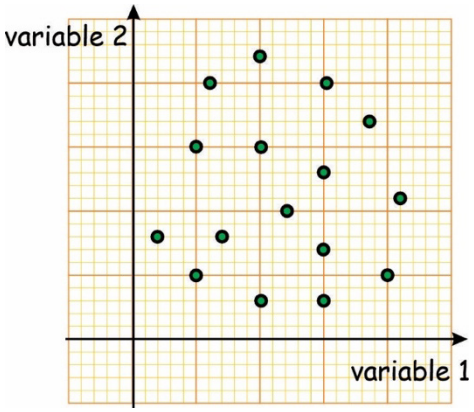


Figure 5.7: A no correlation scatter plot

How to construct a scatter plot manually

Steps:

1. Identify the variables. Since we are dealing with bivariate data, it will involve two variables, an independent variable and a dependent variable. The independent variable goes on the x-axis, with the dependent variable on the y-axis.
2. Label the axes and scale them.
3. Convert each data point into (x, y) coordinates and plot them on the graph. If two or more points fall on the same point, place them side-by-side.

How to construct a scatter diagram using Excel

Step 1: Open a new document in the Excel spreadsheet. A section of the interface should look like the picture below

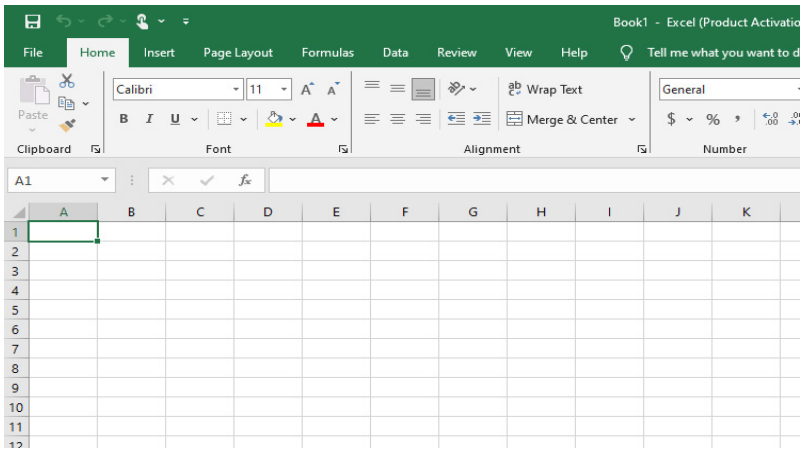


Figure 5.8: Excel spreadsheet page

Step 2: Input your data in the first and second columns of the spreadsheet. As an example, let us input the data in ***Table 5.4.***

Table 5.4: Temperature versus number of people wearing jacket.

Temperature (x)	35	6	17	15	21	20	10	7
Number wearing jackets (y)	11	45	22	30	20	25	32	42

Your spreadsheet should be similar to the one below

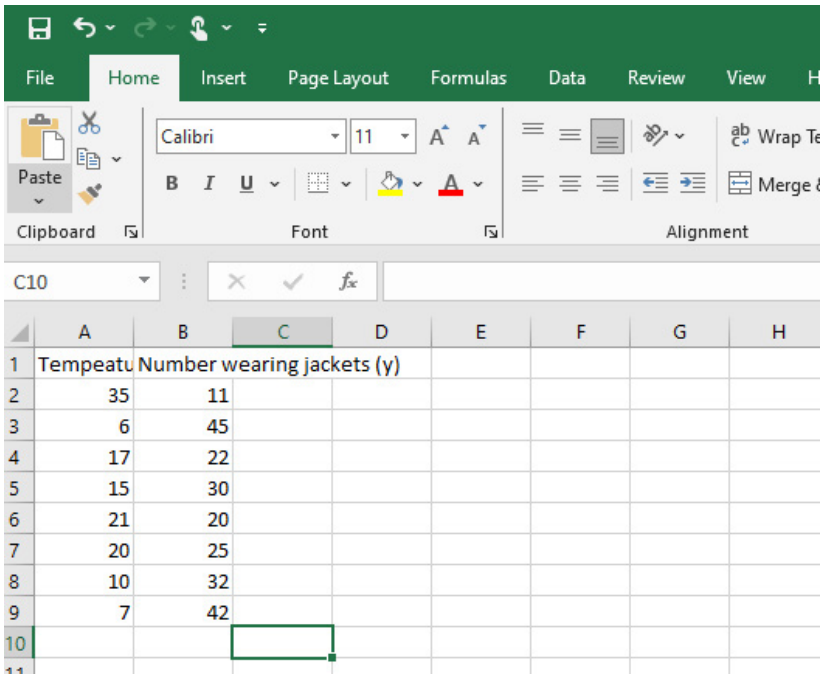


Figure 5.9: Excel spread sheet page showing data of temperature and number of people wearing jackets.

Step 3: Select the data, go to **Insert**, and choose **Scatter** from the drop-down menu.

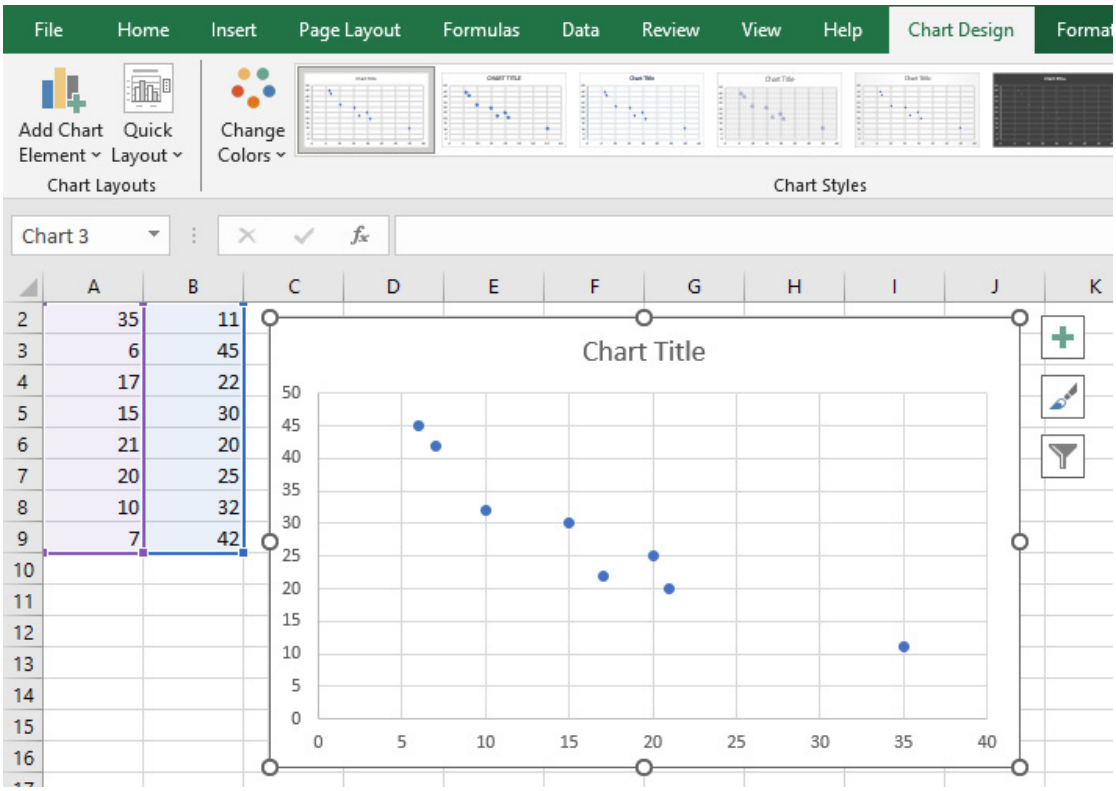


Figure 5.10: Excel spreadsheet page showing bivariate data and a scatter plot of temperature and number of people wearing jackets.

Step 4: Insert the title and the description of the x and y axes. To do this, click on the plus sign beside the scatter graph. This will open a dropdown menu titled Chart Elements. Choose "**Axis Titles**" from the dropdown menu.

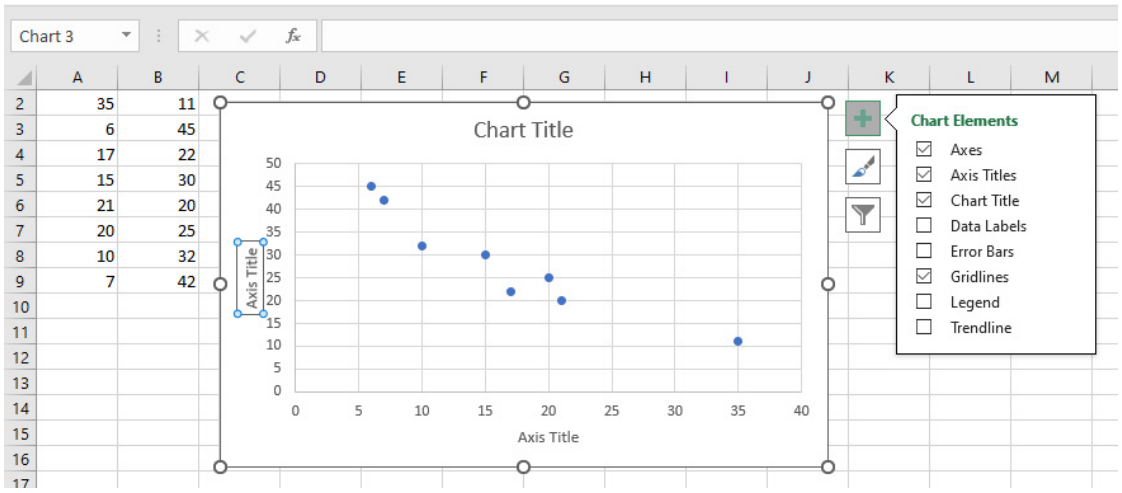


Figure 5.11: Excel spreadsheet page showing bivariate data and a scatter plot

Step 5: Your final plot should be similar to the chart below.

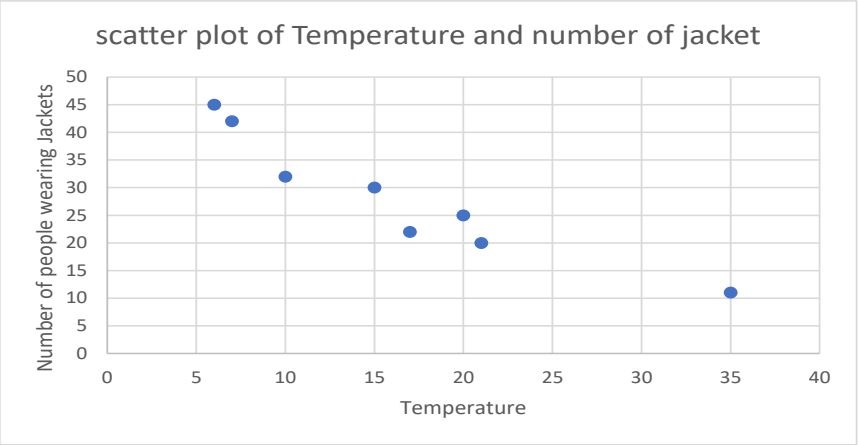


Figure 5.12: Scatter plot of temperature and number of people wearing jackets.

Example 5.2

Table 5.5: A farmer records the amount of fertilizer used and the crop yield:

Fertilizer (kg)	0	2	4	6	8	10
Crop Yield (kg)	100	150	200	250	300	450

a. Draw a scatter diagram to illustrate this data

b. If the farmer uses 5 kg of fertilizer, what is the expected crop yield?

Solution

Steps involved in drawing a scatter diagram

1. Draw two perpendicular lines to form the $x - y$ plane on a sheet of graph paper.
2. Choose an appropriate scale for the axes (x – axis representing the independent variable – the fertiliser and y– axis representing the dependent variable, the crop yield)
3. The ordered pairs (0, 100), (2, 150), (4, 200), (6, 250), (8, 300) and (10, 450) are plotted in the
4. $x - y$ plane.

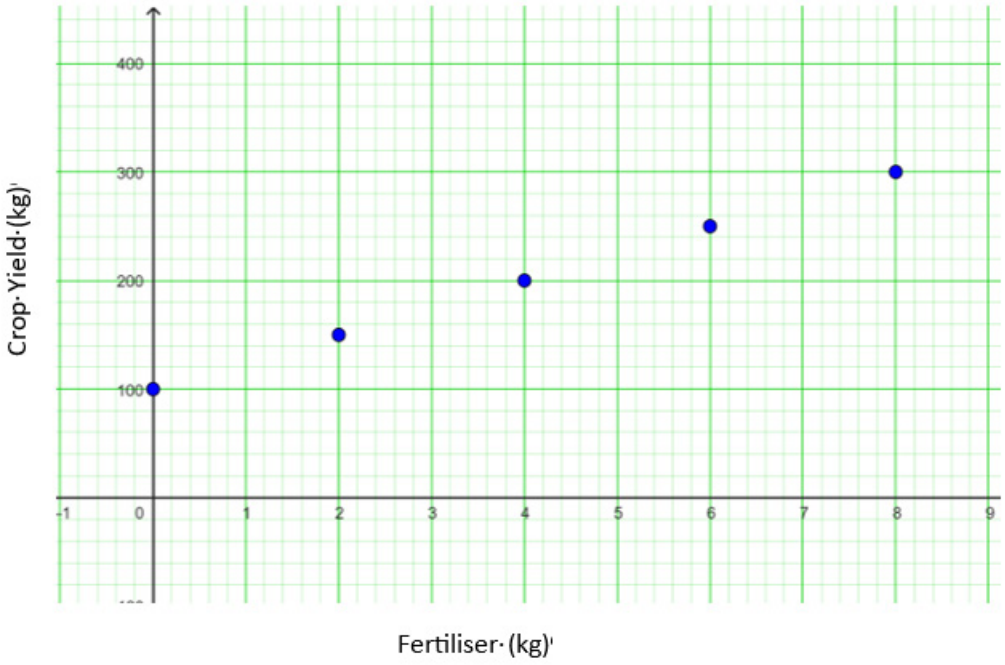


Figure 5.13: A scatter graph showing quantity of fertilizer vs crop yield

Activity 5.5 Social Media Use vs. Sleep Duration

Instructions

In pairs, or individually, create a scatter graph for the data in the table below which shows the number of hours of social media use against the duration of sleep.

Learner	Social Media Use (hours/day)	Sleep Duration (hours/night)
Ama	1	8
Kojo	2	7
Afia	3	6
Kwame	4	5
Esi	5	4

Use graph paper to plot the scatter graph:

- ***x – axis:*** Social media use (independent variable)
- ***y – axis:*** Sleep duration (dependent variable)

Analyse the graph and discuss:

- What type of correlation does it show? (Positive, negative, or no correlation)
- Is there a visible trend or pattern?
- What happens to sleep time as social media time increases?

Discuss with your classmates:

- Is this bivariate data? Why?
- Can we assume a cause-and-effect relationship here?

Follow-Up Questions:

- What does your graph suggest about the effect of social media on sleep?
- What other variables might influence a student's sleep besides social media?

Extension Task:

- Interview five classmates and collect real data (anonymously) on how many hours they spend on social media and how long they sleep.
- Plot the results and compare them with the data above.

EXPERIMENTAL DATA COLLECTION AND ANALYSIS

In many scientific and social studies, we often want to find out whether a certain treatment or intervention has an effect. One common way of doing this is by using **experimental studies**, where we compare two groups:

- A **treatment group**, which receives the intervention or condition being studied.
- A **control group**, which does not receive the treatment.

Data from such experiments can be analysed and represented using **scatter plots** to observe the relationship between the variables.

What is an Experimental Study?

An experimental study is a method of collecting data by intentionally changing one variable (called the independent variable) to observe how it influences another variable (called the dependent variable).

Table 5.6: Examples of experimental study.

Experiment Topic	Independent Variable	Dependent Variable	Variable that a researcher may choose to control
Dropping a ball to find out the number of times it bounces.	Drop Height	Number of bounces.	Ball type, surface, drop height.
Finding out how different amounts of sunlight affect plant growth	Hours of Sunlight	Plant Height	Water, soil, and plant type

How study time affects test scores	Study Time	Test Score	Test type, room conditions
How temperature of water affect dissolving time?	Water Temperature	Dissolving Time	Amount of sugar, volume of water
How the weight applied to a rubber band affects the stretch.	Weight Applied	Length Stretched	Rubber band type, air temperature

Structure of an Experimental Study

The structure of an experimental study includes.

- **Independent Variable**: The variable you change.
- **Dependent Variable**: The variable you measure.
- **Controlled Condition(s)**: Factor(s) kept constant for fairness.
- **Repeated Trials**: Performing an activity multiple times for accuracy.

Let us consider a simple experiment conducted in a Senior High School in Ghana. The purpose is to determine whether attending an extra one-hour mathematics revision session each week improves students' performance in mathematics.

Experiment Title: Effect of Extra Study Time on Learners' Mathematics Performance

Step 1: Collecting the Data

Twenty learners were selected randomly and divided into two equal groups:

- Treatment Group (10 students): Received an additional 1-hour mathematics revision lesson each week.
- Control Group (10 students): Did not receive any additional lessons.

After six weeks, both groups took the same mathematics test marked out of 100. The average number of hours each student studied per week (including class lessons, the additional revision lesson and private study) was also recorded.

Table 5.7: Effect of Extra Study Time on Learners' Mathematics Performance

Group	Learner	Hours Studied per Week (X)	Test Score (Y)
Treatment	T1	7	82
Treatment	T2	6	76
Treatment	T3	8	85
Treatment	T4	9	90
Treatment	T5	5	72
Treatment	T6	7	80

Treatment	T7	6	78
Treatment	T8	8	84
Treatment	T9	7	81
Treatment	T10	5	70
Control	C1	5	66
Control	C2	6	68
Control	C3	4	60
Control	C4	5	65
Control	C5	6	69
Control	C6	7	72
Control	C7	6	67
Control	C8	5	64
Control	C9	4	58
Control	C10	6	68

Step 2: Plotting the Scatter Graph

Plot the points (X, Y) for each student on the same scatter plot. Use different symbols or colours to represent the treatment group and the control group.

Use the horizontal axis (x – axis) for "Hours Studied per Week" and the vertical axis (y – axis) for "Test Scores".

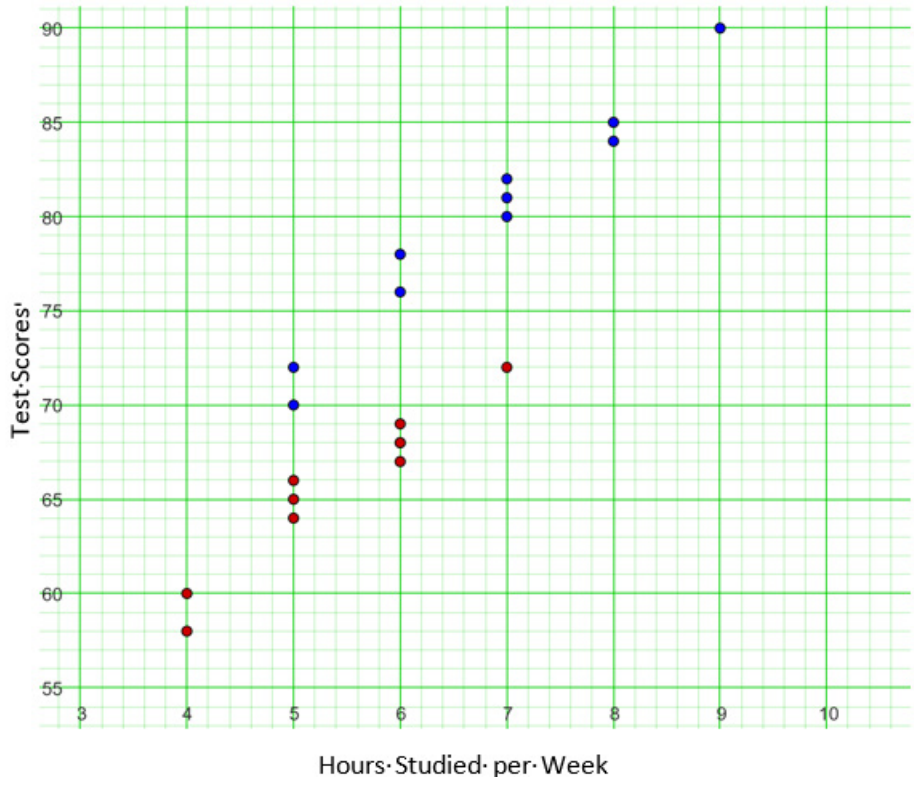


Figure 5.14: A scatter graph showing the number of hours leaners studied against their test scores. The red represents the control group, the blue the treatment group

The blue points are for the treatment group and red for control group.

Step 3: Interpreting the Scatter Plot

a. **Trend Observation:** By examining the scatter plot, you may observe a general upward trend — as the number of study hours increases, test scores tend to increase.

b. **Treatment vs Control Comparison:** Students in the treatment group generally scored higher than those in the control group who studied for the same number of hours.

c. **Relationship Between Variables:** There is a **positive correlation** between hours studied and test scores in both groups. However, the treatment group shows slightly higher test scores for the same number of study hours, suggesting that the extra revision sessions had a positive effect on performance.

Describing The Relationship Between Two Variables

The scatter plot helps us to visualise the relationship between two variables. An alternative way to describe the relationship is to use a linear function (a straight line). Note that this is used to approximate the relationship between the variables being compared, as the data points do not form a straight line in most cases. Hence, the approximated line is called a **line of best fit**. The line of best fit should either pass through most of the points or be closest to most of the points. Because of this, the line of best fit is usually used to predict data values within the given data. This is known as *interpolation*. When used to predict data values outside a given data, it is called *extrapolation*. Pairs of data values (points) relatively far from the line of best fit are called outliers. We will obtain the line of best fit through observation.

Data Collection in Experimental Studies

Planning your experiment

Before collecting data, you need to:

1. Define your research question: What are you trying to find out?
2. Identify your variables:
   - Independent variable (what you manipulate)
   - Dependent variable (what you measure)
3. Select your sample: Choose participants for both treatment and control groups
4. Plan your experimental procedure: Ensure it is consistent for all participants

   Let us use the example below to enhance our understanding of how to conduct a simple experimental study.

Example 5.3

Research Question:

Investigating the Relationship Between Drop Height and Bounce Height of a Ball.

Suggested Solution:

Step 1: Identify the Research Question

Research question: How does the height from which a ball is dropped affect how high it bounces?

Step 2: Identify the Variables:

Independent variable = Drop height (we will determine the heights from which we drop the ball)

Dependent variable = Bounce height (we have no control over the highest point the bounce will reach)

So, in this research, we will control the height from which we will drop the ball, the type of floor we want to drop the ball on, the type of ball we want to drop and maybe the temperature.

When a ball is dropped on a hard surface, all things being equal, it will bounce more than once. We will be only interested in the first bounce, as this gives the highest bounce.

Step 3: Plan the Experiment

Materials:

A rubber ball, a metre rule, a flat hard surface and a notebook for recording data

We will undertake 7 trials. We will drop the ball in intervals of 30cm and repeat each trial three times, and take the average bounce height. This is summarised in the procedure below.

Procedure:

- Drop the ball from different heights (30cm, 60cm, 90cm, 120cm, 150cm, 180cm, 210cm).
- Measure the bounce height each time using a ruler. Recording the results in a table.
- Repeat each measurement 3 times and record the average.

Step 4: Collect the Data

Table 5.8: Drop height versus average bounce height data.

Drop Height(cm)	Bounce Height (cm)			Average bounce height.
	First trial	Second trial	Third trial	
30	19	17	18	18
60	39	42	39	40
90	58	59	60	59
120	72	72	72	72
150	83	86	83	84
180	103	103	109	105
210	146	144	142	144

Step 5: Present the data using a scatter plot.

a. Label the **x-axis** as Drop Height (independent variable).

b. Label the **y-axis** as Bounce Height (dependent variable).

c. Plot the data points from your table.

d. Look for a **trend** or pattern.

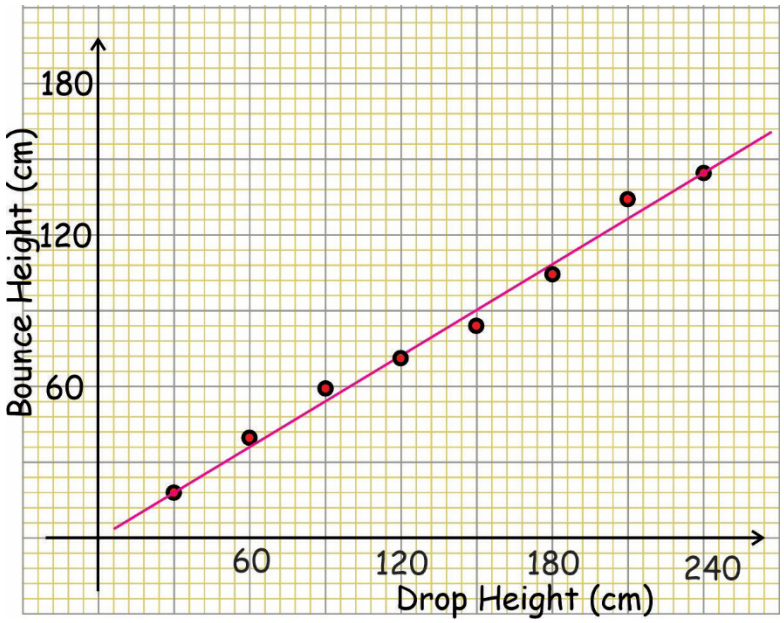


Figure 5.15: Scatter plot of drop height and average bounce height

Step 6: Interpret the Scatter Plot

From the graph, we can observe:

- As the drop height increases, the bounce height also increases.
- This shows a positive correlation.

- A line of best fit can help identify the relationship and predict unknown values.

Step 7: Using the graph to solve problems, if required:

Problem: If a ball is dropped from 70 cm, what is the expected bounce height?

Solution: Use the graph or line of best fit to estimate

To do this, read up from 70cm to the line of best fit and then read across to the bounce height for a predicted bounce height.

Conducting an Observational Study

Steps:

1. **Choose two variables** you want to study.
2. **Decide when and where** you will observe (e.g., during break time, in the school library).
3. **Use a table** to collect data at regular intervals.
4. **Draw a scatter plot** to visualise the relationship.
5. **Interpret** the pattern you observe.

Example 5.4

A reading test was given to 10 learners in Basic 6. The learners then took part in an extensive reading programme. After participating in the programme they were retested. The data collected was organised and plotted as a scatterplot as follows.

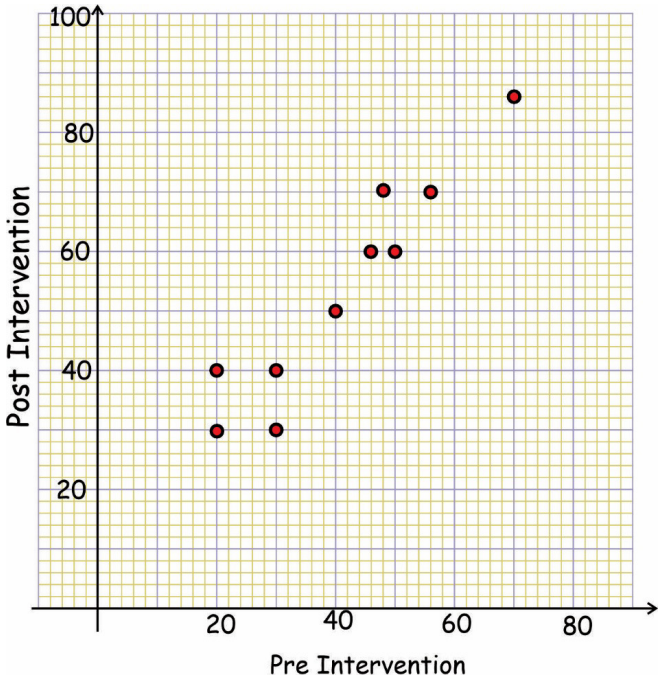


Figure 5.16: Scatter plot of pre and post intervention data.

Find the relationship between Pre-intervention Reading Test Scores and Post-intervention Reading Test Scores. For this, do a comparison, draw a conclusion and justify your conclusion.

Solution

Construct a distribution table for the graph

Table 5.9: Pre and post intervention data.

Pre-intervention scores	20	20	30	30	40	46	48	50	56	70
Post-intervention scores	30	40	30	40	50	60	70	60	70	86

The scatter graph shows a positive correlation between pre-intervention scores and post-intervention scores. The intervention increased the score

Observation:

- Generally, as pre-intervention scores increase, post-intervention scores also increase.
- This suggests a positive correlation.
- The values are not perfectly aligned, but the upward trend is clear.

Interpretation:

- Low pre-scores (e.g., 20, 30) had variable post-scores (30–40), showing some improvement.
- Higher pre-scores (e.g., 50, 56, 70) had higher post-scores (60, 70, 86), showing a consistent increase.
- Although there is a bit of spread in the lower scores, the overall pattern supports a moderate to strong positive correlation.

Conclusion:

There is a positive correlation between pre-intervention and post-intervention scores.

Learners with higher scores before the intervention tended to score even higher after the intervention. This suggests the intervention may have improved performance across all levels, but the effect appears more consistent among higher-performing learners.

Activity 5.6 Conduct Your Own Experiment

Materials

- Seeds (beans or maize)
- Containers

- Soil
- Measuring tape/ruler
- Graph paper or computer with spreadsheet software

Procedure

1. Divide your seeds into two equal groups
2. Plant all seeds in similar containers with the same amount of soil
3. Apply your treatment to one group (e.g., different watering schedule, fertilizer, light exposure)
4. Measure and record the growth every 3-5 days in a data table
5. Draw a scatter plot of your results
6. Analyse the relationship between your variables

Reflect on these questions, with a classmate, or in a small group

1. What is the purpose of having a control group in an experiment?
2. How would you describe the relationship shown in a scatter plot where points move from bottom-left to top-right?
3. In an experiment measuring the effect of study time on test scores, what would be the independent and dependent variables?
4. If a scatter plot shows points scattered randomly with no pattern, what can you conclude about the relationship between the variables?
5. How does the slope of a line of best fit help us understand the relationship between variables?

COMPARING AND INTERPRETING DATA SETS

Comparing Different Data Sets

In statistics, we often collect data from different groups or situations and need to decide which one gives a better picture of reality. To do this, we use **measures of central tendency** (mean, median, mode) and **measures of dispersion** (range, interquartile range, mean deviation, variance, standard deviation).

- **Central Tendency** helps us know the “average” behaviour of the data.
- **Dispersion** tells us how spread out the data is. A data set with very high spread may be less reliable in representing a situation.

Example 5.5: Comparing Exam Scores

Consider the following exam scores from two different classes in a senior High school:

Class A: 65, 70, 72, 75, 78, 80, 82, 85

Class B: 50, 60, 75, 78, 80, 85, 90, 95

Compare these data sets to draw conclusions about the two classes.

Solution

Measures of Central Tendency

Class A:

- Mean $= \frac{65+70+72+75+78+80+82+85}{8} = \frac{607}{8} \approx 75.9$
- Median $= \frac{75+78}{2} = 76.5$
- Mode = None (no repeated values)

Class B:

- Mean $= \frac{50+60+75+78+80+85+90+95}{8} = \frac{613}{8} = 76.6$
- Median $= \frac{78+80}{2} = 79$
- Mode = None (no repeated values)

Measures of Dispersion

Class A:

- Range = 85 – 65 = 20
- Standard Deviation ≈ 6.6

Class B:

- Range = 95 – 50 = 45
- Standard Deviation ≈ 15.1

Interpretation

Although Class B has a slightly higher mean (76.6 vs 75.9), the scores in Class A are more consistent (smaller standard deviation). Class B shows greater variability with some very high scores but also some very low scores.

Example 5.6

Two bicycle sales representatives, Naa and Yaw, are being considered for a promotion.

Their weekly performance over 5 weeks is shown below.

Table 5.10: Naa's sales performance

Hours worked	30	35	40	45	50
Sales made	12	13	16	19	20

Table 5.11: Yaw's sales performance

Hours worked	30	35	40	45	50
Sales made	15	16	17	17	18

Based on this data who do you consider worthy of the promotion?

Solution

Analysis and Interpretation:

Step 1: Plot the data for both Naa and Yaw using scatter graphs.

Scatter plot showing Naa's sales performance

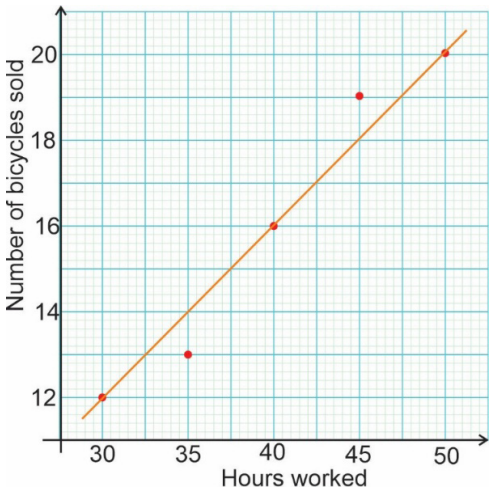


Figure 5.17: Scatter plot of Naa's sales performance

Scatter plot showing Yaw's sales performance

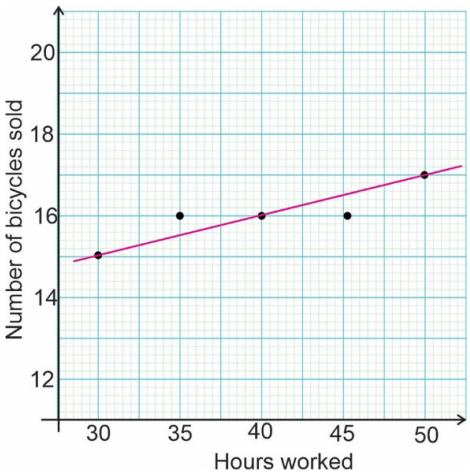


Figure 5.18: Scatter plot of Yaw's sales performance

Step 2: Identify the type of relationship.

Naa: Shows a strong positive correlation, as hours increase, so do sales of bicycles.

Yaw: Shows a less strong correlation, sales do not increase that much despite working more hours.

Step 3: Compare the trends and consistency.

Naa has a steady and consistent increase in sales with more work.

Yaw starts high, but the increase in sales is minimal, showing reduced efficiency with more hours.

Step 4: Conclusion

Naa's data better represents a productive relationship between work and results. The linear growth in sales with effort justifies selecting Naa for a promotion.

Example 5.7

Two mathematics classes wrote the same test. Their results were as follows:

- Arts 1: Mean score = 65%, Standard Deviation = 5
- Arts 2: Mean score = 65%, Standard Deviation = 15

Interpret this data.

Solution

Although both classes have the same mean score, Arts 1's performance is more consistent (less spread out), which means the scores of most students are closer to the average. Arts 2 shows that they have a lot more spread in their data, so some of these students will have scored more highly, but some will also have more low scores.

Justifying Which Data Set Better Represents a Given Situation

When deciding which data set better represents a situation, consider:

1. The purpose of the analysis
2. The context of the data
3. The relevant statistical measures
4. Potential outliers and their impact

Example 5.8

The tables below present data on tree growth observed after applying two distinct fertilizer brands: *Surge* and *Boom*

Table 5.12: Tree growth using Surge fertilizer

Week	1	2	3	4	5
Height (mm)	75	100	100	145	155

Table 5.13: Tree growth using Boom fertilizer

Week	1	2	3	4	5
Height (mm)	50	80	120	160	200

Which of the fertilizers will you recommend? Justify your choice of fertilizer.

Solution

Step 1: Plot the data for both Surge fertilizer and Boom fertilizer using scatter graphs.

A scatter graph showing the growth of trees after applying surge fertilizer

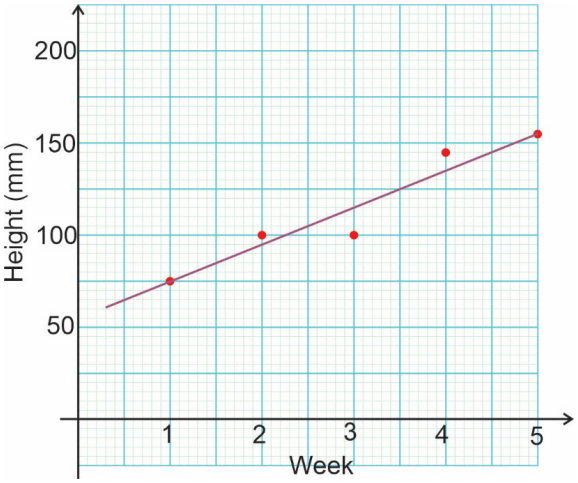


Figure 5.19: Scatter plot of tree growth using surge fertilizer

A scatter graph showing the growth of trees after applying boom fertilizer

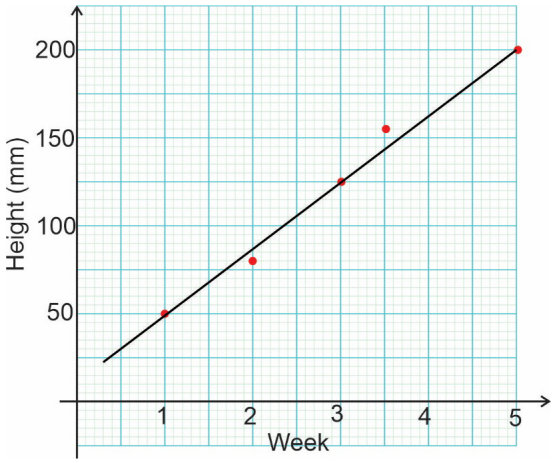


Figure 5.20: Scatter plot of tree growth using boom fertilizer

Step 2: Identify the type of relationship

***Surge*:** Shows a weak positive correlation – the growth of the trees is not that consistent.

***Boom*:** Shows a strong positive correlation – it shows a consistent growth as the weeks go by.

Step 3: Compare the trends and consistency

Surge: shows steady growth as it is consistent and effective over time.

Boom: gives quick but limited growth since growth stalls early.

Step 4: Conclusion

Surge Fertilizer is more effective for long-term plant development.

Example 5.9

Two farming techniques were tested in the Western Region of Ghana, with yields (in kg per hectare) recorded as follows:

Technique A: 1200, 1250, 1300, 1350, 1400, 1450, 1500

Technique B: 900, 1100, 1400, 1500, 1600, 1700, 2000

Which technique would you recommend to cocoa farmers?

Solution

Analysis:

Technique A:

- Mean = 1350 kg/ha
- Range = 300 kg/ha
- Standard Deviation ≈ 104 kg/ha

Technique B:

- Mean = 1457 kg/ha
- Range = 1100 kg/ha
- Standard Deviation ≈ 370 kg/ha

Conclusion

If a farmer wants consistent, reliable yields with minimal risk, Technique A would be more appropriate as it shows less variability (smaller standard deviation).

If a farmer can tolerate risk and wants the possibility of higher yields, Technique B might be preferred as it has a higher mean yield, though with much greater variability.

For most small-scale farmers who cannot afford significant crop failures, Technique A would likely be the better representation of a sustainable farming approach.

Activity 5.7

The following data shows the daily sales (in Ghana Cedis) for two fruit vendors at Makola Market over a week:

Vendor A: 120, 150, 145, 160, 155, 140, 130

Vendor B: 100, 180, 90, 200, 110, 190, 130

a. Calculate the mean, median, range and standard deviation for each vendor's sales.

b. Which vendor has more consistent sales? Justify your answer.

c. If you were advising someone who wants to start a fruit selling business, which sales pattern would you recommend they aim for? Why?

d. What factors might explain the differences in the sales patterns?

Interpreting Data Presented on Media Platforms

In today's world, information is everywhere. Media platforms such as local and international TV stations, newspapers, online journals and social media publish statistical data about health, education, the economy, sports and the environment.

Examples of data in the media include:

- Election results published by the Electoral Commission of Ghana.
- Ghana Statistical Service reports on unemployment or inflation.
- West African Examinations Council (WAEC) reports on student performance.
- World Health Organization (WHO) reports on global health issues like malaria or COVID-19.

Key skills in interpreting published data

1. **Check the source:** Is it credible (e.g., Ghana Statistical Service, World Health Organization, Ministry of Education)?
2. **Understand the representation:** Is the data shown in tables, graphs, or percentages?
3. **Look for patterns or trends:** Is there an increase, decrease, or fluctuation?
4. **Make inferences:** What does the data suggest about the situation?
5. **Draw conclusions and make recommendations:** What action should be taken based on the data?

Example 5.10: Data from Local Media (Ghana)

Interpret the newspaper report below, and consider what recommendations you should make.

- In 2023, 80% of students in Accra had access to internet-based learning resources.
- In 2023, only 35% of students in rural Northern Ghana had access.

Solution

Interpretation: This shows a digital divide between urban and rural students. The difference in access to digital learning may affect performance in online-based learning activities.

Recommendation: Government and stakeholders should provide affordable internet services and ICT facilities to rural areas to bridge the gap.

Example 5.11: Data from International Media

An international journal reports:

- Ghana's inflation rate in 2022 was 31%, while the average inflation rate across West Africa was 15%.

What inferences can you draw from this and what recommendations would you make?

Solution

Inference: Ghana's inflation was more than double the regional average, suggesting higher cost of living pressures on citizens compared with neighbouring countries.

Recommendation: Policymakers should focus on stabilizing the local currency and promoting local production to reduce dependency on imports.

Example 5.12: Screen Time and Sleep

An online health article reported data collected from teens aged 15–18 on average daily screen time and hours of sleep.

Table 5.14: Screen time and sleep time

Screen time(hrs)	3	6	8	5	1	2	1	6	4	7
Sleep time (hrs)	7.5	6.5	5.2	6.7	8.2	8	8.5	6.1	7	5.7

a. Present the data using a scatter plot.

b. What kind of correlation exists between screen time and sleep duration?

c. What might be the implication of this trend?

d. Is this data likely observational or experimental? Explain your answer.

Solution

a.

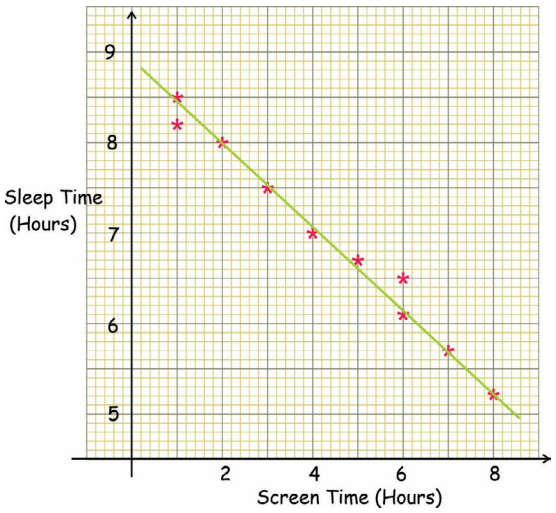


Figure 5.21: Scatter plot of screen time versus sleep time.

b. There is a negative correlation between screen time and sleep duration. As screen time increases, sleep decreases.

c. Excess screen time may lead to reduced sleep time.

d. The data is observational as no variable was controlled.

Example 5.13

The number of goals scored versus the number of goals conceded by a sample of seven football clubs in the Ghana Premier League for the 2024/2025 season is shown in the bar chart.

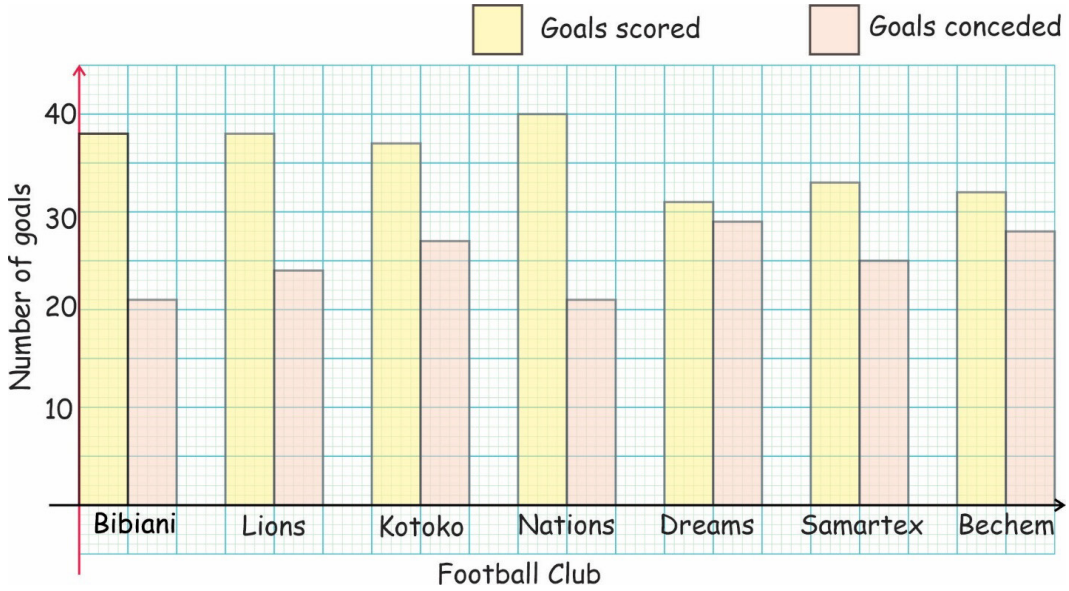


Figure 5.22: Bar chat showing football clubs and number of goals scored in the 2024/2025 season.

a. Copy and complete the table

Table 5.15: A partially completed table showing goals scored versus goals conceded

Football Club	Bibiani	Lions	Kotoko	Nations	Dreams	Samartex	Bechem
Goals Scored	38					33	
Goals Conceded						25	

b. Present the data using a scatter plot.

c. What kind of correlation exists between goals scored and goals conceded?

d. What might be the implication of this trend?

Solution

a.

Table 5.16: A completed table showing goals scored versus goals conceded.

Football Club	Bibiani	Lions	Kotoko	Nations	Dreams	Samartex	Bechem
Goals Scored	38	38	37	40	31	33	32
Goals Conceded	21	24	27	21	29	25	28

b. Present the data using a scatter plot.

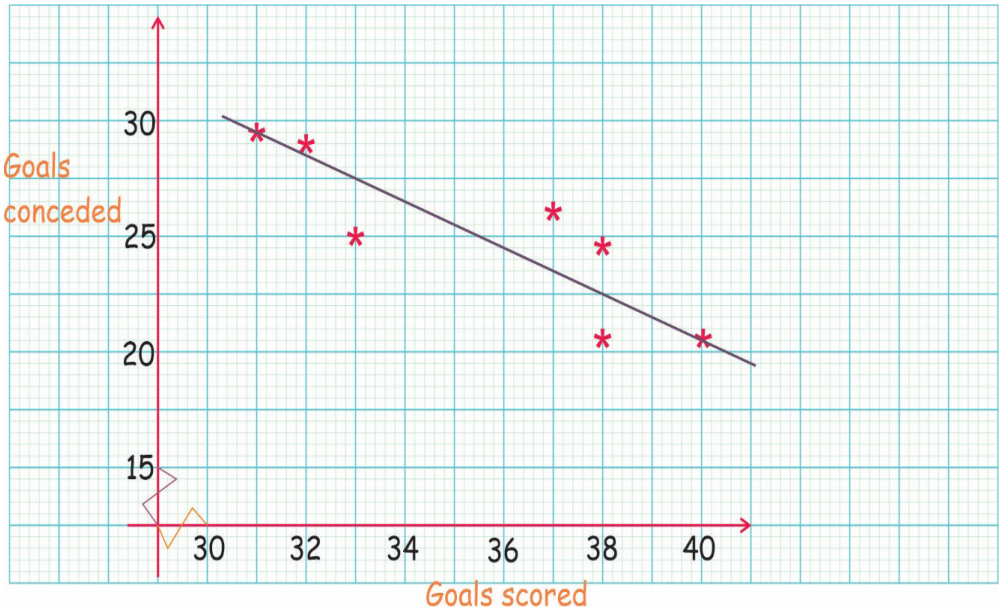


Figure 5.23: Scatter of goals scored versus goals conceded.

c. What kind of correlation exists between goals scored and goals conceded?

There is a negative correlation between goals scored and goals conceded.

d. Football clubs that score more goals concede fewer goals.

Activity 5.8

In pairs, or small groups, work through these questions and compare your answers with others.

1. The table below shows the monthly rainfall (in mm) recorded in two towns, Kumasi and Tamale, over the same period.

Town	Mean Rainfall (mm)	Standard Deviation (mm)
Kumasi	120	10
Tamale	118	25

- Which town has more consistent rainfall?

- Which data set best represents a stable rainfall pattern?
- Give reasons for your choice.

2. A news report shows that:
   - 65% of graduates from Technical Universities in Ghana are employed within two years of graduation.
   - 82% of graduates from Teacher Training Colleges are employed within two years of graduation.
   - Which group of graduates is more likely to get jobs faster?
   - What policy recommendations would you suggest to reduce graduate unemployment in Ghana?
3. A newspaper publishes the following data about two Ghana Premier League teams after 10 matches:

Team	Average Goals per Match	Standard Deviation
Hearts of Oak	2.3	0.5
Asante Kotoko	2.5	1.8

   - Which team is more consistent in scoring goals?
   - Which team is less predictable?
   - What inference can you make about their performance?
4. According to the Ghana Statistical Service (GSS), the unemployment rate increased from 8.5% in 2021 to 13.4% in 2023.
   - What does this trend suggest about job opportunities in Ghana?
   - Suggest two measures the government can take to address this problem.
5. The World Bank reported that in 2022, 33% of Sub-Saharan Africans had access to electricity, while in Ghana, 74% of the population had access.
   - What conclusion can you draw about Ghana compared to the Sub-Saharan average?
   - What recommendation would you make to ensure 100% electricity access in Ghana?

PROBABILITY IN REAL LIFE AND THE MEDIA

We already know that probability is the measure of the likelihood that an event will occur. It is widely used in daily life, decision-making, business, and the media. Media platforms (such as newspapers, television, radio, and the internet) frequently report probability to help individuals, governments and companies make informed choices.

Probability in the Media

Weather Forecasts (Television, Radio, Internet)

Example: The Ghana Meteorological Agency announces on TV:

- *"There is a 70% probability of rainfall in Accra tomorrow."*

Explanation:

- This means there is a high chance (7 out of 10 times) that rain will fall.
- People hearing this forecast may decide to carry umbrellas, change their travel plans, or postpone outdoor events.

Influence on Decisions:

- Farmers may delay planting or harvesting.
- Schools may cancel sports activities.
- Traders may prepare to protect goods in the market.

Sports Predictions (Newspapers, Internet, TV Sports Channels)

Example: A sports analyst reports in the Daily Graphic that:

- *"Based on past performance, Hearts of Oak has a 65% probability of winning their next match against Kotoko."*

Explanation:

- This prediction is based on previous data such as goals scored, player form and head-to-head results.

Influence on Decisions:

- Supporters may place bets on betting platforms.
- Coaches may adjust strategies knowing their probability of success.
- Fans may decide whether to attend the match or watch from home.

Health Reports (Newspapers, WHO Reports, Online Platforms)

Example: A health report states:

- *"There is a 30% probability of contracting malaria in areas without mosquito nets compared to a 10% probability in areas where nets are used."*

Explanation:

- This shows that the likelihood of malaria is three times higher without mosquito nets.

Influence on Decisions:

- Families are encouraged to use insecticide-treated nets.
- Government and NGOs plan health campaigns to reduce the risk.

Election Polls (TV, Newspapers, Internet)

Example: Before Ghana's general elections, a poll published in the Daily Guide states:

- *"Candidate A has a 55% chance of winning, while Candidate B has a 45% chance."*

Explanation:

- Probability is used here to estimate the likely winner, based on sampled voters' opinions.

Influence on Decisions:

- Voters may be motivated to turn out in large numbers to support their candidate.
- Political parties may change campaign strategies in areas where they are less popular.

Business and Insurance (Internet, Newspapers)

Example: An insurance company advertises:

- *"There is only a 5% probability that your car will be involved in an accident in a given year. Protect yourself with insurance."*

Explanation:

- The small but possible risk is highlighted to encourage car owners to buy insurance.

Influence on Decisions:

- Drivers choose to purchase insurance policies for security.
- Businesses prepare contingency plans for unexpected losses.

Why probability is important in media reports

- It helps individuals make informed choices (e.g., whether to carry an umbrella or buy insurance).
- It allows policymakers and leaders to plan ahead (e.g., preparing for floods or disease outbreaks).
- It provides businesses with risk assessments (e.g., banks assessing loan repayment probability).
- It gives the public a scientific basis for decision-making rather than relying on guesswork.

Carry out the following activities individually but compare your answers with a classmate.

Activity 5.7: Weather Report

Watch a local TV news weather forecast.

Write down one probability statement given (e.g., "60% chance of thunderstorms in Koforidua").

- How might this forecast influence a farmer?
- How might it influence a taxi driver?

Activity 5.8 Sports Probability

A betting company reports

- Chelsea has a 40% chance of winning,
- Manchester United has a 35% chance,
- A draw has a 25% chance.

Consider which outcome is most likely?

How might this probability influence fans or bettors?

Activity 5.9 Health Probability

A health journal reports

- The probability of contracting COVID-19 without wearing a mask is 25%,
- With a mask, the probability reduces to 10%.

What decision should an individual make based on this data?

Explain why probability helps in health campaigns.

PROBABILITY OF DEPENDENT AND INDEPENDENT EVENTS

Dependent and Independent Events

a. **Independent Events:** Two events are **independent** if the occurrence of one does **not** affect the probability of the other.

Examples:

- Tossing a coin twice. (The first toss does not affect the second.)
- Rolling a die and then tossing a coin.

- Rolling a single die twice or two dice once.
- Drawing two or more items with replacement.

If **A** and **B** are independent, then:

$P(A \text{ and } B) = P(A) \times P(B)$

b. Dependent Events: Two events are **dependent** if the outcome of one event affects the probability of the other.

Examples:

- Picking a card from a deck **without replacement**.
- Selecting two students from a class for a task, one after the other, without returning the first.
- Selecting individuals to form a group or a committee.
- Selecting individuals for a position where one person cannot occupy multiple positions

If **A** and **B** are dependent, then:

$P(A \text{ and } B) = P(A) \times P(B|A)$

where $P(B|A)$ means the probability of B given that A has already happened.

Addition and Multiplication Laws

a. Addition Law: The **addition law** is used when we want the probability of **either event A or event B** happening.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

If events A and B are **mutually exclusive** (they cannot happen together), then:

$P(A \text{ or } B) = P(A) + P(B)$

b. Multiplication Law: The **multiplication law** is used when we want the probability of **both events happening**.

For independent events:

$P(A \text{ and } B) = P(A) \times P(B)$

For dependent events:

$P(A \text{ and } B) = P(A) \times P(B|A)$

Examples 5.14

A coin is tossed and a die is rolled. Find the probability of obtaining a Head and a 4.

Solution

- $P(\text{Head}) = \frac{1}{2}$
- $P(\text{rolling a } 4) = \frac{1}{6}$

Since the events are independent:

$$P(\text{Head and } 4) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Example 5.15

Two cards are drawn from a standard deck of 52 cards without replacement. Find the probability that both are Aces.

Solution:

- $P(\text{1st Ace}) = \frac{4}{52} = \frac{1}{13}$
- After one Ace is removed, $P(\text{2nd Ace}) = \frac{3}{51} = \frac{1}{17}$

$$P(\text{Two Aces}) = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$$

Example 5.16

In a class of 40 students, 25 like Mathematics, 18 like Science, and 10 like both subjects. If a student is chosen at random, find the probability that the student likes Mathematics or Science.

Solution:

$$P(\text{Math or Science}) = P(\text{Math}) + P(\text{Science}) - P(\text{Math and Science})$$
$$= \frac{25}{40} + \frac{18}{40} - \frac{10}{40}$$
$$= \frac{33}{40}$$

Example 5.17

A building contractor submitted bids for two independent contracts, X and Y. The probability that he will win contract X is 0.5, and the probability that he will not win contract Y is 0.3. Calculate the probability that the contractor will:

a. Win both contracts

b. Win exactly one of the contracts

c. Win neither of the contracts

Solution

$P(\text{Win X}) = 0.5$, $P(\text{Not Win X}) = 1 - 0.5 = 0.5$

$P(\text{Not Win Y}) = 0.3$, $P(\text{Win Y}) = 1 - 0.3 = 0.7$

Contracts X and Y are independent

a. Probability of winning both contracts:

Since the events are independent:

$P(\text{Win X and Win Y}) = P(\text{Win X}) \times P(\text{Win Y})$

$= 0.5 \times 0.7$

$= 0.35$

Therefore, the probability of winning both contracts is

b. Probability of winning exactly one contract:

There are two ways to win exactly one contract:

- $P(\text{Win X and not win Y}) = 0.5 \times 0.3 = 0.15$
- $P(\text{Not win X and Win Y}) = (1 - 0.5) \times 0.7 = 0.5 \times 0.7 = 0.35$

$P(\text{Win exactly one contract}) = 0.15 + 0.35 = 0.50$

c. Probability of winning neither contract:

$P(\text{winning neither contract}) = P(\text{Not win X}) \times P(\text{Not win Y})$

$= 0.5 \times 0.3 = 0.15$

Example 5.18

A basket at the Makola Market contains 5 oranges and 3 mangoes. If two fruits are selected randomly without replacement, what is the probability of selecting an orange followed by a mango?

Solution

$P(\text{Orange then Mango}) = P(\text{Orange}) \times P(\text{Mango} \mid \text{Orange})$

$P(\text{Orange}) = \frac{5}{8}$

After selecting an orange, there are 4 oranges and 3 mangoes left out of 7 total fruits:

$P(\text{Mango/Orange}) = \frac{3}{7}$

Therefore: $P(\text{Orange then Mango}) = \frac{5}{8} \times \frac{3}{7}$

$= \frac{15}{56} \approx 0.268$

There is approximately a 26.8% chance of selecting an orange followed by a mango.

Example 5.19

A social media quiz app offers a 30% chance to win a prize per spin. If you spin twice, what is the probability that you win both times?

Solution

The question is an example of an independent event. The 30% chance of winning does not change regardless of the number of spins.

$$\text{P(Win twice)} = \frac{30}{100} \times \frac{30}{100}$$
$$= \frac{900}{10000} = \frac{9}{100}$$
$$= 0.09 = 9\%$$

Activity 5.10 Real-Life Probability Problem Solving

Work with a partner; consider this task.

A fruit seller at Kejetia Market has 6 apples and 4 pears.

If two fruits are selected one after the other without replacement:

a. What is the probability of selecting an apple followed by a pear?

b. What is the probability of selecting two pears?

Solve using the multiplication law for dependent events.

After solving, discuss how probability could help the seller in planning (e.g., predicting sales or fruit combinations).

REVIEW QUESTIONS

1. The following are the daily sales (in cedis) made by a kenkey seller over 5 days:

   80, 95, 100, 85, 90

   a. Find the mean and median

   b. Draw a bar chart to represent the data

2. The number of children in each of 8 households in Sunyani are: 2, 3, 4, 2, 5, 3, 2, 4

   a. What is the mode?

   b. What is the range?

3. A teacher recorded the number of hours learners studied and their scores:

Learner	Study Time (hrs)	Score (%)
A	2	55
B	4	65
C	6	75
D	8	85

   a. Plot a scatter diagram

   b. What is the relationship between study time and grades?

4. The Regional Director of education is studying the relationship between the number of teachers in a school and the number of learners:

School	Teachers	Learners
A	10	200
B	15	350
C	8	150
D	20	400

   a. Which school has the highest learner-teacher ratio?

   b. Is there a correlation between number of teachers and number of learners?

5. Identify the independent and dependent variables:

   i. A researcher examines how sleep affects exam performance.

   ii. A shopkeeper records daily temperature and sales of cold drinks.

6. Plot a scatter graph for the data below and describe the relationship:

Hours of Exercise	1	2	3	4	5	6
Weight Loss (kg)	0.5	1.0	1.5	2.5	3.0	5.0

7. The Ghana Meteorological Service reports a 65% probability of rain in Accra tomorrow.
   - **a.** What common misunderstanding might people have about this forecast?
   - **b.** What does this 65% probability actually mean?
   - **c.** Based on this information, what actions might someone choose to take?
8. A recent study found that a new malaria vaccine is 87% effective.
   - **a.** How would you explain this 87% effectiveness to a friend in simple terms?
   - **b.** Why is this information important for individuals and the public?
   - **c.** What is a possible misunderstanding someone might have about this result?
9. A sports website predicts that "Medeama FC has a 70% chance of winning their next match."
   - **a.** What does this 70% chance mean in context?
   - **b.** How might fans or gamblers respond to this information?
   - **c.** What is the possible misunderstanding someone might have about the prediction?
10. A bag contains 6 red bottle tops and 4 blue bottle tops. Two are picked one after the other with replacement.

    Find the probability that:
    - **a.** Both are red.
    - **b.** One is red and one is blue.
11. A survey shows that the probability a person reads the Daily Graphic is 0.4, and the probability a person reads the Ghanaian Times is 0.3. These events can be taken as independent.

    What is the probability that a person reads at least one of the two newspapers?
12. The probability that a football team wins a home match is 0.7 and the probability that it wins an away match is 0.4. Assuming independence, find the probability that:
    - **a.** The team wins both an away and a home match.
    - **b.** The team wins at least one match from a home and an away match.
13. In a class at Achimota School, the probability that a student selected at random plays football is 0.35, and the probability that a student plays basketball is 0.25.

If no student plays both sports, what is the probability that a randomly selected student plays either football or basketball?

14. In a survey of 200 Ghanaian high school students, 120 enjoy fufu, 100 enjoy jollof rice, and 60 enjoy both.

If a student is selected at random, what is the probability that they enjoy either fufu or jollof rice or both?

15. Two cards are drawn at random from a standard deck of 52 playing cards without replacement.

Find the probability of selecting:

a. A king, followed by a red card. Give your answer as a fraction in its simplest form.

b. A diamond, followed by a black card. Give your answer as a fraction, in its simplest form.

c. Two odd-numbered cards. Give your answer as a fraction, in its simplest form.

ANSWERS TO REVIEW QUESTIONS

SECTION ONE

1. True/False/Statements

i. True

ii. True

iii. False

iv. True

v. Debatable

vi. True

vii. False

viii. True

ix. True

2. Conjunctions and Disjunctions

a. **Conjunction**: $P \wedge Q \rightarrow$ False

Disjunction: $P \vee Q \rightarrow$ True

b. **Conjunction**: $R \wedge S \rightarrow$ True

Disjunction: $R \vee S \rightarrow$ True

c. **Conjunction**: $T \wedge U \rightarrow$ True

Disjunction: $T \vee U \rightarrow$ True

3. Acreage Decision for Maximum Yield

4 acres of groundnuts and 8 acres of yam

Because yam yields more per acre (2.5 tons vs 1.5 tons)

4. Conclusions and Labels

a. Possibly True

b. Possibly True

c. Possibly True

5. GH¢160

6. 300 km

7. Double the workers (i.e., twice as many)

8. 4 hours

9. GH¢90

10. 10 hours

11. Total cost of journey = 10 + 2k, where k = the number of kilometres travelled
= GH¢24

12. GH¢30

SECTION TWO

1. The graph is shown below.

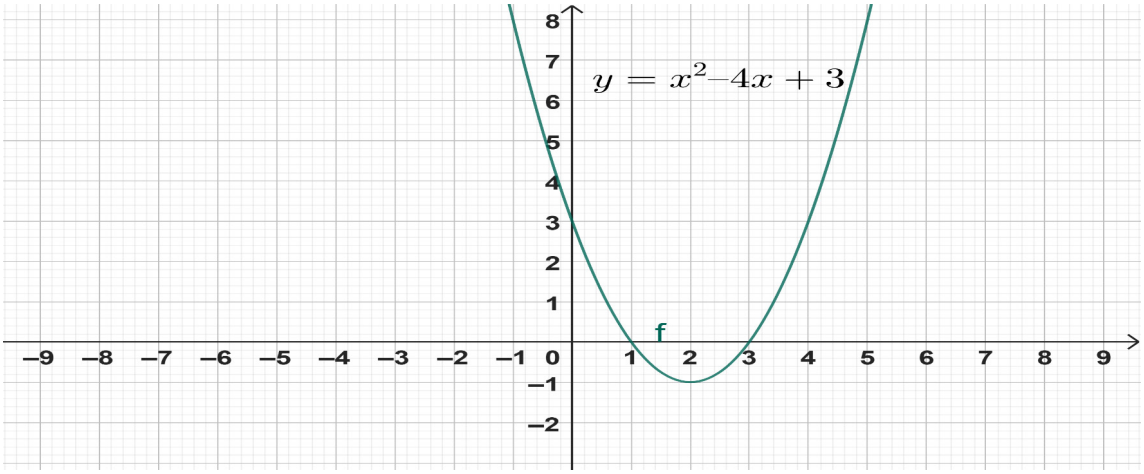


Figure 2.30: Graph of y = $x^2 - 4x + 3$

2. The maximum value is 4 and it occurs when $x = 3$.

3. The graph of y = $x^2 - 2x - 8$ is shown below.

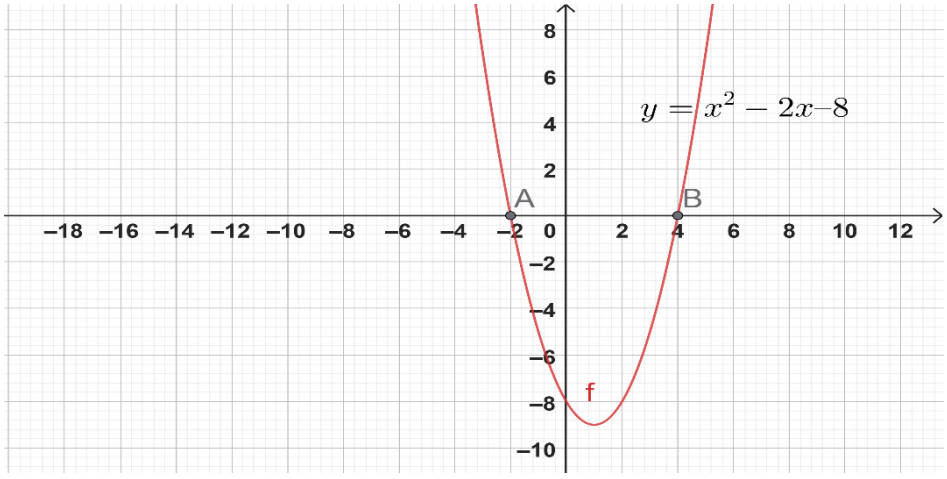


Figure 2.31: Graph of y = $x^2 - 2x - 8$

The roots of the equation, y = $x^2 - 2x - 8$ are $x = -2$ and $x = 4$.

The graph intersects the x-axis when $y = 0$ and these intersections are the roots.

4. The graph of $2x^2 - 3x + 4 = 0$ is shown below.

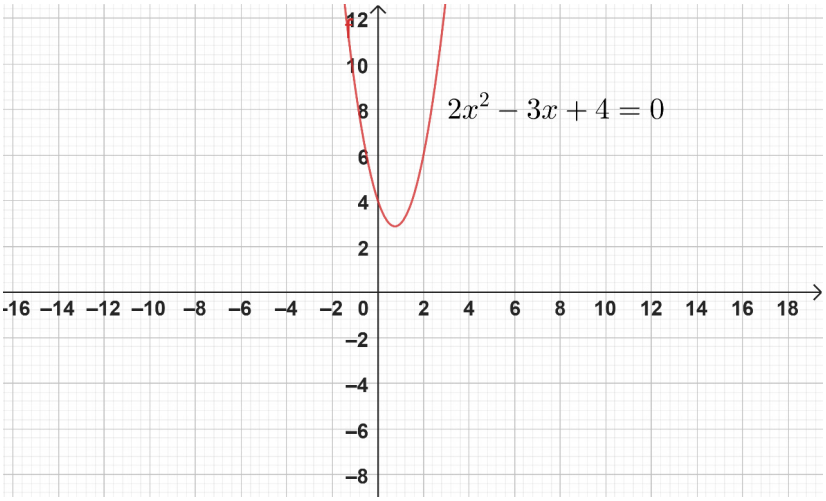


Figure 2.32: Graph of $2x^2 - 3x + 4 = 0$

As the graph does not cross the x-axis, it has no real roots, so it has complex roots.

5.

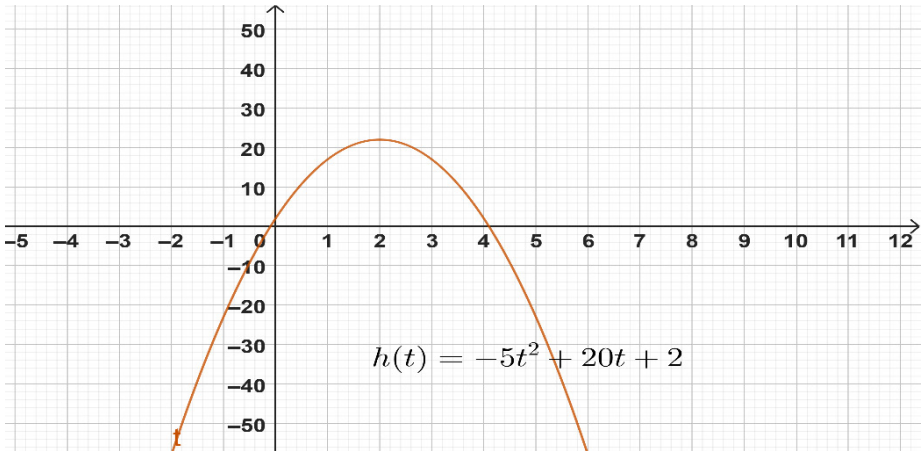


Figure 2.33: Graph of $h(t) = -5t^2 + 20t + 2$

The maximum height is 22 metres and it occurs at 2 seconds.

6. The graphs of $y = x^2 - 4x + 3$ and $y = x^2 + 4x - 3$ are shown below:

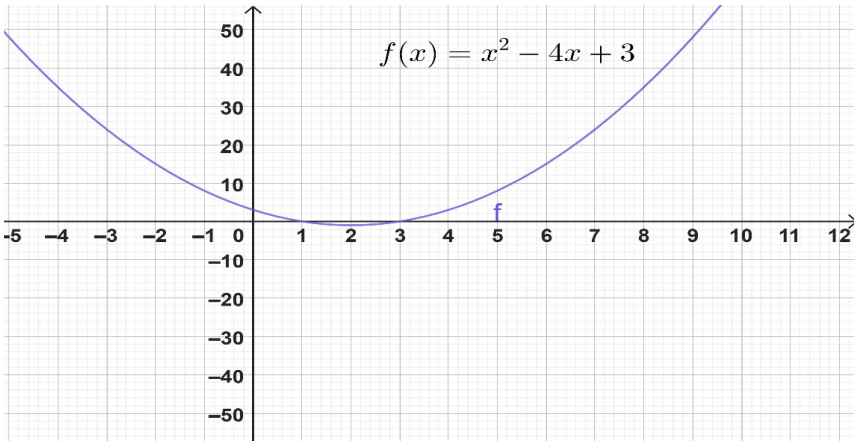


Figure 2.34: Graph of $f(x) = x^2 - 4x + 3$

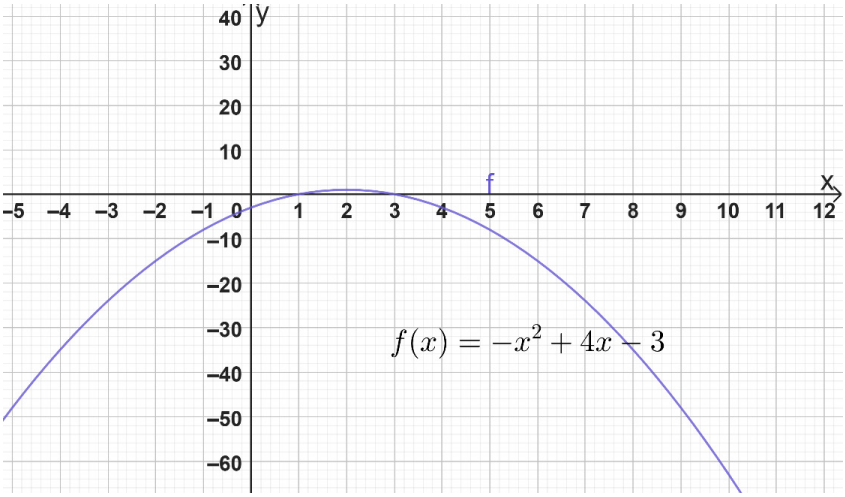


Figure 2.35: Graph of $f(x) = x^2 - 4x + 3$

- The function, $x^2 - 4x + 3$ is the reflection of the function, $y = x^2 + 4x - 3$, with the x axis being the line of reflection.
- $y = x^2 - 4x + 3$ has an upward parabola (minimum point) with the roots, (1, 3).
- $y = x^2 + 4x - 3$ has a downward parabola (maximum point) with the roots (1, 3).

7. $x = 2$ or $x = 3$

8. $x = -2$ or $x = \frac{1}{2}$

9. Maximum point = (2, 5)

10. Your sketch should show an upward opening parabola with a minimum point at (3, −1) with its roots at (2, 0) and (4, 0) and its y-intercept at (0,8)

11. Axis of symmetry is $x = 2$

12.

a. Maximum height is 21 metres (when t = 2 seconds

b. It hits the ground when t = 4.05 seconds

13. y is positive when $1 < x < 3$

14.

a. y is decreasing on the interval: $(-\infty, 3)$

b. y is increasing on the interval: $(3, \infty)$

15.

a. Number of items that will maximize profit: $x = \frac{-12}{2(-2)} = 3 = 300$ items

b. Maximum profit = P(3)= $-2(3)2 + 12(3) - 5 = -18 + 36 - 5 = 13 =$ GH¢13000.00

c. Solve for x in $2x^2 - 12x + 5 < 0$

Therefore, the company is in profit when $\approx 0.45 < x < \approx 5.55$, ie, between 45 and 555 items are produced.

16.

a. $L = 60 - 2x$ where $x =$ width and $L =$ length

b. $A(x) = 60x - 2x^2$ where A is the area of the garden

c. Value of x that maximises the area $= 15$

$A_{max} = 900 - 450 = 450m^2$, where A_{max} = maximum area

d. **Interpretation**:

The garden will have its largest possible area when the width is 15 metres.

Then the length will be, $L = 60 - 2(15) = 30m$

So, the dimensions for the maximum area are:

Width = 15m, Length = 30m, giving a maximum area of 450m2

17.

a. Time at maximum height $\approx$ 1.43s

Maximum height $\approx$ 13.0m

b. Total time of flight $\approx$ 3.06s

c. The projectile passes the sensor at approximately 0.65 s and 2.20 s

d. Graphical verification & shape of graph

- **Graph type:** Downward-opening parabola (because $a = -4.9 < 0$)
- **Key features:**

y-intercept: At $t = 0$, $h(0) = 3 \rightarrow$ starts 3 m above ground.

Vertex: At $t \approx 1.43$s, $h \approx 13.0$m $\rightarrow$ maximum height.

x-intercepts: At $t \approx 0$s (start slightly above ground) and $t \approx 3.06$s (hits ground).

Symmetry: Parabola is symmetric about $t \approx 1.43$s.

Verification method:

Plot t vrs. $H(t)$ from $t = 0$ to $t \approx 3.1$s

Check that:

Peak occurs at $t \approx 1.43$s with $h \approx 13$m.

Crosses $h = 0$ at $t \approx 3.06$s.

Crosses $h = 10$ at $t \approx 0.65$s and $t \approx 2.20$s.

18.

a. Plot the graphs for both functions

b. Intersection points (when they have the same cost):

We solve $25x + 100 = 5x^2 + 10x$

$5x^2 - 15x - 100 = 0$

$x^2 - 3x - 20 = 0$

$x = 6$ hours

Both companies charge the same amount when $t = 6$ hours.

c. When Company B is cheaper:

At x = 0 Company A charges $y = 100$ and Company B charges y = 0, so Company B is cheaper at $x = 0$. Thus, the range is: : $0 \leq x < 6$ hours

d. **Practicality**:

i. **Company A (Linear):** The cost increases at a constant rate of GH¢25.00 per hour, starting with a GH¢100.00 flat fee. This is better for longer hours because the rate does not increase as rapidly as Company B's quadratic rate.

ii. **Company B (Quadratic):** The cost starts low (GH¢0 at 0 hours) but increases rapidly as hours increase due to the x^2 term. This is better for short durations where the total cost is lower than Company A's flat fee plus linear rate.

Practical Scenarios:

- *Short-term use (less than approximately 6 hours):* Company B is cheaper. ***For example***, at 1 hour, Company A charges GH¢125.00, while Company B charges GH¢15.00.
- *Long-term use (e.g., more than ~6.2 hours):* Company A becomes cheaper. ***For example***, at 7 hours, Company A charges GH¢275.00, while Company B charges GH¢315.00
- *At approximately 6 hours:* Both companies charge the same amount (GH¢255.38).

Conclusion

- Choose Company B for short-term or one-time uses (less than approximately 6 hours).
- Choose Company A for long-term or extended uses (more than approximately 6 hours).
- At approximately 6 hours, both are equally priced.
- This analysis assumes that the cost is the only deciding factor; other considerations like service quality or additional fees could also influence the decision.

SECTION THREE

1.

i. 46°

ii. 82°

iii. 98°

2. Parallelogram

3. A kite

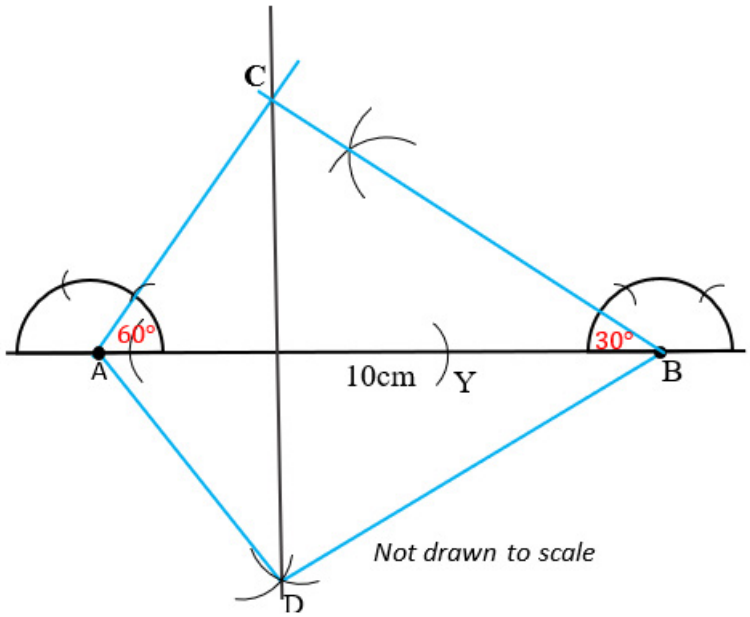


Figure 3.85: Solution to question 3

4.

i. $\angle AEC = 110°$

ii. $\angle AOC = 140°$

iii. $\angle ADC = 40°$

5.

i. $\angle AOB = 64°$

ii. $\angle AOC = 234°$

SECTION FOUR

1.

a. Completed table for the relation the graph of $y = 2\sin x$ for $0° \leq x \leq 360°$

Table 4.17:

x (degrees)	0°	45°	90°	135°	180°	225°	270°	315°	360°
y = 2sinx	0	1.41	2	1.41	0	-1.41	-2	-1.41	0

b.

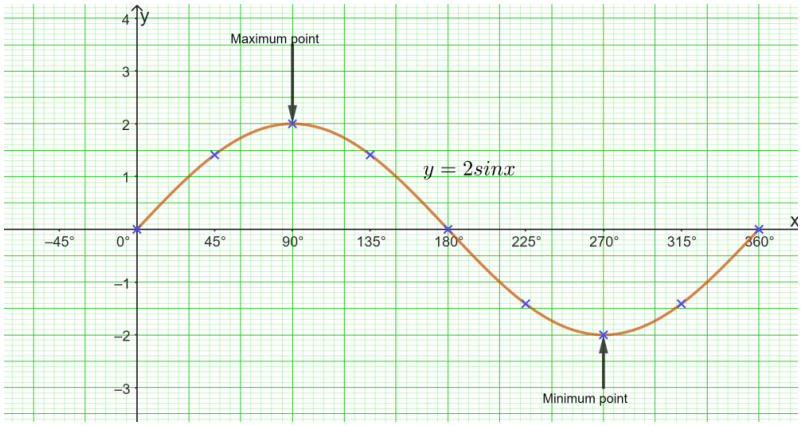


Figure 4.16: *Graph of y = 2sinx for 0° ≤ x ≤ 360°.*

From the graph above:

c.

i. Maximum point is (90°, 2)

ii. Minimum point is (270°,−2)

2.

a. Completed table for the relation $y = 3\cos x + 2\sin x$

Table 4.18

x (degrees)	0°	30°	60°	90°	120°	150°	180°
y = 3cosx + 2sinx	3.00	3.60	3.23	2.00	0.23	-1.60	-3.00

b.

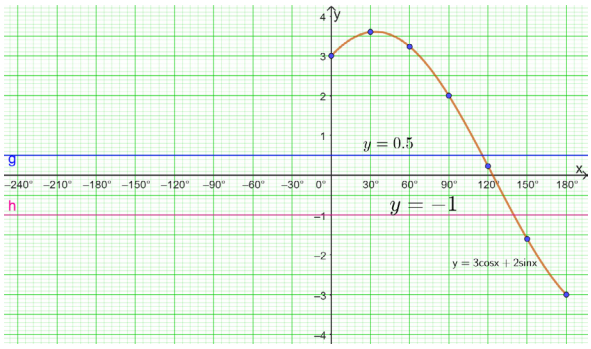


Figure 4.17: Graph for the relation $y = 3\cos x + 2\sin x$

c.

i. From the graph, the curve intersects the line $y = 0.5$ at $x \approx 115.7°$

Hence, the truth set of $6\cos x + 4\sin x = 1$ is given by: $\{x : x = 115.7°\}$

ii. From the graph, the curve, and the line $y = -1$ intersect at $x = 139.8°$

Hence, the truth set of $3\cos x + 2\sin x = -1$ is given by: $\{x : x = 139.8°\}$

3.

a.

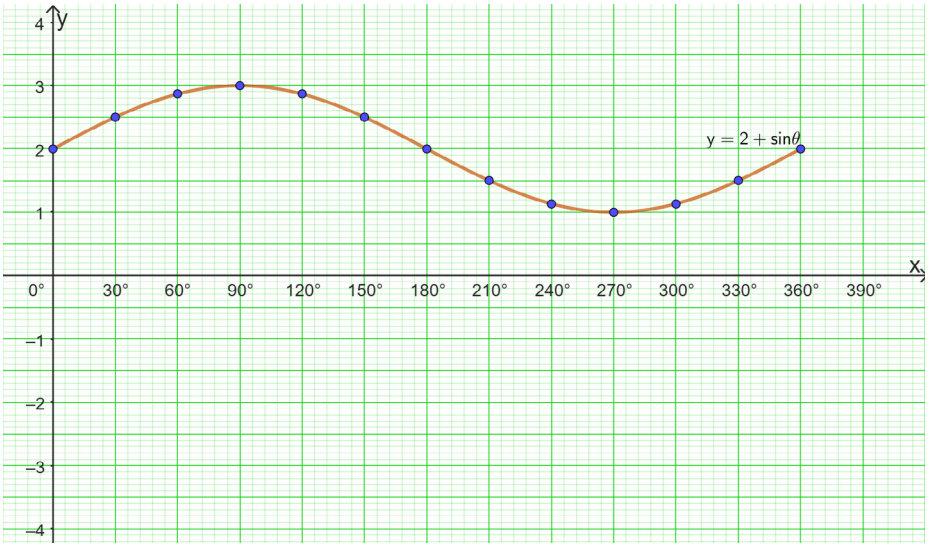


Figure 4.18: Graph of the relation y = 2 + sinθ for 0 0° ≤ θ ≤ 360°

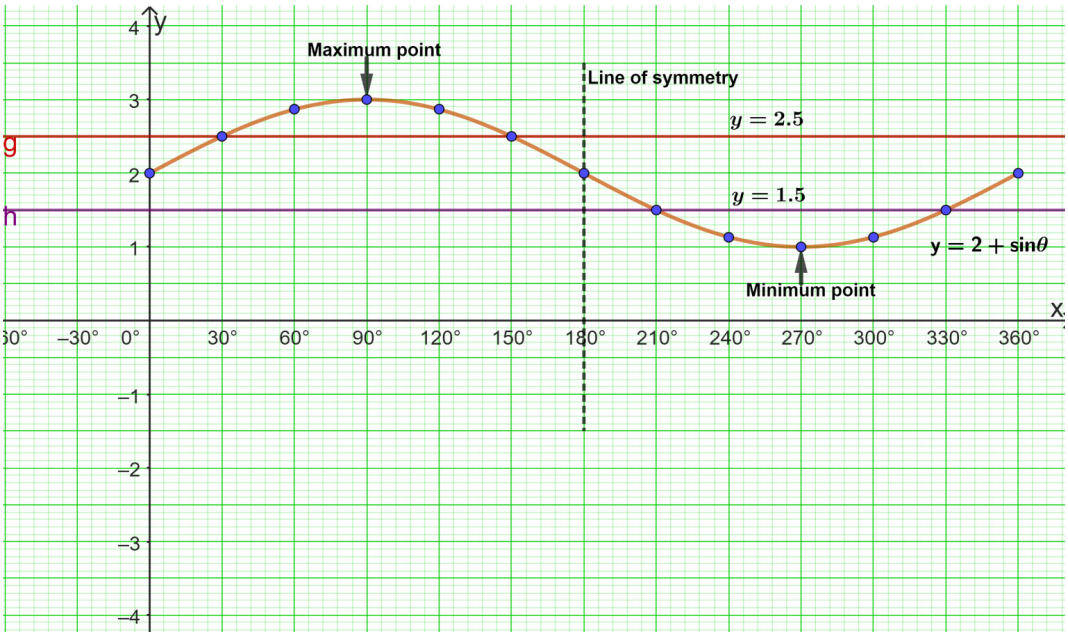


Figure 4.19: Graph of the relation y = 2 + sinθ for 0 0° ≤ θ ≤ 360° *(indicating various parts of the curve)*

b.

From your graph,

i. The maximum point is (90°, 3).

ii. The minimum point is (270°, 1).

c.

i. From the graph, the curve intersects the line $y = 2.5$ at $x \approx 30°$ and 150°

Hence, the truth set of the equation $y = 2.5$ is given by: $\{x : x = 30°$ and $150°\}$

ii. From the graph, the curve, and the line $y = 1.5$ intersect at $x = 230°$ and 330°

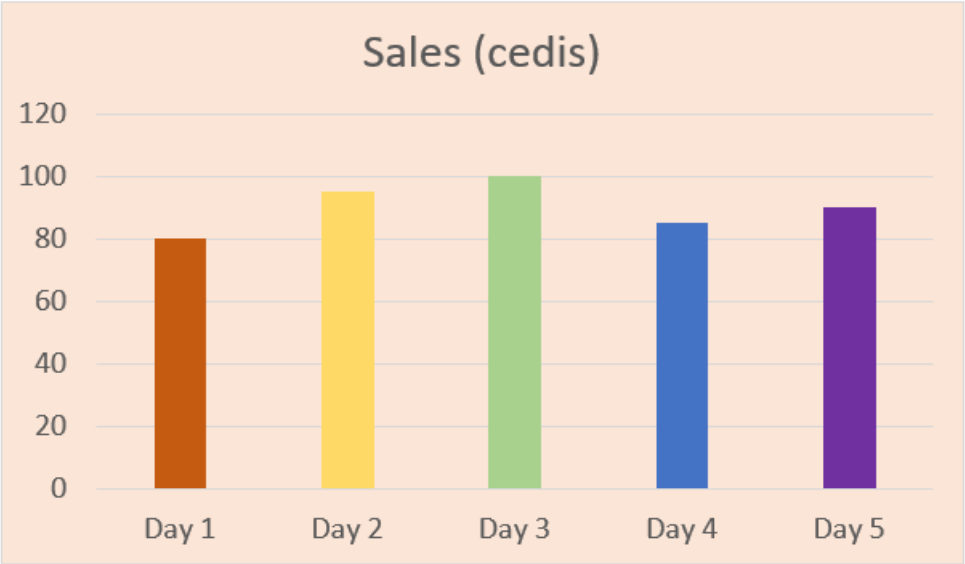
Hence, the truth set of the equation is $y = 1.5$ given by: $\{x : x = 230°$ and $330°\}$

SECTION FIVE

1.

a. Mean = 90; Median = 90.

b. Bar chart



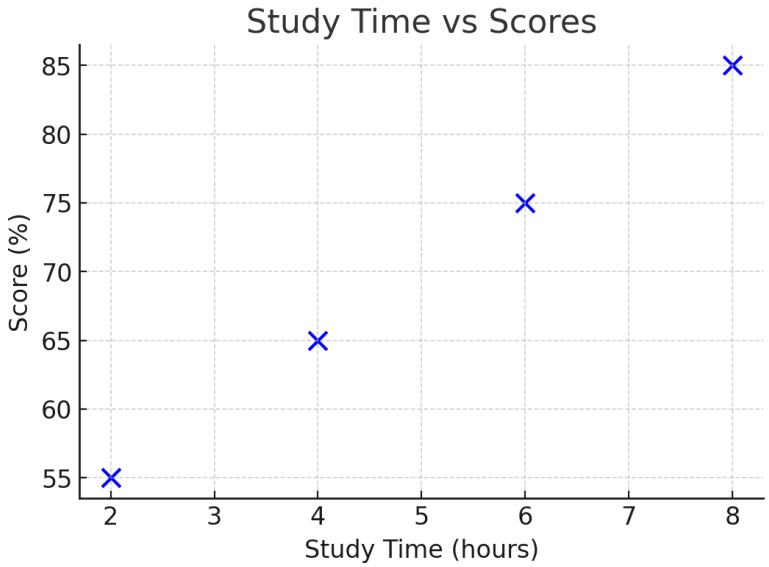
2.

a. Mode = 2.

b. Range = 3.

3.

a. Scatter points: **(2,55), (4,65), (6,75), (8,85)**.



b. **Relationship**: Strong positive linear correlation. This shows that the score increases by about 5 percentage points per hour of study (line: score = 5 × hours + 45).

4.

a. Learner:teacher ratios: A = 20:1, B = **23.33:1**, C = 18.75:1, D = 20:1 → **School B** has the highest learner:teacher ratio.

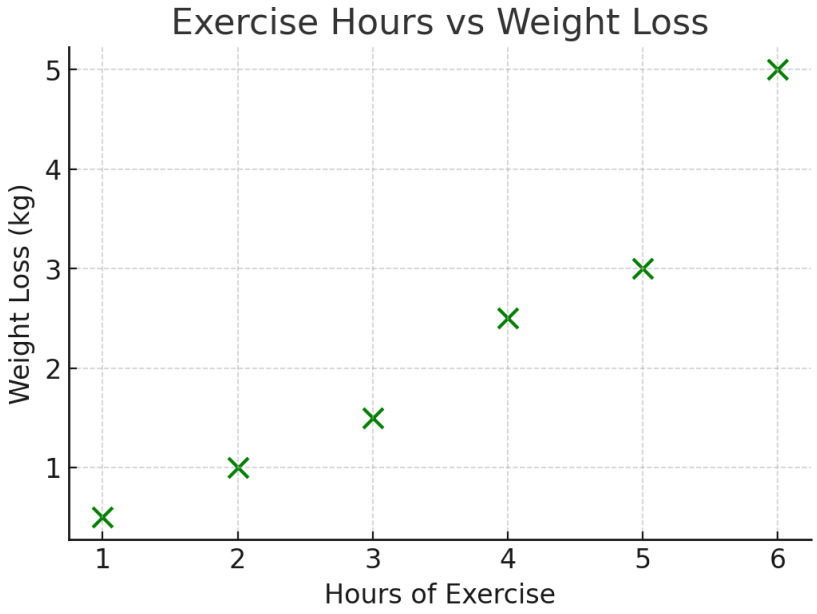
b. Yes there is a **positive correlation**, meaning that more teachers tend to be associated with more learners.

5.

i. Independent = **sleep**; Dependent = **exam performance**.

ii. Independent = **daily temperature**; Dependent = **sales of cold drinks**.

6. **Scatter points:** (1,0.5), (2,1.0), (3,1.5), (4,2.5), (5,3.0), (6,5.0)**.**



Relationship: Positive correlation, (non-linear - accelerating) meaning that weight loss rises as exercise hours increase, with a larger jump at higher hours.

7.

a. Thinking it means it will rain for 65% of the day or over 65% of Accra.

b. On days like tomorrow, rain happens about 65 times out of 100 (not guaranteed for you).

b. Carry an umbrella/raincoat, plan indoor backups, leave earlier, protect items from getting wet.

8.

a. Vaccinated people have about an 87% lower risk of getting malaria compared with unvaccinated people.

b. It guides personal protection and helps reduce illness and strain on hospitals across the community.

c. Believing it means 87% are completely immune or that 13% will definitely get malaria.

9.

a. In many similar matches, Medeama would win about 70% of them — winning this game isn't guaranteed.

b. Fans might feel confident; gamblers may favour Medeama or adjust bets/odds accordingly.

c. Assuming 70% means a sure win.

10.

a. $\frac{9}{25}$

b. $\frac{12}{25}$

11. 0.58

12.

a. 0.28

b. 0.72

13. 0.60

14. 0.80 (i.e., $\frac{4}{5}$)

15.

a. $\frac{1}{26}$

b. $\frac{13}{102}$

c. $\frac{95}{663}$

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GLOSSARY

Abductive Reasoning	Drawing conclusions based on the best available evidence, even if incomplete.
Addition rule	Used when finding the probability of one event *or* another.
Analysis	Breaking down data to understand patterns, trends or differences.
Angle	The space between two intersecting lines or surfaces, measured in degrees.
Asymptote	A line that a curve approaches but never touches, seen in tangent graphs.
Axis of symmetry	A vertical line that divides a parabola into two mirror-image halves.
Bisect	To divide something into two equal parts.
Chance	Everyday term for probability.
Circumcentre	The point where the perpendicular bisectors of a triangle meet; it is the centre of its circumscribed circle.
Circumscribed	A figure drawn around another, touching it at certain points (e.g., a circle around a polygon).
Completing the square	A method for solving quadratic equations by rewriting them in perfect square form.
Concentric Circles	Circles with the same centre but different radii.
Congruent	Having the same shape and size.
Conjunction	A compound statement formed by combining two statements with “and” ($\wedge$).
Control group	The group that does not get the treatment; used for comparison.

Correlation	How strongly and in what direction two variables are related (positive, negative, none).
Cosine Function	A trigonometric function that starts at its maximum value when the angle is 0°.
Cycle	One complete repetition of a trigonometric graph pattern.
Data collection	Gathering facts or numbers from observations or measurements.
Data set	A collection of related information or numbers.
Decision-making	Choosing actions based on the probability of outcomes.
Declarative Statement	A sentence that makes a factual claim or assertion.
Dependent events	Events where the outcome of one affects the outcome of another (e.g., picking cards without replacement).
Direct Variation	A relationship where one variable changes proportionally with another ($y = kx$).
Disjunction	A compound statement formed by combining two statements with "or" ($\vee$).
Domain	The set of all possible input values (x-values) for a function.
Domain	All possible input values (angles) for which a function is defined.
Equidistant	Being at the same distance from two or more points.
Event	An outcome or result in probability (e.g., tossing heads, drawing a red card).
Experiment	A planned study to test an idea or hypothesis.
Factorisation	Solving a quadratic equation by expressing it as a product of two binomial expressions.

General quadratic formula	A formula used to solve any quadratic equation: $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Graphical method	Solving equations by plotting their graphs and identifying key points such as intersections.
Independent events	Two events where one outcome does not affect the other (e.g., rolling a die twice).
Inference	A conclusion or judgment made from evidence in the data.
Inscribed Circle	A circle drawn inside a figure, touching all its sides.
Interpretation	Explaining what the data means in simple terms.
Interrogative Statement	A sentence that asks a question.
Inverse Variation	A relationship where one variable increases as the other decreases ($y = \frac{k}{x}$).
Joint Variation	A relationship where a variable depends on two or more others ($y = kxz$).
Line	A straight path with no thickness, extending endlessly in both directions.
Line of symmetry	Same as axis of symmetry; divides a graph (like a parabola) into symmetrical parts.
Line of Symmetry	A line dividing a graph into two identical mirror-image parts.
Logical Reasoning	The process of using structured steps to reach valid conclusions.
Maximum point	The highest point on a downward-opening parabola.
Measure of central tendency	Numbers like mean, median, mode that describe "average" behaviour.
Media	Communication channels like TV, newspapers, radio, internet.

Midline	The horizontal line halfway between the maximum and minimum values of a graph.
Minimum point	The lowest point on an upward-opening parabola.
Multiplication rule	Used when finding the probability of two events *both* happening.
Non-collinear	Points that do not lie on the same straight line.
Pair of Compass	A drawing tool used to make circles or arcs.
Parabola	The U-shaped or ∩-shaped curve formed by a quadratic function.
Partial Variation	A relationship with both a fixed and a variable component ($y = kx + c$).
Patterns	Recognisable or predictable arrangements in numbers or shapes, often used to identify relationships.
Period	The horizontal length over which a trigonometric graph completes one full cycle.
Perpendicular	Lines or surfaces meeting at a right angle (90°).
Phase Shift	A horizontal shift of a trigonometric graph from its usual position.
Phenomenon	An observable event or fact in society or nature.
Point	An exact location in space with no size or dimension.
Prediction	A statement about what is likely to happen based on probability.
Probability	The likelihood of an event happening, expressed as a fraction, decimal, or percentage.
Proportional Reasoning	Using ratios to compare quantities and solve problems.

Published data	Information shared publicly (e.g., survey results, health stats).
Quadratic equation	An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.
Quadratic function	A function defined by an equation of the form: $y = ax^2 + bx + c$.
Quality of data	How reliable, accurate, and useful a data set is for conclusions.
Range	All possible output values of a function.
Relationship	A link or connection between two variables (e.g., more study → higher marks).
Relationship	Finding if the treatment made a difference by comparing with the control.
Risk	The possibility of loss or harm depending on how probable an event is.
Roots	The solutions of a quadratic equation; the x-values where the graph intersects the x-axis.
Scatter graph (scatter plot)	A graph that shows data points for two variables to look for patterns.
Simultaneous equations	A set of equations with multiple variables that are solved together.
Sine Function	A trigonometric function that starts at zero and oscillates between -1 and 1.
Spread/Variation	How spread out or consistent the data is (range, variance, standard deviation).
Syllogism	A form of deductive reasoning with two premises and a conclusion.
Symbolic Logic	Representing logical statements with symbols for analysis.
Tangent Function	A trigonometric function with repeating patterns every 180° and vertical asymptotes.

Transformation	A change to the position or shape of a graph, such as shifting, stretching or reflecting.
Treatment group	The group that receives the special condition or change (e.g., new medicine).
Trend/Pattern	The overall direction data points seem to follow.
Truth Set	All the values that satisfy a given equation.
Variable	A quantity that can change (e.g., hours studied, test score).
Variation	Describes how one quantity changes in relation to another.
Venn Diagram	A visual tool to represent relationships between sets
Vertical Shift	An upward or downward movement of a graph